

Categorification of Dijkgraaf-Witten theory

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The goal of the paper is to categorify Dijkgraaf-Witten (DW) theory, aiming at providing foundation for a direct construction of DW theory as an Extended Topological Quantum Field Theory. The main tool is cohomology with coefficients in a Picard groupoid, namely the Picard groupoid of hermitian lines.

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1. Introduction

R. Dijkgraaf and E. Witten in [DW90] constructed a gauge theory with a finite gauge group G as a “toy model,” a tool for studying more general gauge theories with compact gauge groups. Their goal was to describe this theory, known as *DW theory*, as a Topological Quantum Field Theory (TQFT), *i.e.*, a functor on the category of 3-dimensional (3d) cobordisms to that of vector

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spaces, starting with an action given by a cocycle $\alpha \in Z^3(G; U(1))$. Dijkgraaf and Witten indicated that the vector space $\Phi(Y)$ corresponding to a closed oriented 2d manifold Y was closely related to the set $\text{Hom}(\pi_1(Y), G)/G$ of equivalence classes of principal G -bundles over Y and that it could be constructed by cutting the surface Y into pairs of pants, as Φ was expected to be a functor. The linear map $\Phi(X) : \partial_- X \rightarrow \partial_+ X$ corresponding to a 3d oriented cobordism X between closed manifolds $\partial_- X$ and $\partial_+ X$ depended on such choices as the choice of a map $\text{Hom}(\pi_1(X, x_0), G) \rightarrow \text{Map}(X, BG)$, the choice of a basepoint x_0 , the choice of a chain, via triangulation, representing the relative fundamental cycle $[X] \in H_3(X, \partial X; \mathbb{Z})$, which was interpreted as “lattice gauge theory.” One can say that, from the categorical point of view, Dijkgraaf and Witten constructed a TQFT functor on a certain subcategory of cobordisms decorated with appropriate extra structure utilized in their constructions. They used an orbifold approach to taking the homotopy quotient by G , that is to say, worked with the G -set $\text{Hom}(\pi_1(Y), G)$.

D. Freed and F. Quinn in [FQ93, Fre94] streamlined the construction of the TQFT functor Φ , so that $\Phi(X)$ would no longer depend on the choice of a representative of the fundamental cycle $[X]$ and thereby would produce a TQFT functor on the category of cobordisms. They also generalized the construction to n -dimensional cobordisms. Their main tool was to define pairings between cocycles in $Z^{n+1}(Y, U(1))$ and cycles $Z_n(Y, \mathbb{Z})$ and between $Z^{n+1}(X, U(1))$ and cycles $Z_{n+1}(X, \partial X; \mathbb{Z})$, resembling but certainly different from cap product, which would not even be defined because of dimension considerations. Freed and Quinn introduced the idea of an *invariant section* of a flat hermitian line bundle over a groupoid. This is a particular case of the idea of the limit of a functor, and in this context, is akin to taking a global section.

J. Lurie in [HL14] sketched a different construction of Dijkgraaf-Witten theory. Rather than using the orbifold $\text{Hom}(\pi_1(Y), G)/G$, he modeled the set of equivalence classes of principal G -bundles on the mapping space $\text{Map}(Y, BG)$. Given a cohomology class $\alpha \in H^{n+1}(BG; U(1))$ and a closed oriented n -manifold Y , he used a “push-pull” construction $\pi_* \text{ev}^* \alpha \in H^1(\text{Map}(Y, BG); U(1))$ for the diagram

$$\begin{array}{ccc} Y \times \text{Map}(Y, BG) & \xrightarrow{\text{ev}} & BG \\ \pi \downarrow & & \\ \text{Map}(Y, BG) & & \end{array}$$

to obtain a hermitian line bundle \mathcal{L}_Y over Y . Then he defined the TQFT functor Φ on objects by taking the space

$$\Phi(Y) := H^0(\text{Map}(Y, BG), \mathcal{L}_Y)$$

of global sections. He used *ambidexterity*, a natural isomorphism

$$H^0(\text{Map}(Y, BG), \mathcal{L}_Y) \xrightarrow{\sim} H_0(\text{Map}(Y, BG), \mathcal{L}_Y),$$

to produce a linear map

$$\Phi(X) : \Phi(\partial_- X) \rightarrow \Phi(\partial_+ X),$$

using push-pull again, now along the diagram

$$\text{Map}(\partial_- X, BG) \xleftarrow{p_-} \text{Map}(X, BG) \xrightarrow{p_+} \text{Map}(\partial_+ X, BG).$$

Lurie’s construction deliberately avoided the following subtlety. The hermitian line bundle \mathcal{L}_Y is determined by the cohomology class α only up to isomorphism. Starting with a cocycle $\alpha \in Z^{n+1}(BG; U(1))$ would partially fix the problem, because the resulting cocycle $\pi_* \text{ev}^* \alpha \in Z^1(\text{Map}(Y, BG); U(1))$ is not quite the same as a hermitian line bundle: isomorphic, but different hermitian line bundles may correspond to the same cocycle, whereas the cocycle is determined by a hermitian line bundle only up to condoundary. Moreover, the push-pull cocycle $\pi_* \text{ev}^* \alpha$ will depend on the choice of a cycle representing the fundamental class $[Y] \in H_n(Y; \mathbb{Z})$.

In the current paper, we replace the coefficient group $U(1)$ with an equivalent Picard groupoid, namely the Picard groupoid \mathcal{L} of hermitian lines, and notice that an object of $H^0(M, \mathcal{L})$ is exactly a flat hermitian line bundle over M , see Section 3.

The paper [FHLT10] attempted the construction of an Extended Topological Quantum Field Theory (ETQFT), which is defined on cobordisms with corners, rather than boundary, and a generalization of the DW theory to the case of a compact group G . The construction utilizes the Cobordism Hypothesis, which asserts that an ETQFT is determined by its value on zero-dimensional manifolds. The two-dimensional case of the cobordism hypothesis was proved by C. J. Schommer-Pries in [SP09], and the full version was proven by Lurie in [Lur09]. However, Freed, Hopkins, Lurie, and Teleman emphasize the importance of a direct construction, which has not been done yet.

This paper arose from the authors' trying to find an approach to this hypothetical direct construction of an ETQFT. In the process we have realized that Freed and Quinn's pairing makes sense as a cohomological operation, cap product, if the group $H^{n+1}(Y; U(1))$ is replaced with cohomology $H^n(Y; \mathcal{L})$ with coefficients in the Picard groupoid \mathcal{L} of hermitian line bundles. Categorifying the coefficients goes along with lowering the cohomological degree, thus opening a way to defining cap products as well as extending the TQFT to an ETQFT by further categorification to higher Picard groupoids and higher gerbes.

Another novel feature of our approach is that we do not use ambidexterity, but rather a transfer map in the context of cohomology with coefficients in Picard groupoids. In principle, one can view the transfer map as an avatar of ambidexterity, but it might be argued that using an avatar is less demanding than engaging the full power of a deity.

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2. Setup

We will consider (flat) hermitian line gerbes over simplicial sets. To deal with gerbes over manifolds and topological spaces, we will associate simplicial sets to them in a standard way: by taking singular simplices or the nerve of an open cover. Flat hermitian line gerbes are analogous to more traditional gerbes over topological spaces with the constant sheaf $U(1)$ as the band, whether given as stacks of groupoids, via gluing (descent) data, or as higher bundles, [BM05, Bry93, Moe02, Mur10]. We will take the liberty of omitting the adjective "flat" when referring to flat hermitian line bundles and gerbes.

We will describe cohomology with coefficients in Picard groupoids over simplicial sets and later apply this construction to cobordisms, which are manifolds, rather than simplicial sets. This may be done by working with the simplicial set of singular chains associated to the cobordism or by using the nerve of a sufficiently fine open covering, see examples in Section 3.

2.1. Cohomology with coefficients in Picard groupoids

A *Picard groupoid* is a symmetric monoidal groupoid in which every object is invertible, up to isomorphism, with respect to the tensor product, which, by a slight abuse of notation, we denote $+$. More precisely, for each object s of a Picard groupoid \mathcal{A} , the functors $t \mapsto s + t$, and $t \mapsto t + s$ define autoequivalences of \mathcal{A} as a category. In this case, one can define a functor $\mathcal{A} \rightarrow \mathcal{A}$, $s \mapsto -s$, and natural isomorphisms

$$m = m_s : s + (-s) \rightarrow 0, \quad n = n_s : (-s) + s \rightarrow 0$$

such that $l_s(m_s + \text{id}_s) = r_s(\text{id}_s + n_s)\alpha_{s,-s,s}$ for all objects s of \mathcal{A} , where 0 is the zero (also known as unit) object of \mathcal{A} and

- (1) $\alpha_{s,t,u} : (s + t) + u \rightarrow s + (t + u)$ and
- (2) $l_s : 0 + s \rightarrow s, \quad r_s : s + 0 \rightarrow s$

are the natural transformations of the monoidal structure on \mathcal{A} . We will assume that $-0 = 0$, $m_0 = r_0$, and $n_0 = l_0$. Another structure natural transformation is a symmetry:

$$\beta_{s,t} : s + t \rightarrow t + s,$$

making \mathcal{A} to be a symmetric monoidal category. Given a Picard groupoid \mathcal{A} , let $\pi_0(\mathcal{A})$ denote the abelian group of its connected components and $\pi_1(\mathcal{A})$ denote the abelian group of automorphisms of the zero object.

A *homomorphism* between two Picard groupoids \mathcal{A} and \mathcal{B} is a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ and an assignment of a *coherence morphism* which is an arrow of \mathcal{B} , $\phi_{s,t}^F : F(s) + F(t) \rightarrow F(s + t)$, to every pair of objects $s, t \in \mathcal{A}$ which is natural in both variables s and t such that the assignment respects the symmetry natural transformations β of \mathcal{A} and \mathcal{B} in the following sense:

$$F(\beta_{s,t}) \circ \phi_{s,t}^F = \phi_{t,s}^F \circ \beta_{F(s),F(t)}$$

and also respects the associativity in the following sense:

$$\phi_{s,t+u}^F \circ (\text{id}_{F(s)} + \phi_{t,u}^F) \circ \alpha_{F(s),F(t),F(u)}^{\mathcal{B}} = F(\alpha_{s,t,u}^{\mathcal{A}}) \circ \phi_{s+t,u}^F \circ (\phi_{s,t}^F + \text{id}_u),$$

for each triple of objects $s, t, u \in \mathcal{A}$ and where $\alpha^{\mathcal{A}}$ and $\alpha^{\mathcal{B}}$ are the associativity natural transformations of \mathcal{A} and \mathcal{B} respectively.

A homomorphism between two Picard groupoids $F : \mathcal{A} \rightarrow \mathcal{B}$ will be called a *strict homomorphism* if the coherence morphisms $\phi_{s,t}^F$ are identities for all pairs $s, t \in \mathcal{A}$ and $F(0) = 0$.

Given two homomorphisms F and $F' : \mathcal{A} \rightarrow \mathcal{B}$, a *monoidal natural transformation from F to F'* is a natural transformation $\theta : F \Rightarrow F'$ which is compatible with the coherence morphisms of both homomorphisms F and F' in the following sense:

$$\phi_{s,t}^{F'} \circ (\theta_s + \theta_t) = \theta_{s+t} \circ \phi_{s,t}^F,$$

for all pairs $s, t \in \mathcal{A}$.

Given any two Picard groupoids \mathcal{A} and \mathcal{B} , the category whose objects are all homomorphisms from \mathcal{A} to \mathcal{B} and whose morphisms are monoidal natural transformations between these homomorphisms has the structure of a Picard groupoid which we denote by $[\mathcal{A}, \mathcal{B}]$, see [Sch08] for a detailed proof of this assertion. One can associate another Picard groupoid with \mathcal{A} and \mathcal{B} which we denote by $\mathcal{A} \otimes \mathcal{B}$, and which will be called the *tensor product*. We will not recall its construction, which is rather elaborate, see [Sch08], but mention that the tensor product 2-functor is determined by an adjunction

$$[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \xrightarrow{\sim} [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$$

in the bicategory of Picard groupoids. This bicategory also has a unit object I for the monoidal structure. The bicategory of Picard groupoids, not only has an internal hom as indicated above, but it has the structure of a *Pic*-category, see appendix B for a definition of a *Pic*-category. More precisely, Picard groupoids, homomorphisms between Picard groupoids and monoidal natural transformations between homomorphisms form a *Pic*-category which we denote by $\mathcal{P}ic$. Further, $\mathcal{P}ic$ is the archetype example of a *Pic*-category. Our point of view on $\mathcal{P}ic$ is that it is the analog of the category of Abelian groups, \mathbf{Ab} , in the world of bicategories.

The groupoid of lines, i.e., one-dimensional vector spaces, and G -torsors for a given abelian group G have natural structures of Picard groupoids with respect to tensor products and the product of torsors over G , respectively. We will later focus our attention on the Picard groupoid \mathcal{L} of hermitian lines, where the hermitian form on the tensor product of hermitian lines is the tensor product of the hermitian forms one each line.

Let X_\bullet be a simplicial set and \mathcal{A} be a Picard groupoid. We will define cohomology $H^\bullet(X_\bullet, \mathcal{A})$ of X_\bullet with values in \mathcal{A} , following [CMM04] and [dRMMV05]. Similar cohomology may be defined for topological spaces and, more generally, with coefficients in sheaves of Picard groupoids.

Let us associate with X_\bullet and \mathcal{A} a *cosimplicial Picard groupoid*, that is to say, a cosimplicial object in the category of Picard groupoids, defined as the “mapping space” $\mathcal{A}^{X_\bullet} := \text{Map}(X_\bullet, \mathcal{A})$: for each $n \geq 0$, we define the Picard groupoid \mathcal{A}^{X_n} whose objects are maps $X_n \rightarrow \text{Ob } \mathcal{A}$, morphisms are maps $X_n \rightarrow \text{Mor } \mathcal{A}$, and the tensor product and morphism composition are defined “point-wise.” The cosimplicial structure is comprised of homomorphisms of Picard groupoids:

$$\mathcal{A}^{X_0} \begin{array}{c} \xleftarrow{s_0^*} \\ \xrightarrow{d_0^*} \\ \xrightarrow{d_1^*} \end{array} \mathcal{A}^{X_1} \begin{array}{c} \xleftarrow{s_1^*} \\ \xleftarrow{s_0^*} \\ \xrightarrow{d_0^*} \\ \xrightarrow{d_2^*} \end{array} \mathcal{A}^{X_2} \begin{array}{c} \xleftarrow{s_2^*} \\ \xleftarrow{s_1^*} \\ \xleftarrow{s_0^*} \\ \xrightarrow{d_0^*} \\ \xrightarrow{d_3^*} \end{array} \cdots,$$

where the coface and codegeneracy homomorphisms $d_i^* : \mathcal{A}^{X_n} \rightarrow \mathcal{A}^{X_{n+1}}$ and $s_j^* : \mathcal{A}^{X_{n+1}} \rightarrow \mathcal{A}^{X_n}$ are obtained by composition with the face maps $d_i : X_{n+1} \rightarrow X_n$ and degeneracy maps $s_j : X_n \rightarrow X_{n+1}$ of the simplicial set X_\bullet , respectively.

Now, by taking alternating sums, we obtain a (cochain) complex of Picard groupoids:

$$C^\bullet(X_\bullet, \mathcal{A}) : 0 \longrightarrow \mathcal{A}^{X_0} \xrightarrow{d} \mathcal{A}^{X_1} \xrightarrow{d} \mathcal{A}^{X_2} \xrightarrow{d} \mathcal{A}^{X_3} \xrightarrow{d} \dots$$

$\begin{array}{c} \overset{0}{\curvearrowright} \xrightarrow{\chi \uparrow} \\ \downarrow \chi \\ \underset{0}{\curvearrowleft} \end{array}$

with $d = \sum_{i=0}^{n+1} (-1)^i d_i^* : \mathcal{A}^{X_n} \rightarrow \mathcal{A}^{X_{n+1}}$ and a monoidal transformation $\chi : d^2 \Rightarrow 0$, obtained in a unique way from the structure isomorphisms α , m and n . This system of coboundary homomorphisms d and monoidal transformations χ is *coherent*, i.e., $\chi d = d\chi$ as 2-cells $d^3 \Rightarrow 0$. Let $\text{Kom}(\mathcal{P}ic)$ denote the *Pic*-category of complexes of Picard groupoids. The objects of $\text{Kom}(\mathcal{P}ic)$ are complexes of Picard groupoids. A 1-morphism between

$\mathcal{A}^\bullet, \mathcal{B}^\bullet \in Ob(\text{Kom}(\mathcal{P}ic))$, pictured below:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \chi_{\mathcal{A}} \uparrow \! \! \! \uparrow & & & \\
 & & \dots & \mathcal{A}^{n-1} & \xrightarrow{d_{\mathcal{A}}} & \mathcal{A}^n & \xrightarrow{d_{\mathcal{A}}} & \mathcal{A}^{n+1} & \dots & , \\
 & & f^{n-1} \downarrow & \phi^n \swarrow \! \! \! \swarrow & f^n \downarrow & \phi^{n+1} \swarrow \! \! \! \swarrow & f^{n+1} \downarrow & & \\
 & & \dots & \mathcal{B}^{n-1} & \xrightarrow{d_{\mathcal{B}}} & \mathcal{B}^n & \xrightarrow{d_{\mathcal{B}}} & \mathcal{B}^{n+1} & \dots & \\
 & & & \chi_{\mathcal{B}} \uparrow \! \! \! \uparrow & & & & & & \\
 & & & 0 & & & & & &
 \end{array}$$

is a pair $F = (f, \phi)$, where f is a sequence of homomorphisms $f^n : \mathcal{A}^n \rightarrow \mathcal{B}^n$ and ϕ is a sequence of monoidal natural transformations $\phi^n : f^n d_{\mathcal{A}} \Rightarrow d_{\mathcal{B}} f^{n-1}$ in $\mathcal{P}ic$, satisfying the following coherence conditions $\phi^{n+1} d_{\mathcal{A}} = d_{\mathcal{B}} \phi^n$ and $(f^{n+1} \chi_{\mathcal{A}}) \circ (\phi^{n+1} d_{\mathcal{A}}) \circ (d_{\mathcal{B}} \phi^n) = \chi_{\mathcal{B}} f^{n-1}$. A 2-morphism $(f, \phi) \Rightarrow (f', \phi')$ is a sequence $\{\gamma_n\}_{n \in \mathbb{Z}}$, where $\gamma_n : f^n \Rightarrow f'^n$ is a monoidal natural transformation, for all $n \in \mathbb{Z}$, and the following coherence condition is satisfied: $(\gamma_{n+1} d_{\mathcal{A}}) \circ \phi_n = \phi'_n \circ (d_{\mathcal{B}} \gamma_n)$. It would be useful to describe an alternative, equivalent, notion of a 2-morphism in $\text{Kom}(\mathcal{P}ic)$ which is a generalization of *cochain homotopy* to the Picard groupoid context. In this notion, a 2-morphism is also a pair $H = (h, \psi)$, where $h^n : \mathcal{A}^n \rightarrow \mathcal{B}^{n-1}$ and ψ is a sequence of monoidal natural transformations $\psi^n : d_{\mathcal{B}} h^n + f^n \Rightarrow f'^n + h^{n+1} d_{\mathcal{A}}$ satisfying an obvious coherence condition. We leave the establishment of an equivalence between the two notions of 2-morphisms in $\text{Kom}(\mathcal{P}ic)$ as an exercise for an interested reader.

The cohomology $H^\bullet(X_\bullet, \mathcal{A})$ of X_\bullet with coefficients in a Picard groupoid \mathcal{A} is defined as the cohomology of the complex $(\mathcal{A}^{X_\bullet}, d, \chi)$ of Picard groupoids. The cohomology of a complex of Picard groupoids may be defined as follows. In principle, to define the n th cohomology $H^n(X_\bullet, \mathcal{A})$, we want to take the kernel $\text{Ker } d$ of the homomorphism $d : \mathcal{A}^{X_n} \rightarrow \mathcal{A}^{X_{n+1}}$ and then the cokernel of the homomorphism $d' : \mathcal{A}^{X_{n-1}} \rightarrow \text{Ker } d$ induced by $d : \mathcal{A}^{X_{n-1}} \rightarrow \mathcal{A}^{X_n}$, but these need to be defined in a suitable categorified sense. In particular, the kernels, cokernels, and cohomology will depend on two subsequent coboundary homomorphisms d as well as χ and be Picard groupoids. The objects of the category $\text{Ker}(d, \chi)$ (of *n-cocycles*) are pairs (a, ϕ) in which a is an object of \mathcal{A}^{X_n} and $\phi : da \rightarrow 0$ is a morphism in $\mathcal{A}^{X_{n+1}}$ satisfying a *cocycle condition*:

$$d(\phi) = \chi_a : d^2(a) \rightarrow 0.$$

A morphism $(a, \phi) \rightarrow (a', \phi')$ in $\text{Ker}(d, \chi)$ is given by a morphism $f : a \rightarrow a'$ in \mathcal{A}^{X_n} such that $\phi' \circ d(f) = \phi$. The monoidal structure on $\text{Ker}(d, \chi)$ is inherited from that of \mathcal{A} . The kernel $\text{Ker}(d, \chi)$ naturally participates in a complex of Picard groupoids, as follows:

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \chi' & \nearrow & \\ \mathcal{A}^{X_{n-2}} & \xrightarrow{d} & \mathcal{A}^{X_{n-1}} & \xrightarrow{d'} & \text{Ker}(d, \chi) \end{array}$$

The cohomology $H^n(X_\bullet, \mathcal{A})$ is defined as the cokernel $\text{Coker}(d', \chi')$ in this complex. The cokernel $\text{Coker}(d', \chi')$ is a Picard category whose objects are the same as those of $\text{Ker}(d, \chi)$, i.e., of the type (a, ϕ) , where a is an object of \mathcal{A}^{X_n} and $\phi : da \rightarrow 0$ is a morphism in $\mathcal{A}^{X_{n+1}}$ satisfying the cocycle condition above. A morphism $(a, \phi) \rightarrow (a', \phi')$ in $\text{Coker}(d', \chi')$ is given by an equivalence class of pairs (b, f) , where b is an object of $\mathcal{A}^{X_{n-1}}$ and $f : (a, \phi) \rightarrow (d'b + a', \chi_b + \phi')$ is a morphism in $\text{Ker}(d, \chi)$. Two morphisms (b, f) and $(b', f') : (a, \phi) \rightarrow (a', \phi')$ are *equivalent*, if there is a pair (c, g) with c being an object of $\mathcal{A}^{X_{n-2}}$ and $g : b \rightarrow dc + b'$ a morphism in $\mathcal{A}^{X_{n-1}}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} a & \xrightarrow{f} & d'b + a' & \xrightarrow{d'(g)+\text{id}} & (d'dc + d'b') + a' & & \\ f' \downarrow & & & & \downarrow \alpha & & \\ d'b' + a' & \xleftarrow{l} & 0 + (d'b' + a') & \xleftarrow{\chi'_c + \text{id} + \text{id}} & dd'c + (d'b' + a') & & \end{array}$$

One can check that $\pi_0(H^n(X_\bullet, \mathcal{A})) \cong \pi_1(H^{n+1}(X_\bullet, \mathcal{A}))$.

The *simplicial homology of a simplicial set X_\bullet with coefficients in a Picard groupoid \mathcal{A}* may be defined similarly by looking at the *simplicial Picard groupoid $\mathcal{A}X_\bullet$* whose n -simplices are formal “linear combinations” $a_1s_1 + \dots + a_ks_k$ of pairwise distinct elements s_1, \dots, s_k in X_n with coefficients a_1, \dots, a_k in \mathcal{A} . Perhaps, a better way of looking at $\mathcal{A}X_\bullet$ is to view it as \mathcal{A} -valued functions on X_\bullet with finite support and apply the same treatment to it as that for \mathcal{A}^{X_\bullet} . In particular, summing up the face homomorphisms gives rise to a *chain complex of Picard groupoids $C_\bullet(X_\bullet, \mathcal{A})$* , which determines the homology Picard groupoids $H_n(X_\bullet, \mathcal{A})$ for $n \geq 0$.

When A is an abelian group, we will think of it as a *discrete* Picard groupoid, denoted $A[0]$, with A being the set of objects and identities being the only morphisms, so as $\pi_0(A[0]) = A$ and $\pi_1(A[0]) = 0$. Then the (co)homology with coefficients in the Picard groupoid $A[0]$ will be related to

the usual simplicial (co)homology with coefficients in the group A as follows:

$$\begin{aligned} \pi_0 H^\bullet(X_\bullet; A[0]) &= H^\bullet(X_\bullet; A), \\ \pi_0 H_\bullet(X_\bullet; A[0]) &= H_\bullet(X_\bullet; A). \end{aligned}$$

2.2. Relative cohomology

Let $\mathcal{A} \in \mathcal{P}ic$, let X_\bullet be a simplicial set, let $Y_\bullet \subset X_\bullet$ be a simplicial subset. There is an inclusion map $Y_\bullet \hookrightarrow X_\bullet$ in that category of simplicial sets. This inclusion induces a 1-morphism

$$i_\bullet : C_\bullet(Y_\bullet, \mathcal{A}) \hookrightarrow C_\bullet(X_\bullet, \mathcal{A})$$

in $\text{Kom}(\mathcal{P}ic)$. We define relative homology $H_\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ to be the homology of the 2-chain complex given by the cokernel of i_\bullet in $\text{Kom}(\mathcal{P}ic)$. We call this 2-chain complex, given by the cokernel, a *relative 2-chain complex* so $H_\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ is the homology of the relative 2-chain complex $C_\bullet(X_\bullet, Y_\bullet, \mathcal{A})$. The n th. degree of the relative 2-chain complex is the Picard groupoid given by the cokernel, in the category of Picard groupoids, of the map $i_n : C_n(Y_\bullet, \mathcal{A}) \hookrightarrow C_n(X_\bullet, \mathcal{A})$. Relative cohomology is defined similarly, $H^\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ is the cohomology of the relative 2-cochain complex given by the cokernel of the following map, induced by the inclusion $Y_\bullet \hookrightarrow X_\bullet$.

$$i^\bullet : C^\bullet(Y_\bullet, \mathcal{A}) \rightarrow C^\bullet(X_\bullet, \mathcal{A}).$$

The objects of $i^\bullet(C^n(Y_\bullet, \mathcal{A}))$ are those functions, $X_n \rightarrow \text{Ob}(\mathcal{A})$, which vanish outside of Y_n . $C^n(X_\bullet, Y_\bullet, \mathcal{A})$ is a Picard subgroupoid of $C^n(X_\bullet, \mathcal{A})$ whose objects are the same as those of $C^n(X_\bullet, \mathcal{A})$. A morphisms in $C^n(X_\bullet, Y_\bullet, \mathcal{A})$ is a certain equivalence class of morphisms in $C^n(X_\bullet, \mathcal{A})$. The cokernel also gives a 1-morphism, in $\text{Kom}(\mathcal{P}ic)$, $p^\bullet : C^\bullet(X_\bullet, \mathcal{A}) \rightarrow C^\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ and a 2-morphism $\phi^\bullet : p^\bullet \circ i^\bullet \Rightarrow 0 : C^\bullet(Y_\bullet, \mathcal{A}) \rightarrow C^\bullet(X_\bullet, Y_\bullet, \mathcal{A})$, where 0 is the zero homomorphism. If $\alpha \in \text{Ob}(C^n(Y_\bullet, \mathcal{A}))$ then the natural transformation ϕ^n assigns to α , a morphism $i(\alpha) \rightarrow 0$ in $C^n(X_\bullet, Y_\bullet, \mathcal{A})$. In other words those objects of $C^n(Y_\bullet, \mathcal{A})$ are isomorphic to the zero object in $C^n(X_\bullet, Y_\bullet, \mathcal{A})$.

2.3. Functoriality

The n th cohomology (and n th homology) defined above is a functor of $\mathcal{P}ic$ -categories, see appendix B, $H^n : \text{Kom}(\mathcal{P}ic) \rightarrow \mathcal{P}ic$. Moreover, every $F \in \text{Mor}_{\text{Kom}(\mathcal{P}ic)}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ determines a morphism $H^\bullet(F) \in \text{Mor}_{\text{Kom}(\mathcal{P}ic)}(H^\bullet(\mathcal{A}^\bullet),$

$H^\bullet(\mathcal{B}^\bullet)$) on cohomology. This fact follows from properties of relative kernels and cokernels; for a direct proof of this fact see [dRMMV05].

Note that the described cohomology and homology are (strictly) functorial with respect to simplicial maps. If \mathcal{A} is Picard groupoid and $f : X_\bullet \rightarrow Y_\bullet$ is a simplicial map, then we get a strict morphism between the corresponding cochain complexes of Picard groupoids $f^* : C^\bullet(Y_\bullet, \mathcal{A}) \rightarrow C^\bullet(X_\bullet, \mathcal{A})$, which yields a strict morphism on cohomology $f^* : H^n(Y_\bullet, \mathcal{A}) \rightarrow H^n(X_\bullet, \mathcal{A})$ for $n \geq 0$. Moreover, a simplicial homotopy between two simplicial maps induces a monoidal natural transformation on cohomology, cf. [BCC93, Proposition 2.1] and [CMM04, Proposition 2.3(i)] and the discussion of 2-morphisms in $\text{Kom}(\mathcal{P}ic)$ in Section 2.1. The same statements are true for homology.

2.4. The long 2-exact sequence

We begin this subsection by recalling the notion of a short 2-exact sequence of Picard groupoids. Here we will only recall this notion in a subcategory of $\mathcal{P}ic$ which has the same objects as $\mathcal{P}ic$ and whose morphisms are homomorphisms which preserve the unit of addition. For the general case see [BV02, Rou03]. A complex

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \curvearrowright & & & \\
 & & & \phi \uparrow\uparrow & & & \\
 & & & \curvearrowleft & & & \\
 0 & \longrightarrow & \mathcal{A} & \xrightarrow{F} & \mathcal{C} & \xrightarrow{G} & \mathcal{B} \longrightarrow 0
 \end{array}$$

is called a *short 2-exact sequence of Picard groupoids* if the unique morphism $\overline{G} : \text{Coker}(F, \text{id}_0) \rightarrow \mathcal{B}$ is full and faithful and further, $\pi_1(\text{Ker}(F, \phi)) = 0$ and $\pi_0(\text{Coker}(G, \phi)) = 0$.

Definition 2.1. A *2-exact sequence of complexes of Picard groupoids* is a diagram

$$(3) \quad \begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \curvearrowright & & & \\
 & & & \phi^\bullet \uparrow\uparrow & & & \\
 & & & \curvearrowleft & & & \\
 0 & \longrightarrow & \mathcal{A}^\bullet & \xrightarrow{F^\bullet} & \mathcal{B}^\bullet & \xrightarrow{G^\bullet} & \mathcal{C}^\bullet \longrightarrow 0,
 \end{array}$$

where F^\bullet and G^\bullet are 1-morphisms and ϕ^\bullet is a 2-morphism in $\text{Kom}(\mathcal{P}ic)$, such that in every degree, the above diagram in $\text{Kom}(\mathcal{P}ic)$, reduces to a short 2-exact sequence of Picard groupoids.

The following example of a short 2-exact sequence is of particular interest and would be referenced frequently.

Example 2.2. Let X_\bullet be a simplicial set and let $Y_\bullet \subset X_\bullet$ be a simplicial subset. Then for any Picard groupoid \mathcal{A} , there is a morphism $i : \mathcal{C}^\bullet(Y_\bullet; \mathcal{A}) \rightarrow \mathcal{C}^\bullet(X_\bullet; \mathcal{A})$ of (cochain) complexes of Picard groupoids. This morphism determines a short 2-exact sequence of complexes of Picard groupoids:

$$(4) \quad 0 \longrightarrow \mathcal{C}^\bullet(X_\bullet, Y_\bullet; \mathcal{A}) \longrightarrow \mathcal{C}^\bullet(X_\bullet; \mathcal{A}) \xrightarrow{i} \mathcal{C}^\bullet(Y_\bullet; \mathcal{A}) \longrightarrow 0.$$

$\begin{array}{c} 0 \\ \curvearrowright \\ \pi \uparrow \uparrow \\ \mathcal{C}^\bullet(X_\bullet, Y_\bullet; \mathcal{A}) \longrightarrow \mathcal{C}^\bullet(X_\bullet; \mathcal{A}) \end{array}$

The inclusion of simplicial sets induces another morphism of (chain) complexes of Picard groupoids, $i : \mathcal{C}_\bullet(Y_\bullet; \mathcal{A}) \rightarrow \mathcal{C}_\bullet(X_\bullet; \mathcal{A})$. This morphism determines a short 2-exact sequence of (chain) complexes of Picard groupoids:

$$(5) \quad 0 \longrightarrow \mathcal{C}_\bullet(Y_\bullet; \mathcal{A}) \xrightarrow{i} \mathcal{C}_\bullet(X_\bullet; \mathcal{A}) \longrightarrow \mathcal{C}_\bullet(X_\bullet, Y_\bullet; \mathcal{A}) \longrightarrow 0.$$

$\begin{array}{c} 0 \\ \curvearrowright \\ \pi \uparrow \uparrow \\ \mathcal{C}_\bullet(Y_\bullet; \mathcal{A}) \longrightarrow \mathcal{C}_\bullet(X_\bullet; \mathcal{A}) \end{array}$

A short 2-exact sequence of complexes (3) has an associated *long 2-exact sequence* of cohomology

$$(6) \quad \begin{array}{ccccccc} & & & 0 & & & 0 \\ & & & \curvearrowright & & & \curvearrowright \\ & & & H^n(\phi) \uparrow \uparrow & & & \Psi^n \uparrow \uparrow \\ \dots \rightarrow & H^n(\mathcal{A}^\bullet) & \xrightarrow{H^n(F)} & H^n(\mathcal{B}^\bullet) & \xrightarrow{H^n(G)} & H^n(\mathcal{C}^\bullet) & \xrightarrow{\partial^n} & H^{n+1}(\mathcal{A}^\bullet) & \xrightarrow{H^{n+1}(F)} & H^{n+1}(\mathcal{B}^\bullet) & \rightarrow \dots \\ & & & \downarrow \Sigma^n & & & & & & & \\ & & & 0 & & & & & & & \end{array}$$

We will briefly outline the construction of the 1-morphism ∂^n and the 2-morphism Ψ^n here. For a more elaborate description of the various components of this long exact sequence, we refer the interested reader to Section 4

of [dRMMV05]. For an outline we will refer to the following diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \phi^n \uparrow\uparrow & & & \\
 \dots \rightarrow & \mathcal{A}^n & \xrightarrow{f^n} & \mathcal{B}^n & \xrightarrow{g^n} & \mathcal{C}^n & \rightarrow \dots \\
 & \downarrow d_{\mathcal{A}}^n & \nearrow \lambda^n & \downarrow d_{\mathcal{B}}^n & \nearrow \mu^n & \downarrow d_{\mathcal{C}}^n & \\
 \dots \rightarrow & \mathcal{A}^{n+1} & \xrightarrow{f^{n+1}} & \mathcal{B}^{n+1} & \xrightarrow{g^{n+1}} & \mathcal{C}^{n+1} & \rightarrow \dots \\
 & & & \phi^{n+1} \downarrow\downarrow & & & \\
 & & & 0 & & &
 \end{array}$$

Let $(C_n, c_n : d_{\mathcal{C}}^n(C_n) \rightarrow 0)$ be an object in $Ker(d_{\mathcal{C}}^n)$; since $\pi_1(\text{Coker}(g_n, \phi^n)) = 0$, there is a $B_n \in \mathcal{B}^n$ and $i : g^n(B_n) \rightarrow C_n$. Since the following pair

$$(d_{\mathcal{B}}^n(B_n), c_n \circ d_{\mathcal{C}}^n(i) \circ \mu_n(B_n) : g^{n+1}(d_{\mathcal{B}}^n(B_n)) \rightarrow d_{\mathcal{C}}^n(g^n(B_n)) \rightarrow d_{\mathcal{C}}^n(C_n) \rightarrow 0)$$

is an object of $Ker(g^{n+1})$ and the factorization of $f^{n+1} : \mathcal{A}^{n+1} \rightarrow \mathcal{B}^{n+1}$ through $Ker(g^{n+1})$ is an equivalence, there are $A_{n+1} \in \mathcal{A}^{n+1}$ and $j : f^{n+1}(A_{n+1}) \rightarrow d_{\mathcal{B}}^n(B_n)$ such that

$$g^{n+1}(j) \circ c_n \circ d_{\mathcal{C}}^n(i) \circ \mu_n(B_n) = \phi^{n+1}(A_{n+1}).$$

We now need an arrow $a_{n+1} : d_{\mathcal{A}}^{n+1} \rightarrow 0$. Since the factorization f^{n+2} of f^{n+2} through $Ker(g^{n+2})$ is an equivalence of categories, it is enough to find an arrow $f^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})) \rightarrow f^{n+2}(0)$. This is given by the following:

$$\chi_{\mathcal{B}}(B_n) \circ d_{\mathcal{B}}^{n+1}(j) \circ \lambda^{n+1}(A_{n+1}) : f^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})) \rightarrow 0 \cong f^{n+2}(0).$$

We put $\partial^n(C_n, c_n) := (A_{n+1}, a_{n+1})$. This is an object of $H^{n+1}\mathcal{A}^\bullet$: the condition $d_{\mathcal{A}}^{n+1}(a_{n+1}) = \chi_{\mathcal{A}}(A_{n+1})$ can be easily checked by applying the faithful functor f^{n+3} . The arrow function of the functor ∂^n has a much more elaborate description and moreover it is not used in the construction of our theory. We will refer the interested reader to [dRMMV05].

Before we can describe a construction of the 2-morphism Ψ^n , we need another description of $H^n(\mathcal{C}^\bullet)$. Since (f^n, ϕ^n, g^n) is a 2-short exact sequence, \mathcal{C}^n is equivalent to the cokernel of f^n , we get the following alternative description of $H^n(\mathcal{C}^\bullet)$. An object is a pair

$$(B_n \in \mathcal{B}^n, [A_{n+1} \in \mathcal{A}_{n+1}, a_{n+1} : d_{\mathcal{B}}^n(B_n) \rightarrow f^{n+1}(A_{n+1})]),$$

where $[A_{n+1}, a_{n+1}] \in \text{Mor}_{\text{Coker}(f^{n+1}, \text{id}_0)}(d_{\mathcal{B}}^n(B_n), 0)$, such that there exists an arrow $t^{n+2} : d_{\mathcal{A}}^n(A_{n+1}) \rightarrow 0$ making the following diagram commutative

$$\begin{array}{ccc}
 d_{\mathcal{B}}^{n+1}(d_{\mathcal{B}}^n(B_n)) & \xrightarrow{d_{\mathcal{B}}^{n+1}(a_{n+1})} & d_{\mathcal{B}}^{n+1}(f^{n+1}(A_{n+1})) \\
 \chi_{\mathcal{B}}(B_n) \downarrow & & \downarrow (\lambda^{n+1})^{-1}(A_{n+1}) \\
 0 & \xleftarrow{f^{n+2}(t^{n+2})} & f^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})).
 \end{array}$$

Note that t^{n+2} is necessarily unique because f^{n+2} is faithful. Now we begin the construction of Ψ^n , given an object

$$(B_n \in \mathcal{B}^n, [A_{n+1} \in \mathcal{A}^{n+1}, a_{n+1} : d_{\mathcal{B}}^n(B_n) \rightarrow f^{n+1}(A_{n+1})]),$$

in $H^n(\mathcal{C}^\bullet)$, we apply ∂^n and $H^{n+1}(f)$ and obtain the following object of $H^{n+1}(\mathcal{B}^\bullet)$:

$$(f^{n+1}(A_{n+1}), f^{n+2}(t^{n+2}) \circ \lambda_{n+1}^{-1}(A_{n+1}) : d_{\mathcal{B}}^{n+1}(f^{n+1}(A_{n+1})) \rightarrow f^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})) \rightarrow 0).$$

This object is naturally isomorphic to the unit of the addition $0 \in H^{n+1}(\mathcal{B}^\bullet)$ via the following morphism which we take as the definition of Ψ^n on the object $(B_n, [A_{n+1}, a_{n+1}])$

$$\Psi^n(B_n, [A_{n+1}, a_{n+1}]) := [B_n \in \mathcal{B}^n, a_{n+1}^{-1} : f^{n+1}(A_{n+1}) \rightarrow d_{\mathcal{B}}^n(B_n)].$$

The following example describes the images under the morphism ∂^n and the natural transformation Ψ^n of an object in degree n of the cohomology sequence associated to the 2-short exact sequence (5).

Example 2.3. Let X be a compact finite-dimensional manifold with boundary ∂X . We denote by $H_n(X, \partial X; \mathcal{L})$ the n th homology Picard groupoid of the chain complex $C_\bullet(X_\bullet, \partial X_\bullet; \mathcal{L})$. Let $(X_n, [X'_{n-1}, x'_{n-1}]) \in \text{Ob } H_n(X, \partial X; \mathcal{L})$, where $X_n \in \text{Ob } C_n(X_\bullet; \mathcal{L})$ and the morphism $[X'_{n-1}, x'_{n-1}] : d(X_n) \rightarrow 0$ in $\text{Mor } C_{n-1}(X_\bullet; \partial X_\bullet; \mathcal{L})$ consists of an object $X'_{n-1} \in C_{n-1}(\partial X_\bullet; \mathcal{L})$ and a morphism $x'_{n-1} : d^n(X_n) \rightarrow X'_{n-1} \in \text{Mor } C_{n-1}(X_\bullet; \mathcal{L})$. The coboundary $\partial_n(X_n, [X'_{n-1}, x'_{n-1}])$ is the pair $(X'_{n-1}, x'_{n-1}) \in \text{Ob } H_{n-1}(\partial X; \mathcal{L})$. The natural transformation $\Psi_n(X_n, [X'_{n-1}, x'_{n-1}])$ is a morphism in $\text{Hom}_{H_{n-1}(X; \mathcal{L})}((X'_{n-1}, x'_{n-1}), 0)$ given by the equivalence class $[X_n, (x'_{n-1})^{-1}]$. Thus every object in $H_n(X, \partial X; \mathcal{L})$ produces a morphism in $H_{n-1}(X; \mathcal{L})$.

2.5. The Cap Product

In this section we develop a cap product between cohomology with coefficients in a Picard groupoid and homology with coefficients in the Picard groupoid $\mathbb{Z}[0]$:

$$\cap : H_{\bullet}(X_{\bullet}, \mathbb{Z}[0]) \otimes H^{\bullet}(X_{\bullet}, \mathcal{A}) \rightarrow H_{\bullet}(X_{\bullet}, \mathcal{A}).$$

In order to do that, we will define a chain map i.e a morphism in $\text{Kom}(\mathcal{P}ic)$

$$H_{\bullet}(X_{\bullet}, \mathbb{Z}[0]) \rightarrow [H^{\bullet}(X_{\bullet}, \mathcal{A}), H_{\bullet}(X_{\bullet}, \mathcal{A})].$$

We start by defining the following chain map

$$(7) \quad \cap^{ch} : C_{\bullet}(X_{\bullet}, \mathbb{Z}[0]) \rightarrow [C^{\bullet}(X_{\bullet}, \mathcal{A}), C_{\bullet}(X_{\bullet}, \mathcal{A})]$$

where the right hand side is the chain complex $[C^{\bullet}(X_{\bullet}, \mathcal{A}), C_{\bullet}(X_{\bullet}, \mathcal{A})]_{\bullet}$ defined in Appendix 2. We define the map in degree p as follows: On objects the chain map is given by

$$\sigma_q \mapsto \prod_{q \geq p} F_q,$$

where $F_q \in \text{Mor}(\mathcal{P}ic)$ is defined on objects by

$$F_q : \alpha \mapsto \alpha(d_f^p(\sigma_q))d_l^{q-p}(\sigma_q),$$

where d_f^p and d_l^{q-p} are the restrictions of $d : C_{q+1}(X_{\bullet}, \mathbb{Z}[0]) \rightarrow C_q(X_{\bullet}, \mathbb{Z}[0])$ to the simplex determined by the first $p + 1$ vertices and the last $q - p + 1$ vertices of σ_q respectively. On morphisms, F_q given by

$$F_q : \{\{\alpha \rightarrow \beta\}\} \mapsto \{\alpha(d_f^p(\sigma_q))d_l^{q-p}(\sigma_q) \rightarrow \beta(d_f^p(\sigma_q))d_l^{q-p}(\sigma_q)\}$$

The map on the right side is determined by the natural transformation $\alpha \rightarrow \beta$. This chain map induces a map on homology

$$(8) \quad H_{\bullet}(X_{\bullet}, \mathbb{Z}[0]) \rightarrow H_{\bullet}([C^{\bullet}(X_{\bullet}, \mathcal{A}), C_{\bullet}(X_{\bullet}, \mathcal{A})]).$$

Composition with the following obvious morphism gives us the desired chain map

$$(9) \quad H_{\bullet}([C^{\bullet}(X_{\bullet}, \mathcal{A}), C_{\bullet}(X_{\bullet}, \mathcal{A})]) \rightarrow [H^{\bullet}(X_{\bullet}, \mathcal{A}), H_{\bullet}(X_{\bullet}, \mathcal{A})].$$

2.6. Relative cap product

We now construct a relative version of the cap product. The 2-functor $[C^\bullet(X_\bullet; \mathcal{A}), -] : \text{Kom}(\mathcal{P}ic) \rightarrow \text{Kom}(\mathcal{P}ic)$ and the chain map (7), determine a composite 1-morphism and a 2-morphism ϕ_\bullet in $\text{Kom}(\mathcal{P}ic)$

$$\begin{array}{ccc}
 & & 0 \\
 & \curvearrowright & \\
 C_\bullet(Y_\bullet; \mathbb{Z}[0]) & \xrightarrow{i_\bullet} & C_\bullet(X_\bullet; \mathbb{Z}[0]) \xrightarrow{\cap_{rel}^{ch}} [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(X_\bullet, Y_\bullet; \mathcal{A})] \\
 & & \uparrow \phi_\bullet
 \end{array}$$

In order to define the 1-morphism \cap_{rel}^{ch} and the 2-morphism ϕ_\bullet in the above diagram, we need to define a restriction of the chain map 7. The image of the restriction of this chain map to $i_\bullet(C_\bullet(Y_\bullet; \mathbb{Z}[0]))$ is contained in the 2-(chain) complex $[C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(Y_\bullet; \mathcal{A})]$, this determines the following commutative diagram

$$\begin{array}{ccc}
 i_\bullet(C_\bullet(Y_\bullet; \mathbb{Z}[0])) & \xrightarrow{\cap_{i_\bullet(C_\bullet(Y_\bullet; \mathbb{Z}[0]))}^{ch}} & [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(X_\bullet; \mathcal{A})] \\
 & \searrow \cap_{Y_\bullet}^{ch} & \uparrow [C^\bullet(X_\bullet; \mathcal{A}), i_\bullet] \\
 & & [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(Y_\bullet; \mathcal{A})]
 \end{array}$$

The following composite chain map will be called the *restricted relative cap product chain map*.

$$(10) \quad \cap_{res}^{ch} : C_\bullet(Y_\bullet; \mathbb{Z}[0]) \xrightarrow{i_\bullet} i_\bullet(C_\bullet(Y_\bullet; \mathbb{Z}[0])) \xrightarrow{\cap_{Y_\bullet}^{ch}} [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(Y_\bullet; \mathcal{A})]$$

The 2-morphism ϕ_\bullet is the composition $\cap_{res}^{ch} \circ ([C^\bullet(X_\bullet; \mathcal{A}), \pi_\bullet^{\mathcal{A}}])$ as described in the following diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \curvearrowright & & \curvearrowright & \\
 C_\bullet(Y_\bullet; \mathbb{Z}[0]) & \xrightarrow{i_\bullet} & C_\bullet(X_\bullet; \mathbb{Z}[0]) & \xrightarrow{p_\bullet} & C_\bullet(X_\bullet, Y_\bullet; \mathbb{Z}[0]) \\
 \downarrow \cap_{res}^{ch} & & \downarrow \cap^{ch} & \swarrow \lambda_\bullet & \downarrow u \\
 [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(Y_\bullet; \mathcal{A})] & \rightarrow & [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(X_\bullet; \mathcal{A})] & \xrightarrow{\cap_{rel}^{ch}} & [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(X_\bullet, Y_\bullet; \mathcal{A})] \\
 & & \downarrow [C^\bullet(X_\bullet; \mathcal{A}), \pi_\bullet^{\mathcal{A}}] & & \\
 & & 0 & &
 \end{array}$$

where $\pi_\bullet^{\mathcal{A}}$ is the following 2-morphism

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \searrow & \curvearrowright & \swarrow & \\
 C_\bullet(Y_\bullet; \mathcal{A}) & \xrightarrow{i_\bullet} & C_\bullet(X_\bullet; \mathcal{A}) & \xrightarrow{p_\bullet^{\mathcal{A}}} & C_\bullet(X_\bullet, Y_\bullet; \mathcal{A}) \\
 & & \uparrow \pi_\bullet^{\mathcal{A}} & & \\
 & & 0 & &
 \end{array}$$

The universality of the cokernel determines a unique pair consisting of a 1-morphism in $\text{Kom}(\mathcal{P}ic)$

$$u : C_\bullet(X_\bullet, Y_\bullet; \mathbb{Z}[0]) \rightarrow [C^\bullet(X_\bullet; \mathcal{A}), C_\bullet(X_\bullet, Y_\bullet; \mathcal{A})],$$

and a 2-morphism in $\text{Kom}(\mathcal{P}ic)$, $\lambda_\bullet : u \circ p_\bullet \Rightarrow \cap_{rel}^{ch}$ such that the following diagram commutes

$$\begin{array}{ccc}
 u \circ p_\bullet \circ i_\bullet & \xrightarrow{u \cdot \pi_\bullet} & u \circ 0 \\
 \lambda_\bullet \cdot i_\bullet \downarrow & & \downarrow \\
 \cap_{rel}^{ch} \circ i_\bullet & \xrightarrow{\phi_\bullet} & 0
 \end{array}$$

For more details on the universality of this cokernel we refer the interested reader to [KV00]. This unique 1-morphism, u , induces the following 1-morphism on passing to homology

$$(11) \quad H_\bullet(X_\bullet, Y_\bullet, \mathbb{Z}[0]) \rightarrow H_\bullet([C^\bullet(X_\bullet, \mathcal{A}), C_\bullet(X_\bullet, Y_\bullet, \mathcal{A})]).$$

Composition with the following chain map

$$(12) \quad H_\bullet([C^\bullet(X_\bullet, \mathcal{A}), C_\bullet(X_\bullet, Y_\bullet, \mathcal{A})]) \rightarrow [H^\bullet(X_\bullet, \mathcal{A}), H_\bullet(X_\bullet, Y_\bullet, \mathcal{A})].$$

and the adjointness of the tensor product gives us the desired chain map

$$(13) \quad \cap : H_\bullet(X_\bullet, Y_\bullet, \mathbb{Z}[0]) \otimes H^\bullet(X_\bullet, \mathcal{A}) \rightarrow H_\bullet(X_\bullet, Y_\bullet, \mathcal{A}).$$

which we will call the relative cap product.

As in the classical case, the boundary map in homology is natural with respect to relative cap product, in a sense made precise below.

Proposition 2.4. *The following diagram of Picard groupoids is commutative up to natural isomorphism for $p - 1 \geq q \geq 0$:*

$$\begin{array}{ccc}
 H_p(X_\bullet, Y_\bullet; \mathbb{Z}[0]) \otimes H^q(X_\bullet, \mathcal{A}) & \xrightarrow{\cap} & H_{p-q}(X_\bullet, Y_\bullet; \mathcal{L}) \\
 \partial \otimes i^* \downarrow & \swarrow & \downarrow \partial \\
 H_{p-1}(Y_\bullet; \mathbb{Z}[0]) \otimes H^q(Y_\bullet, \mathcal{A}) & \xrightarrow{\cap} & H_{p-q-1}(Y_\bullet; \mathcal{L})
 \end{array}$$

3. Hermitian line gerbes

In this section we describe geometric objects which we call (*flat*) *hermitian line n -gerbes*. Then we give an example describing a flat hermitian line 2-gerbe over the simplicial set BG , where G is a (discrete) group. We move on to describe certain geometric objects over a topological space X which are classified by the Čech cohomology of X with coefficients in $U(1)$ and which we call *flat hermitian line 1-gerbes* over X . We describe these in two ways. For the first description, we define a category $C(X; \mathcal{U}_1)$, associated to an open cover of X and show that hermitian line 0-cocycles on the simplicial set $N(C(X; \mathcal{U}_1))$ represent (flat) hermitian line 0-gerbes. Our second description is that a flat hermitian line 0-gerbe can be represented by a functor from the first fundamental groupoid of X into the Picard groupoid of hermitian lines \mathcal{L} . Finally, we move on to describe higher hermitian line gerbes over X .

Definition 3.1. A *hermitian line n -gerbe* on a simplicial set X_\bullet is an n -cocycle on the simplicial set X_\bullet with values in \mathcal{L} i.e. an object $K \in \text{Ob } H^n(X_\bullet, \mathcal{L})$ of degree n cohomology Picard groupoid of X_\bullet with coefficients in the Picard groupoid \mathcal{L} of hermitian lines.

Remark. A n -gerbe should be properly defined as a 0-cocycle with coefficients in an appropriate Picard $(n + 1)$ -groupoid but this would be out of scope of this paper.

The following example shows that a 2-gerbe on BG , where G is a finite group, is exactly the same as a 2-cocycle with values in hermitian lines as defined in [FHLT10].

Example 3.2. Any group G can be viewed as a category with a single object. The simplicial set BG is the nerve of this category. An object of $H^2(BG; \mathcal{L})$ consists of a pair $(\beta, \phi : d\beta \rightarrow 0)$, where $\beta \in \text{Ob } C^2(BG; \mathcal{L})$ and ϕ is an arrow in $C^3(BG; \mathcal{L})$ and d is the differential of the 2-complex

$C^\bullet(BG; \mathcal{L})$. β is a set function whose domain is the underlying set of $G \times G$ and codomain is $\text{Ob } \mathcal{L}$ i.e. it assigns to each pair $(g_1, g_2) \in G \times G$ a hermitian line $l_{g_2, g_1} \in \text{Ob } \mathcal{L}$. The arrow ϕ gives, for every triple $(g_1, g_2, g_3) \in G \times G \times G$, a following isomorphism in \mathcal{L}

$$t_{g_3, g_2, g_1} : l_{g_3, g_2} - l_{g_3, g_2 g_1} + l_{g_3 g_2, g_1} - l_{g_2, g_1} \rightarrow \mathbb{C}.$$

The morphism $d(\phi) : d^2(\beta) \rightarrow 0$ gives, for every quadruple $(g_1, g_2, g_3, g_4) \in G \times G \times G \times G$, the following isomorphism

$$t_{g_4, g_3, g_2} - t_{g_4, g_3, g_2 g_1} + t_{g_4, g_3 g_2, g_1} - t_{g_4 g_3, g_2, g_1} + t_{g_3, g_2, g_1},$$

which is the canonical isomorphism

$$\chi_\beta((g_1, g_2, g_3, g_4)) : d^2\beta((g_1, g_2, g_3, g_4)) \rightarrow \mathbb{C}.$$

By the definition above, a flat hermitian line 0-gerbe on a simplicial set X_\bullet is just an object of the Picard groupoid $H^0(X_\bullet; \mathcal{L})$. Thus, given a topological space X , we may look at hermitian line 0-gerbes over X in two ways: associating simplicial sets $\text{Sing}_\bullet X$ and $N(C(X; \mathfrak{U}_I))$ to X , where $N(C(X; \mathfrak{U}_I))$ is the simplicial sets obtained by taking the nerve of a category associated to the cover of X , $C(X; \mathfrak{U}_I)$, which we now define:

Definition 3.3. We define $C(X; \mathfrak{U}_I)$ to be a category whose object set is the collection $\mathfrak{U}_I = \{U_i : i \in I\}$, which is a chosen open cover of the topological space X . If the set $U_{i,j} \neq \emptyset$, then $\text{Hom}_{C(X; \mathfrak{U}_I)}(U_i, U_j) = \{U_{i,j}\}$, otherwise the set $\text{Hom}_{C(X; \mathfrak{U}_I)}(U_i, U_j) = \emptyset$. Composition in $C(X; \mathfrak{U}_I)$ is defined as follows: $U_{i,j} \circ U_{j,k} := U_{i,j,k} := U_{i,j} \cap U_{j,k}$. $id_{U_i} := U_{ii}$. The source of an arrow $U_{i,j}$ is U_i and its target is U_j .

This leads to two interpretations of flat hermitian line 0-gerbes: Definitions 3.4 and 3.5 below.

Definition 3.4. A flat hermitian line 0-gerbe over X , $\mathcal{G}^0(\Lambda, \theta)$, is defined by the following data

1. A function $\Lambda_0 : (\text{Sing}_\bullet X)_0 \rightarrow \text{Ob}(\mathcal{L})$, i.e. an assignment of a hermitian line to each point of X .
2. A function $\Lambda_1 : (\text{Sing}_\bullet X)_1 \rightarrow \text{Mor}(\mathcal{L})$ which assigns to each $f \in (\text{Sing}_\bullet X)_1$, a linear isometry $\Lambda_1(f) : \Lambda_0 \partial_1(f) \rightarrow \Lambda_0 \partial_0(f)$ in \mathcal{L} such that for all $f, g \in (\text{Sing}_\bullet X)_1$ satisfying $\partial_1(f) = \partial_0(g)$, $\Lambda_1(g \circ f) = \Lambda(g) \circ \Lambda(f)$.

This data is subject to the following condition. For each $n \geq 2$, there exists a function $\Lambda_n : (Sing_{\bullet} X)_n \rightarrow \text{Mor}(\mathcal{L})$ such that for all $\sigma_n \in (Sing_{\bullet} X)_n$, $\Lambda_n(\sigma_n) = \Lambda_{n-1}(\partial_0 \sigma_n) \circ \Lambda_{n-1}(\partial_1 \sigma_n) \circ \cdots \circ \Lambda_{n-1}(\partial_{n-1} \sigma_n)$.

Remark. The above definition assigns a hermitian line to each point of X . Further, two homotopic paths in X , relative to endpoints, are assigned the same linear isometry. In other words the above data is equivalent to defining a functor from the first fundamental groupoid of the space X , $\Pi_1(X)$, to \mathcal{L} .

Definition 3.5. A flat hermitian line 0-gerbe over X , $\mathcal{G}^0(\Lambda, \theta)$, is defined by the following data

1. A constant hermitian line bundle Λ_i over every open set U_i for all $i \in I$.
2. For each ordered pair of distinct indices $(i, j) \in I \times I$, a constant, non-zero section

$$\theta_{i,j} \in \Gamma(U_{i,j}; \Lambda_i \otimes \Lambda_j)$$

This data is subject to a cocycle condition, on $U_{i,j,k}$ which we denote by $\delta\theta \Rightarrow 0$. The cocycle condition is that over any three fold intersections $U_{i,j,k}$, we can tensor the three sections of the coboundary to give a trivialization of the following hermitian line bundle

$$(\Lambda_i \otimes \Lambda_j) \otimes (\Lambda_i \otimes \Lambda_k)^{-1} \otimes (\Lambda_j \otimes \Lambda_k).$$

over $U_{i,j,k}$. Notice that the above hermitian line bundle is canonically trivial, so the cocycle condition is the requirement that the following

$$\theta_{i,j} - \theta_{i,k} + \theta_{j,k}$$

be the canonical section of this trivial hermitian line bundle over $U_{i,j,k}$.

Remark. Each point $x \in X$ has a neighborhood U_i such that the hermitian line bundle Λ_i is isomorphic to the trivial hermitian line bundle $U_i \times \mathbb{C}$. Further, the specification of constant, non-zero section $\theta_{i,j}$ is the same as specifying a hermitian line bundle isomorphism $g_{i,j} : \Lambda_i|_{U_{i,j}} \rightarrow \Lambda_j|_{U_{i,j}}$, which restricts to the same linear isometry on every fiber. These two observations along with the data in the definition above are sufficient to construct a (flat) hermitian line bundle over the space X .

Now we move on to define higher hermitian line gerbes. Our definition of a flat hermitian line 1-gerbe closely follows the definition of a “1-gerb” developed in [Cha98].

Definition 3.6. A flat hermitian line 1-gerbe over X , $\mathcal{G}^1(\Lambda, \theta)$, is defined by the following data

1. A constant hermitian line bundle Λ_i^j over the intersection $U_{i,j}$ for every ordered pair $(i, j) \in I \times I$ and $i \neq j$, such that Λ_i^j and Λ_j^i are dual to each other.
2. For each ordered triple of distinct indices $(i, j, k) \in I \times I \times I$, a nowhere zero section

$$\theta_{i,j,k} \in \Gamma(U_{i,j,k}; \Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i)$$

such that the sections of reorderings of triples (i, j, k) are related in the natural way.

This data is subject to a cocycle condition, on $U_{i,j,k,l}$ which we denote by $\delta\theta \Rightarrow 0$. The cocycle condition is that over any four fold intersections $U_{i,j,k,l}$, we can tensor the four sections of the coboundary to give a trivialization of the following hermitian line bundle

$$(14) \quad (\Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i) \otimes (\Lambda_i^j \otimes \Lambda_j^l \otimes \Lambda_l^i)^{-1} \otimes (\Lambda_i^k \otimes \Lambda_k^l \otimes \Lambda_l^i) \otimes (\Lambda_j^k \otimes \Lambda_k^l \otimes \Lambda_l^j)^{-1}.$$

over $U_{i,j,k,l}$. Notice that the above hermitian line bundle is canonically trivial, so the cocycle condition is the requirement that the following

$$\theta_{i,j,k} - \theta_{i,j,l} + \theta_{i,k,l} - \theta_{j,k,l}$$

be the canonical section of this trivial hermitian line bundle over $U_{i,j,k,l}$.

The tensor product of two flat hermitian line 1-gerbes is obtained by tensoring line bundles and sections in an obvious way.

Let $(\alpha, \phi) \in \text{Ob } H^1(N(C(X; \mathfrak{A}_1); \mathcal{L}))$. To each $U_{i,j}$, the cochain α assigns a hermitian line l_i^j and the morphism ϕ specifies a linear isometry for each $U_{i,j,k}$

$$\phi(U_{i,j,k}) : l_i^j - l_k^i + l_j^k \rightarrow \mathbb{C}.$$

Equivalently, the specification of this linear isometry is the specification of a constant function $t_{i,j,k} : U_{i,j,k} \rightarrow U(1)$. In other words

$$t_{i,j,k}(x) = \phi(U_{i,j,k}),$$

$\forall x \in U_{i,j,k}$. Put constant hermitian line bundles $\Lambda_i^j = U_{i,j} \times l_i^j$ over each $U_{i,j}$. Then $t_{i,j,k}$ gives a trivialization of the coboundary line bundle $\Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i$. We define the section

$$\theta_{i,j,k}(x) = t_{i,j,k}^{-1}(x, e_1).$$

The morphism

$$dt_{i,j,k} = d\phi(U_{i,j,k,l}) = t_{i,j,k} - t_{i,j,l} + t_{i,k,l} - t_{j,k,l}$$

gives a trivialization of the line bundle (14) over $U_{i,j,k,l}$. The following section corresponds to the above trivialization of the the hermitian line bundle (14)

$$\theta_{i,j,k} - \theta_{i,j,l} + \theta_{i,k,l} - \theta_{j,k,l}.$$

Clearly this is the canonical section.

Conversely, a flat hermitian line 1-gerbe over X defines a 1-cocycle in $H^1(N(C(X; \mathfrak{U}_1); \mathcal{L}))$. We leave the easy verification of this fact as an exercise for the reader.

Definition 3.7. Let $\mathcal{G}^1(\Lambda, \theta)$ and $\mathcal{H}^1(\Upsilon, \eta)$ be two flat hermitian line 1-gerbes over X and let (g, ϕ) and (h, ψ) the two 1-cocycles in $H^1(N(C(X; \mathfrak{U}_1); \mathcal{L}))$ determined by them, then the the two gerbes $\mathcal{G}^1(\Lambda, \theta)$ and $\mathcal{H}^1(\Upsilon, \eta)$ are equivalent if there exists a morphism $(g, \phi) \rightarrow (h, \psi)$ in $H^1(N(C(X; \mathfrak{U}_1)); \mathcal{L})$.

If $\mathcal{G}^1(\Lambda, \theta)$ and $\mathcal{H}^1(\Upsilon, \eta)$ are equivalent, then there are hermitian line bundle isomorphisms

$$\Lambda_i^j \cong \Upsilon_i^j,$$

over each $U_{i,j}$, such that the isomorphisms induce a mapping

$$\theta_{i,j,k} \mapsto \eta_{i,j,k}.$$

Definition 3.8. A flat hermitian line 1-gerbe $\mathcal{G}^1(\Lambda, \theta)$ is *globally trivialized* by displaying a basis λ_i^j for each line bundle Λ_i^j such that on each $U_{i,j,k}$, we can express the sections on three fold intersections, in terms of coordinates

specified by the data and the ring $C^\infty(U_{i,j,k}; U(1))$, as follows:

$$\theta_{i,j,k} = 1(x)\lambda_i^j \otimes \lambda_j^k \otimes \lambda_k^i,$$

where $1(x) \in C^\infty(U_{i,j,k}; U(1))$ is the constant function which assigns to each point $x \in U_{i,j,k}$, the identity of the group $U(1)$.

Remark. Let \mathcal{G} be a globally trivial flat hermitian line 1-gerbe and (α, ϕ) be the hermitian line 1-cocycle determined by \mathcal{G} . Then for $U_{i,j,k}$

$$\phi(U_{i,j,k}) : l_i^j - l_j^k + l_k^i \rightarrow \mathbb{C}$$

is the canonical isomorphism.

Definition 3.9. A flat hermitian line 1-gerbe is *trivial* if it is equivalent to the *zero 1-gerbe over X* , which is the flat hermitian line 1-gerbe determined by the cocycle $(0, id_0) \in H^1(N(C(X; \mathfrak{U}_1)); \mathcal{L})$.

The notion of a trivial Čech hermitian line 1-gerbe can equivalently be defined by a geometric entity called an *object*, which we define next.

Definition 3.10. Given a flat hermitian line 1-gerbe $\mathcal{G}^1(\Lambda, \theta)$, an *object compatible with \mathcal{G}^1* , denoted $\mathcal{O}(L, m)$ is specified by the following data

1. Constant hermitian line bundles L_i over each U_i ;
2. Hermitian line bundle isomorphisms over each intersection $U_{i,j}$

$$m_i^j : L_i \cong \Lambda_i^j \otimes L_j;$$

such that the composition on three fold intersection

$$L_i \longrightarrow (\Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i) \otimes L_i$$

is exactly

$$(id \otimes m_k^i) \circ (id \otimes m_j^k) \circ m_i^j \equiv \theta_{i,j,k} \otimes id.$$

Here we are abusing notation by denoting the trivialization determined by the section $\theta_{i,j,k}$ also by $\theta_{i,j,k}$.

Proposition 3.11. Let \mathcal{G}^1 be a flat hermitian line 1-gerbe over X and let (α, ϕ) be the Čech hermitian line 1-cocycle determined by \mathcal{G}^1 . Then \mathcal{G}^1

has an object, $\mathcal{O}(L, m)$, compatible with it iff there is a Čech hermitian line 0-chain $\beta \in \text{Ob } C^0(N(C(X; \mathfrak{U}_1)); \mathcal{L})$ and a morphism $f : (\alpha, \phi) \rightarrow (d\beta, \chi_\beta)$, in $\text{Ker}(d, \chi)$, such that (β, f) is a representative of a morphism $[(\beta, f)] : (\alpha, \phi) \rightarrow (0, id_0)$ in $H^1(N(C(X; \mathfrak{U}_1)); \mathcal{L})$.

Proof. Let \mathcal{G}^1 be a trivial flat hermitian line 1-gerbe over X as above. Then there exists a morphism $[(\beta, f)] : (\alpha, \phi) \rightarrow (0, id_0)$ in $H^1(N(C(X; \mathfrak{U}_1)); \mathcal{L})$. Choose a representative (β, f) of this morphism. Now we define the constant line bundle, L_i , over each U_i as follows: $L_i := U_i \times \beta(U_i)$. The linear isometry $f(U_{i,j}) : \alpha(U_{i,j}) \rightarrow (\beta(U_i) - \beta(U_j))$ determines a morphism of hermitian line bundles

$$m_i^j : L_i \rightarrow \Lambda_i^j \otimes L_j$$

over each $U_{i,j}$. The condition over three fold intersections, in definition 3.10, follows from the equation $\chi_\beta \circ d(f) = \phi$. Conversely, given an object compatible with a trivial flat hermitian line 1-gerbe \mathcal{G}^1 , one can define the isomorphism $[(\beta, f)] : (\alpha, \phi) \rightarrow (0, id_0)$ in $H^1(N(C(X; \mathfrak{U}_1)); \mathcal{L})$. □

Finally, we are ready to define a *flat hermitian line 2-gerbe* over X .

Definition 3.12. A *flat hermitian line 2-gerbe* over X , $\mathcal{G}^2(\mathcal{G}, \mathcal{O}, \theta)$, is defined by the following data

1. A flat hermitian line 1-gerbe \mathcal{G}_i^j over the intersection $U_{i,j}$ for every ordered pair $(i, j) \in I \times I$ and $i \neq j$ such that \mathcal{G}_i^j and \mathcal{G}_j^i are dual to each other.
2. For each ordered triple of distinct indices $(i, j, k) \in I \times I \times I$, an object $\mathcal{O}_{i,j,k}$ compatible with the coboundary gerbe

$$\mathcal{G}_i^j \otimes \mathcal{G}_j^k \otimes \mathcal{G}_k^i$$

such that the sections of reorderings of triples (i, j, k) are related in the natural way.

3. For each ordered quadruple of distinct indices $(i, j, k, l) \in I \times I \times I \times I$, trivializations $\theta_{i,j,k,l}$ of coboundaries of objects

$$\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,l}^{-1} \otimes \mathcal{O}_{i,k,l} \otimes \mathcal{O}_{j,k,l}^{-1}$$

on $U_{i,j,k,l}$. Notice that each pair $(\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,l}^{-1})$ is a line bundle over $U_{i,j,k,l}$ so asking for a *trivialization* of the object is legitimate.

This data is subject to a cocycle condition, on $U_{i,j,k,l,m}$ which we denote by $\delta\theta \Rightarrow 0$. The cocycle condition is that over any five fold intersections $U_{i,j,k,l,m}$, we can tensor the five sections of the coboundary objects to give a trivialization of the following hermitian object

$$\begin{aligned}
 & (\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,l}^{-1} \otimes \mathcal{O}_{i,k,l} \otimes \mathcal{O}_{j,k,l}^{-1}) \bigotimes (\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,m}^{-1} \otimes \mathcal{O}_{i,k,m} \otimes \mathcal{O}_{j,k,m}^{-1})^{-1} \\
 & \quad \bigotimes (\mathcal{O}_{i,j,l} \otimes \mathcal{O}_{i,j,m}^{-1} \otimes \mathcal{O}_{i,l,m} \otimes \mathcal{O}_{j,l,m}^{-1}) \\
 & \quad \bigotimes (\mathcal{O}_{i,k,l} \otimes \mathcal{O}_{i,k,m}^{-1} \otimes \mathcal{O}_{i,l,m} \otimes \mathcal{O}_{k,l,m}^{-1})^{-1} \\
 & \quad \bigotimes (\mathcal{O}_{j,k,l} \otimes \mathcal{O}_{j,k,m}^{-1} \otimes \mathcal{O}_{j,l,m} \otimes \mathcal{O}_{k,l,m}^{-1})
 \end{aligned}$$

Notice that the above object is canonically trivial, so the cocycle condition is that the following

$$\theta_{i,j,k,l} - \theta_{i,j,k,m} + \theta_{i,j,l,m} - \theta_{i,k,l,m} + \theta_{j,k,l,m}$$

is the canonical section.

A hermitian line 2-cocycle (α, ϕ) represents a flat hermitian line 2-gerbe over X . We outline a construction of a flat hermitian line 2-gerbe starting from the 2-cocycle (α, ϕ) . A flat hermitian line 1-gerbe, $\mathcal{G}_j^i(\Lambda, \theta)$ over $U_{i,j}$ for every pair $(i, j) \in I \times I$, is determined by the 2-cocycle (α, ϕ) as follows: Over each three-fold intersection, $U_{i,j,k}$, a constant hermitian line bundle $\Lambda_{i,j,k}$ is defined by $\Lambda_{i,j,k} := U_{i,j,k} \times \alpha(U_{i,j,k})$. On every four-fold intersection $U_{i,j,k,l}$, the section $\theta_{i,j,k,l}$ is determined by the linear isometry

$$\phi(U_{i,j,k,l}) : \alpha(U_{i,j,k}) - \alpha(U_{i,k,l}) + \alpha(U_{i,j,l}) - \alpha(U_{j,k,l}) \rightarrow \mathbb{C}.$$

This section satisfies the cocycle condition $\delta\theta \Rightarrow 0$ over five-fold intersections, thus defining a flat hermitian line 1-gerbe over $U_{i,j}$. Notice that the coboundary flat hermitian line 1-gerbe $\mathcal{G}_j^i \otimes \mathcal{G}_k^j \otimes \mathcal{G}_k^i$ over $U_{i,j,k}$, is trivial, therefore there exists an object $\mathcal{O}_{i,j,k}$ compatible with this trivial coboundary gerbe. This object $\mathcal{O}_{i,j,k}$ is specified by the 2-chain $\alpha \in C^2(N(C(X; \mathfrak{U}_I)); \mathcal{L})$. Over each four-fold intersection $U_{i,j,k,l}$, a section $\theta_{i,j,k,l}$ of the coboundary Object $\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,l}^{-1} \otimes \mathcal{O}_{i,k,l} \otimes \mathcal{O}_{j,k,l}^{-1}$ is specified by the linear isometry $\phi(U_{i,j,k,l})$.

4. Dijkgraaf-Witten theory

We would like to recover Dijkgraaf-Witten’s construction [DW90] of a TQFT. In principle, we follow their construction, using Freed-Quinn’s hermitian-line

incarnation [FQ93], and placing it further within the framework of cohomology with coefficients in the Picard groupoid of hermitian lines.

4.1. Hermitian line corresponding to a closed n -manifold

We start with an n -cocycle α which is an object of the Picard groupoid $H^n(BG; \mathcal{L})$. For each map $f : Y \rightarrow BG$ from a closed n -manifold Y , we take the pullback $f^*\alpha$. Consider the cap product

$$\cap : H^n(Y; \mathcal{L}) \otimes H_n(Y; \mathbb{Z}[0]) \rightarrow H_0(Y; \mathcal{L}),$$

which is a morphism of Picard groupoids. If we substitute the given cocycle α in the first factor, we will get a morphism

$$(15) \quad f^*\alpha \cap - : H_n(Y; \mathbb{Z}[0]) \rightarrow H_0(Y; \mathcal{L}).$$

What we would like to do is to apply this morphism to the fundamental cycle of Y . However, in the homology with coefficients in a Picard groupoid, be it a discrete one, such as $\mathbb{Z}[0]$, no single object represents the fundamental cycle canonically. It is rather a full subgroupoid (not monoidal) C_Y formed by all possible cycles representing the fundamental cycle and connected by equivalence classes of morphisms given by n -boundaries modulo $(n + 1)$ -boundaries: a morphism $y \rightarrow y'$ is given by an $(n + 1)$ -chain x such that $y' = y + dx$; two morphisms $x : y \rightarrow y'$ and $x' : y \rightarrow y'$ are equivalent if there is an $(n + 2)$ -chain w such that $x = dw + x'$. Thus, we can restrict the above morphism (15) to this *fundamental-cycle groupoid* C_Y and get a functor

$$f^*\alpha \cap - : C_Y \rightarrow H_0(Y; \mathcal{L}).$$

If we compose this functor with the *degree map*

$$H_0(Y; \mathcal{L}) \rightarrow \mathcal{L}$$

which takes each linear combination $a_1y_1 + \dots + a_ky_k$ of points y_1, \dots, y_k in Y with coefficients a_1, \dots, a_k in \mathcal{L} to the sum $a_1 + \dots + a_k$, which is an object in \mathcal{L} , we obtain a functor

$$(16) \quad F : C_Y \rightarrow \mathcal{L}$$

from the fundamental-cycle groupoid to the groupoid of hermitian lines. Now we take the limit of this functor. The existence of the limit is guaranteed by the following fact.

Proposition 4.1. *The functor*

$$F : C_Y \rightarrow \mathcal{L},$$

which represents the cap product of the cocycle α with the fundamental-cycle groupoid C_Y , has a limit,

$$\lim_{C_Y} F,$$

in the category \mathcal{L} of hermitian lines.

Proof. The limit of the functor F may be realized by Freed-Quinn’s *invariant-section construction*: an invariant section is a collection of elements in $\{s(y) \in F(y) \mid y \in \text{Ob } C_Y\}$ such that for each morphism $x : y \rightarrow y'$ in C_Y , we have $F(x)s(y) = s(y')$. The space of invariant sections is a hermitian line, in other words, the limit of F exists, if the functor has *no holonomy*, *i.e.*, $F(x) = \text{id}$ for each automorphism $x : y \rightarrow y$. This is indeed the case, due to the following argument.

Being an object of $H^n(Y; \mathcal{L})$, the cocycle α is represented by a pair (a, ϕ) , where a is an object of $C^n(Y; \mathcal{L})$, *i.e.*, a function $a : S^n(Y) \rightarrow \text{Ob } \mathcal{L}$, and $\phi : da \rightarrow 0$ is a morphism in $C^{n+1}(Y; \mathcal{L})$, *i.e.*, a function $S^{n+1}(Y) \rightarrow \text{Mor } \mathcal{L}$. The functor $F : C_Y \rightarrow \mathcal{L}$ acts in the following way on objects and morphisms of the groupoid C_Y :

$$F(y) = a(y) \quad \text{for } y \in \text{Ob } C_Y,$$

and

$$F(x) : a(y) \rightarrow a(y') \quad \text{for } x \in \text{Mor } C_Y, y' = y + dx,$$

is defined by $\phi(x) : a(y') - a(y) = a(dx) = da(x) \rightarrow 0$ as a composition of it with the structure natural transformations (1)–(2) and their inverses.

Now suppose we have an automorphism $x : y \rightarrow y$, which in particular means that we have a chain $x \in \text{Ob } C_{n+1}(Y; \mathbb{Z}[0])$, such that $dx = 0$. Since $H_{n+1}(Y; \mathbb{Z}[0])$ is trivial whenever $\dim Y = n$, the cycle x must be a boundary: $x = dw$ for some w . This renders the equivalence class of the morphism x to be trivial. □

4.2. Linear isometry corresponding to an $(n + 1)$ -cobordism

Now let X be a compact $n + 1$ -manifold with boundary $i : \partial X = \partial X_- \amalg \partial X_+ \subset X$. As a starting point, we use the same n -cocycle α , which is an object of the Picard groupoid $H^n(BG; \mathcal{L})$. For any continuous function $f : X \rightarrow BG$,

a pullback of α along f gives an n -cocycle $f^*\alpha$, which is an object of the Picard groupoid $H^n(X; \mathcal{L})$. Consider the relative cap product

$$\cap : H^n(X; \mathcal{L}) \otimes H_{n+1}(X, \partial X; \mathbb{Z}[0]) \rightarrow H_1(X, \partial X; \mathcal{L}),$$

which is a morphism of Picard groupoids. If we substitute $f^*\alpha$ in the first factor, we will get a functor

$$(17) \quad f^*\alpha \cap - : H_{n+1}(X, \partial X; \mathbb{Z}[0]) \rightarrow H_1(X, \partial X; \mathcal{L}).$$

As above, we restrict this functor to the *relative fundamental-cycle groupoid* $C_{X, \partial X}$ which is the full subgroupoid of $H_{n+1}(X, \partial X; \mathbb{Z}[0])$ whose objects are all possible relative cycles representing the relative fundamental class of X . The restriction gives us a functor

$$f^*\alpha \cap - : C_{X, \partial X} \rightarrow H_1(X, \partial X; \mathcal{L}).$$

We compose this functor first with the 2-morphism Ψ_1 from (the chain version of) the long 2-exact sequence (6) and then the degree map

$$(18) \quad C_{X, \partial X} \xrightarrow{f^*\alpha \cap -} H_1(X, \partial X; \mathcal{L}) \xrightarrow{\partial_1} H_0(\partial X; \mathcal{L}) \xrightarrow{H_0(i)} H_0(X; \mathcal{L}) \xrightarrow{\text{deg}} \mathcal{L}.$$

$\begin{array}{c} 0 \\ \curvearrowright \\ \Psi_1 \uparrow \end{array}$

This diagram gives us a 2-morphism $t : F \Rightarrow 0$, where $F : C_{X, \partial X} \rightarrow \mathcal{L}$ is the composite functor in the lower row. Consider the following diagram:

(19)

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \curvearrowright & & \\
 & & & & \Psi_1 \uparrow & & \\
 H_{n+1}(X, \partial X; \mathbb{Z}[0]) & \xrightarrow{f^*\alpha \cap -} & H_1(X, \partial X; \mathcal{L}) & \xrightarrow{\partial_1} & H_0(\partial X; \mathcal{L}) & \xrightarrow{H_0(i)} & H_0(X; \mathcal{L}) \xrightarrow{\text{deg}} \mathcal{L}, \\
 \downarrow \partial & \nearrow & & \nearrow & & & \\
 H_n(\partial X; \mathbb{Z}[0]) & & & & f|_{\partial X}^* \alpha \cap - & &
 \end{array}$$

where the bottom 2-morphism is comes from Proposition 2.4. When we restrict the boundary 1-morphism $\partial : H_{n+1}(X, \partial X; \mathbb{Z}[0]) \rightarrow H_n(\partial X; \mathbb{Z}[0])$ to

the full subcategory $C_{X,\partial X}$, we get the following commutative diagram:

$$\begin{array}{ccc}
 C_{X,\partial X} & \longrightarrow & H_{n+1}(X, \partial X; \mathbb{Z}[0]) , \\
 \partial \downarrow & \circlearrowleft & \downarrow \partial \\
 -C_{\partial-X} \times C_{\partial+X} & \longrightarrow & H_n(\partial X; \mathbb{Z}[0])
 \end{array}$$

where $-C_{\partial-X}$ is the negative fundamental-cycle groupoid of $\partial-X$, the full subcategory in $H_n(\partial X; \mathbb{Z}[0])$ made up by representatives of the negative fundamental class of $\partial-X$ in $H_n(\partial-X; \mathbb{Z}) \subset H_n(\partial X; \mathbb{Z})$. By stacking together the last two diagrams, we obtain the following diagram:

$$(20) \quad \begin{array}{ccc}
 & \overset{0}{\curvearrowright} & \\
 & \Psi_F \uparrow \! \! \! \uparrow & \\
 C_{X,\partial X} & \xrightarrow{F} & \mathcal{L} , \\
 \partial \downarrow & \nearrow X_F & \nearrow F_- + F_+ \\
 -C_{\partial-X} \times C_{\partial+X} & &
 \end{array}$$

where $F_- := f|_{\partial-X}^* \alpha \cap -$ and $F_+ := f|_{\partial+X}^* \alpha \cap -$ appended by $H_0(i)$ and deg as in (19).

Applying the limit functor, we get canonical morphisms

$$- \lim_{C_{\partial-X}} F_- + \lim_{C_{\partial+X}} F_+ \rightarrow \lim_{-C_{\partial-X} \times C_{\partial+X}} (F_- + F_+) \rightarrow \lim_{C_{X,\partial X}} F \rightarrow 0$$

in \mathcal{L} , whence a morphism

$$l_f : \lim_{C_{\partial-X}} F_- \rightarrow \lim_{C_{\partial+X}} F_+ ,$$

which translates into a canonical linear isometry between hermitian lines.

4.3. The Dijkgraaf-Witten theory TQFT functor

Given a finite group G , for each $\alpha \in H^n(BG; \mathcal{L})$, we construct the Dijkgraaf-Witten theory TQFT functor,

$$Z^\alpha : \mathbf{Cob}(n+1) \rightarrow \mathbf{Vect} ,$$

from the category $\mathbf{Cob}(n+1)$ of cobordisms to the category \mathbf{Vect} of complex vector spaces, using the ingredients developed in the preceding sections.

We first construct the values of the functor on objects. Observe that for every $Y \in \text{Ob } \mathbf{Cob}(n + 1)$, Proposition 4.1 delivers a canonical hermitian line for each $f \in \text{Map}(Y, BG)$. We claim that these lines glue into a flat hermitian line bundle over $\text{Map}(Y, BG)$, or a *local system* with values in \mathcal{L} , *i.e.*, a functor

$$\mathcal{L}_Y : \Pi_1 \text{Map}(Y, BG) \rightarrow \mathcal{L}$$

from the fundamental groupoid of the mapping space $\text{Map}(Y, BG)$ to \mathcal{L} .

A morphism in $\Pi_1 \text{Map}(Y, BG)$ is a homotopy class $[f]$ of a map $f : Y \times I \rightarrow BG$. We can think of $Y \times I$ as the identity cobordism between two copies of Y . Applying the construction of Section 4.2, we get a morphism in \mathcal{L} ,

$$l_f : \lim_{C_Y} F_0 \rightarrow \lim_{C_Y} F_1.$$

We define $\mathcal{L}_Y([f]) := l_f$. The cocycle $f^* \alpha$ does depend on the representative f of the homotopy class $[f]$, see Section 2.3, however the difference disappears at the homology level after applying the cap product with $f^* \alpha$ and the “boundary homomorphism” $\partial_1 : H_1(Y \times I, \partial(Y \times I); \mathcal{L}) \rightarrow H_0(\partial(Y \times I); \mathcal{L})$ in (19). Note that $f|_{\partial(Y \times I)}^* \alpha$ does not depend on the representative of the homotopy class $[f]$, because the homotopy is supposed to be relative to the boundary. Thus, the diagram (20) does not depend of the choice of a representative of the homotopy class $[f]$, and the local system \mathcal{L}_Y is well defined.

One can view the construction of a local system \mathcal{L}_y as “integration of $\text{ev}^* \alpha$ along fibers” of π or a construction of the push-pull in cohomology with values in Picard groupoids along the following diagram:

$$\begin{array}{ccc} Y \times \text{Map}(Y, BG) & \xrightarrow{\text{ev}} & BG \\ \pi \downarrow & & \\ \text{Map}(Y, BG), & & \end{array}$$

$$H^n(BG; \mathcal{L}) \xrightarrow{\text{ev}^*} H^n(Y \times \text{Map}(Y, BG); \mathcal{L}) \xrightarrow{\pi_*} H^0(\text{Map}(Y, BG); \mathcal{L}),$$

where $\pi_* \text{ev}^* \alpha := \mathcal{L}_Y$, by definition, and we recall that objects of $H^0(\text{Map}(Y, BG); \mathcal{L})$ are identified with local systems or 0-gerbes, see Section 3.4.

For any $Y \in \text{Ob } \mathbf{Cob}(n + 1)$, we define the value $Z^\alpha(Y)$ of the *TQFT functor* to be the space of global sections of the local system \mathcal{L}_Y over

$\text{Map}(Y, BG)$ constructed above:

$$Z^\alpha(Y) := H^0(\text{Map}(Y, BG); \mathcal{L}_Y) := \lim \mathcal{L}_Y \in \mathbf{Vect},$$

where the limit is taken for a natural extension $\Pi_1 \text{Map}(Y, BG) \xrightarrow{\mathcal{L}_Y} \mathcal{L} \rightarrow \mathbf{Vect}$ of the functor \mathcal{L}_Y , denoted by the same symbol. The limit exists, because the category \mathbf{Vect} is complete.

Now we construct the arrow function of the TQFT functor. This can also be viewed as a construction of “fiberwise integral.” Let X be an $(n + 1)$ -dimensional cobordism from $\partial_- X$ to $\partial_+ X$. We get two local systems $\mathcal{L}_{\partial_- X}$ and $\mathcal{L}_{\partial_+ X}$ over the mapping spaces $\text{Map}(\partial_- X, BG)$ and $\text{Map}(\partial_+ X, BG)$, respectively. Let $p_\pm : \text{Map}(X, BG) \rightarrow \text{Map}(\partial_\pm X, BG)$ denote the natural restriction morphisms. We start with constructing a morphism $\mathcal{L}_X : p_-^* \mathcal{L}_{\partial_- X} \rightarrow p_+^* \mathcal{L}_{\partial_+ X}$ of local systems on $\text{Map}(X, BG)$. *i.e.*, a natural transformation between functors $p_-^* \mathcal{L}_{\partial_- X}$ and $p_+^* \mathcal{L}_{\partial_+ X} : \Pi_1(\text{Map}(X, BG)) \rightarrow \mathcal{L}$. For each $f \in \text{Map}(X, BG)$, by invoking the construction of Section 4.2 once again, we get two functors $F_\pm : C_{\partial_\pm X} \rightarrow \mathcal{L}$ and the following morphism

$$l_f : \lim_{C_{\partial_- X}} F_- \rightarrow \lim_{C_{\partial_+ X}} F_+$$

in \mathcal{L} . Note that the fiber of each pull-back local system $p_\pm^* \mathcal{L}_{\partial_\pm X}$ over $f \in \text{Map}(X, BG)$ is by definition the fiber of $\mathcal{L}_{\partial_\pm X}$ over $p_\pm(f)$, and that fiber is $\lim_{C_{\partial_\pm X}} F_\pm$ by the construction of Section 4.1. We define $\mathcal{L}_X(f)$ to be $l_f : p_-^* \mathcal{L}_{\partial_- X}|_f \rightarrow p_+^* \mathcal{L}_{\partial_+ X}|_f$ on objects $f \in \text{Map}(X, BG)$ of $\Pi_1(\text{Map}(X, BG))$. A morphism $f \rightarrow g$ in the fundamental groupoid $\Pi_1(\text{Map}(X, BG))$ is represented by a homotopy $h \in \text{Map}(X \times I, BG)$ between maps f and $g \in \text{Map}(X, BG)$. To see that \mathcal{L}_X constitutes a natural transformation, we need to see that the diagram

$$(21) \quad \begin{array}{ccc} p_-^* \mathcal{L}_{\partial_- X}|_f & \xrightarrow{l_f} & p_+^* \mathcal{L}_{\partial_+ X}|_f \\ p_-^* l_{h|_{\partial_- X \times I}} \downarrow & & \downarrow p_+^* l_{h|_{\partial_+ X \times I}} \\ p_-^* \mathcal{L}_{\partial_- X}|_g & \xrightarrow{l_g} & p_+^* \mathcal{L}_{\partial_+ X}|_g \end{array}$$

commutes. Indeed, the homotopy gives a morphism $H : f^* \alpha \rightarrow g^* \alpha$ in the Picard groupoid $H^n(X; \mathcal{L})$. Using the bifactoriality of the cap product, we get a 2-morphism $f^* \alpha \cap - \Rightarrow g^* \alpha \cap -$ added to Diagram (18), resulting in

a commutative triangle

$$\begin{array}{ccc}
 F & \xrightarrow{\Psi_H} & G \\
 \searrow \Psi_F & & \swarrow \Psi_G \\
 & 0 &
 \end{array}$$

on top of the upper part of Diagram (20) and, similarly, a commutative square

$$\begin{array}{ccc}
 (F_- + F_+) \circ \partial & \xrightarrow{\Psi_{\partial H}} & (G_- + G_+) \circ \partial \\
 X_F \downarrow & & \downarrow X_G \\
 F & \xrightarrow{\Psi_H} & G
 \end{array}$$

on top of the lower part of Diagram (20), with $\Psi_{\partial H}$ coming from the 2-morphism $f|_{\partial X}^* \alpha \cap - \Rightarrow g|_{\partial X}^* \alpha \cap -$ added to the bottom triangle in (18). Passing to the limits, we see that (21) is commutative.

Now, after the morphism $\mathcal{L}_X : p_-^* \mathcal{L}_{\partial_- X} \rightarrow p_+^* \mathcal{L}_{\partial_+ X}$ of local systems on $\text{Map}(X, BG)$ is constructed, we are ready to construct a linear map

$$Z^\alpha(X) : Z^\alpha(\partial_- X) \rightarrow Z^\alpha(\partial_+ X)$$

or

$$Z^\alpha(X) : H^0(\text{Map}(\partial_- X, BG); \mathcal{L}_{\partial_- X}) \rightarrow H^0(\text{Map}(\partial_+ X, BG); \mathcal{L}_{\partial_+ X}).$$

The plan is to describe a push-pull along the diagram of spaces:

$$\text{Map}(\partial_- X, BG) \xleftarrow{p_-} \text{Map}(X, BG) \xrightarrow{p_+} \text{Map}(\partial_+ X, BG).$$

The pullback

$$p_-^* : H^0(\text{Map}(\partial_- X, BG); \mathcal{L}_{\partial_- X}) \rightarrow H^0(\text{Map}(X, BG); p_-^* \mathcal{L}_{\partial_- X})$$

is easy. So is an intermediate map:

$$H^0(\mathcal{L}_X) : H^0(\text{Map}(X, BG); p_-^* \mathcal{L}_{\partial_- X}) \rightarrow H^0(\text{Map}(X, BG); p_+^* \mathcal{L}_{\partial_+ X}).$$

The pushforward

$$(p_+)_* : H^0(\text{Map}(X, BG); p_+^* \mathcal{L}_{\partial_+ X}) \rightarrow H^0(\text{Map}(\partial_+ X, BG); \mathcal{L}_{\partial_+ X})$$

is not straightforward, and its existence relies on the specifics of the topology of mapping spaces to BG for a finite group G .

Recall that the space $\text{Map}(X, BG)$ may naturally be realized at the classifying space for principal G -bundles over X . This leads to a natural homotopy equivalence

$$\text{Map}(X, BG) \sim \coprod_{[P \rightarrow X]} B \text{Aut}(P),$$

where the disjoint union is taken over isomorphism classes $[P \rightarrow X] \simeq \pi_0 \text{Map}(X, BG)$ of principal G -bundles $P \rightarrow X$. The map

$$p_+ : \text{Map}(X, BG) \rightarrow \text{Map}(\partial_+ X, BG)$$

is homotopy equivalent to the natural restriction map

$$p'_+ : \coprod_{[P \rightarrow X]} B \text{Aut}(P) \rightarrow \coprod_{[P_+ \rightarrow \partial_+ X]} B \text{Aut}(P_+),$$

which is a finite covering map over each connected component $B \text{Aut}(P_+)$, sometimes with empty fiber.

We will define the pushforward

$$(p_+)_* : H^0(\text{Map}(X, BG); p_+^* \mathcal{L}_{\partial_+ X}) \rightarrow H^0(\text{Map}(\partial_+ X, BG); \mathcal{L}_{\partial_+ X})$$

as a transfer map

$$(p'_+)_* : H^0 \left(\coprod_{[P \rightarrow X]} B \text{Aut}(P); (p'_+)^* \mathcal{L}_{\partial_+ X} \right) \rightarrow H^0 \left(\coprod_{[P_+ \rightarrow \partial_+ X]} B \text{Aut}(P_+); \mathcal{L}_{\partial_+ X} \right),$$

which will be constructed using the definition of H^0 as a limit over the fundamental groupoid. Indeed, for every path γ_+ in $B \text{Aut}(P_+)$, we take all its lifts to the component $B \text{Aut}(P)$ over $B \text{Aut}(P_+)$, which is a finite, possibly zero, number. For each such path γ , we have a linear isometry $(p'_+)^* \mathcal{L}_{\partial_+ X}(\gamma) : (p'_+)^* \mathcal{L}_{\partial_+ X}(\gamma(0)) \rightarrow (p'_+)^* \mathcal{L}_{\partial_+ X}(\gamma(1))$, which, by definition of $(p'_+)^*$, is equal to the isometry $\mathcal{L}_{\partial_+ X}(\gamma_+) : \mathcal{L}_{\partial_+ X}(\gamma_+(0)) \rightarrow \mathcal{L}_{\partial_+ X}(\gamma_+(1))$. Since $H^0 \left(\coprod_{[P \rightarrow X]} B \text{Aut}(P); (p'_+)^* \mathcal{L}_{\partial_+ X} \right)$ is a limit of the functor $(p'_+)^* \mathcal{L}_{\partial_+ X}$, we have a canonical commutative diagram of linear maps:

$$\begin{array}{ccc}
 & H^0 \left(\coprod_{[P \rightarrow X]} B \text{Aut}(P); (p'_+)^* \mathcal{L}_{\partial_+ X} \right) & \\
 \swarrow & & \searrow \\
 (p'_+)^* \mathcal{L}_{\partial_+ X}(\gamma(0)) & \xrightarrow{\quad} & (p'_+)^* \mathcal{L}_{\partial_+ X}(\gamma(1)) \\
 \parallel & & \parallel \\
 \mathcal{L}_{\partial_+ X}(\gamma_+(0)) & \xrightarrow{\quad} & \mathcal{L}_{\partial_+ X}(\gamma_+(1)).
 \end{array}$$

If, given γ_+ , we add the linear maps $H^0 \left(\coprod_{[P \rightarrow X]} B \text{Aut}(P); (p'_+)^* \mathcal{L}_{\partial_+ X} \right) \rightarrow \mathcal{L}_{\partial_+ X}(\gamma_+(0))$ over all possible γ 's covering γ_+ and do the same for maps to $\mathcal{L}_{\partial_+ X}(\gamma_+(1))$, we will get a commutative diagram

$$\begin{array}{ccc}
 & H^0 \left(\coprod_{[P \rightarrow X]} B \text{Aut}(P); (p'_+)^* \mathcal{L}_{\partial_+ X} \right) & \\
 \swarrow & & \searrow \\
 \mathcal{L}_{\partial_+ X}(\gamma_+(0)) & \xrightarrow{\hspace{10em}} & \mathcal{L}_{\partial_+ X}(\gamma_+(1)).
 \end{array}$$

Since $H^0 \left(\coprod_{[P_+ \rightarrow \partial_+ X]} B \text{Aut}(P_+); \mathcal{L}_{\partial_+ X} \right)$ is a limit of the functor $\mathcal{L}_{\partial_+ X}$, we get a canonical linear map

$$H^0 \left(\coprod_{[P \rightarrow X]} B \text{Aut}(P); (p'_+)^* \mathcal{L}_{\partial_+ X} \right) \rightarrow H^0 \left(\coprod_{[P_+ \rightarrow \partial_+ X]} B \text{Aut}(P_+); \mathcal{L}_{\partial_+ X} \right),$$

which we declare to be the *transfer* $(p'_+)_*$.

Finally, the TQFT functor

$$Z^\alpha(X) : H^0(\text{Map}(\partial_- X, BG); \mathcal{L}_{\partial_- X}) \rightarrow H^0(\text{Map}(\partial_+ X, BG); \mathcal{L}_{\partial_+ X})$$

is defined as the composition of $(p_+)_*$, $H^0(\mathcal{L}_X)$, and p_-^* .

The invariance of Z^α under diffeomorphisms $X' \rightarrow X''$ of cobordisms is obvious, as a diffeomorphism induces an isomorphism of simplicial sets $\text{Sing}(X')$ and $\text{Sing}(X'')$ representing the cobordisms and leads to isomorphic diagrams (19) and (20) in a strict sense, thus giving the same isometry l_f of hermitian lines in Section 4.2.

Appendix A. The Hom 2-chain complex

In this section we define the Hom 2 - chain complex and a tensor product in $\mathbf{2Ch}(SCG)$. We recall that given any two Picard groupoids $\mathcal{A}, \mathcal{B} \in \text{Ob}(SCG)$, $\text{Hom}_{SCG}(\mathcal{A}, \mathcal{B})$ inherits a Picard groupoid structure, i.e. the category (SCG) is enriched over itself. Let $\mathcal{A}_\bullet, \mathcal{B}_\bullet \in \text{Ob}(\mathbf{2Ch}(SCG))$. Then, $(\text{Hom}_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet), d, \phi)$ is a chain complex whose *n*th. degree is defined as follows:

$$\text{Hom}_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)_n = \prod_p \text{Hom}_{SCG}(\mathcal{A}_p, \mathcal{B}_{p+n}).$$

The differential $d : \text{Hom}_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)_n \rightarrow \text{Hom}_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)_{n-1}$ is given by $(df)_p = df_p + (-1)^{p+1} f_{p-1} d$ and a composition of 2-morphisms $dd(f) \Rightarrow$

$d^2 f + f d^2 \Rightarrow 0$, where the first 2-morphism comes from the distributivity law on each degree of the Hom complex which is the consequence of the enrichment of SCG over itself and the second 2-morphism in the composition is obvious. Similarly, we may define the Hom 2-chain complex of chain maps between a 2-cochain complex and a 2-chain complex. Let \mathcal{C}^\bullet be a 2-cochain complex of Picard groupoids, let $\mathcal{B}_\bullet \in \text{Ob}(\mathbf{2Ch}(SCG))$, then the 2-chain complex $([\mathcal{C}^\bullet, \mathcal{B}_\bullet], d, \phi)$ is defined as the 2-chain complex $(\text{Hom}_{\mathbf{2Ch}(SCG)}(\mathcal{C}^{-\bullet}, \mathcal{B}_\bullet), d, \phi)$, where $\mathcal{C}^{-\bullet} \in \text{Ob}(\mathbf{2Ch}(SCG))$ is the 2-chain complex obtained by negatively regrading \mathcal{C}^\bullet , its degree n is $[\mathcal{C}^\bullet, \mathcal{B}_\bullet]_n = \coprod_p \text{Hom}_{SCG}(\mathcal{C}^{-p}, \mathcal{B}_{p+n})$. The tensor product of two chain complexes could be defined similarly.

Appendix B. *Pic* categories

In this appendix we give the definition of a mathematical structure which is built on a bicategory but whose mapping categories have the structure of Picard groupoids.

Definition B.1. A *Pic*-category \mathcal{C} consists of the following data

- 1) A small set, $\text{Ob}(\mathcal{C})$, whose elements will be called the *objects* of \mathcal{C} .
- 2) A function $\mathcal{C}(-, -) : \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{Pic})$, where $\text{Ob}(\mathcal{Pic})$ is the set of all Picard groupoids.
- 3) For each object $s \in \text{Ob}(\mathcal{C})$, a homomorphism $id_s : * \rightarrow \mathcal{C}(s, s)$, where $*$ is the terminal Picard groupoid.
- 4) For each triple of objects $s, t, u \in \text{Ob}(\mathcal{C})$, a *composition bifunctor* $- \circ - : \mathcal{C}(t, u) \times \mathcal{C}(s, t) \rightarrow \mathcal{C}(s, u)$ which is subject to the following conditions
 - a) For each $h \in \text{Ob}(\mathcal{C}(t, u))$, the functor

$$h \circ - : \mathcal{C}(s, t) \rightarrow \mathcal{C}(s, u).$$

is a homomorphism

- b) For each $g \in \text{Ob}(\mathcal{C}(s, t))$, the functor

$$- \circ g : \mathcal{C}(t, u) \rightarrow \mathcal{C}(s, u).$$

is a homomorphism.

- 5) For each triple of objects $s, t, u \in \text{Ob}(\mathcal{C})$ and each pair of morphisms $g_1, g_2 \in \text{Ob}(\mathcal{C}(s, t))$, a monoidal natural transformation $\phi_{g_1, g_2}^- : - \circ g_1 + - \circ g_2 \Rightarrow - \circ g_1 + g_2$, where the homomorphism $- \circ g_1 + - \circ g_2 : \mathcal{C}(t, u) \rightarrow \mathcal{C}(s, u)$ is defined pointwise.
- 6) For each triple of objects $s, t, u \in \text{Ob}(\mathcal{C})$ and each pair of morphisms $h_1, h_2 \in \text{Ob}(\mathcal{C}(t, u))$, a monoidal natural transformation $\psi_{h_1, h_2}^- : h_1 \circ - + h_2 \circ - \Rightarrow h_1 + h_2 \circ -$, where the homomorphism $h_1 \circ - + h_2 \circ - : \mathcal{C}(s, t) \rightarrow \mathcal{C}(s, u)$ is defined pointwise.
- 7) For each quadruple of objects $s, t, u, v \in \text{Ob}(\mathcal{C})$, a natural transformation, α called the *associator*, between functors defined in the following diagram

$$\begin{array}{ccc}
 \mathcal{C}(u, v) \times \mathcal{C}(t, u) \times \mathcal{C}(s, t) & \xrightarrow{id \times - \circ -} & \mathcal{C}(u, v) \times \mathcal{C}(s, u) , \\
 \downarrow - \circ - \times id & \xleftarrow{\alpha} & \downarrow - \circ - \\
 \mathcal{C}(t, v) \times \mathcal{C}(s, t) & \longrightarrow & \mathcal{C}(s, v) .
 \end{array}$$

and which is subject to the following conditions:

- a) For each pair $(g, h) \in \text{Ob}(\mathcal{C}(t, u)) \times \text{Ob}(\mathcal{C}(u, v))$, the natural transformation $\alpha_{(h, g, -)}$ as in the following diagram

$$\begin{array}{ccccc}
 & & (h \circ g) \circ - & & \\
 & \swarrow & \uparrow \alpha_{(h, g, -)} & \searrow & \\
 \mathcal{C}(s, t) & \xrightarrow{g \circ -} & \mathcal{C}(s, u) & \xrightarrow{h \circ -} & \mathcal{C}(s, v)
 \end{array}$$

is a monoidal natural transformation.

- b) For each pair $(f, h) \in \text{Ob}(\mathcal{C}(s, t)) \times \text{Ob}(\mathcal{C}(u, v))$, the natural transformation $\alpha_{(h, -, f)}$ as in the following diagram

$$\begin{array}{ccc}
 \mathcal{C}(t, u) & \xrightarrow{h \circ -} & \mathcal{C}(t, v) \\
 \downarrow - \circ f & \xleftarrow{\alpha_{(f, -, h)}} & \downarrow - \circ f \\
 \mathcal{C}(s, u) & \xrightarrow{h \circ -} & \mathcal{C}(s, v)
 \end{array}$$

is a monoidal natural transformation.

- c) For each pair $(f, g) \in \text{Ob}(\mathcal{C}(s, t)) \times \text{Ob}(\mathcal{C}(t, u))$, the natural transformation $\alpha_{(-,g,f)}$ as in the following diagram

$$\begin{array}{ccccc}
 \mathcal{C}(u, v) & \xrightarrow{-\circ g} & \mathcal{C}(t, v) & \xrightarrow{-\circ f} & \mathcal{C}(s, v) \\
 & \searrow & \uparrow \alpha_{(-,g,f)} & \nearrow & \\
 & & -\circ(g \circ f) & &
 \end{array}$$

is a monoidal natural transformation.

- 8) For each pair of objects $s, t \in \text{Ob}(\mathcal{C})$, two monoidal natural transformations

$$\begin{array}{ccc}
 \mathcal{C}(s, t) & \xlongequal{\quad} & \mathcal{C}(s, t) \\
 \searrow & \lambda \uparrow & \nearrow \\
 & id_t \circ - &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{C}(s, t) & \xlongequal{\quad} & \mathcal{C}(s, t) \\
 \searrow & \rho \uparrow & \nearrow \\
 & -\circ id_s &
 \end{array}$$

Let \mathcal{C} and \mathcal{D} be two *Pic*-categories, A functor of *Pic*-categories $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor of bicategories which respects the additional structure on the morphism categories of \mathcal{C} and \mathcal{D} . We will skip a precise definition of a functor of *Pic*-categories but an interested reader can define these functors rigorously using our definition of *Pic*-categories.

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