An analytic formula for numbers of restricted partitions from conformal field theory

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We study the correlators of irregular vertex operators in two-dimensional conformal field theory (CFT) in order to propose an exact analytic formula for calculating numbers of partitions, that is:

1) for given N, k, finding the total number $\lambda(N|k)$ of length k partitions of N: $N = n_1 + \cdots + n_k; 0 < n_1 \leq n_2 \leq \cdots \leq n_k$.

2) finding the total number $\lambda(N) = \sum_{k=1}^{N} \lambda(N|k)$ of partitions of a natural number N

We propose an exact analytic expression for $\lambda(N|k)$ by relating two-point short-distance correlation functions of irregular vertex operators in $c = 1$ conformal field theory (the form of the operators is established in this paper): with the first correlator counting the partitions in the upper half-plane and the second one obtained from the first correlator by conformal transformations of the form $f(z) = h(z)e^{-\frac{i}{z}}$ where $h(z)$ is regular and non-vanishing at $z = 0$.

The final formula for $\lambda(N|k)$ is given in terms of regularized (ϵ-ordered) finite series in the higher-derivative Schwarzians and incomplete Bell polynomials of the above conformal transformation at $z = i\epsilon$ ($\epsilon \to 0$)

1. Introduction

Let

$$N = n_1 + n_2 + \cdots + n_k (1 \leq k \leq N)$$

(1.1)
$$0 < n_1 \leq n_2 \leq \cdots \leq n_k$$

be the length k partition of a natural number N, $\lambda(N|k)$ be the number of such length k partitions of N and

(1.2)
$$\lambda(N) = \sum_{k=1}^{N} \lambda(N|k)$$

be the total number of partitions. Physically, $\lambda(N|k)$ and $\lambda(N)$ count the number of Young diagrams with N cells and k rows, and the total number $\lambda(N)$ of the diagrams with N cells, and therefore are related to counting irreducible representations for higher-spin fields with spin value N. As it is well-known from number theory, obtaining exact analytic expressions for $\lambda(N)$ and especially for $\lambda(N|k)$ (say, in terms of some finite series) is a hard long-standing problem. For $\lambda(N)$, various asymptotic formulae are known for the large N limit. The oldest and perhaps the best-known formula for $\lambda(N)$ was obtained by Ramanujan and Hardy in 1918 [1] and is given by:

$$\lambda(N) \sim \frac{1}{4N\sqrt{3}} e^{\pi\sqrt{\frac{2N}{3}}}$$

There are several improvements of this formula, notably by Rademacher [2], [3] who expressed $\lambda(N)$ in terms of infinite convergent series:

$$\lambda(N) = \frac{1}{\pi\sqrt{2}} \sum_{n=1}^{\infty} \sqrt{n}\alpha_n(N) \frac{d}{dN} \left\{ \frac{sinh[\frac{\pi}{n}\sqrt{\frac{2}{3}(N - \frac{1}{24})}]}{\sqrt{N - \frac{1}{24}}} \right\}$$

where

$$\alpha_n(N) = \sum_{0 \leq m \leq n;[m|n]} e^{i\pi(s(m,n) - \frac{2Nm}{n})}$$

with the notation $[m|n]$ implying the sum over m taken over the values of m relatively prime to n and

$$s(m,n) = \frac{1}{4n} \sum_{k=1}^{n-1} cot\left(\frac{\pi k}{n}\right) cot\left(\frac{\pi k m}{n}\right)$$

is the Dedekind sum for co-prime numbers.

The problem of finding $\lambda(N|k)$ is well-known to be even more tedious (see e.g. [14–16] for the discussion of Ramanujan-Rademacher type asymptotics for the restricted partitions). In this paper, we study the two-point short-distance correlator of irregular vertex operators in Conformal Field Theory [4, 9] that counts the number $\lambda(N|k)$ of restricted partitions, reproducing the well-known generating function for the partitions, when computed in the upper half-plane. One of these operators is the special case of rank one irregular vertex operators [5–8], that can be physically interpreted as a "dipole" in the Liouville theory (in the same sense that regular vertex operators, or

primary fields, are the "charges"); another is related to a class of analytic solutions in open string field theory [10, 11], interpolating between flat and AdS backgrounds.

Next, we investigate the behaviour of this correlator under the peculiar class of conformal transformations that shrink the dipole's size to zero and reduce the correlator to contribution from zero modes of the irregular vertices. This leads to nontrivial identities involving the restricted partitions, expressing them in terms of generalized higher-derivative Schwarzians of these conformal transformations. In particular, this allows to express the number $\lambda(N|k)$ of the restricted partitions in terms of the finite series of the generalized Schwarzians and incomplete Bell polynomials of the conformal transformations considered, leading to the main result of this work. Taking the short-distance limit in the correlation functions is necessary in order to be able to integrate the Ward identities, accounting for the non-global part of the two-dimensional conformal symmetry (or physically, the "spontaneous breaking" of the conformal symmetry for transformations with non-zero Schwarzians, considered in our work, i.e. other than fractional-linear). In general, such an integration is hard to perform and the correlators, computed in different coordinates (related by the transformation) differ by the infinite sum over Schwarzians and their higher-derivative counterparts. This difference, however, becomes controllable in the short-distance limit and, for the conformal transformations with the asymptotics, considered in this paper, can be compensated by a relatively simple factor, derived in our work.

2. Generating function for partitions: the correlator

As it is well-known, $\lambda(N)$ and $\lambda(N|k)$ can be realized as expansion coefficients of the following (respectively) generating functions:

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{N=0}^{\infty} \lambda(N)x^N$$

$$F(x,y) = \prod_{n=1}^{\infty} \frac{1}{1-yx^n} = \sum_{N=0,k=0;k\leq N}^{\infty} \lambda(N|k)x^N y^k$$

Unfortunately, these generating functions by themselves are not very helpful for elucidating explicit expressions for the partition numbers: taking their derivatives just gives trivial identities of the form $\lambda(N|k) = \lambda(N|k)$. For this reason, our strategy in this work will be to

1) identify the two-point correlators in Conformal Field Theory (CFT) counting the partitions (reproducing the generating function $F(x, y)$) in certain coordinates, namely, an upper half-plane

2) using the conformal symmetry and suitable conformal transformations (identified below), derive the identities for the generating function, casting it in terms of an expression, making it possible to obtain an exact analytic expression $\lambda(N|k)$ in terms of the finite series. For simplicity, in this paper we shall concentrate on $c = 1$ CFT (free massless bosons in two dimensions). With some effort, it is straightforward to identify the two-point correlator, counting the partitions on the upper half-plane. This correlator is given by

$$(2.1) \qquad G(\alpha, \beta, \epsilon) = <U_\alpha(z_1)V_\beta(z_2)>|_{z_1=i\epsilon; z_2=0}, \quad \epsilon > 0$$

where

$$U_\alpha(z) =: \prod_{n=0}^{\infty} \frac{1}{1 - \frac{\alpha^n \partial^n \phi}{n!}} : (z)$$
$$V_\beta(w) =: e^{\beta \partial \phi} : (w)$$

where the :: symbol stands for the normal ordering of operators in two-dimensional CFT, ϕ is $D = 2$ boson (e.g. a Liouville field, an open string's target space coordinate or a bosonized ghost), $\epsilon \to 0$, α and β are the parameters that are introduced to control the generating function for the partitions. In this work, both U_α and V_β are understood in terms of formal series in α and β, with each term in the series being normally ordered by definition. To simplify notations, here and below we shall often use the partial derivative symbol for z-derivatives, even though in our case it coincides with ordinary derivative, since we only consider holomorphic sector.

Indeed, expanding in α:

$$U_\alpha = \sum_{k=1}^{\infty} \sum_{n_1 \leq \cdots \leq n_k = 0}^{\infty} \sum_{p_1, \ldots, p_k} \frac{\alpha^{p_1 n_1 + p_2 n_2 + \cdots + p_k n_k}}{(n_1!)^{p_1} \cdots (n_k!)^{p_k}}$$
$$\times : (\partial^{n_1} \phi)^{p_1} \cdots (\partial^{n_k} \phi)^{p_k} :$$

using the operator product expansion (OPE):

$$\partial^n \phi(z) : e^{\beta \partial \phi} : (w) \sim \frac{(-1)^{n+1} n! \beta : e^{\beta \partial \phi} : (w)}{(z - w)^{n+1}} + \cdots$$

and introducing $N = \sum p_k n_k$ one easily calculates

$$G(\alpha, \beta | \epsilon) =< U_\alpha(z) V_\beta(w) > |_{z=i\epsilon; w=0}$$

$$= \sum_{[N|n_1 \cdots n_k]=0}^{\infty} \sum_{k=0}^{N} \frac{\alpha^N \beta^k \lambda(N|k)}{(w-z)^{N+k}} = \prod_{n=1}^{\infty} \frac{1}{1 - \tilde{\alpha}^n \tilde{\beta}},$$

$$\tilde{\alpha} = \frac{\alpha}{w-z}; \quad \tilde{\beta} = \frac{\beta}{w-z}$$

i.e. G is the generating function for restricted partitions with

$$\lambda(N|k) = \frac{(-i\epsilon)^{N+k}}{N!k!} \partial_\alpha^N \partial_\beta^k G(\alpha, \beta | \epsilon)|_{\alpha, \beta=0}$$

Now that we have identified the correlator generating $\lambda(N|k)$, the next step is to identify the suitable conformal transformation. Note that the operator V_β is the special case of rank one irregular vertex operator [8], creating a simultaneous eigenstate of Virasoro generators L_1 and L_2 (with eigenvalues 0 and $\frac{\beta^2}{2}$ respectively) and physically can be understood as a dipole with the size β. For this reason, it is natural to choose the transformation such that the dipole's size shrinks to zero in the new coordinates. So we will consider the conformal transformations of the form

(2.2) $$z \rightarrow f(z) = h(z)e^{-\frac{i}{z}}$$

where $h(z)$ is regular and at 0, $h(0) \neq 0$ and it it is smooth and analytic in the upper half-plane (perhaps except for infinity) In particular, it is instructive to consider $h(z) = 1$ and $h(z) = cos(z)$. Now we have to:

   1. Compute infinitezimal transformations of U_α and V_β.

   2. Integrate them to get the finite transformations for U_α and V_β under $f(z)$.

   3. Since $f(z)$ is not a fractional-linear transformation, and its Schwarzian is singular at 0, to match the correlators in different coordinates, one has to take into account the "spontaneous symmetry breaking" of the conformal symmetry (with higher Virasoro modes playing the role of "Goldstone modes"), by integrating the Ward identities for $f(z)$ and regularizing the final expression, in order to ensure that the correlators computed in two coordinates match upon $f(z)$.

3. Partition-counting correlator: the conformal transformations

An important building block in our computation involves the finite conformal transformation laws for the operators of the form $T^{(n_1,n_2)} = \frac{1}{n_1!n_2!} :$
$\partial^{n_1}\phi\partial^{n_2}\phi :$ of conformal dimensions $n_1 + n_2$ - in fact, the final answer for the number of partitions will be expessed in terms of the finmite series in the higher-derivative Schwarzians of $f(z) = e^{-\frac{i}{z}}$.

In case of $n_1 = n_2 = 1$ the operator $T^{(1,1)}$ (up to normalization constant of $\frac{1}{2}$) is just the stress-energy tensor, and both its infinitezimal and finite transformation laws are well-known.

The infinitezimal transformation of $T^{(1,1)}$ under $z \to z + \epsilon(z)$ is

$$(3.1) \qquad \delta_\epsilon T^{(1,1)}(z) = \epsilon\partial T^{1,1}(z) + 2\partial\epsilon T^{(1,1)}(z) + \frac{1}{6}\partial^3\epsilon(z)$$

(here and everywhere below the infinitezimal conformal transformation parameter $\epsilon(u)$ is not to be confused with the $i\epsilon$ for the location of z which hopefully will always be clear from the context; in our notations the former always will appear with the argument, while the latter will not). This infinitezimal transformation can be integrated to give the finite conformal transformation law for any $f(z)$ according to

$$(3.2) \qquad T^{(1,1)}(z) \to \left(\frac{df(z)}{dz}\right)^2 T^{(1,1)}(f(z)) + S_{1|1}(f;z)$$

where $S_{1|1}$ is (up to the conventional normalization factor of $\frac{1}{6}$) the Schwarzian derivative, defined according to:

$$(3.3) \qquad S_{1|1}(f;z) = \frac{1}{6}\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{12}\left(\frac{f''(z)}{f'(z)}\right)^2$$

The integrated transformation (3.3) can be obtained from (3.2) e.g. by requiring that (3.2) is reproduced from (3.3) in the infinitezimal limit and that the composition of two transformations $f(z)$ and $g(z)$ gives again the conformal transformation with $f(g(z))$. Likewise, we can derive the transformation rules for an arbitrary $T^{(n_1,n_2)}$. The infinitezimal transformation

is

$$(3.4) \qquad \delta_\epsilon T^{(n_1, n_1)}(z) = \left[\frac{1}{2} \oint \frac{du}{2i\pi} \epsilon(u) : (\partial \phi)^2 : (u); T^{(n_1, n_1)}(z) \right]$$

$$= \frac{1}{n_2!} : \partial^{n_1}(\epsilon \partial \phi) \partial^{n_2} \phi : (z)$$

$$+ \frac{1}{n_1!} : \partial^{n_1} \phi \partial^{n_2}(\epsilon \partial \phi) : (z)$$

$$+ \frac{1}{(n_1 + n_2 + 1)!} \partial^{n_1 + n_2 + 1} \epsilon(z)$$

and can be integrated, by imposing the similar requirements, to give:

$$(3.5) \qquad T^{(n_1, n_2)}(z) \to \frac{1}{n_1! n_2!} : \delta_{n_1}(\phi; f(z)) \delta_{n_2}(\phi; f(z)) : + S_{n_1 | n_2}(f; z)$$

where the operators

$$(3.6) \qquad \delta_n(\phi, f) = \frac{\partial^n f}{\partial z^n} \partial \phi(f(z))$$

$$+ \sum_{k=1}^{n-1} \sum_{l=1}^{k} \frac{(n-1)!}{(n-1-k)!} \frac{\partial^{n-k} f}{\partial z^{n-k}} B_{k|l}(f(z); z) \partial^{l+1} \phi(f(z))$$

$$= \sum_{k=1}^{n} B_{n|k}(f(z); z) \partial^k \phi$$

are defined by the conformal transformation for $\partial^n \phi$. Here $B_{n|k}$ are the incomplete Bell polynomials in the derivatives (expansion coefficients) of f (see (3.8) and the formula below for the explicit definition) and $S_{n_1 | n_2}(f; z)$ are the generalized higher-derivative Schwarzians. To calculate $S_{n_1 | n_2}$ we cast the normal ordering according to:

$$\frac{1}{n_1! n_2!} : \partial^{n_1} \phi \partial^{n_2} \phi : (z)$$

$$= lim_{\epsilon \to 0} \left\{ \partial^{n_1} \phi \left(z + \frac{\epsilon}{2} \right) \partial^{n_2} \phi \left(z - \frac{\epsilon}{2} \right) + \frac{(-1)^{n_1} (n_1 + n_2 - 1)!}{n_1! n_2! \epsilon^{n_1 + n_2}} \right\}$$

(again, the regularization parameter ϵ here is not to be confused with $i\epsilon$ for the location of U_α and/or infinitezimal transformation parameter $\epsilon(z)$)

Under the conformal map $z \to f(z)$, this expression transforms according to

$$\frac{1}{n_1!n_2!} : \partial^{n_1}\phi\partial^{n_2}\phi : (z)$$

$$\to \frac{1}{n_1!n_2!} lim_{\epsilon\to 0}\left\{ \sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} B_{n_1|k_1}\left(f\left(z+\frac{\epsilon}{2}\right);z\right)B_{n_k|k_2}\left(f\left(z-\frac{\epsilon}{2}\right);z\right)\right.$$

$$\times\left(\partial^{k_1}\phi\left(z+\frac{\epsilon}{2}\right)\partial^{k_2}\phi\left(z-\frac{\epsilon}{2}\right)\right.$$

$$\left.\left.+\frac{(-1)^{k_1}(k_1+k_2-1)!}{(f(z+\frac{\epsilon}{2})-f(z-\frac{\epsilon}{2}))^{k_1+k_2}}+\frac{(-1)^{n_1}(n_1+n_2-1)!}{n_1!n_2!\epsilon^{n_1+n_2}}\right)\right\}$$

Expanding in ϵ, we extract (upon cancellations of the divergent terms) the higher-order Schwarzians to be given by:

$$(3.7)\qquad S_{n_1|n_2}(f;z)$$

$$= \frac{1}{n_1!n_2!}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\sum_{m_1\geq 0}\sum_{m_2\geq 0}\sum_{p\geq 0}\sum_{q=1}^{p}(-1)^{k_1+m_2+q}2^{-m_1-m_2}(k_1+k_2-1)!$$

$$\times\frac{\partial^{m_1}B_{n_1|k_1}(f(z);z)\partial^{m_2}B_{n_2|k_2}(f(z);z)B_{p|q}(g_1,\ldots,g_{p-q+1})}{m_1!m_2!p!(f'(z))^{k_1+k_2}}$$

$$g_s = 2^{-s-1}(1+(-1)^s)\frac{\frac{d^{s+1}f}{dz^{s+1}}}{(s+1)f'(z)};\quad s=1,\ldots,p-q+1$$

with the sum over the non-negative numbers m_1, m_2 and p taken over all the combinations satisfying

$$m_1+m_2+p = k_1+k_2.$$

Here, in (3.6) and (3.7) $B_{n|k}(g_1,\ldots,g_{n-k+1})$ are the incomplete Bell polynomials, defined according to:

$$(3.8)\qquad B_{n|k}(g_1,\ldots,g_{n-k+1})$$

$$= n!\sum\frac{1}{p_1!\cdots p_{n-k+1}!}\left(\frac{g_1}{1!}\right)^{p_1}\cdots\left(\frac{g_{n-k+1}}{(n-k+1)!}\right)^{p_{n-k+1}}$$

with the sum taken over all the non-negative $p_1,\ldots,p_{n-k+1}$ satisfying

$$p_1+\cdots+p_{n-k+1} = k$$

$$p_1+2p_2+\cdots+(n-k+1)p_{n-k+1} = n$$

In particular, the incomplete Bell polynomials $B_{n|k}(f; z)$ in the derivatives (or the expansion coefficients) of $f(z)$, are given by:

$$B_{n|k}(f(z); z) = n! \sum_{n|n_1 \cdots n_k} \frac{\partial^{n_1} f(z) \cdots \partial^{n_k} f(z)}{n_1! \cdots n_k! q(n_1)! \cdots q(n_k)!}$$

with the sum $n|n_1 \cdots n_k$ taken over all ordered $0 < n_1 \le n_2 \cdots \le n_k$ length k partitions of n and with $q(n_j)$ denoting the multiplicity of n_j element of the partition (e.g. for the partition $7 = 2 + 2 + 3$ we have $q(2) = 2, q(3) = 1$, so the appropriate term would read $\sim \frac{\partial^2 f \partial^2 f \partial^3 f}{2!2!3! \times 2!1!}$). Note that, just as the ordinary Schwarzian satisfies the well-known composite relation for any combination of conformal transformations $z \to f(z) \to g(f)$:

$$S_{1|1}(g(z); z) = \left(\frac{df}{dz}\right)^2 S_{1|1}(g(f); f) + S_{1|1}(f(z); z),$$

the generalized Schwarzians $S_{n_1|n_2}$ also satisfy the composite relations

$$S_{1|1}(g(z); z) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} B_{n_1|k_1}(f(z); z) B_{n_2|k_2}(f(z); z) S_{k_1|k_2}(g(f); f)$$
$$+ S_{n_1|n_2}(f(z); z).$$

(see also [12, 13] who considered the alternative types of higher-order Schwarzians in a rather different context).

Now let us apply the same procedure to the irregular vertex operators U_α and V_β in the partition-counting correlator (2.3). The straightforward computation of the infinitezimal transforms gives:

$$(3.9) \quad \delta_\epsilon V_\beta = \left[\oint \frac{du}{2i\pi} \epsilon(u) T(u); V_\beta(z) \right] =: \left(\beta \partial \epsilon \partial \phi + \frac{1}{12} \beta^2 \partial^3 \epsilon \right) V_\beta : (z)$$

and

$$(3.10) \quad \delta_\epsilon U_\alpha(z) = \left[\oint \frac{du}{2i\pi} \epsilon(u) T(u); U_\alpha(z) \right]$$
$$= \sum_{n=1}^{\infty} \left\{ : \frac{\alpha^n (\partial^n (\epsilon \partial \phi))}{n! (1 - \frac{\alpha^n \partial^n \phi}{n!})} \prod_{N=1}^{\infty} \frac{1}{1 - \frac{\alpha^N \partial^N \phi}{N!}} : \right.$$
$$+ : \frac{\alpha^{2n} \partial^{2n+1} \epsilon}{(2n+1)! (1 - \frac{\alpha^n \partial^n \phi}{n!})^2} \prod_{N=1}^{\infty} \frac{1}{1 - \frac{\alpha^N \partial^N \phi}{N!}} : \right\}$$
$$+ \sum_{0 \le n_1 < n_2 < \infty} : \frac{\alpha^{n_1 + n_2} \partial^{n_1 + n_2 + 1} \epsilon}{(1 - \frac{\alpha^{n_1} \partial^{n_1} \phi}{n_1!})(1 - \frac{\alpha^{n_2} \partial^{n_2} \phi}{n_2!})} \prod_{N=1}^{\infty} \frac{1}{1 - \frac{\alpha^N \partial^N \phi}{N!}} :$$

Integrating these infinitezimal transformations, we obtain the transformations of U_α and V_β for the finite conformal transformation $f(z) = h(z)e^{-\frac{i}{z}}$. For V_β, we get

$$(3.11) \qquad V_\beta(w)|_{w=i\epsilon} \rightarrow :e^{\beta \frac{\partial f}{\partial z}\phi(f(z)) + \frac{\beta^2}{2}S_{1|1}(f(z);z)}: \big|_{f(z)=h(z)e^{-\frac{i}{z}}}$$

To determine the transformation law for U_α, it is convenient to cast U_α as

$$(3.12) \qquad U_\alpha = 1 + \sum_{N=1}^{\infty} \sum_{\{m_i\}} \alpha^N \prod_{n=1}^{N} \left(\frac{\partial^n \phi}{n!}\right)^{m_n}$$

with the sum over $\{m_i\}; i = 1, \ldots, N$ being taken over all the combinations of non-negative $\{m_i\}$, satisfying

$$\sum_{n=1}^{N} n m_n = N$$

Now introduce the *exchange numbers* $\nu_{ij} \geq 0 (i, j = 0, \ldots, N)$ satisfying

$$\nu_{00} = 0$$
$$\nu_{ij} = \nu_{ji}$$
$$(3.13) \qquad \nu_{jj} + \sum_{i=0}^{N} \nu_{ij} = m_j$$

in order to parametrize the internal normal ordering procedure for U_α as follows:

1. $\nu_{ij} = \nu_{ji}(i, j \neq 0)$ defines the number of internal couplings between $(\partial^i \phi)^{m_i}$ and $\partial^j \phi^{m_j}$ factors, creating internal singularities prior to the normal ordering;

2. ν_{ii} defines the number of intrinsic same-derivative couplings between $\partial^i \phi$'s inside each factor $(\partial^i \phi)^{m_i}$.

3. ν_{i0} counts the numbers of $\partial^i \phi$-operators left inside $(\partial^i \phi)^{m_i}$-block, that do not participate in the contractions.

Since each coupling between $\partial^i \phi$ and $\partial^j \phi$ contributes the factor $i!j!S_{i|j}(f;z)$ to the transformation law under $f(z)$, the overall transformation law for U_α is

$$(3.14) \quad U_\alpha(z) = \prod_n : \frac{1}{1 - \frac{\alpha^n \partial^n \phi}{n!}} :$$

$$\to 1 + \sum_{N=1}^\infty \sum_{q=1}^N \sum_{m_1,\dots,m_q;\{\nu_{ij}\}} \alpha^N \prod_{p=1}^q \frac{(\delta_p(\phi;f))^{\nu_{p0}} m_p! (S_{p|p})^{\nu_{pp}}}{\nu_{p_0}! (2\nu_{pp})!!}$$

$$\times \prod_{1 \leq i < j \leq q} \frac{(S_{i|j})^{\nu_{ij}}}{\nu_{ij}!}$$

where

$$(3.15) \qquad \delta_p(\phi;f) = \sum_{k=1}^p B_{p|k}(f(z);z)\partial^k \phi(z)$$

Finally, since the Schwarzian of the conformal transformation $f(z)$ iz nonzero, we need to account for the spontaneous breaking of the conformal symmetry by integrating the Ward identities, in order to match the partition-counting correlators $< U_\alpha V_\beta >$ in different coordinates. For that, we first have to integrate the infinitezimal "overlap" deformation of the correlator, emerging from the contraction of one of $\partial\phi$'s in the stress-energy tensor with U_α and another with V_β. The infinitezimal overlap deformation is given by the integral:

$$(3.16) \qquad \delta_{overlap} < U_\alpha(z)V_\beta(w) > |_{z=i\epsilon;w=0}$$

$$= \sum_{N=1}^\infty \oint \frac{du}{2i\pi} \frac{\epsilon(u)}{(u-z)^{N+1}(u-w)^2}$$

$$\times : \frac{\alpha^N \beta}{(1 - \frac{\alpha^N \partial^N \phi}{N!})^2} \prod_{n=1;n\neq N}^\infty \frac{1}{1 - \frac{\alpha^n \partial^n \phi}{n!}} : (z) : e^{\beta\partial\phi} : (w)|_{z=i\epsilon;w=0}$$

$$= \sum_{N=1}^\infty \left(\partial_z^N \left[\frac{\epsilon(z)}{(z-w)^2} \right] + (-1)^{N+1}\partial_w \left[\frac{\epsilon(w)}{(z-w)^{N+1}} \right] \right)$$

$$\times : \frac{\alpha^N \beta}{(1 - \frac{\alpha^N \partial^N \phi}{N!})^2} \prod_{n=1;n\neq N}^\infty \frac{1}{1 - \frac{\alpha^n \partial^n \phi}{n!}} : (z) : e^{\beta\partial\phi} : (w)|_{z=i\epsilon;w=0}$$

This infinitezimal deformation is straightforward to integrate for the class of the conformal transformations (2.2). The overall integrated transformation for U_α under $f(z)$, with the overlap deformation included, is then

given by

(3.17)

$$U_\alpha(z) = \prod_n : \frac{1}{1 - \frac{\alpha^n \partial^n \phi}{n!}} : \rightarrow$$

$$\left(1 + \sum_{N=1}^\infty \sum_{q=1}^N \sum_{m_1,\ldots,m_q;\{\nu_{ij}\}} \alpha^N \prod_{p=1}^q \frac{(\delta_p(\phi;f))^{\nu_{p0}} m_p! (S_{p|p})^{\nu_{pp}}}{\nu_{p_0}! (2\nu_{pp})!!} \prod_{1 \le i < j \le q} \frac{(S_{i|j})^{\nu_{ij}}}{\nu_{ij}!}\right)$$

$$\times \prod_{n=1}^\infty \frac{1}{1 - \frac{\alpha^n \beta}{n!} \frac{D_n(f(z);z)}{(1 - \alpha^n \delta_n(\phi,f))}}$$

where

$$(3.18) \qquad D_n(f(z);z) \equiv D(n) = -i \sum_{k=1}^n (-1)^{k+1} k! \frac{B_{n|k}(f(z);z)}{(f(z) - f(0))^k}$$

(to abbreviate notations, below we will also use the symbol $D(n)$ for $D_n(f(z);z)$) For the conformal transformations of the form $f(z) = h(z)e^{-\frac{i}{z}}$ the overall transformation law for $V_\beta(z)$ remains the same up to terms that vanish identically at $z = 0$:

$$(3.19) \qquad V_\beta(w)|_{w=i0} \rightarrow e^{\beta \frac{df}{dz} \partial \phi(f(z)) + \frac{\beta^2}{2} S_{1|1}(f(z);z)}\Big|_{f(z)=h(z)e^{-\frac{i}{z}}}$$

Now that we are prepared to calculate the partition-counting correlator in the new coordinates, here comes the crucial part. The dipole's size in the new coordinates is

$$(3.20) \qquad \beta \frac{df}{dz}|_{z \rightarrow 0} \rightarrow 0$$

and shrinks to zero with our choice of $f(z)$. This drastically simplifies the calculation. While the operator U_α looks extremely cumbersome in the new coordinates (particularly, because of the complexities involving $\delta_n(\phi;f)$-operators), any contractions of derivatives of ϕ in U_α with V_β bring down the factors proportional to $f'(z)$ and therefore vanish for the conformal transformations of the form (2.2). As a result, only the zero modes of U_α and V_β contribute to the correlator in the new coordinates. Technically, this implies $\nu_{0j} = 0$ for all j. The correlator is then easily computed to give the generating function for the restricted partitions in terms of higher-derivative

Schwarzians and incomplete Bell polynomials:

(3.21) $G(\alpha, \beta | \epsilon) \equiv\ <U_\alpha(z)V_\beta(w)> |_{z=i\epsilon, w=0}$

$$= e^{\frac{1}{2}\beta^2 S_{1|1}(f(w);w)}|_{w=0}$$

$$\times \left(1 + \sum_{N=1}^{\infty}\sum_{q=1}^{N}\sum_{m_1,\ldots,m_q;\{\nu_{ij}\}} \alpha^N \prod_{p=1}^{q} \frac{m_p!(S_{p|p})^{\nu_{pp}}}{(2\nu_{pp})!!} \prod_{1\le i<j\le q} \frac{(S_{i|j})^{\nu_{ij}}}{\nu_{ij}!}\right)$$

$$\times \prod_{n=1}^{\infty} \frac{1}{1 - \frac{\alpha^n\beta}{n!}D(n)}\Big|_{z=i\epsilon}$$

The overall constant (ϵ-independent) factor of $e^{\frac{1}{2}\beta^2 S_{1|1}(f;w)}|_{w=0}$ is related to the Casimir energy associated with the conformal transformation $f(w)$. It is irrelevant and disappears when the correlator is normalized with the inverse of the partition function of the system. The generation function for the partition numbers is then simply obtained by replacing this factor with 1.

Now the final step is to take the derivatives of G in α, β and to ϵ-order the result, retaining the finite terms as ϵ is set to 0. Straightforward calculation gives:

(3.22)

$$\lambda(N|Q) = (-i\epsilon)^{N+Q} : \sum_{N_1=0}^{N}\sum_{N_2=0}^{N-N_1}\sum_{\{m_j\ge 0\}}\sum_{\{n_j;p_j\ge 0\}}\sum_{\{\nu_{ij}\ge 0;\}}$$

$$\prod_{k=1}^{N_1}\prod_{q,r;1\le q<r\le N_1} : \frac{m_k!(S_{k|k})^{\nu_{kk}}(S_{q|r})^{\nu_{qr}}}{(2\nu_{kk})!\nu_{qr}!}(D(n_1))^{p_1}\cdots(D(n_Q))^{p_Q} :_{N+Q}$$

with the summations/products taken over the non-negative integer values of

$$N_1, N_2, N_3; m_1, \ldots, m_{N_1}; n_1, \ldots, n_Q; p_1, \ldots, p_Q; \{\nu_{ij}\}; 1\le i,j\le N_1$$

satisfying:

$$N_1 + N_2 = N, \quad \sum_{j=1}^{N_1} jm_j = N_1, \quad \sum_{j=1}^{s} n_jp_j = N_2$$

(3.23) $$\nu_{ii} + \sum_{j=1}^{N_1}\alpha_{ij} = m_i; 1\le i\le m_i, \quad \nu_{ij} = \nu_{ji}$$

The ϵ-ordering symbol $: \cdots :_{N+Q}$ in each monomial term of the sum $\sim:$ $S \cdots S D \cdots D :_{N+Q}$ by definition only retains the terms of the order of ϵ^{-N-Q} upon the evaluation of each product $S \cdots S D \cdots D$, in order to ensure that the overall contribution is finite, upon multiplication by $(-i\epsilon)^{N+Q}$ (we refer to this procedure as ϵ-ordering to distinguish it from the usual normal ordering defined for operators in CFT). Let us stress that, since each S or D has the finite and definite singularity order in ϵ, the overall result for $\lambda(N|Q)$ is the exact analytic expression, given by the finite series, uniquely determined by the structures of S and D for each f (with f satisfying the constraints described above). This concludes our derivation of counting the restricted partitions, expressed in terms of finite series in the incomplete Bell polynomials and the generalized higher-derivative Schwarzians of the defining conformal transformation $f(z) = h(z)e^{-\frac{i}{z}}$.

4. Conclusion. Tests and comments

Having presented the exact analytic expressionb for the number of the partitions, in this section we will provide some checks and examples of how the expression (3.22), constituting the main result of this paper, works in practice. First of all, it is quite straightforward to demonstrate that the expression (3.22) leads to the correct answer for any partition number in the case of the conformal transformation $f(z)$ with the simplest choice $h(z) = 1$. Let us start from the most elementary case of the maximal length partition where obviously $\lambda(N|N) = 1.$ for any N. Indeed, according to (3.22), one has

$$(4.1) \qquad \lambda(N|N) = (-i\epsilon)^{2N} : (D(1))^N :_{2N} = (-i\epsilon)^{2N}(-i)^N(-\frac{i}{\epsilon^2})^N = 1$$

Similarly, it is easy to verify the case $1 = \lambda(N-1|N)$ Indeed,

$$(4.2) \qquad \lambda(N|N-1) = (-i\epsilon)^{2N-1} : (D(1))^{N-2}D(2) :_{N-1}$$
$$=: (-i\epsilon)^{2N-1}(-\frac{i}{\epsilon})^{2N-4}(\frac{i}{\epsilon^3} - \frac{1}{2\epsilon^4}) + O(\epsilon) :_{2N-1} = 1$$

Note (although irrelevant to our result) the appearance of the singular term $\sim \frac{1}{\epsilon}$ at this level (which was absent in the case of $\lambda(N|N)$). This term disappears upon the ϵ-ordering procedure and is of no significance for our purposes, but the very reason for its emergence is also related to the Schwarzian singularities at $\epsilon \to 0$. It is easy to check that the results (4.1), (4.2) actually hold for any smooth regular $h(z)$ satisfying the conditions

defined above, not just for $h = 1$. However, the case of $h = 1$ is the easiest one to verify the correctness of (3.22) for any partition, as this can be done by simple analysis of the ϵ-dependence. Indeed, in general, each term for the partition number $\Lambda(N|Q)$ in (3.22) typically consists of Q $D(n)$-factors and R S-factors (Schwarzians of all kinds and orders), where R can in principle vary as $0 \leq R \leq [\frac{N-Q}{2}]$ However, in case of $h(z) = 1$ only the terms with $R = 0$ contribute. Indeed, the terms in each of the Schwarzians $S_{n_1|n_2}$, least singular in ϵ, are of the order of $\frac{1}{\epsilon^{n_1+n_2+2}}$ (e.g. $S_{1|1}(i\epsilon) = \frac{1}{12\epsilon^4}$. On the other hand, the $D(n)$-factors, consisting of combinations of incomplete Bell polynomials $B_{n|k}(f(z); z)$ with various k's, have the lowest singularity order $\sim \frac{1}{\epsilon^{n+1}}$ for $h = 1$. Thus it is clear that each contribution with nonzero R, upon multiplication by $(i\epsilon)^{N+Q}$ has the singularity order of at least $\frac{1}{\epsilon^R}$ and will disappear upon the normal ordering $: \cdots :_{N+Q}$. Furthermore, the only source of the lowest singularity terms in $D(n)$ of the order of $\frac{1}{\epsilon^{n+1}}$ is $B_{n|1}$, as all other $B_{n|k}$ with $k > 1$ are more singular, as it is easy to check. These terms stem from the derivatives $\partial^n(e^{-\frac{i}{z}})|_{z=i\epsilon}$ and are easily computed to be given by (skipping terms with the higher order singularities, not contributing to the ϵ-ordering)

$$-\frac{i}{n!}(e^{\frac{i}{z}})\partial^n(e^{-\frac{i}{z}})|_{z=i\epsilon} = \left(\frac{i}{\epsilon}\right)^{n+1} + h.s.$$

Thus each of the terms with $R = 0$:

$$\sim (-i\epsilon)^{N+Q} : D(n_1) \cdots D(n_Q) :_{N+Q} (N = n_1 + \cdots + n_Q)$$
$$= (-i\epsilon)^{N+Q}(\frac{i}{\epsilon^{N+Q}}) = 1$$

contributes 1 to the sum. But the number of such terms obviously equals the number of partitions $\lambda(N|Q)$, hence this constitutes the proof that the formula (3.22) works correctly with the conformal transformation $f(z) = e^{-\frac{i}{z}}; h(z) = 1$. Although the case of $h(z) = 1$ is somewhat simplistic (e.g. with no Schwarzians entering the game), this by itself is already a non-trivial check of how the conformal invariance works in (3.22). Of course, with $h(z) \neq 1$ things change significantly and the Schwarzians of all orders contribute nontrivially to the expression (3.22) for the partitions. For example, consider $h(z) = cos(z)$ and $\lambda(4|2) = 2$. In this case, the Schwarzian $S_{1|1}$ is given by

(4.3) $$6S_{1|1}|_{z=i\epsilon} = \frac{1}{2z^4} + \frac{2i}{z} = \frac{1}{2\epsilon^4} + \frac{2}{\epsilon} + O(\epsilon)$$

and does of course contribute to the normal ordering in general (the same is true for other $S_{n_1|n_2}$'s). According to (3.22) we have

$$(4.4) \quad \lambda(4|2) = (-i\epsilon)^6[: S_{1|1}(D(1))^2 :_6 + : (D(2))^2 :_6 + : D(1)D(3) :_6]$$

Straightforward calculation gives:

$$(-i\epsilon)^6 : S_{1|1}(D(1))^2 :_6 = 0$$
$$(-i\epsilon)^6 : (D(2))^2 :_6 = 1$$
$$(-i\epsilon)^6 : D(3)D(1) :_6 = 1$$
$$(4.5) \qquad \lambda(4|2) = 2$$

For this partition, the Schwarzian related term still does not contribute, although its vanishing is not automatic but is related to the particular ϵ-structure of the Schwarzian $S_{1|1}(f(z); z)$ for $h(z) = cos(z)$. For $\lambda(5|2)$ one calculates:

$$\lambda(5|2) = (-i\epsilon)^7[: S_{1|2}(D(1))^2 :_7 + : S_{1|1}D(1)D(2) :_7$$
$$+ : D(1)D(4) :_7 + : D(2)D(3) :_7]$$
$$(-i\epsilon)^7 : S_{1|1}D(1)D(2) :_7 = \frac{7}{12}$$
$$(-i\epsilon)^7 : S_{1|2}(D(1))^2 :_7 = -\frac{1}{4}$$
$$(-i\epsilon)^7 : D(1)D(4) :_7 = \frac{1}{4}$$
$$(-i\epsilon)^7 : D(2)D(3) :_7 = \frac{17}{12}$$
$$(4.6) \qquad \lambda(5|2) = 2$$

so for this partition both S-type and D-type terms contribute nontrivially to λ. One can perform some similar tests to show that (3.22) works correctly. In general, however, the complexity of the manifest expressions for $\lambda(N|Q)$ grows dramatically with N and especially with the difference $N - Q$, as not only the structure of higher order Schwarzians becomes increasingly cumbersome, but also the ϵ-ordering procedure of the terms gets quite tedious. For this reason, the formula (3.22), although exact, is in practice less convenient for numerical computations of the partitions, compared to using the standard generating functions. Nevertheless, it casts the partition numbers in terms of exact finite analytic expressions in terms of the conformal transformation (2.2), which demonstrates the power of conformal symmetry and constitutes the main result of this work.

Acknowledgements

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