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Scattering theory for the Dirac equation in Schwarzschild-Anti-de Sitter space-time

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We show asymptotic completeness for linear massive Dirac fields on the Schwarzschild-Anti-de Sitter spacetime. The proof is based on a Mourre estimate. We also construct an asymptotic velocity for this field.

1. Introduction

The aim of this paper is to show asymptotic completeness for the massive Dirac equation on the Anti-de Sitter Schwarzschild space-time.

When studying a physical system for which the dynamics is described by a Hamiltonian, one of the fundamental properties we want to prove is asymptotic completeness. Roughly speaking, it states that, for large time, our dynamics behave, modulo possible eigenvalues, like the well-understood dynamics described by what we call a free Hamiltonian.

The first asymptotic completeness results in General Relativity were obtained by J. Dimock and B. Kay in 1986 and 1987 ([23],[24], [25]) for classical and quantum scalar fields. This study was pursued in the 1990's by A. Bachelot for classical fields. He obtains scattering theories for Maxwell fields in 1991 [3] and Klein-Gordon fields in 1994 [4]. After that, J-P. Nicolas obtained a scattering theory for massless Dirac fields in 1995 [67] and F. Melnyk obtained a complete scattering for massive charged Dirac fields [62] in 2003. In all these works, the authors used trace class perturbation methods. On the other hand, new techniques, using Mourre estimates, were applied to the wave equation on the Schwarzschild space-time in 1992 by S. De Bièvre, P. Hislop and I.M Sigal [20]. Using this method, a complete scattering theory for the wave equation on stationary asymptotically flat space-times was obtained by D. Häfner in 2001 [43] and D. Häfner and J-P. Nicolas obtained a scattering theory for massless Dirac fields outside slowly rotating Kerr black holes in 2004 [46], making use of a positive conserved quantity which exists for the Dirac equation and not for the Klein-Gordon equation. In 2004, T. Daudé obtains a scattering theory for Dirac fields on Reissner-Nordström black holes [19] and on Kerr-Newman black holes in [18]. Using an integral representation for the Dirac propagator, D. Batic gives a new approach to the time-dependent scattering for massive Dirac fields on the Kerr metric in 2007. Recently, V. Georgescu, C. Gérard and D. Häfner obtained an asymptotic completeness result for the Klein-Gordon equation in the De-Sitter Kerr black hole, see [39]. See also M. Dafermos, G. Holzegel and I. Rodnianski for scattering results for the Einstein equations [16] and M. Dafermos, I. Rodnianski and Y. Shlapentokh-Rothman for a scattering theory for the wave equation on Kerr black holes exteriors [17]. One of the principal motivation for all these works is the study of the Hawking effect. That kind of results are needed to give a mathematically rigorous description of the Hawking effect, see [6] and [45].

In our work, we are concerned with problems that arise from the Anti-de Sitter background. Indeed, the Schwarzschild Anti-de Sitter space-time is a solution of the Einstein vacuum equations with cosmological constant $\Lambda < 0$ containing a spherically symmetric black hole. This space-time has a nontrivial causality. In fact, it is not globally hyperbolic, that is to say, Cauchy data defined on a slice $\{t = constant\}\times]r_{SAdS}, +\infty[\times\mathbb{S}^2$ (where r_{SAdS} correspond to the horizon) do not uniquely determine the evolution of the field in all the space-time. So, first of all, there's a difficulty in defining the dynamic.

This is due to the fact that, when studying the geodesics in Boyer-Lindquist coordinates, null geodesics can reach timelike infinity in finite time. This suggests that we will need to put asymptotic conditions as $r \to +\infty$ in order to determine the dynamic uniquely. This problem was first studied by Breitenlohner and Freedman ([13],[14]) for scalar fields. They showed that the need to put boundary conditions depends on the comparison between the mass of the field and the cosmological constant and discovered two critical values known as B-F bounds. More recently, A. Bachelot ([8]) showed a similar bound for the Dirac equation in the Anti-de Sitter space-time using a spectral approach. This approach uses the fact that, in an appropriate coordinate system, the equation can be written as $i\partial_t\psi = iH_m\psi$ with H_m independent of t. We thus have to construct a self-adjoint extension of H_m. In order to put the right boundary condition, we will understand the asymptotic behavior of the states in the natural domain of H_m. This kind of method was also used by Ishibashi and Wald ([56],[57]) for integer spin fields.

Using other techniques, there has been some recent advances concerning scalar fields. We first mention the works of G. Holzegel and J. Smulevici who proved, using vectorfield methods, a result of asymptotic stability of the Schwarzschild-AdS space-time with respect to spherically symmetric perturbations thanks to an exponential decay rate of the local energy [49]. However, looking at the solutions of the linear wave equation on the Schwarzschild-AdS black hole with arbitrary angular momentum l, resonances with imaginary part $e^{-\frac{C}{l}}$ appear (see [38] for details) and the local energy only decays logarithmically. The same phenomenon appear in the Kerr-AdS space-time, see [48]. Thus Kerr-AdS is supposed to be unstable. In these papers, it was supposed that the Dirichlet boundary condition holds. More recently, G. Holzegel and C. M. Warnick considered other boundary conditions for the wave equation on asymptotically AdS black hole [51]. This includes some boundary conditions considered in the context of AdS-CFT correspondence. This correspondence was also in mind of A. Bachelot in his paper about the Klein-Gordon equation in the AdS^5 space-time [9] and of A. Enciso and N. Kamran when they study the Klein-Gordon equation in $\mathrm{AdS}^5 \times Y^{p,q}$ where $Y^{p,q}$ is a Sasaki-Einstein 5-manifold [32].

We now present our results. We denote the natural domain of H_m by

$$D(H_m) = \{\phi \in \mathcal{H};\ \ H_m\phi \in \mathcal{H}\},$$

and we will use $l^2 = -\frac{3}{\Lambda}$ where $\Lambda < 0$ is the cosmological constant. We obtain:

Proposition 1.1. *For $2ml \geqslant 1$, the operator H_m is self-adjoint on $D(H_m)$.*

For the case $2ml < 1$, we will put MIT boundary conditions. This defines an operator H_m^{MIT} with natural domain $D\left(H_m^{MIT}\right)$. Then we obtain:

Proposition 1.2. *The operator H_m^{MIT} is self-adjoint on $D\left(H_m^{MIT}\right)$.*

The Cauchy problem is then well-posed by Stone's theorem.

We then turn our attention to the scattering theory. By means of a Mourre estimate, we are able to prove velocity estimates. We then introduce the comparison operator $H_c = i\gamma^0\gamma^1\partial_x$ with domain

$$D\left(H_c\right) = \left\{\varphi \in \mathcal{H}_{s,n}; H_c\varphi \in \mathcal{H}_{s,n},\ \varphi_1\left(0\right) = -\varphi_3\left(0\right),\ \varphi_2\left(0\right) = \varphi_4\left(0\right)\right\}.$$

Making use of the velocity estimates, we obtain the following asymptotic completeness result:

Theorem 1.3 (Asymptotic completeness). *For all $m > 0$ and all $\varphi \in \mathcal{H}$, the limits:*

$$\lim_{t\to\infty} e^{itH_c}e^{-itH_m}\varphi \tag{1.1}$$

$$\lim_{t\to\infty} e^{itH_m}e^{-itH_c}\varphi \tag{1.2}$$

exist. If we denote these limits by $\Omega\varphi$ and $W\varphi$ respectively, then we have $\Omega^* = W$.

We eventually study the asymptotic velocity. We will say that $B = \mathrm{s} - \mathrm{C}_\infty - \lim\limits_{n\to\infty} B_n$ if, for all $J \in C_\infty\left(\mathbb{R}\right)$, we have $J\left(B\right) = \mathrm{s} - \lim\limits_{t\to\infty} J\left(B_n\right)$ (where $C_\infty\left(\mathbb{R}\right)$ is the set of continuous functions which go to 0 at $\pm\infty$). Then, we obtain the following:

Theorem 1.4 (Asymptotic velocity for H_m). *Let $J \in C_\infty\left(\mathbb{R}\right)$ and $A = -\gamma^0\gamma^1 x$ where γ^0, γ^1 are Dirac matrices. Then, for all $m > 0$, the limit:*

$$\mathrm{s} - \lim_{t\to\infty} e^{itH_m} J\left(\frac{A}{t}\right) e^{-itH_m} \tag{1.3}$$

exists. Moreover, if $J\left(0\right) = 1$, then

$$\mathrm{s} - \lim_{R\to\infty}\left(\mathrm{s} - \lim_{t\to\infty} e^{itH_m} J\left(\frac{A}{Rt}\right) e^{-itH_m}\right) = \mathbb{1}. \tag{1.4}$$

If we define

$$\text{s} - \text{C}_\infty - \lim_{t\to\infty} e^{itH_m} \frac{A}{t} e^{-itH_m} =: P_m^+, \tag{1.5}$$

then the self-adjoint operator P_m^+ *is densely defined and commute with* H_m*. The operator* P_m^+ *is called the asymptotic velocity and is in fact the identity operator.*

The paper is organized as follows.

In Section 2, we present the Schwarzschild-AdS geometry and, due to the lack of global hyperbolicity, the fact that radial null geodesics go to infinity in finite time. Using the Newman-Penrose formalism, we then obtain the Dirac equation on this space-time and give a spectral formulation of this equation for a coordinate system (t, x, θ, φ) where the horizon corresponds to x goes to $-\infty$ and the Anti-de Sitter infinity corresponds to $x = 0$. We eventually generalize this equation by giving asymptotic behaviors of the potentials and we ensure that the Dirac equation in the Schwarzschild-AdS space-time is part of our generalization. In the rest of the paper, we will work with this generalization.

In Section 3, we obtain the self-adjointness of our operator for all $m > 0$. First, we present the spinoidal spherical harmonics and then we use this tool to decompose our operator (in fact, we diagonalize the Dirac operator on the sphere) which leads us to a $1+1$ dimensional problem for the operator now denoted $H_m^{s,n}$. Then we study the states in the natural domain $D\left(H_m^{s,n}\right) = \{\varphi \in \mathcal{H}_{s,n} | H_m^{s,n}\varphi \in \mathcal{H}_{s,n}\}$. The problem is coming from the Anti-de Sitter infinity where the potential behaves badly. Nevertheless, the potential behaves like in the result of A. Bachelot on the Anti-de Sitter space. After a unitary transform we can use his result. In this way, we see that the states behave well when $2ml \geqslant 1$ but it degenerates at 0 when $2ml < 1$. When $2ml \geqslant 1$, we prove that our operator is essentially self-adjoint on $C_0^\infty\left(]-\infty, 0[\right)$ and, using an elliptic estimate and a Hardy-type inequality, we give a precise description of the domain. In the case $2ml < 1$, we need to put a boundary condition to obtain the self-adjointness of our operator. In this paper, we have chosen the MIT boundary condition. This allows us to solve the Cauchy problem. We finally prove the absence of eigenvalues for this operator.

In Section 4, we prove a compactness result. We use an approximation of our resolvent, separating the behavior close to the black hole horizon and close to $x = 0$. We then obtain that $f(x)\left(H_m^{s,n} - \lambda\right)^{-1}$ is compact if f goes to 0 at the horizon and has a finite limit at $x = 0$.

In Section 5, we obtain a Mourre estimate for $H_m^{s,n}$ using $A = \Gamma_1 x$, where Γ_1 is the matrix diag $(1, -1, -1, 1)$, as conjugate operator.

In Section 6, we obtain some propagation estimates. First, making use of the Mourre estimate and of an abstract result about minimal velocity estimates, we prove that the minimal velocity is 1. Then, using a standard observable and a general result which uses Heisenberg derivative to obtain velocity estimates, we prove that the maximal velocity is also 1.

In Section 7, we are now able to prove asymptotic completeness for our hamiltonian. This result is first proved for fixed harmonics and then we prove that we can sum over all harmonics. It is proved by making use of the two velocity estimates and a similar reasoning as in the propagation estimates.

In Section 8, we first prove the existence of the asymptotic velocity for H_c and then deduce the same result for H_m using the wave operators. We see that the asymptotic velocity operator is the identity.

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2. The Schwarzschild Anti-de Sitter space-time and the Dirac equation

In this section, we present the Schwarzschild Anti-de Sitter space-time and give the coordinate system that we will work with in the rest of the paper. We quickly study the radial null geodesics and then formulate the Dirac equation as a system of partial differential equations which are derived from the two spinor component expression of this equation by use of the Newman-Penrose formalism. We finally give a generalization of our equation by just considering a potential that have the same asymptotic behavior as in the case of the Schwarzschild Anti-de Sitter space-time.

2.1. The Schwarzschild Anti-de Sitter space-time

Let $\Lambda < 0$. We define $l^2 = \frac{-3}{\Lambda}$. We denote by M the black hole mass.

In Boyer-Lindquist coordinates, the Schwarzschild-Anti-de Sitter metric is given by:

(2.1)
$$g_{ab} = \left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

We define $F(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$. We can see that F admits two complex conjugate roots and one real root $r = r_{SAdS}$. We deduce that the singularities of the metric are at $r = 0$ and $r = r_{SAdS} = p_+ + p_-$ where $p_\pm = \left(Ml^2 \pm \left(M^2 l^4 + \frac{l^6}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}$. (See [49]) The exterior of the black hole will be the region $r \geqslant r_{SAdS}$ and our spacetime is then seen as $\mathbb{R}_t \times]r_{SAdS}, +\infty[\times S^2$. It is well-know that the metric can be extended for $r \leqslant r_{SAdS}$ by a coordinate change which gives the maximally extended Schwarschild-Anti-de Sitter spacetime. In this paper, we are only interested in the exterior region.

In order to have a better understanding of this geometry, we study the outgoing (respectively ingoing) radial null geodesics (that is to say for which $\frac{dr}{dt} > 0$ (respectively $\frac{dr}{dt} < 0$)). Using the form of the metric we can see that along such geodesics, we have:

$$\frac{dt}{dr} = \pm F\left(r\right)^{-1}. \tag{2.2}$$

We thus introduce a new coordinate r_* such that $t - r_*$ (respectively $t + r_*$) is constant along outgoing (respectively ingoing) radial null geodesics. In other words:

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = F(r)^{-1}. \tag{2.3}$$

The coordinate system $(t, r_*, \theta, \varphi)$ is called Regge-Wheeler coordinates. r_* is given by:

$$r_*(r) = ln\left((r - r_{SAdS})^{\alpha_1}\left(r^2 + r_{SAdS} r + r_{SAdS}^2 + l^2\right)^{-\frac{\alpha_1}{2}}\right) + C \arctan\left(\frac{2r + r_{SAdS}}{\left(3r_{SAdS}^2 + 4l^2\right)^{\frac{1}{2}}}\right). \tag{2.4}$$

where:

$$\alpha_1 = \frac{r_{SAdS} l^2}{3r_{SAdS}^2 + l^2} = \frac{1}{2\kappa}; \qquad C = \frac{l^2\left(3r_{SAdS}^2 + 2l^2\right)}{\left(3r_{SAdS}^2 + l^2\right)\left(3r_{SAdS}^2 + 4l^2\right)^{\frac{1}{2}}} \tag{2.5}$$

We obtain $\lim_{r \to r_{SAdS}} r_*(r) = -\infty$ and $\lim_{r\to\infty} r_*(r) = C\frac{\pi}{2}$. We will consider the coordinate $x = r_* - C\frac{\pi}{2}$ rather than r_*. We thus have:

$$\lim_{r \to r_{SAdS}} x\left(r\right) = -\infty \tag{2.6}$$

$$\lim_{r \to \infty} x\left(r\right) = 0. \tag{2.7}$$

This limit proves that, along radial null geodesic, a particle goes to timelike infinity in finite Boyer-Lindquist time (recall that along these geodesic, $t - r_*$ and $t + r_*$ are constants). This geometric property will be a major issue in our problem. This implies that our space-time is not globally hyperbolic, so that we cannot use the standard result by Leray about the global existence of solution of hyperbolic equations. A similar situation has been encountered by A.Bachelot in his article [8] concerning the Dirac equation on the Anti-de Sitter space-time. We expect to do a similar study concerning the self-adjoint extension.

2.2. The Dirac equation on Schwarzschild Anti-de Sitter space-time

In the two components spinor notation, the Dirac equation takes the following form:

$$\begin{cases} \nabla_{AA'}\phi^A = -\mu\chi_{A'} \\ \nabla_{AA'}\chi^{A'} = -\mu\phi_A \end{cases} \tag{2.8}$$

where $\nabla_{AA'}$ is the Levi-Civita connection, ϕ^A is a two-spinor, $\mu = \frac{m}{\sqrt{2}}$ and $m \geqslant 0$ is the mass of the field.

Thanks to the Newman-Penrose formalism, we can obtain the equation in the form of a system of partial differential equations. In this formalism, we introduce a null tetrad $(l^a, n^a, m^a, \bar{m}^a)$, that is

$$l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = l_a m^a = n_a m^a = 0, \tag{2.9}$$

which is a basis of the complexified of the tangent space. We'll say that the tetrad is normalized if:

$$l_a n^a = 1 \qquad m_a \bar{m}^a = -1. \tag{2.10}$$

The two vectors l^a and n^a correspond to the directions along which the light goes to infinity (we can choose l^a as an outgoing null vector and n^a as an ingoing null vector). The vector m^a admits bounded integral curves. The vectors m^a and $\bar{m}^a$ will generate rotations. In our case, we will consider:

$$l^a\partial x_a = \frac{1}{\sqrt{2}}F(r)^{-\frac{1}{2}}\left(\partial_t + \partial_x\right), \quad n^a\partial x_a = \frac{1}{\sqrt{2}}F(r)^{-\frac{1}{2}}\left(\partial_t - \partial_x\right)$$
$$m^a\partial x_a = \frac{1}{\sqrt{2}r}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi\right), \quad \bar{m}^a\partial x_a = \frac{1}{\sqrt{2}r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi\right).$$

We remark that this tetrad is normalized and since $t \pm x$ is constant along null geodesics, the vector $l^a \partial x_a$ and $n^a \partial x_a$ are null. Moreover, using the equation of radial null geodesics with λ as our affine parameter, we deduce that $\frac{dt}{dr} = \frac{dt}{d\lambda}\frac{d\lambda}{dr} = F(r)^{-1}$ which gives us an outgoing real null vector. We see as well that m^a is linked to rotations. We give the associated dual vectors:

$$
\begin{aligned}
l_a dx^a &= \frac{1}{\sqrt{2}} F(r)^{\frac{1}{2}} (dt - dx), \quad n_a dx^a = \frac{1}{\sqrt{2}} F(r)^{\frac{1}{2}} (dt + dx) \\
m_a dx^a &= \frac{r}{\sqrt{2}} (-d\theta + i\sin(\theta) d\varphi), \quad \bar{m}_a dx^a = \frac{r}{\sqrt{2}} (-d\theta - i\sin(\theta) d\varphi).
\end{aligned}
$$

Using this tetrad, it is then possible to decompose the covariant derivative in directional derivatives along these directions. We introduce the following symbols:

$$D = l^a \nabla_a, \;\; D' = n^a \nabla_a, \;\; \delta = m^a \nabla_a, \;\; \delta' = \bar{m}^a \nabla_a.$$

We have twelve spin coefficients that are defined by the following expressions:

$$
\begin{aligned}
&\hat{\kappa} = m^a D l_a, \quad \rho = m^a \delta' l_a, \quad \sigma = m^a \delta l_a, \quad \tau = m^a D' l_a, \\
&\epsilon = \frac{1}{2}\left(n^a D l_a + m^a D \bar{m}_a\right), \quad \alpha = \frac{1}{2}\left(n^a \delta' l_a + m^a \delta' \bar{m}^a\right), \\
&\beta = \frac{1}{2}\left(n^a \delta l_a + m^a \delta \bar{m}^a\right), \quad \gamma = \frac{1}{2}\left(n^a D' l_a + m^a D' \bar{m}_a\right), \\
&\pi = -\bar{m}^a D n_a, \quad \lambda = -\bar{m}^a \delta' n_a, \quad \mu = -\bar{m}^a \delta n_a, \quad \nu = -\bar{m}^a D' n_a,
\end{aligned}
$$

where $\hat{\kappa}$ is the spin coefficient usually denoted κ, since κ is the surface gravity in our convention. We can now give the equation (2.8) as a system of partial differential equations. These equations act on the components of the spinor $\phi^A, \chi^{A'}$ in a normalized spinorial basis (o^A, ι^A) (that is such that $o_A \iota^A = 1$). To choose our spinorial basis, we use the null tetrad above. Indeed, we can define the spinorial basis (o^A, ι^A), uniquely up to an overall sign, using the following conditions:

$$o^A \bar{o}^{A'} = l^a, \quad \iota^A \bar{\iota}^{A'} = n^a, \quad o^A \bar{\iota}^{A'} = m^a, \quad \iota^A \bar{o}^{A'} = \bar{m}^a, \quad o_A \iota^A = 1.$$

The dual basis is $\epsilon^0_A = -\iota_A$, $\epsilon^1_A = o_A$. Let $\phi^0, \phi^1, \chi^{0'}, \chi^{1'}$ such that $\phi^A = \phi^0 o^A + \phi^1 \iota^A$ and $\chi^{A'} = \chi^{0'} o^{A'} + \chi^{1'} \iota^{A'}$ where $(o^{A'}, \iota^{A'})$ is the conjugate basis of (o^A, ι^A). In this basis, the components of ϕ_A and $\chi_{A'}$ are respectively:

$$\phi_0 = -\phi^1, \quad \phi_1 = \phi^0, \quad \chi_{0'} = -\chi^{1'}, \quad \chi_{1'} = \chi^{0'}.$$

We obtain the following system of partial differential equations:

$$(2.11)\quad \begin{cases} l^a \partial x_a \phi_1 - \bar{m}^a \partial x_a \phi_0 + (\epsilon - \rho)\,\phi_1 - (\pi - \alpha)\,\phi_0 = \frac{m}{\sqrt{2}} \chi^{1'} \\ m^a \partial x_a \phi_1 - n^a \partial x_a \phi_0 + (\beta - \tau)\,\phi_1 - (\mu - \gamma)\,\phi_0 = -\frac{m}{\sqrt{2}} \chi^{0'} \\ l^a \partial x_a \chi^{0'} + m^a \partial x_a \chi^{1'} + (\bar{\epsilon} - \bar{\rho})\,\chi^{0'} + (\bar{\pi} - \bar{\alpha})\,\chi^{1'} = -\frac{m}{\sqrt{2}} \phi_0 \\ \bar{m}^a \partial x_a \chi^{0'} + n^a \partial x_a \chi^{1'} + \left(\bar{\beta} - \bar{\tau}\right)\chi^{0'} + (\bar{\mu} - \bar{\gamma})\,\chi^{1'} = -\frac{m}{\sqrt{2}} \phi_1. \end{cases}$$

Using the 4-component spinor $\psi = \begin{pmatrix} \phi_A \\ \chi^{A'} \end{pmatrix}$, we obtain:

(2.12)
$$\left(\partial_t + \gamma^0\gamma^1 \left(F(r)\partial_r + \frac{F(r)}{r} + \frac{F'(r)}{4}\right) + \frac{F(r)^{\frac{1}{2}}}{r} \not{D}_{\mathbb{S}^2} + im\gamma^0 F(r)^{\frac{1}{2}}\right)\psi = 0.$$

where m is the mass of the field and $\not{D}_{\mathbb{S}^2}$ is the Dirac operator on the sphere. In the coordinate system given by $(\theta, \varphi) \in [0; 2\pi] \times [0; \pi]$, we obtain: $\not{D}_{\mathbb{S}^2} = \gamma^0\gamma^2 \left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \gamma^0\gamma^3 \frac{1}{\sin\theta}\partial_\varphi$ where singularities appear, but we just have to change our chart in this case. We will now work in these coordinates.

Recall that Dirac matrices γ^μ, $0 \leqslant \mu \leqslant 3$, unique up to unitary transform, are given by the following relations:

$$(2.13)\quad \begin{aligned} &\gamma^{0^*} = \gamma^0; \quad \gamma^{j^*} = -\gamma^j, \quad 1 \leqslant j \leqslant 3; \\ &\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}, \quad 0 \leqslant \mu, \nu \leqslant 3. \end{aligned}$$

In our representation, the matrices take the form:

$$(2.14)\quad \gamma^0 = i\begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \gamma^k = i\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

where the Pauli matrices are given by:

$$(2.15)\quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We thus obtain:

$$(2.16)\quad \gamma^0\gamma^1 = \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}; \quad \gamma^0\gamma^2 = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}; \quad \gamma^0\gamma^3 = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.$$

We introduce the matrix:

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.17}$$

which satisfies the relations:

$$\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0, \qquad 0 \leqslant \mu \leqslant 3. \tag{2.18}$$

We make the change of spinor $\phi(t,x,\theta,\varphi) = rF(r)^{\frac{1}{4}}\psi(t,r,\theta,\varphi)$ and obtain the following equation:

$$\partial_t\phi = i\left(i\gamma^0\gamma^1\partial_x + i\frac{F(r)^{\frac{1}{2}}}{r}\not{D}_{\mathbb{S}^2} - m\gamma^0F(r)^{\frac{1}{2}}\right)\phi. \tag{2.19}$$

We set:

$$H_m = i\gamma^0\gamma^1\partial_x + i\frac{F(r)^{\frac{1}{2}}}{r}\not{D}_{\mathbb{S}^2} - m\gamma^0F(r)^{\frac{1}{2}}. \tag{2.20}$$

We introduce the Hilbert space:

$$\mathcal{H} := \left[L^2\left(]-\infty,0[_x \times S^2_\omega, dxd\omega\right)\right]^4 \tag{2.21}$$

2.3. Generalization

Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$, and define the spaces $T^{q,n}$ by:

$$T^{q,n} = \left\{ f \in C^\infty\left(]-\infty;0[\right) \;\middle|\; \forall\alpha\in\mathbb{N},\ |\partial_x^\alpha f(x)| \lesssim \begin{cases} e^{qx} & \text{, when } x\to-\infty \\ (-x)^n & \text{, when } x\to 0 \end{cases} \right\} \tag{2.22}$$

We consider two smooth functions A_0, B_0 such that:

$$A_0 = \begin{cases} 0 & \text{if } x \leqslant -2 \\ \frac{1}{l} & \text{if } x \geqslant -1 \end{cases}; \qquad B_0 = \begin{cases} 0 & \text{if } x \leqslant -2 \\ \frac{l}{-x} & \text{if } x \geqslant -1. \end{cases}$$

We will consider the following operator:

$$H_m = \Gamma^1 D_x + A(x)\not{D}_{\mathbb{S}^2} - m\gamma^0 B(x) \tag{2.23}$$

where m is the mass of the field and, for two positive numbers ϑ, β:

$$A - A_0 \in T^{\vartheta,2} \tag{2.24}$$

$$B - B_0 \in T^{\beta,1}. \tag{2.25}$$

We also recall that $\Gamma^1 = -\gamma^0\gamma^1 = \mathrm{diag}(1,-1,-1,1)$ and $D_x = \frac{1}{i}\partial_x$.

We then check that the Schwarzschild Anti-de Sitter case enters in our abstract model. For x going to $-\infty$, we have:

$$\begin{aligned} r - r_{SAdS} &= \left(3r_{SAdS}^2 + l^2\right)^{\frac{1}{2}} e^{-2\kappa C \arctan\left(\frac{3r_{SAdS}}{\left(3\left(r_{SAdS}\right)^2+4l^2\right)}\right)+C\pi\kappa} e^{2\kappa x} \\ &\quad - C_1 e^{4\kappa x} + o\left(e^{4\kappa x}\right) \\ F(r)^{\frac{1}{2}} &= \frac{\left(3r_{SAdS}^2 + l^2\right)^{\frac{3}{4}} D_4^{\frac{1}{2}}}{r_{SAdS}^{\frac{1}{2}} l} e^{\kappa x} + C_2 e^{3\kappa x} + o\left(e^{3\kappa x}\right), \\ \frac{F(r)^{\frac{1}{2}}}{r} &= \frac{\left(3r_{SAdS}^2 + l^2\right)^{\frac{3}{4}} D_4^{\frac{1}{2}}}{r_{SAdS}^{\frac{3}{2}} l} e^{\kappa x} + C_3 e^{3\kappa x} + o\left(e^{3\kappa x}\right) \end{aligned}$$

where C_1, C_2, C_3 are constants. Then, for x in a neighbourhood of 0, we have:

$$\begin{aligned} r &= -\frac{l^2}{x} + \frac{1}{3}(x) + o(-x) \\ F(r)^{\frac{1}{2}} &= -\frac{l}{x} - \frac{x}{6l} + o(x) \\ \frac{F(r)^{\frac{1}{2}}}{r} &= \frac{1}{l} + \frac{x^2}{2l^3} + o\left(x^2\right). \end{aligned}$$

The Schwarzschild Anti-de Sitter model is thus a particular case of our generalized model with $A = \frac{F(r)^{\frac{1}{2}}}{r}$ and $B = F(r)^{\frac{1}{2}}$.

3. Study of the hamiltonian

In this section, we first present the spinoidal spherical harmonics. This allows us to reduce our problem to the study of a $1+1$ dimensional equation with a new hamiltonian denoted $H_m^{s,n}$. We then use the fact that, at AdS infinity, the potential looks like the one considered by A. Bachelot in [8]. By means of a unitary transform and a cut-off near AdS infinity, we are able to make use of his result and obtain the asymptotic behavior of the elements in the natural domain of our operator. As in [8], the need or not to put a boundary condition is linked to the comparison between the mass of the field and the cosmological constant. For $2ml \geqslant 1$ (where m is the mass of the field and l is linked to the cosmological constant), there's no need to put boundary conditions. When $2ml < 1$, we consider the generalized MIT-bag boundary condition in order to determine the dynamic uniquely. We then prove the self-adjointness of our operators. Using an elliptic inequality, we are able to give the domain of our operator for $2ml > 1$. Using Stone's theorem, we can solve the Cauchy problem for our equation. At last, we give a proof of the absence of eigenvalue for all $m > 0$ which will be useful for the propagation estimates.

3.1. Description of the domain

3.1.1. The spinoidal spherical harmonics. In the rest of this paper, we will often make use of spinoidal spherical harmonics (we can refer to [8] for a more complete presentation of these harmonics) which will permit us to decompose $\mathcal{H}$ as follows:

$$\mathcal{H} = \bigoplus_{(s,n)\in I} \left(\left(L^2(x,dx)\right)^4 \otimes \begin{pmatrix} T^s_{-\frac{1}{2},n} \\ T^s_{\frac{1}{2},n} \\ T^s_{-\frac{1}{2},n} \\ T^s_{\frac{1}{2},n} \end{pmatrix} \right) \tag{3.1}$$

where:

$$I := \left\{ (s,n); \;\; s \in \mathbb{N} + \frac{1}{2}, \;\; n \in \mathbb{Z} + \frac{1}{2}, \;\; s - |n| \in \mathbb{N} \right\}. \tag{3.2}$$

These functions satisfy the following relations:

$$\left(\frac{\partial}{\partial \theta} + \frac{1}{2\tan\theta} \right) T^s_{\pm\frac{1}{2},n} = \pm \frac{n}{\sin\theta} T^s_{\pm\frac{1}{2},n} - i \left(s + \frac{1}{2} \right) T^s_{\mp\frac{1}{2},n}, \tag{3.3}$$

$$\frac{\partial}{\partial \varphi} T^s_{\pm\frac{1}{2},n} = -in T^s_{\pm\frac{1}{2},n}. \tag{3.4}$$

Since $\left(T^s_{\frac{1}{2},n}\right)_{(s,n)\in I}$ and $\left(T^s_{-\frac{1}{2},n}\right)_{(s,n)\in I}$ both span $L^2\left(\mathcal{S}^2\right)$, we can decompose $f \in L^2(S^2)$ as follows:

$$f(\theta,\varphi) = \sum_{(s,n)\in I} u^s_{\pm,n}(f) T^s_{\pm\frac{1}{2},n}(\theta,\varphi), \;\; u^s_{\pm,n}(f) \in \mathbb{C}.$$

Let us introduce the Hilbert spaces $W_{\pm}^{d}$ for $d \in \mathbb{R}$ as the closure of the space:

$$W_f^{\pm} := \left\{ \sum_{finite} u_{\pm,n}^{s} T_{\pm\frac{1}{2},n}^{s};\ \ u_{\pm,n}^{s} \in \mathbb{C} \right\} \tag{3.5}$$

for the norm

$$||f||_{W_{\pm}^{d}}^{2} := \sum_{(s,n)\in I} \left(s+\frac{1}{2}\right)^{2d} |u_{\pm,n}^{s}(f)|^{2}.$$

Using Plancherel's formula, $L^2\left(S^2\right)$ is just W^0. We give some properties of these spaces (for a more complete presentation, we refer to [8]). We have:

$$d \geqslant 0 \Longrightarrow W_{\pm}^{d} = \left\{ f \in L^2\left(S^2\right);\ \ ||f||_{W_{\pm}^{d}} < \infty \right\},$$
$$\left(W_{\pm}^{d}\right)' = W_{\pm}^{-d} \quad \text{and} \quad C_0^{\infty}\left(]0,\pi[_{\theta}\times]0,2\pi[_{\varphi}\right) \subset W_{\pm}^{d}.$$

We must remark that $T_{\pm\frac{1}{2},n}^{s}(\theta,2\pi) = -T_{\pm\frac{1}{2},n}^{s}(\theta,0) \neq 0$. Consequently, these functions are not smooth on the sphere S^2. In correspondence with the decomposition (3.1), we introduce the Hilbert spaces:

$$\mathcal{W}^{d} = W_{-}^{d} \times W_{+}^{d} \times W_{-}^{d} \times W_{+}^{d} \tag{3.6}$$

equipped with the norm:

$$\|\Phi\|_{\mathcal{W}^{d}}^{2} = \sum_{j=1}^{4} \sum_{(s,n)\in I} \left(s+\frac{1}{2}\right)^{2d} \left|u_{j,n}^{s}\right|^{2} \tag{3.7}$$

where:

$$\Phi\left(\theta,\varphi\right) = \sum_{(s,n)\in I} \begin{pmatrix} u_{1,n}^{s} T_{-\frac{1}{2},n}^{s}\left(\theta,\varphi\right) \\ u_{2,n}^{s} T_{+\frac{1}{2},n}^{s}\left(\theta,\varphi\right) \\ u_{3,n}^{s} T_{-\frac{1}{2},n}^{s}\left(\theta,\varphi\right) \\ u_{4,n}^{s} T_{+\frac{1}{2},n}^{s}\left(\theta,\varphi\right) \end{pmatrix}.$$

3.1.2. A result due to A.Bachelot. We recall a result obtained by A.Bachelot (see [8]). In this article, the hamiltonian considered was:

(3.8)
$$H_m^B = i\gamma_B^0\gamma_B^1 \left(F_B(r)\partial_r + \frac{F_B\left(r\right)}{r} + \frac{F_B'\left(r\right)}{4} \right) + i\frac{F_B(r)^{\frac{1}{2}}}{r} \not{D}_{\mathbb{S}^2} - m\gamma_B^0 F_B(r)^{\frac{1}{2}}$$

in (r,θ,φ) coordinates where $F_B(r) = 1 + \frac{r2}{l^2}$. Here, m is $\tilde{m}\sqrt{\frac{3}{\Lambda}}$ with $\tilde{m}$ the mass of the field and $-\Lambda$ the cosmological constant. Moreover, the space $\mathcal{L}^2$ is defined by $\mathcal{L}^2 := \left[L^2\left([0,\frac{\pi}{2}[_\zeta \times [0,\pi]_\theta \times [0,2\pi[_\varphi, \sin\theta \mathrm{d}\zeta\mathrm{d}\theta\mathrm{d}\varphi\right)\right]^4$ where $\zeta = \arctan\left(\sqrt{\frac{\Lambda}{3}}r\right)$.. Using a change of spinor and a change of coordinates such that $\phi(t,\zeta,\theta,\varphi) = rF_B(r)^{\frac{1}{4}}\psi(t,r,\theta,\varphi)$, he obtains:

$$\begin{aligned}(3.9)\quad H_m^B := i\gamma_B^0\gamma_B^1\frac{\partial}{\partial\zeta} + \frac{i}{\sin\zeta}\left[\gamma_B^0\gamma_B^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \frac{1}{\sin\theta}\gamma_B^0\gamma_B^3\frac{\partial}{\partial\varphi}\right] \\ - \frac{m}{\cos\zeta}\gamma_B^0.\end{aligned}$$

where he uses the natural domain:

$$(3.10)\qquad D(H_m^B) := \left\{\Phi \in \mathcal{L}^2; H_m^B\Phi \in \mathcal{L}^2\right\}.$$

At last, we recall that the Dirac matrices $\gamma_B^0, \gamma_B^1, \gamma_B^2, \gamma_B^3$ take the form:

$$(3.11)\qquad \gamma_B^0 = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}, \quad \gamma_B^k = \begin{pmatrix} 0 & \sigma_B^k \\ -\sigma_B^k & 0\end{pmatrix}, \quad k = 1,2,3$$

where the Pauli matrices are given by:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad \sigma_B^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \quad \sigma_B^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \sigma_B^3 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$$

The result is then the following (see Theorem V.1 in [8]):

Theorem 3.1. *For all $\Phi \in D(H_m^B)$, we have:*

$$(3.12)\qquad \Phi \in C^0\left(\left[0,\frac{\pi}{2}\right[_\zeta; \mathcal{W}^{\frac{1}{2}}\right) \quad \textit{with} \quad ||\Phi(\zeta,.)||_{\mathcal{W}^{\frac{1}{2}}} = O(\sqrt{\zeta}),\ \zeta \to 0,$$

and for $m > 0$, we have

$$(3.13)\qquad \int_0^{\frac{\pi}{2}} ||\Phi(\zeta,.)||^2_{\mathcal{W}^1}\frac{d\zeta}{\sin\zeta} \leqslant ||H_m\Phi||^2_{\mathcal{L}^2}.$$

For $m > \frac{1}{2}$, we have

$$(3.14)\qquad ||\Phi(\zeta,.)||_{L^2(S^2)} = O\left(\sqrt{\frac{\pi}{2} - \zeta}\right),\ \zeta \to \frac{\pi}{2}.$$

For $m = \frac{1}{2}$, we have

$$||\Phi(\zeta,.)||_{L^2(S^2)} = O\left(\sqrt{\left(\zeta - \frac{\pi}{2}\right) ln\left(\frac{\pi}{2} - \zeta\right)}\right), \ \ \zeta \to \frac{\pi}{2}. \tag{3.15}$$

For $0 < m < \frac{1}{2}$, there exist functions $\psi_- \in W_-^{\frac{1}{2}}$, $\chi_- \in W_+^{\frac{1}{2}}$, $\psi_+, \chi_+ \in L^2(S^2)$ and $\phi \in C^0\left([0, \frac{\pi}{2}]_\zeta; L^2(S^2;\mathbb{C}^4)\right)$ satisfying

$$\Phi(\zeta,\theta,\varphi) = \left(\frac{\pi}{2} - \zeta\right)^{-m} \begin{pmatrix} \psi_-(\theta,\varphi) \\ \chi_-(\theta,\varphi) \\ -i\psi_-(\theta,\varphi) \\ i\chi_-(\theta,\varphi) \end{pmatrix} + \left(\frac{\pi}{2} - \zeta\right)^{m} \begin{pmatrix} \psi_+(\theta,\varphi) \\ \chi_+(\theta,\varphi) \\ i\psi_+(\theta,\varphi) \\ -i\chi_+(\theta,\varphi) \end{pmatrix} + \phi(\zeta,\theta,\varphi), \tag{3.16}$$

$$||\phi(\zeta,.)||_{L^2(S^2)} = o\left(\sqrt{\frac{\pi}{2} - \zeta}\right), \quad x \to \frac{\pi}{2}. \tag{3.17}$$

Conversely, for all $\psi_- \in W_-^{\frac{1}{2}+m}$, $\chi_- \in W_+^{\frac{1}{2}+m}$, $\psi_+ \in W_-^{\frac{1}{2}-m}$, $\chi_+ \in W_+^{\frac{1}{2}-m}$ there exists $\Phi \in D(H_m^B)$ satisfying (3.16) *and* (3.17)*.*

Remark. This result concerning the asymptotic behavior of elements in the domain of the operator H_m^B is first proved for fixed harmonics (i.e fixed $(s,n) \in I$). In the next sections, we will often make use of the result obtained for fixed harmonics.

The condition on the mass is a consequence of the fact that the states in the natural domain of our operator have to be in L^2. When the mass is sufficiently large, the term $\left(\frac{\pi}{2} - \zeta\right)^{-m}$ in (3.16) is not in L^2 so it cannot appear in the development of the states near $\frac{\pi}{2}$. In this case, we do not need to put boundary conditions to obtain the self-adjointness of this operator and well-posedness of the Cauchy problem.

Unfortunately, for a mass too small compared to the cosmological constant, we see that the term $\left(\frac{\pi}{2} - \zeta\right)^{-m}$ in (3.16) is in L^2 which is problematic for the symmetry of our operator. We thus need to put boundary conditions to get rid of this term and solve the Cauchy problem.

3.1.3. Unitary transform of H_m. Let us introduce the following domains:

- If $2ml \geqslant 1$:

$$D(H_m) = \{\phi \in \mathcal{H};\ \ H_m\phi \in \mathcal{H}\}\,. \tag{3.18}$$

- If $2ml < 1$, we consider the operator equipped with the domain whose elements satisfy a generalized MIT-bag condition (where $\alpha \in \mathbb{R}$ is called the Chiral angle and $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ (see [8])):

$$D(H_m) = \left\{\phi \in \mathcal{H};\ \ H_m\phi \in \mathcal{H},\ \left\|\left(\gamma^1 + ie^{i\alpha\gamma^5}\right)\phi\right\|_2 = o\left(\sqrt{-x}\right),\ x \to 0\right\}. \tag{3.19}$$

First, we'll try to remove α in the case $2ml < 1$. We introduce the following operator:

$$H_m^\alpha = e^{i\frac{\alpha}{2}\gamma^5} H_m e^{-i\frac{\alpha}{2}\gamma^5}. \tag{3.20}$$

Since $e^{i\alpha\gamma^5}$ is unitary and $e^{i\alpha\gamma^5}\gamma^1 = \gamma^1 e^{-i\alpha\gamma^5}$, we see that $\varphi \in D\left(H_m\right)$ if and only if $e^{i\frac{\alpha}{2}\gamma^5}\varphi \in D\left(H_m^\alpha\right)$ where:

$$D\left(H_m^\alpha\right) = \left\{\phi \in \mathcal{H};\ \ H_m^\alpha\phi \in \mathcal{H},\ \left\|\left(\gamma^1 + i\right)\phi\right\|_2 = o\left(\sqrt{-x}\right),\ x \to 0\right\}.$$

So we can restrict to the case $\alpha = 0$ which we will do in the following.

We will now modify our hamiltonian in order to exploit the result of A. Bachelot. We introduce a new time variable $\tilde{t} = -t$ (and we will continue to denote by t) which gives:

$$\partial_t\phi = i\left(-H_m\right)\phi. \tag{3.21}$$

Let:

$$\tilde{H}_m = \gamma_B^5 P^{-1}(-H_m)P\gamma_B^5 \tag{3.22}$$

where:

$$P = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\begin{pmatrix}\mathrm{Id} & \mathrm{Id}\\ -i\mathrm{Id} & i\mathrm{Id}\end{pmatrix},\quad P^* = P^{-1} = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\begin{pmatrix}\mathrm{Id} & i\mathrm{Id}\\ \mathrm{Id} & -i\mathrm{Id}\end{pmatrix},$$
$$\gamma_B^5 = \begin{pmatrix}0 & \mathrm{Id}\\ \mathrm{Id} & 0\end{pmatrix},$$

and Id is the identity matrix of order 2. The matrix P satisfies the following relations:

$$\gamma^0 = P\gamma_B^0 P^{-1};\quad \gamma^j = -P\gamma_B^j P^{-1},\quad 1 \leqslant j \leqslant 3. \tag{3.23}$$

where the Dirac matrices are defined by (3.11) and (2.14). The matrix γ_B^5 satisfies the same relations as γ^5 in (2.18). We obtain:

$$\tilde{H}_m = i\gamma_B^0\gamma_B^1\partial_x + i\gamma_B^0\gamma_B^2 A(x)\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + i\gamma_B^0\gamma_B^3 A(x)\frac{1}{\sin\theta}\partial_\varphi - m\gamma_B^0 B(x). \tag{3.24}$$

3.1.4. Asymptotic behavior of elements of the domain. We introduce the projection $P_{s,n}$ from $\mathcal{H}$ to $\mathcal{H}_{s,n}$ and the operators $\tilde{H}_m^{s,n} = \tilde{H}_{m|\mathcal{H}_{s,n}}$, $H_m^{s,n,B} = H^B_{m|\mathcal{H}_{s,n}}$ for $(s,n) \in I$. We denote $\psi_{s,n} = P_{s,n}(\psi)$ with components $\psi_{i,n}^s$ for $i = 1, \ldots, 4$. Furthermore, the domain of $H_m^{s,n,B}$ is given by:

- If $2ml \geqslant 1$:

$$D\left(H_m^{s,n,B}\right) = \left\{\varphi_{s,n} \in \mathcal{H}_{s,n};\ H_m^{s,n,B}\varphi_{s,n} \in \mathcal{H}_{s,n}\right\}$$

- If $2ml < 1$, we add the condition that $\left\|\left(\gamma_B^1 + i\right)\varphi_{s,n}(x,.)\right\|_{\mathcal{W}^0} = o\left(\sqrt{-x}\right)$ when x goes to 0.

We then have the:

Lemma 3.2. *Let $\psi \in D\left(\tilde{H}_m\right)$ and $\chi \in C_0^\infty\left(]-2\epsilon, 0]\right)$ such that $\chi = 1$ on $]-\epsilon, 0]$ with $\epsilon > 0$. Then $\chi\psi \in D\left(H_m^B\right)$.*

Proof. Recall that the operator obtained by A. Bachelot in [8] is given by (3.8) where $F_B(r) = 1 + \frac{r^2}{l^2}$. This operator has the same form as in (2.12). Moreover, when $r >> r_{SAdS}$, F_B and F have the same behavior (F is defined by $F(r) = 1 + \frac{r^2}{l^2} - \frac{2M}{r}$). We make the change of variable $r \to x$ where $\frac{dx}{dr} = F(r)^{-1}$ and F is defined on $]r_{SAdS}, +\infty[$. We obtain:

$$H_m^B = i\gamma_B^0\gamma_B^1 g(x)\partial_x + i\gamma_B^0\gamma_B^1\left(\frac{F(r)}{r} + \frac{F'(r)}{4}\right) + \frac{3M}{2r^2} + A_B(x) D_{S^2} - m\gamma_B^0 B_B(x)$$

where r is understood as a function of x and:

$$g(x) = 1 + \frac{2M}{l^4}(-x)^3 + o\left((-x)^3\right), \quad A_B(x) = \frac{1}{l} + \frac{1}{2l^3}(-x)^2 + o\left((-x)^2\right)$$
$$B_B(x) = \frac{l}{-x} + \frac{1}{6l}(-x) + o(-x), \quad \frac{F(r)}{r} = \frac{1}{-x} + \frac{2}{3l^2}(-x) + o(-x)$$
$$F'(r) = \frac{2}{-x} - \frac{2}{3l^2}(-x) + o(-x)$$

when x goes to 0. Since $P_{s,n}(\chi\psi) = \chi\psi_{s,n}$, we have:

$$\begin{aligned} H_m^{s,n,B} P_{s,n}(\chi\psi) = {} & g(x)\tilde{H}_m^{s,n} P_{s,n}(\chi\psi) \\ & + i\gamma_B^0\gamma_B^1 \left(\frac{F(r)}{r} + \frac{F'(r)}{4}\right)(1-g(x))\,\chi\psi_{s,n} \\ & + \frac{3M}{2r^2}\chi\psi_{s,n} + \gamma_B^0\gamma_B^2 \left(A_B(x) - g(x)A(x)\right)\left(s+\frac{1}{2}\right)\chi\psi_{s,n} \\ & - m\gamma_B^0 \left(B_B(x) - g(x)B(x)\right)\chi\psi_{s,n} \end{aligned} \tag{3.25}$$

Since $\psi \in D(\tilde{H}_m)$, g is bounded in a neighborhood of 0 and $\chi \in C_0^\infty(]-1,0]_x)$, the first term is in $L^2(x,dx)$. Using the behavior at 0 of g, the terms $A_B(x) - g(x)A(x)$, $B_B(x) - g(x)B(x)$ and $\left(\frac{F(r)}{r} + \frac{F'(r)}{4}\right)(1-g(x))$ are bounded near 0. We deduce that $H_m^{s,n,B} P_{s,n}(\chi\psi) \in \mathcal{H}_{s,n}$. In particular, $\chi\psi_{s,n} \in D\left(H_m^{s,n,B}\right)$.

To be able to sum over (s,n), we need to know that

$$\left(s+\frac{1}{2}\right)^2 \|(\chi\psi_{s,n})\|^2_{L^2\left(-\frac{1}{2},0\right)}$$

is summable. Since $\psi \in D\left(\tilde{H}_m\right)$, $f = \tilde{H}_m\psi$ admits a decomposition $f = \sum_{(s,n)\in I} f_n^s$. We denote $f_{i,n}^s$ $(i = 1,\dots,4)$ the components of f_n^s. We obtain four differential equations:

$$\begin{aligned} & i\overline{\chi\psi_{4,n}^s}\left(\chi\psi_{3,n}^s\right)' + \left(s+\frac{1}{2}\right)A(x)\left|\chi\psi_{4,n}^s\right|^2 - B(x)\,\overline{\chi\psi_{4,n}^s}\chi\psi_{1,n}^s = \overline{\chi\psi_{4,n}^s} f_{1,n}^s, \\ & -i\overline{\chi\psi_{3,n}^s}\left(\chi\psi_{4,n}^s\right)' + \left(s+\frac{1}{2}\right)A(x)\left|\chi\psi_{3,n}^s\right|^2 - B(x)\,\overline{\chi\psi_{3,n}^s}\chi\psi_{2,n}^s = \overline{\chi\psi_{3,n}^s} f_{2,n}^s, \\ & i\overline{\chi\psi_{2,n}^s}\left(\chi\psi_{1,n}^s\right)' + \left(s+\frac{1}{2}\right)A(x)\left|\chi\psi_{2,n}^s\right|^2 + B(x)\,\overline{\chi\psi_{2,n}^s}\chi\psi_{3,n}^s = \overline{\chi\psi_{2,n}^s} f_{3,n}^s, \\ & -i\overline{\chi\psi_{1,n}^s}\left(\chi\psi_{2,n}^s\right)' + \left(s+\frac{1}{2}\right)A(x)\left|\chi\psi_{2,n}^s\right|^2 + B(x)\,\overline{\chi\psi_{1,n}^s}\chi\psi_{4,n}^s = \overline{\chi\psi_{1,n}^s} f_{4,n}^s. \end{aligned}$$

where we have multiply by $\overline{\chi\psi_{j,n}^s}$ for $j = 1,\dots,4$. Adding these equations and taking the real part, we obtain:

$$\begin{aligned} & \frac{d}{dx}\Im\left(\chi\psi_{1,n}^s\overline{\chi\psi_{2,n}^s} + \chi\psi_{3,n}^s\overline{\chi\psi_{4,n}^s}\right) + \left(s+\frac{1}{2}\right)A(x)\sum_{j=1}^4\left|\chi\psi_{j,n}^s\right|^2 \\ & = \Re\left(\overline{\chi\psi_{4,n}^s} f_{1,n}^s + \overline{\chi\psi_{3,n}^s} f_{2,n}^s + \overline{\chi\psi_{2,n}^s} f_{3,n}^s + \overline{\chi\psi_{1,n}^s} f_{4,n}^s\right). \end{aligned} \tag{3.26}$$

Using that:

$$\lim_{x\to 0}\Im\left(\chi\psi_{1,n}^s\overline{\chi\psi_{2,n}^s}+\chi\psi_{3,n}^s\overline{\chi\psi_{4,n}^s}\right)=0. \tag{3.27}$$

and that $\chi\psi_{j,n}^s$ is 0 at 1 for all $j=1,\dots,4$, we obtain:

$$\begin{aligned}&\left(s+\frac{1}{2}\right)\int_{-\frac{1}{2}}^{0}A\left(x\right)\sum_{j=1}^{4}\left|\chi\psi_{j,n}^s\right|^2dx\\ =&\int_{-\frac{1}{2}}^{0}\Re\left(\overline{\chi\psi_{4,n}^s}f_{1,n}^s+\overline{\chi\psi_{3,n}^s}f_{2,n}^s+\overline{\chi\psi_{2,n}^s}f_{3,n}^s+\overline{\chi\psi_{1,n}^s}f_{4,n}^s\right)dx.\end{aligned}$$

After some calculations, this gives:

$$\left(s+\frac{1}{2}\right)^2\int_{-\frac{1}{2}}^{0}\left(2lA\left(x\right)-1\right)\sum_{j=1}^{4}\left|\chi\psi_{j,n}^s\right|^2dx\leqslant\int_{-\frac{1}{2}}^{0}\sum_{j=1}^{4}l^2\left|f_{j,n}^s\right|^2dx.$$

Using the asymptotic behavior of A (see (2.24)), we can prove that $2lA\left(x\right)-1\geqslant 1$ on the support of χ (for ϵ sufficiently small). Finally, we obtain:

$$\left(s+\frac{1}{2}\right)^2\int_{-\frac{1}{2}}^{0}\sum_{j=1}^{4}\left|\chi\psi_{j,n}^s\right|^2dx\leqslant l^2\int_{-\frac{1}{2}}^{0}\sum_{j=1}^{4}\left|f_{j,n}^s\right|^2dx \tag{3.28}$$

and the right hand side is summable because $f\in\mathcal{H}$. This gives the lemma. □

We can know apply Theorem 3.1 to $\chi\psi$ and obtain the asymptotic behavior of ψ:

Proposition 3.3. *If $2ml>1$, we have:*

$$||\psi(\zeta,.)||_{L^2(S^2)}=O\left(\sqrt{-x}\right),\ \ x\to 0. \tag{3.29}$$

If $2ml=1$, we have:

$$||\psi(x,.)||_{L^2(S^2)}=O\left(\sqrt{(-x)\,ln\,(-x)}\right),\ \ x\to 0. \tag{3.30}$$

If $0<2ml<1$, there exists functions $\psi_-\in W_-^{\frac{1}{2}}$, $\chi_-\in W_+^{\frac{1}{2}}$, ψ_+, $\chi_+\in L^2(S^2)$ and $\phi\in C^0\left(]-\infty,0]_x;L^2(S^2;\mathbb{C}^4)\right)$ satisfying (3.16) and (3.17) with $\frac{\pi}{2}-\zeta$ replaced by $(-x)^l$.

Conversely, for all $\psi_- \in W_-^{\frac{1}{2}+m}$, $\chi_- \in W_+^{\frac{1}{2}+m}$, $\psi_+ \in W_-^{\frac{1}{2}-m}$, $\chi_+ \in W_+^{\frac{1}{2}-m}$, *there exists* $\psi \in D(H_m)$ *satisfying* (3.16) *and* (3.17) *with the same replacement as before.*

Remark. By restriction to $\mathcal{H}_{s,n}$, we obtain the same result for s, n fixed. Moreover, if $\varphi_{s,n} \in D\left(H_m^{s,n}\right)$, then it is in $H^1\left(\left]-\infty, -c\right[\right)$ for a constant $c > 0$. We conclude that $\varphi_{s,n} \in C^0\left(\left]-\infty, -c\right[\right) \cap L^2\left(\left]-\infty, -c\right[\right)$ and:

$$\left\|\varphi_{s,n}(x, .)\right\|_{\mathcal{W}^0} \to 0, \quad x \to -\infty. \tag{3.31}$$

3.1.5. Description of the domain. We now give a description of the domain of H_m for fixed $(s, n) \in I$. Recall that H_m and $\tilde{H}_m$ are linked by a unitary transform, so it does not change the norm of the observables. We obtain:

(3.32)

$$-D\left(H_m^{s,n}\right) = \left\{\psi_{s,n} \in \mathcal{H}_{s,n};\ H_m^{s,n}\psi_{s,n} \in \mathcal{H}_{s,n}\right\}, \text{ if } 2ml \geqslant 1;$$

$$-D\left(H_m^{s,n}\right) = \left\{ \psi_{s,n} \in \mathcal{H}_{s,n};\ H_m^{s,n}\psi_{s,n} \in \mathcal{H}_{s,n},\ \psi_{s,n} = (-x)^{-ml} \begin{pmatrix} \psi^s_{-,n}(\theta,\varphi) \\ i\chi^s_{-,n}(\theta,\varphi) \\ -\psi^s_{-,n}(\theta,\varphi) \\ i\chi^s_{-,n}(\theta,\varphi) \end{pmatrix} \right.$$

$$\left. + \phi_n^s(x, \theta, \varphi), \left\|\phi_n^s(x, ., .)\right\|_{\mathcal{W}^0} = o\left(\sqrt{-x}\right) \right\}, \text{ if } 2ml < 1. \tag{3.33}$$

3.2. Self-adjointness for fixed harmonic

In this section, s and n are fixed.

3.2.1. The case $2ml \geqslant 1$.

Lemma 3.4 (Elliptic estimate). *We suppose that* $2ml > 1$*. Then, there exists a constant* $C > 0$ *such that, for all* $\varphi \in C_0^\infty\left(\left]-\infty, 0\right[\right)$*, we have:*

$$\left\|-i\partial_x\varphi\right\|^2 \leqslant C\left(\left\|H_m^{s,n}\varphi\right\|^2 + \left\|\varphi\right\|^2\right) \tag{3.34}$$

Proof. We write $D_x = -i\partial_x$ and $\Gamma^1 = -\gamma^0\gamma^1$. Recall that:

$$H_m^{s,n} = \Gamma^1 D_x + \left(s + \frac{1}{2}\right) A(x)\gamma^0\gamma^2 - mB(x)\gamma^0.$$

We will often denote $V(x) = \left(s + \frac{1}{2}\right) A(x)\gamma^0\gamma^2 - mB(x)\gamma^0$. Choose a partition of unity χ_1, χ_2 such that $\chi_1 + \chi_2 = 1$, $\operatorname{supp}(\chi_1) \subset]-\infty, -\epsilon[$ and $\chi_1 =$

1 on $]-\infty,-2\epsilon[$, $\operatorname{supp}(\chi_2)\subset]-2\epsilon,0[$ and $\chi_2=1$ on $]-\epsilon,0[$. We choose $\epsilon>0$ sufficiently small so that, if γ_B^5 and P are unitary matrices defined as in (3.22), $\gamma^5P^{-1}\chi_2\varphi\in D\left(H_m^B\right)$ when $\varphi\in D\left(H_m^{s,n}\right)$ (it is possible by lemma 3.2). Recall that m is the mass of the field and l correspond to the cosmological constant. Using equation $III.32$ in theorem $III.4$ of [8], (3.22) and (3.25), we obtain:

$$\left\|D_x\left(\gamma_B^5P^{-1}\chi_2\varphi\right)\right\|\leqslant C_{m,l}\left\|g\left(x\right)H_m^{s,n}\left(\gamma_B^5P^{-1}\chi_2\varphi\right)\right\|+\tilde{C}_{m,l}\left\|\chi_2\varphi\right\|,$$

where $C_{m,l}$ and $\tilde{C}_{m,l}$ are constants depending on m and l. Since $\gamma_B^5P^{-1}$ is unitary and commute with D_x and g is bounded near 0, we obtain:

$$\left\|D_x\left(\chi_2\varphi\right)\right\|\leqslant C_{m,l,\epsilon}\left\|H_m^{s,n}\left(\chi_2\varphi\right)\right\|+\tilde{C}_{m,l}\left\|\chi_2\varphi\right\|. \tag{3.35}$$

On the other hand, with $C_{V,\epsilon}$ constant, we have:

$$\left\|D_x\left(\chi_1\varphi\right)\right\|\leqslant\left\|H_m^{s,n}\left(\chi_1\varphi\right)\right\|+C_{V,\epsilon}\left\|\varphi\right\|.$$

Since χ_1,χ_2 commute with V and are bounded as are their derivatives, we obtain:

$$\begin{aligned}\left\|D_x\varphi\right\|^2&\leqslant C\left(\left\|H_m^{s,n}\left(\chi_1\varphi\right)\right\|^2+\left\|H_m^{s,n}\left(\chi_2\varphi\right)\right\|^2\right)+C'\left\|\varphi\right\|^2\\&\leqslant\tilde{C}\left\|H_m^{s,n}\varphi\right\|^2+\tilde{C}'\left\|\varphi\right\|^2.\end{aligned}$$

□

Proposition 3.5. *For $2ml\geqslant 1$, the operator $\tilde{H}_m^{s,n}$ is essentially self-adjoint on $C_0^\infty\left(]-\infty,0[\right)$. Moreover, if $2ml>1$, the domain of this operator is given by $H_0^1\left(]-\infty,0[\right)$.*

Proof. Recall that:

$$\tilde{H}_m^{s,n}=i\gamma_B^0\gamma_B^1\partial_x+\gamma_B^0\gamma_B^2\left(s+\frac{1}{2}\right)A(x)-m\gamma_B^0B(x)$$

with domain $D\left(\tilde{H}_m^{s,n}\right)=\left\{\psi_{s,n}\in\mathcal{H}_{s,n};\ \tilde{H}_m^{s,n}\psi_{s,n}\in\mathcal{H}_{s,n}\right\}$ and if $\psi_{s,n}\in D\left(\tilde{H}_m^{s,n}\right)$, then we have:

$$\left\|\psi_{s,n}(x,.)\right\|_{L^2(S^2)}=O\left(\sqrt{(-x)}\right),\quad x\to 0,\quad\text{if } 2ml>1; \tag{3.36}$$

$$\left\|\psi_{s,n}(x,.)\right\|_{L^2(S^2)}=O\left(\sqrt{x\ln(-x)}\right),\quad x\to 0,\quad\text{if } 2ml=1; \tag{3.37}$$

$$\left\|\psi_{s,n}\left(x,.\right)\right\|_{\mathcal{W}^0}\to 0,\quad x\to-\infty. \tag{3.38}$$

Let us prove that $\tilde{H}_m^{s,n}$ is symmetric on its domain. We remark that $\left(\gamma_B^0\gamma_B^2\right)^* = \gamma_B^0\gamma_B^2$, $\left(\gamma_B^0\gamma_B^1\right)^* = \gamma_B^0\gamma_B^1$ and $\left(\gamma_B^0\right)^* = \gamma_B^0$. So:

$$\left\langle \gamma_B^0\gamma_B^2 A(x)\left(s+\frac{1}{2}\right)\phi_{s,n}, \psi_{s,n}\right\rangle_{\mathcal{H}_{s,n}} = \left\langle \phi_{s,n}, \left(\gamma_B^0\gamma_B^2\right) A(x)\left(s+\frac{1}{2}\right)\psi_{s,n}\right\rangle_{\mathcal{H}_{s,n}},$$
$$\left\langle \gamma_B^0 B(x)\phi_{s,n}, \psi_{s,n}\right\rangle_{\mathcal{H}_{s,n}} = \left\langle \phi_{s,n}, \gamma_B^0 B(x)\psi_{s,n}\right\rangle_{\mathcal{H}_{s,n}}.$$

Thus, in the calculation of $\left\langle \tilde{H}_m^{s,n}\phi_{s,n}, \psi_{s,n}\right\rangle_{\mathcal{H}_{s,n}} - \left\langle \phi_{s,n}, \tilde{H}_m^{s,n}\psi_{s,n}\right\rangle_{\mathcal{H}_{s,n}}$, it remains only the boundary term due to integration by parts. Using (3.37), this gives the symmetry of our operator on its domain.

We then use the same trick as in [8]. Let us consider a new operator H with the same expression as $\tilde{H}_m^{s,n}$ but defined on $D(H) = C_0^\infty(]-\infty,0[)$. Then H^* is $\tilde{H}_m^{s,n}$ with domain $D(H^*)$ included in $D\left(\tilde{H}_m^{s,n}\right)$. Let $\phi_\pm \in \ker(H^* \pm iId)$. Then, using the symmetry of $\tilde{H}_m^{s,n}$ and that $H^* = \tilde{H}_m^{s,n}$, we have:

$$\begin{aligned} 0 &= \left\langle \tilde{H}_m^{s,n}\phi_\pm, \phi_\pm\right\rangle - \left\langle \phi_\pm, \tilde{H}_m^{s,n}\phi_\pm\right\rangle \\ &= \langle H^*\phi_\pm, \phi_\pm\rangle - \langle \phi_\pm, H^*\phi_\pm\rangle = \mp 2i\,\|\phi_\pm\|^2_{\mathcal{H}_{s,n}}. \end{aligned} \tag{3.39}$$

We conclude that $\phi_\pm = 0$. This proves that $\tilde{H}_m^{s,n}$ is essentially self-adjoint on $C_0^\infty(]-\infty,0[)$.

For the last part, using the last lemma, we see that, for $2ml > 1$, we have: $D\left(H_m^{s,n}\right) \subset H_0^1(]-\infty,0[)$. Indeed, if we take $\varphi \in D\left(H_m^{s,n}\right)$, it is the limit of a sequence $(\varphi_n)_{n\in\mathbb{N}} \in \left(C_0^\infty\right)^{\mathbb{N}}$ for the graph norm. The last lemma gives that $\partial_x\varphi_n$ is a Cauchy sequence so that it converges in H_0^1. A distribution argument gives that this limit is $\partial_x\varphi$ which is in L^2 by the lemma.

Moreover, we have $H_m^{s,n} = i\gamma^0\gamma^1\partial_x + \gamma^0\gamma^2\left(s+\frac{1}{2}\right)A(x) - m\gamma^0 B(x)$ with A having the behavior as in (2.24) and B as in (2.25). Using the fact that B and B_B have the same behavior when $x \to 0$ and the unitary transform, we can use the proof of Theorem $III.4$ in [8] to prove a Hardy type inequality of the form:

$$\left\|B\chi_2^2\varphi\right\| \leqslant c\left(\|\varphi\| + \|-i\partial_x\varphi\|\right). \tag{3.40}$$

Using the fact that A is bounded, we have a similar estimate for

$$\gamma^0\gamma^2\left(s+\frac{1}{2}\right)A(x) - m\gamma^0 B(x).$$

Thus $H_0^1 \subset D\left(H_m^{s,n}\right)$. This proves the proposition. □

3.2.2. The case $2ml<1$. Recall that if $0<2ml<1$, then, for all $\psi_{s,n}\in D\left(\tilde{H}_m^{s,n}\right)$, there exists functions $\psi_-\in W_-^{\frac{1}{2}}$, $\chi_-\in W_+^{\frac{1}{2}}$, $\psi_+,\chi_+\in L^2(S^2)$ and $\phi\in C^0\left([0,\frac{\pi}{2}]_x;L^2(S^2;\mathbb{C}^4)\right)$ such that $\|\sigma_n^s(x,\theta,\varphi)\|_{\mathcal{W}^0}=o\left(\sqrt{(-x)}\right)$ as x goes to 0 and:

$$\begin{aligned}(3.41)\quad \psi_{s,n}(x,\theta,\varphi)&=(-x)^{-ml}\begin{pmatrix}\psi^s_{-,n}(\theta,\varphi)\\ \chi^s_{-,n}(\theta,\varphi)\\ -i\psi^s_{-,n}(\theta,\varphi)\\ i\chi^s_{-,n}(\theta,\varphi)\end{pmatrix}+(-x)^{ml}\begin{pmatrix}\psi^s_{+,n}(\theta,\varphi)\\ \chi^s_{+,n}(\theta,\varphi)\\ i\psi^s_{+,n}(\theta,\varphi)\\ -i\chi^s_{+,n}(\theta,\varphi)\end{pmatrix}\\ &\quad+\sigma_n^s(x,\theta,\varphi),\\ &:=(-x)^{-ml}\,\Psi^s_{-,n}(\theta,\varphi)+(-x)^{ml}\,\Psi^s_{+,n}(\theta,\varphi)\\ &\quad+\sigma_n^s(x,\theta,\varphi)\end{aligned}$$

We denote by $\tilde{H}_{s,n}^{MIT}$ the operator $\tilde{H}_m^{s,n}$ with domain:

$$(3.42)\quad D(\tilde{H}_{s,n}^{MIT})=\left\{\psi_{s,n}\in\mathcal{H}_{s,n};\ \tilde{H}_m^{s,n}\psi_{s,n}\in\mathcal{H}_{s,n},\ \psi^s_{+,n}=\chi^s_{+,n}=0\right\}.$$

which is a consequence of the discussion after proposition $VI.2$ in [8]. We have the:

Proposition 3.6. *The operator $\tilde{H}_{s,n}^{MIT}$ is self-adjoint on $D\left(\tilde{H}_{s,n}^{MIT}\right)$.*

Proof. Let $\phi_{s,n},\psi_{s,n}\in D(\tilde{H}_{s,n}^{MIT})$. As in the proof of Proposition 3.5, when calculating

$$\left\langle \tilde{H}_{s,n}^{MIT}\phi_{s,n},\psi_{s,n}\right\rangle_{\mathcal{H}_{s,n}}-\left\langle\phi,\tilde{H}_{s,n}^{MIT}\psi\right\rangle_{\mathcal{H}_{s,n}},$$

only boundary values of $\phi_{s,n},\psi_{s,n}$ are left. Using that

$$\begin{aligned}\phi_{s,n}(x,\theta,\varphi)&=(-x)^{-ml}\begin{pmatrix}\phi^s_{-,n}(\theta,\varphi)\\ \xi^s_{-,n}(\theta,\varphi)\\ -i\phi^s_{-,n}(\theta,\varphi)\\ i\xi^s_{-,n}(\theta,\varphi)\end{pmatrix}+\varphi_n^s(x,\theta,\varphi)\\ &:=(-x)^{-ml}\,\Phi^s_{-,n}(\theta,\varphi)+\varphi_n^s\\ \|\varphi_{s,n}\|_{L^2(S^2)}&=o\left(\sqrt{(-x)}\right),\ x\to 0,\end{aligned}$$

and a similar formula for $\psi_{s,n}$ with $\Phi^s_{-,n},\varphi^s_n$ replaced by $\Psi^s_{-,n},\sigma^s_n$ respectively, we can calculate these boundary values in a neighbourhood of 0 (with the scalar product being the one of $L^2\left(S^2\right)$ and we write $\phi_{s,n}(x)$ for $\phi_{s,n}(x,.)$):

$$\left\langle \phi_{s,n}(x), \gamma_B^0\gamma_B^1\psi_{s,n}(x)\right\rangle = (-x)^{-ml}\left(\left\langle \Phi^s_{-,n}, \sigma_n^s(x)\right\rangle + \left\langle \varphi_n^s(x), \Psi^s_{-,n}\right\rangle\right) + \left\langle \varphi_n^s(x), \sigma_n^s(x)\right\rangle_{\mathcal{W}^0}.$$

Indeed, $\gamma_B^0\gamma_B^1$ arranges the terms such that

$$\left\langle (-x)^{-ml}\,\Phi^s_{-,n}, (-x)^{-ml}\,\Psi^s_{-,n}\right\rangle_{\mathcal{W}^0} = 0.$$

Using the behavior at 0 of φ_n^s, σ_n^s and at $-\infty$ of $\phi_{s,n}, \psi_{s,n}$, we deduce that $\tilde{H}^{MIT}_{s,n}$ is symmetric.

Let $\psi_{s,n} \in D(\tilde{H}_m^{s,n,MIT,*})$. Then, since $D(\tilde{H}_m^{s,n,MIT,*}) \subset D(\tilde{H}_m^{s,n})$, ψ admits a decomposition, in a neighbourhood of 0, as in (3.41). Moreover, $\tilde{H}_m^{s,n,MIT,*} = \tilde{H}_m^{s,n}$ on $D\left(\tilde{H}_m^{s,n,MIT,*}\right)$ (using distributions). We have:

$$\begin{aligned} 0 &= \left\langle \tilde{H}_m^{s,n,MIT}\phi_{s,n}, \psi_{s,n}\right\rangle - \left\langle \phi_{s,n}, \tilde{H}_m^{s,n,MIT,*}\psi_{s,n}\right\rangle \\ &= \lim_{x\to 0}\left\langle (-x)^{-ml}\,\Phi^s_{-,n}, (-x)^{ml}\,\Psi^s_{+,n}\right\rangle, \end{aligned}$$

for all $\phi_{s,n} \in D\left(\tilde{H}_m^{s,n,MIT}\right)$ and $\psi_{s,n} \in D\left(\tilde{H}_m^{s,n,MIT,*}\right)$. In other words, we have:

$$2\left\langle \begin{pmatrix}\phi^s_{-,n}\\ \xi^s_{-,n}\end{pmatrix}, \begin{pmatrix}\psi^s_{+,n}\\ \chi^s_{+,n}\end{pmatrix}\right\rangle = 0. \tag{3.43}$$

But, for all $\phi^s_{-,n}, \xi^s_{-,n} \in C_0^\infty(Y_{s,n})$, we can find $\phi \in D(\tilde{H}_m^{s,n,MIT})$ admitting these components as coordinates. Thus $\psi^s_{+,n} = \chi^s_{+,n} = 0$. We conclude that $D\left(\tilde{H}_m^{s,n,MIT,*}\right) \subset D\left(\tilde{H}_m^{s,n,MIT}\right)$ and that $\tilde{H}_m^{s,n,MIT}$ is self-adjoint on his domain. □

3.3. Self-adjointness of $\tilde{H}_m$

3.3.1. The case $2ml \geqslant 1$. We equip $\tilde{H}_m$ with the domain:

$$D(\tilde{H}_m) = \left\{u \in \mathcal{H};\ \ \tilde{H}_m u \in \mathcal{H}\right\}$$

$$= \left\{ \sum_{(s,n)\in I} \begin{pmatrix} u^s_{1,n} T^s_{-\frac{1}{2},n} \\ u^s_{2,n} T^s_{\frac{1}{2},n} \\ u^s_{3,n} T^s_{-\frac{1}{2},n} \\ u^s_{4,n} T^s_{\frac{1}{2},n} \end{pmatrix};\ \forall (s,n) \in I,\ \ u^s_n \in L^2\left(\left]-\infty, 0\right[_x, dx\right),\right.$$

(3.44)

$$\left. \tilde{H}^{s,n}_m \begin{pmatrix} u^s_{1,n} T^s_{-\frac{1}{2},n} \\ u^s_{2,n} T^s_{\frac{1}{2},n} \\ u^s_{3,n} T^s_{-\frac{1}{2},n} \\ u^s_{4,n} T^s_{\frac{1}{2},n} \end{pmatrix} \in L^2,\ \sum_{(s,n)\in I} \left\| (\tilde{H}^{s,n}_m \pm i) \begin{pmatrix} u^s_{1,n} T^s_{-\frac{1}{2},n} \\ u^s_{2,n} T^s_{\frac{1}{2},n} \\ u^s_{3,n} T^s_{-\frac{1}{2},n} \\ u^s_{4,n} T^s_{\frac{1}{2},n} \end{pmatrix} \right\|^2_{L^2} < \infty \right\}.$$

We then have:

Proposition 3.7. *Suppose that $2ml \geqslant 1$. Then the operator $\tilde{H}_m$ is self-adjoint on its domain.*

Proof. $\tilde{H}_m$ is symmetric. Indeed, let $\varphi, \psi \in D\left(\tilde{H}_m\right)$. We can decompose $\varphi = \sum_{(s,n)\in I} \varphi_{s,n}$ and the same for ψ. Then:

$$\left\langle \tilde{H}_m \varphi, \psi \right\rangle = \sum_{(s,n)\in I} \left\langle \tilde{H}^{s,n}_m \varphi_{s,n}, \psi_{s,n} \right\rangle \tag{3.45}$$
$$= \sum_{(s,n)\in I} \left\langle \varphi_{s,n}, \tilde{H}^{s,n}_m \psi_{s,n} \right\rangle = \left\langle \varphi, \tilde{H}_m \psi \right\rangle$$

since $\tilde{H}^{s,n}_m$ is symmetric. We can prove that $\tilde{H}_m$ is closed in the same way. Let $x = \sum_{(s,n)\in I} x_{s,n} \in \mathcal{H}$. Since $\tilde{H}^{s,n}_m$ is self-adjoint, there exists $y_{s,n} \in D\left(\tilde{H}^{s,n}_m\right)$ such that $(\tilde{H}_m \pm i) y_{s,n} = (\tilde{H}^{s,n}_m \pm i) y_{s,n} = x_{s,n}$. Thus

$$x = \sum_{(s,n)\in I} (\tilde{H}_m \pm i) y_{s,n} = \left(\tilde{H}_m \pm i\right) y$$

where $y = \sum_{(s,n)\in I} y_{s,n} \in D\left(\tilde{H}_m\right)$ since:

$$\sum_{(s,n)\in I} \left\|\tilde{H}_m^{s,n} y_{s,n}\right\|^2 + \|y_{s,n}\|^2 = \sum_{(s,n)\in I} \left\|(\tilde{H}_m^{s,n} \pm i) y_{s,n}\right\|^2 = \sum_{(s,n)\in I} \|x_{s,n}\|^2 < \infty.$$

Consequently, $(y_{s,n})_{(s,n)\in I}$ is summable and $x \in Im(\tilde{H}_m \pm i)$ so $Im(\tilde{H}_m \pm i) = \mathcal{H}$ and $\tilde{H}_m$ is self-adjoint. □

3.3.2. The case $2ml < 1$. Let us denote $\tilde{H}_m^{MIT}$ the operator $\tilde{H}_m$ with domain:

$$D\left(\tilde{H}_m^{MIT}\right) = \left\{ \sum_{(s,n)\in I} \begin{pmatrix} u_{1,n}^s T_{-\frac{1}{2},n}^s \\ u_{2,n}^s T_{\frac{1}{2},n}^s \\ u_{3,n}^s T_{-\frac{1}{2},n}^s \\ u_{4,n}^s T_{\frac{1}{2},n}^s \end{pmatrix} ;\ \forall (s,n) \in I,\ u_n^s \in L^2\left(]-\infty, 0[_x, dx\right), \right.$$

$$\tilde{H}_m^{s,n} \begin{pmatrix} u_{1,n}^s T_{-\frac{1}{2},n}^s \\ u_{2,n}^s T_{\frac{1}{2},n}^s \\ u_{3,n}^s T_{-\frac{1}{2},n}^s \\ u_{4,n}^s T_{\frac{1}{2},n}^s \end{pmatrix} \in L^2,\ \sum_{(s,n)\in I} \left\| (\tilde{H}_m^{s,n} \pm i) \begin{pmatrix} u_{1,n}^s T_{-\frac{1}{2},n}^s \\ u_{2,n}^s T_{\frac{1}{2},n}^s \\ u_{3,n}^s T_{-\frac{1}{2},n}^s \\ u_{4,n}^s T_{\frac{1}{2},n}^s \end{pmatrix} \right\|_{L^2}^2 < \infty$$

$$\left. \sum_{(s,n)\in I} \left\| \left(\gamma_B^1 + i\right) \begin{pmatrix} u_{1,n}^s T_{-\frac{1}{2},n}^s \\ u_{2,n}^s T_{\frac{1}{2},n}^s \\ u_{3,n}^s T_{-\frac{1}{2},n}^s \\ u_{4,n}^s T_{\frac{1}{2},n}^s \end{pmatrix} \right\|_{L^2}^2 = o\left(\sqrt{-x}\right),\ x \to 0 \right\} \tag{3.46}$$

Proposition 3.8. *Suppose that $2ml < 1$. Then the operator $\tilde{H}_m^{MIT}$ is self-adjoint with domain $D\left(\tilde{H}_m^{MIT}\right)$.*

Proof. Let us remark that, if the boundary condition is fulfilled for $\phi \in D\left(\tilde{H}_m^{MIT}\right)$, then it is fulfilled for $\phi_{s,n} \in D\left(\tilde{H}_m^{s,n,MIT}\right)$. We can now prove, as in the proof of Proposition 3.7, that $\tilde{H}_m^{MIT}$ is symmetric on its domain. Show that $\tilde{H}_m^{MIT}$ is closed will require more effort. Choose a sequence $(\psi_j)_{j\in\mathbb{N}}$ of elements of $D\left(\tilde{H}_m^{MIT}\right)$ such that $\psi_j \to \psi$ and $\tilde{H}_m^{MIT}\psi_j \to \varphi$ where $\psi, \varphi \in \mathcal{H}$ and the convergence is understood in the norm of $\mathcal{H}$. Using distributions, we have $\tilde{H}_m^{MIT}\psi = \varphi \in \mathcal{H}$ and we have to show that ψ satisfies the boundary condition. We can write:

$$\psi_j = \sum_{(s,n)\in I} \psi_j^{s,n}, \quad \psi = \sum_{(s,n)\in I} \psi^{s,n}, \quad \varphi = \sum_{(s,n)\in I} \varphi^{s,n}, \tag{3.47}$$

and we obtain:

$$\psi_j^{s,n} \to \psi^{s,n}; \quad \tilde{H}_m^{s,n,MIT}\psi_j^{s,n} \to \varphi^{s,n}$$

in the norm of $\mathcal{H}_{s,n}$. Thus, $\psi^{s,n} \in D\left(\tilde{H}_m^{s,n,MIT}\right)$ since $\tilde{H}_m^{s,n,MIT}$ is closed and $\psi^{s,n}$ admits a decomposition as in (3.33) where:

$$\sum_{(s,n)\in I} \left(\left\|\phi_{1,n}^s\right\|_{\mathcal{W}^0}^2 + \left\|\phi_{2,n}^s\right\|_{\mathcal{W}^0}^2 + \left\|\phi_{3,n}^s\right\|_{\mathcal{W}^0}^2 + \left\|\phi_{4,n}^s\right\|_{\mathcal{W}^0}^2 \right) = o\left(-x\right)$$

when x goes to 0, using the proof of theorem $V.1$ in [8] and the fact that φ is in the natural domain of H_m. Since $\gamma_B^1 + i$ eliminates the terms containing $(-x)^{-ml}$, we have:

$$\begin{aligned} &\left\|\left(\gamma_B^1 + i\right)\varphi\left(x,.\right)\right\|_{L^2(S^2)}^2 \\ &\leqslant C \sum_{(s,n)\in I} \left(\left\|\phi_{1,n}^s\right\|_{\mathcal{W}^0}^2 + \left\|\phi_{2,n}^s\right\|_{\mathcal{W}^0}^2 + \left\|\phi_{3,n}^s\right\|_{\mathcal{W}^0}^2 + \left\|\phi_{4,n}^s\right\|_{\mathcal{W}^0}^2 \right) \end{aligned} \tag{3.48}$$

where the last term is $o\left(-x\right)$. This proves that the boundary condition is fulfilled and that the operator $\tilde{H}_m^{MIT}$ is closed. To prove the self-adjointness of $\tilde{H}_m^{MIT}$, we follow the same argument as in Proposition 3.7 where we have to prove that $y = \sum_{(s,n)\in I} y_{s,n} \in D\left(\tilde{H}_m^{MIT}\right)$. The only difference is that the boundary condition has to be fulfilled. Since $y_{s,n} \in D\left(\tilde{H}_m^{s,n,MIT}\right)$, we can decompose $y_{s,n}$ as for $\varphi^{s,n}$ just above. A similar argument shows that y satisfies the boundary condition. Thus $\tilde{H}_m^{MIT}$ is self-adjoint on $D\left(\tilde{H}_m^{MIT}\right)$. □

3.3.3. Self-adjointness of H_m.

Recall that the domain of H_m is:

- If $2ml \geqslant 1$:

$$D(H_m) = \left\{\phi \in \mathcal{H};\ H_m\phi \in \mathcal{H}\right\}.$$

- If $0 < m < \frac{1}{2l}$, we will denote by H_m^{MIT} the operator H_m with domain:

$$D(H_m^{MIT}) = \left\{\phi\in\mathcal{H};\ H_m\phi\in\mathcal{H},\ \left\|\left(\gamma^1 + i\right)\phi\left(x,.\right)\right\|_{L^2(\mathbb{S}^2)} = o\left(\sqrt{-x}\right),\ x\to 0\right\}.$$

We obtain the following theorem:

Theorem 3.9. - *For all $m \geqslant \frac{1}{2l}$, the operator H_m with domain $D(H_m)$ is self-adjoint.*

- *For all $m < \frac{1}{2l}$, the operator H_m^{MIT} with domain $D(H_m^{MIT})$ is self-adjoint.*

Proof. Recall that $\tilde{H}_m = \gamma_B^5 P^{-1}(-H_m) P\gamma_B^5$ where γ_B^5 and P are unitary matrices. Thus $H_m = P\gamma_B^5\left(-\tilde{H}_m\right)\gamma_B^5 P^{-1}$. This is clear that $\psi \in D(H_m)$ if and only if $\gamma_B^5 P^{-1}\psi \in D\left(\tilde{H}_m\right)$ for $m \geqslant \frac{1}{2l}$. Moreover, recall that $\gamma^1 = -P\gamma_B^1 P^{-1}$ and $\gamma_B^1\gamma_B^5 = -\gamma_B^5\gamma_B^1$ using (3.23) and (2.18). We then obtain:

$$\left\|\left(\gamma^1 + i\right)\psi\right\| = \left\|\left(\gamma_B^1 + i\right)\gamma_B^5 P^{-1}\psi\right\|.$$

Thus $\psi \in D(H_m)$ if and only if $\gamma_B^5 P^{-1}\psi \in D\left(\tilde{H}_m\right)$ for all $m > 0$. This shows that H_m is self-adjoint equipped with the convenient domain. □

3.4. The Cauchy problem

Using Stone theorem, we obtain:

Theorem 3.10. *Let $\psi_0 \in \mathcal{H}$, there exists a unique solution ψ to the equation:*

$$\partial_t\psi = iH_m\psi \tag{3.49}$$

such that

$$\psi \in C^0\left(\mathbb{R}_t; \mathcal{H}\right) \tag{3.50}$$

and satisfying:

$$\psi(t = 0, .) = \psi_0(.) \tag{3.51}$$

$$\forall t \in \mathbb{R},\ \|\psi(t, .)\|_{\mathcal{H}} = \|\psi_0(.)\|_{\mathcal{H}}. \tag{3.52}$$

3.5. Absence of eigenvalues

Proposition 3.11. *For all $m > 0$, the Dirac operator H_m, defined in* (2.23), *does not admit any eigenvalues.*

Proof. Let us first show the absence of eigenvalues for $H_m^{s,n}$ for all $m > 0$ and all $(s,n) \in I$. Since $H_m^{s,n}$ is self-adjoint on its domain, the eigenvalues (if they exist) are all real. So, suppose that there exists $\lambda \in \mathbb{R}$ and $\varphi \in D\left(H_m^{s,n}\right)$ such that $H_m^{s,n}\varphi = \lambda\varphi$.

We define:

$$w(x) = e^{i\lambda\gamma^0\gamma^1 x}\varphi(x)$$

such that

$$w'(x) = i\lambda\gamma^0\gamma^1 w(x) + e^{i\lambda\gamma^0\gamma^1 x}\varphi'(x).$$

But, with $V(x) = \gamma^0\gamma^2 A(x)\left(s + \frac{1}{2}\right) - m\gamma^0 B(x)$, we have:

$$\begin{aligned} H_m^{s,n}\varphi - \lambda\varphi = 0 &\Leftrightarrow i\gamma^0\gamma^1\varphi'(x) = (\lambda - V(x))\,\varphi(x) \\ &\Leftrightarrow \varphi'(x) = i\gamma^0\gamma^1\,(V(x) - \lambda)\,\varphi(x) \end{aligned}$$

So, we obtain:

$$w'(x) = i\gamma^0\gamma^1 e^{i\lambda\gamma^0\gamma^1 x} V(x) e^{-i\lambda\gamma^0\gamma^1 x} w(x). \tag{3.53}$$

Write: $W(x) = i\gamma^0\gamma^1 e^{i\lambda\gamma^0\gamma^1 x} V(x) e^{-i\lambda\gamma^0\gamma^1 x}$. Let $T \in]-\infty, 0[$, we can then solve the preceding equation by:

$$w(x) = e^{\int_T^x W(t)dt} w(T).$$

As in the remark after Proposition 3.3, each component of φ goes to 0 at $-\infty$. Consequently, $w(x) \underset{x\to-\infty}{\to} 0$.

On the other hand, for all $x < 0$, $\displaystyle\int_{-\infty}^x |W(t)|\,dt < \infty$ so:

$$\lim_{T\to-\infty} e^{\int_T^x W(t)dt} = e^{\int_{-\infty}^x W(t)dt}$$

exists and is finished. As a consequence, we have:

$$\lim_{T\to-\infty} e^{\int_T^x W(t)dt} w(T) = 0.$$

We then deduce that $w(x) = 0$ for all $x < 0$ so it is the same for φ. Consequently, $H_m^{s,n}$ admits no eigenvalues.

We can now consider H_m. If $\lambda \in \mathbb{R}$ is an eigenvalue of H_m then there exists $\varphi \in D(H_m)$ such that $(H_m - \lambda)\,\varphi = 0$. Using the decomposition of φ in spherical harmonics, if φ is non zero, there exists $(s,n) \in I$ such that $\varphi_{s,n} \neq 0$ and $\varphi_{s,n}$ satisfies $\left(H_m^{s,n} - \lambda\right)\varphi_{s,n} = 0$. This is impossible since $H_m^{s,n}$

does not admit eigenvalues. Thus φ is identically 0. We deduce that H_m does not admit any eigenvalue for all $m > 0$. □

4. Compactness results

The purpose of this section is to prove that, for a well chosen function f, the operator $f(x)\left(H_m^{s,n} - z\right)^{-1}$ is compact for all $z \in \rho\left(H_m^{s,n}\right)$. We will make use of this result for proving Mourre estimates in the following section. The key point here for the Mourre estimate is that f only admits a finite limit at 0.

This result is proved by separating our operator in two operators denoted H_+ and H_-. The operator H_+ has a potential which behaves like the one in $H_m^{s,n}$ at 0 and is extended so that the potential becomes confining. Hence the resolvent of this operator is itself compact. For H_-, we preserve the behavior near the horizon of the black hole and extend it so that it decreases to 0 at 0. By extending the states and the potential, we are thus able to view the resolvent as the restriction of a resolvent on the entire line. For this last resolvent, we are able to use standard results about Hilbert-Schmidt operators.

We now enter into the details. We have:

$$H_m^{s,n} = \Gamma^1 D_x + \left(s + \frac{1}{2}\right) A(x)\gamma^0\gamma^2 - m\gamma^0 B(x). \tag{4.1}$$

where A and B behave like:

$$A - A_0 \in T^{\kappa,2}; \quad B - B_0 \in T^{\tilde{\kappa},1}$$

with $\kappa, \tilde{\kappa} > 0$. Moreover, $\Gamma^1 = -\gamma^0\gamma^1$ where $\gamma^0\gamma^1$ is given in (2.16). The main result of this section is:

Proposition 4.1. *Let $f \in C\left(]-\infty, 0]\right)$ such that f goes to 0 at $-\infty$. Let $z \in \rho(H_m^{s,n})$ where $\rho(H_m^{s,n})$ is the resolvent set of $H_m^{s,n}$. Then the operator $f(x)\left(H_m^{s,n} - z\right)^{-1}$ is compact on $\mathcal{H}$ for all $m > 0$.*

4.1. Asymptotic operators

4.1.1. Operator $\boldsymbol{H_-}$. Let us first introduce the operator $H_c = i\gamma^0\gamma^1\partial_x$ where $\gamma^0\gamma^1 = \operatorname{diag}(-1, 1, 1, -1)$. We can thus prove the:

Proposition 4.2. *The operator $H_c = i\gamma^0\gamma^1\partial_x$ is self-adjoint on the domain defined by:*

$$D(H_c) = \{\varphi \in \mathcal{H}_{s,n}; H_c\varphi \in \mathcal{H}_{s,n},\ \varphi_1(0) = -\varphi_3(0),\ \varphi_2(0) = \varphi_4(0)\}$$

Proof. Since $D(H_c) \subset H^1(]-\infty,0[) \subset C^0(]-\infty;0[)$, we can deduce that the elements of $D(H_c)$ go to 0 at $-\infty$ and from the boundary condition, we deduce the symmetry of H_c on $D(H_c)$. The closedness is also proven using the fact that $D(H_c) \subset C^0(]-\infty;0[)$.

On the other hand, since $C_0^\infty(]-\infty,0[) \subset D(H_c)$, we can prove (using distribution) that $H_c^* = H_c$ on $D(H_c^*)$. We then study the default spaces. Let $\psi \in \ker(H_c^* + i)$. Since $x \to e^{-x}$ is not in $L^2(]-\infty,0[)$, we obtain:

$$\ker(H_c^* + i) = \text{vect}\left\{\begin{pmatrix} e^x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^x \end{pmatrix}\right\} \cap D(H_c^*).$$

But, if $\psi \in D(H_c^*)$, then, for all $\varphi \in D(H_c)$, we have:

$$\begin{aligned} 0 = \langle H_c\varphi, \psi\rangle - \langle \varphi, H_c^*\psi\rangle = \lim_{x\to 0} \big(&-i\varphi_1(x)\overline{\psi_1}(x) + i\varphi_2(x)\overline{\psi_2}(x) \\ &+ i\varphi_3(x)\overline{\psi_3}(x) - i\varphi_4(x)\overline{\psi_4}(x)\big). \end{aligned}$$

Choosing φ such that $\varphi_1(0) \neq 0$, we see that $\ker(H_c^* + i) = \{0\}$. The same is true for $H_c^* - i = \{0\}$. This shows that H_c is self-adjoint on $D(H_c)$. ☐

Now, let us define the operator H_- by:

$$H_- = H_c + V_-(x) \tag{4.2}$$

where

$$V_-(x) = \begin{cases} xId, & for\ x \geqslant d \\ \gamma^0\gamma^2 A(x)\left(s + \frac{1}{2}\right) - m\gamma^0 B(x), & for\ x \leqslant c. \end{cases} \tag{4.3}$$

with c, d two negative constants such that $c \leqslant d$. We remark that V_- is bounded on $\mathbb{R}_-^*$. Using the Kato-Rellich theorem, we obtain:

Corollary 4.3. *The operator H_- equipped with $D(H_c)$ is self-adjoint.*

Remark. Note that the potential of H_- equals the potential of $H_m^{s,n}$ for x negative and $|x|$ large.

4.1.2. Operator H_+. Let us define the operator H_+ by:

$$H_+ = \Gamma^1 D_x + V_+(x) \tag{4.4}$$

where

$$V_+(x) = \begin{cases} \gamma^0\gamma^2 A(x)\left(s+\frac{1}{2}\right) - m\gamma^0 B(x), & for\ x \geqslant b. \\ -x^2\gamma^0, & for\ x \leqslant a. \end{cases} \tag{4.5}$$

This time, the potential behaves like the potential in $H_m^{s,n}$ at 0 and increases at $-\infty$. We then have a confining potential. This type of potential has been encountered in the article of A.Bachelot [8]. For proving the self-adjointness of his operator, he uses the method we have recovered for proving the self-adjointness of our operator H_m. We just indicate the differents stages of the proof. We introduce the domain:

$$D(H_+) = \Big\{\varphi \in L^2(\mathbb{R}_-^*, \mathbb{C}^4);\ H_+\varphi \in L^2(\mathbb{R}_-^*), \\ \left\|(\gamma^1 + i)\varphi(x,.)\right\|_{L^2(S^2)} = o\left(\sqrt{x}\right),\ x \to 0\Big\}$$

if $2ml < 1$ and we remove the boundary condition for $2ml \geqslant 1$. In the following proof of compactness of $(H_+ - z)^{-1}$, we obtain estimates that allow us to prove the symmetry of this operator for $ml \geqslant \frac{1}{2}$. As before, we can do a unitary transform and obtain a result similar as lemma 3.2. We then obtain the asymptotic behavior of φ. This allows us to conclude in the case $ml \geqslant \frac{1}{2}$. If $ml < \frac{1}{2}$, we introduce the MIT boundary condition and a suitable partition of unity in order to separate the behavior at 0 from the one at $-\infty$. We then obtain:

Proposition 4.4. *The operator H_+ equipped with $D(H_+)$ is self-adjoint.*

4.2. Compactness of $f(x)(H_- - z)^{-1}$

Lemma 4.5. *Let $f \in C^0(]-\infty, 0])$ such that $\lim_{x\to-\infty} f(x) = 0$ and $z \in \rho(H_-)$. Then $f(.)(H_- - z)^{-1}$ is compact.*

Proof. Let $\varphi \in D(H_c)$ and $g = (H_c - z)\varphi$ be defined on $]-\infty, 0[$. Denote by φ_i and g_i, $i = 1, \ldots, 4$, their components. We will extend these functions to $\mathbb{R}$ in the following way:

$$\tilde{\varphi_1}(x) = \begin{cases} \varphi_1(x) \; if \; x \leqslant 0, \\ -\varphi_3(-x) \; if \; x \geqslant 0 \end{cases} ; \quad \tilde{\varphi_2}(x) = \begin{cases} \varphi_2(x) \; if \; x \leqslant 0, \\ \varphi_4(-x) \; if \; x \geqslant 0 \end{cases}$$

$$\tilde{\varphi_3}(x) = \begin{cases} \varphi_3(x) \; if \; x \leqslant 0, \\ -\varphi_1(-x) \; if \; x \geqslant 0 \end{cases} ; \quad \tilde{\varphi_4}(x) = \begin{cases} \varphi_4(x) \; if \; x \leqslant 0, \\ \varphi_2(-x) \; if \; x \geqslant 0. \end{cases}$$

The components are thus in $H^1(\mathbb{R})$. We also extend g into $\tilde{g} \in \left[L^2(\mathbb{R})\right]^4$ in the same way. Here, we have put $\tilde{H}^c$ for the operator with the same formula as H_c but acting on functions defined on $\mathbb{R}$. Some calculation gives that $(H_c - z)\varphi = g$ if and only if $\left(\tilde{H}^c - z\right)\tilde{\varphi} = \tilde{g}$ for all z in the resolvent set of H_c.

Let $f \in C^0(] - \infty, 0])$ such that $\lim_{x \to -\infty} f(x) = 0$. We consider a sequence $(g_n)_{n\in\mathbb{N}} \in \left(L^2\left(\mathbb{R}_-^*\right)\right)^{\mathbb{N}}$ such that $g_n \rightharpoonup 0$ and we want to prove that

$$f(x)(H_c - z)^{-1} g_n$$

goes to 0 strongly in L^2. We introduce $u_n = (H_c - z)^{-1} g_n$ and extend g_n and u_n into $\tilde{g_n}$ and $\tilde{u_n}$ as before. Consequently, $\tilde{g_n} \rightharpoonup 0$ in $L^2(\mathbb{R})$ and $\tilde{u_n} = \left(\tilde{H}^c - z\right)^{-1} \tilde{g_n}$. We mention here a consequence of theorem $IX.29$ in [72] which say that if $f, g \in L^\infty(\mathbb{R}^n)$ and:

$$\lim_{|x|\to\infty} f(x) = 0, \quad \lim_{|\xi|\to\infty} g(\xi) = 0,$$

then the operator $f(x) g(-i\nabla)$ is compact. Since $x \to (x-z)^{-1} \in L^\infty$ and $|x - z|^{-1} \underset{|x|\to\infty}{\to} 0$, we deduce that:

$$\tilde{f}(x)\left(\tilde{H}^c - z\right)^{-1} \tilde{g_n} \underset{n\to\infty}{\overset{L^2(\mathbb{R})}{\to}} 0,$$

where $\tilde{f}$ is the extension of f by symmetry on $\mathbb{R}_+$. Therefore, we have:

$$\begin{aligned} \mathbb{1}_{]-\infty,0[}(x)\tilde{f}(x)\left(\tilde{H}^c - z\right)^{-1} \tilde{g_n} &= \mathbb{1}_{]-\infty,0[}(x) f(x)\tilde{u_n} \\ &= f(x)u_n = f(x)(H_c - z)^{-1} g_n. \end{aligned}$$

So $f(x)(H_c - z)^{-1} g_n \underset{n\to\infty}{\overset{L^2(\mathbb{R}_-^*)}{\to}} 0$ and the operator $f(x)(H_c - z)^{-1}$ is compact.

Since V_- goes to 0 at $-\infty$ and 0 and using the identity:

$$f(x)\left(H_- - z\right)^{-1} = -f(x)\left(H_- - z\right)^{-1} V_-(x)\left(H_c - z\right)^{-1} + f(x)\left(H_c - z\right)^{-1},$$

we deduce that $\left(H_- - z\right)^{-1} - \left(H_c - z\right)^{-1}$ is compact and consequently that $f(x)\left(H_- - z\right)^{-1}$ is also compact. □

4.3. Compactness of $(H_+ - z)^{-1}$

Lemma 4.6. *The operator* $\left(H_+ - z\right)^{-1}$ *is compact.*

Proof. We follow the proof of the compactness result in [8]. Let us show that the set:

$$K = \{\varphi \in D(H_+);\ \|\varphi\| + \|H_+\varphi\| \leqslant 1\} \tag{4.6}$$

is compact. We consider a sequence $(\varphi_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$. Using the Banach-Alaoglu theorem and distributions, we obtain the existence of a sub-sequence (φ_ν) such that:

$$\varphi_\nu \underset{\nu\to\infty}{\rightharpoonup} \varphi; \qquad f_\nu =: H_+\varphi_\nu \underset{\nu\to\infty}{\rightharpoonup} H_+\varphi := f.$$

Let:

$$W(x) = \begin{cases} mB(x) = -\frac{ml}{x} + O\left(x\right), & for\ x \geqslant b. \\ x^2, & for\ x \leqslant a, \end{cases}$$

so that W is smooth on $]a, b[$. The equation $H_+\varphi_\nu = f_\nu$ can be written:

$$\left(\Gamma^1 D_x - \gamma^0 W\left(x\right)\right)\varphi_\nu = -\gamma^0\gamma^2\left(s + \frac{1}{2}\right)A\left(x\right)\varphi_\nu + f_\nu.$$

We denote g_ν the right hand side of this equation. Then g_ν is in $L^2(]-\infty, 0[)$ and $g_\nu \rightharpoonup g$ where g is defined by replacing φ_ν, f_ν by φ, f respectively. We thus obtain four differential equations:

$$\begin{cases} \partial_x\left(\varphi_\nu^1 + \varphi_\nu^3\right) + W(x)\left(\varphi_\nu^1 + \varphi_\nu^3\right) = i\left(g_\nu^1 - g_\nu^3\right) \\ \partial_x\left(\varphi_\nu^2 - \varphi_\nu^4\right) + W(x)\left(\varphi_\nu^2 - \varphi_\nu^4\right) = -i\left(g_\nu^2 + g_\nu^4\right) \\ \partial_x\left(\varphi_\nu^1 - \varphi_\nu^3\right) - W(x)\left(\varphi_\nu^1 - \varphi_\nu^3\right) = i\left(g_\nu^1 + g_\nu^3\right) \\ \partial_x\left(\varphi_\nu^2 + \varphi_\nu^4\right) - W(x)\left(\varphi_\nu^2 + \varphi_\nu^4\right) = i\left(g_\nu^4 - g_\nu^2\right) \end{cases} \tag{4.7}$$

For some constants λ_ν^j, $j = 1, \ldots, 4$, the solutions are:

$$\left(\varphi_\nu^1 + \varphi_\nu^3\right)(x) = \lambda_\nu^1 e^{-\int_{-1}^x W(u)\mathrm{d}u} + i\int_{-\infty}^x \left(g_\nu^1 - g_\nu^3\right) e^{\int_{-1}^t W(u)\mathrm{d}u - \int_{-1}^x W(u)\mathrm{d}u}\mathrm{d}t, \tag{4.8}$$

$$\left(\varphi_\nu^2 - \varphi_\nu^4\right)(x) = \lambda_\nu^2 e^{-\int_{-1}^x W(u)\mathrm{d}u} - i\int_{-\infty}^x \left(g_\nu^2 + g_\nu^4\right) e^{\int_{-1}^t W(u)\mathrm{d}u - \int_{-1}^x W(u)\mathrm{d}u}\mathrm{d}t, \tag{4.9}$$

$$\left(\varphi_\nu^1 - \varphi_\nu^3\right)(x) = \lambda_\nu^3 e^{\int_{-1}^x W(u)\mathrm{d}u} + i\int_{0}^x \left(g_\nu^1 + g_\nu^3\right) e^{-\int_{-1}^t W(u)\mathrm{d}u + \int_{-1}^x W(u)\mathrm{d}u}\mathrm{d}t, \tag{4.10}$$

$$\left(\varphi_\nu^2 + \varphi_\nu^4\right)(x) = \lambda_\nu^4 e^{\int_{-1}^x W(u)\mathrm{d}u} + i\int_{0}^x \left(g_\nu^4 - g_\nu^2\right) e^{-\int_{-1}^t W(u)\mathrm{d}u + \int_{-1}^x W(u)\mathrm{d}u}\mathrm{d}t. \tag{4.11}$$

Proof of the pointwise convergence of the integral terms.

We have:

$$\int_{-1}^x W(u)\mathrm{d}u = \begin{cases} -ml\ln(-x) + \int_{-1}^x O(u)\mathrm{d}u, & \textit{for } x \geqslant b. \\ \frac{x^3}{3} - \frac{a^3}{3} + \int_{-1}^a W(u)\mathrm{d}u, & \textit{for } x \leqslant a. \end{cases} \tag{4.12}$$

where $\int_{-1}^x O(u)\mathrm{d}u$ is bounded on $[b;0[$. We obtain:

$$e^{\int_{-1}^x W(u)\mathrm{d}u} = \begin{cases} (-x)^{-ml} e^{\int_{-1}^x O(u)\mathrm{d}u}, & \textit{for } x \geqslant b. \\ C_1 e^{\frac{x^3}{3}}, & \textit{for } x \leqslant a. \end{cases}$$

$$e^{-\int_{-1}^x W(u)\mathrm{d}u} = \begin{cases} (-x)^{ml} e^{-\int_{-1}^x O(u)\mathrm{d}u}, & \textit{for } x \geqslant b. \\ C_2 e^{-\frac{x^3}{3}}, & \textit{for } x \leqslant a. \end{cases}$$

where C_1, C_2 are positive constants. We thus see that $e^{\int_{-1}^t W(u)\mathrm{d}u}$ is square integrable on $]-\infty, x[$ and that $e^{-\int_{-1}^t W(u)\mathrm{d}u}$ is square integrable on $]x, 0[$. Consequently, since g_ν is weakly convergent, we deduce that:

$$\int_{-\infty}^x \left(g_\nu^1 - g_\nu^3\right) e^{\int_{-1}^t W(u)\mathrm{d}u - \int_{-1}^x W(u)\mathrm{d}u}\mathrm{d}t \underset{\nu\to\infty}{\to} \int_{-\infty}^x \left(g^1 - g^3\right) e^{\int_{-1}^t W(u)\mathrm{d}u - \int_{-1}^x W(u)\mathrm{d}u}\mathrm{d}t$$

when $\nu \to \infty$. The same is true for the integral with $g_\nu^1 + g_\nu^3$.

Majorations of integral terms by L^2 functions independent of ν.

In the following, we will only treat $\left(\varphi_\nu^1 + \varphi_\nu^3\right)$ and $\left(\varphi_\nu^1 - \varphi_\nu^3\right)$. The other functions can be treated in the same way. When $a \leqslant x \leqslant b$, the functions are smooth hence integrable. We study the other cases:

a) First, using the Cauchy-Schwarz inequality and that $g_\nu^1 + g_\nu^3$ is bounded in L^2, we obtain:

$$\left|\int_0^x \left(g_\nu^1 + g_\nu^3\right) e^{\int_t^x W(u)\mathrm{d}u} \mathrm{d}t\right|^2 \lesssim \left|\int_0^x e^{-2\int_{-1}^t W(u)\mathrm{d}u + 2\int_{-1}^x W(u)\mathrm{d}u} \mathrm{d}t\right| \tag{4.13}$$

Therefore, we prove that the right hand side is integrable:

i) If $x \geqslant b$, using the expression of W, the right hand side is integrable since:

$$\left|\int_x^0 e^{-2\int_{-1}^t W(u)\mathrm{d}u + 2\int_{-1}^x W(u)\mathrm{d}u} \mathrm{d}t\right| \leqslant e^{2C} \left|\int_x^0 \left(-\frac{1}{x}\right)^{2ml} (-t)^{2ml}\, \mathrm{d}t\right| = e^{2C} \frac{-x}{1+2ml}.$$

ii) If $x \leqslant a$, we have:

$$\left|\int_x^0 e^{2\int_t^x W(u)\mathrm{d}u} \mathrm{d}t\right| = (C_1)^2 e^{2\frac{x^3}{3}} \left(\int_x^a (C_2)^2 e^{-2\frac{t^3}{3}}\, \mathrm{d}t + \int_a^0 e^{-2\int_{-1}^t W(u)\mathrm{d}u} \mathrm{d}t\right).$$

The function $(C_1)^2 e^{2\frac{x^3}{3}} \left(\int_a^0 e^{-2\int_{-1}^t W(u)\mathrm{d}u} \mathrm{d}t\right)$ is integrable on $]-\infty, a]$ and:

$$\int_x^a e^{-2\frac{t^3}{3}}\, \mathrm{d}t \leqslant -\frac{1}{2a^2} e^{-2\frac{a^3}{3}} + \frac{1}{2x^2} e^{-2\frac{x^3}{3}} - \frac{1}{a^3}\int_x^a e^{-2\frac{t^3}{3}}\, \mathrm{d}t,$$

by integration by parts. Choosing a such that $1 + \frac{1}{a^3} > 0$, we deduce that $e^{2\frac{x^3}{3}} \int_x^a e^{-2\frac{t^3}{3}} \mathrm{d}t$ is integrable on $]-\infty, a]$ and goes to 0 at $-\infty$.

b) Secondly, as above, we study the integrability of

$$\int_{-\infty}^x e^{2\int_{-1}^t W(u)\mathrm{d}u - 2\int_{-1}^x W(u)\mathrm{d}u} \mathrm{d}t.$$

i) If $x \geqslant b$, using the expression of W and separating the integral from $-\infty$ to b and from b to x, we have to study

$$(-x)^{2ml}\, e^{-2\int_{-1}^{x} T(u)\mathrm{d}u} \int_{-\infty}^{b} e^{2\int_{-1}^{t} W(u)\mathrm{d}u}\mathrm{d}t$$

and

$$(-x)^{2ml}\, e^{-2\int_{-1}^{x} T(u)\mathrm{d}u} \int_{b}^{x} \left(-\frac{1}{t}\right)^{2ml} e^{2\int_{-1}^{t} T(u)\mathrm{d}u}\mathrm{d}t.$$

The first term is clearly integrable and since $e^{2\int_{-1}^{t} T(u)\mathrm{d}u}$ is bounded on $[b, 0[$, we can perform the second integral to see that it is also integrable.

ii) If $x \leqslant a$, since $\displaystyle\int_{-\infty}^{x} \frac{1}{t^3} e^{\frac{2t^3}{3}}\,\mathrm{d}t \leqslant 0$, by integration by part, we have:

$$\int_{-\infty}^{x} e^{2\int_{-1}^{t} W(u)\mathrm{d}u - 2\int_{-1}^{x} W(u)\mathrm{d}u}\mathrm{d}t \leqslant C_2^2 C_1^2 \frac{1}{2x^2}.$$

This ends the proof of the integrability.

Convergence in L^2 of integral terms.

We can use the dominate convergence theorem to obtain:

$$\begin{aligned} &\int_{0}^{x} \left(g_\nu^1 + g_\nu^3\right) e^{-\int_{-1}^{t} W(u)\mathrm{d}u + \int_{-1}^{x} W(u)\mathrm{d}u}\mathrm{d}t \\ \underset{\nu\to\infty}{\overset{L^2}{\to}} &\int_{0}^{x} \left(g^1 + g^3\right) e^{-\int_{-1}^{t} W(u)\mathrm{d}u + \int_{-1}^{x} W(u)\mathrm{d}u}\mathrm{d}t. \end{aligned} \tag{4.14}$$

and the same for the integral with $g_\nu^1 - g_\nu^3$.

Study of the sequences λ_ν^i, $i = 1, \ldots, 4$.

a) Let us study the convergence of λ_ν^3 in (4.10) (we can do the same for λ_ν^4).
- If $ml < \frac{1}{2}$, using that $e^{\int_{-1}^{x} W(u)\mathrm{d}u} \in L^2$, $\varphi_\nu \rightharpoonup \varphi$ and (4.14), the term:

$$\begin{aligned} &\left(\lambda_\nu^3 - \lambda^3\right) \left\| e^{\int_{-1}^{x} W(u)\mathrm{d}u} \right\|_{L^2}^2 \\ = &\left\langle \left(\left(\varphi_\nu^1 - \varphi_\nu^3\right) - \int_0^x \left(g_\nu^1 + g_\nu^3\right) e^{-\int_{-1}^{t} W(u)\mathrm{d}u + \int_{-1}^{x} W(u)\mathrm{d}u}\mathrm{d}t \right), e^{\int_{-1}^{x} W(u)\mathrm{d}u} \right\rangle_{L^2} \\ - &\left\langle \left(\left(\varphi^1 - \varphi^3\right) - \int_0^x \left(g^1 + g^3\right) e^{-\int_{-1}^{t} W(u)\mathrm{d}u + \int_{-1}^{x} W(u)\mathrm{d}u}\mathrm{d}t \right), e^{\int_{-1}^{x} W(u)\mathrm{d}u} \right\rangle_{L^2} \end{aligned}$$

goes to 0 as $\nu \to -\infty$. We deduce that $\lambda_\nu^3 \underset{\nu\to\infty}{\to} \lambda^3$.

- If $ml \geqslant \frac{1}{2}$, $e^{\int_{-1}^x W(u)\mathrm{d}u} \notin L^2$ and $\lambda_\nu^3 = 0$.

b) We then study the convergence of λ_ν^1 and λ_ν^2.
Since $\varphi_\nu^1 + \varphi_\nu^3 \in L^2$, $e^{-\int_{-1}^x W(u)\mathrm{d}u} \notin L^2$ and the other terms are in L^2, we deduce that $\lambda_\nu^1 = \lambda_\nu^2 = 0$ for all $\nu \in \mathbb{N}$.

Convergence in L^2 of the sequences $\varphi_\nu^1 - \varphi_\nu^3$, $\varphi_\nu^2 + \varphi_\nu^4$, $\varphi_\nu^1 + \varphi_\nu^3$, $\varphi_\nu^2 - \varphi_\nu^4$.

Using the dominate convergence theorem, we deduce that $\varphi_\nu^1 - \varphi_\nu^3 \underset{\nu\to\infty}{\overset{L^2}{\to}} \varphi^1 - \varphi^3$. The same is true for the other functions. Thus, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ admits a converging sub-sequence which proves that K is compact. Consequently, $(H_+ + i)^{-1}$ is compact and so is $(H_+ - z)^{-1}$ for all $z \in \rho(H_+)$ using a resolvent identity. □

4.4. Proof of Proposition 4.1

Proof. Let $j_-, j_+ \in C^\infty$ such that $j_-^2 + j_+^2 = 1$, $\mathrm{supp}(j_-) \subset]-\infty, c[$ and $\mathrm{supp}(j_+) \subset]b, 0[$. We define:

$$Q(z) = j_-(x)\,(H_- - z)^{-1}\,j_-(x) + j_+(x)\,(H_+ - z)^{-1}\,j_+(x).$$

Since $H_m^{s,n} - z = H_- - z$ on $]-\infty, c[$ and $H_m^{s,n} - z = H_+ - z$ on $]b, 0[$, we have:

$$(H_m^{s,n} - z)\,Q(z) = 1 - w(z)$$

where:

$$w(z) = -\Big(\left[(H_m^{s,n} - z)\,, j_-(x)\right](H_- - z)^{-1}\,j_-(x) + \left[(H_m^{s,n} - z)\,, j_+(x)\right](H_+ - z)^{-1}\,j_+(x)\Big).$$

Since $\left[(H_m^{s,n} - z)\,, j_-(x)\right] = i\gamma^0\gamma^1 j_-'(x)$ and $\left[(H_m^{s,n} - z)\,, j_+(x)\right] = i\gamma^0\gamma^1 j_+'(x)$ and j_-', j_+' have compact support, we deduce that $w(z)$ is compact for all $z \in \rho(H)$ using the last two sections. Moreover, $w : \rho(H) \to \mathcal{L}\left(L^2\right)$ is analytic.

Since j_-', j_+', j_-, j_+ are bounded, for some constant $C > 0$, we have:

$$\|w(z)\varphi\|_2 \leqslant \frac{C}{|\Im z|}\,\|\varphi\|_2\,,$$

for all $\varphi \in L^2$. We then choose z such that the imaginary part satisfies $\frac{C}{|\Im z|} < 1$. Therefore, $1 - w(z)$ is invertible. Using the analytic Fredholm theorem, we have that $1 - w(z)$ is invertible for all $z \in \rho(H) \setminus S$ where S is a discrete set without accumulation points.

For these z, we deduce that:

$$(H_m^{s,n} - z)^{-1} = Q(z)\,(1 - w(z))^{-1}\,. \tag{4.15}$$

Let f be a continuous function going to 0 at $-\infty$ and admitting a finite limit at 0. Then $f(x)Q(z)$ is compact. Thus for $z \in \rho(H) \setminus S$, $f(x)\,(H_m^{s,n} - z)^{-1}$ is compact. Using the analyticity of $z \to (H_m^{s,n} - z)^{-1}$, we obtain the compactness for all $z \in \rho(H_m^{s,n})$. □

5. Mourre estimates

5.1. Mourre theory

We recall here some facts about the Mourre theory. Let $\mathcal{A}$ be a self-adjoint operator. We say that the pair $(\mathcal{A}, H)$ satisfies the Mourre conditions if

$$D(\mathcal{A}) \cap D(H) \text{ is dense in } D(H) \tag{5.1}$$

$$e^{it\mathcal{A}} \text{ preserves } D(H) \text{ for t¿0, } \sup_{|t| \leqslant 1} \left\| H e^{it\mathcal{A}} u \right\| < \infty, \ \forall u \in D(H) \tag{5.2}$$

$$\begin{aligned}&[iH, \mathcal{A}] \text{ defined as quadratic form on } D(H) \cap D(\mathcal{A})\\ &\text{extend to a bounded operator from } D(H) \text{ into } \mathcal{H}.\end{aligned} \tag{5.3}$$

The Mourre conditions are stronger than $C^1(\mathcal{A})$ regularity. We recall the definition of $C^k(\mathcal{A})$:

Definition 5.1. We say that $H \in C^k(\mathcal{A})$ if there exists $z \in \mathbb{C} \setminus \sigma(H)$ such that

$$\mathbb{R} \ni t \mapsto e^{it\mathcal{A}}\,(z - H)^{-1}\,e^{-it\mathcal{A}} \tag{5.4}$$

is C^k for the strong topology of $\mathcal{L}(\mathcal{H})$.

We then have the following lemma (see [1, Proposition 5.1.2, Theorem 6.3.4]):

Lemma 5.2. *Suppose that $(H, \mathcal{A})$ satisfies the Mourre conditions. Then $H \in C^1(\mathcal{A})$.*

We also recall a lemma concerning the $C^2(\mathcal{A})$ regularity:

Lemma 5.3. *Suppose that $H \in C^1(\mathcal{A})$ and that the commutator $[i\mathcal{A}, H]$ extends to a bounded operator from $D(H)$ into $\mathcal{H}$. We denote $[i\mathcal{A}, H]_0$ this extension. If, in addition, the commutator $[i\mathcal{A}, [i\mathcal{A}, H]_0]$ defined as a quadratic form on $D(\mathcal{A}) \cap D(H)$ extends to a bounded operator from $D(H)$ into $D(H)^*$, then $H \in C^2(\mathcal{A})$.*

5.2. Mourre estimate

We will use $\mathcal{A} = \Gamma x$ as conjugate operator where

$$\Gamma = -\gamma^0\gamma^1 = diag\,(1, -1, -1, 1)\,.$$

The operator $\mathcal{A}$ is self-adjoint when equipped with domain

$$D(\mathcal{A}) = \{\varphi \in \mathcal{H}_{s,n};\ \mathcal{A}\varphi \in \mathcal{H}_{s,n}\}. \tag{5.5}$$

Lemma 5.4. *For all $m > 0$, the pair $(H_m^{s,n}, \mathcal{A})$ satisfies the Mourre conditions. Consequently, $H_m^{s,n} \in C^1(\mathcal{A})$*

Proof. Let us check (5.1):

Case $2ml < 1$:

Let χ be a C^∞ function such that $\chi = 1$ on $[-1, 0]$, $\operatorname{supp}\chi \subset]-2, 0]$. We set $\chi_k(x) = \chi\left(\frac{x}{k}\right)$ for all $k \in \mathbb{N} \setminus \{0\}$. This implies that $\operatorname{supp}\chi_k(x) = 1$ on $]-k, 0]$. We have $\chi_k'(x) = \frac{1}{k}\chi'\left(\frac{x}{k}\right)$ so that it is bounded. Using these facts, we see that $\chi_k\varphi \in D(\mathcal{A}) \cap D\left(H_m^{s,n}\right)$ if $\varphi \in D\left(H_m^{s,n}\right)$.

We now show that $\chi_k\varphi \underset{k\to\infty}{\to} \varphi$ for the norm:

$$\|\varphi\|_{H_m^{s,n}} = \|\varphi\|_{\mathcal{H}^{s,n}} + \|H_m^{s,n}\varphi\|_{\mathcal{H}_{s,n}}\,.$$

By the dominate convergence theorem we have $\chi_k\varphi \overset{\mathcal{H}_{s,n}}{\underset{k\to\infty}{\longrightarrow}} \varphi$. Moreover, $|\chi_k'(x)| \leqslant \frac{1}{k}C$, so:

$$\|H_m^{s,n}\varphi - H_m^{s,n}\chi_k\varphi\| \leqslant \frac{C_0}{k}\|\varphi\| + \|H_m^{s,n}\varphi - \chi_k H_m^{s,n}\varphi\|\,.$$

which gives the desired result when k goes to infinity for $\varphi \in D\left(H_m^{s,n}\right)$. We deduce (5.1).

We denote $D\left(H_m^{s,n}\right)_c = \{\chi_k\varphi;\ \varphi \in D\left(H_m^{s,n}\right),\ k \in \mathbb{N} \setminus \{0\}\}$.

Case $2ml \geqslant 1$:

In this case, $C_0^\infty\left(]-\infty, 0[\right)$ is a subset of $D\left(\mathcal{A}\right) \cap D\left(H_m^{s,n}\right)$ and is dense in $D\left(H_m^{s,n}\right)$.

Let us check (5.2):
For all $t > 0$,

$$e^{it\mathcal{A}} = \mathrm{diag}(e^{itx}, e^{-itx}, e^{-itx}, e^{itx}).$$

Let $\varphi \in D\left(H_m^{s,n}\right)$, then:

- $e^{it\mathcal{A}}\varphi \in \mathcal{H}_{s,n}$.
- $H_m^{s,n} e^{it\mathcal{A}}\varphi = e^{it\mathcal{A}} H_m^{s,n}\varphi + te^{it\mathcal{A}}\varphi$. So

$$H_m^{s,n} e^{it\mathcal{A}}\varphi \in \mathcal{H}_{s,n} \quad \text{and} \quad \sup_{|t|\leqslant 1} \left\| H_m^{s,n} e^{it\mathcal{A}}\varphi \right\| < \infty.$$

We need to check the boundary condition in the case $2ml < 1$. We have:

$$\left\| \left(\gamma^1 + i\right) e^{it\mathcal{A}}\varphi(x,.) \right\|_{\mathcal{W}^0} = \left\| \begin{pmatrix} ie^{itx}\varphi_1 + ie^{-itx}\varphi_3 \\ ie^{-itx}\varphi_2 - ie^{itx}\varphi_4 \\ ie^{itx}\varphi_1 + ie^{-itx}\varphi_3 \\ -ie^{-itx}\varphi_2 + ie^{itx}\varphi_4 \end{pmatrix} \right\|_{[L^2(S^2)]^4}.$$

Let's consider: $\left\| ie^{itx}\varphi_1 + ie^{-itx}\varphi_3 \right\|_{L^2(S^2)}$ when x goes to 0. By Taylor expansion, we must check that $-x\left(\|\varphi_1(x,.)\|_{L^2(S^2)} + \|\varphi_3(x,.)\|_{L^2(S^2)}\right)$ is $o\left((-x)^{\frac{1}{2}}\right)$. Since $\varphi \in D\left(H_m^{s,n}\right)$, there exists functions $\psi_- \in W_-^{\frac{1}{2}}$, $\chi_- \in W_+^{\frac{1}{2}}$ and a function $\phi \in C^0\left([0, \frac{\pi}{2}]_x; L^2(S^2;\mathbb{C}^4)\right)$, such that $\|\phi_n^s(r_*,\theta,\varphi)\|_{\mathcal{W}^0} = o\left(\sqrt{(-x)}\right)$ as $x \to 0$, satisfying:

$$\psi_{s,n} = \left(-x^{-ml}\right) \begin{pmatrix} \psi_{-,n}^s(\theta,\varphi) \\ \chi_{-,n}^s(\theta,\varphi) \\ -i\psi_{-,n}^s(\theta,\varphi) \\ i\chi_{-,n}^s(\theta,\varphi) \end{pmatrix} + \phi_n^s(r_*,\theta,\varphi).$$

We thus obtain:

$$-x\left\|\varphi_1(x,.)\right\|_{L^2(S^2)} \leqslant C_{s,n}\left(-x\right)^{1-ml} - x\left(o\left((-x)^{\frac{1}{2}}\right)\right).$$

Since $1-ml > \frac{1}{2}$ when $ml < \frac{1}{2}$, we have that $-2x\left\|\varphi_1(x,.)\right\|_{L^2(S^2)} = o\left((-x)^{\frac{1}{2}}\right)$. Since $\varphi \in D\left(H_m^{s,n}\right)$, this proves that the boundary condition is fulfilled and then (5.2).

Let us check (5.3):

First, we see that $xA(x)$ and $xB(x)$ are bounded functions on $]-\infty,0[$. Let $u,v \in D\left(H_m^{s,n}\right)_c$ in the case $2ml<1$ and $u,v\in C_0^\infty\left(]-\infty,0[\right)$ in the case $2ml \geqslant 1$, we have:

$$(5.6)\quad \left[H_m^{s,n}, i\mathcal{A}\right](u,v) = \left\langle u + 2i\left(s+\frac{1}{2}\right)xA(x)\gamma^2\gamma^1 u + 2imxB(x)\gamma^1 u, v\right\rangle.$$

This shows that:

$$\left|\left[H_m^{s,n}, i\mathcal{A}\right](u,v)\right| \leqslant C_1 \|u\|_{\mathcal{H}_{s,n}} \|v\|_{\mathcal{H}_{s,n}}$$

for some constant C_1 and consequently, (5.3) is satisfied. □

We then have the following:

Proposition 5.5. *Recall that $\mathcal{A}=\Gamma x$. Let $I\subset\mathbb{R}$ be a compact non-empty interval. Then, for all $m>0$, we have:*

$$(5.7)\quad \mathbb{1}_I\left(H_m^{s,n}\right)\left[H_m^{s,n}, i\mathcal{A}\right]\mathbb{1}_I\left(H_m^{s,n}\right) \geqslant \mathbb{1}_I^2\left(H_m^{s,n}\right) + \mathbb{1}_I\left(H_m^{s,n}\right)K\mathbb{1}_I\left(H_m^{s,n}\right)$$

where $\mathbb{1}_I$ is the characteristic function of I and K is a compact operator.

Proof. We remark that $xA(x) \underset{x\to-\infty,0}{\to} 0$, that $xB(x) \underset{x\to-\infty}{\to} 0$ and that $xB(x) \underset{x\to 0}{\to} -l$ using the asymptotic behavior of A and B described in (2.24) and (2.25). We obtain

$$\left[H_m^{s,n}, i\mathcal{A}\right] \geqslant \mathrm{Id} - (2s+1)\,xA(x)\gamma^2\gamma^1 - 2mxB(x)\gamma^1.$$

Consider a compact non-empty interval $I\subset\mathbb{R}$ and $\tilde{I}$ a compact interval strictly containing I. Let $\varsigma\in C_0^\infty\left(\tilde{I}\right)$ such that $\varsigma\equiv 1$ on I. We have:

$$(5.8)\quad \varsigma\left(H_m^{s,n}\right)\left[H_m^{s,n}, i\mathcal{A}\right]\varsigma\left(H_m^{s,n}\right) \geqslant \varsigma^2\left(H_m^{s,n}\right) + K.$$

where $K=\varsigma\left(H_m^{s,n}\right)\left(-(2s+1)\,xA(x)\gamma^2\gamma^1 - 2mxB(x)\gamma^1\right)\varsigma\left(H_m^{s,n}\right)$ is compact. Indeed, by Proposition 4.1 and the use of Helffer-Sjöstrand formula, we see that $\varsigma\left(H_m^{s,n}\right)$ multiplied by a good function will be compact. The asymptotic behavior of A and B gives that $xA(x)$ and $xB(x)$ are bounded near 0 and goes to 0 at $-\infty$. This gives the compacity of K. Multiplying both sides by $\mathbb{1}_I\left(H_m^{s,n}\right)$, this gives the desired result since $\mathbb{1}_I\varsigma=\mathbb{1}_I$. □

Using the absence of eigenvalues, we deduce the following corollary:

Corollary 5.6. *For all $m > 0$, all $\lambda \in \mathbb{R}$ and all $0 < \epsilon < 1$, there exists a compact non-empty interval $I' \subset \mathbb{R}$ containing λ such that:*

$$\mathbb{1}_{I'}\left(H_m^{s,n}\right)\left[H_m^{s,n}, i\mathcal{A}\right]\mathbb{1}_{I'}\left(H_m^{s,n}\right) \geqslant (1-\epsilon)\,\mathbb{1}_{I'}^2\left(H_m^{s,n}\right). \tag{5.9}$$

Recall that $\mathbb{1}_{I'}$ is the characteristic function of I'.

Proof. We have the Mourre estimate with I such that $\lambda \in I$. Let $I' \subset I$ such that $\lambda \in I'$. We can multiply both sides by $\mathbb{1}_{I'}\left(H_m^{s,n}\right)$ to obtain the same inequality with I replaced by I'. Since λ is not an eigenvalue of $H_m^{s,n}$, $\mathbb{1}_{I'}\left(H_m^{s,n}\right)$ tends strongly to 0 when the size of I' decreases. Then $\mathbb{1}_{I'}\left(H_m^{s,n}\right) K \mathbb{1}_{I'}\left(H_m^{s,n}\right)$ goes to 0 in the operator norm (K is compact). We can thus choose I' sufficiently small such that the desired inequality holds. □

6. Propagation estimates

In this section, we first present abstract results about propagation estimates and the minimal velocity estimate. Then, we apply this to prove that our minimal and maximal velocity is 1. This will be useful in the proof of asymptotic completeness.

6.1. Abstract propagation estimates

We present the abstract theory of propagation estimates. Proofs can be found in [21].

Consider a Hilbert space $\mathcal{H}$ and $(H, D(H))$ a self-adjoint operator on $\mathcal{H}$. Let $\Phi(t)$ be a C^1 uniformly bounded function with values in $\mathcal{L}(\mathcal{H})$ defined on $\mathbb{R}^+$. We define the Heisenberg derivative of Φ by:

$$\mathbb{D}\Phi(t) := \frac{d}{dt}\Phi(t) + i\left[H, \Phi(t)\right].$$

6.1.1. Basic principle.

Lemma 6.1. *[21, Lemma B.4.1, B.4.2] Let $\Phi(t)$ be a C^1 uniformly bounded function with values in $\mathcal{L}(\mathcal{H})$ and defined on $\mathbb{R}^+$.*

i) *If there exists measurables functions with values in $\mathcal{L}(\mathcal{H})$ $B(t), B_i(t)$, $i = 1, \ldots, n$ with*

$$\mathbb{D}\Phi(t) \geqslant C_0 B^*(t) B(t) - \sum_{i=1}^{n} B_i^*(t) B_i(t)$$

such that for all $i \in \{1, \ldots, n\}$

$$\int_1^{\infty} \left\| B_i(t) e^{-itH} u \right\|^2 dt \leqslant C \|u\|^2, \ \ \forall u \in \mathcal{H}$$

then there exists a constant $C_1 > 0$ *such that*

$$\int_1^{\infty} \left\| B(t) e^{-itH} u \right\|^2 dt \leqslant C_1 \|u\|^2, \ \ \forall u \in \mathcal{H}.$$

ii) *Suppose that* $B_{2,i}(t)$ *and* $B_{1,i}(t)$ *are mesurable functions with value in* $\mathcal{L}(\mathcal{H})$ *and that the function* Φ *satisfies*

$$|\langle \psi_2, \mathbb{D}\Phi(t) \psi_1 \rangle| \leqslant \sum_{i=1}^{n} \|B_{2,i}(t) \psi_2\| \, \|B_{1,i}(t) \psi_1\|,$$

for all $\psi_1, \psi_2 \in \mathcal{H}$, *with*

$$\int_1^{\infty} \left\| B_{2,i}(t) e^{-itH} u \right\|^2 dt \leqslant C_1 \|u\|^2, \ \ \forall u \in \mathcal{H}$$

and

$$\int_1^{\infty} \left\| B_{1,i}(t) e^{-itH} u \right\|^2 dt \leqslant C_1 \|u\|^2, \ \ \forall u \in \mathcal{D},$$

where $\mathcal{D}$ *is a dense subset of* $\mathcal{H}$. *Then the limit*

$$s - \lim_{t \to \infty} e^{itH} \Phi(t) e^{-itH}$$

exists.

6.1.2. Abstract minimal velocity estimates.

Proposition 6.2. *[41, Proposition A.1] Let* $H \in C^{1+\epsilon}(\mathcal{A})$ *for* $\epsilon > 0$. *Let* Δ *be an interval such that*

$$\mathbf{1}_{\Delta}(H) [H, i\mathcal{A}] \mathbf{1}_{\Delta}(H) \geqslant c_0 \mathbf{1}_{\Delta}(H).$$

Then, for all $g \in C_0^{\infty}(\mathbb{R})$, $\operatorname{supp} g \subset (-\infty, c_0)$ *and for* $f \in C_0^{\infty}(\Delta)$, *we have*

$$\int_1^{\infty} \left\| g\left(\frac{\mathcal{A}}{t}\right) f(H) e^{-itH} u \right\|^2 \frac{dt}{t} \leqslant C \|u\|^2, \ \ \forall u \in \mathcal{H},$$

$$s - \lim_{t \to \infty} g\left(\frac{\mathcal{A}}{t}\right) f(H) e^{-itH} = 0.$$

6.2. Propagation estimates

We have seen that $[H_m^{s,n}, i\mathcal{A}]$ admits a bounded extension from $D(\mathcal{A}) \cap D(H_m^{s,n})$ to $D(H_m^{s,n})$. We denote this extension by $[H_m^{s,n}, i\mathcal{A}]_0$. We have:

$$[[H_m^{s,n}, i\mathcal{A}]_0, i\mathcal{A}] = 4\left(\left(s+\frac{1}{2}\right)x^2 A(x)\gamma^2\gamma^0 + mx^2 B(x)\gamma^0\right) \tag{6.1}$$

so $[[H_m^{s,n}, i\mathcal{A}]_0, i\mathcal{A}]$ extends to a bounded operator to $D(H_m^{s,n})$ with values in $\mathcal{H}_{s,n}$. Using lemma 5.3, we deduce that $H \in C^2(\mathcal{A})$. Using the Mourre estimate and a partition of unity argument, this gives:

Proposition 6.3. *For all $m > 0$, $g \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}(g) \subset (-\infty, 1-\delta)$ and $f \in C_0^\infty(\mathbb{R})$, we have:*

$$\int_1^\infty \left\| g\left(\frac{\mathcal{A}}{t}\right) f(H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{\mathrm{d}t}{t} \leqslant C\|u\|^2, \ \ \forall u \in \mathcal{H}_{s,n}, \tag{6.2}$$

$$s - \lim_{t\to\infty} g\left(\frac{\mathcal{A}}{t}\right) e^{-itH_m^{s,n}} = 0. \tag{6.3}$$

Proof of Proposition 6.3. Using the corollary 5.6 where we denote I our interval, we obtain

$$\int_1^\infty \left\| g\left(\frac{\mathcal{A}}{t}\right) f(H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{\mathrm{d}t}{t} \leqslant C\|u\|^2, \ \ \forall u \in \mathcal{H}_{s,n},$$
$$s - \lim_{t\to\infty} g\left(\frac{\mathcal{A}}{t}\right) f(H_m^{s,n}) e^{-itH_m^{s,n}} = 0,$$

for $f \in C_0^\infty(I)$ by the abstract velocity estimate. For $f \in C_0^\infty(\mathbb{R})$, we can cover $\operatorname{supp}(f)$ by a finite number of intervals $I_1, \ldots, I_n$ where a Mourre estimate holds. Then, we consider a partition of unity subordinate to this cover $\eta_1, \ldots, \eta_n$ and we note $f_i = \eta_i f$ for all $i = 1, \ldots, n$. Then:

$$\begin{aligned}&\int_1^\infty \left\| g\left(\frac{\mathcal{A}}{t}\right) f(H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{\mathrm{d}t}{t}\\ &\leqslant \sum_{i=1}^n \int_1^\infty \left\| g\left(\frac{\mathcal{A}}{t}\right) f_i(H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{\mathrm{d}t}{t}\\ &\leqslant C_n\|u\|^2, \ \ \forall u \in \mathcal{H}_{s,n},\end{aligned}$$

and:

$$s-\lim_{t\to\infty} g\left(\frac{\mathcal{A}}{t}\right) f\left(H_m^{s,n}\right) e^{-itH_m^{s,n}} = \sum_{i=1}^{n} s-\lim_{t\to\infty} g\left(\frac{\mathcal{A}}{t}\right) f_i\left(H_m^{s,n}\right) e^{-itH_m^{s,n}} = 0.$$

Thanks to a density argument, we obtain the desired limit. □

Proposition 6.3 allows us to obtain:

Lemma 6.4. *Let $J_- \in C^\infty$ such that* $\operatorname{supp}(J_-) \subset]-\infty, 1-\epsilon[$ *and $J_-(x) = 1$ for all $x \in]-\infty, 1-2\epsilon[$ and let $\chi \in C_0^\infty$. Then, for all $m > 0$, we have:*

$$\int_1^\infty \left\| J_- \left(\frac{\mathcal{A}}{t}\right) \chi\left(H_m^{s,n}\right) e^{-itH_m^{s,n}} u \right\|^2 \frac{dt}{t} \leqslant C \|u\|^2, \ \forall u \in \mathcal{H}_{s,n} \tag{6.4}$$

$$\lim_{t\to\infty} J_- \left(\frac{\mathcal{A}}{t}\right) e^{-itH_m^{s,n}} u = 0, \ \forall u \in \mathcal{H}_{s,n}. \tag{6.5}$$

Proof. 1) Let $\theta_1, \theta_2 \in C^\infty$ such that $\operatorname{supp}(\theta_1) \subset]-\infty, -1-\frac{\epsilon}{2}[$, $\operatorname{supp}(\theta_2) \subset]-1-\epsilon, 1-\epsilon[$ and $\theta_1 + \theta_2 = 1$. Then, using the triangular inequality and the minimal velocity estimate, we only need to prove the integral estimate for $\theta_1 J_-$.

So suppose that $K \in C^\infty$ such that $\operatorname{supp}(K) \subset]-\infty, -1-\frac{\epsilon}{2}[$ and $K(x) = 1$ for all $x \in]-\infty, -1-\epsilon[$. We define $F(s) = \int_s^\infty K^2(t)\,dt$ and

$$\Phi(t) = \chi\left(H_m^{s,n}\right) F\left(\frac{\mathcal{A}}{t}\right) \chi\left(H_m^{s,n}\right)$$

such that Φ is C^1 uniformly bounded. We have:

$$\begin{aligned}\mathbb{D}\Phi(t) = &\frac{1}{t}\chi\left(H_m^{s,n}\right) \frac{\mathcal{A}}{t} K^2\left(\frac{\mathcal{A}}{t}\right) \chi\left(H_m^{s,n}\right) \\ &+ i\chi\left(H_m^{s,n}\right)\left[H_m^{s,n}, F\left(\frac{\mathcal{A}}{t}\right)\right]\chi\left(H_m^{s,n}\right),\end{aligned}$$

where

$$\begin{aligned}\left[H_m^{s,n}, F\left(\frac{\mathcal{A}}{t}\right)\right] = &\frac{i}{t}K^2\left(\frac{\mathcal{A}}{t}\right) + \left(s+\frac{1}{2}\right) A(x)\left(F\left(-\frac{x}{t}\right) - F\left(\frac{x}{t}\right)\right)\gamma^1\gamma^2 \\ &- mB(x)\left(F\left(-\frac{x}{t}\right) - F\left(\frac{x}{t}\right)\right)\gamma^1,\end{aligned}$$

with

$$\left| F\left(\frac{-x}{t}\right) - F\left(\frac{x}{t}\right)\right| \leqslant -\frac{2x}{t} \sup_{y\in\left[\frac{x}{t}, -\frac{x}{t}\right]} K^2(y) \leqslant -\frac{2x}{t}\mathbb{1}_{\left\{x \leqslant \left(-1-\frac{\epsilon}{2}\right)t\right\}},$$

where $\mathbb{1}$ is the characteristic function and $\sup_{y\in\left[\frac{x}{t},-\frac{x}{t}\right]} K^2(y)$ is thought as a function depending on the variables x and t. We know that for $x<0$ and $|x|$ sufficiently large, the functions A and B are exponentially decaying. If we fix T sufficiently large, then, since $e^x \leqslant \frac{1}{-x^3}$ for x sufficiently small, for all $t \geqslant T$, we have:

$$\left|A(x)\left(F\left(-\frac{x}{t}\right)-F\left(\frac{x}{t}\right)\right)\right| \leqslant \frac{C}{t^2}\zeta_{\left\{x\leqslant\left(-1-\frac{\epsilon}{2}\right)T\right\}}.$$

We can do the same thing with B. We obtain:

$$\begin{aligned}-\mathbb{D}\Phi(t) &= \frac{1}{t}\chi\left(H_m^{s,n}\right)\left(1-\frac{\mathcal{A}}{t}\right)K^2\left(\frac{\mathcal{A}}{t}\right)\chi\left(H_m^{s,n}\right)+O\left(t^{-2}\right)\\ &\geqslant \frac{2+\frac{\epsilon}{2}}{t}\chi\left(H_m^{s,n}\right)K^2\left(\frac{\mathcal{A}}{t}\right)\chi\left(H_m^{s,n}\right)+O\left(t^{-2}\right),\end{aligned}$$

since $\frac{\mathcal{A}}{t} \leqslant -1-\frac{\epsilon}{2}$ on the support of K^2. By lemma 6.1.1, this shows that:

$$\int_1^\infty \left\|K\left(\frac{\mathcal{A}}{t}\right)\chi\left(H_m^{s,n}\right)e^{-itH_m^{s,n}}u\right\|^2\frac{dt}{t} \leqslant C\|u\|^2 \tag{6.6}$$

for all $u \in \mathcal{H}_{s,n}$. This proves the first statement of the lemma.

2) We next set:

$$\Phi(t) = \chi\left(H_m^{s,n}\right)J_-^2\left(\frac{\mathcal{A}}{t}\right)\chi\left(H_m^{s,n}\right).$$

So, we have:

$$\mathbb{D}\Phi(t) \leqslant \frac{4\epsilon}{t}\chi\left(H_m^{s,n}\right)\left(J_-^{'}J_-\right)\left(\frac{\mathcal{A}}{t}\right)\chi\left(H_m^{s,n}\right)+O\left(t^{-2}\right)$$

where $\operatorname{supp}\left(J_-'J_-\right) \subset]1-2\epsilon, 1-\epsilon[$ so it is integrable by the minimal velocity estimate. Using lemma 6.1.1 and the integrability in 6.6, this gives

$$\lim_{t\to\infty} e^{itH_m^{s,n}}\chi\left(H_m^{s,n}\right)J_-^2\left(\frac{\mathcal{A}}{t}\right)\chi\left(H_m^{s,n}\right)e^{-itH_m^{s,n}}u = 0,\ \forall u \in \mathcal{H}_{s,n}.$$

Using the last lemma, we obtain the desired limit by a density argument. □

Proposition 6.5. *Let* $g \in C^{\infty}$ *such that* $\operatorname{supp}(g) \subset]1+\epsilon, \infty[$ *with* $\epsilon > 0$ *and such that* $g(x) = 1$ *for all* $x \in]1+2\epsilon, \infty[$. *Let* $\zeta \in C_0^{\infty}(\mathbb{R})$. *Then, for all* $m > 0$, *we have:*

$$\int_1^{\infty} \left\| g\left(\frac{\mathcal{A}}{t}\right) e^{-itH_m^{s,n}} \zeta\left(H_m^{s,n}\right) u \right\|^2 \frac{\mathrm{d}t}{t} \leqslant C \|u\|^2, \quad \forall u \in \mathcal{H}_{s,n} \tag{6.7}$$

$$s - \lim_{t\to\infty} g\left(\frac{A}{t}\right) e^{-itH_m^{s,n}} = 0. \tag{6.8}$$

Proof of the Proposition 6.5. Let $J \in C^{\infty}(\mathbb{R})$ such that

$$\operatorname{supp}(J) \subset (1+\epsilon, +\infty)$$

with $\epsilon > 0$ and $J(x) = 1$ for all $x \in]1+2\epsilon, +\infty[$. Let $\zeta \in C_0^{\infty}(\mathbb{R})$. We define

$$F(s) = \int_{-\infty}^{s} J^2(u)\,\mathrm{d}u$$

and

$$\Phi(t) = \zeta\left(H_m^{s,n}\right) F\left(\frac{\mathcal{A}}{t}\right) \zeta\left(H_m^{s,n}\right)$$

so that Φ is C^1 uniformly bounded. As in the last proof, we calculate the Heisenberg derivative of Φ and thanks to the support of J, we obtain:

$$\begin{aligned} -\mathbb{D}\Phi(t) \geqslant\ & \frac{\epsilon}{t}\zeta\left(H_m^{s,n}\right) J^2\left(\frac{\mathcal{A}}{t}\right)\zeta\left(H_m^{s,n}\right) \\ & + \zeta\left(H_m^{s,n}\right)\left(i\left(s+\frac{1}{2}\right) A(x) \left(F\left(\frac{-x}{t}\right) - F\left(\frac{x}{t}\right)\right)\gamma^2\gamma^1 \right. \\ & \left. + imB(x)\left(F\left(\frac{-x}{t}\right) - F\left(\frac{x}{t}\right)\right)\gamma^1\right)\zeta\left(H_m^{s,n}\right), \end{aligned} \tag{6.9}$$

and we have:

$$\left| F\left(\frac{-x}{t}\right) - F\left(\frac{x}{t}\right)\right| \leqslant \frac{-2x}{t} \sup_{y\in\left[\frac{x}{t}, \frac{-x}{t}\right]} J^2(y)\, \mathbb{1}_{\left\{1+\epsilon \leqslant \frac{-x}{t}\right\}}.$$

Using the exponential decay of A and B, we obtain:

(6.10)
$$\zeta\left(H_m^{s,n}\right)\left(i\left(s+\frac{1}{2}\right)A(x)\left(F\left(\frac{-x}{t}\right)-F\left(\frac{x}{t}\right)\right)\gamma^2\gamma^1 + imB(x)\left(F\left(\frac{-x}{t}\right)-F\left(\frac{x}{t}\right)\right)\gamma^1\right)\zeta\left(H_m^{s,n}\right)=O\left(e^{-\frac{\kappa}{2}t}\right)$$

for t sufficiently large. We deduce that:

$$\int_1^\infty \left\| J\left(\frac{\mathcal{A}}{t}\right)e^{-itH_m^{s,n}}\zeta\left(H_m^{s,n}\right)u\right\|^2\frac{\mathrm{d}t}{t}\leqslant C\left\|u\right\|^2,\quad \forall u\in\mathcal{H}_{s,n}. \tag{6.11}$$

Next, we use:

$$\Phi(t)=\zeta\left(H_m^{s,n}\right)J^2\left(\frac{\mathcal{A}}{t}\right)\zeta\left(H_m^{s,n}\right),$$

and obtain:

$$\begin{aligned}\mathbb{D}\Phi(t)=&\frac{2}{t}\zeta\left(H_m^{s,n}\right)\frac{-\mathcal{A}}{t}J\left(\frac{\mathcal{A}}{t}\right)J'\left(\frac{\mathcal{A}}{t}\right)\zeta\left(H_m^{s,n}\right)\\ &+\frac{2}{t}\zeta\left(H_m^{s,n}\right)J\left(\frac{\mathcal{A}}{t}\right)J'\left(\frac{\mathcal{A}}{t}\right)\zeta\left(H_m^{s,n}\right)\\ &+\zeta\left(H_m^{s,n}\right)\left(i\left(s+\frac{1}{2}\right)A(x)\left(J^2\left(\frac{-x}{t}\right)-J^2\left(\frac{x}{t}\right)\right)\gamma^2\gamma^1\right.\\ &\left.+imB(x)\left(J^2\left(\frac{-x}{t}\right)-J^2\left(\frac{x}{t}\right)\right)\gamma^1\right)\zeta\left(H_m^{s,n}\right).\end{aligned}$$

The first two terms are integrable due to the support of J and (6.11). The last two are also integrable using the support of J. Consequently:

$$s-\lim_{t\to\infty}J\left(\frac{\mathcal{A}}{t}\right)e^{-itH_m^{s,n}}\zeta\left(H_m^{s,n}\right)$$

exists and is zero by (6.11). The proposition follows by density. □

7. Asymptotic completeness

7.1. Comparison operator

Our comparison operator will be H_c defined by:

$$H_c = i\gamma^0\gamma^1\partial_x \tag{7.1}$$

where $\gamma^0\gamma^1 = \mathrm{diag}\,(-1,1,1,-1)$ and with domain:

$$D(H_c) = \{\varphi \in \mathcal{H}_{s,n}; H_c\varphi \in \mathcal{H}_{s,n},\ \varphi_1(0) = -\varphi_3(0)\,,\ \varphi_2(0) = \varphi_4(0)\} \tag{7.2}$$

By Proposition 4.2, this is a self-adjoint operator on its domain.

7.2. Asymptotic completeness

Recall that $\mathcal{A} = \Gamma x$ where $\Gamma = -\gamma^0\gamma^1$. We have:

Theorem 7.1 (Asymptotic completeness for fixed harmonics). *For all $m > 0$ and all $\varphi \in \mathcal{H}_{s,n}$, the limits*

$$\lim_{t\to\infty} e^{itH_c}e^{-itH_m^{s,n}}\varphi \tag{7.3}$$

$$\lim_{t\to\infty} e^{itH_m^{s,n}}e^{-itH_c}\varphi \tag{7.4}$$

exist. If we denote them by:

$$\Omega_{s,n}\varphi = \lim_{t\to\infty} e^{itH_c}e^{-itH_m^{s,n}}\varphi \tag{7.5}$$

$$W_{s,n}\varphi = \lim_{t\to\infty} e^{itH_m^{s,n}}e^{-itH_c}\varphi \tag{7.6}$$

for all $\varphi \in \mathcal{H}_{s,n}$, we have $\Omega_{s,n}^* = W_{s,n}$.

Proof. Let $J_-, J_0, J_+ \in C^\infty$ such that $J_- + J_0 + J_+ = 1$, the supports of J_-, J_+ are as in 6.5 and 6.4, and $J_0 = 1$ on $]1-\epsilon, 1+\epsilon[$, $\mathrm{supp}\,(J_0) \subset]1-2\epsilon, 1+2\epsilon[$ with $\epsilon > 0$. Using Proposition 6.5 and lemma 6.4, it suffices to prove that, for all $\varphi \in \mathcal{H}_{s,n}$, the limit:

$$\lim_{t\to\infty} e^{itH_c}J_0\left(\frac{\mathcal{A}}{t}\right)e^{-itH_m^{s,n}}\varphi$$

exists. We remark that $J_0\left(\frac{x}{t}\right) \neq 0$ if and only if $x \geqslant (1-2\epsilon)\,t > 0$. Since $x < 0$, $J_0\left(\frac{x}{t}\right) = 0$, for all $t > 0$ and $x < 0$. We thus have:

$$J_0\left(\frac{\mathcal{A}}{t}\right) = J_0\left(\frac{-x}{t}\right) M_0$$

where $M_0 = \operatorname{diag}(0,1,1,0)$. We then define:

$$\Phi(t) = \chi(H_c)\, J_0\left(\frac{\mathcal{A}}{t}\right) \chi\left(H_m^{s,n}\right),$$

and, denoting $V(x) = \left(s+\frac{1}{2}\right) A(x)\,\gamma^1\gamma^2 - mB(x)\,\gamma^0$, we have:

$$\begin{aligned}
\mathbb{D}\Phi(t) &= \frac{d}{dt}\Phi(t) + i\left(H_c\Phi(t) - \Phi(t)\,H_m^{s,n}\right)\\
&= \frac{2}{t}\chi(H_c)\left(\frac{x}{t}+1\right)\left(J_0'J_0\right)\left(\frac{-x}{t}\right) M_0\chi\left(H_m^{s,n}\right)\\
&\quad - i\chi(H_c)\, J_0^2\left(\frac{-x}{t}\right) M_0 V(x)\,\chi\left(H_m^{s,n}\right).
\end{aligned}$$

On the support of $J_0'J_0$, we have $\frac{x}{t}+1 \leqslant 2\epsilon$. Moreover, $J_0\left(\frac{-x}{t}\right) \neq 0$ if and only if $-(1+2\epsilon)\,t \leqslant x \leqslant -(1-2\epsilon)\,t$. Since A, B are exponentially decreasing at $-\infty$, we obtain:

$$\mathbb{D}\Phi(t) \leqslant \frac{4\epsilon}{t}\chi(H_c)\left(J_0'J_0\right)\left(\frac{\mathcal{A}}{t}\right)\chi\left(H_m^{s,n}\right) + O\left(t^{-2}\right).$$

Using the support of $J_0'J_0$, minimal and maximal velocity estimates, the right hand side is integrable. Hence the limit exists. We can show that the second limit exists in the same way. Finally, for all $t > 0$ and $\varphi, \psi \in \mathcal{H}_{s,n}$, we have $\left\langle e^{itH_c}e^{-itH_m^{s,n}}\varphi, \psi\right\rangle = \left\langle \varphi, e^{itH_m^{s,n}}e^{-itH_c}\psi\right\rangle$ which proves the last statement. □

Therefore, we obtain:

Theorem 7.2 (Asymptotic completeness). *For all $m > 0$ and all $\varphi \in \mathcal{H}$, the limits:*

$$\lim_{t\to\infty} e^{itH_c}e^{-itH_m}\varphi \tag{7.7}$$

$$\lim_{t\to\infty} e^{itH_m}e^{-itH_c}\varphi \tag{7.8}$$

exist. If we denote these limits by $\Omega\varphi$ and $W\varphi$ respectively, we have $\Omega^* = W$.

Proof. We can decompose $\varphi = \sum_{(s,n)\in I} \varphi_{s,n}$ where

$$\varphi_{s,n} \in \mathcal{H}_{s,n} \quad \text{and} \quad \sum_{(s,n)\in I} \|\varphi_{s,n}\|^2_{\mathcal{H}_{s,n}} < \infty.$$

We have:

$$e^{itH_c} e^{-itH_m} \varphi = \sum_{(s,n)\in I} e^{itH_c} e^{-itH_m^{s,n}} \varphi_{s,n}.$$

Since $\lim_{t\to\infty} e^{itH_c} e^{-itH_m^{s,n}} \varphi_{s,n} = \Omega_{s,n}\varphi_{s,n}$ exists for all $(s,n) \in I$ and $e^{itH_c} e^{-itH_m^{s,n}}$ is unitary, we deduce, using the dominate convergence theorem, that the limit in the theorem exists. We can do the same for the other limit. The last statement follows as in the last proof. □

8. Asymptotic velocity

8.1. Abstract theory

In this section, we follow the appendix $B.2$ in [21]. We consider a sequence $(B_n)_{n\in\mathbb{N}}$ of vectors of self-adjoint operators which commute in a Hilbert space $\mathcal{H}$. More precisely:

$$B_n = \left(B_n^1, \ldots, B_n^m\right), \quad \left[B_n^i, B_n^j\right] = 0, \quad 0 \leqslant i,j \leqslant m, \quad n = 1, 2, \ldots.$$

We have the following proposition:

Proposition 8.1. *Suppose that, for all $g \in C_\infty(\mathbb{R}^m)$, there exists*

$$\text{s} - \lim_{n\to\infty} g(B_n). \tag{8.1}$$

Then there exists a unique vector of self-adjoint operators

$$B = \left(B^1, \ldots, B^m\right) \tag{8.2}$$

such that (8.1) *is equal to $g(B)$. B is densely defined if, for some $g \in C_\infty(\mathbb{R}^m)$ such that $g(0) = 1$, we have:*

$$\text{s} - \lim_{R\to\infty} \left(\text{s} - \lim_{t\to\infty} g\left(R^{-1} B_n\right)\right) = \mathbb{1}. \tag{8.3}$$

We then define:

Definition 8.2. Under the hypotheses of the preceding proposition, we will write:

$$B = \mathrm{s} - C_\infty - \lim_{n \to \infty} B_n. \tag{8.4}$$

8.2. Asymptotic velocity for H_c

Theorem 8.3 (Asymptotic velocity for H_c). *Let $J \in C_\infty(\mathbb{R})$. Then the limit:*

$$\mathrm{s} - \lim_{t \to \infty} e^{itH_c} J\left(\frac{\mathcal{A}}{t}\right) e^{-itH_c} \tag{8.5}$$

exists and is equal to $J(1)\mathbb{1}$ where $\mathbb{1}$ is the identity. Moreover, if $J(0) = 1$, then

$$\mathrm{s} - \lim_{R \to \infty} \left(\mathrm{s} - \lim_{t \to \infty} e^{itH_c} J\left(\frac{\mathcal{A}}{Rt}\right) e^{-itH_c} \right) = \mathbb{1}. \tag{8.6}$$

If we define

$$\mathrm{s} - \mathrm{C}_\infty - \lim_{t \to \infty} e^{itH_c} \frac{\mathcal{A}}{t} e^{-itH_c} =: P_c^+, \tag{8.7}$$

then the self-adjoint operator P_c^+ is densely defined and it commutes with H_c. P_c^+ is called the asymptotic velocity.

Proof. Recall that $\mathcal{A} = -\gamma^0\gamma^1 x$ where $-\gamma^0\gamma^1 = \mathrm{diag}(1,-1,-1,1)$. Thus, for $J \in C_\infty(\mathbb{R})$, we have $J\left(\frac{\mathcal{A}}{t}\right) = \mathrm{diag}\left(J\left(\frac{x}{t}\right), J\left(-\frac{x}{t}\right), J\left(-\frac{x}{t}\right), J\left(\frac{x}{t}\right)\right)$. Moreover, we have $H_c = i\gamma^0\gamma^1\partial_x$. Let $\psi^0 \in D(H_c)$, we wish to solve the equation

$$\begin{aligned} &\partial_t \psi(t,x) = iH_c \psi(t,x), \\ &\psi(0,.) = \psi^0(.) = \left(\psi_1^0(.), \psi_2^0(.), \psi_3^0(.), \psi_4^0(.)\right) \end{aligned}$$

where $iH_c = \mathrm{diag}(1,-1,-1,1)\,\partial_x$. We will prove that the formula:

$$\psi(t,x) = \begin{pmatrix} \psi_1^0(x+t)\mathbb{1}_{\mathbb{R}^-}(x+t) - \psi_3^0(-(x+t))\mathbb{1}_{\mathbb{R}^+}(x+t) \\ \psi_2^0(x-t)\mathbb{1}_{\mathbb{R}^-}(x-t) + \psi_4^0(-x+t)\mathbb{1}_{\mathbb{R}^+}(x-t) \\ \psi_3^0(x-t)\mathbb{1}_{\mathbb{R}^-}(x-t) - \psi_1^0(-x+t)\mathbb{1}_{\mathbb{R}^+}(x-t) \\ \psi_4^0(x+t)\mathbb{1}_{\mathbb{R}^-}(x+t) + \psi_2^0(-(x+t))\mathbb{1}_{\mathbb{R}^+}(x+t) \end{pmatrix}.$$

gives an explicit solution for this problem. Since $x < 0$ in our case, $\mathbb{1}_{\mathbb{R}^+}(x-t) = 0$ for all $t > 0$, but we need this term for the group property of this solution.

We first prove that our formula gives a solution of the desired equation. Indeed, for all $t > 0$, we see that $\psi_3(t,0) = \psi_3^0(-t)$ and $\psi_1(t,0) = -\psi_3^0(-t)$ since $\mathbb{1}_{\mathbb{R}^-}(t) = 0$ for $t > 0$. Thus $\psi_3(t,0) = -\psi_1(t,0)$. On the other hand, we have $\psi_2(t,0) = \psi_2^0(-t)$ and $\psi_4(t,0) = \psi_2^0(-t)$ which gives us $\psi_2(t,0) = \psi_4(t,0)$. The boundary conditions are thus satisfied. It remains to prove that it satisfies the equation. For the first component of our formula, using the boundary consition and the derivation in the distributional sense, we obtain:

$$\partial_t \psi_1(t,x) = \psi_1^{0\,\prime}(x+t)\,\mathbb{1}_{\mathbb{R}^-}(x+t) + \psi_3^{0\,\prime}(-(x+t))\,\mathbb{1}_{\mathbb{R}^+}(x+t).$$

We also have:

$$\partial_x \psi_1^0(t,x) = \psi_1^{0\,\prime}(x+t)\,\mathbb{1}_{\mathbb{R}^-}(x+t) + \psi_3^{0\,\prime}(-(x+t))\,\mathbb{1}_{\mathbb{R}^+}(x+t)$$

which gives $\partial_t \psi_1(t,x) = \partial_x \psi_1(t,x)$. For the second and third components, $\mathbb{1}_{\mathbb{R}^-}(x-t)$ is constant so its derivative is 0 and we can check that $\partial_t \psi_2(t,x) = -\partial_x \psi_2(t,x)$ and $\partial_t \psi_3(t,x) = -\partial_x \psi_3(t,x)$. For the fourth component, we obtain:

$$\partial_t \psi_4(t,x) = \psi_4^{0\,\prime}(x+t)\,\mathbb{1}_{\mathbb{R}^-}(x+t) - \psi_2^{0\,\prime}(-(x+t)).$$

We have the same for $\partial_x \psi_4(t,x)$ so that $\partial_t \psi_4(t,x) = \partial_x \psi_4(t,x)$. So $\partial_t \psi(t,x) = iH_c \psi(t,x)$ in the sense of distribution. Since $\psi^0 \in D(H_c)$, the derivatives are, in fact, well defined in $\mathcal{H}_{s,n}$ and the equality is satisfied in $\mathcal{H}_{s,n}$. We thus have a solution.

We then turn our attention to the asymptotic velocity. We have:

$$e^{itH_c} J\left(\frac{\mathcal{A}}{t}\right) e^{-itH_c}\psi^0$$
$$= \begin{pmatrix} J\left(\frac{x}{t}+1\right)\left(\psi_1^0(x)\,\mathbb{1}_{\mathbb{R}^-}(x)\,\mathbb{1}_{\mathbb{R}^-}(x+t) + \psi_1^0(x)\,\mathbb{1}_{\mathbb{R}^+}(-x)\,\mathbb{1}_{\mathbb{R}^+}(x+t)\right) \\ J\left(-\frac{x}{t}+1\right)\left(\psi_2^0(x)\,\mathbb{1}_{\mathbb{R}^-}(x)\,\mathbb{1}_{\mathbb{R}^-}(x-t)\right) \\ J\left(-\frac{x}{t}+1\right)\left(\psi_3^0(x)\,\mathbb{1}_{\mathbb{R}^-}(x)\,\mathbb{1}_{\mathbb{R}^-}(x-t)\right) \\ J\left(\frac{x}{t}+1\right)\left(\psi_4^0(x)\,\mathbb{1}_{\mathbb{R}^-}(x)\,\mathbb{1}_{\mathbb{R}^-}(x+t) + \psi_4^0(x)\,\mathbb{1}_{\mathbb{R}^+}(-x)\,\mathbb{1}_{\mathbb{R}^+}(x+t)\right) \end{pmatrix}.$$

This last term converges pointwise to $J(1)\,\psi^0(x)$ as $t \to \infty$. Since J, $\mathbb{1}_{\mathbb{R}^-}$, $\mathbb{1}_{\mathbb{R}^+}$, $\mathbb{1}_{\mathbb{R}^-}$ are bounded and $\psi^0 \in \mathcal{H}_{s,n}$, we can use the dominate convergence theorem to conclude that:

$$\lim_{t\to\infty} e^{itH_c} J\left(\frac{\mathcal{A}}{t}\right) e^{-itH_c}\psi^0 = J(1)\,\psi^0.$$

If $J \in C_\infty(\mathbb{R})$ with $J(0) = 1$, then

$$\lim_{t\to\infty} e^{itH_c} J\left(\frac{\mathcal{A}}{Rt}\right) e^{-itH_c}\psi^0 = J\left(\frac{1}{R}\right)\psi^0,$$

and the last term goes to $J(0)\psi^0 = \psi^0$. So

$$\mathrm{s}-\lim_{R\to\infty}\left(\mathrm{s}-\lim_{t\to\infty} e^{itH_c} J\left(\frac{\mathcal{A}}{Rt}\right) e^{-itH_c}\right) = \mathbb{1}.$$

The last part of the theorem follows from the abstract theory. $\square$

We can know study the spectrum of P_c^+:

Proposition 8.4. $\sigma\left(P_c^+\right) = \{1\}$

Proof. Let $J \in C_\infty(\mathbb{R})$ such that $J(1) = 0$. We can approach J by a sequence $(J_n)_{n\in\mathbb{N}}$ of $C_0^\infty(\mathbb{R})$ functions which are zero in a neighbourhood of 1 in L^∞. By density, we can suppose that $J \in C_0^\infty(\mathbb{R})$ and J is zero in a neighbourhood of 1. Using minimal and maximal velocity estimates, we obtain:

$$J\left(P_c^+\right) = \mathrm{s}-\lim_{t\to\infty} e^{itH_c} J\left(\frac{\mathcal{A}}{t}\right) e^{-itH_c} = 0 \tag{8.8}$$

Now, if we have $J(1) \neq 0$, we can suppose that $J \in C_0^\infty(\mathbb{R})$ is constant, non zero, in a neighbourhood of 1. Then, for all $\varphi \in \mathcal{H}$, we have:

$$J\left(P_c^+\right)\varphi - J(1)\varphi = \mathrm{s}-\lim_{t\to\infty} e^{itH_c}\left(J\left(\frac{\mathcal{A}}{t}\right) - J(1)\right) e^{-itH_c}\varphi.$$

Since $J(x) - J(1)$ is zero in a neighbourhood of 1, we obtain $J\left(P_c^+\right)\varphi = J(1)\varphi \neq 0$. This ends the proof. $\square$

The following consequence is immediate:

Corollary 8.5. $P_c^+ = \mathbb{1}$

8.3. Asymptotic velocity for H_m

Theorem 8.6 (Asymptotic velocity for H_m). *Let $J \in C_\infty(\mathbb{R})$. Then, for all $m > 0$, the limit:*

$$\text{s} - \lim_{t\to\infty} e^{itH_m} J\left(\frac{\mathcal{A}}{t}\right) e^{-itH_m} \tag{8.9}$$

exists. Moreover, if $J(0) = 1$, then

$$\text{s} - \lim_{R\to\infty} \left(\text{s} - \lim_{t\to\infty} e^{itH_m} J\left(\frac{\mathcal{A}}{Rt}\right) e^{-itH_m}\right) = 1 \tag{8.10}$$

If we define

$$\text{s} - \text{C}_\infty - \lim_{t\to\infty} e^{itH_m} \frac{\mathcal{A}}{t} e^{-itH_m} =: P_m^+, \tag{8.11}$$

then the self-adjoint operator P_m^+ is densely defined and commutes with H_m. The operator P_m^+ is called the asymptotic velocity.

Proof. We can write

$$e^{itH_m} J\left(\frac{\mathcal{A}}{t}\right) e^{-itH_m} = e^{itH_m} e^{itH_c} e^{itH_c} J\left(\frac{\mathcal{A}}{t}\right) e^{-itH_c} e^{itH_c} e^{-itH_m}$$

Using uniform boundedness of our operators and introducing Ω and W at the right place, this limit is equal to $WJ(P_c^+)\,\Omega$ where W, Ω are defined in theorems 7.2. We can use the same argument for the second limit and the existence of P_m^+ follows by the abstract theory and we have:

$$J\left(P_m^+\right) = WJ\left(P_c^+\right)\Omega \tag{8.12}$$

□

We deduce:

Proposition 8.7. *For all $m > 0$, $\sigma(P_m^+) = \{1\}$*

Proof. Using the last proof, we have:

$$J\left(P_m^+\right) = WJ\left(P_c^+\right)\Omega$$

for all $J \in C_\infty(\mathbb{R})$ where Ω, W are unitary and $\Omega^{-1} = W$. □

We then have the following consequence:

Corollary 8.8. *For all* $m > 0$, $P_m^+ = \mathbb{1}$.

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