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On Calabi-Yau generalized complete intersections from Hirzebruch varieties and novel K3-fibrations

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We consider the construction of Calabi-Yau varieties recently generalized to where the defining equations may have negative degrees over some projective space factors in the embedding space [1]. Within such "generalized complete intersection" Calabi-Yau ("gCICY") three-folds, we find several sequences of distinct manifolds. These include both novel elliptic and K3-fibrations and involve Hirzebruch surfaces and their higher dimensional analogues. *En route*, we generalize the standard techniques of cohomology computation to these generalized complete intersection Calabi-Yau varieties.

1. Introduction, results and synopsis

Ever since the discovery [2] that compact Calabi-Yau 3-folds provide string vacua with possibly realistic phenomenology, the systematic construction of such varieties and computation of their physically relevant numerical characteristics has grown from the initial attempts [3–6] to the impressive catalogue of some half a billion or more examples [7, 8]. Besides providing an incredible haystack of models in which to search for one that can describe the vacuum of our own Universe, this collection also provides a "laboratory" in which to explore both mathematical and physical properties of string theory and its M- and F-theory extensions, such as mirror symmetry [9–13].

Recently, a novel class of "generalized complete intersection Calabi-Yau" ("gCICY") 3-folds was introduced [1], constructed as solutions to systems of algebraic equations in products of projective spaces where some of the defining equations may have negative degrees over some of the projective spaces. The Laurent polynomials of these equations are "tuned" so that their poles avoid the common zero-locus of the system, and this considerably enlarges the original pool of complete intersection Calabi-Yau (CICY) varieties [3–5]. In fact, we find that the construction of gCICYs, many of which are K3-fibrations, provides for even more distinct Calabi-Yau manifolds than reported in Ref. [1].

In particular, the exploratory collection and preliminary classification in [1] lists several sequences of K3-fibrations[1] such as

$$(1.1)\qquad X_m \in \left[\begin{array}{c\|c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2{-}m \end{array}\right]^{(2,86)}_{-168}, \qquad m = 0,1,2,3,\dots;$$
$$X_m \subset F_m \in \left[\begin{array}{c\|c} \mathbb{P}^4 & 1 \\ \mathbb{P}^1 & m \end{array}\right],$$

where $h^{1,1}(X_m) = 2$, $h^{2,1}(X_m) = 86$ and $\chi(X_m) = -168$. This defines the intermediate 4-fold F_m as a degree-$\binom{1}{m}$ hypersurface $p(x, y) = 0$ embedded in $A := \mathbb{P}^4 \times \mathbb{P}^1$, with $p(x, y)$ a holomorphic section of $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{O}\binom{1}{m}$ and with $h^{1,1}(F_m) = 2$ and $\chi(F_m) = 4$. Then, $X_m \subset F_m$ is a degree-$\binom{4}{2-m}$ hypersurface $q(x, y) = 0$, with $q(x, y)$ a holomorphic section of the $\mathcal{Q} \stackrel{\text{def}}{=} \mathcal{O}\binom{4}{2-m}$ line-bundle on F_m. For $m > 2$, $q(x, y)$ is a Laurent-polynomial over $\mathbb{P}^4 \times \mathbb{P}^1$,

[1]The sequences in question are labeled as "Type III" in Tables 1–4 of [1]; see also Eqs. (5.28)–(5.32) therein. Following Ref. [14], we write $X \in [A\|\mathbb{D}]$ to signify that X is a member of the deformation family of varieties embedded in the *embedding space* A by means of degree-$\mathbb{D}$ holomorphic constraints.

but may be chosen ("tuned" [1]) so its poles avoid the zeros of $p(x,y)$, making $X_m = \{p(x,y)=0\} \cap \{q(x,y)=0\} \subset \mathbb{P}^4 \times \mathbb{P}^1$ well-defined for every $m \geqslant 0$. In particular, this also proves that although $q(x,y)$ has a Laurent representative over $\mathbb{P}^4 \times \mathbb{P}^1$, it is holomorphic over $F_m \subset \mathbb{P}^4 \times \mathbb{P}^1$.

For the configurations (1.1) with $m \leqslant 2$ and other configurations in Appendix C with non-negative degrees, the classical analysis has been shown [15] to relate directly to the BRST treatment of constraints in the (world-sheet) field theory of superstrings compactified on so-defined Calabi-Yau 3-folds, and is also well known to correspond to Landau-Ginzburg orbifolds [16] and Witten's gauged linear σ-model (GLSM) [17]. For $m \geqslant 3$ however, the superpotentials in these world-sheet field theories necessarily include Laurent polynomials in the fields, and the correspondingly generalized GLSMs are discussed in Ref. [18]. Herein, we focus on the classical geometry and its physics implications, and defer the quantum aspects of compactification on such generalized complete intersections to separate efforts such as Refs. [18–20].

In Section 2, we show that the Calabi-Yau 3-folds constructed in (1.1) and similar semi-infinite sequences[2] are in fact distinct from each other, and in physically relevant ways: Although all members within a sequence have the same Hodge numbers and even the same $\dim H^1(X_m, \mathrm{End}\, T)$, the *classical* triple intersections and the Pontryagin (Chern) evaluations of $H^{1,1}(X_m) \approx H^2(X_m,\mathbb{Z}) \approx H_4(X_m,\mathbb{Z})$ elements vary within each sequence. However, we find that this m-dependence of *classical* topology characteristics is periodic in such "Type III" sequences of K3-fibrations: in (1.1) they depend on m (mod 4). In fact, this periodicity in the topological data of the Calabi-Yau 3-folds X_m in (1.1) is inherited from the 4-fold F_m. Analogous phenomena are shown below to exist also in lower dimensions, generalizing the well-known [m (mod 2)]-diffeomorphism of Hirzebruch surfaces $\mathfrak{F}_m$. A theorem by C.T.C. Wall [21] then guarantees that the sequence (1.1) contains four distinct diffeomorphism classes of Calabi-Yau 3-folds, of which X_3 is a novel construction; see Figure 1 for a partial roadmap. In particular, distinct configurations in this network have conifold transitions to distinct $h^{1,1}=1$ models, as indicated in Figure 1. As we show below, this distinction is related to the fact that the Hirzebruch surface $\mathfrak{F}_1$ *can* be blown down to $\mathbb{P}^2$, while $\mathfrak{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ cannot.

Also, we provide a homological algebra explanation and general prescription for the specially tuned rational (Laurent-polynomial) sections $q(x,y)$

[2] Appendix C presents more examples, including terminating sequences that nevertheless contain novel constructions.

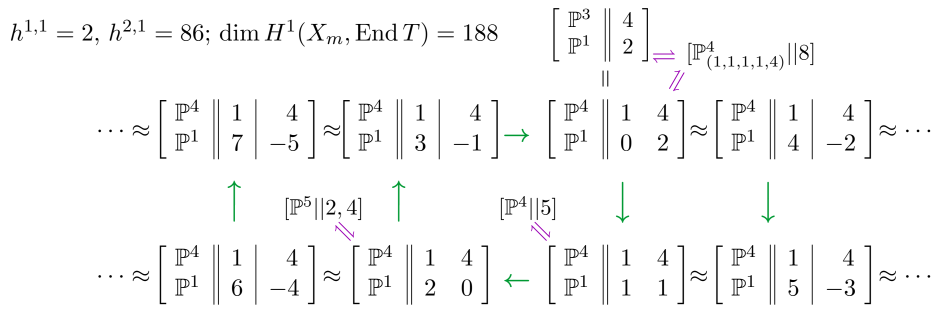

Figure 1: The $m \to [m+1 \pmod 4]$ "pinwheel" network of various models related in this article; see Section 2. Here, "$\approx$" denotes diffeomorphism (homotopy equivalence) as per Wall's theorem [21], while "$\leftrightharpoons$" denotes conifold transitions such as those discussed in Refs. [22, 23].

used to define generalized complete intersections such as (1.1) for $m \geqslant 3$. This reconstructs the results of the iterative method as reported in Ref. [1] for the specific cases considered, indicating a substantial generalization of the "linear algebra" methods [14, 24], which is however outside our present scope. The sequence (1.1) involves the 4-folds $F_m = \mathscr{F}_m^{(4)} \in \left[\begin{smallmatrix}\mathbb{P}^4\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\right]$, while other of the "Type III" sequences of Ref. [1] involve its 3- and 2-dimensional analogues. Adopting the name from the well-known 2-dimensional case $\mathfrak{F}_m = \mathscr{F}_m^{(2)}$, we dub $\mathscr{F}_m^{(n)}$ (m-twisted) "Hirzebruch n-folds."

In Section 3, we analyze the so-constructed Calabi-Yau 3-folds X_m as K3-fibrations, elliptic fibrations, and even iteratively nested fibrations; several of these features have been noted in Ref. [1]. We also identify Calabi-Yau gCICY configurations which support this periodicity, and provide a geometric interpretation of this periodicity.

Just as the Hirzebruch surfaces $\mathfrak{F}_m$, Hirzebruch n-folds $\mathscr{F}_m^{(n)}$ are no longer Fano for $m \geqslant 2$. This explains the absence of such constructions from previous efforts, and the addition of some novel Calabi-Yau 3-folds even to such comprehensive databases as the Kreuzer-Skarke catalogue [7]. For example, the anticanonical bundle $\mathcal{Q} = \mathcal{O}\binom{4}{2-m}$ of F_m in (1.1) is no longer positive over $\mathbb{P}^1$ for $m \geqslant 2$, and fails to be ample for $m \geqslant 3$. Nevertheless, sequences such as (1.1) do contain smooth and often novel Calabi-Yau 3-folds.

Finally, we summarize our results and their implications in Section 4, and comment on the *quantum* cohomology of such 3-folds as well as finding their analogues in toric constructions; see also Refs. [18–20]. Technical details are deferred to the appendices: in particular, Appendices A and B collect the requisite cohomology computations for the sequence of 3-folds (1.1), while Appendix C contains some further interesting examples, some of which have doubly periodic topological data.

Note Added. After the completion of this paper and its submission for publication, "generalized complete intersection" varieties (Calabi-Yau or not) have been given a rigorous scheme-theoretic formulation within the Čech cohomology framework [25]. Detailed comparison of the computations for the class of models discussed herein, and upon "clearing denominators," shows perfect agreement not only for the defining polynomials (2.1) and (2.3), but also the auxiliary Laurent polynomials

$$f(x,y) = f_{(abc)}^{(j_1\cdots j_{2m-4})}\,\frac{x^a\,x^b\,x^c}{y^{j_1}\cdots y^{j_{2m-4}}}$$

in (2.4), the analogue of which specifies the level of clearing denominators [25].

2. A curiously periodic sequence

We first explore the generalized complete intersections in projective spaces (1.1), and begin with a few key properties of the 4-folds F_m. To this end, we use the classical methods of algebraic geometry to compute the required cohomology of $X_m \subset F_m \subset A = \mathbb{P}^4 \times \mathbb{P}^1$ iteratively; technical details are deferred to the Appendices A and B. First, the 4-fold F_m is defined:

$$F_m \subset A = \mathbb{P}^4 \times \mathbb{P}^1 : \quad p(x,y) = p_{a\,(i_1\cdots i_m)}\,x^a\,y^{i_1}\cdots y^{i_m} = 0, \tag{2.1}$$

where $(x^0 : \cdots : x^4) \in \mathbb{P}^4$ and $(y^0 : y^1) \in \mathbb{P}^1$ are the usual respective homogeneous coordinates, and the coefficients $p_{a\,(i_1\cdots i_m)}$ represent the defining tensor of F_m. Holomorphic sections and forms on F_m are obtained by restricting those on the ambient space A by means of the Koszul resolution *monad*[3]:

$$\mathcal{O}_A\big(\begin{smallmatrix}-1\\-m\end{smallmatrix}\big) \overset{p}{\hookrightarrow} \mathcal{O}_A \overset{\rho_F}{\twoheadrightarrow} \mathcal{O}_{F_m}, \tag{2.2}$$

stating that sheaf of holomorphic functions on F_m may be identified with the sheaf of holomorphic functions on A, taken modulo $p(x,y)$-multiples of

[3]Throughout, "$\hookrightarrow$" denotes injections (1–1 linear maps which annihilate no nontrivial domain element), while "$\twoheadrightarrow$" denotes surjections ("onto" linear maps which omit nothing in the codomain).

$\mathcal{O}_A\big({-1 \atop -m}\big)$-valued functions on A. We also use that $\mathbb{P}^n = \frac{U(n+1)}{U(1)\times U(n)}$, whereby Bott-Borel-Weil's theorem guarantees that bundles over $\mathbb{P}^n$ furnish $U(1)\times U(n)$-representations and all the cohomology groups valued in those bundles furnish $U(n+1)$-representations, so that the maps in (2.2) and between the associated cohomology groups are completely represented by linear algebra with "direct image" $U(n+1)$-tensors [24]. For example, (2.1) defines the tensor representative $p_{a\,(i_1\cdots i_m)}$ of sections of $\mathcal{O}_A\big({1 \atop m}\big)$. This computational framework [14, 24] is also closely related to the Atiyah-Bott-Gårding-Candelas residue formulae [15, 26, 27], *i.e.*, the Grothendieck local residue symbol [28, 29], as well as the BRST treatment of constraints and gauge-equivalence classes; see Ref. [15].

2.1. Viability of X_m

To verify that the anticanonical bundle $\mathcal{K}^*_{F_m} = \mathcal{Q}$ of the hypersurface (2.1) does have global holomorphic sections with which to define the Calabi-Yau hypersurface X_m, we compute the cohomology groups $H^*(F_m, \mathcal{Q})$ and find that $\dim H^0(F_m, \mathcal{Q}) \geqslant 105$ for all $m \geqslant 0$; see Appendix A. With that many linearly independent holomorphic sections to use for the defining equation of $X_m \subset F_m$, we expect that generic members of (1.1) are smooth for each $m \geqslant 0$, but we are not aware of a suitable generalization of Bertini's theorem to guarantee this also for the $m > 2$ cases.[4] We verified by direct computation that the analogous construction of 2-tori as hypersurfaces in Hirzebruch surfaces do provide smooth models for all $m \geqslant 0$, and that the inclusion of the Laurent monomials is crucial to this end when $m \geqslant 3$.

Given our lower-dimensional explicit computations and the computer-aided assurances from Ref. [1], we work on the assumption that the system of 105 sections (A.1) and (A.10) does suffice to construct smooth models $X_m \subset F_m \subset \mathbb{P}^4 \times \mathbb{P}^1$ for every $m \geqslant 0$. In turn, the existence of holomorphic anticanonical sections for other gCICY's is certainly not a foregone conclusion: there do exist similarly constructed gCICY sequences that terminate, see Appendix C.1.

[4] Ref. [1] discusses computer-aided case-by-case methods of analysis which could do so for any fixed m, and cite the "Type III" configuration (1.1) as containing smooth models.

2.2. Hodge numbers

The Calabi-Yau 3-folds X_m in (1.1) are defined by intersecting the hypersurface (2.1) with a second hypersurface, defined as the vanishing locus of:

$$\begin{aligned}
&(2.3\text{a})\quad q(x,y) = q_{(abcd)(ij)}\, x^a \cdots x^d\, y^i y^j \quad \& \quad q_{(abcd)\,i}\, x^a \cdots x^d\, y^i, && m = 0,1;\\
&(2.3\text{b})\qquad\quad = \Big(q_{(abcd)} + q_{(abcd)\,k}^{\;j}\,\frac{y^k}{y^j}\Big)\, x^a \cdots x^d, && m = 2;\\
&(2.3\text{c})\qquad\quad = q_{(abcd)}^{(j_1\cdots j_{m-2})}\,\frac{x^a \cdots x^d}{g^{(j_1\cdots j_{m-2})}(y)}, && m \geqslant 3.
\end{aligned}$$

As shown in Appendix A.1, the defining tensors for the Laurent polynomials are parametrized by auxiliary $\mathbb{P}^4$-cubics $f_{(abc)}^{\cdots}\, x^a x^b x^c$:

$$\begin{aligned}
&(2.4\text{a})\qquad q_{(abcd)\,k}^{\;j} \stackrel{\text{def}}{=} \varepsilon^{ij} f_{(abc}\, p_{d)(ik)}, && m = 2;\\
&(2.4\text{b})\qquad q_{(abcd)}^{(j_1\cdots j_{m-2})} \stackrel{\text{def}}{=} \varepsilon^{i(j_1} f_{(abc}^{j_2\cdots j_{m-2} j_{m-1}\cdots j_{2m-3})}\, p_{d)(i\, j_{m-1}\cdots j_{2m-3})}, && m \geqslant 3,\\
&(2.4\text{c})\qquad \text{where}\quad f_{(abc}^{(j_1\cdots j_{2m-4})}\, p_{d)(j_{m-3}\cdots j_{2m-4})} = 0, && m \geqslant 4.
\end{aligned}$$

The $(m{-}1)$ degree-$(m{-}2)$ generic $\mathbb{P}^1$-polynomials $g^{(j_1\cdots j_{m-2})}(y)$ used in the denominator in (2.3c) provide for the convenience of moving and separating the poles of $q(x,y)$ to $(m{-}1)(m{-}2)$ distinct locations, and so minimally extend the "direct image" linear algebra methods [14, 24].

Using the adjunction relation $T_{X_m} \hookrightarrow T_A|_{X_m} \stackrel{\mathrm{d}q}{\twoheadrightarrow} [\mathcal{P} \oplus \mathcal{Q}]_{X_m}$, we compute the cohomology groups $H^*(X_m, T) = H^*(X_m, \wedge^2 T^*)$. Deferring the technical details of the computation to Appendix B, we quote here that

$$(2.5)\qquad \begin{aligned} h^{1,2} &= \dim H^1(X_m, T) = 86 \qquad \text{and}\\ h^{1,1} &= h^{2,2} = \dim H^2(X_m, T) = 2, \end{aligned}$$

uniformly for all $m \geqslant 0$. The same techniques also compute

$$\dim H^1(X_m, \operatorname{End} T) \geqslant 188.$$

In particular, the results (2.5) computed in Appendix B.1 also prove that $H^1(X_m, T^*) = H^{1,1}(X_m)$, the dual of $H^2(X_m, T)$, is generated by (the pullbacks of) the Kähler classes J_1 of $\mathbb{P}^4$ and J_2 of $\mathbb{P}^1$ for all $m \geqslant 0$. The standard computation of the Chern class then gives:

$$\begin{aligned}(2.6)\quad c(X_m) &= \frac{(1{+}J_1)^5(1{+}J_2)^2}{(1{+}J_1{+}mJ_2)(1{+}4J_1+(2{-}m)J_2)},\\ &= 1+\big(6J_1^2+(8{-}3m)J_1J_2\big)+\big({-}20J_1^3-(32{+}15m)J_1^2J_2\big),\end{aligned}$$

confirming that the Euler number is independent of m:

$$(2.7)\qquad \chi_{_E} = \int_{X_m} c_3 = \int_A \big(J_1{+}mJ_2\big)\big(4J_1+(2{-}m)J_2\big)\,c_3 = -168.$$

2.3. Classical topology

Wall's theorem [21] guarantees that the diffeomorphism class of compact and orientable real 6-dimensional manifolds X is determined by the Betti numbers b_2 and b_3, the cubic intersections (classical Yukawa couplings) and the (first Pontryagin class) p_1-evaluation of $H^2(X,\mathbb{Z}) \approx H_4(X,\mathbb{Z})$ elements; see the full discussion below. The standard relation $p_1 = c_1{}^2{-}2c_2$ simplifies for Calabi-Yau 3-folds to $p_1 = -2c_2$, and we also have that $b_2 = h^{1,1} = 2$ and $b_3 = 2+2h^{2,1} = 174$ are m-independent.

As shown in Appendix B.1 and discussed below, $H^{1,1}(X_m) \approx H^2(X_m,\mathbb{Z})$ is generated by (the pullbacks to X_m of) the Kähler classes of $\mathbb{P}^4$ and $\mathbb{P}^1$, so the classical Yukawa couplings in $H^{1,1}(X_m)$ are the standard classical (topological) intersection numbers:

$$(2.8)\quad [(aJ_1+bJ_2)^3]_{X_m} = 2a^3+3a^2(\underline{4b{+}ma}),\quad \text{i.e.,}\ \begin{cases}\kappa_{111} = 2{+}3m,\\ \kappa_{112} = 4,\\ \kappa_{122} = 0 = \kappa_{222}.\end{cases}$$

Also,

$$(2.9)\qquad C_2[aJ_1+bJ_2] = 44a+6(\underline{4b{+}ma}),\quad \text{i.e.,}\ \begin{cases}C_2[J_1] = 44+6m,\\ C_2[J_2] = 24.\end{cases}$$

In turn, the topological invariants (2.8) and the Chern evaluation (2.9) do depend on m, proving that the sequence (1.1) does contain topologically distinct Calabi-Yau 3-folds.

Given that the Hodge diamond and the Euler characteristic are independent of m, and the topological intersections (2.8) and Chern evaluations (2.9) depend on bJ_2 and m only through the (underlined) hallmark combination

$(4b{+}am)$, it follows that all topological invariants remain unchanged by transforming $(a,b,m)\to(a,b{-}ac,m{+}4c)$, which is the integral basis-change relation:

$$\text{(2.10)}\qquad \begin{bmatrix}J_1\\J_2\end{bmatrix}_m \overset{\approx}{\longleftrightarrow} \begin{bmatrix}1&-c\\0&1\end{bmatrix}\begin{bmatrix}J_1\\J_2\end{bmatrix}_{m+4c},\qquad c\in\mathbb{Z},\quad \det\begin{bmatrix}1&-c\\0&1\end{bmatrix}=1.$$

That is, the topological invariants (2.8) and (2.9) of X_m and of X_{m+4c} for $c\in\mathbb{Z}$ differ only by an integral basis change, and Wall's theorem guarantees that X_m is diffeomorphic to X_{m+4c} for all $c\in\mathbb{Z}$. Through this $[m \pmod 4]$-dependence of the topological data (2.8) and (2.9), Wall's theorem guarantees that the sequence (1.1) contains precisely four distinct diffeomorphism classes of Calabi-Yau 3-folds, counted by $m \pmod 4$. It is this $[m \pmod 4]$-periodicity that provides the "pinwheel" diagram in Figure 1 with the characteristic cyclicality.

We close here with a remark on the use of Wall's theorem. (**1**) As the zero set of ample and positive line bundles, all m-twisted Hirzebruch n-folds are directly subject to the Lefschetz hyperplane theorem: $H^r(\mathscr{F}^{(n)},\mathbb{Z}) = H^r(\mathbb{P}^n\times\mathbb{P}^1,\mathbb{Z})$ for $r\neq n$, and has no torsion. (**2**) For $r=n$, the independent computation of the Euler number and the use of the universal coefficient theorem [30] jointly insure that also $H^n(\mathscr{F}^{(n)}_m,\mathbb{Z}) = H^n(\mathbb{P}^n\times\mathbb{P}^1,\mathbb{Z})$, and has no torsion. (**3**) The torsion-free (co)homology of all m-twisted Hirzebruch n-folds exhibits the $[m \pmod n]$-periodicity in the (classical) ring structure of $H^*(\mathscr{F}^{(n)}_m,\mathbb{Z})$; see Appendices A.2 and A.3. (**4**) The Calabi-Yau $(n{-}1)$-folds embedded as anticanonical hypersurfaces in the m-twisted Hirzebruch n-folds exhibit exactly the same $[m \pmod n]$-periodicity — generalizing (2.8)–(2.9), which therefore cannot possibly be the consequence of any torsion element. Finally, (**5**) since the 2nd Stiefel-Whitney class w_2 is a $\mathbb{Z}_2$ reduction of the 1st Chern class — which vanishes for Calabi-Yau $(n{-}1)$-folds by definition — the $w_2=0$ condition of Wall's theorem is also satisfied.

3. Calabi-Yau 3-folds from Hirzebruch n-folds

All the "Type-III" sequences of Ref. [1] involve Hirzebruch n-folds: Just as our main example (1.1) involves the Hirzebruch 4-fold $F_m\in\left[\begin{smallmatrix}\mathbb{P}^4\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\right]$, the sequence [1]

$$\text{(3.1)}\qquad X'_m\in\left[\begin{array}{c||c|c}\mathbb{P}^1&0&2\\ \hdashline \mathbb{P}^3&1&3\\ \mathbb{P}^1&m&2{-}m\end{array}\right]^{(3,75)}_{-144}$$

$$[(aJ_1+bJ_2+cJ_3)^3]_{X'_m} = 6ab^2 + 2b(3a+b)(\underline{3c+bm}),$$
$$C_2[(aJ_1+bJ_2+cJ_3)] = 24a+36b + 2(\underline{3c+bm})],$$

involves $\mathcal{F}_m \in \left[\begin{smallmatrix}\mathbb{P}^3\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\right]$ while the "Type III" sequences [1]

(3.2)
$$X''_m \in \left[\begin{array}{c||c|c} \mathbb{P}^2 & 0 & 3 \\ \hdashline \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(3,75)}_{-144}$$
$$[(aJ_1+bJ_2+cJ_3)^3]_{X''_m} = 6a^2b + 3a(a+3b)(\underline{2c+bm}),$$
$$C_2[(aJ_1+bJ_2+cJ_3)] = 36a+24b + 12(\underline{2c+bm}),$$

(3.3)
$$X'''_m \in \left[\begin{array}{c||c|c} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \hdashline \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(4,68)}_{-128}$$
$$[(aJ_1+bJ_2+cJ_3+dJ_4)^3]_{X'''_m} = 12abc + 6(ab+ac+bc)(\underline{2d+cm}),$$
$$C_2[(aJ_1+bJ_2+cJ_3+dJ_4)] = 24(a+b+c) + 12(\underline{2d+cm}),$$

both involve $\mathfrak{F}_m \in \left[\begin{smallmatrix}\mathbb{P}^2\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\right]$, which are well-known as Hirzebruch surfaces [14, 31] as well as "rational ruled surfaces" [28].

3.1. Fibrations

Each X'_m in the sequence (3.1) is a generalized "double solid" [32]: the small resolution[5] of a double-cover of the Hirzebruch 3-fold $\mathcal{F}_m$, branched over a degree-$\binom{6}{4-2m}$ hypersurface $\mathcal{B} \subset \mathcal{F}_m$. Notice that the branching locus is itself a generalized complete intersection from the configuration $\left[\begin{smallmatrix}\mathbb{P}^3\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\middle|\begin{smallmatrix}6\\ 4-2m\end{smallmatrix}\right]$, where the second constraint becomes negative over $\mathbb{P}^1$ for $m \geqslant 3$. It is amusing to think of (3.1) also as a deformation family of "see-saw twisted"[6] double-point fibrations over $\mathcal{F}_m$, since $[\mathbb{P}^1\|2] = \{2 \text{ pts}\}$ is the Calabi-Yau 0-fold. In turn, we may also regard (3.1) as a deformation family of fibrations of K3 $\in \left[\begin{smallmatrix}\mathbb{P}^3\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\middle|\begin{smallmatrix}3\\ 2-m\end{smallmatrix}\right]$ over $\mathbb{P}^1$, where now fibers are K3 surfaces, the well-known Calabi-Yau 2-folds.

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[5] The branching locus is itself typically singular even if the whole 3-fold is smooth [14, p. 141].

[6] For $m \geqslant 3$, the twist of the fibration is positive over $\mathbb{P}^3$ but negative over $\mathbb{P}^1$.

In turn, the two sequences of Calabi-Yau 3-folds (3.2) may be regarded as "ordinary" elliptic fibrations over the Fano (del Pezzo) bases $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, however with the fibers being "generalized complete intersection" tori in $\left[\begin{smallmatrix}\mathbb{P}^2\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\middle|\begin{smallmatrix}2\\ 2-m\end{smallmatrix}\right]$. In turn, the same sequences may also be regarded as (see-saw complementarily twisted for $m > 2$) elliptic (torus) fibrations over the Hirzebruch surfaces $\mathfrak{F}_m$, where the fibers are familiar tori from the configurations $[\mathbb{P}^2\|3]$ and $\left[\begin{smallmatrix}\mathbb{P}^1\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}2\\ 2\end{smallmatrix}\right]$, respectively. Viewed this way and since the Hirzebruch surfaces $\mathfrak{F}_m$ are themselves fibrations, the sequences (3.2) are in fact iterated fibrations.

In agreement with Ref. [1], we find that each of these Calabi-Yau 3-folds may be regarded as a fibration of a Calabi-Yau n-fold over a $(3-n)$-dimensional base in at least two different ways. As compared with the constructions studied until Ref. [1], the novelty in (1.1), (3.1) and (3.2) stems either from: (**1**) using a decidedly non-Fano base such as the 3-folds $\mathcal{F}_m$ and the 2-folds $\mathfrak{F}_m$ for $m \geqslant 3$, or from (**2**) fibering generalized complete intersection Calabi-Yau n-folds such as

$$T^2 \in \left[\begin{array}{c\|c|c}\mathbb{P}^2 & 1 & 2\\ \mathbb{P}^1 & m & 2-m\end{array}\right] \quad\text{and}\quad \mathrm{K3} \in \left[\begin{array}{c\|c|c}\mathbb{P}^3 & 1 & 3\\ \mathbb{P}^1 & m & 2-m\end{array}\right], \quad \text{with } m \geqslant 3. \tag{3.4}$$

The novelty of these fibrations is seen already in our main example, the sequence (1.1), which contains four distinct diffeomorphism classes of Calabi-Yau 3-folds represented by the configurations

$$\left[\begin{array}{c\|c|c}\mathbb{P}^4 & 1 & 4\\ \hdashline \mathbb{P}^1 & 0 & 2\end{array}\right] = \left[\begin{array}{c\|c}\mathbb{P}^3 & 4\\ \hdashline \mathbb{P}^1 & 2\end{array}\right], \quad \left[\begin{array}{c\|c|c}\mathbb{P}^4 & 1 & 4\\ \hdashline \mathbb{P}^1 & 1 & 1\end{array}\right], \quad \left[\begin{array}{c\|c|c}\mathbb{P}^4 & 1 & 4\\ \hdashline \mathbb{P}^1 & 2 & 0\end{array}\right], \quad \left[\begin{array}{c\|c|c}\mathbb{P}^4 & 1 & 4\\ \hdashline \mathbb{P}^1 & 3 & -1\end{array}\right]. \tag{3.5}$$

In the first of these, the first, degree-$\binom{1}{0}$ defining equation is a simple, $\mathbb{P}^1$-constant hyperplane, $[\mathbb{P}^4||1] \approx \mathbb{P}^3$, which induces a global isomorphism indicated by the "=" sign. The resulting configuration, $\left[\begin{smallmatrix}\mathbb{P}^3\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}4\\ 2\end{smallmatrix}\right]$ may be regarded as a deformation family of quadratic $\mathrm{K3} \in [\mathbb{P}^3||4]$-fibrations over $\mathbb{P}^2$. In turn, the remaining three representatives may also be regarded as K3-fibrations over $\mathbb{P}^1$ but in subtly different ways — which explains the m-dependence in the intersection numbers (2.8) and Chern evaluations (2.9).

In particular, the second configuration is a fibration of a $\mathrm{K3} \in [\mathbb{P}^4||1,4]$ surface defined as the intersection of a hyperplane and a quartic in $\mathbb{P}^4$ — both of which vary non-trivially (linearly) over the base $\mathbb{P}^1$. Although a hyperplane in $\mathbb{P}^4$ is always isomorphic to a $\mathbb{P}^3$, now both the hyperplane and the quartic vary over the base $\mathbb{P}^1$. Thus, the isomorphism $[\mathbb{P}^4||1] \approx \mathbb{P}^3$ keeps also varying over the base $\mathbb{P}^1$, so that this is not a fibration (over $\mathbb{P}^1$) of a quartic hypersurface in a fixed $\mathbb{P}^3$, but in a similarly $\mathbb{P}^1$-variable $\mathbb{P}^3 \in [\mathbb{P}^4||1]$.

The third configuration now has the quartic $[\mathbb{P}^3||4]$ held constant over $\mathbb{P}^1$, and is being intersected by a (quadratically) $\mathbb{P}^1$-variable hyperplane in $\mathbb{P}^4$.

Finally, the fourth configuration again has both the hyperplane and the quartic vary over the base $\mathbb{P}^1$, but differently than in the second configuration: the hyperplane now varies cubically, while the quartic varies "inverse-linearly" (of degree-(-1)) over $\mathbb{P}^1$.

Viewing the succession of these various types of fibration, the classical [m (mod 4)]-periodicity (2.10) is rather surprising. For example, the configurations

$$(3.6)\qquad \left[\begin{array}{c||c} \mathbb{P}^3 & 4 \\ \hdashline \mathbb{P}^1 & 2 \end{array}\right] = \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \hdashline \mathbb{P}^1 & 0 & 2 \end{array}\right], \quad \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \hdashline \mathbb{P}^1 & 4 & -2 \end{array}\right], \quad \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \hdashline \mathbb{P}^1 & 8 & -6 \end{array}\right], \quad \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \hdashline \mathbb{P}^1 & 12 & -10 \end{array}\right], \quad \ldots$$

are all deformation families of Calabi-Yau 3-folds that are diffeomorphic to each other by virtue of Wall's theorem, in spite of the increasingly higher degree of $\mathbb{P}^1$-fibration of the hyperplane in $\mathbb{P}^4$ complemented by the increasingly more negative degree of $\mathbb{P}^1$-fibration of the quartic in $\mathbb{P}^4$.

3.2. Periodicity

The peculiar [m (mod 4)]-periodic diffeomorphisms $X_m \approx X_{m+4}$ (2.10) of the Calabi-Yau 3-folds in (1.1) in fact stem from the same diffeomorphisms (A.17) between the Hirzebruch 4-folds F_m; see Appendix A for more detail. Indeed, the [m (mod n)]-periodic diffeomorphisms of the Hirzebruch n-folds induce the same periodicity in all "Type-III" sequences:

$$(3.7)\quad [m \text{ (mod 4)}]:\quad F_m \in \left[\begin{array}{c||c} \mathbb{P}^4 & 1 \\ \mathbb{P}^1 & m \end{array}\right] \quad\text{and}\quad X_m \in \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & \textcircled{4} \\ \mathbb{P}^1 & m & 2-m \end{array}\right];$$

$$(3.8)\quad [m \text{ (mod 3)}]:\quad \mathcal{F}_m \in \left[\begin{array}{c||c} \mathbb{P}^3 & 1 \\ \mathbb{P}^1 & m \end{array}\right] \quad\text{and}\quad X'_m \in \left[\begin{array}{c||c|c} \mathbb{P}^1 & 0 & 2 \\ \hdashline \mathbb{P}^3 & 1 & \textcircled{3} \\ \mathbb{P}^1 & m & 2-m \end{array}\right];$$

$$(3.9)\qquad [m\ (\mathrm{mod}\ 2)]:\quad \mathfrak{F}_m \in \begin{bmatrix} \mathbb{P}^2 \\ \mathbb{P}^1 \end{bmatrix}\!\!\begin{Vmatrix} 1 \\ m \end{Vmatrix} \quad \text{and} \quad X''_m \in \left[\begin{array}{c||c|c} \mathbb{P}^2 & 0 & 3 \\ \hdashline \mathbb{P}^2 & 1 & \textcircled{2} \\ \mathbb{P}^1 & m & 2-m \end{array}\right];$$

$$(3.10)\qquad \text{and} \quad X'''_m \in \left[\begin{array}{c||c|c} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \hdashline \mathbb{P}^2 & 1 & \textcircled{2} \\ \mathbb{P}^1 & m & 2-m \end{array}\right].$$

The circles highlight the particular degree necessary for the periodicity of the Hirzebruch n-fold (appearing below the dashed horizontal line in three of the examples) to be inherited by the Calabi-Yau 3-fold. In turn, the example (C.1) does not satisfy this condition, the sequence therein terminates and exhibits none of the periodicity of the Hirzebruch surface in which those $\widetilde{X}_m$ are embedded.

This regularity persists generally, throughout the examples constructed from Hirzebruch n-folds, as demonstrated by several more complicated examples in Appendix C.

3.3. Discrete deformations and extremal transitions

As detailed in Appendix A.4, it is known that Hirzebruch surfaces of the same homotopy type, $\mathfrak{F}_m \approx \mathfrak{F}_{m+2}$, may be regarded as discrete deformations of one another [14, 33]. The direct computations in Appendix A are consistent with our conjecture A.1, that the same is true of the straightforward higher-dimensional generalizations,

$$(3.11)\qquad \mathscr{F}^{(n)}_m \in \begin{bmatrix} \mathbb{P}^n \\ \mathbb{P}^1 \end{bmatrix}\!\!\begin{Vmatrix} 1 \\ m \end{Vmatrix}, \qquad 2 \leqslant n \in \mathbb{Z} \ \text{ and } \ 0 \leqslant m \in \mathbb{Z}.$$

It therefore seems natural to propose:

Conjecture 3.1. ***(i)*** *The deformation spaces of Calabi-Yau 3-folds X_m and X_{m+n} which belong to an $[m\ (\mathrm{mod}\ n)]$-periodic sequence of configurations the periodicity of which stems from the same periodicity of a Hirzebruch n-fold factor in the embedding space are "separate but infinitesimally near," so that X_m is a discrete deformation of X_{m+n}.*

(ii) *In any classical field theory, the use of X_m and X_{m+n} should produce identical models; however, some quantum effects may well distinguish X_m from X_{m+n}; see Section 4 and Refs. [19, 20].*

In particular, the outward emanating sequences of configurations in Figure 1, of which the upper right-hand side quarter $(X_0, X_4, X_8, X_{12}\ldots)$ is reproduced in (3.6), are in fact sequences of such discrete deformations; see Figure 2. The *local* Kodaira-Spencer deformation spaces $H^1(X_m, T)$,

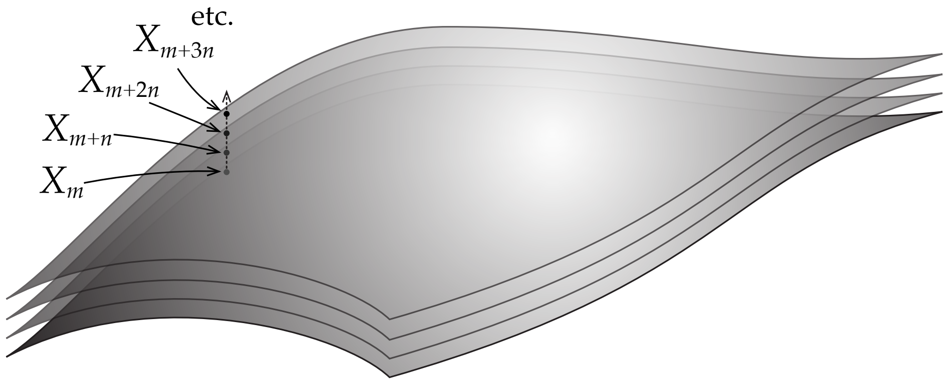

Figure 2: The Calabi-Yau $(n-1)$-folds $X_{m+kn} \subset \mathscr{F}^{(n)}_{m+kn}$ that are diffeomorphic to each other for $k = 0, 1, 2, \ldots$ have deformation spaces that are infinitesimally close; see Conjecture 3.1.

$H^1(X_{m+n}, T)$, $H^1(X_{m+2n}, T)$, $H^1(X_{m+3n}, T)$ etc., are of course all isomorphic. Whether this isomorphism extends to the entire moduli spaces as suggested in Figure 2, to the cohomology rings defined by the Yukawa couplings, and also away from the "large radius limit," remain open questions. As the Calabi-Yau 3-folds X_{m+kn} are all diffeomorphic for $k = 0, 1, 2, \ldots$ and so represent the same real manifold, the situation in Figure 2 would imply that the complex structure moduli space of such real manifolds comes in disjoint "sheets," possibly distinguishable by quantum effects as per Conjecture 3.1; see also Section 4 and Refs. [19, 20]. Of course, these "sheets" of the moduli space may well connect at certain singular limit points.

The first three models in the sequence (1.1), X_0, X_1 and X_2, are in fact "ordinary" CICYs [4, 5], for all of which it has long since been known that they are connected by way of extremal transitions [34]. That proof does not extend to the $m \geqslant 3$ members of the sequence (1.1). However, it can be shown that $X_0, \cdots, X_5$ in (1.1) can also be representated as hypersurfaces in toric 4-folds [7, 8] or their "flops," and that those toric representations of X_m are connected by way of extremal transitions; these and related matters will be discussed elsewhere [20].

4. Summary and outlook

The gCICYs introduced in [1] and further studied in the present paper are providing a promising new class of Calabi-Yau manifolds which extends beyond the current complete intersection (and hypersurfaces) in Fano toric varieties. The novel construction allows for interesting new K3 and elliptically fibered Calabi-Yau manifolds which are important in string duality scenarios and F-theory considerations. In particular, in that context, the α'-perturbative $d = 4$, $N = 2$ string vacua, with (often) a dual heterotic K3 $\times T^2$ compactification with some choice of $SU(2)$ instanton embedding in the vector bundle, provide equivalent, perturbative, low-energy effective field theories in terms of several classically isomorphic K3 fibered Calabi-Yau manifolds. However, non-perturbative effects, in particular the world-sheet instanton effects, in fact do turn out to be different and hence provide different non-perturbative completions [18, 20].

This issue was studied for hypersurfaces in toric varieties already more than twenty years ago [35]; see also [36]. Several examples were found where the Kähler moduli space has a large radius limit which is a K3-fibration with the Hodge numbers $(2, 86)$ and the same topological data as the $m = 0, 1, 2, 3, 4, 5$ gCICY models presented here [35]. For $m = 3, 4, 5$ the extended Kähler moduli space has multiple large radius phases in which the K3 fibered Calabi-Yau phase is obtained by a novel flop [20, 35]. With the mod 4 periodicity of the current work we may then be able to test our conjecture by comparing the Gromow-Witten (GW) invariants that can be calculated for the above mod 4 related Calabi-Yau manifolds. Specifically, let us consider the case of $m = 0$ (mod 4) realized as hypersurfaces in toric varieties. For $m = 0$ the Calabi-Yau has identical GW invariants to the $m = 0$ gCICY, and hence also exhibits a conifold transition to $[\mathbb{P}^4_{(1:1:1:1:4)}||8]$, see Figure 1. However, the $m = 4$ Calabi-Yau hypersurface has GW invariants which differ from those of its mod 4 cousin apart from the invariants associated to the identical K3-fiber. In particular, there is a conifold transition to a different $h_{1,1} = 1$ Calabi-Yau 3-fold, to $[\mathbb{P}^4_{(1:1:1:1:2)}||6]$.

This phenomenon extends beyond the particular K3-fibration we have focused on in this paper. Consider the heterotic string compactified on $T^2 \times$ K3 with $SU(2)$ instanton embedding $(4, 10, 10)$ in the $SU(2) \times E_8 \times E_8$ gauge bundle at the $SU(2)$ symmetric point of the T^2, which was conjectured to be dual to type IIA theory on the K3-fibered Calabi-Yau hypersurface $[\mathbb{P}^4_{(1:1:2:2:6)}||12]$ with Hodge numbers $(2, 128)$ [37]. However, there are in fact multiple Calabi-Yau three-folds with Hodge numbers $(2, 128)$ [7, 38], with instances of an extended moduli space with several Calabi-Yau phases [20, 35], analogous to our earlier discussion of the manifolds with Hodge numbers $(2, 86)$. In this case it can be shown that there is a mod 3 periodicity and we once more have multiple diffeomorphic representatives with the same classical topological data, but where the GW invariants differ even after the integral change of basis [20].

Thus, because these latter type IIA vacua have heterotic duals we then have several different non-perturbative completions of the same perturbative heterotic vacuum. It would be interesting to explore how this can be understood from the heterotic perspective, which we leave for future investigations.

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Note added in proof. Forced symmetrization $\varepsilon^{i(j_1} f^{j_2 \cdots j_M)} = f^{i(j_1 j_2 \cdots j_M)}$ for $M \geqslant 2$ renders such expressions *not* antisymmetric in any pair of indices. Thereby, e.g.,

$$\varepsilon^{i(j} f^{kl)} \cdot p_{(i\,l)} / y^j y^k = f^{i(jkl)} \cdot p_{(i\,l)} / y^j y^k \neq 0$$

although $\varepsilon^{i(j} f^{kl)} / y^i y^j y^k y^k = 0 = f^{i(jkl)} / y^i y^j y^k y^k$ since $f^{i(jkl)}$ is defined as the kernel of *total symmetrization*.

Appendix A. Hirzebruch n-folds

We compute various useful properties of the 4-folds F_m appearing in (1.1), and then discuss their analogues in different dimensions.

A.1. Anticanonical sections

As the configuration (1.1) embeds the Calabi-Yau 3-folds X_m as hypersurfaces in the 4-folds F_m, it is imperative to prove that the anticanonical bundle of F_m does have holomorphic sections from which to construct the defining equation of X_m.

A.1.1. Counting and tensor structure. The anticanonical bundle of the hypersurface $F_m \in \left[\begin{smallmatrix}\mathbb{P}^4\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\right]$ is $\mathcal{K}^*_{F_m} = \mathcal{Q} = \mathcal{O}\big({4 \atop 2-m}\big)$. To determine $H^*(F_m, \mathcal{Q})$, we tensor the monad (2.2) by $\mathcal{O}\big({4 \atop 2-m}\big)$ and obtain the Koszul resolution given in the header row of the tabulation of the so-valued cohomology:

(A.1)

	$\mathcal{O}_A\big({3 \atop 2-2m}\big)$	$\xhookrightarrow{p}$	$\mathcal{Q} = \mathcal{O}_A\big({4 \atop 2-m}\big)$	$\xrightarrow{\rho_F}$	$\mathcal{Q}\vert_{F_m}$
0.	$\theta^1_m\{\varphi_{(abc)(i_1\cdots i_{2-2m})}\}$	$\xrightarrow{p}$	$\theta^2_m\{\phi_{(abcd)(i_1\cdots i_{2-m})}\}$	$\xrightarrow{\rho_F}$	$H^0(F_m,\mathcal{Q}) \xrightarrow{\mathrm{d}}$
1.	$\theta^m_2\{\varepsilon^{i(j}\varphi_{(abc)}^{k_1\cdots k_{2m-4})}\}$	$\xrightarrow{p}$	$\theta^m_4\{\varepsilon^{i(j}\phi_{(abcd)}^{k_1\cdots k_{m-4})}\}$	$\xrightarrow{\rho_F}$	$H^1(F_m,\mathcal{Q}) \xrightarrow{\mathrm{d}}$
2.	0		0		$H^2(F_m,\mathcal{Q}) = 0$
$\vdots$	$\vdots$		$\vdots$		$\vdots$

The "direct image" tensor representatives [14, 24] of the so-valued cohomology groups are tabulated underneath the corresponding sheaves. The appearances of the step-function

$$\theta^n_m = \begin{cases} 1 & m \leqslant n, \\ 0 & m > n. \end{cases} \tag{A.2}$$

indicate that there are four separate cases:

$\boldsymbol{m = 0, 1.}$ All the contributions are in the top, 0^{th} cohomology row, and produce 105 equivalence classes of polynomials (see (A.10) below):

$$H^0(F_m,\mathcal{Q}) = \big\{\big(\phi_{(abcd)(ij)} \,/\, p_{(a}f_{bcd)(ij)}\big)\, x^a x^b x^c x^d\, y^i y^j\big\}, \quad m = 0; \tag{A.3}$$

$$= \big\{\big(\phi_{(abcd)\,i} \,/\, f_{(abc}\,p_{d)\,i}\big)\, x^a x^b x^c x^d\, y^i\big\}, \quad m = 1. \tag{A.4}$$

These are the familiar deformations of the degree-$\big({4 \atop 2-m}\big)$ polynomials $\phi(x,y)$, taken modulo degree-$\big({3 \atop 2-2m}\big)$ $f(x,y)$-multiples of the degree-$\big({1 \atop m}\big)$ defining polynomial $p(x,y)$.

$\boldsymbol{m = 2.}$ There are now two separate contributions, $H^0\big(A, \mathcal{O}\big({4 \atop 0}\big)\big) = \{\phi_{(abcd)}\}$ in the middle of the 0^{th} cohomology row and $H^1\big(A, \mathcal{O}\big({3 \atop -2}\big)\big) = \{\varepsilon^{ij} f_{(abc)}\}$ on the left of the 1^{st} cohomology row. This results in:

$$0 \to H^0(A,\mathcal{Q}) \xrightarrow{\rho_F} H^0(F_2,\mathcal{Q}) \xrightarrow{\mathrm{d}} H^1\big(A, \mathcal{O}\big({3 \atop -2}\big)\big) \to 0, \tag{A.5a}$$

$$H^0(F_2,\mathcal{Q}) = \big\{\big(\phi_{(abcd)} + \gamma^j_{(abcd)\,k}\,\tfrac{y^k}{y^j}\big)\, x^a x^b x^c x^d\big\}, \tag{A.5b}$$

$$\gamma^j_{(abcd)\,k} \stackrel{\text{def}}{=} \varepsilon^{ij} f_{(abc}\,p_{d)(ik)}$$

The Laurent polynomial $\gamma^{j}_{(abcd)\,k}\, x^a x^b x^c x^d\, \frac{y^k}{y^j}$ is one of the equivalent representatives generated by the defining equation (2.1) of F_2. As shown explicitly by (A.12c) and (A.12d) below, this F_2-equivalence class of 35 holomorphic sections of $\mathcal{Q}|_{F_2}$ contains representatives that are well-defined over every point of $\mathbb{P}^1$.

$\boldsymbol{m=3.}$ The only nonzero contribution is now $H^1\big(A, \mathcal{O}\big({}^{\;3}_{-4}\big)\big)$, in the second row of the left column in (A.1), producing:

$$\text{(A.6a)} \qquad 0 \to H^0(F_3, \mathcal{Q}) \xrightarrow{\text{d}} H^1\big(A, \mathcal{O}\big({}^{\;3}_{-4}\big)\big) \to 0,$$

$$\text{(A.6b)} \qquad H^0(F_3, \mathcal{Q}) = \big\{\gamma^{i}_{(abcd)} \tfrac{x^a x^b x^c x^d}{y^i}\big\}, \quad \gamma^{j}_{(abcd)} \stackrel{\text{def}}{=} \varepsilon^{i(j} f^{kl)}_{(abc}\, p_{d)(ikl)}.$$

As in (A.12a) below, the inclusion of the $\varepsilon^{ij}\, p_{a(ikl)}$ factors and the vanishing of $p(x,y)$ turns these Laurent polynomials into an F_3-equivalence class of 105 holomorphic sections of $\mathcal{Q}|_{F_3}$, with well-defined representatives over every point of $\mathbb{P}^4 \times \mathbb{P}^1$.

$\boldsymbol{m \geqslant 4.}$ Now both contributions in the second row in (A.1) are nonzero, and fit into the sequence:

(A.7)

$$0 \to H^0(F_m, \mathcal{Q}) \xrightarrow{\text{d}} H^1\big(A, \mathcal{O}\big({}^{\;\;3}_{2-2m}\big)\big) \xrightarrow{p} H^1\big(A, \mathcal{O}\big({}^{\;\;4}_{2-m}\big)\big) \xrightarrow{\rho} H^1(F_m, \mathcal{Q}) \to 0.$$

This specifies elements of $H^0(F_m, \mathcal{Q})$ by the tensor:

$$\text{(A.8a)} \qquad \gamma^{(j_1 \cdots j_{m-2})}_{(abcd)} \stackrel{\text{def}}{=} \varepsilon^{i(j_1} \varphi^{j_2 \cdots j_{2m-3})}_{(abc} \, p_{d)(i\, j_{m-1} \cdots j_{2m-3})},$$

$$\text{(A.8b)} \qquad \text{where} \quad \varphi^{(j_1 \cdots j_{2m-4})}_{(abc} \, p_{d)(j_{m-3} \cdots j_{2m-4})} = 0.$$

which is the preimage of the "direct image" within $H^1\big(A, \mathcal{O}\big({}^{\;\;3}_{2-2m}\big)\big)$ by the differential d-map, and which is used to construct the Laurent polynomials for $H^0(F_m, \mathcal{Q})$:

$$\text{(A.8c)} \qquad \gamma(x,y) := \varepsilon^{i(j_1} \varphi^{j_2 \cdots j_{2m-3})}_{(abc} \, p_{d)(i\, j_{m-1} \cdots j_{2m-3})} \, \frac{x^a x^b x^c x^d}{g^{(j_1 \cdots j_{m-2})}(y)}.$$

The form of the condition (A.8b) is dictated by the only covariant way to contract the tensor representatives $\varepsilon^{i(j} f^{k_1 \cdots k_{2m-4})}_{(abc)}$ of $H^1\big(A, \mathcal{O}\big({}^{\;\;3}_{2-2m}\big)\big)$ with $p_{a(i_1 \cdots i_m)}$ so as to produce the tensor coefficients of a degree-$\big({}^{\;\;4}_{2-m}\big)$ polynomial. The $(m-1)$ degree-$(m-2)$ generic $\mathbb{P}^1$-polynomials $g^{(j_1 \cdots j_{m-2})}(y)$ allow separating the poles of $\gamma(x,y)$ to $(m-1)(m-2)$ distinct locations and minimally extends the "direct image" linear algebra methods [14, 24] to accommodate the manifestly non-linear nature of the generalized complete intersections (1.1) for $m \geqslant 3$. It also facilitates using the defining equation (A.11) of F_m to construct well-defined of holomorphic sections (A.8c) of $\mathcal{O}\binom{4}{2-m}$ over F_m for every $m \geqslant 0$, the (A.12a)-like *equivalence classes* of which reproduce the explicit case-by-case constructions of the type given in Ref. [1]. While this suggests a corresponding Laurent generalization of the "direct image" homological algebra methods [14, 24], we are not aware of general theorems to this effect.

The $\binom{4+4}{4}\cdot\binom{(m-4)+1}{1} = 70(m-3)$ constraints in the system (A.8b) must leave *at least* 105 of the $\binom{3+4}{4}\cdot\binom{(2m-4)+1}{1} = 35(2m-3)$ tensor coefficients $\varepsilon^{i(j}\varphi_{(abc)}^{k_1\cdots k_{2m-4})}$ free to span $H^0(F_m,\mathcal{Q})$. In fact, this is an undercount for $m \geqslant 4$, and the exact result is

$$H^0(F_m,\mathcal{K}^*) = 105 + \delta_m^{(4)}, \quad H^1(F_m,\mathcal{K}^*) = \delta_m^{(4)}, \qquad \text{where} \quad \delta_m^{(4)} := \theta_3^m\, 15(m-3). \tag{A.9}$$

The computation of $\delta_m^{(4)}$ is given in (A.27)–(A.28) below, for general Hirzebruch n-folds. Stated differently and for $m \geqslant 4$, 105 is the index of the cohomology map generated by multiplication with the defining polynomial $p(x,y)$ in degree-1 row of (A.1).

To summarize, we have obtained:

(A.10)

m	$H^0(F_m,\mathcal{Q}),\ \ \dim F_m = 4$	Number	Sections
0	$\{\phi_{(abcd)(ij)}/p_{(a}\varphi_{bcd)(ij)}\}$	$\binom{4+4}{4}\binom{2+1}{1}-\binom{3+4}{4}\binom{2+1}{1} = 105$	ordinary
1	$\{\phi_{(abcd)\,i}/\varphi_{(abc}\,p_{d)\,i}\}$	$\binom{4+4}{4}\binom{1+1}{1}-\binom{3+4}{4}\binom{0+1}{1} = 105$	ordinary
2	$\{\phi_{(abcd)}\}$ $\{\varepsilon^{ij}\varphi_{(abc}\,p_{d)(ik)}\}$	$\binom{4+4}{4}\binom{0+1}{1} = 70$ $\binom{3+4}{4}\binom{0+1}{1} = 35$	ordinary Laurent
3	$\{\varepsilon^{i(j}\varphi_{(abc}^{kl)}\,p_{d)(ikl)}\}$	$\binom{3+4}{4}\binom{2+1}{1} = 105$	Laurent
$\geqslant 4$	$\{\varepsilon^{i(j_1}\varphi_{(abc}^{j_2\cdots j_{2m-3})}\,p_{d)(ij_{m-1}\cdots j_{2m-3})}\}$ $\varphi_{(abc}^{(j_1\cdots j_{2m-4})}\,p_{d)(j_1\cdots j_m)} = 0$	$\binom{3+4}{4}(2m-3)$ $-\binom{4+4}{4}(m-3) \leqslant 105 + \delta_m^{(4)\ddagger}$	Laurent

‡ The "excess" number of sections $\delta_m^{(4)} = \theta_3^m\, 15(m-3)$ is computed in (A.27)–(A.28).

A.1.2. Being well-defined. Holomorphic degree-$\binom{a}{b}$ sections on F_m are, by (2.2), equivalence classes of functions on A modulo $p(x,y)$-multiples of sections of $\mathcal{O}_A\binom{a-1}{b-m}$. This is crucial in showing that the above-obtained Laurent polynomials are well-defined on F_m. Suffice it here to show this for

$m=2$: Without loss of generality, we may write the defining equation of F_2 as

$$p(x,y) = p_{00}(x)\,(y^0)^2 + 2p_{01}(x)\,y^0y^1 + p_{11}(x)\,(y^1)^2 \;= 0. \tag{A.11}$$

In turn, the second, $\gamma^i_{(abcd)\,k}$-parametrized term in (A.5b) results in the Laurent polynomial

(A.12a)

$$\gamma(x,y) = \varepsilon^{ij}\varphi_{(abc\,p_{d)(ik)}}\,x^ax^bx^cx^d\,\frac{y^k}{y^j} = \varphi(x)\left(p_{00}(x)\frac{y^0}{y^1} - p_{11}(x)\frac{y^1}{y^0}\right).$$

The vanishing (A.11) of $p(x,y)$ on F_2 implies that this is equivalent to:

(A.12b)

$$\gamma(x,y) = \varphi(x)\left[p_{00}(x)\frac{y^0}{y^1} - p_{11}(x)\frac{y^1}{y^0} + \lambda\underbrace{\left(p_{00}(x)\,\frac{y^0}{y^1} + 2p_{01}(x) + p_{11}(x)\,\frac{y^1}{y^0}\right)}_{=0 \text{ on } F_2 \text{ owing to (A.11)}}\right].$$

In particular, this λ-*continuum* of equivalent representatives includes:

(A.12c)

$$\overset{(A.11)}{\simeq} -2\,\varphi(x)\left(p_{01}(x) + p_{11}(x)\,\frac{y^1}{y^0}\right), \quad \text{where } y^0 \neq 0, \text{ choose } \lambda \to -1;$$

(A.12d)

$$\overset{(A.11)}{\simeq} +2\,\varphi(x)\left(p_{01}(x) + p_{00}(x)\,\frac{y^0}{y^1}\right), \quad \text{where } y^1 \neq 0, \text{ choose } \lambda \to +1,$$

and which are holomorphic in the indicated regions. This *equivalence class* of degree-$\binom{4}{0}$ rational polynomials over $\mathbb{P}^4\times\mathbb{P}^1$ then provides sections of $\mathcal{O}\binom{4}{2-m}$ that are well-defined and holomorphic everywhere on F_2. This irresistibly reminds of the well-known Wu-Yang construction of the magnetic monopole, since: (**1**) neither of the expressions (A.12a) is well-defined everywhere on $\mathbb{P}^1$, (**2**) at any point of $\mathbb{P}^1$ at least one of (A.12a) is well-defined, (**3**) wherever both of (A.12c) and (A.12d) are well-defined on $\mathbb{P}^1$, they are equivalent owing to (A.11). Together with the "ordinary" $\mathbb{P}^4$-quadrics $\phi(x) = \phi_{abcd}\,x^ax^bx^cx^d$, the Laurent polynomials $\gamma(x,y)$ provide $\binom{4+3}{3}+\binom{3+3}{3} = 70 + 35 = 105$ sections for $H^0(F_m,\mathcal{Q})$ with which to define Calabi-Yau 3-folds $X_2 \subset F_2$.

Conversely, the degree-$\binom{3}{2-2m}$ Laurent polynomials $\varphi(x,y)$ which in (A.8c) parametrize the anticanonical sections $\gamma(x,y)|_{F_m} \in H^0(F_m,\mathcal{Q})$ are localized to the hypersurface 4-fold $F_m \overset{\text{def}}{=} \{(x,y)\in\mathbb{P}^4\times\mathbb{P}^1 : p(x,y)=0\}$ by means of the residue formula [15]:

$$\text{(A.13a)}\qquad \varphi(x,y)|_{F_m} \stackrel{\text{def}}{=} \oint_{\Gamma(F_m)} \frac{(y\,\mathrm{d}y)}{p(x,y)}\,\gamma(x,y)$$
$$= \Omega^{(i_1\cdots i_{m-2})}(x)\big[\partial_{i_1}\cdots\partial_{i_{m-2}}\gamma(x,y)\big]_{F_m},$$

$$\text{(A.13b)}\qquad \Omega^{(i_1\cdots i_{m-2})}(x) \stackrel{\text{def}}{=} \oint_{\Gamma(F_m)} \frac{(y\,\mathrm{d}y)}{[\partial_{i_1}\cdots\partial_{i_{m-2}}p(x,y)]}.$$

Here, $\Gamma(F_m)$ is the $(S^1\times F_m)$-like "Gaussian" boundary enclosing a sufficiently "thin" tubular neighborhood of $F_m\subset\mathbb{P}^4\times\mathbb{P}^1$, $\Omega^{(i_1\cdots i_{m-2})}(x)$ are degree-$\binom{-1}{0}$ holomorphic $\mathbb{P}^1$-constant 0-forms on F_m. This type of residue formula has been shown to represent the cohomology elements in all complete intersections in (even weighted) projective spaces [15], and it is gratifying to find that (A.13) also extends to the generalized complete intersections of Ref. [1].

A.2. Other properties of F_m

The computation (A.1) may be generalized to produce the *plurigenera* $\mathfrak{P}_{-k} = \dim H^0\big(F_m,(\mathcal{K}^*_{F_m})^{\otimes k}\big)$ and the Euler number

$$\text{(A.14)}\qquad \chi\big((\mathcal{K}^*_{F_m})^{\otimes k}\big) \stackrel{\text{def}}{=} \sum_{r=0}^{4}(-1)^r \dim H^r\big(F_m,(\mathcal{K}^*_{F_m})^{\otimes k}\big)$$
$$= \tfrac{1}{3}(2k+1)^2(4k+1)(4k+3),$$

where $\mathcal{K}^*_{F_m} = \mathcal{O}\binom{4}{2-m}\,\mathcal{Q}$, and which is independent of m for all k; also, $\mathfrak{P}_{-k}=\mathfrak{P}_{k+1}$.

The Lefschetz hyperplane theorem [14, 39] is applicable to F_m for all $m>0$, while $F_0=\mathbb{P}^3\times\mathbb{P}^1$ straightforwardly. Together with Poincaré duality, the Künneth formula and the universal coefficient theorem [30], this guarantees [14, p. 44] that $H^r(F_m,\mathbb{Z})\approx H^r(\mathbb{P}^4\times\mathbb{P}^1,\mathbb{Z})$ for all $r\neq 4$. Moreover, the surjection $H^4(F_m,\mathbb{Z})\twoheadrightarrow H^4(\mathbb{P}^4\times\mathbb{P}^1,\mathbb{Z})$ is in fact an isomorphism since $\chi_{_E}(F_m)=8$, so that $H^*(F_m,\mathbb{Z})=H^*(\mathbb{P}^4\times\mathbb{P}^1,\mathbb{Z})$ and with no torsion. In particular, $H^{1,1}(F_m)\approx H^2(F_m,\mathbb{Z})\approx H^2(\mathbb{P}^4\times\mathbb{P}^1,\mathbb{Z})$ is generated by the pull-backs of the Kähler forms of $\mathbb{P}^4$ and $\mathbb{P}^1$—which are thus guaranteed to generate the Chern class of F_m for all $m\geqslant 0$. This agrees with the direct computation using the adjunction relation for F_m:

$$\text{(A.15)}\qquad \begin{array}{rcccccc} F_m\text{-resolution} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & (\mathcal{Q}^*_A)^{\oplus 2} & \xrightarrow{\mathrm{d}p} & T_A \otimes \mathcal{Q}^*_A & & \\ & \downarrow p & & \downarrow p & & \\ & \mathcal{Q}^*_A & \xrightarrow{\mathrm{d}p} & T_A & & \\ & \downarrow \rho & & \downarrow \rho & & \\ \text{dual adjunction :} & \mathcal{Q}^*|_{F_m} & \stackrel{\mathrm{d}p}{\hookrightarrow} & T^*_A|_{F_m} & \twoheadrightarrow & T^*_{F_m} \end{array}$$

Having determined that (the pullbacks of) the Kähler forms J_1 of $\mathbb{P}^n$ and J_2 of $\mathbb{P}^1$ generate $H^{1,1}(F_m) \cap H^2(F_m, \mathbb{Z})$ for all $m \geqslant 0$, the straightforward Chern class, the intersection and various Chern evaluation computations produce:

$$\text{(A.16a)}\qquad \begin{aligned} c = \big(4J_1 + (2{-}m)J_2\big) + \big(6{J_1}^2 + (8{-}3m)J_1J_2\big) \\ + \big(4{J_1}^3 + (12{-}3m){J_1}^2J_2\big) + \big({J_1}^4 + (8{-}m){J_1}^3J_2\big), \end{aligned}$$

$$\text{(A.16b)}\qquad \begin{gathered} {C_1}^4 = 512, \qquad {C_1}^2{\cdot}C_2 = 224, \qquad C_1{\cdot}C_3 = 56, \\ {C_2}^2 = 96, \qquad C_4 = \chi_{_E} = 8, \\ {C_1}^3[aJ_1{+}bJ_2] = 16[6a + (\underline{4b{+}am})], \\ C_1{\cdot}C_2[aJ_1{+}bJ_2] = 2[22a + 3(\underline{4b{+}am})], \\ C_3[aJ_1{+}bJ_2] = 12a + (\underline{4b{+}am}), \\ {C_1}^2[(aJ_1{+}bJ_2)^2] = 8a[2a + (\underline{4b{+}am})], \\ C_2[(aJ_1{+}bJ_2)^2] = a(8a + 3(\underline{4b{+}am})], \\ C_1[(aJ_1{+}bJ_2)^3] = a^2(2a + 3(\underline{4b{+}am})], \\ [(aJ_1{+}bJ_2)^4]_{F_m} = a^3(\underline{4b{+}am}). \end{gathered}$$

All the Chern numbers are m-independent (A.16b), and all the various Chern evaluations on $H^{1,1}(F_m)$ depend on m and bJ_2 only through the (underlined) combination $(4b{+}ma)$. This indicates an $[m \pmod 4]$-relation:

$$\text{(A.17)}\qquad F_m \approx F_{m+4c}: \qquad \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}_m \stackrel{\approx}{\longleftrightarrow} \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}_{m+4c} \quad \text{iff} \quad c \in \mathbb{Z}.$$

The *differences* in the topological data (A.16) insure that there are *at least* four distinct diffeomorphism classes. While we are not aware of a 4-fold classification result as straightforwardly precise as Wall's theorem [14, 21] that classifies the diffeomorphism class of Calabi-Yau 3-folds, we will assume that the relation (A.17) between $F_m \approx F_{m+4}$ is in fact a diffeomorphism. That is, we assume that the above topological data insures that the sequence of Hirzebruch 4-folds F_m forms precisely four diffeomorphism classes, $[F_m] \approx$

$F_{m \,(\mathrm{mod}\,4)}$ for $0 \leqslant m \in \mathbb{Z}$. The first two of these four diffeomorphism classes each have a Fano representative: $F_0 = \mathbb{P}^3 \times \mathbb{P}^1$ and $F_1 = \left[\begin{smallmatrix}\mathbb{P}^4\\ \mathbb{P}^1\end{smallmatrix} \middle\| \begin{smallmatrix}1\\1\end{smallmatrix}\right]$, while the anticanonical bundle of F_2 and F_3 are evidently non-positive over $\mathbb{P}^1$ and are not Fano. The two-dimensional analogues of these results are well-known [28]; see below.

A.3. Other dimensions

Analogously to the sequence (1.1), sequences (3.1) and (3.2) involve 3- and 2-dimensional analogues of the 4-fold F_m. Listing them side-by-side,

$$\text{(A.18)}\qquad \mathfrak{F}_m \in \left[\begin{smallmatrix}\mathbb{P}^2\\ \mathbb{P}^1\end{smallmatrix} \middle\| \begin{smallmatrix}1\\m\end{smallmatrix}\right], \qquad \mathcal{F}_m \in \left[\begin{smallmatrix}\mathbb{P}^3\\ \mathbb{P}^1\end{smallmatrix} \middle\| \begin{smallmatrix}1\\m\end{smallmatrix}\right],$$
$$F_m \in \left[\begin{smallmatrix}\mathbb{P}^4\\ \mathbb{P}^1\end{smallmatrix} \middle\| \begin{smallmatrix}1\\m\end{smallmatrix}\right], \quad \cdots \quad \mathscr{F}^{(n)}_m \in \left[\begin{smallmatrix}\mathbb{P}^n\\ \mathbb{P}^1\end{smallmatrix} \middle\| \begin{smallmatrix}1\\m\end{smallmatrix}\right],$$

makes it obvious that these degree-$\left(\begin{smallmatrix}1\\m\end{smallmatrix}\right)$ hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^1$ are at every point of $\mathbb{P}^1$ simple hyperplanes in $\mathbb{P}^n$, i.e., $[\mathbb{P}^n \| 1] \approx \mathbb{P}^{n-1}$. Varying then the base-point over $\mathbb{P}^1$, each such hypersurface forms an m-twisted $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$. It is worthwhile noting that the $n = 1$-dimensional case of such varieties,

$$\text{(A.19)}\qquad \mathfrak{f}_m \in \left[\begin{smallmatrix}\mathbb{P}^1_x\\ \mathbb{P}^1_y\end{smallmatrix} \middle\| \begin{smallmatrix}1\\m\end{smallmatrix}\right]$$

are m-twisted 1-point fibrations over (simple covers of) $\mathbb{P}^1_y$, where the 1-point fiber is the hyperplane $[\mathbb{P}^1_x \| 1]$. Alternatively, they may also be understood "the other way around," as an m-fold ramified cover of $\mathbb{P}^1_x$: at each point $x_* \in \mathbb{P}^1_x$, the defining equation $p(x_*, y)$ is a degree-m polynomial over $\mathbb{P}^1_y$. The zero-locus of this degree-m polynomial consists of m $\mathbb{P}^1_y$-points, thus producing an m-fold cover of $\mathbb{P}^1_x$, ramified (branched) at the $\mathbb{P}^1_y$-locations where the zeros of $p(x, y)$ coalesce.

Hodge numbers. Just as for the 4-fold F_m above, the Lefschetz hyperplane theorem applies for all $m, n > 0$, while $\mathscr{F}^{(n)}_0 = \mathbb{P}^{n-1} \times \mathbb{P}^1$, straightforwardly. Together with Poincaré duality, the Künneth formula and the universal coefficient theorem [30], this guarantees that

$$H^r(\mathscr{F}^{(n)}_m, \mathbb{Z}) \approx H^r(\mathbb{P}^n \times \mathbb{P}^1, \mathbb{Z})$$

for all $r \neq n$, and $H^n(\mathscr{F}^{(n)}_m, \mathbb{Z}) \twoheadrightarrow H^n(\mathbb{P}^n \times \mathbb{P}^1, \mathbb{Z})$ is an isomorphism precisely if $h^{n,n}(\mathbb{P}^4 \times \mathbb{P}^1) = h^{n,n}(\mathscr{F})$. This last condition is in turn guaranteed by the standard computation of $\chi_{_E}(\mathscr{F}^{(n)}_m) = 2n$, whereby $H^*(\mathscr{F}^{(n)}_m, \mathbb{Z}) =$

$H^*(\mathbb{P}^4\times\mathbb{P}^1,\mathbb{Z})$ and with no torsion. In particular,

$$H^{1,1}(\mathscr{F}_m^{(n)}) \approx H^2(\mathscr{F}_m^{(n)},\mathbb{Z}) \approx H^2(\mathbb{P}^n\times\mathbb{P}^1,\mathbb{Z}),$$

and is generated by the pull-backs of the Kähler forms of $\mathbb{P}^n$ and $\mathbb{P}^1$—which are thus guaranteed to generate the Chern class of $\mathscr{F}_m^{(n)}$—for all $n\geqslant 2$ and all $m\geqslant 0$. As above, this result may be verified by direct computation for all $n\geqslant 2$ using (the dual of) the (F_m-resolved adjunction) relation (A.15).

Chern and intersection numbers. Just as was done for the 4-fold F_m above, we readily compute the Chern numbers and the various Chern evaluations for all $n\geqslant 2$. For the 2-fold $\mathfrak{F}_m$ we compute:

$$\text{(A.20a)}\qquad \begin{gathered} c = \big(2J_1 + (2{-}m)J_2\big) + \big({J_1}^2 + (4{-}m)J_1J_2\big),\\ {C_1}^2 = 8, \qquad C_2 = \chi_{_E} = 4,\end{gathered}$$

$$\text{(A.20b)}\qquad \begin{gathered} C_1[aJ_1{+}bJ_2] = 2a + (\underline{2b{+}am}),\\ [(aJ_1{+}bJ_2)^2]_{\mathfrak{F}_m} = a(\underline{2b{+}am}).\end{gathered}$$

The tandem of facts: (**1**) the Chern numbers are m-independent, and (**2**) the intersections and Chern evaluations depend on bJ_2 and m only through the (underlined) hallmark combination $(2b{+}am)$, demonstrates the known homotopy type $[m\ (\mathrm{mod}\ 2)]$-periodicity of Hirzebruch surfaces:

$$\text{(A.21)}\qquad \mathfrak{F}_m \approx \mathfrak{F}_{m+2c}:\quad \begin{bmatrix} J_1\\ J_2\end{bmatrix}_m \overset{\approx}{\longleftrightarrow} \begin{bmatrix} 1 & -c\\ 0 & 1\end{bmatrix}\begin{bmatrix} J_1\\ J_2\end{bmatrix}_{m+2c}\quad \text{iff}\quad c\in\mathbb{Z}.$$

That is, the sequence of Hirzebruch surfaces $\mathfrak{F}_m$ forms two diffeomorphism classes, $[\mathfrak{F}_{2k}]\approx\mathfrak{F}_0$ and $[\mathfrak{F}_{2k+1}]\approx\mathfrak{F}_1$, for $0\leqslant k\in\mathbb{Z}$; both of these have a Fano (del Pezzo) representative: $\mathfrak{F}_0=\mathbb{P}^1\times\mathbb{P}^1$ and $\mathfrak{F}_1=\left[\begin{smallmatrix}\mathbb{P}^2\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]$.

Similarly, for the 3-fold $\mathcal{F}_m$ we compute:

(A.22a)

$$c = \big(3J_1 + (2{-}m)J_2\big) + \big(3{J_1}^2 + (6{-}2m)J_1J_2\big) + \big({J_1}^3 + (6{-}m){J_1}^2J_2,$$

$$\text{(A.22b)}\qquad \begin{gathered} {C_1}^3 = 54, \qquad C_1{\cdot}C_2 = 24, \qquad C_3 = \chi_{_E} = 6,\\ {C_1}^2[aJ_1{+}bJ_2] = 12a + 3(\underline{3b{+}am}),\\ C_2[aJ_1{+}bJ_2] = 6a + (\underline{3b{+}am}),\\ C_1[(aJ_1{+}bJ_2)^2] = 2a^2 + 2a(\underline{3b{+}am}),\\ [(aJ_1{+}bJ_2)^3]_{\mathcal{F}_m} = a^2(\underline{3b{+}am}).\end{gathered}$$

Again, the tandem of facts: (**1**) the Chern numbers are m-independent, and (**2**) the intersections and Chern evaluations depend on bJ_2 and m only through the (underlined) hallmark combination $(3b{+}am)$, demonstrates the homotopy type $[m \pmod 3]$-periodicity of Hirzebruch 3-folds[7]:

$$\text{(A.23)}\qquad \mathcal{F}_m \approx \mathcal{F}_{m+3c}: \quad \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}_m \overset{\approx}{\longleftrightarrow} \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}_{m+3c} \quad \text{iff}\quad c \in \mathbb{Z}.$$

That is, the sequence of Hirzebruch 3-folds $\mathcal{F}_m$ forms three diffeomorphism classes, $[\mathcal{F}_{3k}] \approx \mathcal{F}_0$, $[\mathcal{F}_{3k+1}] \approx \mathcal{F}_1$ and $[\mathcal{F}_{3k+2}] \approx \mathcal{F}_2$, for $0 \leqslant k \in \mathbb{Z}$; the first two of these have a Fano representative: $\mathcal{F}_0 = \mathbb{P}^2 \times \mathbb{P}^1$ and $\mathcal{F}_1 = \left[\begin{smallmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \end{smallmatrix} \middle\| \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, while the anticanonical bundle of $\mathcal{F}_2$ is non-positive over $\mathbb{P}^1$, as computed explicitly in (A.26), below.

Anticanonical sections. The computation (A.1) easily adapts to all n: the anticanonical bundle becomes a restriction of $\mathcal{K}^*_{\mathscr{F}_m} = \mathcal{O}\binom{n}{2-m}$, and we have:

(A.24)

$n \geqslant 2$	$\mathcal{O}_A\binom{n-1}{2-2m}$	$\overset{p}{\hookrightarrow}$	$\mathcal{O}_A\binom{n}{2-m}$	$\overset{\rho_{\mathscr{F}}}{\twoheadrightarrow}$	$\mathcal{K}^*_{\mathscr{F}^{(n)}_m}$
0.	$\theta^1_m\{f_{(a_1\cdots a_{n-1})(i_1\cdots i_{2-2m})}\}$	$\overset{p}{\to}$	$\theta^2_m\{\phi_{(a_1\cdots a_n)(i_1\cdots i_{2-m})}\}$	$\overset{\rho_{\mathscr{F}}}{\longrightarrow}$	$H^0(\mathscr{F}^{(n)}_m, \mathcal{K}^*) \overset{\mathrm{d}}{\to}$
1.	$\theta^m_2\{\varepsilon^{i(j}f^{k_1\cdots k_{2m-4})}_{(a_1\cdots a_{n-1})}\}$	$\overset{p}{\to}$	$\theta^m_4\{\varepsilon^{i(j}\phi^{k_1\cdots k_{m-4})}_{(a_1\cdots a_n)}\}$	$\overset{\rho_{\mathscr{F}}}{\longrightarrow}$	$H^1(\mathscr{F}^{(n)}_m, \mathcal{K}^*) \overset{\mathrm{d}}{\to}$
2.	0		0		$H^2(\mathscr{F}^{(n)}_m, \mathcal{K}^*) = 0$
$\vdots$	$\vdots$		$\vdots$		$\vdots$

Akin to the 4-fold F_m case (A.10), this produces for the familiar $(n = 2)$ Hirzebruch surfaces:

(A.25)

$\boldsymbol{m}$	$\boldsymbol{H^0(\mathfrak{F}_m, \mathcal{K}^*)}$, $\mathbf{dim}\,\boldsymbol{\mathfrak{F}_m = 2}$	**Number**	**Sections**
0	$\{\phi_{(ab)(ij)}/p_{(a}\varphi_{b)(ij)}\}$	$\binom{2+2}{2}\binom{2+1}{1} - \binom{1+2}{2}\binom{2+1}{1} = 9$	ordinary
1	$\{\phi_{(ab)\,i}/\varphi_{(a}\,p_{b)\,i}\}$	$\binom{2+2}{2}\binom{1+1}{1} - \binom{1+2}{2}\binom{0+1}{1} = 9$	ordinary
2	$\{\phi_{(ab)\,i}\}$ $\{\varepsilon^{ij}\varphi_{(a}\,p_{b)(ik)}\}$	$\binom{2+2}{2}\binom{0+1}{1} = 6$ $\binom{1+2}{2}\binom{0+1}{1} = 3$	ordinary Laurent
3	$\{\varepsilon^{i(j}\varphi^{kl)}_{(a}\,p_{b)(ikl)}\}$	$\binom{1+2}{2}\binom{2+1}{1} = 9$	Laurent
$\geqslant 4$	$\{\varepsilon^{i(j_1}\varphi^{j_2\cdots j_{2m-3})}_{(a}\,p_{b)(ij_{m-1}\cdots j_{2m-3})}\}$ $\varphi^{(j_1\cdots j_{2m-4})}_{(a}\,p_{b)(j_1\cdots j_m)} = 0$	$\binom{1+2}{2}(2m{-}3)$ $-\binom{2+2}{2}(m{-}3) \leqslant 9 + \delta^{(2)}_m$ ‡	Laurent

‡ The "excess" number of sections $\delta^{(2)}_m = \theta^m_3\,(m{-}3)$ is computed in (A.27)–(A.28).

[7]For 3-folds, Wall's theorem [14, 21] does imply that the relationship in (A.23) is a diffeomorphism.

Similarly, in the $n=3$-dimensional case, we have:

(A.26)

m	$H^0(\mathcal{F}_m,\mathcal{K}^*)$, $\dim\mathcal{F}_m=3$	Number	Sections
0	$\{\phi_{(abc)(ij)}/p_{(a}f_{bc)(ij)}\}$	$\binom{3+3}{3}\binom{2+1}{1}-\binom{2+3}{3}\binom{2+1}{1}=30$	ordinary
1	$\{\phi_{(abc)\,i}/f_{(ab}\,p_{c)\,i}\}$	$\binom{3+3}{3}\binom{1+1}{1}-\binom{2+3}{3}\binom{0+1}{1}=30$	ordinary
2	$\{\phi_{(abc)\,i}\}$ $\{\varepsilon^{ij}f_{(ab}\,p_{c)(ik)}\}$	$\binom{3+3}{3}\binom{0+1}{1}=20$ $\binom{2+3}{3}\binom{0+1}{1}=10$	ordinary Laurent
3	$\{\varepsilon^{i(j}f^{kl)}_{(ab}\,p_{c)(ikl)}\}$	$\binom{2+3}{3}\binom{2+1}{1}=30$	Laurent
$\geqslant 4$	$\{\varepsilon^{i(j_1}\varphi^{j_2\cdots j_{2m-3})}_{(ab}\,p_{c)(ij_{m-1}\cdots j_{2m-3})}\}$ $\varphi^{(j_1\cdots j_{2m-4})}_{(ab}\,p_{c)(j_1\cdots j_m)}=0$	$\binom{2+3}{3}(2m{-}3)$ $-\binom{3+3}{3}(m{-}3)\leqslant 30+\delta^{(3)}_m{}^{\ddagger}$	Laurent

‡ The "excess" number of sections $\delta^{(3)}_m=\theta^m_3\,4(m{-}3)$ is computed in (A.27)–(A.28).

The "excess" number of anticanonical sections. For completeness, the m-twisted Hirzebruch n-fold $\mathscr{F}^{(n)}_m$ may be identified with the projectivization $\mathscr{F}^{(n)}_m=P(E)$ of the rank-n bundle $E=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(m)^{\oplus(n-1)}$. The push-forward (to the base-$\mathbb{P}^1$, where all vector bundles decompose as direct sums of line-bundles) of the anticanonical bundle of $\mathscr{F}^{(n)}_m$ is then computed[8] as

(A.27a)
$$\pi_*(\mathcal{K}^*_{\mathscr{F}^{(n)}_m})=(E^*)^n\otimes\big(\mathcal{K}^*_{\mathbb{P}^1}\otimes\det(E)\big),\qquad E=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(m)^{\oplus(n-1)};$$

(A.27b)
$$=\Big(\bigoplus_{k=0}^{n}\tbinom{n+k-2}{k}\mathcal{O}_{\mathbb{P}^1}(-km)\Big)\otimes\Big(\mathcal{O}_{\mathbb{P}^1}(2)\otimes\mathcal{O}_{\mathbb{P}^1}\big((n{-}1)m\big)\Big),$$

(A.27c)
$$=\bigoplus_{k=0}^{n}\tbinom{n+k-2}{k}\mathcal{O}_{\mathbb{P}^1}\big(2{+}(n{-}k{-}1)m\big).$$

This produces the number of sections, which when combined with the long exact cohomology sequence (A.24) guarantees that:

(A.28)
$$\begin{aligned}\dim H^0(\mathscr{F}^{(n)}_m,\mathcal{K}^*)&=\sum_{k=0}^{n}\theta^{3+nm}_{(k+1)m}\tbinom{n+k-2}{k}\big(3{+}(n{-}k{-}1)m\big)+\delta^{(n)}_m,\\ \dim H^1(\mathscr{F}^{(n)}_m,\mathcal{K}^*)&=\delta^{(n)}_m:=\theta^m_3\tbinom{2n-2}{n}\big(m{-}3\big),\end{aligned}$$

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[8] We thank Richard Wentworth for alerting us to this independent and standard algebro-geometric computation for Hirzebruch 2-folds, which we generalize here for all Hirzebruch n-folds.

and $\dim H^i(\mathscr{F}_m^{(n)}, \mathcal{K}^*) = 0$ for $i > 1$. Note that

$$\text{(A.29)} \qquad \chi(\mathcal{K}^*) = \sum_{i=0}^{n} \dim H^i(\mathscr{F}_m^{(n)}, \mathcal{K}^*) = 9, \quad 30, \quad 105 \quad \text{for} \quad n = 2, 3, 4.$$

A.4. Discrete deformations

In fact, not only is it known that the Hirzebruch surfaces $\mathfrak{F}_m$ and $\mathfrak{F}_{m+2}$ are abstractly diffeomorphic, one can construct an explicit deformation family of Hirzebruch surfaces that includes both $\mathfrak{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathfrak{F}_2$ [33] and [14, Section 3.1.2]. This construction provides a complex 1-parameter family such that $\mathfrak{F}_0$ is fibered over $\epsilon \neq 0$, while $\mathfrak{F}_2$ fits at $\epsilon = 0$: The deformation family is the configuration

$$\text{(A.30)} \qquad \left[\begin{matrix}\mathbb{P}^3\\ \mathbb{P}^1\end{matrix}\right\Vert\left.\begin{matrix}1 & 1\\ 1 & 1\end{matrix}\right]: \quad \begin{bmatrix}x^0 & x^1\\ x^2 & \left(\sum_{i=0}^{2} a_i x^i + \epsilon x^3\right)\end{bmatrix}\begin{bmatrix}y^0\\ y^1\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

For generic choices of (a_0, a_1, a_2) and $\epsilon \neq 0$, the determinant of the system is a smooth quadric in $\mathbb{P}^3$, known to be the Segré embedding of $\mathfrak{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 = [\mathbb{P}^3||2]$. At $\epsilon = 0$, the determinant of the system develops a singularity, which is "blown-up" in the smooth 2-fold $\mathfrak{F}_2$ defined in the $\epsilon \to 0$ limit of (A.30).

The deformation space of (A.30) is explicitly parametrized by $(a_0, a_1, a_2, \epsilon) \in \mathbb{C}^4$ and reduces through $\mathbb{P}^3$-reparametrizations to two distinct but infinitesimally close points: $\{\epsilon \neq 0\}$ and $\{\epsilon = 0\}$. By this explicit construction, $\lim_{\epsilon \to 0} \mathfrak{F}_0 = \mathfrak{F}_2$ is a discrete deformation.[9] Whereas $\mathfrak{F}_0$ and $\mathfrak{F}_2$ are diffeomorphic to each other, we note that there do exist subtle differences: $\mathfrak{F}_2$ has an exceptional curve of self-intersection -2 and so may be blown down to $\mathbb{P}^2$, while $\mathfrak{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ cannot be blown down to any 2-fold. Owing to the diffeomorphism $\mathfrak{F}_2 \approx \mathfrak{F}_0$, we do not expect any classical field theory model using these spaces to be able to detect such a subtle difference, but conjecture that quantum field theory can do so.

We are not aware of any explicit demonstration that $\mathscr{F}_m^{(n)}$ and $\mathscr{F}_{m+n}^{(n)}$ are also discrete deformations of each other for $n \neq 2$. However, explicit

[9] Ref. [14] calls this a "jumping deformation."

computation using

$$
\text{(A.31)}\qquad
\begin{array}{ccccccl}
 & & T_A\otimes\mathcal{P}^*_A & \xrightarrow{\mathrm{d}p} & \mathcal{O}_A & \\
 & & \downarrow p & & \downarrow p & \\
 & & T_A & \xrightarrow{\mathrm{d}p} & \mathcal{P}_A & \Bigg\}\ \mathscr{F}^{(n)}_m\text{-resolution}\\
 & & \downarrow\rho & & \downarrow\rho & \\
T_{\mathscr{F}^{(n)}_m} & \hookrightarrow & T_A\big|_{\mathscr{F}^{(n)}_m} & \overset{\mathrm{d}p}{\twoheadrightarrow} & \mathcal{P}\big|_{\mathscr{F}^{(n)}_m} & :\ \text{adjunction}
\end{array}
$$

produces:

$$
\text{(A.32)}\qquad \dim H^0(\mathscr{F}^{(n)}_m,T)=n^2{+}2+\Delta^{(n)}_m \quad\text{and}\quad \dim H^1(\mathscr{F}^{(n)}_m,T)=\Delta^{(n)}_m.
$$

Here $\Delta^{(n)}_m$ is the net number of Kodaira-Spencer deformations [40] of $\mathscr{F}^{(n)}_m$ represented, by the tensor components $\phi_{a\,(i_1\cdots i_m)}$ that cannot be gauged away by the combined transformation

$$
\text{(A.33)}\qquad
\begin{aligned}
\delta\,\phi_{a\,(i_1\cdots i_m)} &= \lambda\,p_{a\,(i_1\cdots i_m)}+\lambda_a{}^b\,p_{b\,(i_1\cdots i_m)}+p_{a\,j(i_1\cdots i_{m-1}}\,\lambda_{i_m)}{}^j\\
&\quad+\theta^m_2\,p_{a\,j(i_1\cdots i_{m-1}|}\,p_{b\,|i_m)(k_1\cdots k_{m-1})}\,\lambda^{b\,(k_1\cdots k_{m-2}}\varepsilon^{k_{m-1})j}.
\end{aligned}
$$

Contracting this equivalence class of tensors with $x^a\,y^{i_1}\cdots y^{i_m}$ provides this tensorial relation with the familiar interpretation of reparametrizations of the polynomial deformations — except for the higher cohomology contribution in the last term containing the step-function θ^m_2.

In turn, the tensor components $\{\lambda,\ \lambda_a{}^b,\ \lambda_i{}^j,\ \lambda^{a\,(i_1\cdots i_{m-2})}:\ \lambda_a{}^a=0=\lambda_i{}^i\}$ that cannot be used up in the transformation (A.33) span $H^0(\mathscr{F}^{(n)}_m,T)$ — representing the coordinate reparametrizations of $\mathscr{F}^{(n)}_m$. Exceptionally for $m=0$, the $\mathbb{P}^n$ reparametrization generators $\lambda_a{}^b$ are themselves subject to an additional equivalence relation generated by multiplication by the defining tensor p_a, and in addition to the constraints (A.33). It is gratifying to note that the dual constraint sub-system for $m=0$:

$$
\text{(A.34)}\qquad \{\lambda_a{}^b\simeq\lambda_a{}^b+p_a\,\vartheta^b\}:\ \lambda_a{}^b\,p_b=0
$$

leaves $\{[(n{+}1)^2{-}1]-[n{+}1]\}-n=n^2{-}1$ free components of $\lambda_a{}^b$, as is appropriate for $\mathbb{P}^{n-1}$ coordinate reparametrizations, in $\left[\begin{smallmatrix}\mathbb{P}^n\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\0\end{smallmatrix}\right]=\mathbb{P}^{n-1}\times\mathbb{P}^1$. The ϑ^a-generated equivalence allows gauging away $n{+}1$ degrees of freedom, but the constraint system $\lambda_a{}^b\,p_b=0$ consists of only n independent equations, since $\lambda_a{}^b\,p_b$ cannot be proportional to p_a itself as $\lambda_a{}^a=0$ and so $\lambda_a{}^b\not\propto\delta^b_a$.

For the Hirzebruch surfaces, $\mathfrak{F}_m$, $\Delta^{(2)}_m:=\dim H^1(\mathfrak{F}_m,T)=\theta^m_1(m{-}1)$ [40, Eq. (6.52)]; see also [41]. For higher-dimensional Hirzebruch n-folds $\mathscr{F}^{(n)}_m$

with $m \geqslant 0$ and $n \geqslant 2$, the analysis of the constrained gauge-equivalence system of tensors (A.33) becomes considerably more involved. However, SAGE's result for the toric representation of $\mathscr{F}_m^{(n)}$ [18] gives $\texttt{Aut}(\mathscr{F}_m^{(n)}) = \dim H^0(\mathscr{F}_m^{(n)}, T) = n^2{+}2{+}\theta_1^m(n{-}1)(m{-}1)$, and implies that

$$\Delta_m^{(n)} = \theta_1^m(n{-}1)(m{-}1), \tag{A.35}$$

which agrees with the above-cited standard result for $\mathfrak{F}_m$. We thus conclude that the dimension of the (local) Kodaira-Spencer deformation space $H^1(\mathscr{F}_m^{(n)}, T)$ is $\Delta_m^{(n)} = \theta_1^m(n{-}1)(m{-}1)$. Taken modulo the $n^2{+}2 + \Delta_m^{(n)}$ reparametrizations (A.32), this deformation space becomes discrete.

We have that for all $m \geqslant 0$ and for $n = 2, 3$, the Hirzebruch n-folds $\mathscr{F}_m^{(n)} \approx \mathscr{F}_{m+n}^{(n)}$ are diffeomorphic, and *assume* this to be true also for $n = 4$ at least. Given that $\mathfrak{F}_m$ and $\mathfrak{F}_{m+2}$ in fact are discrete deformations of each other and that the Kodaira-Spencer deformation space of $\mathscr{F}_m^{(n)}$ is discrete, we propose:

Conjecture A.1. ***(i)*** *The $\Delta_m^{(n)}$-dimensional deformation spaces of "adjacent" Hirzebruch n-folds $\mathscr{F}_m^{(n)} = \left[\begin{smallmatrix}\mathbb{P}^n\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\right]$ and $\mathscr{F}_{m+n}^{(n)} = \left[\begin{smallmatrix}\mathbb{P}^n\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m+n\end{smallmatrix}\right]$ to be "separate but infinitesimally near," so that $\mathscr{F}_m^{(n)}$ is a discrete deformation of $\mathscr{F}_{m+n}^{(n)}$.*

(ii) *In any classical field theory, the use of $\mathscr{F}_m^{(n)}$ and $\mathscr{F}_{m+n}^{(n)}$ should produce identical models; however, some quantum effects may well distinguish $\mathscr{F}_m^{(n)}$ from $\mathscr{F}_{m+n}^{(n)}$ [19, 20].*

Appendix B. The refined Koszul resolution for meromorphic intersections

The Koszul resolution (B.1) of the holomorphic sheaf of functions over X_m may be written as

$$\mathcal{P}^* \otimes \mathcal{Q}^* \begin{array}{c} \overset{p}{\nearrow} \\ \\ \underset{q}{\searrow} \end{array} \begin{array}{c} \mathcal{Q}^* \\ \downarrow \varepsilon f \\ \mathcal{P}^* \end{array} \begin{array}{c} \overset{q}{\searrow} \\ \\ \underset{p}{\nearrow} \end{array} \mathcal{O}_A \overset{\rho}{\twoheadrightarrow} \mathcal{O}_{X_m}. \tag{B.1}$$

where the p- and the q-maps are contractions with the defining tensors $p_{a\,(i_1\cdots i_m)}$ from (2.1) and $q^{\cdots}_{(abcd)\cdots}$ (2.4). Generalizing (2.2), this identifies holomorphic objects $\mathcal{B}$ on X_m as the analogous objects $\mathcal{B}$ defined on $A = \mathbb{P}^4 \times \mathbb{P}^1$, taken however modulo p-multiples of $\mathcal{B} \otimes \mathcal{P}^*$ and q-multiples of $\mathcal{B} \otimes \mathcal{Q}^*$, and taking into account the "double-counting" of objects that are $p{\cdot}q$-multiples of $\mathcal{B} \otimes \mathcal{P}^* \otimes \mathcal{Q}^*$.

However, we must also include the *additional* bundle map, εf, induced by the contraction (2.4a) and (2.4b): linear maps generated by contracting with the q-tensor (2.4) equal the sequential contraction with $p_{a(i_1\cdots i_m)}$- and $\varepsilon^{ij} f_{(abc)}^{(k_1\cdots k_{2m-4})}$-tensors, in either order. However, since the corresponding cohomology εf-map involves contracting with (the d-preimage of) an element of $H^1(A, \mathcal{Q}\times\mathcal{P}^*)$, it can be nonzero only for $m\geqslant 2$ when $H^1(A,\mathcal{Q}\times\mathcal{P}^*)\neq 0$, and must act $H^r(A,\mathcal{B}) \xrightarrow{\varepsilon f} H^{r+1}(A,\mathcal{B}\otimes\mathcal{Q}\otimes\mathcal{P}^*)$. In this way, we extend the standard treatment of algebraic systems of constraints [14] to gCICYs.

We then tensor (B.1) by each of the components of the tangent and normal bundles,

(B.2a) $$T_{\mathbb{P}^4\oplus\mathbb{P}^1} = T_{\mathbb{P}^4}\oplus T_{\mathbb{P}^1},$$

(B.2b) $$N = \left(\mathcal{P} = \mathcal{O}\binom{1}{m}\right)\oplus\left(\mathcal{Q} = \mathcal{O}\binom{4}{2-m}\right),$$

to compute the so-valued cohomology on X_m, for use in the cohomology sequence associated with the adjunction monad (B.3).

B.1. The *T*-valued cohomology

Being formed from sections of $\mathcal{Q} = \mathcal{O}\binom{4}{2-m}$, $q(x,y)$ is non-positive over $\mathbb{P}^1$ for $m>2$, and we cannot use the Lefschetz hyperplane theorem [14, 39] to compute the Hodge numbers $h^{*,*}(X_m)$. However, we can compute $H^*(X_m,T) = H^{2,*}(X_m)$ and $H^*(X_m,T^*) = H^{1,*}(X_m)$ using respectively the "adjunction" sequence and its dual:

(B.3)
$$T_{X_m}\hookrightarrow T_A|_{X_m} \overset{\mathrm{d}q}{\twoheadrightarrow} [\mathcal{P}\oplus\mathcal{Q}]_{X_m} \qquad\text{and}\qquad [\mathcal{P}^*\oplus\mathcal{Q}^*]_{X_m} \overset{\mathrm{d}q}{\hookrightarrow} T^*_A|_{X_m}\twoheadrightarrow T^*_{X_m}.$$

The restrictions $\mathcal{B}|_{X_m} = \rho(\mathcal{B})$ are obtained using the codimension-2 Koszul resolution (B.1) of $X_m\subset A$. In particular, the Koszul resolutions of $\mathcal{P} = \mathcal{O}\binom{1}{m}$ is (crossed-out sheaves have no cohomology):

(B.4)

	$\cancel{\mathcal{O}\binom{-4}{m-2}}$	$\cancel{\mathcal{O}\binom{-3}{2m-2}}$ $\mathcal{O}\binom{0}{0} \xrightarrow{p}$	$\mathcal{O}\binom{1}{m}$	$\mathcal{P}\big\vert_{X_m}$
0.	0	$\{\varphi\}\xrightarrow{p}$	$\{\phi_{a(i_1\cdots i_m)}\}$	$H^0(X_m,\mathcal{P})$
1.	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$H^0(X_m,\mathcal{P})$
$\sim \{\phi_{a(i_1\cdots i_m)}/p_{a(i_1\cdots i_m)}\varphi\}$
$\dim = \binom{1+4}{4}\cdot\binom{m+1}{1} - 1$
$= 5(m+1) - 1$

The Koszul resolution of $\mathcal{Q} = \mathcal{O}\binom{4}{2-m}$ is more involved:

(B.5)

	$\cancel{\mathcal{O}\binom{-1}{-m}}$	$\mathcal{O}\binom{0}{0} \xrightarrow{q}$ $\downarrow \varepsilon f$ $\mathcal{O}\binom{3}{2-2m} \xrightarrow{p}$	$\mathcal{O}\binom{4}{2-m}$	$\mathcal{Q}\vert_{X_m}$
0.	0	$\theta^1_m\{\varphi_{(abc)(i_1\cdots i_{2-2m})}\} \xrightarrow{p}$ $\{\vartheta\} \xrightarrow{q}$ $\downarrow \varepsilon f$	$\theta^2_m\{\phi_{(abcd)(i_1\cdots\, i_{2-m})}\}$	$H^0(X_m, \mathcal{Q})$
1.	0	$\theta^m_2\{\varepsilon^{ij}\varphi^{(k_1\cdots\, k_{2m-4})}_{(abc)}\} \xrightarrow{p}$	$\theta^m_4\{\gamma^{(i_1\cdots\, i_{m-2})}_{(abcd)}\}$	$H^1(X_m, \mathcal{Q})$
2.	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The $H^0(X, \mathcal{Q})$ cohomology group varies with m and is represented by tensors in the following m-dependent fashion:

(B.6)

$\boldsymbol{m}$	**Tensor Representative of $\boldsymbol{H^0(X, \mathcal{Q})}$**
0	$\{\phi_{(abcd)(ij)} / [p_{(a}\,\varphi_{bcd)(ij)} \oplus \vartheta\, q_{(abcd)(ij)}]\}$
1	$\{\phi_{(abcd)\,i} / [\varphi_{(abc}\, p_{d)\,i} \oplus \vartheta\, q_{(abcd)\,i}]\}$
2	$\{[\phi_{(abcd)} \oplus \varepsilon^{ij}\varphi_{(abc}\, p_{d)ik}] / \vartheta\,[q_{(abcd)} \oplus \varepsilon^{ij} f_{(abc}\, p_{d)ik}]\}$
3	$\{[\varepsilon^{i(j}\varphi^{kl)}_{(abc} \,/\, \vartheta\, \varepsilon^{i(j} f^{kl)}_{(abc}]\, p_{d)(jkl)}\}$
$\geqslant 4$	$\{[\varepsilon^{i(j_1}\phi^{j_2\cdots j_{2m-3})}_{(abc} / \varphi\, \varepsilon^{i(j_1} f^{j_2\cdots j_{2m-3})}_{(abc}]\, p_{d)(ij_{m-1}\cdots j_{2m-3})},$ $\text{s.t.}: f^{(j_1\cdots j_{2m-4})}_{(abc}\, p_{d)(j_{m-3}\cdots j_{2m-4})} = 0\}$

Denoting for brevity $T_i\binom{d_1}{d_2} \stackrel{\text{def}}{=} T_i \otimes \mathcal{O}\binom{d_1}{d_2}$, we tensor (B.1) by $T_{\mathbb{P}^4}$:

(B.7)

	$T_{\mathbb{P}^4}\binom{-5}{-2}$	$\cancel{T_{\mathbb{P}^4}\binom{-4}{m-2}}$ $T_{\mathbb{P}^4}\binom{-1}{-m} \xrightarrow{p}$	$T_{\mathbb{P}^4}$	$T_{\mathbb{P}^4}\vert_{X_m}$
0.	0	$\delta_{m,0}\,\{\varkappa^a\} \xrightarrow{p}$	$\{\lambda_a{}^b\}$	$H^0(X_m, T_{\mathbb{P}^4})$
1.	0	$\theta^m_2\{\varepsilon^{i(j_1}\varkappa^{j_2\cdots\, j_{m-1})\,a}\}$	0	0
2.	0	0	0	$H^2(X_m, T_{\mathbb{P}^4})$
3.	0	0	0	0
4.	$\{\varepsilon^{abcde}\varepsilon^{ij}\Lambda_1\}$	0	0	—
5.	0	0	0	—

This lets us represent

$$
\begin{aligned}
&\text{(B.8)} \qquad H^0(X_m, T_{\mathbb{P}^4}): \quad && \{\lambda_a{}^b/\delta_{m,0}(p_a\varkappa^b)\}\\
& && \oplus\, \theta_2^m\{\tilde{\varkappa}_k{}^i \stackrel{\text{def}}{=} \varepsilon^{i(j_1}\varkappa^{j_2\cdots j_{m-1})\,a}p_{a\,(j_1\cdots j_{m-1}k)}\},\\
&\text{(B.9)} \qquad H^2(X_m, T_{\mathbb{P}^4}): \quad && \{\varepsilon^{abcde}\varepsilon^{ij}\Lambda_1\}.
\end{aligned}
$$

Except for $m=0$, the first contribution to $H^0(X_m, T_{\mathbb{P}^4})$ represents the standard linear, traceless $\mathbb{P}^4$-reparametrizations $\{\lambda_a{}^b\}$. Although the second contribution acts as a standard linear, traceless $\mathbb{P}^1$-reparametrization, it is parametrized by the $\binom{(m-2)+1}{1}\binom{1+4}{4} = 5(m-1)$-component tensor $\varkappa^{(j_1\cdots j_{m-2})\,a}$. We note that $\varepsilon^{abcde}\varepsilon^{ij}\Lambda_1$ represents the Serre dual of the Kähler form $J_1 \in H^2(\mathbb{P}^4, \mathbb{Z})$.

Tensoring (B.1) by $T_{\mathbb{P}^1}$, we obtain:

(B.10)

	$T_{\mathbb{P}^1}\binom{-5}{-2}$	~~$T_{\mathbb{P}^1}\binom{-4}{m-2}$~~ ~~$T_{\mathbb{P}^1}\binom{-1}{-m}$~~	$T_{\mathbb{P}^1}$	$T_{\mathbb{P}^1}\big\vert_{X_m}$
0.	0	0	$\{\lambda_i{}^j\}$	$H^0(X_m, T_{\mathbb{P}^1})$
1.	0	0	0	0
2.	0	0	0	$H^2(X_m, T_{\mathbb{P}^1})$
3.	0	0	0	0
4.	$\{\varepsilon^{abcde}\varepsilon^{ij}\Lambda_2\}$	0	0	—
5.	0	0	0	—

This represents $H^0(X_m, T_{\mathbb{P}^1})$ by the standard linear, traceless $\mathbb{P}^1$-reparametrization $\{\lambda_i{}^j\}$, and $\varepsilon^{abcde}\varepsilon^{ij}\Lambda_2$ represents $H^2(X_m, T_{\mathbb{P}^1})$ and the Serre dual of the Kähler form $J_2 \in H^2(\mathbb{P}^1, \mathbb{Z})$.

Putting (B.4), (B.10), (B.7) and (B.5) together, we obtain:

(B.11)

T_{X_m}	$T_{\mathbb{P}^4\times\mathbb{P}^1}\big\vert_{X_m}$		$[\mathcal{P}\oplus\mathcal{Q}]_{X_m}$
$H^0(X_m, T)$	$\{\lambda_a{}^b/\delta_{m,0}(p_a\varkappa^b)\}$ $\{\lambda_i{}^j\}$ $\theta_2^m\{\tilde{\varkappa}_k{}^i\}_{(\text{B.8})}$	$\xrightarrow[\mathrm{d}q]{\mathrm{d}p}$	$\{\phi_{a(i_1\cdots i_m)}\,/\,\vartheta\, p_{a(i_1\cdots i_m)}\}$ $\underbrace{\left\{\begin{matrix}\theta_m^2\{\phi_{(abcd)(i_1\cdots i_{2-m})}\,/\,\ldots\}\oplus\\ \theta_2^m\{\gamma_{(abcd)}^{(j_1\cdots j_{m-2})}\,/\,\ldots\}\end{matrix}\right\}}_{\text{see (A.8a) and (B.6)}}$
$H^1(X_m, T)$	0		0
$H^2(X_m, T)$	$\{\varepsilon^{abcde}\varepsilon^{ij}(\Lambda_1\oplus\Lambda_2)\}$		0
$H^3(X_m, T)$	0		0

The combined mapping denoted "$\overset{\mathrm{d}p}{\underset{\mathrm{d}q}{\to}}$" maps the contributions from the $T_{\mathbb{P}^4\times\mathbb{P}^1}|_{X_m}$ column to those in the $[\mathcal{P}\oplus\mathcal{Q}]_{X_m}$ column by means of a simple contraction with $p_{a(i_1\cdots i_m)}$, or $q_{(abcd)(i_1\cdots i_{2-m})}$ for $m\leqslant 2$, or $q_{(abcd)}^{(i_1\cdots i_{m-2})}$ for $m\geqslant 3$ and generates the equivalence relations such as:

$$\text{(B.12a)}\qquad \lambda_a{}^b:\quad \phi_{a(k_1\cdots k_m)}\simeq \phi_{a(k_1\cdots k_m)}+\lambda_a{}^b\, p_{b(k_1\cdots k_m)},$$

$$\text{(B.12b)}\qquad \lambda_i{}^j:\quad \phi_{a(i_1\cdots i_m)}\simeq \phi_{a(k_1\cdots k_m)}+\lambda_{k_1}{}^j\, p_{b(jk_2\cdots k_m)},$$

$$\text{(B.12c)}\qquad \text{for } m\geqslant 2,\ \tilde{\varkappa}_i{}^j:\quad \phi_{a(i_1\cdots i_m)}\simeq \phi_{a(k_1\cdots k_m)}+\tilde{\varkappa}_{k_1}{}^j\, p_{b(jk_2\cdots k_m)},$$

and similarly with the other terms in $H^0(X_m,\mathcal{P}\oplus\mathcal{Q})$. Notice that the $\tilde{\varkappa}^i{}_j$-reparametrizations acting on the $m\geqslant 2$ contributions to $H^0(X_m,\mathcal{Q})$ are given as:

$$\text{(B.12d)}\qquad \tilde{\varkappa}^i{}_j:\quad \gamma_{(abcd)}^{(j_1\cdots j_{m-2})}\simeq \gamma_{(abcd)}^{(j_1\cdots j_{m-2})}+\tilde{\varkappa}_i{}^{j_1}\, q_{(abcd)}^{(ij_2\cdots j_{m-2})},$$

and involve a contraction with the defining tensor $p_{a\,(i_1\cdots i_m)}$ twice: both within the definition (B.8) of $\tilde{\varkappa}_i{}^{j_1}$ and also within the definition (2.4) of $q_{(abcd)}^{(ij_2\cdots j_{m-2})}$. Although acting as a standard linear, traceless $\mathbb{P}^1$-reparametrization, $\tilde{\varkappa}_i{}^{j_1}\overset{\text{def}}{=}\varepsilon^{i(j_1}\varkappa^{j_2\cdots j_{m-1})\,a}\,p_{a\,(j_1\cdots j_{m-1}k)}$ is parametrized by the $5(m{-}1)$ independent components of $\varkappa^{(i_1\cdots i_{m-2})\,a}$, and can "gauge away" that many degrees of freedom via relations such as (B.12d).

For generic choices of $p(x,y)$ and $q(x,y)$, defined respectively in (2.1) and (2.4), the combined mapping $\overset{\mathrm{d}p}{\underset{\mathrm{d}q}{\longrightarrow}}$ in (B.11) is of maximum rank, i.e., has no kernel so that $H^0(X_m,T)=0$ and the tangent bundle of X_m is simple, as expected for Calabi-Yau n-folds on general grounds. In turn, that implies that

$$\text{(B.13)}\qquad H^1(X_m,T)\approx H^0(X_m,\mathcal{P}\oplus\mathcal{Q})\big/\{\tfrac{\mathrm{d}p}{\mathrm{d}q}\}\cdot H^0(X_m,T_{\mathbb{P}^4\times\mathbb{P}^1}),$$

so that $h^{2,1}=\dim H^1(X_m,T)$ is given, for various m as:

$$\begin{aligned}\text{(B.14)}\qquad h^{2,1}=&\big\{5(m{+}1)-1\big\}\\ &+\big\{\theta^2_m[70(3{-}m)-\theta^1_m\,35(3{-}2m)]\\ &\quad+\theta^m_2[35(2m{-}3)-\theta^m_4\,70(m{-}3)]-1\big\}\\ &-\big\{(24-\delta_{m,0}5)+(3)+\theta^m_2[5(m{-}1)]\big\},\end{aligned}$$

which evaluates to $h^{2,1}=86$ for all $m\geqslant 0$.

Finally, (B.11) also shows that $H^2(X_m, T) \approx H^2(X_m, T_{\mathbb{P}^4\times\mathbb{P}^1})$ which in turn was shown in (B.7) and (B.10) to be generated by the duals of the Kähler forms of $\mathbb{P}^4$ and $\mathbb{P}^1$. Therefore, $H^{1,1}(X_m, T^*) \approx H^2(X_m, \mathbb{Z})$ is generated by the direct images of those Kähler forms.

While rather involved, the above computation is considerably swifter than the monad-by-monad calculation following through the system of eleven monads in Figure B1 and the corresponding web of long exact cohomology sequences [14]. We have however verified that these two computational frameworks perfectly agree, and in particular that the (vertical) εf-map modification of the Koszul resolution (B.1) is both necessary and sufficient in all the cases considered herein.

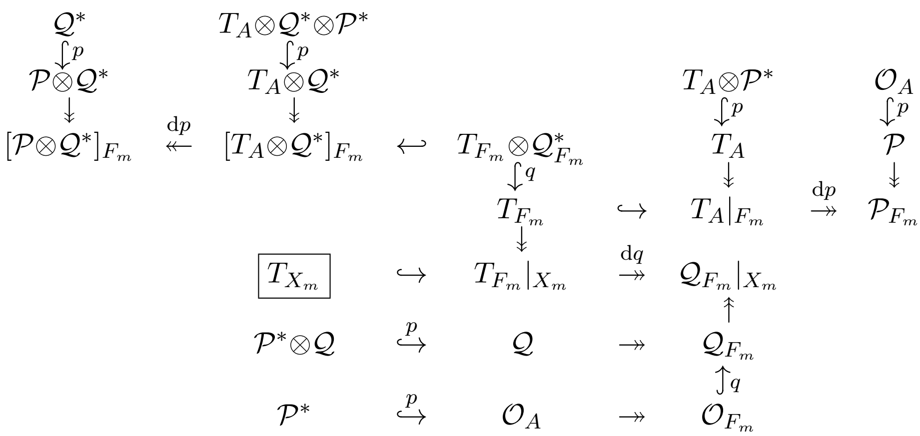

Figure B1: The network of eleven monads ($A \hookrightarrow B \twoheadrightarrow C$, i.e., $C = B/A$) that determine the tangent bundle T_{X_m} on $X_m \subset \mathbb{P}^4 \times \mathbb{P}^1$ in terms of various bundles and sheaves on $A = \mathbb{P}^4 \times \mathbb{P}^1$.

Appendix C. Further examples

C.1. A terminating sequence

To demonstrate the relevance of checking the number and tensorial structure of anticanonical sections as done in (A.10), as well as in (A.25) and (A.26), consider the common zero-locus of the system of equations $p(x, y) = 0 =$

$q(x,y)$:

(C.1)
$$\widetilde{X}_m \in \left[\begin{array}{c\|c|c} \mathbb{P}^4 & 2 & 3 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,56-15m)}_{-108+30m}, \qquad m=0,1,2,3,\dots;$$
$$\widetilde{X}_m \subset \widetilde{F}_m \in \left[\begin{array}{c\|c} \mathbb{P}^4 & 2 \\ \mathbb{P}^1 & m \end{array}\right].$$

For $m>2$, the second, degree-$\binom{3}{2-m}$ equation is negative over $\mathbb{P}^1$ and we need to verify that the anticanonical bundle $\mathcal{Q}=\mathcal{O}_{\widetilde{F}_m}\binom{3}{2-m}$ has sections. As in Appendix A.1, the Koszul resolution for $\widetilde{F}_m$ provides the required restriction:

(C.2)

	$\mathcal{O}\binom{1}{2-2m}$	$\overset{p}{\hookrightarrow}$	$\mathcal{Q}=\mathcal{O}\binom{3}{2-m}$	$\overset{\rho_G}{\twoheadrightarrow}$	$\mathcal{Q}\vert_{\widetilde{F}_m}$
0.	$\theta^1_m\{\varphi_{a\,(i_1\cdots i_{2-2m})}\}$	$\xrightarrow{p}$	$\theta^2_m\{\phi_{(abc)(i_1\cdots i_{2-m})}\}$	$\xrightarrow{\rho_G}$	$H^0(\widetilde{F}_m,\mathcal{Q})\xrightarrow{\mathrm{d}}$
1	$\theta^m_2\{\varepsilon^{i(j}\varphi_a^{k_1\cdots k_{2m-4})}\}$	$\xrightarrow{p}$	$\theta^m_4\{\varepsilon^{i(j}\phi_{(abc)}^{k_1\cdots k_{m-4})}\}$	$\xrightarrow{\rho_G}$	$H^1(\widetilde{F}_m,\mathcal{Q})\xrightarrow{\mathrm{d}}$
2	0		0		$H^2(\widetilde{F}_m,\mathcal{Q})=0$
$\vdots$	$\vdots$		$\vdots$		$\vdots$

The p-map is generated by contraction with $p_{(ab)(i_1\cdots i_m)}$

A section $q(x,y)\in H^0(\widetilde{F}_m,\mathcal{Q})$ is to be used for the defining equation of $\widetilde{X}_m\subset\widetilde{F}_m$. For $m\leqslant 3$, $H^0(\widetilde{F}_m,\mathcal{Q})$ is nonzero:

(C.3)

m	$H^0(\widetilde{F}_m,\mathcal{Q})$	**Number**	**Sections**
0	$\{\phi_{(abc)(ij)}/p_{(ab}\varphi_{c)}(ij)\}$	$\binom{3+4}{4}\binom{2+1}{1}-\binom{1+4}{4}\binom{2+1}{1}=90$	ordinary
1	$\{\phi_{(abc)\,i}/\varphi_{(a}\,p_{bc)\,i}\}$	$\binom{3+4}{4}\binom{1+1}{1}-\binom{1+4}{4}\binom{0+1}{1}=65$	ordinary
2	$\{\phi_{(abc)}\}$ $\{\varepsilon^{ij}\varphi_{(a}\,p_{bc)(ik)}\}$	$\binom{3+4}{4}\binom{0+1}{1}=35$ $\binom{1+4}{4}\binom{0+1}{1}=5$	ordinary Laurent
3	$\{\varepsilon^{i(j}\varphi_{(a}^{kl)}\,p_{bc)(ikl)}\}$	$\binom{1+4}{4}\binom{2+1}{1}=15$	Laurent

However, for $m\geqslant 4$ the cohomology group $H^0(\widetilde{F}_m,\mathcal{Q})$ vanishes when the tensor coefficients in the defining sections $p(x,y)$ and $q(x,y)$ are chosen so as to insure bundle maps of maximal rank, so that

(C.4)
$$H^1(\widetilde{F}_m,\mathcal{Q})\sim\{\varepsilon^{i(j}\phi_{(abc)}^{k_1\cdots k_{m-4})}/\varepsilon^{i(j}\varphi_a^{k_1\cdots k_{2m-4})}\},$$
$$\dim H^1(\widetilde{F}_m,\mathcal{Q})=\binom{3+4}{4}\binom{m-4+1}{1}-\binom{1+4}{4}\binom{2m-4+1}{1}=5(13m-18).$$

Unlike (1.1), the sequence of Calabi-Yau 3-folds (C.1) thereby terminates with $\widetilde{X}_3$: for $m \geqslant 4$, $\left[\begin{smallmatrix}\mathbb{P}^4\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}2\\ m\end{smallmatrix}\right]$ has no holomorphic sections of $\mathcal{Q} = \mathcal{O}\binom{3}{2-m}$ with which to define $\widetilde{X}_m$. It is logically possible that *special* choices of the defining sections $p(x,y)$ and $q(x,y)$ may be found to provide for extending the sequence (C.1), but we do not explore this herein. In turn and assuming maximum rank for the p- and q-bundle maps, the computation analogous to the one presented in Appendix B.1 insures that $H^{1,1}(\widetilde{X}_m)$ is 2-dimensional and is generated by (the pullbacks of) the Kähler classes of $\mathbb{P}^4$ and $\mathbb{P}^1$ for all $m \geqslant 2$. This justifies the standard Chern-class computation and produces χ_E, $h^{1,1}$ and so also $h^{2,1}$ as displayed in (C.1).

In particular, $h^{2,1}$ would become (nonsensically) negative for $m \geqslant 4$ — when in fact there are no holomorphic sections to define $\widetilde{X}_m \subset \widetilde{F}_m$ in the first place. In this way, the termination of the series (C.1) is signaled already by the standard Chern-class computation.

C.2. More complicated configurations

Even just permitting the Hirzebruch n-folds of arbitrary twist as a factor in the embedding space generalizes the constructions of both Refs. [3–5] as well as Ref. [33] with doubly periodic examples such as:

$$\text{(C.5a)}\qquad [m \pmod 2]\cdot[n \pmod 2]:\quad \left[\begin{array}{c||cc|c} \mathbb{P}^2 & 1 & 0 & \textcircled{2} \\ \mathbb{P}^1 & m & 0 & 2-m \\ \hdashline \mathbb{P}^2 & 0 & 1 & \textcircled{2} \\ \mathbb{P}^1 & 0 & n & 2-n \end{array}\right]^{(4,68)}_{-128},$$

$$\text{(C.5b)}\qquad [m \pmod 2]\cdot[n \pmod 3]:\quad \left[\begin{array}{c||ccc|c} \mathbb{P}^2 & 1 & 0 & 0 & \textcircled{2} \\ \mathbb{P}^1 & m & 0 & 1 & 1-m \\ \hdashline \mathbb{P}^3 & 0 & 1 & 0 & \textcircled{3} \\ \mathbb{P}^1 & 0 & n & 1 & 1-n \end{array}\right]^{(4,76)}_{-144},$$

which are both doubly periodic. By direct computation as in Appendices A and B, we verify that the last constraint is viable, i.e., that the regular complete intersection 4-fold defined by imposing all but this last constraint has sections of the degrees specified in the last column: (C.5a) has 81, while (C.5b) has 90 sections with which to define this last constraint and so the indicated Calabi-Yau 3-folds.

Among several different ways to regard these configurations and as indicated by the horizontal dashed lines, the configuration (C.5a) may be regarded as a fibration of the "upper" torus $\left[\begin{smallmatrix}\mathbb{P}^2\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\middle|\begin{smallmatrix}2\\ 2-m\end{smallmatrix}\right]$ over the "lower" $\mathfrak{F}_n$ or as a fibration of the "lower" torus $\left[\begin{smallmatrix}\mathbb{P}^2\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ n\end{smallmatrix}\middle|\begin{smallmatrix}2\\ 2-n\end{smallmatrix}\right]$ over the "upper" $\mathfrak{F}_m$. Similarly, the configuration (C.5b) may be regarded as a fibration of the "upper" 2-points $\left[\begin{smallmatrix}\mathbb{P}^2\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ m\end{smallmatrix}\middle|\begin{smallmatrix}0 & 2\\ 1 & 2-m\end{smallmatrix}\right]$ over the "lower" $\mathcal{F}_n$ or as a fibration of the "lower" torus $\left[\begin{smallmatrix}\mathbb{P}^3\\ \mathbb{P}^1\end{smallmatrix}\middle\|\begin{smallmatrix}1\\ n\end{smallmatrix}\middle|\begin{smallmatrix}0 & 3\\ 1 & 2-n\end{smallmatrix}\right]$ over the "upper" $\mathfrak{F}_m$.

To demonstrate that the particular degrees indicated by circles in (C.5) are necessary for the Hirzebruch n-folds' periodicity to be inherited by the embedded Calabi-Yau 3-fold, consider modifying (C.5b) by "splitting" the [m (mod 2)]-periodicity preserving degree "2" into:

(C.6)
$$\widetilde{Y}_{m,n} \in \left[\begin{array}{c||ccc|c} \mathbb{P}^2 & 1 & 0 & \textcircled{1} & \textcircled{1} \\ \mathbb{P}^1 & m & 0 & 1 & 1-m \\ \hdashline \mathbb{P}^3 & 0 & 1 & 0 & \textcircled{3} \\ \mathbb{P}^1 & 0 & n & 1 & 1-n \end{array}\right]^{(4,58-9m)}_{18(m-6)}, \quad \widetilde{Y}_{m,n} \subset \widetilde{G}_{m,n} \in \left[\begin{array}{c||ccc} \mathbb{P}^2 & 1 & 0 & 1 \\ \mathbb{P}^1 & m & 0 & 1 \\ \hdashline \mathbb{P}^3 & 0 & 1 & 0 \\ \mathbb{P}^1 & 0 & n & 1 \end{array}\right].$$

The resulting (m,n)-sequence of Calabi-Yau 3-folds is however only simply, [n (mod 3)]-periodic. The m-periodicity is broken by the lack of the first $\mathbb{P}^2$-degree of "2" in the last constraint — having split "2"→"1 | 1" as highlighted by the oval; this also makes the Euler number and $h^{2,1}$ m-dependent. In fact, techniques detailed in Appendices A and B permit computing $\dim H^0(\widetilde{G}_{m,n}, \mathcal{Q}) = 10(7-m)$. That is, $\widetilde{G}_{m,n}$ has holomorphic anti-canonical sections with which to define $\widetilde{Y}_{m,n}$ only for $0 \leqslant m \leqslant 6$, and the double sequence (C.6) terminates in the m-direction while remaining infinite but [n (mod 3)]-periodic in the n-direction. Of the so-defined $\widetilde{Y}_{m,n}$, most (those with $2 \leqslant m \leqslant 6$) are gCICYs.

Finally, "spoiling" also the n-periodicity in the same way, we have the double sequence

$$\text{(C.7)} \qquad \breve{Y}_{m,n} \in \left[\begin{array}{c||ccc|c} \mathbb{P}^2 & 1 & 0 & \textcircled{1} & \textcircled{1} \\ \mathbb{P}^1 & m & 0 & 1 & 1-m \\ \hdashline \mathbb{P}^3 & 0 & 1 & \textcircled{1} & \textcircled{2} \\ \mathbb{P}^1 & 0 & n & 1 & 1-n \end{array}\right]^{(4,44+(m-2)(n-3))}_{-2(40+(m-2)(n-3))},$$
$$\breve{Y}_{m,n} \subset \breve{G}_{m,n} \in \left[\begin{array}{c||ccc} \mathbb{P}^2 & 1 & 0 & 1 \\ \mathbb{P}^1 & m & 0 & 1 \\ \hdashline \mathbb{P}^3 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & n & 1 \end{array}\right],$$

which is periodic in neither m nor n, and the Euler number of which depends on both m and n. By direct computation as in Appendices A and B, we find that the number of sections available for constructing the last constraint varies with both m and n and equals

(C.8) $$\dim H^0(\breve{G}_{m,n}, \mathcal{Q}) = 3(15-3m-2n) \geqslant 0 \quad \text{for} \quad 3m+2n \leqslant 15,$$

where $\mathcal{Q} = \mathcal{O}(1, 1-m, 2, 1-n) = \mathcal{K}^*_{\breve{G}_{m,n}}$. Although this is an aperiodic (m,n)-sequence of configurations and terminates so $3m+2n \leqslant 15$ (beyond which there are no $\breve{G}_{m,n}$-sections of $\mathcal{Q}$ to define $\breve{Y}_{m,n}$), it contains 24 models, 18 of which are gCICYs.

C.3. A higher codimension subtlety

As a further illustration of the prescriptive power of the homological algebra encoded by the exact and spectral sequences — and the formulae (2.3) and (2.4) in particular — consider the following configuration:

(C.9)
$$Z \in \left[\begin{array}{c\|cc|c} \mathbb{P}^4 & 0 & 1 & 4 \\ \mathbb{P}^2 & 2 & 2 & -1 \end{array}\right]^{(2,86)}_{-168} : \quad \begin{cases} s(y) := s_{(ij)}\, y^i y^j, \\ p(x,y) := p_{(ij)}(x)\, y^i y^j = p_{a\,(ij)}\, x^a\, y^i y^j, \\ q(x,y) := \frac{q^i(x)}{y^i} = q^i_{(abcd)} \frac{x^a x^b x^c x^d}{y^i}. \end{cases}$$

The intermediate — and regular — complete intersection $M = \{s(y) = 0 = p(x,y)\} \subset \mathbb{P}^4 \times \mathbb{P}^2$ has the standard Koszul resolution of its anticanonical bundle $\mathcal{Q} = \mathcal{O}\binom{4}{-1}$, and we list it with the associated spectral sequence:

(C.10)

	$\mathcal{O}\binom{3}{-5}$ $\xrightarrow{p}$ / $\xrightarrow{s}$	$\mathcal{O}\binom{4}{-3}$ $\xrightarrow{s}$ / $\mathcal{O}\binom{3}{-3}$ $\xrightarrow{p}$	~~$\mathcal{O}\binom{4}{-1}$~~	$\mathcal{Q}_M$
0.	0	0	0	$H^0(M,\mathcal{Q})$
1.	0	0	0	$H^1(M,\mathcal{Q})$
2.	$\{\varepsilon^{ij(k}\varphi^{l_1 l_2)}_{(abc)}\}$	$\xrightarrow{p} \{\varepsilon^{ijk}\phi_{(abcd)}\}$, $\xrightarrow{s} \{\varepsilon^{ijk}\phi_{(abc)}\}$	0	0
3.	0	0	0	0
⋮	⋮	⋮	⋮	⋮

$H^1(M,\mathcal{Q}) = 0$ for sufficiently generic $s(y)$ and $p(x,y)$.

If $s_{(ij)}$ and $p_{a\,(ij)}$ in (C.9) are chosen sufficiently generically so the combined $p+s$-mapping is surjective, nothing is mapped to $H^1(M,\mathcal{Q})$. The kernel of this mapping (domain elements that are annihilated) is then 105-dimensional and is identified with $H^0(M, \mathcal{Q})$, which is thus parametrized by the tensor coefficients $\varphi_{(abc)}^{(ij)}$ satisfying the kernel constraints:

$$\{\varphi_{(abc)}^{(jk)}: \quad \varphi_{(abc)}^{(jk)} s_{(jk)} = 0 = f_{(abc}^{(jk)} p_{d)(jk)}\} \tag{C.11}$$

These coefficients are then used to define the degree-$\binom{4}{-1}$ rational sections by means of a double contraction

$$\gamma_{(abcd)}^{\;i} = p_{(j_1 j_2)(a}\, \varepsilon^{j_1 k_1 (i} \varphi_{bcd)}^{j_2 k_2)}\, s_{(k_1 k_2)}. \tag{C.12}$$

The appearance of the ε^{ijk}-symbol again introduces relative signs, much as ε^{ij} does in (A.8c).

A quick comparison of this prescription with (2.4), and a corresponding comparison of the chart (C.10) with (A.1) reveals the following:

1) The ε-symbols in (2.4) and (C.12) are dual to the volume-forms of $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively, owing to the fact that these sections stem, respectively, from 1- and 2-forms in (A.1) and (C.10), respectively.
2) The representatives (2.4) are obtained by a single contraction with $p_{a\,(ijk_1\cdots k_n)}$—as dictated by the resolution map indicated in the "header" of the chart (A.1) and since F_m is a codimension-1 hypersurface in $\mathbb{P}^4 \times \mathbb{P}^1$. By contrast, the representatives (C.12) include a double "pull-back" contraction, using each of the two tensor coefficients $s_{(ij)}$ and $p_{a\,(ij)}$. Again, this is as dictated by the sequence of resolution maps indicated in the "header" row of the chart (C.10) and owing to M being a codimension-2 intersection $\{s(x,y) = 0 = p(x,y)\}$ of hypersurfaces in $\mathbb{P}^4 \times \mathbb{P}^2$.

Owing to Corollary 2 of Ref. [42] (see also Ref. [14, Lemma 2.1, p. 54]), we have the sequence of relations

$$\left[\begin{array}{c\|cc|c} \mathbb{P}^4 & 0 & 1 & 4 \\ \mathbb{P}^2 & 2 & n & 1-n \end{array}\right] \cong \left[\begin{array}{c\|c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 2n & 2-2n \end{array}\right], \tag{C.13}$$

so that the configuration in the left-hand side n-sequence correspond to even-m configurations in our main example sequence (1.1). This relationship derives from the fact that a generic quadric in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. However, one must be careful in using this relation since it does not include an isomorphism of the integral cohomology. The straightforward Chern class computation for the quadric $[\mathbb{P}^2||2]$ and the hyperplane in $[\mathbb{P}^2||1]$ which equals $\mathbb{P}^1$,

$$\text{(C.14)}\quad \big(c[\mathbb{P}^2||2] = 1 + J_3\big) \;\cong\; \big(c[\mathbb{P}^1] = 1 + 2J_2\big) \;\approx\; \big(c[\mathbb{P}^2||1] = 1 + 2J_4\big),$$

implies that $(J_2 = J_4) = \frac{1}{2}J_3$. That is, the generator of $H^2\big([\mathbb{P}^2||2], \mathbb{Z}\big)$ is identified with the double of the generator of $H^2\big([\mathbb{P}^2||1], \mathbb{Z}\big) \approx H^2(\mathbb{P}^1, \mathbb{Z})$. Indeed, using the corresponding *fractional* generator, we calculate the intersection numbers [14, p. 178]:

(C.15)

$$\left[\begin{array}{c||cc|c||ccc} \mathbb{P}^4 & 0 & 1 & 4 & 1 & 1 & 1 \\ \mathbb{P}^1 & 2 & n & 1{-}n & 0 & 0 & 0 \end{array}\right] = 2 + 6n \quad \text{and} \quad \left[\begin{array}{c||cc|c||ccc} \mathbb{P}^4 & 0 & 1 & 4 & 1 & 1 & 0 \\ \mathbb{P}^1 & 2 & n & 1{-}n & 0 & 0 & \frac{1}{2} \end{array}\right] = 4.$$

Using the standard result $c_2 = \big(6{J_1}^2 + (4{-}3n)J_1J_3 + (1{-}n{+}n^2){J_3}^2\big)$, we have

$$\text{(C.16)}\qquad [(aJ_1{+}b(\tfrac{1}{2}J_3))^3] = 2a^3 + 3a^2(\underline{4b{+}2na}),$$

$$\text{(C.17)}\qquad C_2[(aJ_1{+}b(\tfrac{1}{2}J_3))] = 44a + 6(\underline{4b{+}2na}).$$

Comparing (C.16) and (C.17) respectively with (2.8) and (2.9) proves that $m = 2n$, and the left-hand side sequence of configurations in (C.13) indeed captures only the even-m configurations in the original series (1.1). More generally, configuration relations of the form

$$\text{(C.18)}\qquad X \in \left[\begin{array}{c||c:c} \mathbb{X} & 0 & \mathbb{A} \\ \hdashline \mathbb{P}^2_y & 2 & \mathbb{B} \end{array}\right] \cong \left[\begin{array}{c||c} \mathbb{X} & \mathbb{A} \\ \hdashline \mathbb{P}^1_z & 2\mathbb{B} \end{array}\right] \ni \mathring{X} \quad\Rightarrow\quad \tfrac{1}{2}J_y = J_z$$

imply a homeomorphism but not a diffeomorphism between X and $\mathring{X}$; in particular, the classical cohomology rings of X and $\mathring{X}$ agree only upon a (non-integral) rational basis change.

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