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# Super Riemann surfaces, metrics and gravitinos

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The underlying even manifold of a super Riemann surface is a Riemann surface with a spinor valued differential form called gravitino. Consequently infinitesimal deformations of super Riemann surfaces are certain infinitesimal deformations of the Riemann surface and the gravitino. Furthermore the action functional of non-linear super symmetric sigma models, the action functional underlying string theory, can be obtained from a geometric action functional on super Riemann surfaces. All invariances of the super symmetric action functional are explained in super geometric terms and the action functional is a functional on the moduli space of super Riemann surfaces.

## Introduction

Let $|M|$ be a compact closed two dimensional manifold. In super string theory and super gravity one studies a super symmetric extension of the harmonic action functional where both the field $\varphi\colon |M| \to \mathbb{R}$ and the Riemannian metric $g$ on $|M|$ get a super partner. See for example [1, 6]. Let $S$ be a spinor bundle on $|M|$ with respect to a chosen spin structure and $S^\vee$ its dual bundle. Let $\psi$ be a section of $S^\vee$ and $\chi$ a spinor valued differential form, i.e. a section of $T^\vee|M| \otimes_{\mathbb{R}} S$. The super symmetric action functional is

$$(*)\quad A(\varphi, \psi, g, \chi) = \int_{|M|} \Big( \| d\varphi\|_g^2 + \langle \psi, \not{D}\psi\rangle \\ + 2\langle \gamma^a\gamma^b\chi_a, \psi\rangle \partial_{x^b}\varphi + \frac{1}{2}\langle \chi_a, \gamma^b\gamma^a\chi_b\rangle\langle\psi,\psi\rangle \Big) dvol_g.$$

This action is invariant under

- Diffeomorphisms of $|M|$: $A(\varphi\circ f, f^*\psi, f^*g, f^*\chi) = A(\varphi, \psi, g, \chi)$
- Conformal transformations: $A(\varphi, \psi, \lambda^2 g, \chi) = A(\varphi, \psi, g, \chi)$
- Super Weyl transformations: $A(\varphi, \psi, g, \chi + \gamma s) = A(\varphi, \psi, g, \chi)$

- Super symmetry, assuming that the fields $\psi$ and $\chi$ are odd:

$$\begin{aligned} \delta\varphi &= \langle q, \psi\rangle & \delta\psi &= \left(\partial_{x^k}\varphi - \langle\psi, \chi_k\rangle\right)\gamma^k q \\ \delta f_a &= -2\langle\gamma^b q, \chi(f_a)\rangle f_b & \delta\chi_a &= \nabla^S_{f_a} q \end{aligned}$$

  Here $f_a$ is a $g$-orthonormal frame and $\nabla^S$ a particular spin connection with torsion.

The aim of this paper is to clarify the relation of the action functional $(*)$ to super Riemann surfaces. The invariances of the action functional arise from geometric properties of super Riemann surfaces. Super Riemann surfaces are an analogue of Riemann surfaces in super geometry. This mathematical theory was developed already in the seventies for the treatment of super symmetric theories in high energy physics (see e.g. [16, 18, 19]). The concept of a super Riemann surface appeared only a little later and their moduli space was studied, see for example [4, 8, 9, 17, 21]. But the precise connection between the super Riemann surfaces and the metric field $g$ and the gravitino $\chi$ remained unclear even though many conjectured a super Teichmüller theory that would study the moduli space of super Riemann surfaces (or a covering of it) in terms of the metric and the gravitino field. The action functional $(*)$ was claimed to arise from a particular Berezin integral on a super Riemann surface (e.g. [7]). However, no explicit proof of this claim seems to exist.

In this article we argue that the key ingredient to a geometric understanding of the relation between the action functional $(*)$ and super Riemann surfaces is the underlying even manifold of a family of super manifolds. The underlying even manifold is an intermediate concept between the concept of super manifold and its reduced space. More specifically, for a family of super manifolds $M \to B$ of (relative) dimension $m|n$ the underlying even manifold is a family $|M| \to B$ of (relative) dimension $m|0$ together with a topologically trivial embedding $i\colon |M| \to M$ over $B$. In contrast to the reduced space $M_{red}$ the underlying even manifold $|M|$ allows to define odd fields, such as $\chi$ and $\psi$ and different embeddings $i\colon |M| \to M$.

The concept of underlying even manifolds for families of super manifold will be introduced in the first section of this article. It will be shown that such an underlying even manifold $|M|$ exists for all super manifolds $M$. Any Berezin integral on $M$ can then be reduced to an integral on $|M|$.

In the second section we will study the geometric structures induced on manifolds underlying super Riemann surfaces. We will show that the geometry is completely determined by a metric $g$ and a gravitino $\chi$ on an underlying even manifold $|M|$. This opens the possibility for a super differential geometric approach to the moduli of super Riemann surfaces, i.e. a super Teichmüller theory in therms of metrics and gravitinos. As a first step we study the tangent space to the moduli space of super Riemann surfaces, using metrics and gravitinos.

The aim of the third section is to demonstrate how the action functional $(*)$ arises from a Berezin integral on a super Riemann surface. The formulation in terms of the Berezin integral leads to a very clear geometrical interpretation of the symmetries of $(*)$. Consequently the action functional $(*)$ is a functional on the moduli space of super Riemann surfaces. We give an interpretation of its energy momentum tensor and super current in terms of cotangent vectors to the moduli space.

In this paper, we present the main results of the second author's thesis [15]. Some of the results in the last two sections rely on long and complicated computations. In order not to overly burden the presentation we have omitted those and refer instead to [15].

## 1. Super geometry

We use the ringed space approach to super geometry (see, for example, [18]).

**Definition 1.1.** A (smooth) super manifold is a locally ringed space $(\|M\|, \mathcal{O}_M)$ that is locally isomorphic to $\mathbb{R}^{m|n} = (\mathbb{R}^m, C^\infty(\mathbb{R}^m, \mathbb{R}) \otimes_{\mathbb{R}} \Lambda_n)$. Here $\Lambda_n$ is a real Grassmann algebra generated by $n$ elements. A map of super manifolds $f\colon M \to N$ is a map of locally ringed spaces. That is, a pair $(\|f\|, f^\#)$ consisting of a continuous map $\|f\|\colon \|M\| \to \|N\|$ and a sheaf homomorphisms $f^\#\colon \mathcal{O}_N \to \mathcal{O}_M$. It follows that the sheaf of rings $\mathcal{O}_M$ is a super commutative $\mathbb{Z}_2$-graded sheaf of rings. The elements of $\mathcal{O}_M$ will be called functions.

Let $x^a$, $a = 1, \ldots, m$ be the standard coordinate functions on $\mathbb{R}^m$ and $\eta^\alpha$, $\alpha = 1, \ldots, n$ be generators for $\Lambda_n$. Their lift to $\mathcal{O}_{\mathbb{R}^{m|n}}$ will be called coordinates for $\mathbb{R}^{m|n}$. We write $X^A = (x^a, \eta^\alpha)$, using the convention that small Latin letters refer to even objects, small Greek letters to odd ones and capital Latin indices refer to odd and even objects together. Any function on $\mathbb{R}^{m|n}$ can be expanded as

$$f = \sum_{\underline{\alpha}} \eta^{\underline{\alpha}} f_{\underline{\alpha}}(x),$$

where $\underline{\alpha}$ is a $\mathbb{Z}_2$-multiindex and the $f_{\underline{\alpha}}$ are smooth functions that can be expressed in the coordinates $x^a$. According to [18, Theorem 2.17] any morphism between super domains $U \subseteq \mathbb{R}^{m|n}$ and $V \subseteq \mathbb{R}^{p|q}$ can be given in terms of coordinates.

**Example 1.2.** Let $X^A = (x^a, \eta^\alpha)$ be coordinates on $\mathbb{R}^{2|2}$. Any map $\varphi : \mathbb{R}^{2|2} \to \mathbb{R}$ is determined by the pullback of the coordinate $r$ on $\mathbb{R}$:

$$\varphi^{\#} r = f_0(x) + \eta^2 \eta^1 f_{21}(x).$$

Here $f_0(x)$ and $f_{21}(x)$ are smooth functions depending only on $x^a$. Note that there is no term proportional to $\eta^\alpha$ because the ring homomorphisms $\varphi^{\#}$ preserve automatically the $\mathbb{Z}_2$-parity of the super functions.

For the applications we have in mind the full Taylor expansion is required. Therefore we need to work with families of super manifolds.

**Definition 1.3 ([18]).** A submersion $p_M \colon M \to B$ of super manifolds is also called a family of super manifolds over $B$. A morphism $f$ of families of super manifolds from $p_M \colon M \to B$ to $p_N \colon N \to B$ is a morphism $f \colon M \to N$ such that $p_N \circ f = p_M$. Any super manifold is a family over $\mathbb{R}^{0|0}$. Any family is locally a projection $\mathbb{R}^{m|n} \times B \to B$. We call $m|n$ the dimension of the family.

**Example 1.4.** Consider the trivial families of super manifolds given by $\mathbb{R}^{2|2} \times B$ and $\mathbb{R} \times B$. A map $\varphi \colon \mathbb{R}^{2|2} \times B \to \mathbb{R} \times B$ of families over $B$ is now again given by the pullback of the coordinate function $r$ on $\mathbb{R}$, the map on the $B$-factor is determined by the properties of maps of families over $B$. But this time all coefficients in the coordinate expansion can appear (using the Einstein summation convention):

$$\varphi^{\#} r = f_0(x) + \eta^\mu f_\mu(x) + \eta^2 \eta^1 f_{21}(x).$$

Here $f_0(x)$, $f_\mu(x)$ and $f_{21}(x)$ are functions on $\mathbb{R}^{2|0} \times B$. For all open $U$ the ring homomorphisms $\varphi^{\#}\big|_U$ must be even. This implies that $f_0$ and $f_{21}$ are even functions, whereas the functions $f_\mu$ must be odd, but in contrast to Example 1.2 not necessarily zero.

**Lemma 1.5 (Existence of base change, e.g. [5, Remark 2.6.(v)]).** *Let $b \colon B' \to B$ a morphism of super manifolds and $p_M \colon M \to B$ a family of super manifolds over $B$. Then there exists a unique family of super manifolds*

*$p_{M'}\colon M' \to B'$ and a morphism $p\colon M' \to M$ over $b$ such that $p_M \circ p = b \circ p_{M'}$. We will also write $M' = M \times_B B'$ and say that $M' \to B'$ arises from $M \to B$ by base change.*

According to Lemma 1.5 it is not necessary to fix $B$. However $B$ is always supposed to be "big enough", see Example 1.4. Henceforth, all super manifolds and maps of super manifolds are implicitly to be understood as families of super manifolds and morphisms of families of super manifolds. In particular, also $\mathbb{R}^{m|n}$ is to be understood as the trivial family $\mathbb{R}^{m|n} \times B$.

Many geometric concepts known from smooth manifolds carry over to families of super manifolds and are functorial under base change. Examples such as tangent bundles, vector bundles, differential forms and Lie groups can be found in [5]. A construction that has no analogue in differential geometry is that of the underlying even manifold.

**Definition 1.6.** Let $M = (\|M\|, \mathcal{O}_M)$ be a family of super manifolds of dimension $m|n$ over $B$. A family of super manifolds $|M| = (\|M\|, \mathcal{O}_{|M|})$ of dimension $m|0$ together with an embedding of families of super manifolds $i\colon |M| \to M$ that is the identity on the underlying topological space is called an underlying even manifold.

**Example 1.7.** Let $M = (\|M\|, \mathcal{O}_M)$ be a super manifold over $B = \mathbb{R}^{0|0}$ of dimension $m|n$ and $\mathcal{I}_{nil} \subset \mathcal{O}_M$ be the ideal sheaf of nilpotent elements. Then the underlying even manifold is given by the reduced space $M_{red} = (\|M\|, {}^{\mathcal{O}_M}\!/_{\mathcal{I}_{nil}})$, a manifold of dimension $m$. Indeed, the canonical projection $i^\#\colon \mathcal{O}_M \to {}^{\mathcal{O}_M}\!/_{\mathcal{I}_{nil}}$ yields an embedding $i\colon M_{red} \to M$ which is the identity on the underlying topological space $\|M\|$. Any map from a reduced space to $M$ has to factor over $i$; hence the map $i$ is unique as embedding of the underlying even manifold.

Though the concept of reduced space was widely used for super manifolds, it was to our knowledge never extended to families of super manifolds in a functorial way. Here by functorial we understand that for a super manifold $M \to B$ with embedded underlying even manifold $i\colon |M| \to M$ and $b\colon B' \to B$, the manifold $|M| \times_B B'$ is the underlying even manifold for $M \times_B B'$ and $i \times_B \mathrm{id}_{B'}\colon |M| \times_B B' \to M \times_B B'$ is an embedding. However in the case of families the uniqueness of the underlying manifold is lost, as is already seen in the following example:

**Example 1.8 (Underlying even manifolds for $\mathbb{R}^{m|n}$).** Consider an embedding $i\colon \mathbb{R}^{m|0} \times B \to \mathbb{R}^{m|n} \times B$ such that $\|i\|$ is the identity. Denote the standard coordinates on $\mathbb{R}^{m|0}$ by $y^a$ and the standard coordinates on $\mathbb{R}^{m|n}$ by $(x^b, \eta^\beta)$. Then $i$ can be expressed in coordinates:

$$i^\# x^b = g^b(y), \qquad i^\# \eta^\beta = g^\beta(y).$$

If one chooses coordinates $L^C = (l^c, \lambda^\gamma)$ on $B$ one can expand the even function $g^b(y)$ further

$$i^\# x^b = g^b(y) = y^b + \sum_{\underline{\nu} \neq 0} \lambda^{\underline{\nu}} g_{\underline{\nu}}{}^b(y, l).$$

Here the zero order term is given by the fact that $\|i\|$ should be the identity. However, the functions $g_{\underline{\nu}}{}^b(y, l)$ and $g^\beta(y) = g^\beta(y, l, \lambda)$ are arbitrary (with the sole exception of prescribed parity). Consequently, underlying even manifolds of the trivial family $\mathbb{R}^{m|n} \times B$ are not at all unique if $B \neq B_{red}$, in contrast to the reduced space.

It is always possible to find coordinates $(\tilde{x}^b, \tilde{\eta}^\beta)$ on $\mathbb{R}^{m|n}$ and coordinates $\tilde{y}^a$ on $\mathbb{R}^{m|0}$ such that

$$i^\# \tilde{x}^a = \tilde{y}^a, \qquad i^\# \tilde{\eta}^\beta = 0. \tag{1.9}$$

Indeed, using the coordinate transformation

$$\tilde{y}^a = y^b + \sum_{\underline{\nu} \neq 0} \lambda^{\underline{\nu}} g_{\underline{\nu}}{}^b(y, l)$$

on $\mathbb{R}^{m|0} \times B$ and the coordinate change

$$\tilde{x}^b = x^b \qquad \tilde{\eta}^\beta = -g^\beta(x) + \eta^\beta$$

on $\mathbb{R}^{m|n} \times B$ assures the Equations (1.9). Put differently, there are diffeomorphisms $\xi \in \mathrm{Diff}_B(\mathbb{R}^{m|0} \times B)$ and $\Xi \in \mathrm{Diff}_B(\mathbb{R}^{m|n} \times B)$ such that $\Xi \circ i \circ \xi$ coincides with the standard underlying even manifold of $\mathbb{R}^{m|n} \times B$ given by Equation (1.9).

There are automorphisms $\Xi$ of $\mathbb{R}^{m|n} \times B$ such that $i \circ \Xi = i$. Those can best be expressed in the coordinates $\tilde{x}^b, \tilde{\eta}^\beta$ as

$$\Xi^\# \tilde{x}^b = \tilde{x}^b + \tilde{\eta}^\mu f_\mu^b(\tilde{x}, \tilde{\eta}), \qquad \Xi^\# \tilde{\eta}^\beta = \tilde{\eta}^\mu f_\mu^\beta(\tilde{x}, \tilde{\eta}).$$

The functions $f_\mu^B$ are arbitrary functions on $\mathbb{R}^{m|n} \times B$ with appropriate parity.

**Theorem 1.10 (Existence of underlying even manifolds).** *Let* $M = (\|M\|, \mathcal{O}_M)$ *be a family of super manifolds over* $B$*. Also, let* $\|U_1\| \subseteq \|M\|$ *be a subset (which might also be empty) such that there is an underlying even manifold* $|U_1|$ *with given embedding* $i_U\colon |U_1| \to U_1$ *and* $\|U_2\| \subset \|U_1\|$ *an open subset such that its closure is contained in* $U_1$*. There exists an underlying manifold* $|M|$ *and an embedding* $i\colon |M| \to M$ *such that* $|U_1|$ *coincides with* $|M|$ *and* $i$ *with* $i_U$ *over* $\|U_2\|$*.*

*Proof.* Let $V_k$ be an open cover of the family $b_M\colon M \to B$ by adapted coordinate charts $V_k$. As $M$ is paracompact, we may assume that $V_k$ is a countable cover, hence $k = 1, \ldots$. Let us write $V_k = F_k \times b_M(V_k)$ with coordinates $X_k^A = (x_k^a, \eta_k^\alpha)$ on $F_k$. We will denote the coordinate changes as follows:

$$f_{kl}^{\#} X_l^A = f_{kl}^A(X_k) = \sum_{\underline{\nu}} \eta_k^{\underline{\nu}} f_{\underline{\nu}kl}^A(x_k).$$

Here the sum runs over all $\mathbb{Z}_2$-multi-indices $\underline{\nu}$ including zero. The manifold $|M|$ that we are going to construct is covered by the same open sets $\|V_k\| = \|F_k\| \times \|b_M(V_k)\|$ and have adapted coordinates $y_k^a$ such that $(y_k^a)_{red} = (x_k^a)_{red}$. Notice that the coordinate changes $h_{kl}^{\#} y_l^a = h_{kl}^a(y_k)$ need to be constructed in the proof.

We construct a family $b_{|M|}\colon |M| \to B$ of relative dimension $m|0$ and a map $i\colon |M| \to M$ over $B$ inductively. To start the induction we may assume without loss of generality that $U_1$ is covered by the first $j$ open sets, that is

$$U_1 = \bigcup_{k=1}^{j} V_k.$$

Furthermore, we assume that

$$U_2 \cap \bigcup_{k>j} V_k = \emptyset.$$

If $U_1 = \emptyset$ choose an arbitrary embedding $i|_{V_1}\colon |V_1| \to V_1$ over $b_M(V_1)$ as in Example 1.8.

Suppose now that we have the structure of an underlying even manifold together with the embedding $i$ for $\bigcup_{k=0}^{m-1} V_k$. We assume that $i$ is given in the coordinates $X_k^A$ and $y_k^a$ by

$$i^{\#} x_k^a = y_k^a + g_k^a(y_k), \qquad i^{\#} \eta_k^\alpha = g_k^\alpha(y_k),$$

where $g_k^a$ are even nilpotent functions and $g_k^\alpha$ odd nilpotent functions. We will show that we can extend the underlying even manifold structure and the embedding $i$ to $\bigcup_{k=0}^m V_k$. In order to extend the manifold structure we have to give the coordinate changes $h_{km}$. For $g_m^A$ to describe an extension of the given $i$ we need that the following compatibility conditions hold on $V_k \cap V_m$ for all $k < m$:

$$i^\# f_{km}^\# X_m^A = h_{km}^\# i^\# X_m^A \tag{1.11}$$

By what has been discussed in Example 1.8, we may assume that $g_k^A = 0$ for all $k < m$. Hence the compatibility conditions (1.11) read

$$\begin{aligned} i^\# f_{km}^\# x_m^a &= h_{km}^\# i^\# x_m^a = h_{km}^\# \left(y_m^a + g_m^a(y_m)\right), \\ i^\# f_{km}^\# \eta^\alpha &= h_{km}^\# i^\# \eta^\alpha = h_{km}^\# g_m^\alpha(y_m). \end{aligned}$$

For $g_m^a = 0$ the first equation can be read as a definition of $h_{km}$, whereas the second equation specifies $g_m^\alpha$ on $V_m \cap V_k$. However, the function $g_m^\alpha$ may not extend to the whole of $V_m$ because it may be unbounded. Let $\{\sigma, \tau\}$ be a partition of unity subordinate to $\{\bigcup_{k=0}^{m-1} V_k, V_m\}$, see [3, Proposition 4.2.7]. The function $t_m^\alpha = \sigma g_m^\alpha$ defined on the set $V_m \cap \bigcup_{k=0}^{m-1} V_k$ can be extended to $V_m$ by zero. We will now construct $\tilde{h}_{kl}$ and an embedding $j$ that coincide with $h_{kl}$ and $i$ respectively on $\bigcup_{k=0}^{m-1} V_k \setminus V_m$ such that

$$j^\# x_m^a = y_m^a, \qquad j^\# \eta_m^\alpha = t_m^\alpha. \tag{1.12}$$

Hence the manifold structure and the embedding $j$ extend to $\bigcup_{k=0}^m V_k$.

Let $j$ be in the coordinates $X_k^A$ be given by

$$j^\# x_k^a = y_k^a, \qquad j^\# \eta_k^\alpha = \tilde{g}_k^\alpha(y_k).$$

The coordinate changes $\tilde{h}_{kl}$ are then determined by the compatibility conditions (1.11):

$$\sum_{\underline{\nu}} \tilde{g}_k^{\underline{\nu}}(y_k) f_{\underline{\nu}kl}^a(y_k) = j^\# f_{kl}^\# x^a = \tilde{h}_{kl}^\# j^\# x^a = \tilde{h}_{kl}^\# y^a$$

Notice that $\tilde{h}_{kl}$ differs from $h_{kl}$ only by a nilpotent term dependent on $g_k^\alpha$. Furthermore, the functions $\tilde{h}_{kl}$ satisfy the cocycle conditions because $f_{kl}$

satisfy them:

$$\tilde{h}_{kl}^{\#}\tilde{h}_{lp}^{\#}y_p^a = \tilde{h}_{kl}j^{\#}f_{lp}^{\#}x_p^a = j^{\#}f_{kl}^{\#}f_{lp}^{\#}x_p^a = j^{\#}f_{kp}^{\#}x_p^a = \tilde{h}_{kp}^{\#}y_p^a$$

It remains to see that Equation (1.12) determines $\tilde{g}_k^\alpha$ for $k < m$ uniquely. We have to expand $\tilde{h}_{km}^{\#}\left(j^{\#}\eta_m^\alpha - t_m^\alpha\right)$ with respect to coordinates $L_k^A = (l_k^a, \lambda_k^\alpha)$ of the base:

$$\begin{aligned}(1.13)\ 0 = \tilde{h}_{km}^{\#}\left(j^{\#}\eta_m^\alpha - t_m^\alpha\right) &= j^{\#}f_{km}^{\#}\eta_m^\alpha - \tilde{h}_{km}^{\#}t_m^\alpha = \sum_{\underline{\nu}}\tilde{g}_k^{\underline{\nu}}f_{\underline{\nu}km}^\alpha - \tilde{h}_{km}^{\#}t_m^\alpha\\
&= \sum_{\underline{\kappa}\neq 0}\lambda^{\underline{\kappa}}\left((f_{0km}^\alpha)_{\underline{\kappa}} + (\tilde{g}_k{}^{\nu})_{\underline{\kappa}}(f_{\nu km}^\alpha)_0 - (\sigma f_{0km}^\alpha)_{\underline{\kappa}} + R_{\underline{\kappa}}{}^{\alpha}\right)\\
&= \sum_{\underline{\kappa}\neq 0}\lambda^{\underline{\kappa}}\left((\tilde{g}_k{}^{\nu})_{\underline{\kappa}}(f_{\nu km}^\alpha)_0 + (\tau f_{0km}^\alpha)_{\underline{\kappa}} + R_{\underline{\kappa}}{}^{\alpha}\right)\end{aligned}$$

Here the additional bracket and indices around $f_{0km}^\alpha$, $\tilde{g}_k^\nu$ and $\sigma$ indicate the $\lambda$-dependence. The term $R_{\underline{\kappa}}{}^{\alpha}$ contains all terms containing $\tilde{g}_{\underline{\pi}k}{}^{\alpha}$ of order lower than $\underline{\kappa}$. The matrix $(f_{\nu km}^\alpha)_0$ is invertible because the coordinate change $f_{km}$ is invertible. Hence the Equation (1.13) is solvable by recursion. The support of $\tilde{g}_k^\alpha$ is contained in the support of $\tau$. Consequently $j$ and $\tilde{h}_{kl}$ coincide with $i$ and $h_{kl}$ outside of $V_m$. □

**Remark 1.14.** Let $N$ and $N'$ be families of super manifolds with odd dimension zero. It is shown in [15, Proposition 3.3.11] that any diffeomorphism $N_{red} \to N'_{red}$ can be extended to a diffeomorphism $N \to N'$. Consequently, for two embeddings of underlying even manifolds $i_1\colon |M|_1 \to M$ and $i_2\colon |M|_2 \to M$ we have that $|M|_1$ and $|M|_2$ are diffeomorphic. Furthermore, for any diffeomorphism $\Xi\colon M \to M$ and $i\colon |M| \to M$ there exists a diffeomorphism $\xi\colon |M| \to |M|$ and an embedding of the underlying even manifold $j\colon |M| \to M$ such that

$$\Xi \circ i = j \circ \xi.$$

The theory of integration for families of super manifolds is sketched in [5, §3.10], building upon the framework of [18]. For fiberwise compact, oriented families of super manifolds integration is an $\mathcal{O}_B$-linear functional

$$\int_M\colon \operatorname{Ber} T^\vee M \to \mathcal{O}_B$$

from the Berezinian of the cotangent bundle to the functions on $B$. The Berezinian is the generalization of the determinant bundle to super geometry.

Integration is given in local coordinates $(x^a, \eta^\alpha)$ by

$$\int_{\mathbb{R}^{m|n}} g(x,\eta)[dx^1 \cdots dx^m \, d\eta^1 \cdots d\eta^n] = \int_{\mathbb{R}^{m|0}} g_{top}(x) \, dx^1 \cdots dx^m$$

where $g_{top}$ is the $B$-dependent coefficient of $\eta^n \cdots \eta^1$ of in the coordinate expansion of $g(x,\eta)$. The integral on the right-hand side is defined by $\mathcal{O}_B$-linearity and the classical integral over the fibers of $\mathbb{R}^m \times B \to B$. The local expressions are then shown to be coordinate invariant and glued to a global expression on a super manifold with the help of a partition of unity.

An alternative definition of the Berezin integral for $B = \{pt\}$ is given in [11, 20]. There the Berezin integral is reduced to an integral over $M_{red}$ via the unique embedding $i\colon M_{red} \to M$ of Example 1.7 as follows: Denote by $P = \Omega^m(M) \otimes \mathrm{Der}^n_{\mathbb{R}}(\mathcal{O}_M)$ the $\mathbb{R}$-linear differential operators of order at most $n$ with values in $m$-forms on $M$. Here $M$ is of dimension $m|n$ and $\mathrm{Der}^n_{\mathbb{R}}(\mathcal{O}_M)$ is considered as a right module over $\mathcal{O}_M$. For any function $f \in \mathcal{O}_M$ and any $Q \in P$ we denote by $Q[f]$ the differential form obtained by applying the differential operator to $f$. Define an $\mathbb{R}$-linear functional on $P$ by

$$L(Q) = \int_{M_{red}} i^* \left(Q[1]\right).$$

Here $i^* \left(Q[1]\right)$ denotes the top-form on $M$ obtained from $Q[1]$ by pullback along the embedding $i$.

Let $K \subset P$ be the $\mathcal{O}_M$-right module containing all $Q \in P$ such that $i^*Q[f]$ is exact for all $f \in \mathcal{O}_M$. Obviously $K \subset \ker L$. In [11, Theorem 2.2] it has been shown that

$$\mathrm{Ber}\, T^\vee M = {}^{P}\!/_{K}. \tag{1.15}$$

According to [11, 20], for any pre-image $Q \in P$ of $b \in \mathrm{Ber}\, T^\vee M$ it holds that

$$L(Q) = \int_M b. \tag{1.16}$$

In the statements (1.15) and (1.16) the $\mathbb{R}$-linearity can be replaced by $\mathcal{O}_B$-linearity for general $B$. Therefore, $M_{red}$ can be replaced by $|M|$ which proves the following statement:

**Proposition 1.17.** *Let $i\colon |M| \to M$ be the embedding of an underlying even manifold for a family $M$ of fiberwise compact, orientable super manifolds* *over $B$. For any section $b$ of* $\operatorname{Ber} T^\vee M$ *there exists a top form $|b|$ on $|M|$ such that*

$$\int_M b = \int_{|M|} |b|.$$

*Here the integrand $|b|$ coincides with $i^*Q[1]$ up to a global exact term and depends on the embedding $i\colon |M| \to M$.*

## 2. Super Riemann surfaces

**Definition 2.1 (see [17]).** A super Riemann surface is a $1|1$-dimensional complex super manifold $M$ with a $0|1$-dimensional distribution $\mathcal{D} \subset TM$ such that the commutator of vector fields induces an isomorphism

$$\frac{1}{2}[\cdot,\cdot]\colon \mathcal{D} \otimes_{\mathbb{C}} \mathcal{D} \to {}^{TM}\!/_{\mathcal{D}}.$$

**Example 2.2.** Let $(z,\theta)$ be the standard coordinates on $\mathbb{C}^{1|1}$ and define $\mathcal{D} \subset T\mathbb{C}^{1|1}$ by $\mathcal{D} = \langle \partial_\theta + \theta\partial_z \rangle$. The isomorphism $\mathcal{D} \otimes \mathcal{D} \simeq {}^{TM}\!/_{\mathcal{D}}$ is explicitly given by

$$[\partial_\theta + \theta\partial_z, \partial_\theta + \theta\partial_z] = 2\partial_z.$$

This example is generic since any super Riemann surface is locally of this form, see [17, Lemma 1.2].

**Theorem 2.3 ([9]).** *A super Riemann surface is a $2|2$-dimensional real super manifold with a reduction of the structure group to*

$$G = \left\{ \begin{pmatrix} A^2 & B \\ 0 & A \end{pmatrix} \middle| \, A, B \in \mathbb{C} \right\} \subset \mathrm{GL}_{\mathbb{C}}(1|1) \subset \mathrm{GL}_{\mathbb{R}}(2|2)$$

*together with the following integrability conditions. Remember that $\mathbb{C}$ is to be understood as the trivial family $\mathbb{C} \times B$. Denote the $G$-frames by $F_z$ and $F_+$. Their decomposition in real and imaginary part yields frames $F_a, F_\alpha$ for $a = 1,2, \alpha = 3,4$ as follows:*

$$\begin{aligned} F_z &= \frac{1}{2}\left(F_1 - \mathrm{i}F_2\right), & F_+ &= \frac{1}{2}\left(F_3 - \mathrm{i}F_4\right), \\ F_{\overline{z}} &= \overline{F_z}, & F_- &= \overline{F_+}. \end{aligned}$$

*Let us denote the structure coefficients by* $t^{C}_{AB}$*:*

$$[F_A, F_B] = t^{C}_{AB} F_C.$$

*Then the integrability conditions in terms of the complex frames are given by the following* $G$*-invariant equations:*

$$\begin{gathered} t^{\bar z}_{z+} = t^{-}_{z+} = t^{\bar z}_{++} = t^{-}_{++} = t^{z}_{+-} = t^{z}_{+\bar z} = 0, \\ t^{z}_{++} = 2. \end{gathered} \tag{2.4}$$

*The vanishing of the first four structure coefficients guarantees an integrable holomorphic structure and the vanishing of the last two that* $\mathcal{D}$ *is a holomorphic distribution. Furthermore,* $t^{z}_{++} = 2$ *gives the complete non-integrability of* $\mathcal{D}$*.*

Theorem 2.3 leads to two observations. First, since the orthonormal group $\mathrm{O}(2|2)$ is not contained in $G$ it is not possible to describe the geometry of super Riemann surfaces in terms of super Riemannian metrics on $M$. Second, a further reduction to $\mathrm{U}(1)$ is always possible via

$$\begin{aligned} \mathrm{U}(1) &\to G \\ U &\mapsto \begin{pmatrix} U^2 & 0 \\ 0 & U \end{pmatrix}. \end{aligned}$$

Consider now such a $\mathrm{U}(1)$-structure on $M$. It induces a non-degenerate, super symmetric bilinear form $m$ on $TM$, given in the $\mathrm{U}(1)$-frames by

$$m(F_a, F_b) = \delta_{ab}, \qquad m(F_a, F_\beta) = 0, \qquad m(F_\alpha, F_\beta) = \varepsilon_{\alpha\beta}.$$

The projector on $\mathcal{D}$ gives a splitting of the following short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{D} \arrow[r] & TM = \mathcal{D}^{\perp} \oplus \mathcal{D} \arrow[r] & TM/\mathcal{D} \arrow[r] \arrow[l, bend right, "p"'] & 0. \end{tikzcd} \tag{2.5}$$

The pullback of the short exact sequence (2.5) along an embedding $i\colon |M| \to M$

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] & i^{*}TM \arrow[r] & T|M| \arrow[r] \arrow[l, bend right, "\tilde p"'] \arrow[l, bend left, "di"] & 0 \end{tikzcd}$$

possesses a second splitting given by $di$.

**Definition 2.6 (Metric $g$, spinor bundle $S$ and gravitino $\chi$).** By the identification $T|M| = i^*\mathcal{D}^\perp$, the tangent bundle of $|M|$ gets equipped with a metric $g$.

The bundle $S = i^*\mathcal{D}$ is a spinor bundle of the metric $g$ because $i^*\mathcal{D} \otimes_{\mathbb{C}} i^*\mathcal{D} = i^*\left({}^{TM}\!/_{\mathcal{D}}\right) = T|M|$. The identification $S = i^*\mathcal{D}$ induces a non degenerate bilinear form $g_S$ on $S$ that is given in the frames $s_\alpha = i^*F_\alpha$ by

$$g_S(s_\alpha, s_\beta) = \varepsilon_{\alpha\beta}.$$

The difference of the splittings $\tilde{p}$ and $di$ is a section of $T^\vee|M| \otimes S$ which we call gravitino $\chi$:

$$\chi(v) = p_S\,(\tilde{p} - di)\,v \tag{2.7}$$

Here $p_S\colon i^*TM \to S$ is the projector given by the splitting of the short exact sequence by $\tilde{p}$.

Keep in mind that the vector bundle $S$ is of real rank $0|2$ and the frames $s_\alpha$ are odd. Consequently, the coefficients of $\chi(f_b) = \chi_b^\alpha s_\alpha$ are odd functions on $|M|$. Also notice that in general the embedding $i\colon |M| \to M$ is not holomorphic with respect to the complex structure on $|M|$ induced by $g$ (cf. the construction in the proof of the Theorem 1.10).

Different choices of U(1)-structure lead to metrics and gravitinos which differ from $g$ and $\chi$ only by a conformal and super Weyl transformation. Every matrix of $G$ can be decomposed as

$$\begin{pmatrix} A^2 & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} U^2 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} R^2 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \tag{2.8}$$

where $U \in \mathrm{U}(1)$, $R \in \mathbb{R}^+$ and $T \in \mathbb{C}$. The first matrix preserves the U(1)-structure on $M$. Consequently the bilinear forms $m$, $g$ and $g_S$ are preserved. The second matrix in the decomposition (2.8) rescales the frames $F_A$ and changes the U(1) structure. As a result the bilinear form $g$ is rescaled by $i^\# R^2$ and $g_S$ is rescaled by $i^\# R$. The third matrix in the decomposition (2.8) changes the splitting $TM = \mathcal{D} \oplus \mathcal{D}^\perp$. It is easy to see that the induced change on $\chi$ is indeed a super Weyl transformation. However only the functions $i^\# U$, $i^\# R$ and $i^\# T$ effect the metric $g$ and the gravitino. The higher order terms of $R$ and $T$ leave $g$ and $\chi$ invariant.

Having constructed a metric and gravitino on a 2-dimensional surface $|M|$ from a super Riemann surface $M$, we now consider the opposite question. Given a $2|0$-dimensional manifold $|M|$ and a metric $g$ and a gravitino $\chi$, is there a unique super Riemann surface $M$ with an embedding $i\colon |M| \to M$ such that the above construction gives the same metric and gravitino back? In order to affirmatively answer the question, one has to take into account all geometrical degrees of freedom on $M$ that are not fixed by the metric $g$ and the gravitino $\chi$ on $|M|$. An example for such geometrical degree of freedom is given by the higher order terms in the decomposition (2.8).

**Definition 2.9 (Wess–Zumino frames).** A $G$-frame $F_A$ is called Wess–Zumino frame if the following commutator relations hold in addition to the integrability conditions (2.4):

$$i^{\#} t_{+-}^{+} = 0, \qquad i^{\#} F_{+} t_{+-}^{+} = 0, \qquad t_{++}^{+} = 0. \tag{2.10}$$

**Lemma 2.11.** *Let $F_A$ be a* U(1)*-frame. There is a unique Wess–Zumino frame $\tilde{F}_A$ in the same $G$-class such that $i^* F_A = i^* \tilde{F}_A$.*

*Proof.* Apply a transformation $h \in G$ to $F_A$ such that $i^* h = \mathrm{id}$. The conditions (2.10) fix the higher order terms of $U$, $R$ and $T$ in (2.8). ☐

**Definition 2.12 (Wess–Zumino coordinates).** The coordinates $X^A = (x^a, \eta^\alpha)$ are called Wess–Zumino coordinates of the frame $F_A$ if $i^{\#}\eta^\alpha = 0$ and the coordinate expression of the frame $F_\alpha$ is given by

$$F_\alpha = \left(\eta^\mu {F_{\mu\alpha}}^b(x) + \eta^2\eta^1 \cdots\right) \partial_{x^b} + \left(\delta_\alpha^\beta + \eta^\mu {F_{\mu\alpha}}^\beta + \eta^2\eta^1 \cdots\right) \partial_{\eta^\beta}. \tag{2.13}$$

Here the degree one coefficients are symmetric with respect to the lower indices, i.e.

$$\varepsilon^{\mu\alpha} {F_{\mu\alpha}}^C = 0.$$

**Lemma 2.14.** *Given a $G$-frame $F_A$ and coordinates $\tilde{X}^A = (\tilde{x}^a, \tilde{\eta}^\alpha)$ there are unique Wess–Zumino coordinates $X^A = (x^a, \eta^\alpha)$ for $F_A$ such that $i^{\#}\tilde{x}^a = i^{\#}x^a$.*

The notions of “Wess–Zumino frames” and “Wess–Zumino coordinates” are derived from the notion of “Wess–Zumino gauge” used in [7]. They have at least two purposes. The first one is that they reduce the freedom in the local description of super Riemann surfaces. Instead of all super coordinate systems and all $G$-frames, we now only need to consider the Wess–Zumino frames and Wess–Zumino coordinates. As was shown in Lemma 2.11 and Lemma 2.14 they are unique up to a choice of $i^{\#}x^a$ and $i^* F_\alpha$. Second they relate the odd coordinates on $M$ to spinors on $|M|$, as the frames $s_\alpha = i^*F_\alpha = i^*\partial_{\eta^\alpha}$ are frames for $S$.

Let now $F_A$ be a U(1)-frame on $U \subset M$. Consider the coordinate expansion in Wess–Zumino coordinates $X^A = (x^a, \eta^\alpha)$ for $F_A$.

$$\begin{aligned} F_a &= \left({F_{0a}}^b + \eta^\mu {F_{\mu a}}^b + \eta^2\eta^1 {F_{21a}}^b\right)\partial_{x^b} \\ &\quad + \left({F_{0a}}^\beta + \eta^\mu {F_{\mu a}}^\beta + \eta^2\eta^1 {F_{21a}}^\beta\right)\partial_{\eta^\beta} \\ F_\alpha &= \left(\eta^\mu {F_{\mu\alpha}}^b + \eta^2\eta^1 {F_{21\alpha}}^b\right)\partial_{x^b} \\ &\quad + \left(\delta_\alpha^\beta + \eta^\mu {F_{\mu\alpha}}^\beta + \eta^2\eta^1 {F_{21\alpha}}^\beta\right)\partial_{\eta^\beta} \end{aligned} \tag{2.15}$$

The frames $s_\alpha = i^*F_\alpha$ are U(1)-frames for $S$. Furthermore the frame $F_a$ can be expanded

$$i^*F_a = {F_{0a}}^b i^*\partial_{x^b} + {F_{0a}}^\beta s_\beta$$

Then by formula (2.7) we know that $f_a = {F_{0a}}^b\partial_{y^b}$ is a $g$-orthonormal frame and the gravitino is given by

$$\chi(f_a) = {F_{0a}}^\beta s_\beta$$

To complete the local description of super Riemann surfaces in terms of metrics and gravitinos we still need the following lemma, which has been mentioned first in [12] under stronger assumptions:

**Lemma 2.16.** *Let $F_A$ be a Wess–Zumino frame and $X^A = (x^a, \eta^\alpha)$ Wess–Zumino coordinates for $F_A$. All higher order coefficients in* (2.15) *can be expressed in terms of ${F_{0a}}^b$ and ${F_{0a}}^\beta$ and thus in terms of $f_a$ and $\chi$.*

*Proof.* The Equations (2.4), (2.13) and (2.10) are solvable for the unknown coefficient functions. ☐

**Theorem 2.17.** *Given a super manifold $|M|$ over $B$ together with a metric $g$, a spinor bundle $S$ and a gravitino field $\chi$. Then there is a unique super Riemann surface $M$ over $B$ together with an inclusion $i\colon |M| \to M$ such that the above procedure gives back the gravitino and metric up to conformal transformation of $g$ and super Weyl transformation of the gravitino $\chi$.*

*Proof.* Cover $|M|$ by open coordinate sets $(V, y^a)$. Choose a local U(1)-frame $s_\alpha$ of $S$ and $f_a$ of $T|M|$ such that $s_+ \otimes_{\mathbb{C}} s_+ \mapsto f_z$. Construct over the topological space $V$ the super manifold $(V, \mathcal{O}_V)$ by setting $\mathcal{O}_V = \Lambda(\Gamma_V(S^\vee))$ with coordinates $x^a = y^a$ and $\eta^\alpha = s^\alpha$, where $s^\alpha$ is the canonical dual basis to $s_\alpha$. Denote by $F_A$ the Wess–Zumino frame constructed from the coefficients of the frame $f_a$ and the gravitino $\chi$ according to Lemma 2.16. This gives an integrable $G$-reduction of the structure group of $TV$. The map $i$ is locally constructed via its action on the coordinates $(x^a, \eta^\alpha)$, i.e. $i^\# x^a = y^a$ and $i^\# \eta^\alpha = 0$.

It remains to glue different local constructions in order to obtain a well defined super Riemann surface over the same topological space $\|M\|$. The Wess–Zumino frames over different trivializing covers may differ by a $G$-transformation that reduces to U(1)-transformation on $|M|$. The Wess–Zumino coordinates of Wess–Zumino frames that differ by such a transformation are completely fixed by the U(1)-transformation. Details may be found in [15]. ☐

We have shown a one-to-one correspondence

$$\{i\colon |M| \to M, M \text{ super Riemann surface}\} \longleftrightarrow {\{|M|, S, g, \chi\}}\big/_{\text{Weyl, SWeyl}}$$

An advantage of this description is that on the right-hand side there are no integrability conditions to be fulfilled. On the left hand side the integrability conditions (2.4) have to be fulfilled. The presence of the integrability conditions complicates the study of deformations as one needs to assure the integrability of the deformations.

To obtain a description of the moduli space of super Riemann surfaces in terms of metrics and gravitinos one may look for a one to one correspondence (see e.g. [13] and references therein)

$$\begin{aligned} &{\{M \text{ super Riemann surface}\}}\big/_{\mathrm{SDiff}(M)} \\ &\longleftrightarrow {\{|M|, S, g, \chi\}}\big/_{\text{Weyl, SWeyl}, \mathrm{Diff}(|M|), \text{SUSY}} \end{aligned} \tag{2.18}$$

The super symmetry transformations SUSY on the right hand side can conjecturally be identified with the change of embedding $i$, that is, a particular subgroup of $\mathrm{SDiff}(M)$. A precise definition of SUSY and the study of the full quotient must be left for further research. Here, we treat the infinitesimal case. As a preparation we first study the infinitesimal change of embedding.

**Proposition 2.19.** *The normal bundle to the embedding $i\colon |M| \to M$ is given by $i^*\mathcal{D} = S$. Let $i_t\colon |M|_t \to M$ be a smooth family of embeddings such* *that the infinitesimal deformation is*

$$q = \left.\frac{d}{dt}\right|_{t=0} i_t \in \Gamma_{|M|_0}(i_0^*\mathcal{D}).$$

*The derivatives of the corresponding families of local frames* $f(t)_a$ *and gravitinos* $\chi(t)_a$ *are given by*

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} f(t)_a &\simeq -2\langle\gamma^b q, \chi(f_a)\rangle f_b,\\ \left.\frac{d}{dt}\right|_{t=0} \chi(t)_a &\simeq \nabla^S_{f_a} q = \nabla^{LC}_{f_a} q + \langle\gamma^b\chi_b, \chi_a\rangle\gamma^1\gamma^2 q.\end{aligned} \tag{2.20}$$

*Here, the symbol* $\simeq$ *denotes equality up to local* U(1)*-, conformal and super Weyl transformations.* $\nabla^{LC}$ *is the Levi-Civita connection lifted to* $S$*.*

This proposition also justifies that the field $\chi$ defined above was called gravitino, because the transformations (2.20) are the expected super symmetries. Compare [1, 7, 13]. The interpretation of super symmetry given in Proposition 2.19 as a normal variation of the underlying even manifold resembles the one given in [2]. There, super symmetry of four-dimensional super gravity is interpreted as a deformation of a local splitting of the body projection.

**Lemma 2.21.** *The gravitino can be gauged to zero locally. More precisely for every point* $m \in \|M\|$ *there exists an open neighbourhood* $U \subseteq M$ *such that there is a* U(1)*-structure and an embedding* $i\colon |M| \to M$ *such that* $\chi|_{i^{-1}(U)} = 0$*. The gravitino can be gauged to zero globally, if* $M$ *is a trivial family of super Riemann surfaces.*

*Proof.* Choose around $m$ complex coordinates $(z, \theta)$ such that $\mathcal{D} = \langle\partial_\theta + \theta\partial_z\rangle$ (see example 2.2). Let the U(1)-structure be given by the frames $F_z = \partial_z$ and $F_+ = \partial_\theta + \theta\partial_z$ and the embedding by $i^\# \theta = 0$. Then the gravitino vanishes on $U$. ☐

**Theorem 2.22.** *Let the metric* $g$*, the spinor bundle* $S$ *and the gravitino* $\chi = 0$ *on* $|M|$ *determine the super Riemann surface* $M$ *and the embedding* $i\colon |M| \to M$*. The infinitesimal deformations of* $M$ *are given by*

$$H^0(T^\vee|M| \otimes_{\mathbb{C}} T^\vee|M|) \oplus H^0(S^\vee \otimes_{\mathbb{C}} S^\vee \otimes_{\mathbb{C}} S^\vee).$$

*Here* $H^0$ *denotes holomorphic sections.*

*Proof.* Any infinitesimal deformation of $M$ is given by an infinitesimal deformation of metric and gravitino, denoted by $h$ and $\rho$ respectively. However not every infinitesimal deformation of metric and gravitino give rise to an infinitesimal deformation of the super Riemann surface. The infinitesimal deformations of the metric and gravitino induced by Weyl and super Weyl, diffeomorphisms and super symmetry do lead to equivalent super Riemann surfaces. We will thus need to decompose the infinitesimal deformation $h$ of the metric as

$$h = \lambda g + L_X g + \operatorname{susy}(q) + D \tag{2.23}$$

for some infinitesimal Weyl transformation with parameter $\lambda$, a Lie derivative along the vector field $X$ and infinitesimal super symmetry transformation $\operatorname{susy}(q)$ given by the spinor $q$ as in (2.20). The parameters $\lambda$, $X$ and $q$ need to be determined. The remaining part $D$ is a true even infinitesimal deformations of the super Riemann surface. Analogously, the infinitesimal deformation $\rho$ of the gravitino needs to be decomposed in

$$\rho = \gamma t + L_X \chi + \nabla^S q + \mathfrak{D} \tag{2.24}$$

for some spinor $t$, that rests to be determined. The remaining part $\mathfrak{D}$ is a true odd infinitesimal deformation of the super Riemann surface.

We will work in local holomorphic coordinates $z = x^1 + \mathrm{i}x^2$ defined in some open neighbourhood $U$. We consider only the special case $g_{ij} = \delta_{ij}$ and $\chi = 0$. The case of isothermal coordinates works analogously. Let $X = X^k \partial_{x^k}$. The Equation (2.23) simplifies to

$$h_{ij} = \lambda \delta_{ij} + \left(\partial_{x^i} X^k\right) \delta_{kj} + \left(\partial_{x^j} X^k\right) \delta_{ki} + D_{ij}.$$

Letting

$$\lambda = \frac{1}{2} h_{ij} \delta^{ij} - \left(\partial_{x^k} X^k\right)$$

it is possible to assume $D_{ij}$ is symmetric and trace free. As a consequence, the bilinear form $D$ can be identified with a section of $T^\vee|M| \otimes_{\mathbb{C}} T^\vee|M|$:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mapsto (a - \mathrm{i}b)\, dz \otimes dz$$

It is possible to choose the vector field $X$ such that $D$ is a holomorphic quadratic differential. The holomorphicity condition for $D$ is equivalent to the following Laplace equations for $X^k$:

$$\begin{aligned} 0 = \partial_{x^1} a + \partial_{x^2} b &= \frac{1}{2}\partial_{x^1}\left(h_{11} - h_{22}\right) + \partial_{x^2} h_{12} - \partial_{x^1}^2 X^1 - \partial_{x^2}^2 X^1 \\ 0 = -\partial_{x^1} b + \partial_{x^2} a &= \frac{1}{2}\partial_{x^2}\left(h_{11} - h_{22}\right) - \partial_{x^1} h_{12} + \partial_{x^1}^2 X^2 + \partial_{x^2}^2 X^2 \end{aligned}$$

We have decomposed every infinitesimal deformation $h$ of the metric $g$ into an infinitesimal Weyl transformation, a Lie derivative and a holomorphic quadratic differential. The holomorphic quadratic differentials represent the true even deformations of $M$.

In an analogous manner we proceed with the deformation $\rho$ of the gravitino. It will be convenient to consider $\rho$ as a section of $T^\vee|M| \otimes S^\vee$. We choose a complex basis $s_+$ for $S$ such that $s_+ \otimes s_+ = \partial_z$ and let $s_+ = s_3 - \mathrm{i}s_4$. The corresponding dual basis will be denoted $s^+$ and $s^\alpha$ respectively. The vector bundle $T^\vee|M| \otimes S^\vee$ can be decomposed in $S^\vee \oplus S^\vee \otimes_{\mathbb{C}} S^\vee \otimes_{\mathbb{C}} S^\vee$. In the basis we use here, the spinor part of an arbitrary section $\rho$ is given by $s^\alpha \gamma^{a\beta}_{\ \ \alpha} \rho_{a\beta}$. The Equation (2.24) is given in our local coordinates by

$$\rho_{a\beta} = \delta_{ab}\gamma^{b\mu}_{\ \ \beta}\varepsilon_{\mu\nu}t^\nu - \varepsilon_{\beta\mu}\left(\partial_{x^a} q^\mu\right) + \mathfrak{D}_{a\beta}.$$

It is possible to fix the spinor $t$ such that $\mathfrak{D}$ is in $S^\vee \otimes_{\mathbb{C}} S^\vee \otimes_{\mathbb{C}} S^\vee$, i.e.

$$0 = \gamma^{a\beta}_{\ \ \alpha}\left(\rho_{a\beta} + \varepsilon_{\beta\mu}\partial_{x^a} q^\mu\right) - 2\varepsilon_{\alpha\nu}t^\nu.$$

Consequently the coefficients of $\mathfrak{D}$ fulfil

$$\mathfrak{D}_{13} + \mathfrak{D}_{24} = 0, \qquad \mathfrak{D}_{23} - \mathfrak{D}_{14} = 0.$$

The cospinor valued differential form $\mathfrak{D}$ can be identified with

$$\left(\mathfrak{D}_{13} + \mathrm{i}\mathfrak{D}_{14}\right) dz \otimes s^+.$$

The condition, for $\mathfrak{D}$ to be a holomorphic section of $S^\vee \otimes_{\mathbb{C}} S^\vee \otimes_{\mathbb{C}} S^\vee$ is given again by the Cauchy–Riemann equations

$$0 = \partial_{x^1}\mathfrak{D}_{13} - \partial_{x^2}\mathfrak{D}_{14} = \frac{1}{2}\left(\partial_{x^1}\left(\rho_{13} - \rho_{24}\right) + \partial_{x^2}\left(\rho_{14} + \rho_{23}\right)\right) + \partial_{x^1}^2 q^4 + \partial_{x^2}^2 q^4,$$

$$0 = \partial_{x^2}\mathfrak{D}_{13} + \partial_{x^1}\mathfrak{D}_{14} = \frac{1}{2}\left(\partial_{x^2}\left(\rho_{13} - \rho_{24}\right) - \partial_{x^1}\left(\rho_{14} + \rho_{23}\right)\right) - \partial_{x^1}^2 q^3 - \partial_{x^2}^2 q^3.$$

We can thus decompose the infinitesimal deformations of the gravitino in an infinitesimal super Weyl transformation, an infinitesimal super symmetry and a holomorphic section of $S^\vee \otimes_{\mathbb{C}} S^\vee \otimes_{\mathbb{C}} S^\vee$. ☐

Similar statements for trivial families can be found in [4, 17, 21]. However the version given here is more general, as it allows for certain non-trivial families. Furthermore the proof given here shows directly which deformations of metric and gravitino correspond to infinitesimal deformations of the given super Riemann surface.

The complex dimension of the infinitesimal deformation space can be calculated by the theorem of Riemann–Roch in the case of $B = \mathbb{R}^{0|0}$. The dimension was found to be $3p-3|2p-2$ for genus $p \geq 2$.

## 3. The action functional

We now turn to the action functional $(*)$. In this section we assume that $M$ is a fiberwise compact family of super Riemann surfaces with a compatible super metric $m$. Let $N$ be an arbitrary (super) manifold with Riemannian metric $n$ and Levi-Civita covariant derivative $\nabla^{TN}$. For details on Levi-Civita covariant derivatives on super manifolds see [10]. Consider a morphism $\Phi\colon M \to N$. The action

$$A(M,\Phi) = \frac{1}{2}\int_M \parallel d\Phi|_{\mathcal{D}} \parallel^2_{m^\vee|_{\mathcal{D}^\vee}\otimes\Phi^* n}[dvol_m] \tag{3.1}$$

might be seen as a generalization of the harmonic action functional to super Riemann surfaces. Remark that in contrast to the harmonic action functional the tangent map $d\Phi$ is restricted to the subbundle $\mathcal{D}$ in $TM$. Given U(1)-frames $F_A$ the action can be written as

$$A(M,\Phi) = \frac{1}{2}\int_M \varepsilon^{\alpha\beta}\langle F_\alpha\Phi, F_\beta\Phi\rangle_{\Phi^* n}[F^1F^2F^3F^4]. \tag{3.2}$$

The action (3.1) can be found in different forms in the literature, see in particular [7, 9]. In [9] one can find an explicit proof for the $G$-invariance of (3.2). Thus the action functional does not depend on the metric $m$, but rather only on the super Riemann surface structure, i.e. the $G$-structure.

**Proposition 3.3.** *The Euler–Lagrange equation of* (3.1) *for* $\Phi$ *is*

$$0 = \Delta^{\mathcal{D}}\Phi = \varepsilon^{\alpha\beta}\nabla_{F_\alpha}F_\beta\Phi + \varepsilon^{\alpha\beta}\left(\operatorname{div} F_\alpha\right)F_\beta\Phi. \tag{3.4}$$

*We will call the differential operator $\Delta^{\mathcal{D}}$, defined here, the $\mathcal{D}$-Laplace operator.*

*Proof.* Let $\Phi_t : M \times \mathbb{R} \to N$ be a perturbation of $\Phi_0 = \Phi$. One can expand $\Phi_t$ in $t$ around 0 and obtains

$$\Phi_t = \Phi_0 + t\partial_\alpha \Phi_t|_{t=0} + O(t^2).$$

Let us denote $\partial_t \Phi_t|_{t=0} = \Xi \in \Gamma(\Phi^* TN)$ and expand $A$ in $t$ around 0:

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} A(\Phi_t, F_A)\\
&= \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \int_M \varepsilon^{\alpha\beta}\langle F_\alpha \Phi_t, F_\beta \Phi_t\rangle [F^1F^2F^3F^4]\\
&= \frac{1}{2}\int_M \partial_t \varepsilon^{\alpha\beta}\langle F_\alpha \Phi_t, F_\beta \Phi_t\rangle [F^1F^2F^3F^4]\Big|_{t=0}\\
&= \int_M \varepsilon^{\alpha\beta}\langle \nabla^{\Phi_t^* TN}_{\partial_t} F_\alpha \Phi_t, F_\beta \Phi_t\rangle [F^1F^2F^3F^4]\Big|_{t=0}\\
&= \int_M \varepsilon^{\alpha\beta}\langle \nabla^{\Phi_t^* TN}_{F_\alpha} \partial_t \Phi_t, F_\beta \Phi_t\rangle [F^1F^2F^3F^4]\Big|_{t=0}\\
&= \int_M \varepsilon^{\alpha\beta}\langle \nabla^{\Phi^* TN}_{F_\alpha} \Xi, F_\beta \Phi\rangle [F^1F^2F^3F^4]\\
&= -\int_M \varepsilon^{\alpha\beta}\left(\langle \Xi, \nabla^{\Phi^* TN}_{F_\alpha} F_\beta \Phi\rangle [F^1F^2F^3F^4] - \langle \Xi, F_\beta \Phi\rangle L_{F_\alpha}[F^1F^2F^3F^4]\right)
\end{aligned}$$

With the definition of divergence

$$(\operatorname{div} F_\alpha)\,[F^1F^2F^3F^4] = L_{F_\alpha}[F^1F^2F^3F^4],$$

the result follows. Of course the Euler–Lagrange Equation (3.4) is $G$-invariant like the action (3.1). The $\mathcal{D}$-Laplace, however, is only U(1)-invariant. ☐

We now turn to the question how the action (3.1) can be represented on an underlying even manifold $i\colon |M| \to M$.

**Definition 3.5.** Let $\Phi\colon M \to N$ be a morphism and $i\colon |M| \to M$ be an underlying even manifold. We call the fields

$$\begin{aligned}
&\varphi\colon |M| \to N && \psi\colon |M| \to S^\vee \otimes \varphi^* TN && F\colon |M| \to \varphi^* TN\\
&\varphi = \Phi \circ i && \psi = s^\alpha \otimes i^* F_\alpha \Phi && F = \frac{1}{2} i^* \Delta^{\mathcal{D}} \Phi
\end{aligned}$$

component fields of $\Phi$. Recall that $s^\alpha$ is the dual basis to the basis $s_\alpha = i^*F_\alpha$ of the spinor bundle $S = i^*\mathcal{D}$ on $|M|$.

**Remark 3.6.** Suppose that $X^A = (x^a, \eta^\alpha)$ are Wess–Zumino coordinates for the Wess–Zumino frame $F_A$. Let furthermore $Y^B$ be local coordinates on $N$. The map $\Phi\colon M \to N$ is then given by the functions

$$\Phi^\# Y^B = f_0^B + \eta^\mu f_\mu^B + \eta^2\eta^1 f_{21}^B$$

It holds that $f_0^B = \varphi^\# Y^B$ because $i^\#\eta^\mu = 0$. By the properties of Wess–Zumino coordinates we have that $i^*F_\alpha = i^*\partial_{\eta^\alpha}$ and thus $f_\mu^B = \psi_\mu Y^B$. Here $\psi_\mu$ is the coefficient of $\psi$ in the basis $s^\mu$ and consequently a derivation on $\mathcal{O}_N$ with values in $\mathcal{O}_M$. If the target manifold $N = \mathbb{R}^p$ is Euclidean space one can show that $i^*\Delta^{\mathcal{D}} = 2i^*\partial_{\eta^1}\partial_{\eta^2}$. Consequently the map $\Phi$ can be written schematically as

$$\Phi = \varphi + \eta^\mu\psi_\mu + \eta^2\eta^1 F.$$

**Theorem 3.7.** *Let $M$ be a fiberwise compact family of super Riemann surfaces and $i\colon |M| \to M$ an underlying even manifold. We denote by $g$, $\chi$ and $g_S$ respectively the metric, gravitino and spinor metric on $|M|$ constructed in Section 2 for a given* U(1)*-structure on $M$. Let $\Phi\colon M \to N$ be a morphism to a Riemannian super manifold $(N, n)$ and $\varphi$, $\psi$ and $F$ its component fields, as introduced in Definition 3.5. It holds*

$$\begin{aligned}
(3.8)\quad A(M,\Phi) &= A(\varphi, g, \psi, \chi, F)\\
&= \int_{|M|} \Bigg( \|\, d\varphi\|^2_{g^\vee\otimes\varphi^*n} + \langle \psi, \not{D}\psi\rangle_{g_S^\vee\otimes\varphi^*n} - \langle F, F\rangle_{\varphi^*n}\\
&\qquad + 2\langle \chi_a\gamma^b\gamma^a\partial_{x^b}\varphi, \psi\rangle_{g_S^\vee\otimes\varphi^*n}\\
&\qquad + \frac{1}{2}\langle \chi_a, \gamma^b\gamma^a\chi_b\rangle_{g_S}\langle\psi,\psi\rangle_{g_S^\vee\otimes\varphi^*n}\\
&\qquad + \frac{1}{6}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\langle R^{\varphi^*TN}(\psi_\alpha,\psi_\gamma)\psi_\delta, \psi_\beta\rangle_{\varphi^*n}\Bigg)\, dvol_g
\end{aligned}$$

The idea for the proof of Theorem 3.7 is Proposition 1.17. One uses crucially that integration in the odd directions is locally a derivation. In Wess–Zumino coordinates $(x^a, \eta^\alpha)$ for $F_A$ a local expression for the action is given by

$$\begin{aligned}A(M,\Phi) &= \frac{1}{2}\int_M \varepsilon^{\alpha\beta}\langle F_\alpha\Phi, F_\beta\Phi\rangle_{\Phi^*n}(\operatorname{Ber}F)^{-1}[dx^1\,dx^2\,d\eta^1\,d\eta^2]\\ &= \frac{1}{2}\int_{|M|} i^*\partial_{\eta^1}\partial_{\eta^2}\left(\varepsilon^{\alpha\beta}\langle F_\alpha\Phi, F_\beta\Phi\rangle_{\Phi^*n}(\operatorname{Ber}F)^{-1}\right)dx^1\,dx^2\\ &= \frac{1}{4}\int_{|M|} i^*\varepsilon^{\mu\nu}F_\mu F_\nu\left(\varepsilon^{\alpha\beta}\langle F_\alpha\Phi, F_\beta\Phi\rangle_{\Phi^*n}(\operatorname{Ber}F)^{-1}\right)dx^1\,dx^2.\end{aligned}$$

The expansion of the last expression is given in terms of component fields of $\Phi$ (compare Definition 3.5) and commutators of $F_\alpha$ and derivatives of $\operatorname{Ber}F$. By Lemma 2.16 the coordinate expansion of $F_\alpha$, its commutators and the Berezinian are determined by $g$ and $\chi$. The full calculation can be found in [15].

It is now clear how the different symmetries of the action functional $(*)$ arise. Different U(1)-reductions of the given $G$-structure on $M$ induce metrics and gravitinos on $|M|$ that differ only by Weyl and super Weyl transformations. The action functional (3.1) is $G$-invariant and thus in turn the action functional (3.8) is conformally and super Weyl invariant. The action functional (3.1) is formulated without any reference to an embedding of an underlying even manifold, but Theorem 3.7 is. The independence of (3.1) of the embedding $i$ translates into super symmetry of (3.8).

**Proposition 3.9.** *The Euler–Lagrange equations of the action functional* (3.8) *are given by the components of the Euler–Lagrange equation of* (3.1)*:*

$$(3.10)\qquad 0 = i^*\Delta^{\mathcal{D}}\Phi \qquad 0 = s^\alpha\otimes i^*\nabla_{F_\alpha}\Delta^{\mathcal{D}}\Phi \qquad 0 = i^*\Delta^{\mathcal{D}}\Delta^{\mathcal{D}}\Phi$$

*Sketch of proof.* Schematically the infinitesimal variation $\Xi$ of $\Phi$ can be decomposed

$$\Xi = \delta\varphi + \eta^\mu\delta\psi_\mu + \eta^2\eta^1\delta F.$$

The infinitesimal variation of the action is then given by

$$\delta A = -\int_M \langle\Xi, \Delta^{\mathcal{D}}\Phi\rangle[dvol_m].$$

Integration over the odd variables selects the coefficients of highest degree in $\eta$, so that

$$\delta A = -\int_{|M|}\frac{1}{2}\langle\delta\varphi, i^*\Delta^{\mathcal{D}}\Delta^{\mathcal{D}}\Phi\rangle + \langle\delta\psi, s^\alpha\otimes i^*\nabla_{F_\alpha}\Phi\rangle + \langle\delta F, i^*\Delta^{\mathcal{D}}\Phi\rangle\,dvol_g.$$

□

By Theorem 2.3 different super Riemann surfaces are given by different $G$-structures. The functional (3.1) is $G$-invariant but different $G$-structures lead to different values of the functional. Consequently the action functional (3.1) is a functional on the moduli space of super Riemann surfaces for fixed $\Phi\colon M \to N$. The conjectured correspondence (2.18) shows that, in principle, the component action (3.8) should be defined on the moduli space of super Riemann surfaces. As explained earlier the difficulty lies in the correct definition of the right-hand side of (2.18). However, infinitesimal properties of the moduli space of super Riemann surfaces can be studied from (3.8) already.

**Proposition 3.11.** *Let $M$ be a super Riemann surface and $i\colon |M| \to M$ an underlying even manifold. By the construction in Section 2, the geometry of $M$ is determined by a metric $g$ and a gravitino $\chi$ on $|M|$. Define the energy-momentum tensor $T$ of $A(\varphi, g, \psi, \chi, F)$ by*

$$\delta_g A(\varphi, g, \psi, \chi, F) = \int_{|M|} \delta g \cdot T \, dvol_g, \tag{3.12}$$

*and the super current $J$ by*

$$\delta_\chi A(\varphi, g, \psi, \chi, F) = \int_{|M|} \delta\chi \cdot J \, dvol_g. \tag{3.13}$$

*Geometrically, the integrals* (3.12) *and* (3.13) *can be viewed as cotangent vectors of the moduli space of super Riemann surfaces at $M$.*

*Let now $\varphi$, $\psi$ and $F$ fulfil the Euler–Lagrange Equations* (3.10) *and $\chi = 0$. The energy-momentum tensor $T$ is traceless by the conformal symmetry and the Noether current associated to the diffeomorphism invariance. The super current $J$ is a section of $S^\vee \otimes_{\mathbb{C}} S^\vee \otimes_{\mathbb{C}} S^\vee$ by the super Weyl symmetry and the Noether current to super symmetry.*

*Furthermore, as the Noether currents are conserved quantities, they are divergence free. Consequently, the energy-momentum tensor $T$ is a holomorphic quadratic differential and the super current $J$ is a holomorphic section of $S^\vee \otimes_{\mathbb{C}} S^\vee \otimes_{\mathbb{C}} S^\vee$.*

For a similar argument in the case of purely commuting variables see [14].

Similar to the case of Riemann surfaces and the harmonic action functional we hope that the action functional (3.1) may be helpful to derive further results about the moduli space of super Riemann surfaces.

## Summary

We have established the relation between the super symmetric action functional $(*)$ and super Riemann surfaces. That is, we have shown that for a particular underlying even manifold $|M|$ of the super Riemann surface $M$ the integral $A(M, \Phi)$ reduces to the action functional $A(\varphi, g, \psi, \chi, F)$ on $|M|$.

The first step was to define the underlying family of even manifolds $|M| \to B$ of a family of super manifolds $M \to B$. The underlying even manifolds is in between the super manifold $M$ and the completely reduced space of $M$, as it still involves odd functions from the base $B$.

With the help of the underlying even manifold $|M|$ we were able to show that the structure of a super Riemann surface $M$ is completely determined by an underlying even manifold $|M|$ together with a metric $g$, a spinor bundle $S$ and a spinor valued differential form $\chi$, called gravitino. The redundancy in the choice of $g$, $S$ and $\chi$ could be shown to coincide with the conformal, super Weyl and super symmetry invariance of the action $A(\varphi, g, \psi, \chi, F)$. Infinitesimal deformations of the super Riemann surface can be expressed via infinitesimal deformations of $g$ and $\chi$, reproducing the classical result that even infinitesimal deformations of $M$ are given by holomorphic sections of $T^\vee M \otimes T^\vee M$, whereas odd infinitesimal deformations are given by holomorphic sections of $(S^\vee)^{\otimes 3}$.

As an outlook, the similarities of $A(M, \Phi)$ with the functional of harmonic maps on Riemann surfaces, together with the results presented in this paper, give rise to the hope that the action functional $A(M, \Phi)$ and its critical points may be useful to study the moduli space of super Riemann surfaces. On one hand, the definition of super Riemann surfaces and their moduli involve the integrability conditions (2.4). On the other hand, however, the characterization of super Riemann surfaces in terms of metrics and gravitinos is not obstructed. Due to Theorem 3.7, the action functional $A(\varphi, g, \psi, \chi, F)$ in terms of metric and gravitino is well defined on the moduli space of super Riemann surfaces.

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