

# On the algebraic structure of Killing superalgebras

JOSÉ FIGUEROA-O'FARRILL AND ANDREA SANTI

We study the algebraic structure of the Killing superalgebra of a supersymmetric background of 11-dimensional supergravity and show that it is isomorphic to a filtered deformation of a  $\mathbb{Z}$ -graded subalgebra of the Poincaré superalgebra. We are able to map the classification problem for highly supersymmetric backgrounds (i.e., those which preserve more than half the supersymmetry) to the classification problem of a certain class of filtered deformations of graded subalgebras of the Poincaré superalgebra. We show that one can reconstruct a highly supersymmetric background from its Killing superalgebra; in so doing, we relate the bosonic field equations of 11-dimensional supergravity to the Jacobi identity of the Killing superalgebra and show in this way that preserving more than half the supersymmetry implies the bosonic field equations.

<b>1</b>	<b>Introduction</b>	<b>1116</b>
<b>2</b>	<b>Preliminaries on filtered deformations</b>	<b>1121</b>
<b>3</b>	<b>The Killing superalgebra as a filtered deformation</b>	<b>1128</b>
<b>4</b>	<b>The classification problem for Killing superalgebras</b>	<b>1138</b>
<b>5</b>	<b>Towards the field equations</b>	<b>1145</b>
<b>6</b>	<b>Summary and conclusions</b>	<b>1154</b>
	<b>Acknowledgments</b>	<b>1155</b>
	<b>References</b>	<b>1155</b>

## 1. Introduction

Arguably the most interesting open problem in eleven-dimensional supergravity is the classification of (supersymmetric, bosonic) backgrounds. This problem has a long pedigree. It started in the 1980s, where it took the form of the classification problem for Freund–Rubin backgrounds (and generalisations thereof) in the context of Kaluza–Klein supergravity. The substantial progress made during this time is fairly well documented in the review [1]. One problem with Freund–Rubin backgrounds from a Kaluza–Klein perspective is that the spacetime and the compact extra dimensions have commensurate radii of curvature, but they resurfaced in the 1990s as near-horizon geometries of branes, which is perhaps their most popular interpretation today. The advent in the mid-1990s of the “branes and duality” paradigm led to a renewed effort in the study of supersymmetric backgrounds. Many such constructions emerged, but by the end of the decade there was still no systematic approach to the classification. Since the definition of a supersymmetric background entails the existence of Killing spinors, which are parallel with respect to a connection on the spinor bundle, an obvious approach is via the study of the holonomy of that connection. A first step in that direction was taken in [2], which studied purely gravitational supersymmetric backgrounds in terms of the possible lorentzian holonomy groups of eleven-dimensional manifolds admitting parallel spinors, but it was not clear how to re-introduce the flux in that approach. Indeed, since the connection with nonzero flux is not induced from a connection on the spin bundle, there are no theorems concerning the possible holonomy groups, except that the generic (restricted) holonomy group is  $\mathrm{SL}(32, \mathbb{R})$  [3]; although see, e.g., [4–6] for some of the groups that can appear.

One fares a little better starting not from the generic holonomy, but from the trivial holonomy. In [7] the maximally supersymmetric backgrounds — i.e., those with trivial (restricted) holonomy — were classified, recovering the known maximally supersymmetric backgrounds: the Freund–Rubin backgrounds [8, 9] and the gravitational wave of [10], in addition to the trivial Minkowski background. Attempts to extend this classification to sub-maximally supersymmetric backgrounds yielded some negative results: absence of backgrounds with precisely  $n = 31$  [11, 12] and  $n = 30$  [13], but the methods (based on so-called spinorial geometry) become impractical already for  $n = 29$ . In fact, we do not even know the size of the “supersymmetry gap”: the highest known sub-maximal background is a pp-wave with  $n = 26$  [14], but nothing is known about  $n = 27, 28, 29$ . Methods of spinorial geometry (also confusingly known as  $G$ -structures) have also yielded some

information at the opposite end, with local forms of backgrounds for  $n = 1$  [15, 16] in terms of ingredients (such as, Calabi–Yau 5-folds) which offer little hope of classification.

In this paper we would like to propose a different approach to the classification, based on the classification of the Killing superalgebra of the background. Indeed, every supersymmetric supergravity background has an associated Lie superalgebra which is generated by its Killing spinors. Its construction is reviewed in Section 3.1 below. Its origin is lost in the mists of time and probably was already understood, at least in special cases, in the early days of Kaluza–Klein supergravity. In more recent times, it made its appearance in the context of the AdS/CFT correspondence [17, 18], brane solutions [19–21], plane waves [22, 23] and homogeneous backgrounds [24], with the general construction appearing for the first time in [25] for  $d=11$  and [26] for  $d=10$  supergravities. Since then, a number of other supergravity theories have been treated, such as  $d=6$  [27],  $d=10$  conformal in [28] and  $d=4$  (off-shell, minimal) in [29].

The Killing superalgebra has proved to be a very useful invariant of a supersymmetric supergravity background. First of all, it “categorifies” the fraction of supersymmetry preserved by the background. In addition it behaves well under geometric limits, such as asymptotic and near-horizon limits, but also plane-wave limits. It also underlies the (local) homogeneity theorem [25–27, 30, 31] which states that a supergravity background preserving more than half of the supersymmetry is (locally) homogeneous, which is one of the few general structural results known about supersymmetric supergravity backgrounds.

The purpose of this paper is to show that the Killing superalgebra has a very precise algebraic structure — one which had passed unnoticed until recently — and to derive some of its consequences. In particular, we will show that the Killing superalgebra is a filtered deformation of a  $\mathbb{Z}$ -graded subalgebra of the Poincaré superalgebra. Let us explain this statement.

Let  $(V, \eta)$  denote the lorentzian vector space on which Minkowski space is modelled,  $\mathfrak{so}(V)$  the Lie algebra of the Lorentz group and  $S$  its spinor representation. In our conventions the inner product  $\eta$  has signature  $(1, 10)$ , i.e., it is “mostly minus”, and  $S \cong \mathbb{R}^{32}$  is an irreducible module of the Clifford algebra  $C\ell(V) \cong 2\mathbb{R}(32)$ . (There are two such modules up to isomorphism, and they are equivalent as  $\mathfrak{so}(V)$ -representations. We have chosen the module for which the centre acts nontrivially; that is, for which the action of the volume element  $\text{vol} \in C\ell(V)$  is  $\text{vol} \cdot s = -s$  for all  $s \in S$ .) We recall that  $S$

has an  $\mathfrak{so}(V)$ -invariant symplectic structure  $\langle -, - \rangle$  satisfying

$$\langle v \cdot s_1, s_2 \rangle = - \langle s_1, v \cdot s_2 \rangle,$$

for all  $s_1, s_2 \in S$  and  $v \in V$ , where  $\cdot$  refers to the Clifford action.

The *Poincaré superalgebra*  $\mathfrak{p}$  has underlying vector space  $\mathfrak{so}(V) \oplus S \oplus V$  and nonzero Lie brackets given by the following expressions, for  $A, B \in \mathfrak{so}(V)$ ,  $v \in V$  and  $s \in S$ :

$$(1) \quad [A, B] = AB - BA, \quad [A, s] = \sigma(A)s, \quad [A, v] = Av, \quad [s, s] = \kappa(s, s).$$

Here  $\sigma$  is the spinor representation of  $\mathfrak{so}(V)$  and  $\kappa(s, s) \in V$  is the *Dirac current* of  $s$ , defined by

$$(2) \quad \eta(\kappa(s, s), v) = \langle s, v \cdot s \rangle,$$

for all  $v \in V$ . One important property of the Dirac current  $\kappa : \odot^2 S \rightarrow V$  is that its restriction to a subspace  $\odot^2 S'$  is still surjective on  $V$ , provided that the vector subspace  $S' \subset S$  has dimension  $\dim S' > 16$ . We shall refer to this linear algebraic fact as “local homogeneity”, due to the crucial rôle it plays in the proof of the local homogeneity theorem of [31].

If we grade  $\mathfrak{p}$  by declaring  $\mathfrak{so}(V)$ ,  $S$  and  $V$  to have degrees 0,  $-1$  and  $-2$ , respectively, then the above Lie brackets turn  $\mathfrak{p}$  into a  $(\mathbb{Z})$ -graded Lie superalgebra

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2}, \quad \mathfrak{p}_0 = \mathfrak{so}(V), \quad \mathfrak{p}_{-1} = S, \quad \mathfrak{p}_{-2} = V.$$

The  $\mathbb{Z}_2$  grading is compatible with the  $\mathbb{Z}$  grading, in that  $\mathfrak{p}_{\bar{0}} = \mathfrak{p}_0 \oplus \mathfrak{p}_{-2}$  and  $\mathfrak{p}_{\bar{1}} = \mathfrak{p}_{-1}$ ; that is, the parity is the reduction modulo 2 of the  $\mathbb{Z}$  degree.

Let now  $\mathfrak{a} < \mathfrak{p}$  be a graded subalgebra that is,  $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2}$ , with  $\mathfrak{a}_i \subset \mathfrak{p}_i$ . Recall that a Lie superalgebra  $\mathfrak{g}$  is said to be filtered, if it admits a vector space filtration

$$\mathfrak{g}^\bullet : \quad \dots \supset \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 \supset \dots,$$

with  $\cup_i \mathfrak{g}^i = \mathfrak{g}$  and  $\cap_i \mathfrak{g}^i = 0$ , which is compatible with the Lie bracket: that is,  $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ . Associated canonically to every filtered Lie superalgebra  $\mathfrak{g}^\bullet$  there is a graded Lie superalgebra  $\mathfrak{g}_\bullet = \bigoplus_i \mathfrak{g}_i$ , where  $\mathfrak{g}_i = \mathfrak{g}^i / \mathfrak{g}^{i+1}$ . It follows from the fact that  $\mathfrak{g}^\bullet$  is filtered that  $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ , hence  $\mathfrak{g}_\bullet$  is graded. We say that a Lie superalgebra  $\mathfrak{g}$  is a *filtered deformation* of  $\mathfrak{a} < \mathfrak{p}$ , if it is filtered and its associated graded superalgebra is isomorphic (as a graded Lie

superalgebra) to  $\mathfrak{a}$ . If we do not wish to mention the subalgebra  $\mathfrak{a}$  explicitly, we simply say that  $\mathfrak{g}$  is a *filtered subdeformation* of  $\mathfrak{p}$ . The first main result of this paper is the following, which is part of Theorem 12. That theorem is in turn part of the more general Theorem 13 in Section 3.3.

**Theorem.** *The Killing superalgebra  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$  of an 11-dimensional supergravity background  $(M, g, F)$  is a filtered subdeformation of the Poincaré superalgebra  $\mathfrak{p}$ .*

Although by “background” one typically means a solution of the (bosonic) field equations, the above result actually only uses the form of the Killing spinor equation and the fact that  $F \in \Omega^4(M)$  is closed. A natural question of long standing is whether some amount of supersymmetry implies the bosonic field equations. It is known to be the case for maximal supersymmetry: the bosonic field equations are equivalent to the vanishing of the Clifford trace of the gravitino connection, whereas maximal supersymmetry is equivalent to flatness. It is also known to fail for  $\leq \frac{1}{2}$ -BPS backgrounds, but it has long been suspected that there is some critical fraction of supersymmetry which forces the equations of motion. We give a positive answer to this question in this paper, where in Section 5.2 we prove the following theorem (see Theorem 23).

**Theorem.** *Let  $(M, g, F)$  be an 11-dimensional lorentzian spin manifold endowed with a closed 4-form  $F \in \Omega^4(M)$ . If the real vector space*

$$\mathfrak{k}_1 = \{ \varepsilon \in \Gamma(\mathbb{S}) \mid \nabla_X \varepsilon = \frac{1}{24}(X \cdot F - 3F \cdot X) \cdot \varepsilon \}$$

*of Killing spinors has dimension  $\dim \mathfrak{k}_1 > 16$ , then  $(M, g, F)$  satisfies the bosonic field equations of 11-dimensional supergravity.*

The above condition on the dimension of the space of Killing spinors is crucial to many of our results and we have tentatively given it the name of “high supersymmetry”. We will therefore refer to “highly supersymmetric backgrounds” when talking about backgrounds preserving more than half of the supersymmetry. Similarly, we will say that a filtered subdeformation  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  is “highly supersymmetric” if  $\dim \mathfrak{g}_1 > 16$ .

The two theorems quoted above suggest an approach to the classification of highly supersymmetric backgrounds via the classification of their Killing superalgebras, which as mentioned above are (certain) filtered subdeformations of the Poincaré superalgebra. A first step in such a research programme was taken in [32, 33], where we recovered the classification in

[7] of maximally supersymmetric supergravity backgrounds. We also refer to the introduction in [32] and to [34–37] for more details on the underlying geometric interpretation of the Killing superalgebra in the context of “nonholonomic”  $G$ -structures on supermanifolds.

Of course, there is no reason to believe that *any* filtered subdeformation of the Poincaré superalgebra is the Killing superalgebra of a supersymmetric background and one of the aims of this paper is to characterise algebraically those filtered subdeformations which are Killing superalgebras of highly supersymmetric backgrounds. It would be interesting to characterise the filtered deformations which are Killing superalgebras of supersymmetric backgrounds, regardless the amount of supersymmetry preserved, but we don't do that in this paper.

Thus we will narrow down the class of filtered subdeformations  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  to those which are highly supersymmetric ( $\dim \mathfrak{g}_1 > 16$ ) and which satisfy additional criteria set out in Definition 9. These criteria essentially amount to demanding that  $\mathfrak{g}$  should be constructed out of a closed 4-form in a way consistent with supergravity. We will say that those highly supersymmetric filtered subdeformations are (geometrically) realizable.

The Killing superalgebra of a highly supersymmetric background is realizable (see Theorem 12). Conversely, if a highly supersymmetric filtered subdeformation  $\mathfrak{g}$  is realizable, it is then possible to reconstruct a background whose Killing superalgebra contains  $\mathfrak{g}$ . This result is contained in a partial converse of the first quoted theorem above, which also forms part of Theorem 13.

**Theorem.** *Any realizable highly supersymmetric filtered subdeformation  $\mathfrak{g}$  of the Poincaré superalgebra  $\mathfrak{p}$  is a subalgebra of the Killing superalgebra of a highly supersymmetric 11-dimensional supergravity background.*

We make no claims about the existence of a similar result when we drop the “highly supersymmetric” condition: our reconstruction theorem requires (local) homogeneity, which only high supersymmetry guarantees.

As a consequence of these results we will be able to map the classification problem of highly supersymmetric backgrounds of 11-dimensional supergravity to the classification problem of (maximal) realizable highly supersymmetric filtered subdeformations of the Poincaré superalgebra. A refined version of this approach, where we restrict to the classification of Killing ideals (see below), corresponding to classifying realizable highly supersymmetric filtered subdeformations which are odd-generated, seems slightly more tractable.

This paper is organised as follows. In Section 2 we summarise the basic notions and results about filtered deformations of Lie superalgebras and in particular of the Poincaré superalgebra. We also define the notion of a (highly supersymmetric) realizable filtered subdeformation, since those are the ones which can correspond to Killing superalgebras of highly supersymmetric 11-dimensional supergravity backgrounds. In Section 3.1 we review the geometric construction of the Killing superalgebra and in Section 3.2 we prove that the Killing superalgebra is a filtered subdeformation of  $\mathfrak{p}$ . In Section 3.3, we prove our first main result: Theorem 13. In Section 4 we consider the Jacobi identity of the Killing superalgebra and define the classification problem for highly supersymmetric Killing superalgebras as the classification of realizable highly supersymmetric filtered subdeformations. We then describe the latter in terms of simpler objects. In Section 5 we relate the Jacobi identity to the supergravity field equations. We first recall some useful algebraic and differential identities from [15, 16], and then show that high supersymmetry implies the field equations (see Theorem 23). We conclude with some observations.

## 2. Preliminaries on filtered deformations

Let  $\mathfrak{p}$  be the Poincaré superalgebra. In this section we discuss filtered deformations of its  $\mathbb{Z}$ -graded subalgebras.

### 2.1. Basic definitions and results

Let  $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2}$  be a  $\mathbb{Z}$ -graded subalgebra of  $\mathfrak{p}$ , where  $\mathfrak{a}_{-2} = V' \subset V$ ,  $\mathfrak{a}_{-1} = S' \subset S$  and  $\mathfrak{a}_0 = \mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{so}(V)$ . We denote the negatively graded part of  $\mathfrak{a}$  by  $\mathfrak{a}_- = \mathfrak{a}_{-1} \oplus \mathfrak{a}_{-2}$ ; in particular  $\mathfrak{p}_- = \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2}$ ,  $\mathfrak{p}_{-2} = V$  and  $\mathfrak{p}_{-1} = S$ , is the usual (2-step nilpotent) supertranslation ideal of the Poincaré superalgebra.

**Definition 1.** The subalgebra  $\mathfrak{a}$  of  $\mathfrak{p}$  is called *highly supersymmetric* if  $\dim S' > 16$ .

We recall that a  $\mathbb{Z}$ -graded Lie superalgebra  $\mathfrak{a} = \bigoplus \mathfrak{a}_p$  with negatively graded part  $\mathfrak{a}_- = \bigoplus_{p < 0} \mathfrak{a}_p$  is called *fundamental* if  $\mathfrak{a}_-$  is generated by  $\mathfrak{a}_{-1}$  and *transitive* if for any  $x \in \mathfrak{a}_p$  with  $p \geq 0$  the condition  $[x, \mathfrak{a}_-] = 0$  implies  $x = 0$ . It is not hard to exhibit graded subalgebras  $\mathfrak{a}$  of  $\mathfrak{p}$  for which  $S'$  has dimension 16 and  $V'$  is a proper subspace of  $V$ . On the other hand, we have the following

**Lemma 2.** *Let  $\mathfrak{a}$  be a highly supersymmetric graded subalgebra of  $\mathfrak{p}$ . Then  $\mathfrak{a}_{-2} = V$  and  $\mathfrak{a}$  is fundamental and transitive.*

*Proof.* The algebraic fact underlying the local homogeneity theorem in [31] says precisely that the image of  $\kappa$  restricted to  $S' \otimes S'$  equals  $V$  if  $\dim S' > 16$ . It follows that  $\mathfrak{a}_{-2} = V$  and  $\mathfrak{a}$  is fundamental. The transitivity of  $\mathfrak{a}$  follows from the fact that  $V$  is a faithful representation of any Lie subalgebra of  $\mathfrak{so}(V)$ .  $\square$

To proceed further, we first need to recall the definition of an appropriate complex associated with  $\mathfrak{a}$ . It is called the (generalised) Spencer complex and it is a refinement (by degree) of the usual Chevalley–Eilenberg complex of a Lie superalgebra.

Table 1: Even  $q$ -cochains of small degree

deg	$q$				
	0	1	2	3	4
0	$\mathfrak{h}$	$S' \rightarrow S'$ $V' \rightarrow V'$	$\odot^2 S' \rightarrow V'$		
2		$V' \rightarrow \mathfrak{h}$	$\Lambda^2 V' \rightarrow V'$ $V' \otimes S' \rightarrow S'$ $\odot^2 S' \rightarrow \mathfrak{h}$	$\odot^3 S' \rightarrow S'$ $\odot^2 S' \otimes V' \rightarrow V'$	$\odot^4 S' \rightarrow V'$
4			$\Lambda^2 V' \rightarrow \mathfrak{h}$	$\odot^2 S' \otimes V' \rightarrow \mathfrak{h}$ $\Lambda^2 V' \otimes S' \rightarrow S'$ $\Lambda^3 V' \rightarrow V'$	$\odot^4 S' \rightarrow \mathfrak{h}$ $\odot^3 S' \otimes V' \rightarrow S'$

The cochains of the Spencer complex of  $\mathfrak{a}$  are linear maps  $\Lambda^q \mathfrak{a}_- \rightarrow \mathfrak{a}$  or, equivalently, elements of  $\mathfrak{a} \otimes \Lambda^q \mathfrak{a}_-^*$ , where  $\Lambda^\bullet$  is meant here in the super sense. We extend the degree in  $\mathfrak{a}$  to such cochains by declaring that  $\mathfrak{a}_p^*$  has degree  $-p$ . The spaces in the complexes of even cochains for small degree are given in Table 1; although for degree  $d = 4$  there are cochains also for  $q = 5, 6$  which we omit.

Let  $C^{d,q}(\mathfrak{a}_-, \mathfrak{a})$  be the space of  $q$ -cochains of degree  $d$ . The Spencer differential

$$\partial : C^{d,q}(\mathfrak{a}_-, \mathfrak{a}) \rightarrow C^{d,q+1}(\mathfrak{a}_-, \mathfrak{a})$$

coincides with the restriction of the Chevalley–Eilenberg differential for the Lie superalgebra  $\mathfrak{a}_-$  relative to its module  $\mathfrak{a}$  with respect to the adjoint action.



For  $q = 0, 1, 2$  and  $d \equiv 0 \pmod{2}$ , the Spencer differential is explicitly given by the following expressions:

$$\begin{aligned}
 (3) \quad & \partial\phi(x) = [x, \phi] \\
 (4) \quad & \partial\phi(x, y) = [x, \phi(y)] - (-1)^{|x||y|}[y, \phi(x)] - \phi([x, y]) \\
 (5) \quad & \partial\phi(x, y, z) = [x, \phi(y, z)] + (-1)^{|x|(|y|+|z|)}[y, \phi(z, x)] \\
 & \quad + (-1)^{|z|(|x|+|y|)}[z, \phi(x, y)] - \phi([x, y], z) \\
 & \quad - (-1)^{|x|(|y|+|z|)}\phi([y, z], x) - (-1)^{|z|(|x|+|y|)}\phi([z, x], y),
 \end{aligned}$$

where  $|x|, |y|, \dots$  are the parity of elements  $x, y, \dots$  of  $\mathfrak{a}_-$  and  $\phi \in C^{d,q}(\mathfrak{a}_-, \mathfrak{a})$  with  $q = 0, 1, 2$ , respectively.

We say that a  $\mathbb{Z}$ -graded Lie superalgebra  $\mathfrak{a}$  with negatively graded part  $\mathfrak{a}_-$  is a *full prolongation of degree  $k$*  if  $H^{d,1}(\mathfrak{a}_-, \mathfrak{a}) = 0$  for all  $d \geq k$  (see [38]).

**Lemma 3.** *Let  $\mathfrak{a}$  be a highly supersymmetric graded subalgebra of  $\mathfrak{p}$ . Then  $\mathfrak{a}$  is a full prolongation of degree 2 and  $H^{d,2}(\mathfrak{a}_-, \mathfrak{a}) = 0$  for all even  $d \geq 4$ .*

*Proof.* Since  $\mathfrak{a}$  is fundamental, many of the components of the Spencer differential turn out to be injective. For instance every  $\phi \in C^{2,1}(\mathfrak{a}_-, \mathfrak{a}) = \text{Hom}(V, \mathfrak{h})$  satisfies  $\partial\phi(s_1, s_2) = -\phi(\kappa(s_1, s_2))$  for all  $s_1, s_2 \in S'$  so that  $\phi = 0$  is the only cocycle and  $H^{2,1}(\mathfrak{a}_-, \mathfrak{a}) = 0$ . If  $d > 2$  the space of cochains  $C^{d,1}(\mathfrak{a}_-, \mathfrak{a}) = 0$  and first claim follows.

If  $\phi \in C^{4,2}(\mathfrak{a}_-, \mathfrak{a}) = \text{Hom}(\Lambda^2 V, \mathfrak{h})$ , one has  $\partial\phi(s_1, s_2, v) = -\phi(\kappa(s_1, s_2), v)$  where  $s_1, s_2 \in S'$  and  $v \in V$ . In particular  $\ker \partial|_{C^{4,2}(\mathfrak{a}_-, \mathfrak{a})} = 0$  and  $H^{4,2}(\mathfrak{a}_-, \mathfrak{a}) = 0$ . If  $d > 4$  then  $C^{d,2}(\mathfrak{a}_-, \mathfrak{a}) = 0$  and last claim follows.  $\square$

Let  $\mathfrak{g}$  be a filtered deformation of a graded subalgebra  $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V'$  of  $\mathfrak{p}$ . It satisfies  $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ , where the filtration  $\mathfrak{g}^\bullet$  is

$$\mathfrak{g}^\bullet : \quad \mathfrak{g} = \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 \supset 0, \quad \mathfrak{g}^{-1} = \mathfrak{h} \oplus S', \quad \mathfrak{g}^0 = \mathfrak{h},$$

hence its Lie brackets take the following general form

$$\begin{aligned}
 (6) \quad & [A, B] = AB - BA & [s, s] &= \kappa(s, s) + \gamma(s, s) \\
 & [A, s] = \sigma(A)s & [v, s] &= \beta(v, s) \\
 & [A, v] = Av + \delta(A, v) & [v, w] &= \alpha(v, w) + \rho(v, w),
 \end{aligned}$$

for some maps  $\alpha \in \text{Hom}(\Lambda^2 V', V')$ ,  $\beta \in \text{Hom}(V' \otimes S', S')$ ,  $\gamma \in \text{Hom}(\odot^2 S', \mathfrak{h})$  and also  $\delta \in \text{Hom}(\mathfrak{h} \otimes V', \mathfrak{h})$  of degree 2 and a map  $\rho \in \text{Hom}(\Lambda^2 V', \mathfrak{h})$  of degree 4, where  $A, B \in \mathfrak{h}$ ,  $s \in S'$  and  $v, w \in V'$ .

**Definition 4.** The filtered deformation  $\mathfrak{g}$  of  $\mathfrak{a}$  is called *highly supersymmetric* if  $\mathfrak{a}$  is highly supersymmetric; that is, if  $\dim \mathfrak{g}_{\bar{1}} = \dim S' > 16$ .

To introduce the notion of isomorphism between filtered subdeformations of  $\mathfrak{p}$ , we note that the spin group  $\text{Spin}(V)$  naturally acts on  $\mathfrak{p} = \mathfrak{so}(V) \oplus S \oplus V$  by 0-degree Lie superalgebra automorphisms. In particular any element  $g \in \text{Spin}(V)$  sends a graded subalgebra of  $\mathfrak{p}$  into an (isomorphic) graded subalgebra of  $\mathfrak{p}$ .

**Definition 5.** An *isomorphism* of filtered subdeformations  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  of  $\mathfrak{p}$  is a map  $\Phi : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$  such that:

- (i)  $\Phi$  is an isomorphism of Lie superalgebras;
- (ii)  $\Phi$  is compatible with the filtrations; i.e.,  $\Phi(\mathfrak{g}^i) = \tilde{\mathfrak{g}}^i$  for  $i = -2, -1, 0$ ;
- (iii) the induced 0-degree Lie superalgebra isomorphism of associated graded Lie superalgebras  $\mathfrak{a}$  and  $\tilde{\mathfrak{a}}$  is given by the natural action of some  $g \in \text{Spin}(V)$ .<sup>1</sup>

If we do not wish to mention  $\Phi$  explicitly, we simply say that  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  are *isomorphic*.

It is easy to see that an isomorphism  $\Phi : \mathfrak{g} \xrightarrow{\sim} \tilde{\mathfrak{g}}$  takes the following general form, for some  $g \in \text{Spin}(V)$  and  $X' : V' \rightarrow \mathfrak{h}$ :

$$(7) \quad \Phi(A) = g \cdot A, \quad \Phi(s) = g \cdot s, \quad \text{and} \quad \Phi(v) = g \cdot v + X'_v,$$

where  $A \in \mathfrak{h}$ ,  $s \in S'$  and  $v \in V'$ . In the following, we consider isomorphisms of highly supersymmetric filtered subdeformations whose associated 0-degree map is the identity, that is with  $g = e$  in (7). We denote the sum of all components in (6) of degree 2 by the symbol  $\mu = \alpha + \beta + \gamma + \delta : \mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathfrak{a}$ , where  $\alpha, \beta, \gamma, \delta$  are the maps introduced just before Definition 4.

**Proposition 6.** *Let  $\mathfrak{g}$  be a highly supersymmetric filtered deformation of a graded subalgebra  $\mathfrak{a}$  of  $\mathfrak{p}$ . Then:*

---

<sup>1</sup>It is well known that the bosonic field equations of 11-dimensional supergravity are invariant under a homothety which rescales both the metric and the 4-form. With this definition of isomorphism of filtered subdeformations, the Killing superalgebras of two different supergravity backgrounds related by a homothety would *not* be isomorphic. They would be isomorphic, however, were we to modify the definition by replacing  $\text{Spin}(V)$  by  $\text{CSpin}(V)$  in (iii).

1)  $\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$  is a cocycle in  $C^{2,2}(\mathfrak{a}_-, \mathfrak{a})$  and its cohomology class

$$[\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$$

is  $\mathfrak{h}$ -invariant (that is, the cocycle  $\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$  is  $\mathfrak{h}$ -invariant up to coboundaries);

2) if  $\tilde{\mathfrak{g}}$  is another filtered deformation of the same  $\mathfrak{a}$  such that  $[\tilde{\mu}|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}] = [\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}]$  then  $\tilde{\mathfrak{g}}$  is isomorphic to  $\mathfrak{g}$ .

*Proof.* The proof relies on general results on filtered deformations in [38] and the full line of arguments is the same as in [32, Theorem 9] or [29, Proposition 10]. Here we simply record that, since  $V' = V$ , any  $\alpha \in \text{Hom}(\Lambda^2 V, V)$  can be written as

$$\alpha(v, w) = X_v w - X_w v, \quad v, w \in V,$$

for a unique linear map  $X : V \rightarrow \mathfrak{so}(V)$  and that

$$\begin{aligned} \delta(A, v) &= [A, X_v] - X_{Av}, \\ \partial\rho(s, s, v) &= 2\gamma(s, \beta(v, s)) + \delta(\gamma(s, s), v), \end{aligned}$$

for all  $A \in \mathfrak{h}$ ,  $v \in V$  and  $s \in S'$ . It follows from Lemmas 2 and 3 that the components  $\delta$  and  $\rho$  are uniquely determined, once  $\mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-} = \alpha + \beta + \gamma$  has been fixed. In a similar way the components  $\tilde{\delta}$  and  $\tilde{\rho}$  of  $\tilde{\mathfrak{g}}$  are also fixed in terms of  $\tilde{\mu}|_{\mathfrak{a}_- \otimes \mathfrak{a}_-}$ .

By hypothesis  $\tilde{\mu}|_{\mathfrak{a}_- \otimes \mathfrak{a}_-} = \mu|_{\mathfrak{a}_- \otimes \mathfrak{a}_-} - \partial X'$ , with  $X' : V \rightarrow \mathfrak{h}$  giving the required isomorphism. □

We have seen that highly supersymmetric filtered subdeformations with associated graded subalgebra  $\mathfrak{a}$  of  $\mathfrak{p}$  are determined, up to isomorphisms of filtered subdeformations, by the space  $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})^{\mathfrak{h}}$  of  $\mathfrak{h}$ -invariant elements in  $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ . We will obtain improved versions of Proposition 6 in Section 4, in the case of (odd-generated) realizable filtered subdeformations. The concept of realizability is introduced in the next section.

### 2.2. Realizable filtered deformations

In [32, Proposition 7], we determined the group  $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$  when  $\mathfrak{a} = \mathfrak{p}$ , and found that  $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \cong \wedge^4 V$  as an  $\mathfrak{so}(V)$ -module. More precisely any class admits a canonical representative of the form  $\beta^\varphi + \gamma^\varphi$  for a unique  $\varphi \in \Lambda^4 V$ ,

where  $\beta^\varphi : V \otimes S \rightarrow S$  and  $\gamma^\varphi : \odot^2 S \rightarrow \mathfrak{so}(V)$  are given by

$$(8) \quad \begin{aligned} \beta^\varphi(v, s) &= \frac{1}{24}(v \cdot \varphi - 3\varphi \cdot v) \cdot s, \\ \gamma^\varphi(s, s)(v) &= -2\kappa(\beta^\varphi(v, s), s), \end{aligned}$$

for all  $s \in S$  and  $v \in V$ .

Associated to the natural inclusion  $\iota : \mathfrak{a} \rightarrow \mathfrak{p}$  there are chain maps  $\iota_* : C^{\bullet, \bullet}(\mathfrak{a}_-, \mathfrak{a}) \rightarrow C^{\bullet, \bullet}(\mathfrak{a}_-, \mathfrak{p})$  and  $\iota^* : C^{\bullet, \bullet}(\mathfrak{p}_-, \mathfrak{p}) \rightarrow C^{\bullet, \bullet}(\mathfrak{a}_-, \mathfrak{p})$  inducing the corresponding maps in cohomology

$$(9) \quad \begin{aligned} \iota_* : H^{2,2}(\mathfrak{a}_-, \mathfrak{a}) &\rightarrow H^{2,2}(\mathfrak{a}_-, \mathfrak{p}), \\ \iota^* : H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) &\rightarrow H^{2,2}(\mathfrak{a}_-, \mathfrak{p}). \end{aligned}$$

Both maps in (9) are  $\mathfrak{h}$ -equivariant. Moreover we have the following

**Lemma 7.** *Let  $\mathfrak{a}$  be a highly supersymmetric graded subalgebra of  $\mathfrak{p}$ . Then  $\iota_*$  is injective and  $\ker \iota^* \cong \{\varphi \in \Lambda^4 V \mid \beta^\varphi|_{V \otimes S'} = 0\}$ .*

*Proof.* Let  $[\alpha + \beta + \gamma] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$  be such that  $\iota_*[\alpha + \beta + \gamma] = 0$ ; that is,  $\alpha + \beta + \gamma = \partial\phi$  for some  $\phi : V \rightarrow \mathfrak{so}(V)$ . However, in particular  $\phi(\kappa(s_1, s_2)) = -\gamma(s_1, s_2) \in \mathfrak{h}$  for all  $s_1, s_2 \in S'$ , hence  $\phi(v) \in \mathfrak{h}$  for all  $v \in V$  and  $[\alpha + \beta + \gamma] = 0$ . This proves the first claim. The second claim is straightforward.  $\square$

**Remark 8.** We will see that the space  $\ker \iota^*$  parametrises the 4-forms compatible with a highly supersymmetric flat supergravity background, so that  $\iota^*$  too is injective (see Corollary 26). It would be desirable to have an a priori representation-theoretic proof of this fact.

In the following definition we introduce the concept of realizability for highly supersymmetric filtered subdeformations of  $\mathfrak{p}$ . This is an algebraic criterion which, as we will see, singles out the filtered subdeformations which are geometrically realizable as (subalgebras of) Killing superalgebras of supergravity backgrounds. Since the main purpose of this paper is to lay the foundations for the classification of highly supersymmetric backgrounds, we restrict the definition to the highly supersymmetric case. This is not to suggest that the notion of a (geometrically) realizable filtered subdeformation is not interesting if  $\dim \mathfrak{g}_{\bar{1}} \leq 16$ , but the definition would be more involved.

**Definition 9.** A filtered deformation  $\mathfrak{g}$  of a highly supersymmetric  $\mathbb{Z}$ -graded subalgebra  $\mathfrak{a}$  of  $\mathfrak{p}$  is said to be *realizable* if the following conditions are satisfied:

(i) the associated cohomology class  $[\alpha + \beta + \gamma] \in H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$  is of the form

$$(10) \quad \iota_*[\alpha + \beta + \gamma] = \iota^*[\beta^\varphi + \gamma^\varphi]$$

for some  $\varphi \in \Lambda^4 V$  (the collection of such forms is an affine space modeled on the vector space  $\ker \iota^*$ ); and

(ii) there exists a  $\varphi \in \Lambda^4 V \cong \Lambda^4 V^*$  as in (i) which is also  $\mathfrak{h}$ -invariant and closed. The condition for an  $\mathfrak{h}$ -invariant  $\varphi$  to be closed is equivalent to

$$(11) \quad d\varphi(v_0, \dots, v_4) = \sum_{i < j} (-1)^{i+j} \varphi(\alpha(v_i, v_j), v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_4) = 0$$

for all  $v_0, \dots, v_4 \in V$ .

We call *admissible* any  $\varphi \in \Lambda^4 V$  which is  $\mathfrak{h}$ -invariant and satisfies (10) and (11).

**Definition 10.** A filtered deformation  $\mathfrak{g}$  is called *odd-generated realizable* if it is realizable and generated by the odd part, that is  $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_1]$ .

We will see below that the Killing spinors of a highly supersymmetric background generate a filtered Lie superalgebra which is odd-generated and realizable.

We note that the condition of being (odd-generated) realizable is preserved by the isomorphisms of filtered subdeformations of  $\mathfrak{p}$ . In particular, it is worth remarking that even though the condition (11) that  $\varphi$  be closed seems to depend explicitly on  $\alpha$ , it actually only depends on the cohomology class of  $\alpha + \beta + \gamma$  in  $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ . Indeed, if we modify the cocycle by a coboundary  $\partial\phi$ , for some  $\phi : V \rightarrow \mathfrak{h}$ , then  $\alpha$  changes to  $\tilde{\alpha}(v, w) = \alpha(v, w) + \phi_v w - \phi_w v$ , and using the  $\mathfrak{h}$ -invariance of  $\varphi$  one sees that the expression for  $d\varphi$  in equation (11) remains unchanged.

It follows from (10) that the Lie brackets of a realizable filtered deformation  $\mathfrak{g}$  of  $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V$  are as in (6), where  $V' = V$  and

$$(12) \quad \begin{aligned} \alpha(v, w) &= X_v w - X_w v \\ \beta(v, s) &= \beta^\varphi(v, s) + \sigma(X_v) s \\ \gamma(s, s) &= \gamma^\varphi(s, s) - X_{\kappa(s,s)} \\ \delta(A, v) &= [A, X_v] - X_{Av}, \end{aligned}$$

for some linear map  $X : V \rightarrow \mathfrak{so}(V)$ . Here  $A, B \in \mathfrak{h}$ ,  $v, w \in V$ ,  $s \in S'$ . This implies that any highly supersymmetric graded subalgebra  $\bar{\mathfrak{a}} = \bar{\mathfrak{h}} \oplus \bar{S}' \oplus V$  of  $\mathfrak{a}$  which is also closed under the Lie brackets of  $\mathfrak{g}$  inherits a natural structure  $\bar{\mathfrak{g}}$  of realizable filtered subdeformation. This motivates the following

**Definition 11.** An *embedding* of filtered subdeformations of  $\mathfrak{p}$  is an injective map  $\Phi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$  such that  $\Phi : \tilde{\mathfrak{g}} \rightarrow \bar{\mathfrak{g}} = \Phi(\tilde{\mathfrak{g}})$  is an isomorphism of filtered subdeformations, where  $\bar{\mathfrak{g}} \subseteq \mathfrak{g}$  has the natural structure of filtered subdeformation induced by  $\mathfrak{g}$ .

### 3. The Killing superalgebra as a filtered deformation

We will first review in Section 3.1 the construction of the Killing superalgebra  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$  associated to a supersymmetric background  $(M, g, F)$  of 11-dimensional supergravity, following the description in [25]. Actually, it is known that  $\mathfrak{k}$  can be constructed for any 11-dimensional lorentzian spin manifold  $(M, g, F)$  endowed with a closed 4-form  $F \in \Omega^4(M)$ . In other words, the supergravity Einstein and Maxwell equations are of no consequence in what follows in Section 3.

We will then show in Section 3.2 that the Killing superalgebra, as a filtered Lie superalgebra, is (isomorphic to) a filtered subdeformation of the Poincaré superalgebra. The main result of this section deals with the highly supersymmetric case and it is given by Theorem 13 in Section 3.3.

#### 3.1. The Killing superalgebra

Let  $(M, g, F)$  be a connected 11-dimensional lorentzian spin manifold endowed with a closed 4-form  $F \in \Omega^4(M)$ . We denote the Levi-Civita connection by  $\nabla$  and the associated spinor bundle by  $\mathbb{S} \rightarrow M$  (more precisely, this is a bundle of Clifford modules over  $C\ell(TM)$  associated to one of the two non-isomorphic irreducible Clifford modules).

The spinor fields  $\varepsilon \in \Gamma(\mathbb{S})$  which satisfy, for all vector fields  $Z \in \mathcal{X}(M)$ ,

$$\nabla_Z \varepsilon = \frac{1}{24}(Z \cdot F - 3F \cdot Z) \cdot \varepsilon$$

are called *Killing spinors* and they define a real vector space  $\mathfrak{k}_1$ . We also let  $\mathfrak{k}_0$  be the space of  $F$ -preserving Killing vectors.

The Killing superalgebra is a Lie superalgebra structure on  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ ; we will review the construction below. For our purposes, it is convenient to introduce bundle morphisms  $\beta^F : TM \otimes \mathbb{S} \rightarrow \mathbb{S}$  and  $\gamma^F : \odot^2 \mathbb{S} \rightarrow \mathfrak{so}(TM)$

defined by

$$(13) \quad \begin{aligned} \beta^F(Z, \varepsilon) &= \frac{1}{24}(Z \cdot F - 3F \cdot Z) \cdot \varepsilon, \\ \gamma^F(\varepsilon, \varepsilon)(Z) &= -2\kappa(\beta^F(Z, \varepsilon), \varepsilon), \end{aligned}$$

where  $Z \in \mathcal{X}(M)$  and  $\varepsilon \in \Gamma(\mathfrak{S})$ . In particular Killing spinors are exactly those spinors  $\varepsilon$  which satisfy  $\nabla_Z \varepsilon = \beta^F(Z, \varepsilon)$  for all  $Z \in \mathcal{X}(M)$ .

Let  $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$  be the supervector bundle where  $\mathcal{E}_0 = TM \oplus \mathfrak{so}(TM)$  and  $\mathcal{E}_1 = \mathfrak{S}$ . On  $\mathcal{E}$  we have an even connection  $D$  defined by

$$D_Z \varepsilon = \nabla_Z \varepsilon - \beta^F(Z, \varepsilon),$$

for  $\varepsilon \in \Gamma(\mathcal{E}_1)$  and

$$D_Z \begin{pmatrix} \xi \\ \Xi \end{pmatrix} = \begin{pmatrix} \nabla_Z \xi + \Xi(Z) \\ \nabla_Z \Xi - R(Z, \xi) \end{pmatrix},$$

for  $(\xi, \Xi) \in \Gamma(\mathcal{E}_0)$ , where  $R : \Lambda^2 TM \rightarrow \mathfrak{so}(TM)$  is the Riemann curvature. Then

$$\begin{aligned} \mathfrak{k}_1 &= \{\varepsilon \in \Gamma(\mathcal{E}_1) \mid D\varepsilon = 0\} \\ \mathfrak{k}_0 &= \{(\xi, \Xi) \in \Gamma(\mathcal{E}_0) \mid D(\xi, \Xi) = 0 \quad \text{and} \quad \nabla_\xi F + \Xi \cdot F = 0\}, \end{aligned}$$

where  $\Xi \cdot F$  is the natural action of  $\mathfrak{so}(TM)$  on 4-forms. In particular, an element of the Killing superalgebra is determined by the value at a point in  $M$  of the corresponding parallel section of  $\mathcal{E}$ . In other words, given any point  $o \in M$ , the Killing superalgebra defines a vector subspace of  $\mathfrak{so}(T_oM) \oplus \mathfrak{S}_o \oplus T_oM$ .

We will introduce the notation

$$(V, \eta) = (T_oM, g_o), \quad \mathfrak{so}(V) = \mathfrak{so}(T_oM), \quad S = \mathfrak{S}_o,$$

so that  $(V, \eta)$  is an 11-dimensional lorentzian vector space with Lie algebra  $\mathfrak{so}(V)$  of skew-symmetric endomorphisms, and  $S$  an irreducible  $C\ell(V)$ -module. Notice that  $\mathfrak{so}(V) \oplus S \oplus V$  is the vector space underlying the Poincaré superalgebra.

We now describe the Lie brackets of  $\mathfrak{k}$ . Let  $(\xi, X_\xi), (\zeta, X_\zeta) \in \mathfrak{k}_0$ . This means that  $\xi, \zeta$  are  $F$ -preserving Killing vector fields with  $X_\xi = -\nabla\xi$  and  $X_\zeta = -\nabla\zeta$ . Their Lie bracket is given by

$$(14) \quad [(\xi, X_\xi), (\zeta, X_\zeta)] = (X_\xi\zeta - X_\zeta\xi, [X_\xi, X_\zeta] + R(\xi, \zeta)),$$

with the Riemann curvature measuring the deviation of  $\mathfrak{k}_0$  from being a subalgebra of the Poincaré algebra  $\mathfrak{p}_0$ . Now let  $\varepsilon \in \mathfrak{k}_1$  be a Killing spinor.

The action of  $\mathfrak{k}_0$  on  $\mathfrak{k}_1$  is given by the spinorial Lie derivative ([39]; see also, e.g., [18])

$$(15) \quad \mathcal{L}_\xi \varepsilon = \nabla_\xi \varepsilon + \sigma(X_\xi)\varepsilon,$$

where  $\sigma$  is the spinor representation of  $\mathfrak{so}(TM)$ . From the fact that  $D\varepsilon = 0$ , we may rewrite this action without derivatives:

$$(16) \quad [(\xi, X_\xi), \varepsilon] = \beta^F(\xi, \varepsilon) + \sigma(X_\xi)\varepsilon.$$

Lastly, the square of a Killing spinor is its Dirac current, which belongs to  $\mathfrak{k}_0$  ([25]; see also, e.g., Corollary 22 in Section 5.1):

$$[\varepsilon, \varepsilon] = (\kappa(\varepsilon, \varepsilon), -\nabla\kappa(\varepsilon, \varepsilon)).$$

A calculation shows that

$$(17) \quad \begin{aligned} -\nabla\kappa(\varepsilon, \varepsilon)(Z) &= -\nabla_Z\kappa(\varepsilon, \varepsilon) = -2\kappa(\nabla_Z\varepsilon, \varepsilon) \\ &= -\frac{1}{12}\kappa((Z \cdot F - 3F \cdot Z) \cdot \varepsilon, \varepsilon) \\ &= \gamma^F(\varepsilon, \varepsilon)(Z), \end{aligned}$$

for all vector fields  $Z \in \mathcal{X}(M)$ .

In summary, the Lie brackets of  $\mathfrak{k}$  are given, for  $(\xi, X_\xi), (\zeta, X_\zeta) \in \mathfrak{k}_0$  and  $\varepsilon \in \mathfrak{k}_1$ , by the following:

$$(18) \quad \begin{aligned} [(\xi, X_\xi), (\zeta, X_\zeta)] &= (X_\xi\zeta - X_\zeta\xi, [X_\xi, X_\zeta] + R(\xi, \zeta)) \\ [(\xi, X_\xi), \varepsilon] &= \beta^F(\xi, \varepsilon) + \sigma(X_\xi)\varepsilon \\ [\varepsilon, \varepsilon] &= (\kappa(\varepsilon, \varepsilon), \gamma^F(\varepsilon, \varepsilon)). \end{aligned}$$

As in every Lie superalgebra  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ , the odd subspace  $\mathfrak{k}_1$  generates an ideal  $\widehat{\mathfrak{k}} = \widehat{\mathfrak{k}}_0 \oplus \widehat{\mathfrak{k}}_1$  of  $\mathfrak{k}$ , where  $\widehat{\mathfrak{k}}_0 = [\mathfrak{k}_1, \mathfrak{k}_1]$  and  $\widehat{\mathfrak{k}}_1 = \mathfrak{k}_1$ . We refer to it as the *Killing ideal* of the Killing superalgebra.

### 3.2. The Killing superalgebra is a filtered subdeformation of $\mathfrak{p}$

We now show that the Lie superalgebra  $\mathfrak{k}$  described in (18) is isomorphic to a filtered subdeformation of the Poincaré superalgebra  $\mathfrak{p}$ . It is not in general a subalgebra of  $\mathfrak{p}$ .

We will first show that the Killing superalgebra  $\mathfrak{k}$  determines a  $\mathbb{Z}$ -graded vector subspace of  $\mathfrak{p}$ . As above, let us identify  $\mathfrak{p}$  as a vector space with



the fibre  $\mathcal{E}_o$ . Let  $\text{ev}_o^{\bar{0}}: \mathfrak{k}_{\bar{0}} \rightarrow V$  be the composition of evaluation at  $o$  and projection onto  $V = T_oM$ , and let  $\text{ev}_o^{\bar{1}}: \mathfrak{k}_{\bar{1}} \rightarrow S$  be evaluation at  $o$ . Let also  $\text{im ev}_o^{\bar{1}} = S' \subseteq S$  and  $\text{im ev}_o^{\bar{0}} = V' \subseteq V$  be the images of these evaluations.

Let  $\mathfrak{h} = \ker \text{ev}_o^{\bar{0}}$  be the Lie subalgebra of  $\mathfrak{k}_{\bar{0}}$  consisting of elements of  $\mathfrak{k}_{\bar{0}}$  which vanish at  $o \in M$ ; that is, which take the form  $(0, A) \in V \oplus \mathfrak{so}(V)$ . In other words,  $\mathfrak{h}$  defines a subspace of  $\mathfrak{so}(V)$ . From the definition of  $\mathfrak{h}$ , we have a short exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k}_{\bar{0}} \xrightarrow{\text{ev}_o^{\bar{0}}} V' \longrightarrow 0.$$

Since short exact sequences split in the category of vector spaces, we have a vector space isomorphism  $\mathfrak{k}_{\bar{0}} \cong \mathfrak{h} \oplus V'$  and since  $\mathfrak{k}_{\bar{1}} \cong S'$ , we see that (again, as a vector space)  $\mathfrak{k} \cong \mathfrak{h} \oplus S' \oplus V' \subset \mathfrak{so}(V) \oplus S \oplus V = \mathfrak{p}$ . However there is no canonical splitting and hence no preferred isomorphism. If there were, we could simply transport the Lie superalgebra structure in  $\mathfrak{k}$  to  $\mathfrak{h} \oplus S' \oplus V'$ . In our case, however, we will have to choose a splitting. Geometrically, this amounts to choosing (in a linear fashion) for every  $v \in V'$  a Killing vector field  $\xi \in \mathfrak{k}_{\bar{0}}$  with  $\text{ev}_o^{\bar{0}}(\xi) = v$ . Such a choice gives an embedding of  $V'$  into  $\mathfrak{k}_{\bar{0}} \subset V \oplus \mathfrak{so}(V)$  by sending  $v \in V'$  to  $(v, X_v)$ , where  $X_v \in \mathfrak{so}(V)$  is the image of  $v$  under a linear map  $X: V' \rightarrow \mathfrak{so}(V)$ . Any other choice of splitting would result in  $(v, X'_v)$  for some other linear map  $X': V' \rightarrow \mathfrak{so}(V)$ , but where  $X - X': V' \rightarrow \mathfrak{h}$ . (In applications we can fix  $X: V' \rightarrow \mathfrak{h}^\perp$  to take values in a complement  $\mathfrak{h}^\perp$  of  $\mathfrak{h}$  in  $\mathfrak{so}(V)$ , e.g., the orthogonal complement with respect to the Killing form, whenever this exists).

Since  $\mathfrak{h}$  consists of those Killing vectors in  $\mathfrak{k}_{\bar{0}}$  which vanish at  $o$ , the corresponding parallel section of  $\mathcal{E}_{\bar{0}}$  takes the form  $(0, A)$  at  $o \in M$ . Now it is clear from equation (18) that the Lie brackets of  $\mathfrak{k}$  only depend on the value of the sections at the point  $o \in M$ , hence we see that if  $(0, A), (0, B) \in \mathfrak{h}$ , then

$$[(0, A), (0, B)] = (0, [A, B]),$$

so that  $\mathfrak{h}$  defines a Lie subalgebra of  $\mathfrak{so}(V)$ . In addition, when evaluated at  $o \in M$ , the condition  $\mathcal{L}_\xi F = 0$  for all vector fields  $\xi$  with  $\text{ev}_o^{\bar{0}}(\xi) = 0$  becomes

$$A \cdot F_o = 0 \quad \text{for all } A \in \mathfrak{h}.$$

In summary,  $\mathfrak{h}$  defines a Lie subalgebra of  $\mathfrak{so}(V) \cap \text{stab}(\varphi)$ , where  $\varphi = F_o \in \Lambda^4 V^* \cong \Lambda^4 V$  is the value of  $F$  at  $o \in M$ .

It also follows from equation (18) that the action of  $\mathfrak{h}$  on  $\mathfrak{k}_{\bar{1}}$  at  $o \in M$  is the restriction to  $\mathfrak{h} < \mathfrak{so}(V)$  of the action of  $\mathfrak{so}(V)$  on  $S$ :

$$[(0, A), s] = \sigma(A)s.$$

This implies in particular that  $\mathfrak{h}$  preserves the subspace  $S'$ .

Similarly, using the fixed embedding  $V' \subset V \oplus \mathfrak{so}(V)$  given by  $v \mapsto (v, X_v)$ , we find that the remaining brackets are

$$\begin{aligned} [(0, A), (v, X_v)] &= (Av, [A, X_v]) = (Av, X_{Av}) + (0, [A, X_v] - X_{Av}) \\ [(v, X_v), s] &= \beta^\varphi(v, s) + X_v s \\ [s, s] &= (\kappa(s, s), X_{\kappa(s,s)}) + (0, \gamma^\varphi(s, s) - X_{\kappa(s,s)}) \end{aligned}$$

and

$$\begin{aligned} [(v, X_v), (w, X_w)] &= (X_v w - X_w v, [X_v, X_w] + R(v, w)) \\ &= (X_v w - X_w v, X_{X_v w - X_w v}) \\ &\quad + (0, [X_v, X_w] - X_{X_v w - X_w v} + R(v, w)). \end{aligned}$$

In summary, the Killing superalgebra  $\mathfrak{k}$  is isomorphic to a Lie superalgebra structure defined on the graded subspace  $\mathfrak{h} \oplus S' \oplus V'$  of  $\mathfrak{p}$ , where  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$  which preserves  $S'$ . The Lie brackets are the following:

$$(19) \quad \begin{aligned} [A, B] &= AB - BA & [s, s] &= \kappa(s, s) + \gamma^\varphi(s, s) - X_{\kappa(s,s)} \\ [A, s] &= \sigma(A)s & [v, s] &= \beta^\varphi(v, s) + \sigma(X_v)s \\ [A, v] &= Av + \delta(A, v) & [v, w] &= \alpha(v, w) + \rho(v, w), \end{aligned}$$

where  $A, B \in \mathfrak{h}$ ,  $v, w \in V'$ ,  $s \in S'$  and

$$\begin{aligned} \alpha(v, w) &= X_v w - X_w v \\ \delta(A, v) &= [A, X_v] - X_{Av} \\ \rho(v, w) &= [X_v, X_w] - X_{\alpha(v,w)} + R(v, w). \end{aligned}$$

We observe that both  $\delta : \mathfrak{h} \otimes V' \rightarrow \mathfrak{h}$  and  $\rho : \Lambda^2 V' \rightarrow \mathfrak{h}$  take values in  $\mathfrak{h}$ . Moreover the element  $\beta(v, s) = \beta^\varphi(v, s) + \sigma(X_v)s$  is in  $S'$  (and not  $S$ ) whilst the individual terms may not; similarly the sum  $\gamma(s, s) = \gamma^\varphi(s, s) - X_{\kappa(s,s)}$  is in  $\mathfrak{h}$  (and not  $\mathfrak{so}(V)$ ).

From now on we will identify  $\mathfrak{k}$  with the Lie superalgebra structure defined on the graded subspace  $\mathfrak{h} \oplus S' \oplus V'$  of  $\mathfrak{p}$  by (19). The grading of the

Poincaré superalgebra  $\mathfrak{p}$  gives rise to a natural filtration of  $\mathfrak{p}$ :

$$\mathfrak{p}^\bullet : \quad \mathfrak{p} = \mathfrak{p}^{-2} \supset \mathfrak{p}^{-1} \supset \mathfrak{p}^0 \supset 0,$$

where  $\mathfrak{p}^{-1} = \mathfrak{so}(V) \oplus S$ ,  $\mathfrak{p}^0 = \mathfrak{so}(V)$ , and therefore also to a filtration of  $\mathfrak{k}$

$$\mathfrak{k}^\bullet : \quad \mathfrak{k} = \mathfrak{k}^{-2} \supset \mathfrak{k}^{-1} \supset \mathfrak{k}^0 \supset 0,$$

where  $\mathfrak{k}^{-1} = \mathfrak{h} \oplus S'$ ,  $\mathfrak{k}^0 = \mathfrak{h}$ . One checks from the Lie brackets (19) that  $[\mathfrak{k}^i, \mathfrak{k}^j] \subset \mathfrak{k}^{i+j}$ , so that  $\mathfrak{k}^\bullet$  is a filtered Lie superalgebra. Its associated graded Lie superalgebra  $\mathfrak{k}_\bullet$  has graded pieces  $\mathfrak{k}_{-2} = V'$ ,  $\mathfrak{k}_{-1} = S'$ ,  $\mathfrak{k}_0 = \mathfrak{h}$  and, comparing again with the Lie brackets of  $\mathfrak{k}$ , we see that  $\mathfrak{k}_\bullet$  is a subalgebra of  $\mathfrak{p}$ . Indeed, the maps  $\alpha, \beta, \gamma, \delta, \rho$  all have positive filtration degree (compare also with equation (6)).

Of course, it is not an arbitrary filtered subdeformation, since some of its components are prescribed by the supergravity theory via the definition of Killing spinor (compare also with equation (12)).

In summary, we have proved most of the following

**Theorem 12.** *The Killing superalgebra  $\mathfrak{k}$  is a filtered subdeformation of the Poincaré superalgebra and if  $\dim \mathfrak{k}_{\widehat{-1}} > 16$  it is a realizable filtered subdeformation. Moreover the Killing ideal  $\widehat{\mathfrak{k}}$  is odd-generated realizable.*

*Proof.* It remains to prove that  $\mathfrak{k}$  is realizable if  $\dim \mathfrak{k}_{\widehat{-1}} > 16$ , from where it will follow that  $\widehat{\mathfrak{k}}$  is odd-generated and realizable. Property (i) of Definition 9 is immediate, whereas property (ii) follows from the fact that the exterior derivative  $dF$  of a  $\mathfrak{k}_{\widehat{0}}$ -invariant 4-form  $F$  is a  $\mathfrak{k}_{\widehat{0}}$ -invariant 5-form, hence (locally) determined by its value  $(dF)_o$  at  $o \in M$ . □

### 3.3. Highly supersymmetric lorentzian spin manifolds

We will now restrict to the highly supersymmetric case and show that any realizable filtered subdeformation  $\mathfrak{g} = \mathfrak{g}_{\widehat{0}} \oplus \mathfrak{g}_{\widehat{-1}}$  of  $\mathfrak{p}$  can be realised as (a subalgebra of) the Killing superalgebra of a homogeneous  $(M = G_{\widehat{0}}/H, g, F)$ . To this end, it is actually more natural to assume  $\mathfrak{g}$  to be *anti*-isomorphic to a realizable filtered subdeformation; in other words, in this section  $\mathfrak{g}$  has the opposite Lie brackets to those in equations (6) and (12).

We first need to recall some basic definitions. Let  $G$  be a connected Lie supergroup with Lie superalgebra  $Lie(G) = \mathfrak{g}$ . We consider it as a super Harish-Chandra pair [40–42], a pair  $G = (G_{\widehat{0}}, \mathfrak{g})$  consisting of a connected Lie

group  $G_{\bar{0}}$  with Lie algebra  $Lie(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$  and a Lie superalgebra  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  admitting an *adjoint representation*, i.e., a morphism of Lie groups

$$(20) \quad \text{Ad} : G_{\bar{0}} \longrightarrow \text{GL}(\mathfrak{g})$$

such that  $\text{ad}(x)y = \frac{d}{dt}|_{t=0} \text{Ad}_{\exp(tx)} y$  for all  $x \in \mathfrak{g}_{\bar{0}}$  and  $y \in \mathfrak{g}$ . In particular  $V' = V$  and the analytic subgroup  $H$  of  $G_{\bar{0}}$  with Lie algebra  $Lie(H) = \mathfrak{h}$  acts orthogonally on  $V \cong \mathfrak{g}_{\bar{0}}/\mathfrak{h}$  via the natural representation

$$(21) \quad \text{Ad} : H \longrightarrow \text{SO}(V)$$

induced by (20).

If  $H$  is closed in  $G_{\bar{0}}$ , then  $M = (G_{\bar{0}}/H, g)$  is an 11-dimensional lorentzian homogeneous manifold, where  $g$  is the  $G_{\bar{0}}$ -invariant lorentzian metric on  $M$  corresponding to the  $H$ -invariant inner product  $\eta$  on  $V$ . Consider the  $\text{SO}(V)$ -bundle  $P$  on  $M$  of oriented orthonormal frames of  $(M, g)$ . We have

$$P \cong G_{\bar{0}} \times_H \text{SO}(V)$$

and  $TM \cong P \times_{\text{SO}(V)} V \cong G_{\bar{0}} \times_H V$ . In particular the vector fields on  $M$  are identified with the  $H$ -equivariant maps  $\xi : G_{\bar{0}} \rightarrow V$ .

Any lift of the adjoint representation (21) to the spin group  $\text{Spin}(V)$  — i.e., any homomorphism  $H \rightarrow \text{Spin}(V)$  such that the diagram

$$(22) \quad \begin{array}{ccc} H & \longrightarrow & \text{Spin}(V) \\ \parallel & & \downarrow \sigma \\ H & \xrightarrow{\text{Ad}} & \text{SO}(V) \end{array}$$

commutes — allows us to define a spin structure  $Q = G_{\bar{0}} \times_H \text{Spin}(V)$  on  $(M, g)$ , usually referred to as the homogeneous spin structure associated to the lift (22) [43]. If  $G_{\bar{0}}$  is simply connected, the homogeneous spin structures are in one-to-one correspondence with the spin structures [44]. Now, since  $\mathfrak{a}$  is transitive and fundamental, any element  $A \in \mathfrak{h}$  is uniquely determined by its action on  $\mathfrak{g}_{\bar{1}} \simeq S' \subseteq S$  and it is not difficult to see that the restriction of (20) to  $H$  and  $\mathfrak{g}_{\bar{1}}$  determines a unique lift  $\text{Ad} : H \longrightarrow \text{Spin}(V)$ . We call any triple  $(M = G_{\bar{0}}/H, g, Q)$  with  $Q$  determined by (20) as above a *homogeneous lorentzian spin manifold associated with  $\mathfrak{g}$* . For an analogous discussion in the special case of reductive homogeneous manifolds with  $S' = S$  we refer the reader to [42, §§5.1-2] and [24]. The spin bundle on  $M$  is  $\mathcal{S} = Q \times_{\text{Spin}(V)} S \cong G_{\bar{0}} \times_H S$  and the spinor fields on  $M$  are identified with the  $H$ -equivariant maps  $\varepsilon : G_{\bar{0}} \rightarrow S$ .

Finally, it is often convenient to work on  $G_{\bar{0}}$  through the natural projection  $\pi : G_{\bar{0}} \rightarrow M = G_{\bar{0}}/H$ . For instance invariant affine connections on  $M = G_{\bar{0}}/H$  are known to be in a one-to-one correspondence with Nomizu maps; that is, linear maps

$$L : \mathfrak{g}_{\bar{0}} \rightarrow \mathfrak{gl}(V)$$

satisfying [45]:

- (i)  $L(A) = \text{ad}(A)$  for all  $A \in \mathfrak{h}$ ; and
- (ii)  $L$  is  $H$ -equivariant.

Let us consider the natural projection from  $\mathfrak{g}_{\bar{0}}$  to  $V \cong \mathfrak{g}_{\bar{0}}/\mathfrak{h}$  and trivially extend  $\eta$  to the  $H$ -invariant symmetric bilinear map  $(-, -) : \mathfrak{g}_{\bar{0}} \otimes \mathfrak{g}_{\bar{0}} \rightarrow \mathbb{R}$  with kernel  $\mathfrak{h}$ , and let  $U$  be the symmetric bilinear map on  $\mathfrak{g}_{\bar{0}}$  with values into  $V$  uniquely determined by

$$2(U(x, y), z) = (x, [z, y]) + ([z, x], y),$$

where  $x, y, z \in \mathfrak{g}_{\bar{0}}$ . It is not difficult to see that the operator  $\tilde{L} : \mathfrak{g}_{\bar{0}} \rightarrow \text{Hom}(\mathfrak{g}_{\bar{0}}, V)$  given by

$$\tilde{L}(x)y := \frac{1}{2}[x, y] \text{ mod } \mathfrak{h} + U(x, y)$$

factors through a Nomizu map  $L : \mathfrak{g}_{\bar{0}} \rightarrow \mathfrak{gl}(V)$  which satisfies

- (iii)  $\text{im } L \subseteq \mathfrak{so}(V)$ ;
- (iv)  $\tilde{L}(x)y - \tilde{L}(y)x - [x, y] \text{ mod } \mathfrak{h} = 0$  for all  $x, y \in \mathfrak{g}_{\bar{0}}$ .

Indeed this is the Nomizu map associated to the Levi-Civita connection of  $(M, g)$  (cf. [45, Theorem 3.3] for the case of reductive homogeneous manifolds).

The Levi-Civita covariant derivative can be easily described, at least locally. Let  $\xi_i : G_{\bar{0}} \rightarrow V \cong \mathfrak{g}_{\bar{0}}/\mathfrak{h}$  be (locally defined) vector fields on  $M$ ,  $i = 1, 2$ , and choose (locally defined) vector fields  $\tilde{\xi}_i : G_{\bar{0}} \rightarrow \mathfrak{g}_{\bar{0}}$  on  $G_{\bar{0}}$  such that  $\xi_i$  is  $\pi$ -related to  $\tilde{\xi}_i$ , i.e., such that  $\pi_*(\tilde{\xi}_i) = \xi_i$  for  $i = 1, 2$ . Then

$$(23) \quad \nabla_{\xi_1} \xi_2 = \pi_*(\tilde{\xi}_1(\tilde{\xi}_2) + \tilde{L}(\tilde{\xi}_1)(\tilde{\xi}_2)),$$

where  $\tilde{\xi}_1(\tilde{\xi}_2)$  is the derivative of  $\tilde{\xi}_2$  along  $\tilde{\xi}_1$  and  $\tilde{L}$  acts as usual at any fixed  $g \in G_{\bar{0}}$ . For more details, we refer the reader to e.g. [42, §4].

We are now ready to state our main result, which subsumes Theorem 12.

**Theorem 13.** *Let  $(M, g, F)$  be an 11-dimensional lorentzian spin manifold endowed with a closed  $F \in \Omega^4(M)$  and  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$  the associated Killing superalgebra. If  $\dim \mathfrak{k}_1 > 16$  then  $(M, g, F)$  is locally homogeneous and the Killing superalgebra  $\mathfrak{k}$  (resp. Killing ideal  $\widehat{\mathfrak{k}}$ ) is a (resp. odd-generated) realizable filtered subdeformation of  $\mathfrak{p}$ .*

*Conversely, let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be (the opposite Lie superalgebra to) a realizable filtered subdeformation of  $\mathfrak{p}$ , with corresponding 11-dimensional homogeneous lorentzian spin manifold  $(M = G_0/H, g, Q)$ . Then there exist*

- (1) a  $G_0$ -invariant closed 4-form  $F \in \Omega^4(M)$ ;
- (2) an (anti)embedding  $\Phi : \mathfrak{g} \rightarrow \mathfrak{k}$  of realizable filtered subdeformations of  $\mathfrak{p}$  from  $\mathfrak{g}$  in the Killing superalgebra  $\mathfrak{k}$  of  $(M, g, F)$ . If  $\mathfrak{g}$  is odd-generated realizable, then  $\Phi(\mathfrak{g}) \subseteq \widehat{\mathfrak{k}}$ .

*In particular  $\dim \mathfrak{k}_1 > 16$ .*

*Proof.* The first statement is a direct consequence of the local homogeneity theorem in [31] and Theorem 12 in Section 3.2.

Let now  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be the opposite Lie superalgebra to a realizable filtered deformation of a graded subalgebra  $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V$  of  $\mathfrak{p}$  and  $(M = G_0/H, g, Q)$  an associated homogeneous lorentzian spin manifold. Since the Lie brackets of  $\mathfrak{g}$  are the opposite of those in (6) and (12), we have that the map  $L : \mathfrak{g}_0 \rightarrow \mathfrak{so}(V)$ ,

$$(24) \quad \begin{aligned} L(A) &= \text{ad}(A) & (A \in \mathfrak{h}) \\ L(v) &= -X_v & (v \in V), \end{aligned}$$

satisfies properties (i)-(iv) and therefore is the Nomizu map corresponding to the Levi-Civita connection of  $(M, g)$ .

Consider the fundamental vector field

$$(25) \quad \xi^{(x)} : G_0 \rightarrow V, \quad \xi^{(x)}(g) = (\text{Ad}_{g^{-1}} x) \bmod \mathfrak{h}$$

associated to  $x = (A, v) \in \mathfrak{g}_0 \cong \mathfrak{h} \oplus V$ . Clearly  $\xi^{(x)}$  is a Killing vector field and using equations (23), (24) and (25), it can be checked directly that the value of the section  $(\xi^{(x)}, -\nabla \xi^{(x)})$  of  $\mathcal{E}_0 = TM \oplus \mathfrak{so}(TM)$  at  $o = eH \in M$  is  $(v, A + X_v) \in V \oplus \mathfrak{so}(V)$ . This gives the realisation of the abstract Lie algebra  $\mathfrak{g}_0$  as subalgebra of the algebra of Killing vector fields on  $M$ :

$$(26) \quad [\xi^{(x)}, \xi^{(y)}] = -\xi^{([x, y])},$$

for all  $x, y \in \mathfrak{g}_0$ . Given any admissible  $\varphi \in \Lambda^4 V^* \cong \Lambda^4 T_o^* M$ , we let  $F \in \Omega^4(M)$  be the unique  $G_0$ -invariant closed 4-form with value  $F_o = \varphi$  at  $o \in M$ . As for the elements of  $\mathfrak{g}_0$ , every  $s \in \mathfrak{g}_1 \cong S' \subseteq S$  has an associated spinor field

$$(27) \quad \varepsilon^{(s)} : G_0 \rightarrow S, \quad \varepsilon^{(s)}(g) = \text{Ad}_{g^{-1}} s.$$

For any vector field  $\xi : G_0 \rightarrow V$  with  $\pi$ -related  $\tilde{\xi} : G_0 \rightarrow \mathfrak{g}_0$  we compute

$$\begin{aligned} \nabla_\xi \varepsilon^{(s)} &= \pi_*(\tilde{\xi}(\varepsilon^{(s)}) + \sigma(\tilde{L}(\tilde{\xi}))(\varepsilon^{(s)})) \\ &= \pi_*(-\text{ad}(\tilde{\xi})(\varepsilon^{(s)}) + \sigma(\tilde{L}(\tilde{\xi}))(\varepsilon^{(s)})) \\ &= \beta^\varphi(\xi, \varepsilon^{(s)}) \end{aligned}$$

where the last equality follows from the Lie brackets of  $\mathfrak{g}_0$  and (24). This shows that  $\varepsilon^{(s)}$  is a Killing spinor, for all  $s \in S'$ .

The required map  $\Phi : \mathfrak{g} \rightarrow \mathfrak{k}$  is defined by:

$$\Phi(x) = \xi^{(x)} \quad \text{and} \quad \Phi(s) = \varepsilon^{(s)},$$

where  $x = (v, A) \in \mathfrak{g}_0$  and  $s \in \mathfrak{g}_1$ . Note that

$$\begin{aligned} \mathcal{L}_{\xi^{(x)}} \varepsilon^{(s)} &= \nabla_{\xi^{(x)}} \varepsilon^{(s)} - \sigma(\nabla \xi^{(x)}) \varepsilon^{(s)} \\ &= \beta^\varphi(\xi^{(x)}, \varepsilon^{(s)}) - \sigma(\nabla \xi^{(x)}) \varepsilon^{(s)} \end{aligned}$$

so that  $\mathcal{L}_{\xi^{(x)}} \varepsilon^{(s)}$  is the Killing spinor on  $M$  with value  $\beta^\varphi(v, s) + \sigma(A)s + \sigma(X_v)s$  at  $o \in M$ . In other words

$$(28) \quad [\xi^{(x)}, \varepsilon^{(s)}] = -\varepsilon^{([x,s])}$$

and one similarly checks

$$(29) \quad [\varepsilon^{(s)}, \varepsilon^{(s)}] = -\xi^{([s,s])}.$$

Identities (26), (28) and (29) show that  $\Phi$  is a Lie superalgebra anti-homomorphism. The fact that  $\Phi$  is an (anti)embedding of realizable filtered subdeformations of  $\mathfrak{p}$  is immediate, as well as the last two claims of the theorem. □

**Remark 14.** The  $G_0$ -invariant closed  $F \in \Omega^4(M)$  associated to a realizable filtered deformation in Theorem 13 is a priori not unique, since it appears to depend on the choice of an admissible  $\varphi \in \Lambda^4 V$  (recall Definition 9).

However, as already advertised, we will obtain  $\ker \iota^* = 0$  in Corollary 26, so that  $\varphi$  (and  $F$ ) are unique.

**Remark 15.** The Killing superalgebra  $\mathfrak{k}$  of the homogeneous lorentzian spin manifold  $(M, g, Q, F)$  associated to  $\mathfrak{g}$  in Theorem 13 is strictly larger than  $\mathfrak{g}$  in general. (The analogous statement holds for Killing ideals  $\widehat{\mathfrak{k}}$  and odd-generated realizable  $\mathfrak{g}$ .) We do not know of general conditions on  $\mathfrak{g}$  under which equality actually holds.

Theorem 13 and the above remarks say that Killing superalgebras (resp. Killing ideals) of highly supersymmetric  $(M, g, F)$ , up to local equivalence, are in a one-to-one correspondence with *maximal* realizable (resp. odd-generated realizable) filtered subdeformations of  $\mathfrak{p}$ , up to isomorphism of filtered subdeformations.

In Sections 4 and 5 below, we set up the classification problem for the Killing superalgebras of highly supersymmetric 11-dimensional supergravity backgrounds as the classification problem of realizable filtered subdeformations of  $\mathfrak{p}$ . In particular, we show that high supersymmetry implies that the Einstein and Maxwell equations are satisfied; that is, the homogeneous lorentzian spin manifold reconstructed in Theorem 13 from a realizable filtered subdeformation is automatically a supergravity background.

In this regard, we remark that one needs the full datum of a realizable filtered subdeformation of  $\mathfrak{p}$  to reconstruct the supergravity background unambiguously; the assignment of a Lie superalgebra is not sufficient in general. For instance there is an example of a Lie superalgebra with (at least) two *non-isomorphic* structures of realizable filtered subdeformation of  $\mathfrak{p}$ : namely, the Killing superalgebra of a supergravity background with 24 supercharges described in [46] and shown in [47] to be isomorphic *as an abstract Lie superalgebra* to a subalgebra of the Killing superalgebra of the maximally supersymmetric pp-wave of [10].

#### 4. The classification problem for Killing superalgebras

We have just seen that the Killing superalgebra is a filtered subdeformation of the Poincaré superalgebra. In the highly supersymmetric case, Proposition 6 applies and the aim of this section is to improve that result in the case of (odd-generated) realizable filtered subdeformations.



### 4.1. The Jacobi identity of Killing superalgebras

The Lie brackets of a Killing superalgebra are given by equation (19) in terms of the following data.

First we have a graded Lie subalgebra  $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V'$  of the Poincaré superalgebra. In particular, this means that  $\mathfrak{h} < \mathfrak{so}(V)$  stabilises both  $S' \subseteq S$  and  $V' \subseteq V$  and that  $\kappa(S', S') \subseteq V'$ . The rest of the data consists of an  $\mathfrak{h}$ -invariant  $\varphi \in \Lambda^4 V$ ,  $X : V' \rightarrow \mathfrak{so}(V)$  (or, more precisely,  $V' \rightarrow \mathfrak{so}(V)/\mathfrak{h}$ ) and  $R : \Lambda^2 V' \rightarrow \mathfrak{so}(V)$ . In terms of this data, we have the following Lie brackets on the vector space  $\mathfrak{h} \oplus S' \oplus V'$ :

$$\begin{aligned}
 [A, B] &= AB - BA \\
 [A, s] &= \sigma(A)s \\
 [A, v] &= Av + [A, X_v] - X_{Av} \\
 [s, s] &= \kappa(s, s) + \gamma^\varphi(s, s) - X_{\kappa(s,s)} \\
 [v, s] &= \beta^\varphi(v, s) + \sigma(X_v)s \\
 [v, w] &= X_v w - X_w v + [X_v, X_w] - X_{X_v w - X_w v} + R(v, w),
 \end{aligned}
 \tag{30}$$

where  $A, B \in \mathfrak{h}$ ,  $v, w \in V'$  and  $s \in S'$ . It bears reminding that the right-hand sides of the Lie brackets in (30) take values in  $\mathfrak{h} \oplus S' \oplus V'$ , but that the individual terms may not. For example,  $[A, X_v], X_{Av} \in \mathfrak{so}(V)$ , but their difference  $[A, X_v] - X_{Av} \in \mathfrak{h}$ . Similarly,  $\gamma^\varphi(s, s) - X_{\kappa(s,s)} \in \mathfrak{h}$ , but  $\gamma^\varphi(s, s), X_{\kappa(s,s)} \in \mathfrak{so}(V)$ , and the same happens with  $[X_v, X_w] - X_{X_v w - X_w v} + R(v, w) \in \mathfrak{h}$ , even though  $[X_v, X_w], X_{X_v w - X_w v}, R(v, w) \in \mathfrak{so}(V)$ . Also  $X_v w - X_w v \in V'$ ,  $\beta^\varphi(v, s) + \sigma(X_v)s \in S'$ , but  $X_v w \in V$  and  $\beta^\varphi(v, s), \sigma(X_v)s \in S$ .

The only additional conditions come from demanding that the Lie brackets (30) do define a Lie superalgebra. In other words, they come from imposing the Jacobi identity. There are ten components of the Jacobi identity and we summarise the results for each component in turn.

**The  $[\mathfrak{h}\mathfrak{h}\mathfrak{h}]$  Jacobi.** This is automatically satisfied because  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{so}(V)$ .

**The  $[\mathfrak{h}\mathfrak{h}S']$  Jacobi.** This is automatically satisfied because the action of  $\mathfrak{h}$  on  $S'$  is the restriction to  $\mathfrak{h}$  and  $S'$  of the spinor representation  $\sigma$  of  $\mathfrak{so}(V)$  on  $S$ .

**The  $[\mathfrak{h}\mathfrak{h}V']$  Jacobi.** This is also automatically satisfied, extending the adjoint action of  $\mathfrak{h}$  on itself to  $\mathfrak{so}(V)$ .

**The  $[\mathfrak{h}S'S']$  Jacobi.** This is automatically satisfied since  $\mathfrak{h} < \mathfrak{so}(V) \cap \mathbf{stab}(\varphi)$ . Indeed, for  $A \in \mathfrak{h}$  and  $s \in S'$ ,

$$\begin{aligned} [A, [s, s]] &= [A, \kappa(s, s) + \gamma^\varphi(s, s) - X_{\kappa(s,s)}] \\ &= A\kappa(s, s) + [A, \gamma^\varphi(s, s)] - X_{A\kappa(s,s)}, \end{aligned}$$

whereas

$$2[[A, s], s] = 2[\sigma(A)s, s] = 2\kappa(\sigma(A)s, s) + 2\gamma^\varphi(\sigma(A)s, s) - 2X_{\kappa(\sigma(A)s,s)}.$$

Since  $\mathfrak{h} < \mathfrak{so}(V)$ ,  $A\kappa(s, s) = 2\kappa(\sigma(A)s, s)$ , so that the Jacobi identity is satisfied provided that

$$[A, \gamma^\varphi(s, s)] = 2\gamma^\varphi(\sigma(A)s, s).$$

But  $\gamma^\varphi$  only depends on  $\mathfrak{so}(V)$ -equivariant operations like Clifford product and Dirac current, and on  $\varphi$ . It follows that  $\gamma^\varphi$  is equivariant under  $\mathfrak{so}(V) \cap \mathbf{stab}(\varphi)$ , and by restriction also under  $\mathfrak{h}$ .

**The  $[\mathfrak{h}S'V']$  Jacobi.** In this case, for  $A \in \mathfrak{h}$ ,  $v \in V'$  and  $s \in S'$ ,

$$[A, [v, s]] - [[A, v], s] - [v, [A, s]] = \sigma(A)\beta^\varphi(v, s) - \beta^\varphi(Av, s) - \beta^\varphi(v, \sigma(A)s).$$

The Jacobi identity is again satisfied since  $\mathfrak{h} < \mathfrak{so}(V) \cap \mathbf{stab}(\varphi)$ .

**The  $[\mathfrak{h}V'V']$  Jacobi.** A somewhat lengthy calculation shows that, for all  $A \in \mathfrak{h}$  and  $v, w \in V'$ ,

$$[A, [v, w]] - [[A, v], w] - [v, [A, w]] = [A, R(v, w)] - R(Av, w) - R(v, Aw).$$

It follows that the Jacobi identity is satisfied if and only if

$$(31) \quad R : \Lambda^2 V' \rightarrow \mathfrak{so}(V) \text{ is } \mathfrak{h}\text{-equivariant.}$$

**The  $[S'S'S']$  Jacobi.** The Jacobi identity says that  $[[s, s], s] = 0$  for all  $s \in S'$ , and it expands to

$$\sigma(\gamma^\varphi(s, s))s = -\beta^\varphi(\kappa(s, s), s),$$

for all  $s \in S'$ . This identity is known to be automatically satisfied for all  $s \in S$ , cf. [32, Proposition 7].

**The  $[S'S'V']$  Jacobi.** After another somewhat lengthy calculation, and letting

$$\beta_v^\varphi(s) = \beta^\varphi(v, s)$$

for all  $v \in V$  and  $s \in S$ , the Jacobi identity is equivalent to

$$\begin{aligned} (32) \quad \frac{1}{2}R(v, \kappa(s, s))w &= \kappa((X_v\beta^\varphi)(w, s), s) + \gamma^\varphi(\beta_v^\varphi s, s)(w) \\ &= \kappa((X_v\beta^\varphi)(w, s), s) - \kappa(\beta_v^\varphi(s), \beta_w^\varphi(s)) \\ &\quad - \kappa(\beta_w^\varphi\beta_v^\varphi(s), s), \end{aligned}$$

for all  $s \in S'$ ,  $v \in V'$  and  $w \in V$ .

**Remark 16.** If  $\dim S' > 16$ , then by local homogeneity  $V' = V$ , and equation (32) expresses the curvature operator  $R : \Lambda^2V \rightarrow \mathfrak{so}(V)$  in terms of  $X$  and  $\varphi$ . By a further contraction, this determines the Ricci tensor and as we will show in Section 5, it implies the bosonic field equations of 11-dimensional supergravity.

**The  $[S'V'V']$  Jacobi.** This Jacobi identity expands to the following condition

$$(33) \quad R(v, w)s = (X_v\beta^\varphi)(w, s) - (X_w\beta^\varphi)(v, s) + [\beta_v^\varphi, \beta_w^\varphi](s),$$

for all  $s \in S'$  and  $v, w \in V'$ .

**The  $[V'V'V']$  Jacobi.** Finally the last component of the Jacobi identity expands to two Bianchi-like identities, one algebraic

$$(34) \quad R(u, v)w + R(v, w)u + R(w, u)v = 0,$$

and one differential

$$\begin{aligned} (35) \quad R(X_uv - X_vu, w) + R(X_vw - X_wv, u) + R(X_wu - X_uw, v) \\ = [X_w, R(u, v)] + [X_u, R(v, w)] + [X_v, R(w, u)], \end{aligned}$$

for all  $u, v, w \in V'$ . If  $V' = V$ , (34) is precisely the algebraic Bianchi identity for  $R$ , whereas the differential identity simplifies to

$$(36) \quad (X_uR)(v, w) + (X_vR)(w, u) + (X_wR)(u, v) = 0.$$

(Notice that  $X_u \in \mathfrak{so}(V)$ , but unless  $V' = V$ ,  $R \in \text{Hom}(\Lambda^2V', \mathfrak{so}(V))$ , which is not an  $\mathfrak{so}(V)$ -module, but only an  $\mathfrak{h}$ -module.)

#### 4.2. The classification problem for highly supersymmetric Killing superalgebras

Particularly interesting is the highly supersymmetric case, where  $\dim S' > 16$  so that  $V' = V$ . In this case, the classification problem for highly supersymmetric Killing superalgebras breaks up into two main steps:

- 1) classify highly supersymmetric graded subalgebras  $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V$  of the Poincaré superalgebra  $\mathfrak{p}$ ;
- 2) for each such  $\mathfrak{a}$ , find  $\varphi \in (\Lambda^4 V)^\mathfrak{h}$ ,  $R \in \text{Hom}(\Lambda^2 V, \mathfrak{so}(V))^\mathfrak{h}$  which is an algebraic curvature tensor (i.e., satisfying the algebraic Bianchi identity (34)) and  $X : V \rightarrow \mathfrak{so}(V)$  (only its image modulo  $\mathfrak{h}$  matters) such that:
  - (i)  $\varphi$  is closed, cf. (ii) of Definition 9;
  - (ii) the right-hand sides of the expressions in (30) take values in  $\mathfrak{h} \oplus S' \oplus V$ ;
  - (iii) the three equations (32), (33) and (36) are satisfied.

The Jacobi identity (32) determines  $R$  in terms of  $\varphi$  and  $X$  so that the highly supersymmetric Killing superalgebra (or, more generally, any realizable filtered subdeformation of  $\mathfrak{p}$ ) is completely determined by  $(\mathfrak{h}, S', \varphi, X)$ . This result improves Proposition 6 in the case of *realizable* filtered subdeformations.

Step (1) of the classification problem is too broad and not tied to the existence of *nontrivial* filtered subdeformation of a given graded algebra. We can fare better if we restrict the classification problem to the Killing ideals. In the next section, we consider the odd-generated realizable case and we will show that one can fully specify Killing ideals in terms of simpler data than  $(\mathfrak{h}, S', \varphi, X)$  and, at the same time, modify step (1) by the addition of further constraints.

#### 4.3. Killing ideals and Lie pairs

To state the main result of this section, we first need to introduce some preliminary notions. Let  $S$  be the spinor representation of  $\mathfrak{so}(V)$ . It is well-known that

$$(37) \quad \odot^2 S \cong \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V,$$

as  $\mathfrak{so}(V)$ -modules. This decomposition is unique, since all the three summands are  $\mathfrak{so}(V)$ -irreducible and inequivalent, and we may (and in this section will) consider  $\Lambda^q V$  directly as a subspace of  $\odot^2 S$ , for  $q = 1, 2, 5$ . We decompose any element  $\omega \in \odot^2 S$  into  $\omega = \omega^{(1)} + \omega^{(2)} + \omega^{(5)}$  according to (37), where  $\omega^{(q)} \in \Lambda^q V$  for  $q = 1, 2, 5$ .

If  $S'$  is a given linear subspace of  $S$  with  $\dim S' > 16$ , then  $\odot^2 S' \subset \odot^2 S$  projects surjectively on  $\Lambda^1 V$ , through the Dirac current operation. The embedding

$$\odot^2 S' \subset \odot^2 S = \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V$$

is in general diagonal and one cannot expect  $\odot^2 S'$  to contain  $\Lambda^q V$ , not even if  $q = 1$ . This motivates the following.

Let  $S'$  be a subspace of  $S$ ,  $\dim S' > 16$ . Then restricting the Dirac current  $\kappa : \odot^2 S \rightarrow V$  to  $\odot^2 S'$  gives rise to a short exact sequence:

$$0 \longrightarrow \mathfrak{D} \longrightarrow \odot^2 S' \xrightarrow{\kappa} V \longrightarrow 0,$$

where  $\mathfrak{D} = \mathfrak{D}(S')$  is the *Dirac kernel* of  $S'$ ; that is, the subspace of  $\odot^2 S$  given by

$$\begin{aligned} \mathfrak{D} &= \odot^2 S' \cap (\Lambda^2 V \oplus \Lambda^5 V) \\ &= \left\{ \omega \in \odot^2 S' \mid \omega^{(1)} = 0 \right\}. \end{aligned}$$

A splitting of the above short exact sequence — that is, a linear map  $\Sigma : V \rightarrow \odot^2 S'$  such that  $\Sigma(v)^{(1)} = v$  for all  $v \in V$  — is called a *section* associated to  $S'$  and we may write it as  $\Sigma(S')$  if we need to specify  $S'$ . A section  $\Sigma$  associated to  $S'$  always exists and it is unique up to elements in the Dirac kernel.

Let  $(S', \varphi)$  be a pair consisting of a subspace  $S'$  of  $S$  with  $\dim S' > 16$  and  $\varphi \in \Lambda^4 V$ .

**Definition 17.** The *envelope*  $\mathfrak{h}_{(S', \varphi)}$  of  $(S', \varphi)$  is the subspace of  $\mathfrak{so}(V)$  given by

$$\begin{aligned} \mathfrak{h}_{(S', \varphi)} &= \{ \gamma^\varphi(\omega) \mid \omega \in \mathfrak{D} \} \\ &= \left\{ \gamma^\varphi(\omega) \mid \omega \in \odot^2 S' \text{ with } \omega^{(1)} = 0 \right\}. \end{aligned}$$

The pair  $(S', \varphi)$  is called a *Lie pair* if

- (i)  $A \cdot \varphi = 0$  for every  $A \in \mathfrak{h}_{(S', \varphi)}$ ; and

(ii)  $\sigma(A)s \in S'$  for every  $A \in \mathfrak{h}_{(S',\varphi)}$  and  $s \in S'$ .

The name “Lie pair” is motivated by the following

**Lemma 18.** *The envelope  $\mathfrak{h}_{(S',\varphi)}$  of a Lie pair  $(S', \varphi)$  is a Lie subalgebra of  $\mathfrak{so}(V)$ .*

*Proof.* The map  $\gamma^\varphi : \odot^2 S \rightarrow \mathfrak{so}(V)$  is equivariant under  $\mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$ , hence the restriction  $\gamma^\varphi|_{\odot^2 S'} : \odot^2 S' \rightarrow \mathfrak{so}(V)$  to  $S'$  is equivariant under  $\mathfrak{so}(V) \cap \mathfrak{stab}(\varphi) \cap \mathfrak{stab}(S')$ . Now  $\mathfrak{h}_{(S',\varphi)} \subset \mathfrak{so}(V) \cap \mathfrak{stab}(\varphi) \cap \mathfrak{stab}(S')$  by properties (i) and (ii) of Definition 17. In particular, for any  $A \in \mathfrak{h}_{(S',\varphi)}$  and  $\omega \in \mathfrak{D}$ , we have  $[A, \gamma^\varphi(\omega)] = \gamma^\varphi(A \cdot \omega)$ , with  $A \cdot \omega \in \mathfrak{D}$ . In other words  $[\mathfrak{h}_{(S',\varphi)}, \mathfrak{h}_{(S',\varphi)}] \subset \mathfrak{h}_{(S',\varphi)}$ , proving the lemma.  $\square$

The following result gives necessary conditions that are satisfied by any odd-generated realizable filtered subdeformation. We recall that a realizable  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  is called odd-generated realizable if in addition  $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_1]$ .

**Proposition 19.** *Let  $\mathfrak{g}$  be a odd-generated realizable filtered subdeformation of  $\mathfrak{p}$ , with associated graded algebra  $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V$ . Then the associated pair  $(S', \varphi)$  is a Lie pair and*

- (1) *the isotropy  $\mathfrak{h}$  equals the envelope of  $(S', \varphi)$ ; i.e.,  $\mathfrak{h} = \mathfrak{h}_{(S',\varphi)}$ ;*
- (2) *the map  $X : V \rightarrow \mathfrak{so}(V)$  is determined, up to elements in  $\mathfrak{h}$ , by the identity*

$$X = \gamma^\varphi \circ \Sigma,$$

where  $\Sigma$  is any section associated to  $S'$ .

*In particular  $\mathfrak{g}$  is completely determined, up to isomorphisms of filtered subdeformations, by the associated Lie pair  $(S', \varphi)$ .*

*Proof.* Let  $T := [-, -]|_{\odot^2 S'} : \odot^2 S' \rightarrow V \oplus \mathfrak{h}$  be the tensor given by the Lie bracket between odd elements. By equation (30), it has the following explicit expression for all  $\omega = \omega^{(1)} + \omega^{(2)} + \omega^{(5)} \in \odot^2 S'$ :

$$\begin{aligned} (38) \quad T(\omega) &= \kappa(\omega) + \gamma^\varphi(\omega) - X_{\kappa(\omega)} \\ &= \omega^{(1)} + \gamma^\varphi(\omega) - X_{\omega^{(1)}}, \end{aligned}$$

with  $\omega^{(1)} \in V$  and  $\gamma^\varphi(\omega) - X_{\omega^{(1)}} \in \mathfrak{h}$ . The last identity in (38) follows from the fact that the kernel of the Dirac current  $\kappa : \odot^2 S \rightarrow V$  is  $\Lambda^2 V \oplus \Lambda^5 V$ .

The tensor  $T$  is surjective since  $\mathfrak{g}$  is odd-generated. In particular any  $A \in \mathfrak{h}$  is of the form  $A = T(\omega)$ , for some  $\omega \in \odot^2 S'$ . By equation (38), the condition  $T(\omega) \in \mathfrak{h}$  is equivalent to  $\omega^{(1)} = 0$  and hence  $A = \gamma^\varphi(\omega)$  for some  $\omega \in \mathfrak{D}$ . In other words,  $\mathfrak{h} = \mathfrak{h}_{(S', \varphi)}$ , which proves (1).

Surjectivity of  $T$  also allows one to choose (in a linear fashion) for every  $v \in V$  an element  $\Sigma(v) \in \odot^2 S'$  with  $T(\Sigma(v)) = v$ . Note that  $\Sigma(v)^{(1)} = v$  by equation (38), i.e.,  $\Sigma : V \rightarrow \odot^2 S'$  is a section associated to  $S'$ . On the other hand  $\gamma^\varphi(\Sigma(v)) - X_v = 0$  for all  $v \in V$ , i.e.,  $X = \gamma^\varphi \circ \Sigma$ . Since sections associated to  $S'$  differ by elements in  $\mathfrak{D}$ , a different choice of  $\Sigma$  determines  $X$  up to elements in  $\mathfrak{h} = \mathfrak{h}_{(S', \varphi)}$ . This proves (2).

The fact that  $(S', \varphi)$  is a Lie pair is a direct consequence of  $\mathfrak{h} \subset \mathfrak{stab}(S') \cap \mathfrak{stab}(\varphi)$ ; the last claim of the proposition follows from (1), (2) and Section 4.2. □

Proposition 19 improves Proposition 6 in the case of *odd-generated realizable* filtered subdeformations. It also allows to modify step (1) of the classification problem in Section 4.2 with the following step:

- (1') classify Lie pairs  $(S', \varphi)$  (and therefore the corresponding graded algebras  $\mathfrak{a} = \mathfrak{h}_{(S', \varphi)} \oplus S' \oplus V$ ), up to isomorphism.

Here we say that two pairs  $(S', \varphi) \cong (g \cdot S', g \cdot \varphi)$  are isomorphic, where  $g \in \text{Spin}(V)$ . In this case

$$\mathfrak{D}(g \cdot S') = g \cdot \mathfrak{D}(S'), \quad \Sigma(g \cdot S') = g \cdot \Sigma(S'), \quad \mathfrak{h}_{(g \cdot S', g \cdot \varphi)} = g \cdot \mathfrak{h}_{(S', \varphi)}$$

and it is immediate that  $(g \cdot S', g \cdot \varphi)$  is a Lie pair if and only if  $(S', \varphi)$  is a Lie pair.

### 5. Towards the field equations

In this section we explore the possibility of deriving the field equations from the Jacobi identity (32). The main result is Theorem 23 in Section 5.2, which states that if the Killing superalgebra is highly supersymmetric, then the bosonic field equations are satisfied. We begin with some preliminary results. We shall only need some of the formulae in the propositions below, but we record them all for completeness and because one of the identities corrects a small error which has propagated in the literature.

### 5.1. The algebraic and differential relations

Let  $(M, g, F)$  be an 11-dimensional lorentzian spin manifold endowed with a closed 4-form  $F \in \Omega^4(M)$ . Associated to any spinor field  $\varepsilon \in \Gamma(\mathbb{S})$ , there are differential forms on  $M$ , defined as follows:

- (i)  $\omega^{(1)} \in \Omega^1(M)$ , where  $\omega^{(1)}(Z) = \langle \varepsilon, Z \cdot \varepsilon \rangle$ ;
- (ii)  $\omega^{(2)} \in \Omega^2(M)$ , where  $\omega^{(2)}(Z_1, Z_2) = \langle \varepsilon, (Z_1 \wedge Z_2) \cdot \varepsilon \rangle$ ;
- (iii)  $\omega^{(5)} \in \Omega^5(M)$ , where  $\omega^{(5)}(Z_1, \dots, Z_5) = \langle \varepsilon, (Z_1 \wedge \dots \wedge Z_5) \cdot \varepsilon \rangle$ .

The 1-form  $\omega^{(1)}$  is the metric dual of the Dirac current  $K = \kappa(\varepsilon, \varepsilon) \in \mathfrak{X}(M)$  of  $\varepsilon$ . Forms  $\omega^{(q)} \in \Omega^q(M)$  can also be defined in a similar way for  $q = 6, 9, 10$  and it is straightforward to check that they are the Hodge duals  $\omega^{(q)} = \star \omega^{(11-q)}$  of (i)-(iii).

The differential forms defined above satisfy certain algebraic relations which are a consequence of the underlying Clifford algebra. They are usually proved by repeated applications of Fierz rearrangements.

**Proposition 20.** *([15, p. 5], [16, p. 21]) Let  $\varepsilon \in \Gamma(\mathbb{S})$  be a spinor field, with associated Dirac current  $K = \kappa(\varepsilon, \varepsilon) \in \mathfrak{X}(M)$ . Then:*

- (39)  $\|\omega^{(2)}\|^2 = 5\|\omega^{(1)}\|^2$
- (40)  $\|\omega^{(5)}\|^2 = -6\|\omega^{(1)}\|^2$
- (41)  $g(\iota_Z \omega^{(2)}, \iota_W \omega^{(2)}) = -\omega^{(1)}(Z)\omega^{(1)}(W) + g(Z, W)\|\omega^{(1)}\|^2$
- (42)  $g(\iota_Z \omega^{(5)}, \iota_W \omega^{(5)}) = 14\omega^{(1)}(Z)\omega^{(1)}(W) - 4g(Z, W)\|\omega^{(1)}\|^2$
- (43)  $\iota_K \omega^{(1)} = \|\omega^{(1)}\|^2$
- (44)  $\iota_K \omega^{(2)} = 0$
- (45)  $\iota_K \omega^{(5)} = -\frac{1}{2}\omega^{(2)} \wedge \omega^{(2)}$
- (46)  $\iota_K \star \omega^{(5)}(Z_1, \dots, Z_5) = \text{Skew}_{Z_1, \dots, Z_5} g(\iota_{Z_1} \omega^{(2)}, \iota_{Z_2} \dots \iota_{Z_5} \omega^{(5)})$
- (47)  $\|\omega^{(1)}\|^2 \omega^{(2)} \wedge \omega^{(5)} = -\frac{1}{2}\omega^{(1)} \wedge \omega^{(2)} \wedge \omega^{(2)} \wedge \omega^{(2)}$   
 $g(\iota_{Z_1} \omega^{(2)}, \iota_{Z_2} \dots \iota_{Z_6} \star \omega^{(5)}) = 5 \text{Skew}_{Z_2, \dots, Z_6} g(Z_1, Z_2) \iota_K \omega^{(5)}(Z_3, \dots, Z_6)$
- (48)  $-5 \text{Skew}_{Z_2, \dots, Z_6} \omega^{(5)}(Z_1, Z_2, \dots, Z_5) \omega^{(1)}(Z_6)$
- (49)  $\omega^{(2)} \wedge \omega^{(2)}(Z_1, \dots, Z_4) = -\frac{6}{5} \text{Skew}_{Z_1, \dots, Z_4} g(\iota_{Z_1} \iota_{Z_2} \omega^{(5)}, \iota_{Z_3} \iota_{Z_4} \omega^{(5)})$

for all vector fields  $Z, W, Z_i \in \mathfrak{X}(M)$ ,  $i = 1, \dots, 6$ , where *Skew* is skew-symmetrisation with weight one.



Formulae in Proposition 20 are by no means exhaustive. We note that some of our identities differ in sign from those in [15] and [16]; this is due to our conventions on the metric, which is “mostly minus”, and Clifford algebras. Equation (46) corrects equation (2.14) of [15] and equation (B.6) of [16].

The covariant derivative of the differential forms were also calculated in [15] and [16]. They are summarised in the following.

**Proposition 21.** *([15, p. 6], [16, p. 5]) Let  $\varepsilon \in \Gamma(\mathcal{S})$  be a Killing spinor on  $(M, g, F)$ , with associated Dirac current  $K = \kappa(\varepsilon, \varepsilon) \in \mathfrak{X}(M)$ . Then:*

$$\begin{aligned}
 (50) \quad \nabla_W \omega^{(1)}(Z) &= \frac{1}{3} \omega^{(2)}(\iota_Z \iota_W F) - \frac{1}{6} \star \omega^{(5)}(Z \wedge W \wedge F) \\
 \nabla_W \omega^{(2)}(Z_1, Z_2) &= -\frac{1}{3} \omega^{(1)}(\iota_W \iota_{Z_1} \iota_{Z_2} F) - \frac{1}{3} \omega^{(5)}(Z_1 \wedge Z_2 \wedge \iota_W F) \\
 &\quad + \frac{1}{6} \omega^{(5)}(W \wedge Z_1 \wedge \iota_{Z_2} F) - \frac{1}{6} \omega^{(5)}(W \wedge Z_2 \wedge \iota_{Z_1} F) \\
 (51) \quad &\quad - \frac{1}{6} g(W, Z_1) \omega^{(5)}(Z_2 \wedge F) + \frac{1}{6} g(W, Z_2) \omega^{(5)}(Z_1 \wedge F) \\
 \nabla_W \omega^{(5)}(Z_1, \dots, Z_5) &= \frac{5}{3} \text{Skew}_{Z_1, \dots, Z_5} \star \omega^{(5)}(Z_1 \wedge \dots \wedge Z_4 \wedge \iota_{Z_5} \iota_W F) \\
 &\quad - \frac{1}{3} \omega^{(2)} \wedge \iota_W F(Z_1, \dots, Z_5) \\
 &\quad - \frac{1}{6} \star \omega^{(1)}(Z_1 \wedge \dots \wedge Z_5 \wedge W \wedge F) \\
 &\quad - \frac{10}{6} \text{Skew}_{Z_1, \dots, Z_5} \star \omega^{(5)}(Z_1 \wedge Z_2 \wedge Z_3 \wedge \iota_{Z_4} \iota_{Z_5} (W \wedge F)) \\
 (52) \quad &\quad - \frac{5}{6} \text{Skew}_{Z_1, \dots, Z_5} \omega^{(2)}(Z_1 \wedge \iota_{Z_2} \dots \iota_{Z_5} (W \wedge F)).
 \end{aligned}$$

*In particular the exterior differentials of the forms are given by:*

$$\begin{aligned}
 (53) \quad d\omega^{(1)} &= \frac{1}{3} \star (F \wedge \omega^{(5)}) + \frac{2}{3} \star (\star F \wedge \omega^{(2)}) \\
 (54) \quad d\omega^{(2)} &= -\iota_K F \\
 (55) \quad d\omega^{(5)} &= \iota_K \star F - \omega^{(2)} \wedge F.
 \end{aligned}$$

From Propositions 20 and 21 we can immediately deduce the following result; the important identity (57) already appeared in [15, p. 7].

**Corollary 22.** *Let  $(M, g, F)$  be an 11-dimensional lorentzian spin manifold endowed with a closed 4-form  $F \in \Omega^4(M)$ . If  $\varepsilon \in \Gamma(\mathcal{S})$  is a Killing spinor, then the associated Dirac current  $K \in \mathfrak{X}(M)$  is an  $F$ -preserving Killing vector which satisfies*

$$(56) \quad \mathcal{L}_K \omega^{(1)} = \mathcal{L}_K \omega^{(2)} = \mathcal{L}_K \omega^{(5)} = 0$$

and

$$(57) \quad \iota_K(d \star F - \frac{1}{2}F \wedge F) = 0.$$

In particular if the Killing superalgebra  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$  is highly supersymmetric, then  $(M, g, F)$  satisfies the Maxwell equation of 11-dimensional supergravity.

*Proof.* The Dirac current  $K$  is a Killing vector, since (50) is evidently skew-symmetric in  $W$  and  $Z$ . Moreover  $\mathcal{L}_K F = d\iota_K F + \iota_K dF = d\iota_K F = 0$ , by  $dF = 0$  and (54). Equation  $\mathcal{L}_K \omega^{(1)} = 0$  is immediate, whereas

$$\begin{aligned} \mathcal{L}_K \omega^{(2)} &= \iota_K d\omega^{(2)} = -\iota_K \iota_K F = 0 \\ \mathcal{L}_K \omega^{(5)} &= d\iota_K \omega^{(5)} + \iota_K d\omega^{(5)} \\ &= -d\omega^{(2)} \wedge \omega^{(2)} - \iota_K(\omega^{(2)} \wedge F) \\ &= \iota_K F \wedge \omega^{(2)} - \omega^{(2)} \wedge \iota_K F = 0, \end{aligned}$$

using equations (44), (45), (54) and (55). Finally

$$\begin{aligned} 0 = \star \mathcal{L}_K F &= \mathcal{L}_K \star F = d\iota_K \star F + \iota_K d \star F \\ &= d(\omega^{(2)} \wedge F) + \iota_K d \star F = d\omega^{(2)} \wedge F + \iota_K d \star F \\ &= -\frac{1}{2}\iota_K(F \wedge F) + \iota_K d \star F = \iota_K(d \star F - \frac{1}{2}F \wedge F), \end{aligned}$$

using (54) and (55). The last claim is a direct consequence of (57) and the surjectivity of the Dirac current. □

### 5.2. High supersymmetry implies the field equations

The main result of this section is the following

**Theorem 23.** *Let  $(M, g, F)$  be an 11-dimensional lorentzian spin manifold endowed with a closed 4-form  $F \in \Omega^4(M)$ . If the associated Killing superalgebra  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$  is highly supersymmetric (i.e., if  $\dim \mathfrak{k}_1 > 16$ ) then  $(M, g, F)$  is a supergravity background; i.e., it satisfies the bosonic field equations of 11-dimensional supergravity:*

$$(58) \quad \begin{aligned} d \star F &= \frac{1}{2}F \wedge F, \\ \text{Ric}(Z, W) &= -\frac{1}{2}g(\iota_Z F, \iota_W F) + \frac{1}{6}\|F\|^2 g(Z, W), \end{aligned}$$

for all  $Z, W \in \mathfrak{X}(M)$ .

The proof of Theorem 23 will occupy the remainder of this section, but before we start let us remark that the theorem is sharp. Indeed, there exist lorentzian 11-dimensional manifolds  $(M, g)$  with  $F = 0$ , which admit a 16-dimensional space of parallel spinors and which are not Ricci-flat [2, 48].

Let us now turn to the proof of the theorem. From now on, we will use the Einstein summation convention and consider the canonical isomorphism  $\Lambda^\bullet V \cong Cl(V)$  of vector spaces. It sends a  $p$ -polyvector  $\Theta = \frac{1}{p!} \Theta^{a_1 \dots a_p} e_{a_1} \wedge \dots \wedge e_{a_p}$  into  $\frac{1}{p!} \Theta^{a_1 \dots a_p} \Gamma_{a_1 \dots a_p}$ , where  $(e_a)$  is any  $\eta$ -orthonormal basis of  $V$  and  $\Gamma_{a_1 \dots a_p}$  the totally antisymmetric product (with weight one) of the corresponding operators  $\Gamma_{a_i} \in Cl(V)$  of Clifford multiplication by  $e_{a_i} \in V$ . Finally, we denote by  $[\Xi]_p$  the  $p$ -form component of  $\Xi \in Cl(V)$ .

We begin with two useful lemmas.

**Lemma 24.** *Let  $\Theta \in \Lambda^p V$  be a  $p$ -polyvector. Then*

$$(59) \quad u \cdot \Theta = u \wedge \Theta - \iota_u \Theta \quad \text{and} \quad \Theta \cdot u = (-1)^p (u \wedge \Theta + \iota_u \Theta),$$

for all  $u \in V$ . In particular

$$(60) \quad \text{tr}_{v,w} v \cdot \Theta \cdot w = (-1)^{p+1} (11 - 2p) \Theta,$$

where  $\text{tr}_{v,w}$  is the tracing operation over  $v, w \in V$ .

**Lemma 25.** *Let  $\varphi \in \Lambda^4 V$  be a 4-polyvector. Then*

$$(61) \quad u \wedge \varphi = \frac{1}{2} (u \cdot \varphi + \varphi \cdot u) \quad \text{and} \quad \iota_u \varphi = \frac{1}{2} (\varphi \cdot u - u \cdot \varphi),$$

for all  $u \in V$ . Moreover

$$(62) \quad \varphi^2 = \varphi \cdot \varphi = \varphi \wedge \varphi + [\varphi^2]_4 + \|\varphi\|^2 \mathbb{1},$$

where  $[\varphi^2]_4 = -\frac{1}{8} \varphi^{abmn} \varphi_{mn}{}^{cd} \Gamma_{abcd}$ .

The identities in Lemmas 24 and 25 are obtained by routine calculations in  $Cl(V)$ . We omit the proof for the sake of brevity.

So let us now fix a point  $o \in M$  and assume  $\dim S' > 16$ , so that  $\kappa : \odot^2 S' \rightarrow V$  is surjective and  $V' = V$ . We will abuse notation slightly by using  $F$  both for the 4-form as for the value at  $o$ , which is an element of  $\Lambda^4 V \cong \Lambda^4 V^*$ .

We consider the  $[S'S'V]$  Jacobi identity (32) and take the inner product with a vector  $u \in V$  to arrive at

$$\eta(u, R(\kappa(s, s), v)w) = 2 \langle u \cdot \beta_w^F \cdot \beta_v^F \cdot s, s \rangle + 2 \langle u \cdot \beta_w^F \cdot s, \beta_v^F \cdot s \rangle - 2 \langle u \cdot (X_v \beta^F)(w, s), s \rangle,$$

for all  $u, v, w \in V$  and  $s \in S$ . Now the symplectic transpose of  $\beta_v^F = \frac{1}{24}(v \cdot F - 3F \cdot v)$  is  $\widetilde{\beta}_v^F = \frac{1}{24}(3v \cdot F - F \cdot v)$ , so that

$$\eta(u, R(\kappa(s, s), v)w) = 2 \left\langle \left( u \cdot \beta_w^F \cdot \beta_v^F + \widetilde{\beta}_v^F \cdot u \cdot \beta_w^F \right) \cdot s, s \right\rangle - 2 \langle u \cdot (X_v^F \beta)(w, s), s \rangle.$$

We now expand by using the definition of  $\beta^F$  and the fact that

$$(X_v \beta^F)(w, s) = \beta^{X_v F}(w, s) = \frac{1}{24}(w \cdot (X_v F) - 3(X_v F) \cdot w) \cdot s,$$

for all  $v, w \in V$  and  $s \in S$ . Dropping the Clifford multiplication  $\cdot$  from the notation, we arrive at

$$\begin{aligned} & \eta(u, R(\kappa(s, s), v)w) \\ &= \frac{2}{(24)^2} \langle (uwFvF - 3uFwvF - 3uwF^2v + 9uFwFv + 3vFuwF \\ & \quad - 9vFuFw - FvuwF + 3FvuFw) \cdot s, s \rangle \\ & \quad - \frac{1}{12} \langle (uw(X_v F) - 3u(X_v F)w) \cdot s, s \rangle. \end{aligned}$$

The Ricci tensor is obtained by “tracing” over  $v, w$ :

$$\begin{aligned} (63) \quad & \text{Ric}(u, \kappa(s, s)) \\ &= \text{tr}_{v,w} \eta(u, R(\kappa(s, s), v)w) \\ &= \frac{2}{(24)^2} \langle \Upsilon_u \cdot s, s \rangle - \frac{1}{12} \text{tr}_{v,w} \langle (uw(X_v F) - 3u(X_v F)w) \cdot s, s \rangle, \end{aligned}$$

where

$$\begin{aligned} \Upsilon_u = \text{tr}_{v,w} & (uwFvF - 3uFwvF - 3uwF^2v + 9uFwFv \\ & \quad + 3vFuwF - 9vFuFw - FvuwF + 3FvuFw). \end{aligned}$$

We treat the two terms in the RHS of (63) separately and in turn. First we expand the  $\Upsilon$  term by making use of (62) in Lemma 25 and the following

traces, which are a direct consequence of (60) in Lemma 24:

$$\begin{aligned} \text{tr}_{v,w} vFw &= -3F \\ \text{tr}_{v,w} vw &= -11\mathbb{1} \\ \text{tr}_{v,w} vuw &= 9u \\ \text{tr}_{v,w} vF^2w &= 5F \wedge F - 3[F^2]_4 - 11\|F\|^2\mathbb{1}. \end{aligned}$$

Therefore substituting this into  $\Upsilon_u$  we find

$$\begin{aligned} \Upsilon_u &= -12u(F \wedge F) + 12u[F^2]_4 + 36\|F\|^2u + 3(u \wedge F)F + 3F(u \wedge F) \\ &\quad + 15(\iota_u F)F - 15F(\iota_u F) - 9FuF - 9\text{tr}_{v,w} v(FuF)w. \end{aligned}$$

Remember, though, that this expression appears in

$$\langle \Upsilon_u \cdot s, s \rangle = -\langle s, \Upsilon_u \cdot s \rangle = -\langle \widetilde{\Upsilon}_u \cdot s, s \rangle,$$

where  $\widetilde{\Upsilon}_u$  is the symplectic transpose of  $\Upsilon_u$ , so that

$$\langle \Upsilon_u \cdot s, s \rangle = \frac{1}{2} \langle (\Upsilon_u - \widetilde{\Upsilon}_u) \cdot s, s \rangle.$$

Using that for  $\Theta$  a  $p$ -polyvector,  $\widetilde{\Theta} = (-1)^{p(p+1)/2}\Theta$ , we may thus replace  $\Upsilon_u$  by the following term

$$\begin{aligned} -12u(F \wedge F) + 12u[F^2]_4 + 36\|F\|^2u + 6(u \wedge F)F \\ + 30(\iota_u F)F - 9FuF - 9\text{tr}_{v,w} v(FuF)w. \end{aligned}$$

Identities (59) in Lemma 24 allows to further expand this term, and keeping in mind that only the skewsymmetric endomorphisms survive, we arrive at

$$\begin{aligned} \frac{1}{24}(\Upsilon_u - \widetilde{\Upsilon}_u) &= 4u \wedge F \wedge F + u \wedge [F^2]_4 + 3\|F\|^2u - [(u \wedge F)F]_5 \\ &\quad - 7[(u \wedge F)F]_1 + [(\iota_u F)F]_5 - 5[(\iota_u F)F]_1. \end{aligned}$$

Now observe that the 2nd, 4th and 6th terms add to zero, so that

$$\frac{1}{12} \langle \Upsilon_u \cdot s, s \rangle = \langle (4u \wedge F \wedge F + 3\|F\|^2u - 7[(u \wedge F)F]_1 - 5[(\iota_u F)F]_1) \cdot s, s \rangle$$

and, from (61) in Lemma 25, we arrive at

$$\frac{1}{12} \langle \Upsilon_u \cdot s, s \rangle = \langle (4u \wedge F \wedge F + 2\|F\|^2u - 6[FuF]_1) \cdot s, s \rangle.$$

It is clear after a moment's thought that

$$[FuF]_1 = (\alpha \|F\|^2 \eta_{ab} + \beta F_{ab}^2) u^a \Gamma^b,$$

for some  $\alpha, \beta \in \mathbb{R}$ , where

$$F_{ab}^2 = \eta(\iota_{e_a} F, \iota_{e_b} F) = \frac{1}{6} F_{amnp} F_b^{mnp}.$$

By taking  $F = \Gamma_{0123}$  and taking  $u = \Gamma_0$  and  $u = \Gamma_5$  in turn, say, we find that  $\alpha = 1$  and  $\beta = -2$ , so that in the end

$$(64) \quad \frac{2}{(24)^2} \langle \Upsilon_u \cdot s, s \rangle = \frac{1}{6} \langle u \wedge F \wedge F \cdot s, s \rangle + \frac{1}{2} F_{ab}^2 u^a \langle \Gamma^b s, s \rangle - \frac{1}{6} \|F\|^2 \langle u \cdot s, s \rangle.$$

Now we tackle the other terms in (63). We first observe that  $(v, X_v)$  is a Killing vector field which preserves  $F$ , by the geometric interpretation of the Killing superalgebra in Sections 3.1 and 3.2. In particular the Lie derivative  $\mathcal{L}_{(v, X_v)} F = \nabla_v F + X_v F = 0$  and hence  $X_v F = -\nabla_v F$ . Therefore,

$$\text{tr}_{v,w} w \cdot (X_v F) = -dF + \delta F \quad \text{and} \quad \text{tr}_{v,w} (X_v F) \cdot w = -dF - \delta F,$$

where  $dF$  is the exterior derivative and  $\delta F = -\star d \star F$  the divergence. It follows that

$$-\frac{1}{12} \text{tr}_{v,w} \langle (uw(X_v F) - 3u(X_v F)w) \cdot s, s \rangle = -\frac{1}{6} \langle u \cdot (dF + 2\delta F) \cdot s, s \rangle$$

and remembering that only the 1-, 2- and 5-form terms (and their duals) survive, we finally arrive at

$$(65) \quad \begin{aligned} & -\frac{1}{12} \text{tr}_{v,w} \langle (uw(X_v F) - 3u(X_v F)w) \cdot s, s \rangle \\ & = -\frac{1}{6} \langle (u \wedge dF - 2\iota_u \delta F) \cdot s, s \rangle. \end{aligned}$$

In summary, we add equations (64) and (65) to arrive at

$$(66) \quad \begin{aligned} \text{Ric}(u, \kappa(s, s)) &= \frac{1}{2} F_{ab}^2 u^a \langle \Gamma^b s, s \rangle - \frac{1}{6} \|F\|^2 \langle u \cdot s, s \rangle \\ &+ \frac{1}{6} \langle (u \wedge F \wedge F - u \wedge dF + 2\iota_u \delta F) \cdot s, s \rangle. \end{aligned}$$

There are three kinds of terms which depend on  $s$  in equation (66): terms which depend via the Dirac current, terms which depend via the 2-form bilinear  $\omega^{(2)}$  and terms which depend via the 5-form bilinear  $\omega^{(5)}$  (see Section 5.1 for definitions). The embedding  $\odot^2 S' \subset \odot^2 S = \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V$

is in general diagonal, and the fact that (66) has to be true for all  $s \in S'$  does not guarantee a priori that each of these three terms satisfies the equation separately; although they do in the maximally supersymmetric case when  $S' = S$ .

Notice however that the equation for the terms depending on the 5-form bilinear is

$$(67) \quad \langle (u \wedge dF) \cdot s, s \rangle = 0,$$

for all  $u \in V$  and  $s \in S'$ . Similarly the equation for the terms depending on the 2-form (or, dually, the 9-form) bilinear is

$$(68) \quad \langle (u \wedge F \wedge F + 2\iota_u \delta F) \cdot s, s \rangle = 0,$$

for all  $u \in V$  and  $s \in S'$ . By hypothesis  $dF = 0$ , so that (67) is automatically satisfied. By high supersymmetry and Corollary 22, the Maxwell equation of 11-dimensional supergravity is also satisfied and this directly implies equation (68). This then boils down equation (66) to the vanishing of the terms depending just on the Dirac current, namely:

$$(69) \quad \text{Ric}(u, \kappa(s, s)) = \frac{1}{2} F_{ab}^2 u^a \langle \Gamma^b s, s \rangle - \frac{1}{6} \|F\|^2 \langle u \cdot s, s \rangle,$$

which, since  $\kappa$  is surjective, is nothing but the expected Einstein equation  $\text{Ric}_{ab} = -\frac{1}{2} F_{ab}^2 + \frac{1}{6} \|F\|^2 g_{ab}$ . Theorem 23 is hence proved.

As a corollary, we now show that the space  $\ker \iota^*$  given in Lemma 7 vanishes if  $\dim S' > 16$ .

**Corollary 26.** *Let  $\mathfrak{a} = \mathfrak{h} \oplus S' \oplus V$  be a highly supersymmetric graded subalgebra of  $\mathfrak{p}$ . Then  $\ker \iota^* = 0$ . In particular a filtered deformation  $\mathfrak{g}$  of  $\mathfrak{a}$  has at most one admissible  $\varphi \in \Lambda^4 V$ .*

*Proof.* We first note that  $\iota^* : H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \rightarrow H^{2,2}(\mathfrak{a}_-, \mathfrak{p})$  depends only on the negatively graded part  $\mathfrak{a}_- = S' \oplus V$  of  $\mathfrak{a}$ . We can therefore assume without any loss of generality that  $\mathfrak{a} = \mathfrak{a}_-$  from now on, so that  $\mathfrak{h} = 0$ .

Now let  $\varphi \in \Lambda^4 V$  such that the corresponding class  $[\beta^\varphi + \gamma^\varphi] \in H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$  satisfies  $\iota^*[\beta^\varphi + \gamma^\varphi] = 0$ . In other words  $\beta^\varphi|_{V \otimes S'} = \gamma^\varphi|_{S' \otimes S'} = 0$ . Also let  $\mathfrak{g}$  be the filtered deformation of  $\mathfrak{a}$  determined by the brackets (6) and (12) with  $X = \rho = 0$ ; by construction  $\mathfrak{g}$  is a trivial realizable filtered subdeformation of  $\mathfrak{p}$ , with associated admissible 4-polyvector  $\varphi$ . Triviality here refers to the fact that  $\mathfrak{g} \cong \mathfrak{a}$  is actually a graded Lie subalgebra of  $\mathfrak{p}$ .

It follows from Theorem 13 that the associated homogeneous lorentzian spin manifold  $(M, g, Q, F)$ , where  $F_o = \varphi$ , has vanishing Riemann curvature. It is also highly supersymmetric so that, by Theorem 23, it solves the bosonic field equations. In particular, the Einstein equation says

$$(70) \quad 0 = -\frac{1}{2}g(\iota_Z F, \iota_W F) + \frac{1}{6}\|F\|^2 g(Z, W),$$

for all  $Z, W \in \mathfrak{X}(M)$ . Taking the trace over  $Z, W$  yields  $0 = -\frac{1}{6}\|F\|^2$  so that both terms in (70) have to vanish separately and  $g(\iota_Z F, \iota_W F) = 0$  for all  $Z, W \in \mathfrak{X}(M)$ . Using a Witt basis for  $T_o M$  it is then straightforward to see that this can only happen when  $\varphi = F_o = 0$ .  $\square$

As we have had ample opportunity to see, filtered deformations  $\mathfrak{g}$  of graded subalgebras  $\mathfrak{a}$  of  $\mathfrak{p}$  are not, in general, graded Lie subalgebras of  $\mathfrak{p}$ . By Corollary 26, the unique *highly supersymmetric* background associated to graded subalgebras of  $\mathfrak{p}$  is actually the Minkowski vacuum. In particular, the Minkowski vacuum is also the unique *highly supersymmetric* background with vanishing flux  $F$ .

Corollary 26 fails to hold in the general case. There are indeed other supergravity backgrounds whose associated Killing superalgebras are graded subalgebras of  $\mathfrak{p}$ . This is the case for some  $\frac{1}{2}$ -BPS solutions such as M2 and M5 branes, see e.g., [49], and it also seems to be the case for backgrounds asymptotic to the Minkowski vacuum. Finally, any Ricci-flat 11-dimensional lorentzian spin manifold endowed with a parallel spinor provides a low supersymmetric background with vanishing flux  $F$ , cf. [2, 48].

## 6. Summary and conclusions

In this paper we have elucidated the algebraic structure of the Lie superalgebra generated by the Killing spinors of an 11-dimensional supergravity background. We have shown that it is a filtered deformation of a  $\mathbb{Z}$ -graded subalgebra of the Poincaré superalgebra. (Parenthetically, this is not unique to 11-dimensional supergravity, but it is known to be the case for the Lie algebra of automorphisms of riemannian and conformal manifolds, as well as other supergravity theories. Moreover it is also expected to be the case for conformal supergravities.) Together with the (local) homogeneity theorem, which states that “highly supersymmetric” backgrounds (i.e, those preserving more than half of the supersymmetry) are locally homogeneous, this provides a new approach to the classification problem based on the classification of the Killing superalgebras (or the Killing ideals) of such backgrounds, which we have identified with a class of (odd-generated) realizable



filtered subdeformations of the Poincaré superalgebra. We have outlined in purely algebraic terms the classification problem of Killing ideals of highly supersymmetric supergravity backgrounds. It consists of two steps

- 1) classify all the Lie pairs  $(S', \varphi)$  up to isomorphism; and
- 2) for each such isomorphism class, consider all  $(R, X)$ , where  $R$  is an  $(\mathfrak{h} = \mathfrak{h}_{(S', \varphi)})$ -invariant algebraic curvature tensor and  $X : V \rightarrow \mathfrak{so}(V)/\mathfrak{h}$ , such that
  - (i) the 4-form  $F$  defined by  $\varphi$  is closed;
  - (ii) the right-hand sides of (30) take values in  $\mathfrak{h} \oplus S' \oplus V$ ; and
  - (iii) the three equations (32), (33) and (36) are satisfied.

Among the corollaries derived from this approach is the statement that high supersymmetry (and  $dF = 0$ ) imply the bosonic field equations. Hence we can be sure that classifying (maximal, odd-generated) realizable filtered subdeformations one classifies highly supersymmetric backgrounds.

### Acknowledgments

We are grateful to the anonymous referee for comments on a previous version of the paper. We believe that the paper has improved as a result of the peer-review process. The present version of the paper was finished while we were participating in the workshop “Geometry, Gravity and Supersymmetry” at the Mainz ITP. It is our pleasure to thank them for their hospitality and for providing a pleasant collaborative environment.

The research of JMF is supported in part by the grant ST/L000458/1 “Particle Theory at the Higgs Centre” from the UK Science and Technology Facilities Council. The research of AS is fully supported by a Marie-Curie research fellowship of the “Istituto Nazionale di Alta Matematica” (Italy). We are grateful to these funding agencies for their support.

### References

- [1] M. Duff, B. Nilsson, and C. Pope, *Kaluza–Klein supergravity*, Phys. Rep. **130** (1986), 1–142.
- [2] J. M. Figueroa-O’Farrill, *Breaking the M-waves*, Class. Quant. Grav. **17** (2000), 2925–2947. [arXiv:hep-th/9904124](#).
- [3] C. M. Hull, *Holonomy and symmetry in M-theory*, [arXiv:hep-th/0305039](#).

- [4] M. J. Duff and J. T. Liu, *Hidden space-time symmetries and generalized holonomy in M theory*, Nucl. Phys. **B674** (2003), 217–230. [arXiv:hep-th/0303140](#) [hep-th].
- [5] G. Papadopoulos and D. Tsimpis, *The Holonomy of the supercovariant connection and Killing spinors*, JHEP **07** (2003) 018, [arXiv:hep-th/0306117](#) [hep-th].
- [6] I. A. Bandos, J. A. de Azcarraga, J. M. Izquierdo, M. Picon, and O. Varela, *On BPS preons, generalized holonomies and  $D = 11$  supergravities*, Phys. Rev. **D69** (2004) 105010. [arXiv:hep-th/0312266](#) [hep-th].
- [7] J. M. Figueroa-O'Farrill and G. Papadopoulos, *Maximally supersymmetric solutions of ten- and eleven-dimensional supergravity*, J. High Energy Phys. **03** (2003) 048. [arXiv:hep-th/0211089](#).
- [8] P. Freund and M. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. **B97** (1980), 233–235.
- [9] K. Pilch, P. van Nieuwenhuizen, and P. K. Townsend, *Compactification of  $d=11$  supergravity on  $S^4$  (or  $11 = 7 + 4$ , too)*, Nucl. Phys. **B242** (1984) 377.
- [10] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, Phys. Lett. **134B** (1984), 194–196.
- [11] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest,  *$N = 31$ ,  $D = 11$* , J. High Energy Phys. **02** (2007) 043. [arXiv:hep-th/0610331](#).
- [12] J. M. Figueroa-O'Farrill and S. Gadhia, *M-theory preons cannot arise by quotients*, J. High Energy Phys. **06** (2007) 043. [arXiv:hep-th/0702055](#).
- [13] U. Gran, J. Gutowski, and G. Papadopoulos, *M-theory backgrounds with 30 Killing spinors are maximally supersymmetric*, JHEP **1003** (2010) 112. [arXiv:1001.1103](#) [hep-th].
- [14] J. Michelson, *A pp wave with twenty six supercharges*, Class. Quant. Grav. **19** (2002), 5935–5949. [arXiv:hep-th/0206204](#) [hep-th].
- [15] J. P. Gauntlett and S. Pakis, *The geometry of  $D = 11$  Killing spinors*, J. High Energy Phys. **04** (2003) 039. [arXiv:hep-th/0212008](#).
- [16] J. P. Gauntlett, J. Gutowski, and S. Pakis, *The geometry of  $D=11$  null Killing spinors*, J. High Energy Phys. **12** (2003) 049. [arXiv:hep-th/0311112](#).

- [17] B. S. Acharya, J. M. Figueroa-O’Farrill, C. M. Hull, and B. Spence, *Branes at conical singularities and holography*, Adv. Theor. Math. Phys. **2** (1998), 1249–1286. [arXiv:hep-th/9808014](#).
- [18] J. M. Figueroa-O’Farrill, *On the supersymmetries of Anti-de Sitter vacua*, Class. Quant. Grav. **16** (1999), 2043–2055. [arXiv:hep-th/9902066](#).
- [19] J. P. Gauntlett, R. Myers, and P. K. Townsend, *Supersymmetry of rotating branes*, Phys. Rev. **D59** (1999) 025001. [arXiv:hep-th/9809065](#).
- [20] J. P. Gauntlett, R. Myers, and P. K. Townsend, *Black holes of  $D = 5$  supergravity*, Class. Quant. Grav. **16** (1999), 1–21. [arXiv:hep-th/9810204](#).
- [21] P. K. Townsend, *Killing spinors, supersymmetries and rotating intersecting branes*, in: Novelties in string theory (Göteborg, 1998), pp. 177–182. World Sci. Publishing, River Edge, NJ, 1999. [arXiv:hep-th/9901102](#).
- [22] J. M. Figueroa-O’Farrill and G. Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, J. High Energy Phys. **06** (2001) 036. [arXiv:hep-th/0105308](#).
- [23] M. Blau, J. M. Figueroa-O’Farrill, C. M. Hull, and G. Papadopoulos, *A new maximally supersymmetric background of type IIB superstring theory*, J. High Energy Phys. **01** (2002) 047. [arXiv:hep-th/0110242](#).
- [24] N. Alonso-Alberca, E. Lozano-Tellechea, and T. Ortín, *Geometric construction of Killing spinors and supersymmetry algebras in homogeneous spacetimes*, Class. Quant. Grav. **19** (2002), 6009–6024. [arXiv:hep-th/0208158](#).
- [25] J. M. Figueroa-O’Farrill, P. Meessen, and S. Philip, *Supersymmetry and homogeneity of M-theory backgrounds*, Class. Quant. Grav. **22** (2005), 207–226. [arXiv:hep-th/0409170](#).
- [26] J. M. Figueroa-O’Farrill, E. Hackett-Jones, and G. Moutsopoulos, *The Killing superalgebra of ten-dimensional supergravity backgrounds*, Class. Quant. Grav. **24** (2007), 3291–3308. [arXiv:hep-th/0703192](#).
- [27] J. Figueroa-O’Farrill and N. Hustler, *The homogeneity theorem for supergravity backgrounds II: the six-dimensional theories*, JHEP **04** (2014) 131. [arXiv:1312.7509](#) [hep-th].

- [28] P. de Medeiros and J. Figueroa-O'Farrill, *Supersymmetric Yang-Mills theory on conformal supergravity backgrounds in ten dimensions*, J. High Energy Phys. **2016** (2015), no. 3, 1–36. [arXiv:1512.08342](#) [hep-th].
- [29] P. de Medeiros, J. Figueroa-O'Farrill, and A. Santi, *Killing superalgebras for Lorentzian four-manifolds*, J. High Energy Physics **2016** (2016), no. 6, 1–50. [arXiv:1605.00881](#) [hep-th].
- [30] J. M. Figueroa-O'Farrill, *The homogeneity conjecture for supergravity backgrounds*, Journal of Physics: Conference Series **175** (2009) 012002 (15pp). [arXiv:0812.1258](#) [hep-th].
- [31] J. Figueroa-O'Farrill and N. Hustler, *The homogeneity theorem for supergravity backgrounds*, JHEP **1210** (2012) 014. [arXiv:1208.0553](#) [hep-th].
- [32] J. Figueroa-O'Farrill and A. Santi, *Spencer cohomology and eleven-dimensional supergravity*, Comm. Math. Phys. **349** (2017), no. 2, 627–660. [arXiv:1511.08737](#) [hep-th].
- [33] J. Figueroa-O'Farrill and A. Santi, *Eleven-dimensional supergravity from filtered subdeformations of the Poincaré superalgebra*, Journal of Physics A: Mathematical and Theoretical **49** (2016), no. 29, 295204. [arXiv:1511.09264](#) [hep-th].
- [34] A. Santi and A. Spiro, *Super-Poincaré algebras, space-times and supergravities (I)*, Adv. Theor. Math. Phys. **16** (2012), no. 5, 1411–1441. [arXiv:1011.2722](#) [math-ph].
- [35] A. Santi and A. Spiro, *Super-Poincaré algebras, space-times, and supergravities. II*, J. Math. Phys. **53** (2012), no. 3, 032505. [arXiv:1108.6314](#) [math-ph].
- [36] A. Altomani and A. Santi, *Tanaka structures modeled on extended Poincaré algebras*, Indiana Univ. Math. J. **63** (2014), no. 1, 91–117. [arXiv:1201.0555](#) [math.DG].
- [37] A. Altomani and A. Santi, *Classification of maximal transitive prolongations of super-Poincaré algebras*, Adv. Math. **265** (2014) 60–96. [arXiv:1212.1826](#) [math.RA].
- [38] S.-J. Cheng and V. G. Kac, *Generalized Spencer cohomology and filtered deformations of  $\mathbb{Z}$ -graded Lie superalgebras*, Adv. Theor. Math. Phys. **2** (1998), no. 5, 1141–1182. [arXiv:math/9805039](#).

- [39] Y. Kosmann, *Dérivées de Lie des spineurs*, Ann. Mat. Pura Appl. (4) **91** (1972), 317–395.
- [40] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, in: Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), pp. 177–306. Lecture Notes in Math., Vol. 570, Springer, Berlin, 1977.
- [41] J.-L. Koszul, *Graded manifolds and graded Lie algebras*, in: Proceedings of the international meeting on geometry and physics (Florence, 1982), pp. 71–84, Pitagora, Bologna, 1983.
- [42] A. Santi, *Superization of homogeneous spin manifolds and geometry of homogeneous supermanifolds*, Abh. Math. Semin. Univ. Hambg. **80** (2010), no. 1, 87–144. [arXiv:0905.3832](#) [math.DG].
- [43] H. Baum, T. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Vol. 124 of Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1991. With German, French and Russian summaries.
- [44] C. Bär, *The Dirac operator on homogeneous spaces and its spectrum on 3-dimensional lens spaces*, Arch. Math. (Basel) **59** (1992), no. 1, 65–79.
- [45] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1969 original, A Wiley-Interscience Publication.
- [46] J. Michelson, *Twisted toroidal compactification of pp waves*, Phys. Rev. **D66** (2002) 066002. [arXiv:hep-th/0203140](#) [hep-th].
- [47] S. Fernando, M. Günaydin, and S. Hyun, *Oscillator construction of spectra of pp-wave superalgebras in eleven dimensions*, Nucl. Phys. **B727** (2005) 421–460. [arXiv:hep-th/0411281](#) [hep-th].
- [48] R. L. Bryant, *Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor*, in: Global analysis and harmonic analysis (Marseille-Luminy, 1999), J. P. Bourgignon, T. Branson, and O. Hijazi, eds., Vol. 4 of Sémin. Congr., pp. 53–94. Soc. Math. France, Paris, 2000. [arXiv:0004073](#) [math.DG].
- [49] J. M. Figueroa-O’Farrill, *Deformations of M-theory Killing superalgebras*, Class. Quant. Grav. **24** (2007) 5257–5276, [arXiv:0706.2600](#) [hep-th].

MAXWELL INSTITUTE AND SCHOOL OF MATHEMATICS  
THE UNIVERSITY OF EDINBURGH  
JAMES CLERK MAXWELL BUILDING, PETER GUTHRIE TAIT ROAD  
EDINBURGH EH9 3FD, UNITED KINGDOM  
*E-mail address:* [j.m.figueroa@ed.ac.uk](mailto:j.m.figueroa@ed.ac.uk)

DEPARTMENT OF MATHEMATICS, UNIVERSITÀ DI BOLOGNA  
VIA ZAMBONI, 33 – 40126 BOLOGNA, ITALY  
*E-mail address:* [asanti.math@gmail.com](mailto:asanti.math@gmail.com)