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Mixed trTLEP-structures and mixed Frobenius structures

Yota Shamoto

We introduce the notion of mixed trTLEP-structures and prove that a mixed trTLEP-structure with some conditions naturally induces a mixed Frobenius manifold. This is a generalization of the reconstruction theorem of Hertling and Manin. As a special case, we also show that a graded polarizable variation of mixed Hodge structure with H^2-generation condition gives rise to a family of mixed Frobenius manifolds. It implies that there exist mixed Frobenius manifolds associated to local B-models.

1. Introduction

1.1. Frobenius manifolds and their comparison

A Frobenius manifold is a complex manifold whose tangent bundle is equipped with a commutative product, a metric and a global section satisfying a kind of integrability condition. The notion of Frobenius manifold was introduced by Dubrovin [9] for the investigation of 2D topological field theories. (The equivalent structure, called flat structure, was introduced by K. Saito [27].) It has been shown that Frobenius manifolds naturally arise

in many theories: the invariant theory of Weyl groups [28]; singularity theory [7, 8, 26, 28]; Gromov-Witten theory [21]; the deformation theory of A_∞-structures [1], etc.

In some cases, we have an interesting isomorphism of Frobenius manifolds in two different theories. For instance, one of the goals in mirror symmetry is to prove that a Frobenius manifold constructed in a A-model is isomorphic to the one constructed in the corresponding B-model.

However, in general, it is difficult to compare Frobenius manifolds $\mathscr{F}^{(0)}$ and $\mathscr{F}^{(1)}$. A useful strategy is to split the problem into two steps as follows.

Step 1.: Show that each $\mathscr{F}^{(i)}$ is constructed from more restricted set of data $\mathcal{T}^{(i)}$ $(i = 0, 1)$.

Step 2.: Show that $\mathcal{T}^{(0)} \simeq \mathcal{T}^{(1)}$.

The fact in Step 1 is called (re-)construction theorem. It depends on the problem which data we choose. Following [14], let us consider the case of trTLEP-structure.

Let M be a complex manifold and $j_\lambda : \mathbb{P}^1_\lambda \times M \to \mathbb{P}^1_\lambda \times M$ a map defined by $j_\lambda(\lambda, t) := (-\lambda, t)$ where λ is non-homogeneous coordinate on $\mathbb{P}^1_\lambda$ and t is a point in M. For an integer k, trTLEP(k)-structure on M is a tuple $(\mathcal{H}, \nabla, P)$ with following properties (Definition 2.13). $\mathcal{H}$ is a holomorphic vector bundle on $\mathbb{P}^1_\lambda \times M$ trivial along $\mathbb{P}^1_\lambda$, ∇ is a meromorphic flat connection on $\mathcal{H}$;

$$\nabla \colon \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\mathbb{P}^1_\lambda \times M}\big(\log(\{0, \infty\} \times M)\big) \otimes \mathcal{O}_{\mathbb{P}^1_\lambda \times M}(\{0\} \times M). \tag{1.1}$$

P is ∇-flat non-degenerate $(-1)^k$-symmetric pairing

$$P : \mathcal{H} \otimes j_\lambda^* \mathcal{H} \to \mathcal{O}_{\mathbb{P}^1_\lambda \times M}(-k\{0\} \times M + k\{\infty\} \times M). \tag{1.2}$$

Hertling and Manin [14] showed the construction theorem for trTLEP-structure. In other words, they proved that a trTLEP-structure with some condition uniquely induces a Frobenius manifold. Then, they applied the construction theorem to compare the Frobenius manifolds constructed from isolated singularities, Frobenius manifolds associated to variation of polarized Hodge structure of some family of hypersurfaces, and super Frobenius manifolds in the deformation theory of A_∞-structures.

Reichelt [23] defined the notion of logarithmic trTLEP-structure as a generalization of trTLEP-structure and proved the construction theorem for logarithmic Frobenius manifolds. Reichelt and Sevenheck [24] used the construction theorem to refine the mirror symmetry theorem [11] for weak

Fano toric manifolds. Here, we note that in [24], the result of Givental [11] plays an important role at Step 2 of the strategy above.

1.2. Construction theorem for mixed Frobenius manifolds

The first main theorem of this paper is the construction theorem for mixed Frobenius manifolds. We introduce the notion of mixed trTLEP-structure and show that a mixed trTLEP-structure with some condition naturally induces a mixed Frobenius manifold. The notion of mixed Frobenius manifolds was introduced by Konishi and Minabe [19] to understand the local mirror symmetry. Here we shall explain the notions of mixed trTLEP-structures and mixed Frobenius structures, and the statement of the construction theorem. The applications of the construction theorem to the local mirror symmetry will be discussed in §1.4 and §1.5.

1.2.1. Mixed Frobenius structures and mixed trTLEP-structures. Let M be a complex manifold equipped with holomorphic vector fields e and E. Suppose that the tangent bundle Θ_M has an associative commutative product $\circ$, a torsion free flat connection $\boldsymbol{\nabla}$, and a $\boldsymbol{\nabla}$-flat increasing filtration $\mathcal{I} = (\mathcal{I}_k \mid k \in \mathbb{Z})$. If the tuple $(\circ, \boldsymbol{\nabla}, e, E, \mathcal{I})$ and sequence of metrics $g = (g_k \mid k \in \mathbb{Z})$ on $\mathrm{Gr}^{\mathcal{I}}\Theta_M = \bigoplus_{k\in\mathbb{Z}} \mathrm{Gr}^{\mathcal{I}}_k \Theta_M$ satisfies some conditions, we call the tuple $\mathscr{F} := (\circ, \boldsymbol{\nabla}, e, E, \mathcal{I}, g)$ a mixed Frobenius structure (MFS) on M. A complex manifold equipped with a MFS is called mixed Frobenius manifold (Definition 2.16).

Similarly, a mixed trTLEP-structure is a trTLEP-structure with a filtration. Let $\mathcal{H}$ be a holomorphic vector bundle on $\mathbb{P}^1_\lambda \times M$ trivial along $\mathbb{P}^1_\lambda$ and ∇ a meromorphic flat connection on $\mathcal{H}$ as in (1.1). If we are given an increasing filtration $W = (W_k \mid k \in \mathbb{Z})$ of ∇-flat subbundle on $\mathcal{H}$ and pairings $P = (P_k \mid k \in \mathbb{Z})$ on $\mathrm{Gr}^W\mathcal{H}$ such that $(\mathrm{Gr}^W_k(\mathcal{H}), \nabla, P_k)$ is a trTLEP$(-k)$-structure for any k, then we call the tuple $\mathcal{T} := (\mathcal{H}, \nabla, W, P)$ mixed trTLEP-structure on M (Definition 2.14). As we will see in Proposition 2.19, a mixed Frobenius structure $\mathscr{F}$ on M naturally induces a mixed trTLEP-structure $\mathcal{T}(\mathscr{F})$ on M.

Let $f : M_0 \to M_1$ be a holomorphic map between complex manifolds. A mixed trTLEP-structure $\mathcal{T}$ on M_1 naturally induces mixed trTLEP-structure $f^*\mathcal{T}$ on M_0. In particular, if we are given MFS $\mathscr{F}$ on a complex manifold M and closed embedding $\iota : M \hookrightarrow \widetilde{M}$ then we have the mixed trTLEP-structure $\mathcal{T} = \iota^*\mathcal{T}(\mathscr{F})$. This plays the role of the "restricted set of data" in Step 1 in the strategy.

1.2.2. Unfolding of mixed trTLEP-structure and the construction theorem. Let $(M,0)$ be a germ of a complex manifold and $\mathcal{T}$ be a mixed trTLEP-structure on $(M,0)$. An unfolding of $\mathcal{T}$ is a tuple $\big((\widetilde{M},0),\widetilde{\mathcal{T}},\iota,i\big)$ where $\iota:(M,0)\hookrightarrow(\widetilde{M},0)$ is a closed embedding , $\widetilde{\mathcal{T}}$ is a mixed trTLEP-structure on $(\widetilde{M},0)$, and $i:\mathcal{T}\xrightarrow{\sim}\iota^*\widetilde{\mathcal{T}}$ is an isomorphism of mixed trTLEP-structures. We can define the notion of morphisms of unfoldings of $\mathcal{T}$. Hence we get the category of unfoldings of $\mathcal{T}$ denoted by $\mathfrak{Unf}_{\mathcal{T}}$ (Definition 2.21).

If there exists a terminal object in $\mathfrak{Unf}_{\mathcal{T}}$, we call it universal unfolding of $\mathcal{T}$. We show that there is a universal unfolding $\big((\widetilde{M},0),\widetilde{\mathcal{T}},\iota,i\big)$ of $\mathcal{T}$ under some conditions (Theorem 2.27). Moreover, we show that $\widetilde{\mathcal{T}}$ is isomorphic to $\mathcal{T}(\mathscr{F})$ for a MFS $\mathscr{F}$ on $(\widetilde{M},0)$ (Corollary 2.28). Hence we get the first main theorem of this paper as follows.

Theorem 1.1 (Theorem 2.27, Corollary 2.28). *Let $\mathcal{T}$ be a mixed* trTLEP*-structure on a germ of complex manifold $(M,0)$. Assume that $\mathcal{T}$ satisfies "some conditions". Then there exists* (*uniquely up to isomorphisms*) *a mixed Frobenius structure $\mathscr{F}$ on a germ of a complex manifold $(\widetilde{M},0)$ such that the induced mixed* trTLEP*-structure $\mathcal{T}(\mathscr{F})$ gives a universal unfolding of $\mathcal{T}$.*

This is a generalization of Theorem 4.5 in Hertling-Manin [14]. "Some conditions" in this theorem is explained in Definition 2.8 and Definition 2.23. We also give the definition of the equivalent condition for a special case in §1.3.

1.3. Mixed Frobenius manifolds and variations of mixed Hodge structure

Consider a graded polarizable variation of mixed Hodge structure (VMHS) $\mathscr{H}:=(\mathbb{V}_{\mathbb{Q}},F,W)$ on a germ of a complex manifold $(M,0)$. Here, $\mathbb{V}_{\mathbb{Q}}$ is a local system of $\mathbb{Q}$-vector space, $F=(F^\ell\mid\ell\in\mathbb{Z})$ is a Hodge filtration on $K:=\mathbb{V}_{\mathbb{Q}}\otimes\mathcal{O}_{M,0}$, and $W=(W_k\mid k\in\mathbb{Z})$ is a weight filtration on $\mathbb{V}_{\mathbb{Q}}$. If we fix a graded polarization $S=(S_k\mid k\in\mathbb{Z})$ on $\mathrm{Gr}^W V_{\mathbb{Q}}$ and an opposite filtration $U=(U_\ell\mid\ell\in\mathbb{Z})$ (see Definition 3.1 for the definition of opposite filtrations), then we have a mixed trTLEP-structure $\mathcal{T}(\mathscr{H},S,U)$ by Rees construction (Lemma 3.3).

For the mixed trTLEP-structure $\mathcal{T}=\mathcal{T}(\mathscr{H},U,S)$, "some conditions" in Theorem 1.1 can be reformulated as a condition for $\mathscr{H}$. The condition is called H^2-generation condition([14]), motivated by quantum cohomology.

Let ∇ be the flat connection on K induced by $\mathscr{H}$. By Griffiths transversality, we have a Higgs field $\theta := \mathrm{Gr}_F(\nabla) : \mathrm{Gr}_F K \to \mathrm{Gr}_F K \otimes \Omega^1_{M,0}$. Put $w := \max\{\ell \in \mathbb{Z} \mid \mathrm{Gr}_F^\ell K \neq 0\}$. Note that $\mathrm{Gr}_F^w K = F^w$. The H^2-generation condition for $\mathscr{H}$ is the following.

(i) The rank of F^w is 1, and the rank of $\mathrm{Gr}_F^{w-1} K$ is equal to the dimension of $(M, 0)$,

(ii) The map $\mathrm{Sym}\, \Theta_{M,0} \otimes F^w \to \mathrm{Gr}_F K$ induced by θ is surjective.

Let ζ_0 be a non-zero vector in $F^w|_0$. Then the condition (ii) is equivalent to the condition that $\mathrm{Gr}_F K|_0$ is generated by ζ_0 over $\mathrm{Sym}\, \Theta_{M,0}|_0$ and conditions (i) and (ii) imply $\Theta_{M,0} \simeq \mathrm{Gr}_F^{w-1} K$.

The following theorem is an application of Theorem 1.1 in the case of $\mathcal{T} = \mathcal{T}(\mathscr{H}, U, S)$.

Theorem 1.2 (Corollary 3.6). *Let $\mathscr{H} = (\mathbb{V}_\mathbb{Q}, F, W)$ be a VMHS on a germ of a complex manifold $(M, 0)$ with the H^2-generation condition. Take an integer w as above and a non-zero vector $\zeta_0 \in F^w\,|_0$. Fix a graded polarization S and an opposite filtration U on $\mathrm{Gr}^W V_\mathbb{Q}$. Then there exists (uniquely up to isomorphisms) a tuple $\big((\widetilde{M}, 0), \mathscr{F}, \iota, i\big)$ with the following conditions.*

1. *$\mathscr{F} = (\circ, \boldsymbol{\nabla}, e, E, \mathcal{I}, g)$ is a MFS on a germ of a complex manifold $(\widetilde{M}, 0)$.*
2. *$\iota : (M, 0) \hookrightarrow (\widetilde{M}, 0)$ is a closed embedding.*
3. *$i : \mathcal{T}(\mathscr{H}, U, S) \xrightarrow{\sim} \mathcal{T}(\mathscr{F})$ is an isomorphism of mixed* trTLEP*-structure with $i|_{(0,0)}(\zeta_0) = e|_0$.*

1.4. Mixed Frobenius manifolds in local B models

We shall explain some applications of construction theorem to the local mirror symmetry. Konishi and Minabe introduced the notion of mixed Frobenius manifolds in [19, 20] to understand the local mirror symmetry. In [19], they constructed mixed Frobenius manifolds for weak Fano toric surface. It remains to construct mixed Frobenius manifolds for local B-models.

Mixed Frobenius manifolds for local B-models are expected to be constructed from variations of mixed Hodge structure ([18]). Using the results of Batyrev [2] and Stienstra [31], Konishi and Minabe [18] gave a combinatorial description for the VMHS in the local B-models.

Let $\Delta \subset \mathbb{Z}^2$ be a two dimensional reflexive polyhedron. We have the moduli space $\mathcal{M}(\Delta)$ of affine hypersurfaces in $(\mathbb{C}^*)^2$ (Definition 3.17). Let $V_f \subset (\mathbb{C}^*)^2$ be a hypersurface corresponds to $[f] \in \mathcal{M}(\Delta)$. Fix a stable smooth

point $[f_0]$ in $\mathcal{M}(\Delta)$. Then the mixed Hodge structure on the relative cohomology $H^2((\mathbb{C}^*)^2, V_f)$, $(f \in \mathcal{M}(\Delta))$ defines a VMHS $\mathscr{H}_\Delta$ on the germ of complex manifold $(\mathcal{M}(\Delta), [f_0])$. Using the results of [18] and [2], we give a sufficient condition for $\mathscr{H}_\Delta$ to satisfy the H^2-generation condition in terms of the toric data. As a consequence, we have the following theorem.

Theorem 1.3 (Corollary 3.26)**.** *Fix a graded polarization S and an opposite filtration U for $\mathscr{H}_\Delta$, and take a non-zero vector $\zeta_0 \in F^2([f_0])$. Then, there exists a tuple $((\widetilde{M}, 0), \mathscr{F}, \iota, i)$ with the following conditions uniquely up to isomorphisms.*

1. *$\mathscr{F} = (\theta, \nabla, e, E, W, g)$ is a mixed Frobenius structure on $(\widetilde{M}, 0)$.*
2. *$\iota : (\mathcal{M}(\Delta), [f_0]) \hookrightarrow (\widetilde{M}, 0)$ is an embedding.*
3. *$i : \mathcal{T}(\mathscr{H}_\Delta, U, S) \xrightarrow{\sim} \iota^*\mathcal{T}(\mathscr{F})$ is an isomorphism of* trTLEP*-structure with $i|_{(0,[f_0])}(\zeta_0) = e|_0$.*

This theorem gives the mixed Frobenius manifolds associated to local B-models.

1.5. Limit mixed trTLEP-structure and local A-models

We shall give a method to construct mixed trTLEP-structures from logarithmic trTLEP-structures. Let $(M, 0)$ be a germ of complex manifold and $(Z, 0) \subset (M, 0)$ a co-dimension 1 submanifold. If we are given a $\log Z$-trTLEP(0)-structure $\mathcal{T}$ (the definition of $\log Z$-trTLEP-structure is given in Definition 4.2 or [23, Definition 1.8]) satisfies some conditions, we have a mixed trTLEP-structure $\mathcal{T}_{Z,0}$ on $(Z, 0)$, which is called a limit mixed trTLEP-structure (See Definition 4.7).

Let X be a weak Fano toric manifold. Let r be the dimension of $H^2(X, \mathbb{C})$. For an apporopiate open embedding of $H^2(X, \mathbb{C})/2\pi\sqrt{-1}H^2(X, \mathbb{Z})$ to $\mathbb{C}^r$, we have the logarithmic trTLEP-structure $\mathcal{T}_X^{\mathrm{small}}$ on a neighborhood V of $0 \in \mathbb{C}^r$, which is called the small quantum D-module ([24]).

Let S be a weak Fano toric surface and X be the projective compactification of the canonical bundle of S. There is a divisor Z of V which is canonically identified with a locally closed subset of the quotient space $H^2(S, \mathbb{C})/2\pi\sqrt{-1}H^2(S, \mathbb{Z})$. For each $z \in Z$, the logarithmic trTLEP-structure $\mathcal{T}_X^{\mathrm{small}}$ on (V, z) induces a limit mixed trTLEP-structure $(\mathcal{T})_{Z,z}$ on (Z, z). Moreover, if z is close to $0 \in \mathbb{C}^r$ enough, then $(\mathcal{T})_{Z,z}$ induces the mixed Frobenius manifold constructed by Konishi-Minabe [19].

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2. Construction theorem for mixed Frobenius manifolds

The aim of this chapter is to prove the construction theorem for mixed Frobenius manifolds (Corollary 2.28), which is a generalization of [14, Theorem 4.5].

The symbol $\mathbb{P}^1_\lambda$ denotes a projective line with non-homogeneous parameter λ. We identify a holomorphic vector bundle with the associated locally free sheaf. For a holomorphic vector bundle K, we write $s \in K$ to mean that s is a local section of K. We denote the dual vector bundle of K by $K^\vee$. When we consider filtrations, we always assume that the filtrations are exhaustive. Hence we always omit "exhaustive".

2.1. Mixed trTLEP-structures and mixed Frobenius manifolds

We define the notion of mixed trTLEP-structures and mixed Frobenius manifolds. We show that a mixed Frobenius manifold always induces a mixed trTLEP-structures. We also show that a mixed trTLEP-structure induces a mixed Frobenius manifold under certain conditions.

2.1.1. trTLE-structures and Saito structures. Recall the definition of trTLE-structures. Let M be a complex manifold and $p_\lambda : \mathbb{P}^1_\lambda \times M \to M$ a natural projection.

Definition 2.1 ([13, 14]**).** A pair $(\mathcal{H}, \nabla)$ is called a **trTLE-structure** on M if the following properties are satisfied:

1. $\mathcal{H}$ is a holomorphic vector bundle on $\mathbb{P}^1_\lambda \times M$ such that the adjoint morphism $p_\lambda^* p_{\lambda *}\mathcal{H} \to \mathcal{H}$ is an isomorphism,
2. ∇ is a meromorphic flat connection on $\mathcal{H}$ with pole order 1 along $\{0\} \times M$ and logarithmic pole along $(\{\infty\} \times M)$:

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\mathbb{P}^1_\lambda \times M}\big(\log(\{0, \infty\} \times M)\big) \otimes \mathcal{O}_{\mathbb{P}^1_\lambda \times M}(\{0\} \times M).$$

A morphism of trTLE-structures is a flat morphism of holomorphic vector bundles.

Remark 2.2. For a trTLE-structure $(\mathcal{H}, \nabla)$ and a complex number c, the pair $(\mathcal{H}, \nabla + c \cdot \mathrm{id}_{\mathcal{H}}\ \lambda^{-1} \mathrm{d}\lambda)$ is also a trTLE-structure.

We recall the definition of Saito structure (without a metric) in [25]. Let M be a complex manifold and $p_\lambda : \mathbb{P}^1_\lambda \times M \to M$ a natural projection. Suppose that its tangent bundle Θ_M is equipped with a symmetric Higgs field θ, a torsion free flat connection $\boldsymbol{\nabla}$, and two global sections e and E. We have endomorphisms $\boldsymbol{\nabla}_\bullet E$ and θ_E defined by $a \mapsto \boldsymbol{\nabla}_a E$ and $a \mapsto \theta_E(a)$ for $a \in \Theta_M$.

Definition 2.3 ([25, Definition VII. 1.1]). The tuple $\mathcal{S} := (\theta, \boldsymbol{\nabla}, e, E)$ is called a **Saito structure** on M if the following conditions are satisfied.

1. The vector field e is $\boldsymbol{\nabla}$-flat and $\theta_e a = -a$ for all $a \in \Theta_M$.
2. The following meromorphic connection $\hat{\nabla}$ on $p_\lambda^* \Theta_M$ is flat:

$$\hat{\nabla} := p_\lambda^* \boldsymbol{\nabla} + \frac{1}{\lambda} p_\lambda^* \theta - \left(\frac{1}{\lambda} p_\lambda^* \theta_E + p_\lambda^* \boldsymbol{\nabla}_\bullet E \right) \frac{\mathrm{d}\lambda}{\lambda}. \tag{2.1}$$

The vector field e is called the **unit vector field** and E is called the **Euler vector field**.

Remark 2.4.
1. The tangent bundle Θ_M has the structure of a commutative associative $\mathcal{O}_M$-algebra defined by $a \circ b := -\theta_a b$ $(a, b \in \Theta_M)$. The unit vector field e is the global unit section of this algebra.
2. The flatness of $\hat{\nabla}$ is equivalent to the condition that the equations $\boldsymbol{\nabla}(\circ) = 0$, $\boldsymbol{\nabla}(\boldsymbol{\nabla}_\bullet E) = 0$, and $\mathrm{Lie}_E(\circ) = \circ$ hold.

By definition, a Saito structure always induces a trTLE-structure.

Lemma 2.5. *Let $\mathcal{S} := (\theta, \boldsymbol{\nabla}, e, E)$ be a Saito Structure on a complex manifold M. Then the pair $(p_\lambda^* \Theta_M, \hat{\nabla})$ is a* trTLE*-structure on M.* □

We recall the definition of Frobenius type structure.

Definition 2.6 ([13, Definition 5.6]). Let K be a holomorphic vector bundle over a complex manifold M. A **Frobenius type structure** on K consists of a flat connection ∇^{r} on K, a Higgs field $\mathcal{C}$ on K, and endomorphisms $\mathcal{U}, \mathcal{V} \in \mathrm{End}(K)$ such that

$$\nabla^{\mathrm{r}}(\mathcal{C}) = \nabla^{\mathrm{r}}(\mathcal{V}) = [\mathcal{C}, \mathcal{U}] = 0, \tag{2.2}$$

$$\nabla^{\mathrm{r}}(\mathcal{U}) - [\mathcal{C}, \mathcal{V}] + \mathcal{C} = 0. \tag{2.3}$$

We remark that this definition of Frobenius type structure lacks the pairing comparing with [13, Definition 5.6]. There is a correspondence between a trTLE-structure and a Frobenius type structure as follows.

Lemma 2.7 ([13, Theorem 5.7]). *Let $(\mathcal{H},\nabla)$ be a trTLE-structure on a complex manifold M. There exists a unique Frobenius type structure $(\nabla^{\mathrm{r}},\mathcal{C},\mathcal{U},\mathcal{V})$ on $\mathcal{H}|_{\lambda=0}$ such that*

$$\nabla = p_\lambda^*\nabla^{\mathrm{r}} + \frac{1}{\lambda}p_\lambda^*\mathcal{C} + \left(\frac{1}{\lambda}p_\lambda^*\mathcal{U} - p_\lambda^*\mathcal{V}\right)\frac{\mathrm{d}\lambda}{\lambda} \tag{2.4}$$

via the natural isomorphism $\mathcal{H} \simeq p_\lambda^*(\mathcal{H}|_{\lambda=0})$. We call it the Frobenius type structure associated to $(\mathcal{H},\nabla)$.

Proof. Let a be a local section of Θ_M. Extend the section a constantly along $\mathbb{P}^1_\lambda$ and denote it by $\widetilde{a}$. Similarly, take a local section s of $\mathcal{H}|_{\lambda=0}$ and extend it to the local section $\widetilde{s}$ of $\mathcal{H}$. Define $\mathcal{C}_a s$ as the restriction of $\lambda\nabla_{\widetilde{a}}\widetilde{s}$ to $\lambda=0$. Define $\mathcal{U}s$ as the restriction of $\lambda\nabla_{\lambda\partial_\lambda}\widetilde{s}$ to $\{\lambda=0\}$. Since the flat section ∇ is pole order 1 along $\{\lambda=0\}$, the morphism $(a,s)\mapsto\mathcal{C}_a s$ defines a Higgs field on $\mathcal{H}|_{\lambda=0}$ and $s\mapsto\mathcal{U}s$ defines an endomorphism on $\mathcal{H}|_{\lambda=0}$.

Since ∇ is regular singular along $\{\lambda=\infty\}$, we have the residual connection ∇^{res} and the residue endomorphism $\mathrm{Res}_{\lambda=\infty}\nabla$ on $\mathcal{H}|_{\lambda=\infty}$. By the condition 1 in Definition 2.1, we have a natural isomorphism $\mathcal{H}|_{\lambda=\infty}\simeq\mathcal{H}|_{\lambda=0}$. Using the isomorphism, regard ∇^{res} as the connection on $\mathcal{H}|_{\lambda=0}$ and denote it by ∇^{r}. Similarly, regard $\mathrm{Res}_{\lambda=\infty}\nabla$ as a endomorphism on $\mathcal{H}|_{\lambda=0}$ and denote it by $\mathcal{V}$. One can check the equation (2.4). The flatness of ∇ implies (2.2) and (2.3). The uniqueness is trivial by construction. □

Definition 2.8 ([14]). Let $(\mathcal{H},\nabla)$ be a trTLE-structure on M. Let $(\nabla^{\mathrm{r}},\mathcal{C},\mathcal{U},\mathcal{V})$ be the Frobenius type structure associated to $(\mathcal{H},\nabla)$. Assume that there is a ∇^{r}-flat global section ζ of $\mathcal{H}|_{\lambda=0}$.

- The section ζ is said to satisfy the injectivity condition (resp. the identity condition) when the induced morphism

$$\mathcal{C}_\bullet\zeta : \Theta_M \to \mathcal{H}|_{\lambda=0} \tag{2.5}$$

defined by $a\mapsto\mathcal{C}_a\zeta$ $(a\in\Theta_M)$ is an injective morphism (resp. an isomorphism).

- Take a complex number d. The section ζ is said to satisfy the eigenvalue condition for d (with respect to $(\mathcal{H}, \nabla)$) if the following equation holds.

$$\mathcal{V}\zeta = \frac{d}{2}\zeta. \tag{2.6}$$

We denote by (IC), (IdC), and $(\mathrm{EC})_d$ the injectively condition, the identity condition and the eigenvalue condition for d respectively.

Remark 2.9. Let $(\mathcal{H}, \nabla)$, $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$, and ζ be as in Definition 2.8. Fix complex numbers c and d. The Frobenius type structure associated to $(\mathcal{H}, \nabla - c \cdot \mathrm{id}_{\mathcal{H}} \lambda^{-1} \mathrm{d}\lambda)$ is $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V} + c \cdot \mathrm{id})$. If ζ satisfies $(\mathrm{EC})_d$ with respect to $(\mathcal{H}, \nabla)$, then it satisfies $(\mathrm{EC})_{d+2c}$ with respect to $(\mathcal{H}, \nabla - c \cdot \mathrm{id}\lambda^{-1}\mathrm{d}\lambda)$.

Let $\mathcal{S} = (\theta, \boldsymbol{\nabla}, E, e)$ be a Saito structure on a complex manifold M. By Lemma 2.5, we have the trTLE-structure $(p_\lambda^* \Theta_M, \hat{\nabla})$. Comparing the equations (2.1) and (2.4), we can check that the Frobenius type structure associated to $(p_\lambda^* \Theta_M, \hat{\nabla})$ is $(\boldsymbol{\nabla}, \theta, \theta_E, \boldsymbol{\nabla}_\bullet E)$.

Lemma 2.10. *The unit vector field e satisfies* (IC) *and* $(\mathrm{EC})_2$.

Proof. The condition 1 in Definition 2.3 implies that the unit vector field e satisfies (IC). Since $\boldsymbol{\nabla}$ is torsion free and e is $\boldsymbol{\nabla}$-flat, $\boldsymbol{\nabla}_e E = \boldsymbol{\nabla}_E e - [E, e] = -[E, e]$. By Remark 2.4,

$$[E, e \circ e] - [E, e] \circ e - e \circ [E, e] = e \circ e.$$

This implies $-[E, e] = e$. Hence e also satisfies $(\mathrm{EC})_2$. □

Let $(\mathcal{H}, \nabla)$ be a trTLE-structure on M. Let $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$ be the Frobenius type structure associated to $(\mathcal{H}, \nabla)$. Assume that we have a global ∇^{r}-flat section ζ of $\mathcal{H}|_{\lambda=0}$ with (IdC) and $(\mathrm{EC})_2$. Put $\mu := -\mathcal{C}_\bullet \zeta : \Theta_M \xrightarrow{\sim} \mathcal{H}|_{\lambda=0}$. Using this isomorphism, regard ∇ (resp. $\mathcal{C}$) as a flat section (resp. Higgs field) on Θ_M and denote it by the same letter. Put $e := \mu^{-1}(\zeta)$ and $E := \mu^{-1}(\mathcal{U}\zeta)$. The following proposition is essentially proved in [14] and [23].

Proposition 2.11.
- *The tuple $\mathcal{S}_{\mathcal{H},\zeta} := (\mathcal{C}, \nabla^{\mathrm{r}}, e, E)$ is a Saito structure on M.*

- *$\mathcal{S}_{\mathcal{H},\zeta}$ is the unique Saito structure on M such that $\mu(e)=\zeta$ and the morphism*

$$p_\lambda^*(\mu): p_\lambda^*\Theta_M \xrightarrow{\sim} p_\lambda^*(\mathcal{H}|_{\lambda=0}) \simeq \mathcal{H}$$

gives an isomorphism of trTLE*-structures between $(p_\lambda^*\Theta_M, \hat{\nabla})$ and $(\mathcal{H},\nabla)$.* □

We conclude this subsection with the following corollary.

Corollary 2.12. *Fix a complex number d. Put $c:=(2-d)/2$. Let M be a complex manifold and $p_\lambda : \mathbb{P}^1_\lambda \times M \to M$ a natural projection.*

1. *If $\mathcal{S}$ is a Saito structure on M, then $\mathcal{H}_{\mathcal{S},d} := (p_\lambda^*\Theta_M, \hat{\nabla} + c\cdot \mathrm{id}\lambda^{-1}d\lambda)$ is a* trTLE*-structure such that the unit vector field satisfies* (IdC) *and* $(\mathrm{EC})_d$.
2. *Let $(\mathcal{H},\nabla)$ be a trTLE-structure and $(\nabla^{\mathrm{r}},\mathcal{C},\mathcal{U},\mathcal{V})$ the associated Frobenius type structure. Let ζ be a ∇^{r}-flat global section of $\mathcal{H}|_{\lambda=0}$ satisfying* (IdC) *and* $(\mathrm{EC})_d$. *Then there is a unique Saito structure $\mathcal{S}$ on M such that the unit vector e satisfies $-\mathcal{C}_e\zeta=\zeta$ and the morphism*

$$-p_\lambda^*(\mathcal{C}_\bullet\zeta): p_\lambda^*\Theta_M \xrightarrow{\sim} p_\lambda^*(\mathcal{H}|_{\lambda=0}) \simeq \mathcal{H}$$

gives an isomorphism of trTLE*-structures between $\mathcal{H}_{\mathcal{S},d}$ and $(\mathcal{H},\nabla)$.*

Proof. The first assertion is easily checked by using Lemma 2.10 and Remark 2.9. Let $(\mathcal{H},\nabla)$ and ζ be the same as in the second assertion. By Remark 2.9, ζ satisfies (IdC) and $(\mathrm{EC})_2$ with respect to $(\mathcal{H}, \nabla - c\cdot \mathrm{id}_{\mathcal{H}}\lambda^{-1}\mathrm{d}\lambda)$. Hence by Proposition 2.11, there exists a unique Saito structure $\mathcal{S}$ such that $-\mathcal{C}_e\zeta=\zeta$ and $-p_\lambda^*(\mathcal{C}_\bullet\zeta)$ gives an isomorphism of trTLE-structures between $(p_\lambda^*\Theta_M,\hat{\nabla})$ and $(\mathcal{H},\nabla - c\cdot\mathrm{id}_{\mathcal{H}}\lambda^{-1}\mathrm{d}\lambda)$. □

2.1.2. Weight filtrations, graded pairings, and Frobenius filtrations. Let M be a complex manifold. Let $j_\lambda : \mathbb{P}^1_\lambda \times M \to \mathbb{P}^1_\lambda \times M$ be the morphism defined by $j_\lambda(\lambda,t)=(-\lambda,t)$ where λ is the non-homogeneous coordinate on $\mathbb{P}^1_\lambda$ and t is a point in M. For two holomorphic vector bundles $\mathcal{E}$ and $\mathcal{F}$, we denote by σ the natural isomorphism $\mathcal{E}\otimes\mathcal{F}\to\mathcal{F}\otimes\mathcal{E}$ given by

$$e\otimes f \mapsto f\otimes e \quad (e\in\mathcal{E},\ f\in\mathcal{F}).$$

For an integer k, we denote the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_\lambda\times M}(-k\{0\}\times M + k\{\infty\}\times M)$ by $\lambda^k\mathcal{O}_{\mathbb{P}^1_\lambda\times M}$. Let $\mathcal{H}$ be a holomorphic vector bundle on $\mathbb{P}^1_\lambda\times M$. Let

$P : \mathcal{H} \otimes j_\lambda^* \mathcal{H} \to \lambda^k \mathcal{O}_{\mathbb{P}^1_\lambda \times M}$ be a morphism of $\mathcal{O}_{\mathbb{P}^1_\lambda \times M}$-modules. The morphism P is called $(-1)^k$-symmetric if $j_\lambda^* P = (-1)^k P \circ \sigma$, and it is called non-degenerate if the morphism $\mathcal{H} \to (j_\lambda^* \mathcal{H})^\vee$ induced by $\lambda^{-k} P$ is an isomorphism.

Recall the definition of trTLEP(k)-structures.

Definition 2.13 ([13]**).** Let $(\mathcal{H}, \nabla)$ be a trTLE-structure on M and fix an integer k. If a morphism of $\mathcal{O}_{\mathbb{P}^1_\lambda \times M}$-modules $P : \mathcal{H} \otimes j_\lambda^* \mathcal{H} \to \lambda^k \mathcal{O}_{\mathbb{P}^1_\lambda \times M}$ is ∇-flat, $(-1)^k$-symmetric, and non-degenerate, then we call the triple $(\mathcal{H}, \nabla, P)$ a **trTLEP(k)-structure**. We also call the morphism P a **pairing** of the trTLEP(k)-structure.

We introduce the notions of filtered trTLEP-structures and mixed trTLEP-structures.

Definition 2.14. Let $(\mathcal{H}, \nabla)$ be a trTLE-structure on M.

1. An increasing filtration $W = (W_k \mid k \in \mathbb{Z})$ of ∇-flat subbundles in $\mathcal{H}$ is called a **weight filtration** of $(\mathcal{H}, \nabla)$ if the subquotient $\mathrm{Gr}_k^W \mathcal{H} := W_k / W_{k-1}$ is a trTLE-structure for every integer k. We call the triple $\mathcal{T}_{\mathrm{filt}} := (\mathcal{H}, \nabla, W)$ a **filtered trTLE-structure** if $(\mathcal{H}, \nabla)$ is a trTLE-structure and W is a weight filtration of it.
2. Let W be a weight filtration of $(\mathcal{H}, \nabla)$. A sequence of morphisms

$$P := (P_k : \mathrm{Gr}_k^W \mathcal{H} \otimes j_\lambda^* \mathrm{Gr}_k^W \mathcal{H} \to \lambda^{-k} \mathcal{O}_{\mathbb{P}^1_\lambda \times M} \mid k \in \mathbb{Z})$$

   is called a **graded pairing** on the filtered trTLE-structure $\mathcal{T}_{\mathrm{filt}} = (\mathcal{H}, \nabla, W)$ if the triple $(\mathrm{Gr}_k^W(\mathcal{H}), \nabla, P_k)$ is a trTLEP($-k$)-structure for every integer k. We call the pair $\mathcal{T} := (\mathcal{T}_{\mathrm{filt}}, P)$ a **mixed trTLEP-structure** if $\mathcal{T}_{\mathrm{filt}}$ is a filtered trTLE-structure and P is a graded pairing on it.

Isomorphisms of these structures are the isomorphisms of underlying trTLE-structures which preserves the weight filtrations and graded Pairings.

Remark 2.15.

- We can define pull-backs for mixed trTLEP-structures. Let $f : M_0 \to M_1$ be a holomorphic map and $\mathcal{T} = (\mathcal{H}, (W_k)_k, (P_k)_k)$ a mixed trTLEP-structure on M_1. Then $f^*\mathcal{T} := \big((\mathrm{id}_{\mathbb{P}^1_\lambda} \times f)^* \mathcal{H}, \{(\mathrm{id}_{\mathbb{P}^1_\lambda} \times f)^* W_k\}_k, \{(\mathrm{id}_{\mathbb{P}^1_\lambda} \times f)^* P_k\}_k\big)$ is mixed trTLEP-structure on M_0.

- Let $\mathcal{T} = (\mathcal{H}, \nabla, W, P)$ a mixed trTLEP-structure and ℓ a half-integer. Then if we put $W(\ell)_k := W_{k+2\ell}$ and $P(\ell)_k := \lambda^{2\ell} P_{k+2\ell}$, the tuple $\big(\mathcal{H}, \nabla - \ell \mathrm{id}_{\mathcal{H}} \lambda^{-1} \mathrm{d}\lambda, W(\ell), P(\ell)\big)$ is a mixed trTLEP structure. We denote it by $\mathcal{T}(\ell)$ and call it the Tate twist of $\mathcal{T}$ by ℓ.
- A trTLEP(k)-structure $\mathcal{T}$ can be regarded as a mixed trTLEP-structure by the canonical way. The Tate twist $\mathcal{T}(\ell)$ gives a trTLEP($k + 2\ell$)-structure for every half-integer ℓ.

We recall the definition of mixed Frobenius structure introduced in [19, 20]. Let $\mathcal{S} = (\theta, \boldsymbol{\nabla}, e, E)$ be a Saito structure on a complex manifold M. A subbundle $\mathcal{J} \subset \Theta_M$ is called θ-invariant (resp.$\boldsymbol{\nabla}$-flat) if $\theta_x y \in \mathcal{J}$ (resp.$\boldsymbol{\nabla}_x y \in \mathcal{J}$) for all $x \in \Theta_M, y \in \mathcal{J}$. A subbundle $\mathcal{J} \subset \Theta_M$ is called E-closed if $\mathrm{Lie}_E(y) = [E, y] \in \mathcal{J}$ for all $y \in \mathcal{J}$. If $\mathcal{J}$ is $\boldsymbol{\nabla}$-flat, the E-closedness of $\mathcal{J}$ is equivalent to the condition that $\boldsymbol{\nabla}_y E \in \mathcal{J}$ for all $y \in \mathcal{J}$.

Definition 2.16 ([19, Definition 6.2],[20, Definition 4.5]). Fix a complex number d. Let $\mathcal{I} = (\mathcal{I}_k \subset \Theta_M \mid k \in \mathbb{Z})$ be an increasing filtration of θ-invariant, $\boldsymbol{\nabla}$-flat, and E-closed subbundles on Θ_M. Let $g = (g_k : \mathrm{Gr}_k^{\mathcal{I}} \Theta_M \otimes \mathrm{Gr}_k^{\mathcal{I}} \Theta_M \to \mathcal{O}_M \mid k \in \mathbb{Z})$ be a sequence of $\boldsymbol{\nabla}$-flat symmetric morphisms. The pair $(\mathcal{I}, g)$ is called a **Frobenius filtration** on the Saito structure $\mathcal{S}$ of charge d if they satisfy the following equations for all integer k:

$$g_k(\theta_x y, z) = g_k(y, \theta_x z) \quad (x \in \Theta_M, y, z \in \mathrm{Gr}_k^{\mathcal{I}} \Theta_M) \tag{2.7}$$

$$\mathrm{Lie}_E(g_k) = (2 - d + k) g_k. \tag{2.8}$$

A triple $\mathscr{F} := (\mathcal{S}, \mathcal{I}, g)$ is called a **mixed Frobenius structure** (**MFS**) on a complex manifold M if $\mathcal{S}$ is a Saito structure on M and $(\mathcal{I}, g)$ is a Frobenius filtration on $\mathcal{S}$. A complex manifold equipped with a MFS is called **mixed Frobenius manifold**.

Remark 2.17.
- This definition of Frobenius filtrations (and MFS) is slightly different from the one in [19, 20]. A MFS $(\mathcal{S}, \mathcal{I}, g)$ gives a Frobenius structure if $\mathcal{I}_{-1} = 0$ and $\mathcal{I}_0 = \Theta_M$.
- Let $\mathscr{F} = (\mathcal{S}, \mathcal{I}, g)$ be a MFS of charge d and ℓ a half-integer. If we put $\mathcal{I}(\ell)_k := \mathcal{I}_{k+2\ell}$ and $g(\ell)_k := g_{k+2\ell}$, then $\mathscr{F}(\ell) := (\mathcal{S}, \mathcal{I}(\ell), g(\ell))$ is a MFS of charge $d - 2\ell$.

Let $\mathscr{F} = (\mathcal{S}, \mathcal{I}, g)$ be a MFS on a complex manifold M of charge d. The underlying Saito structure $\mathcal{S}$ induces a trTLE-structure $\mathcal{H}_{\mathcal{S},d}$ (Corollary 2.12). Put $W_{\mathcal{I},k} := p_\lambda^* \mathcal{I}_k$ and $W_{\mathcal{I}} := (W_{\mathcal{I},k} \mid k \in \mathbb{Z})$.

Lemma 2.18. *$W_{\mathcal{I}}$ is a weight filtration on $\mathcal{H}_{\mathcal{S},d}$.*

Proof. Recall that $\mathcal{H}_{\mathcal{S},d} = (p_\lambda^*\Theta_M, \hat{\nabla} + c\lambda^{-1}\mathrm{d}\lambda)$ where $\hat{\nabla}$ is given in (2.1) and $c = (2-d)/2$. Since each $\mathcal{I}_k$ is θ-invariant, ∇-flat, and E-closed, $W_{\mathcal{I},k} = p_\lambda^*\mathcal{I}_k$ is $(\hat{\nabla} + c \cdot \mathrm{id}_{p_\lambda^*\Theta_M}\lambda^{-1}\mathrm{d}\lambda)$-flat. The natural isomorphism $\mathrm{Gr}_k^{W_{\mathcal{I}}}(p_\lambda^*\Theta_M) \simeq p_\lambda^*\mathrm{Gr}_k^{\mathcal{I}}\Theta_M$ shows that $W_{\mathcal{I}}$ is a weight filtration. □

We denote $\mathrm{Gr}_k^{W_{\mathcal{I}}}(p_\lambda^*\Theta_M)$ by $\mathrm{Gr}_k^{W_{\mathcal{I}}}(\mathcal{H}_{\mathcal{S},d})$. For each integer k, let $P_{g,k}$ be a morphism given by the following composition:

$$\begin{aligned}\mathrm{Gr}_k^{W_{\mathcal{I}}}(\mathcal{H}_{\mathcal{S},d}) \otimes j_\lambda^*\mathrm{Gr}_k^{W_{\mathcal{I}}}(\mathcal{H}_{\mathcal{S},d}) &\xrightarrow{id\otimes j^*} \mathrm{Gr}_k^{W_{\mathcal{I}}}(\mathcal{H}_{\mathcal{S},d}) \otimes \mathrm{Gr}_k^{W_{\mathcal{I}}}(\mathcal{H}_{\mathcal{S},d}) \\ &\xrightarrow{\lambda^{-k}p_\lambda^*(g_k)} \lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times M}.\end{aligned}$$

Proposition 2.19. *Put $P_g := (P_{g,k} \mid k \in \mathbb{Z})$. Then $\mathcal{T}(\mathscr{F}) := ((\mathcal{H}_{\mathcal{S},d}, W_{\mathcal{I}}), P_g)$ is a mixed trTLEP-structure.*

Proof. We need to show that $(\mathrm{Gr}_k^{W_{\mathcal{I}}}(\mathcal{H}_{\mathcal{S},d}), P_{g,k})$ is a trTLEP$(-k)$-structure for each integer k. Since g_k is symmetric and non-degenerate (and by construction), $P_{g,k}$ is $(-1)^{-k}$-symmetric and non-degenerate. Hence it remains to show that $P_{g,k}$ is $(\hat{\nabla} + c \cdot \mathrm{id}_{p_\lambda^*\Theta_M}\lambda^{-1}\mathrm{d}\lambda)$-flat. This follows from the equations (2.7) and (2.8), the fact that g_k is ∇-flat, and some easy calculations. □

The following proposition describes a correspondence between mixed Frobenius structures and mixed trTLEP-structures.

Proposition 2.20. *Let $\mathcal{T} = (\mathcal{H}, \nabla, W, P)$ be a mixed trTLEP-structure on a complex manifold M. Let $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$ be the Frobenius type structure associated to the underlying trTLE-structure $(\mathcal{H}, \nabla)$. Assume that there is a ∇^{r}-flat global section ζ of $\mathcal{H}|_{\lambda=0}$ satisfying* (IdC) *and* $(\mathrm{EC})_d$ *for a complex number d. Then there exists a unique MFS $\mathscr{F}$ on M of charge d such that the unit vector field e satisfies $-\mathcal{C}_e\zeta = \zeta$ and the morphism*

$$-p_\lambda^*(\mathcal{C}_\bullet\zeta) : p_\lambda^*\Theta_M \xrightarrow{\sim} p_\lambda^*(\mathcal{H}|_{\lambda=0}) \simeq \mathcal{H} \tag{2.9}$$

gives an isomorphism of mixed trTLEP*-structures between $\mathcal{T}(\mathscr{F})$ and $\mathcal{T}$.*

Proof. By Corollary (2.12), we have a unique Saito structure $\mathcal{S}$ such that (2.9) gives an isomorphism between $\mathcal{H}_{\mathcal{S},d}$ and $(\mathcal{H}, \nabla)$. Put $\mu := -\mathcal{C}_\bullet\zeta$ and $\mathcal{I}_{W,k} := \mu^{-1}(W_k|_{\lambda=0})$. Since W_k is ∇-flat, $W_k|_{\lambda=0}$ is ∇^{r}-flat, $\mathcal{C}$-invariant, and closed under $\mathcal{U}$ and $\mathcal{V}$. Recall that the flat connection and Higgs field of $\mathcal{S}$

are identified with ∇^{r} and $\mathcal{C}$ via μ. If E is the Euler vector field of $\mathcal{S}$ and $c = (2-d)/2$, the endomorphism $\nabla^{\mathrm{r}}_{\bullet}E$ is identified with $\mathcal{V} + c \cdot \mathrm{id}_{\Theta_M}$ via μ. Hence $\mathcal{I}_{W,k}$ is E-closed. Let $g_{P,k}$ be the restriction of $\lambda^k P$ to $\{\lambda = 0\}$ and put $g_P := (g_{P,k} \mid k \in \mathbb{Z})$. Then via (2.9) and the natural isomorphism $(p_\lambda^*\Theta_M)|_{\lambda=0} \simeq \Theta_M$, $g_{P,k}$ gives a symmetric, ∇^{r}-flat, non-degenerate pairing on $\mathrm{Gr}_k^{\mathcal{I}}\Theta_M$ satisfying (2.7) and (2.8), which we denote by the same letter. As a conclusion, the pair $(\mathcal{I}_W, g_P)$ is a Frobenius filtration on $\mathcal{S}$ and hence $\mathscr{F} := (\mathcal{S}, \mathcal{I}_W, g_P)$ is a MFS on M. It is easy to check that (2.9) gives the isomorphism $\mathcal{T}(\mathscr{F}) \simeq \mathcal{T}$. The uniqueness of such $\mathscr{F}$ is trivial by construction. □

2.2. Unfoldings and the construction theorem

For a complex manifold M and a point $0 \in M$, we denote by $(M, 0)$ the germ of manifold around 0. Let $(\mathcal{H}, \nabla)$ be a trTLE-structure on $(M, 0)$, and $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$ the associated Frobenius type structure on $\mathcal{H}|_{\lambda=0}$(Lemma 2.7). Let ζ be a ∇^{r}-flat section of $\mathcal{H}|_{\lambda=0}$. Then ζ satisfies (IC) (resp. (IdC)) if and only if the map (2.5) is injective (resp. isomorphism) at $0 \in (M, 0)$.

Let ζ_0 be a non-zero vector in $\mathcal{H}|_{(0,0)}$ and take the ∇^{r}-flat extension $\zeta \in \mathcal{H}|_{\lambda=0}$. The vector ζ_0 is said to satisfy (IC) (resp. (IdC)) if ζ satisfies (IC) (resp. (IdC)).

2.2.1. Statements of the unfolding theorem and the construction theorem. We define the category of unfoldings of mixed trTLEP-structures.

Definition 2.21 (cf. [14, Definition 2.3]). Fix a mixed trTLEP-structure $\mathcal{T}$ on $(M, 0)$.

(*a*) An **unfolding** of $\mathcal{T}$ is a mixed trTLEP-structure $\widetilde{\mathcal{T}}$ on a germ $(\widetilde{M}, 0)$ of complex manifold together with a closed embedding $\iota : (M, 0) \to (\widetilde{M}, 0)$ and an isomorphism $i : \mathcal{T} \simeq \iota^*\widetilde{\mathcal{T}}$.

(*b*) Let $\big((\widetilde{M}, 0), \widetilde{\mathcal{T}}, \iota, i\big)$ and $\big((\widetilde{M}', 0), \widetilde{\mathcal{T}}', \iota', i'\big)$ be two unfoldings of $\mathcal{T}$. A **morphism of unfoldings** of $\mathcal{T}$ from $\big((\widetilde{M}, 0), \widetilde{\mathcal{T}}, \iota, i\big)$ to $\big((\widetilde{M}', 0), \widetilde{\mathcal{T}}', \iota', i'\big)$ is a pair (φ, ϕ) such that
- $\varphi : (\widetilde{M}, 0) \to (\widetilde{M}', 0)$ is a holomorphic map with $\varphi \circ \iota = \iota'$, and
- $\phi : \widetilde{\mathcal{T}} \xrightarrow{\sim} \varphi^*\widetilde{\mathcal{T}}'$ is an isomorphism of mixed trTLEP-structure with $i = \iota'^*(\phi) \circ i'$.

We denote the category of unfoldings by $\mathfrak{Unf}_{\mathcal{T}}$. A terminal object of $\mathfrak{Unf}_{\mathcal{T}}$ is called a universal unfolding of $\mathcal{T}$ if it exists.

Remark 2.22. Let $\big((\widetilde{M},0),\widetilde{\mathcal{T}},\iota,i\big)$ be an unfolding such that $(\widetilde{M},0)=\big(M\times\mathbb{C}^l,(0,0)\big)$ and ι is the inclusion. Then we denote the unfolding by $\big((M\times\mathbb{C}^l,0),\widetilde{\mathcal{T}},i\big)$. Every unfolding is isomorphic to such an unfolding.

Let $(\mathcal{H},\nabla)$ be a trTLE-structure on $(M,0)$, and $(\nabla^{\mathrm{r}},\mathcal{C},\mathcal{U},\mathcal{V})$ the associated Frobenius type structure. Recall that $\mathcal{C}$ is a Higgs field and $\mathcal{U}$ is an endomorphism on $\mathcal{H}|_{\lambda=0}$. Let $\mathcal{A}$ be the sub-algebra in $\mathrm{End}(\mathcal{H}|_{\lambda=0})$ generated by $\{\mathcal{C}_x \mid x\in\Theta_{M,0}\}$ and $\mathcal{U}$. The relation (2.2) implies that $\mathcal{A}$ is a commutative algebra.

Definition 2.23. A ∇^{r}-flat section ζ is said to satisfy the **generation condition** ((GC) for short) if it generates $\mathcal{H}|_{\lambda=0}$ over $\mathcal{A}$, i.e. $\mathcal{A}\zeta=\mathcal{H}|_{\lambda=0}$.

Remark 2.24. A ∇^{r}-flat section ζ satisfies (GC) if and only if its restriction to $(0,0)\in\mathbb{P}^1_\lambda\times M$ generates $\mathcal{H}|_{(0,0)}$ over $\mathcal{A}|_{(0,0)}$. A non-zero vector $\zeta_0\in\mathcal{H}|_{(0,0)}$ is said to satisfy (GC) if its ∇^{r}-flat extension $\zeta\in\mathcal{H}|_{\lambda=0}$ satisfies (GC).

Lemma 2.25. *If a ∇^{r}-flat section ζ satisfies* (GC)*, then the map $\mathcal{A}\to\mathcal{H}|_{\lambda=0},\ a\mapsto a(\zeta)$ is an isomorphism of $\mathcal{O}_{M,0}$-modules.*

Proof. Let a,a' be two endomorphisms in $\mathcal{A}$ such that $a(\zeta)=a'(\zeta)$. For any section $s\in\mathcal{H}|_{\lambda=0}$, we have $b\in\mathcal{A}$ with $b(\zeta)=s$ by (GC). Since $\mathcal{A}$ is a commutative algebra, we have $a(b(\zeta))=b(a(\zeta))$. This implies $a(s)=a'(s)$ and hence $a=a'$. □

Remark 2.26. Let $\big((\widetilde{M},0),\widetilde{\mathcal{T}},\iota,i\big)$ be an unfolding of a mixed trTLEP-structure $\mathcal{T}$ on $(M,0)$. Let $(\widetilde{\mathcal{H}},\widetilde{\nabla})$ (resp. $(\mathcal{H},\nabla)$) be the underlying trTLE-structures of $\widetilde{\mathcal{T}}$ (resp. $\mathcal{T}$). The restriction of i to $(0,0)\in\mathbb{P}^1_\lambda\times(M,0)$ is an isomorphism $i|_{(0,0)}:\mathcal{H}|_{(0,0)}\xrightarrow{\sim}\widetilde{\mathcal{H}}|_{(0,0)}$ of vector spaces. If a non-zero vector $\zeta_0\in\mathcal{H}|_{(0,0)}$ satisfies (GC) or $(\mathrm{EC})_d$, then $i|_{(0,0)}(\zeta_0)$ satisfies the same condition. On the other hand, even if ζ_0 satisfies (IC) or (IdC), $i|_{(0,0)}(\zeta_0)$ does not necessarily satisfy the same condition.

The following theorem is the first main theorem of this paper.

Theorem 2.27 (Unfolding theorem). *Let $\mathcal{T}$ be a mixed* trTLEP*-structure on a germ $(M,0)$ of complex manifold and $(\mathcal{H},\nabla)$ the underlying* trTLE*-structure. Let ζ_0 be a non-zero vector in $\mathcal{H}|_{0,0}$ satisfying* (IC) *and* (GC)*. Then there exists a universal unfolding of $\mathcal{T}$. Moreover, an unfolding $\big((\widetilde{M},0),\widetilde{\mathcal{T}},\iota,i\big)$ is universal if and only if the vector $i|_{(0,0)}(\zeta_0)$ satisfies* (IdC)*.*

This theorem will be proved in §2.2.2. Using this theorem, we have the following.

Corollary 2.28 (Construction theorem). *Let $\mathcal{T}$, $(\mathcal{H},\nabla)$, and ζ_0 be the same as in* Theorem 2.27. *Assume moreover that ζ_0 satisfies* $(\mathrm{EC})_d$ *for a complex number d. Then there exists a tuple $\big((\widetilde{M},0),\mathscr{F},\iota,i\big)$ with the following properties uniquely up to isomorphisms.*

1. *$\mathscr{F}$ is a MFS of charge d on a germ $(\widetilde{M},0)$ of a complex manifold.*
2. *$\iota:(M,0)\hookrightarrow(\widetilde{M},0)$ is a closed embedding.*
3. *$i:\mathcal{T}\xrightarrow{\sim}\iota^*\mathcal{T}(\mathscr{F})$ is an isomorphism of mixed* trTLEP*-structure such that $e(0)=i|_{(0,0)}(\zeta_0)$.*

Proof. By Theorem 2.27, we have a universal unfolding $\big((\widetilde{M},0),\widetilde{\mathcal{T}},\iota,i\big)$ of $\mathcal{T}$. Since $i|_{(0,0)}(\zeta_0)$ satisfies (IdC) and $(\mathrm{EC})_d$ (Remark 2.26), there is a unique MFS $\mathscr{F}$ on $(\widetilde{M},0)$ such that the morphism (2.9) gives the isomorphism $\mathcal{T}(\mathscr{F})\simeq\widetilde{\mathcal{T}}$ (Proposition 2.19). This gives the existence of the tuple $\big((\widetilde{M},0),\mathscr{F},\iota,i\big)$. The universality of the unfolding and the uniqueness of the MFS in Proposition 2.19 imply the uniqueness of the tuple. □

This is a generalization of the Theorem 4.5 in [14].

2.2.2. A proof of the unfolding theorem. In this section, we give a proof of Theorem 2.27. We define the category of unfoldings of a (filtered) trTLE-structure as in the case of mixed trTLEP-structure.

Proposition 2.29. *Let $\mathcal{T}_{\mathrm{filt}}$ be a filtered* trTLE*-structure on a germ of complex manifold $(M,0)$ and $(\mathcal{H},\nabla)$ the underlying trTLE-structure. Let ζ_0 be a vector in $\mathcal{H}|_{(0,0)}$ satisfying* (GC) *and* (IC)*. Then, a universal unfolding of $\mathcal{T}_{\mathrm{filt}}$ exists and is characterized by the same condition as in Theorem* 2.27.

To prove this proposition, let us prepare some notions. Let $(\nabla^{\mathrm{r}},\mathcal{V},\mathcal{C},\mathcal{U})$ be the Frobenius type structure on $\mathcal{H}|_{\lambda=0}$ associated to $(\mathcal{H},\nabla)$. Let $V_{\mathcal{H}}$ the vector space of ∇^{r}-flat sections of $\mathcal{H}|_{\lambda=0}$. The dimension of $V_{\mathcal{H}}$ is equal to the rank of $\mathcal{H}$. In fact, there is a canonical isomorphism $(\mathcal{O}_{M,0}\otimes V_{\mathcal{H}},d\otimes\mathrm{id})\xrightarrow{\sim}(\mathcal{H}|_{\lambda=0},\nabla^r)$ of flat bundles on the germ of manifold $(M,0)$. We also note that for each unfolding $\widetilde{\mathcal{H}}$ of trTLE-structure, the restriction map $V_{\widetilde{\mathcal{H}}}\to V_{\mathcal{H}}$ is an isomorphism.

Fix a ∇^{r}-flat section $\zeta\in\mathcal{H}|_{\lambda=0}$. Since $\nabla^{\mathrm{r}}(\mathcal{C}_\bullet\zeta)=0$ as a section of $\mathcal{H}|_{\lambda=0}\otimes\Omega^1_{M,0}$, there is an unique section $\psi_\zeta\in\mathcal{H}|_{\lambda=0}$ such that $\psi_\zeta(0)=0$ and $\nabla^{\mathrm{r}}\psi_\zeta=$

$\mathcal{C}_\bullet\zeta$. If we consider ψ as a holomorphic function $\psi : (M,0) \to (V_{\mathcal{H}},0)$ via the identification $\mathcal{H}|_{\lambda=0} \simeq V_{\mathcal{H}} \otimes \mathcal{O}_{M,0}$, we have $d\psi = \mathcal{C}_\bullet\zeta$.

When we are given an unfolding $\widetilde{\mathcal{T}}_{\text{filt}}$ of $\mathcal{T}_{\text{filt}}$, we have a unique $\widetilde{\nabla}^{\text{r}}$-flat section $\widetilde{\zeta}$ such that its restriction to $(M,0)$ equals to ζ. Then the restriction of the holomorphic function $\psi_{\widetilde{\zeta}}$ to $(M,0)$ equals to ψ_ζ.

Lemma 2.30. *Let $\mathcal{T}_{\text{filt}} = (\mathcal{H},\nabla,W)$ be a filtered trTLE-structure on a germ of complex manifold $(M,0)$. Let $\zeta \in V_{\mathcal{H}}$ be a ∇^{r}-flat section with* (GC)*. Let ψ be the holomorphic function on $(M\times\mathbb{C}^l,0)$ such that $\psi|_{(M\times\{0\},0)} = \psi_\zeta$. Then, there exists a unique unfolding $\big((M\times\mathbb{C}^l,0),\widetilde{\mathcal{T}}_{\text{filt}},i\big)$ such that $\psi = \psi_{\widetilde{\zeta}}$.*

Remark 2.31.

- We can regard $V_{\mathcal{H}}$ as a vector space of global section of $\mathcal{H}$ whose restriction to $\{\lambda=0\}$ is ∇^{r}-flat. From this point of view, we have a natural isomorphism of holomorphic vector bundles $\mathcal{O}_{\mathbb{P}^1_\lambda\times(M,0)} \otimes V_{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}$.
- Let $\mathcal{T}_{\text{filt}} = (\mathcal{H},\nabla,W)$ be filtered trTLE-structure. Since $(W_k(\mathcal{H}),\nabla)$ is also a trTLE-structure, we have a filtration $W_k(V_{\mathcal{H}}) := V_{W_k(\mathcal{H})}$ on $V_{\mathcal{H}}$. We have a canonical isomorphism of filtered vector bundles

$$(\mathcal{H},W) \simeq \big(V_{\mathcal{H}}\otimes\mathcal{O}_{\mathbb{P}^1_\lambda\times(M,0)}, \{W_k(V_{\mathcal{H}})\otimes\mathcal{O}_{\mathbb{P}^1_\lambda\times(M,0)}\}_k\big). \tag{2.10}$$

- Define an algebra $\mathcal{P}^W_{\mathcal{H}}$ of $\text{End}_{\mathbb{C}}(V_{\mathcal{H}})$ by

$$\mathcal{P}^W_{\mathcal{H}} := \{a \in \text{End}_{\mathbb{C}}(V_{\mathcal{H}}) \mid a(W_k) \subset W_k \text{ for all } k\in\mathbb{Z}\}. \tag{2.11}$$

  Then $\mathcal{A}$ can be regarded as a subalgebra of $\mathcal{P}^W_{\mathcal{H}}\otimes\mathcal{O}_{M,0}$ via the natural isomorphism $\mathcal{H}|_{\lambda=0} \simeq V_{\mathcal{H}}\otimes\mathcal{O}_{M,0}$.

Proof of Lemma 2.30. We may assume $l=1$. Put $(\widetilde{M},0) := (M\times\mathbb{C},0)$, $\widetilde{\mathcal{H}} := \mathcal{O}_{\mathbb{P}^1_\lambda\times(\widetilde{M},0)}\otimes V_{\mathcal{H}}$, and $\widetilde{W}_k := \mathcal{O}_{\mathbb{P}^1_\lambda\times(\widetilde{M},0)}\otimes W_k(V_{\mathcal{H}})$. We will prove the existence and uniqueness of a meromorphic differential form Ω with values in $\text{End}(\widetilde{\mathcal{H}})$ such that $\widetilde{\nabla} := d+\Omega$ defines the desired filtered trTLE-structure $\widetilde{\mathcal{T}}_{\text{filt}} := (\widetilde{\mathcal{H}},\widetilde{\nabla},\widetilde{W})$.

Take a coordinate $(t,y) := (t_1,t_2,\dots,t_m,y)$ on $(M\times\mathbb{C},0)$. Put $\mathcal{P}(n) := \mathcal{P}^W_{\mathcal{H}}\otimes(\mathcal{O}_{M,0}[y]/(y)^{n+1})$ for every non-negative integer n. Let $(\nabla^{\text{r}},\mathcal{C},\mathcal{U},\mathcal{V})$ be the Frobenius type structure associated to $(\mathcal{H},\nabla)$, and put $C_i^{(0)} := \mathcal{C}_{\partial/\partial t_i}$ $(i=1,\dots,m)$, $U^{(0)} := \mathcal{U}$, and $V^{(0)} := \mathcal{V}$. Identifying $\mathcal{H}|_{\lambda=0}$ and $\mathcal{O}_{M,0}\otimes V_{\mathcal{H}}$, we

regard $C_i^{(0)}, U^{(0)}$, and $V^{(0)}$ as a element of $\mathcal{P}(0)$. The meromorphic differential form $\Omega^{(0)} := \nabla - d$ is written as

$$\Omega^{(0)} = \frac{1}{\lambda}\sum_{i=1}^{m} C_i^{(0)} \mathrm{d}t_i + \left(\frac{1}{\lambda}U^{(0)} - V^{(0)}\right)\frac{\mathrm{d}\lambda}{\lambda}. \tag{2.12}$$

Claim 2.32. For every non-negative integer n, there uniquely exist $(C_i^{(n)}, U^{(n)}, V^{(n)}) \subset \mathcal{P}(n)$ and $C_y^{(n-1)} \in \mathcal{P}(n-1)$ with the following properties. Here, we put $\mathcal{P}(-1) := \mathcal{P}(0)$.

- The equations

$$C_y^{(n-1)} = 0, \quad C_i^{(n)} = C_i^{(0)}, \quad U^{(n)} = U^{(0)}, \quad V^{(n)} = V^{(0)} \tag{2.13}$$

  are satisfied in $\mathcal{P}(0)$.

- The equations

$$[C_i^{(n)}, C_j^{(n)}] = \frac{\partial C_i^{(n)}}{\partial t_j} - \frac{\partial C_j^{(n)}}{\partial t_i} = [C_i^{(n)}, U^{(n)}] = \frac{\partial V^{(n)}}{\partial t_i} = 0, \tag{2.14}$$

$$\frac{\partial U^{(n)}}{\partial t_i} = [V^{(n)}, C_i^{(n)}] - C_i^{(n)} \tag{2.15}$$

  are satisfied in $\mathcal{P}(n)$.

- The equations

$$[C_i^{(n)}, C_y^{(n-1)}] = \frac{\partial C_i^{(n)}}{\partial y} - \frac{\partial C_y^{(n-1)}}{\partial t_i} = [C_y^{(n-1)}, U^{(n)}] = \frac{\partial V^{(n)}}{\partial y} = 0, \tag{2.16}$$

$$\frac{\partial U^{(n)}}{\partial y} = [V^{(n-1)}, C_y^{(n-1)}] - C_y^{(n-1)} \tag{2.17}$$

  are satisfied in $\mathcal{P}(n-1)$. Here $\partial/\partial y : \mathcal{P}(n) \to \mathcal{P}(n-1)$ is induced from the differential.

- The equation

$$C_y^{(n-1)}(\zeta) = d\psi\left(\frac{\partial}{\partial y}\right) \tag{2.18}$$

  is satisfied in $V_{\mathcal{H}} \otimes (\mathcal{O}_{M,0}[y]/(y)^n)$

Proof of Claim 2.32. We use an induction on n. In the case $n = 0$, the flatness of ∇ and the equation (2.12) imply (2.14) and (2.15). Since $C_y^{(-1)}$, $\partial U^{(0)}/\partial y$, $\partial C_i^{(0)}/\partial y$, and $\partial V^{(0)}/\partial y$ are zero, the equations (2.16), (2.17), and (2.18) are trivial. The induction step from n to $n+1$ consists of the following three steps:

Step 1.: Construction of $C_y^{(n)} \in \mathcal{P}(n)$ as a lift of $C_y^{(n-1)}$ so that $C_y^{(n)}$ together with $C_i^{(n)}, U^{(n)}$ satisfies the part $[C_i^{(n)}, C_y^{(n)}] = [U^{(n)}, C_y^{(n)}] = 0$ of (2.16) in $\mathcal{P}(n)$, and (2.18) in $V_{\mathcal{H}} \otimes (\mathcal{O}_{M,0}[y]/(y)^{n+1})$.

Step 2.: Construction of $C_i^{(n+1)}, U^{(n+1)}, V^{(n+1)} \in \mathcal{P}(n+1)$ as a lift of $C_i^{(n)}, U^{(n)}, V^{(n)}$ such that conditions (2.17) and the part $\partial C_i^{(n+1)}/\partial y - \partial C_y^{(n+1)}/\partial t_i = \partial V^{(n+1)}/\partial y = 0$ of (2.16) are satisfied in $\mathcal{P}(n)$.

Step 3.: Check that $C_i^{(n+1)}, U^{(n+1)}, V^{(n+1)}$ satisfy the conditions (2.14) and (2.15) in $\mathcal{P}(n+1)$.

Let $\mathcal{A}^{(n)}$ be a commutative subalgebra of $\mathcal{P}(n)$ generated by $C_i^{(n)}$ and $U^{(n)}$. By (GC), the map $\mathcal{A}^{(n)} \to V_{\mathcal{H}} \otimes (\mathcal{O}_{M,0}[y]/(y)^{n+1})$ defined by $a \mapsto a(\zeta)$ is an isomorphism. Take $C_y^{(n)}$ as the inverse image of $d\psi(\partial/\partial y)$ of this isomorphism. This completes Step 1. Step 2 is obvious. To prove Step 3, we use the derivation $\partial/\partial y$, the equations (2.16), (2.17), and the induction hypothesis. For example, in $\mathcal{P}(n)$, we have

$$\begin{aligned}
&\frac{\partial}{\partial y}\left(\frac{\partial U^{(n+1)}}{\partial t_i} - \left[V^{(n+1)}, C_i^{(n+1)}\right] + C_i^{(n+1)}\right) \\
&= \frac{\partial}{\partial t_i}\frac{\partial U^{(n+1)}}{\partial y} - \left[V^{(n)}, \frac{\partial C_i^{(n+1)}}{\partial y}\right] + \frac{\partial C_i^{(n+1)}}{\partial y} \\
&= \frac{\partial}{\partial t_i}\left\{\left[V^{(n)}, C_y^{(n)}\right] - C_y^{(n)}\right\} - \left\{\left[V^{(n)}, \frac{\partial C_y^{(n)}}{\partial t_i}\right] - \frac{\partial C_y^{(n)}}{\partial t_i}\right\} \\
&= 0.
\end{aligned}$$

This implies that the equation (2.15) holds for $n+1$. The equation (2.14) is proved similarly. □

The sequences $\left(C_i^{(n)}\right)_n, \left(U^{(n)}\right)_n, \left(V^{(n)}\right)_n$, and $\left(\mathcal{C}_y^{(n)}\right)_n$ give formal endomorphisms C_i, U, V, and C_y in $\mathcal{P}_{\mathcal{H}}^W \otimes \mathcal{O}_{M,0}[[y]]$. We show that they are actually convergent.

Claim 2.33. The endomorphisms C_i, U, V, and C_y are in $\mathcal{P}_{\mathcal{H}}^W \otimes \mathcal{O}_{\widetilde{M},0}$.

Proof of Claim 2.33. Let e_k $(1 \leq k \leq r = \text{rank } \mathcal{H})$ be a ∇^{r}-flat frame of $\widetilde{\mathcal{H}}|_{\lambda=0}$. Regard the endomorphisms on $\widetilde{\mathcal{H}}|_{\lambda=0}$ (or on its formal completion $\mathcal{H}|_\lambda \otimes \mathcal{O}_{M,0}[[y]]$) as $r \times r$ matrices. Put $N := r^2(m+2)$ and let $X(t,y) \in \mathbb{C}^N \otimes \mathcal{O}_{M,0}[[y]]$ be a N-dimensional vector valued (formal) function whose entries are the entries of $C_1, \ldots, C_m, U$, and V. Similarly, let $X^{(0)} \in \mathbb{C}^N \otimes \mathcal{O}_{M,0}$ be a N-dimensional vector valued holomorphic function whose entries are the entries of $C_1^{(0)}, \ldots, C_m^{(0)}, U^{(0)}$, and $V^{(0)}$. The order of entries are chosen to satisfy $X(t,0) = X^{(0)}(t)$.

Let $\mathscr{A}$ be a subalgebra of $\mathcal{P}_{\mathcal{H}}^W \otimes \mathcal{O}_{M,0}[[y]]$ generated by $C_1, \ldots, C_m$, and U over $\mathcal{O}_{M,0}[[y]]$. By (GC), the map $\mathscr{A} \to \mathcal{H}|_{\lambda=0} \otimes \mathcal{O}_{M,0}[[y]]$ given by $a \mapsto a(\widetilde{\zeta})$ is an isomorphism. Therefore, $\mathscr{A}$ is free $\mathcal{O}_{M,0}[[y]]$-module of rank r. Take monomials $G_1, \ldots, G_r$ in the endomorphisms $C_1, \ldots, C_m, U$ which form an $\mathcal{O}_{M,0}[[y]]$-basis of $\mathscr{A}$. Then, there are formal functions $g_j \in \mathcal{O}_{M,0}[[y]]$ $(1 \leq j \leq r)$ such that

$$C_y = \sum_{j=1}^{r} g_j G_j. \tag{2.19}$$

By (2.18), we have

$$\sum_{j=1}^{r} g_j G_j(\widetilde{\zeta}) = \mathrm{d}\psi\left(\frac{\partial}{\partial y}\right). \tag{2.20}$$

Since G_j are monomials in $C_1, \ldots, C_m$, and U, there exist $r \times r$-matrix valued functions $Q_j(t,x)$ such that the entries are in $\mathbb{C}\{t\}[x_1, x_2, \ldots, x_N]$ and $G_j(t,y) = Q_j\big(t, X(t,y)\big)$. Therefore, by the equation (2.20) and the fact that $\{G_j(\widetilde{\zeta}) \mid 1 \leq j \leq r\}$ form a frame of $\mathcal{H}|_{\lambda=0} \otimes \mathcal{O}_{M,0}[[y]]$, there exist convergent power series $q_j(t,y,x) \in \mathbb{C}\{t,y,x\}$ such that $g_j(t,y) = q_j\big(t,y,X(t,y)\big)$. Put $Q := \sum_j q_j Q_j$. Then by (2.19), we have $C_y = Q(t,y,X)$.

By (2.16) and (2.17), we have the equations

$$\frac{\partial C_i}{\partial y} = \frac{\partial C_y}{\partial t_i}, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial U}{\partial y} = [V, C_y] + C_y.$$

Using the expression $C_y = Q(t, y, X)$ and these equations, we can regard $X(t, y)$ as a formal solution of the following partial differential equation

$$\begin{aligned} \frac{\partial X}{\partial y}(t, y) &= \sum_{i=1}^{m} A_i(t, y, X) \frac{\partial X}{\partial t_i}(t, y) + B(t, y, X) \\ X(t, 0) &= X^{(0)}(t) \end{aligned}$$

where $A_i(t, y, x)$, $(1 \leq i \leq m)$ are $N \times N$ matrix whose entries are in $\mathbb{C}\{t, y, x\}$ and $B_i(t, y, x)$ is N-dimensional vector whose entries are also in $\mathbb{C}\{t, y, x\}$. The theorem of Cauchy-Kovalevski implies that $X(t, y)$ actually converges. Therefore, C_i, U, V are all homomorphic and hence C_y is also holomorphic by (2.19). □

Put

$$\Omega := \frac{1}{\lambda} \left(\sum_{i=1}^{m} C_i \mathrm{d}t_i + C_y \mathrm{d}y \right) + \left(\frac{1}{\lambda} U - V \right) \frac{\mathrm{d}\lambda}{\lambda}, \tag{2.21}$$

and $\widetilde{\nabla} := d + \Omega$. Then the equations (2.13) imply that the restriction of $\widetilde{\nabla}$ to $(M, 0)$ is ∇. The equations (2.14), (2.15), (2.16), and (2.17) imply that $\widetilde{\nabla}$ is flat. And the equation (2.18) implies $\psi = \psi_{\widetilde{\zeta}}$. This proves the existence of the unfolding. The uniqueness in Claim 2.32 implies the uniqueness of the unfolding. □

Proof of Proposition 2.29. By (IC), $\psi_\zeta : (M, 0) \to (V_{\mathcal{H}}, 0)$ is closed embedding. Hence there exist a non-negative integer l and an isomorphism $\psi : (M \times \mathbb{C}^l, 0) \xrightarrow{\sim} (V_{\mathcal{H}}, 0)$ such that $\psi|_{(M \times \{0\}, 0)} = \psi_\zeta$. Applying the lemma 2.30 for this ψ, we have an unfolding $((M \times \mathbb{C}^l, 0), \mathcal{T}_{\mathrm{filt}}, i)$ such that $\psi_{\widetilde{\zeta}} = \psi$ and hence $\widetilde{\mathcal{C}}_\bullet \widetilde{\zeta} : \Theta_{(M \times \mathbb{C}^l, 0)} \to \widetilde{\mathcal{H}}|_{\lambda=0}$ is an isomorphism. It is easy to check that this unfolding is the universal unfolding. □

The following proposition together with Proposition 2.29 proves Theorem 2.27.

Proposition 2.34. *Let $\mathcal{T} = (\mathcal{H}, \nabla, W, P)$ be a mixed* trTLEP*-structure. Assume that there is a vector $\zeta_0 \in \mathcal{H}|_{(0,0)}$ with* (GC)*. Then, for any unfolding $\big((\widetilde{M}, 0), \widetilde{\mathcal{T}}_{\mathrm{filt}}, \iota, i\big)$ of the underlying filtered* trTLE*-structure $(\mathcal{H}, \nabla, W)$, there exists a unique sequence of graded pairings $\widetilde{P}$ on $\widetilde{\mathcal{T}}_{\mathrm{filt}}$ such that $\big((\widetilde{M}, 0), (\widetilde{\mathcal{T}}_{\mathrm{filt}}, \widetilde{P}), \iota, i\big)$ is an unfolding of the mixed* trTLEP*-structure $\mathcal{T}$.*

Proof. We may assume that $(\widetilde{M},0) := (M\times\mathbb{C},0)$. Let $(\widetilde{\mathcal{H}},\widetilde{\nabla})$ be the underlying trTLE-structure in $\widetilde{\mathcal{T}}_{\text{filt}}$ and $\widetilde{W}$ the weight filtration. Fix an arbitrary integer k. The graded pairing P_k uniquely extends to a $\widetilde{\nabla}$-flat section $\widetilde{P}_k$ on $\mathrm{Gr}_k^{\widetilde{W}}(\widetilde{\mathcal{H}})$ over $\mathbb{C}^*_\lambda\times(M\times\mathbb{C},0)$. We need to show that it takes values on $\mathrm{Gr}_k^W(\widetilde{\mathcal{H}})$ in $\lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda\times(\widetilde{M},0)}$. As in Remark 2.31 and in the proof of Lemma 2.30 we can normalize $\widetilde{\mathcal{H}}=\mathcal{O}_{\mathbb{P}^1_\lambda\times(\widetilde{M},0)}\otimes V_{\mathcal{H}}$, $\widetilde{W}_k=\mathcal{O}_{\mathbb{P}^1_\lambda\times(\widetilde{M},0)}\otimes W_k(V_{\mathcal{H}})$. Put $\Omega=\widetilde{\nabla}-d$ and coordinate $(t,y)=(t_1,\dots,t_m,y)$ on $(M\times\mathbb{C},0)$. Then as in the proof of Lemma 2.30, we have equation (2.21) where C_i,C_y, U,V are the elements of $\mathcal{P}^W_{\mathcal{H}}\otimes\mathcal{O}_{\widetilde{M},0}$. Let $C_{i,[k]},C_{y,[k]},U_{[k]},V_{[k]}$ be the image of C_i,C_y,U,V via the morphism $\mathcal{P}^W_{\mathcal{H}}\otimes\mathcal{O}_{\widetilde{M},0}\to\mathrm{End}\big(\mathrm{Gr}^W_k(V_{\mathcal{H}})\big)\otimes\mathcal{O}_{\widetilde{M},0}$. Define $\Omega_{[k]}$ by the following equation:

$$\Omega_{[k]}:=\frac{1}{\lambda}\left(\sum_{i=1}^m C_{i,[k]}\mathrm{d}t_i+C_{y,[k]}\mathrm{d}y\right)+\left(\frac{1}{\lambda}U_{[j]}-V_{[k]}\right)\frac{\mathrm{d}\lambda}{\lambda}.$$

Then $d+\Omega_{[k]}$ is the flat connection on $\mathrm{Gr}_k^{\widetilde{W}}\big(\widetilde{\mathcal{H}}\big)$ induced by $\widetilde{\nabla}$.

Giving $\widetilde{P}_k$ is equivalent to give a morphism

$$\phi_{\widetilde{P}_k}:\big(\mathrm{Gr}_k^{\widetilde{W}}\widetilde{\mathcal{H}}\big)\Big|_{\mathbb{C}^*_\lambda\times(\widetilde{M},0)}\to\big(\mathrm{Gr}_k^{j_\lambda^*\widetilde{W}}j_\lambda^*\widetilde{\mathcal{H}}\big)^\vee\Big|_{\mathbb{C}^*_\lambda\times(\widetilde{M},0)}$$

by $\langle\phi_{\widetilde{P}_k}(u),v\rangle=\widetilde{P}_k(u,v)$ where $\langle\bullet,\bullet\rangle$ is the natural pairing. Since $\widetilde{\mathcal{H}}=\mathcal{O}_{\mathbb{P}^1_\lambda\times(\widetilde{M},0)}\otimes V_{\mathcal{H}}$, and $\widetilde{W}_k=\mathcal{O}_{\mathbb{P}^1_\lambda\times(\widetilde{M},0)}\otimes W_k(V_{\mathcal{H}})$, the morphism $\phi_{\widetilde{P}_k}$ can be regarded as a global section of $E_k\otimes\mathcal{O}_{\mathbb{C}^*_\lambda\times(\widetilde{M},0)}$ where

$$E_k:=\mathrm{Hom}\Big(\mathrm{Gr}_k^W(V_{\mathcal{H}}),\big(\mathrm{Gr}_k^{j^*W}(V_{j^*\mathcal{H}})\big)^\vee\Big).$$

The flatness condition for $\widetilde{P}_k$ is equivalent to

$$d\phi_{\widetilde{P}_k}=\Omega_{[k]}^\vee\circ\phi_{\widetilde{P}_k}+\phi_{\widetilde{P}_k}\circ j_\lambda^*\Omega_{[k]}$$

which means

$$\frac{\partial}{\partial t_i}\phi_{\widetilde{P}_k}=\frac{1}{\lambda}(C_{i,[k]}^\vee\circ\phi_{\widetilde{P}_k}-\phi_{\widetilde{P}_k}\circ j_\lambda^*C_{i,[k]}),\tag{2.22}$$

$$\frac{\partial}{\partial y}\phi_{\widetilde{P}_k}=\frac{1}{\lambda}(C_{y,[k]}^\vee\circ\phi_{\widetilde{P}_k}-\phi_{\widetilde{P}_k}\circ j_\lambda^*C_{y,[k]}),\tag{2.23}$$

$$\lambda\frac{\partial}{\partial\lambda}\phi_{\widetilde{P}_k}=\frac{1}{\lambda}(U_{[k]}^\vee\circ\phi_{\widetilde{P}_k}-\phi_{\widetilde{P}_k}\circ j_\lambda^*U_{[k]})-(V_{[k]}^\vee\circ\phi_{\widetilde{P}_k}+\phi_{\widetilde{P}_k}\circ j_\lambda^*V_{[k]}).\tag{2.24}$$

Let $\phi^{(n)}_{\widetilde{P}_k}$ be the equivalent class in $E_k \otimes \left(\mathcal{O}_{\mathbb{C}^*_\lambda \times (M,0)}[[y]]/(y)^{n+1}\right)$ represented by $\phi_{\widetilde{P}_k}$. Then $\phi^{(0)}_{\widetilde{P}_k}$ is a global section of $E_k \otimes \lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times (M,0)}$ since $(\mathcal{H}, W, P)$ is a mixed trTLEP-structure.

Claim 2.35. The pairing $\phi^{(n)}_{\widetilde{P}_k}$ gives a global section of

$$E_k \otimes \left(\lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times (M,0)}[[y]]/(y)^{n+1}\right)$$

for every non-negative integer n.

Proof of Claim 2.35. We use an induction on n. The case $n = 0$ is explained above. Suppose that $\phi^{(n-1)}_{\widetilde{P}_k}$ is a global section of $E_k \otimes \left(\lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times (M,0)}[y]/(y)^n\right)$. Let $C^{(n)}_{i,[k]}$ be the image of $C_{[k]}$ to $\mathrm{End}(\mathrm{Gr}^W_k V_{\mathcal{H}}) \otimes \left(\mathcal{O}_{\mathbb{P}^1_\lambda \times (M,0)}[[y]]/(y)^{n+1}\right)$. Define $C^{(n)}_{y,[k]}, U^{(n)}_{[k]}, V^{(n)}_{[k]}$ similarly.

By (2.22), and the induction hypothesis, $C^{(n-1)*}_{i,[k]} \circ P^{(n-1)}_k - P^{(n-1)}_k \circ j^*_\lambda C^{(n-1)}_{i,[k]}$ gives a (global) section of $E_k \otimes \left(\lambda^{-k+1}\mathcal{O}_{\mathbb{P}^1_\lambda \times (M,0)}[[y]]/(y)^n\right)$. Similarly, $U^{(n-1)*}_{[k]} \circ P^{(n-1)}_k - P^{(n-1)}_k \circ j^*_\lambda U^{(n-1)}_{[k]}$ also gives a section of the same module. By (GC), $C^{(n-1)}_{y,[k]}$ is an element of the algebra generated by $C^{(n-1)}_{i,[k]}$and $U^{(n-1)}_{[k]}$. Therefore, $C^{\vee(n-1)}_{y,[k]} \circ P^{(n-1)}_k - P^{(n-1)}_k \circ j^*_\lambda C^{(n-1)}_{y,[k]}$ is a section of the same module. By (2.23), this implies that $\phi^{(n)}_{\widetilde{P}_k}$ is a section of

$$E_k \otimes \left(\lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times (M,0)}[[y]]/(y)^{n+1}\right).$$

□

This claim shows that $\phi_{\widetilde{P}_k}$ is a global section of $E_k \otimes \lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times (M,0)}[[y]]$. Since we know that $\phi_{\widetilde{P}_k}$ is analytic along y-direction, we have proved that $\phi_{\widetilde{P}_k}$ is a global section of $E_k \otimes \lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times (\widetilde{M},0)}$. □

3. Application to local B-models

In this section, we give an application of the construction theorem (Corollary 2.28) to local B-models.

First, in Section 3.1, we show that a germ of a variation of mixed Hodge structure with H^2-generation condition ([14, Definition 5.3]) defines a family of mixed Frobenius manifolds. After that, following [18], we recall the settings of local B-models. The VMHS's for local B-models are given by the relative cohomology group of the affine hypersurface in $(\mathbb{C}^*)^d$. By the work

of Batyrev [2], Stienstra [31], and Konishi-Minabe [18], the Hodge filtrations and the weight filtrations are described by a kind of toric data. We recall their results in Section 3.2. Using these results, in Section 3.3, we give a sufficient condition for H^2-generation condition in terms of the toric data and we show that the local B-model mirror to the canonical bundle of a weak Fano toric surface gives rise to a mixed Frobenius manifold (Corollary 3.26).

3.1. Mixed Frobenius manifolds and variations of mixed Hodge structure

We denote by $\mathscr{H} = (V_{\mathbb{Q}}, F, W)$ a graded polarizable variation of mixed Hodge structure (VMHS) on a germ of a complex manifold $(M, 0)$. Here, $V_{\mathbb{Q}}$ is a $\mathbb{Q}$-local system on $(M, 0)$, $W = (W_k \mid k \in \mathbb{Z})$ is a weight filtration on $V_{\mathbb{Q}}$, and $F = (F^\ell \mid \ell \in \mathbb{Z})$ is a Hodge filtration on $K := V_{\mathbb{Q}} \otimes \mathcal{O}_{M,0}$. Recall that if $S = (S_k \mid k \in \mathbb{Z})$ is a graded polarization on $\mathscr{H}$, then we have

$$S_k\big(\mathrm{Gr}_k^W(F^\ell), \mathrm{Gr}_k^W(F^{k-\ell+1})\big) = 0 \tag{3.1}$$

for any integers k and ℓ.

Definition 3.1. Fix a graded polarization $S = (S_k \mid k \in \mathbb{Z})$ on a VMHS $\mathscr{H} = (V_{\mathbb{Q}}, F, W)$. Let $\nabla := \mathrm{id}_{V_{\mathbb{Q}}} \otimes \mathrm{d}$ be the flat connection on $K = V_{\mathbb{Q}} \otimes \mathcal{O}_{M,0}$. Then an increasing filtration $U = (U_\ell \mid \ell \in \mathbb{Z})$ on K is called **opposite filtration** if the following conditions are satisfied :

(a) U_ℓ is ∇-flat subbundle of K for each ℓ.

(b) For any integers k and ℓ,

$$\mathrm{Gr}_k^W(F^\ell) \oplus \mathrm{Gr}_k^W(U_{\ell-1}) = \mathrm{Gr}_k^W(K), \tag{3.2}$$

$$S_k\big(\mathrm{Gr}_k^W(U_\ell), \mathrm{Gr}_k^W(U_{k-\ell+1})\big) = 0. \tag{3.3}$$

Remark 3.2. We can always construct an opposite filtration using the Deligne splitting.

Fix a VMHS $\mathscr{H} = (V_{\mathbb{Q}}, F, W)$, a graded polarization S, and an opposite filtration U. Then we get a mixed trTLEP-structure as follows. First, let $p_\lambda : \mathbb{P}^1_\lambda \times (M, 0) \to (M, 0)$ be the natural projection and take a lattice $\mathcal{H}$ of

the meromorphic flat bundle $\big(p_\lambda^*(K)(*\{0,\infty\}\times M), p_\lambda^*\nabla\big)$ by

$$\mathcal{H}|_{\mathbb{C}_\lambda\times(M,0)} := \sum_{\ell\in\mathbb{Z}} p_\lambda^* F^\ell \otimes \mathcal{O}_{\mathbb{C}_\lambda\times(M,0)}(\ell\{0\}\times(M,0)), \tag{3.4}$$

$$\mathcal{H}|_{(\mathbb{P}^1_\lambda\setminus\{0\})\times(M,0)} := \sum_{\ell\in\mathbb{Z}} p_\lambda^* U_\ell \otimes \mathcal{O}_{(\mathbb{P}^1_\lambda\setminus\{0\})\times(M,0)}(-\ell\{0\}\times(M,0)). \tag{3.5}$$

Then, put $\hat{W}_k := p_\lambda^* W_k \cap \mathcal{H}$ for every integer k. Take a pairing P_k on $\mathrm{Gr}_k^{\hat{W}}(\mathcal{H})$ by the composition of the morphism

$$\mathrm{id}\otimes j_\lambda^* : \mathrm{Gr}_k^{\hat{W}}(\mathcal{H}) \otimes j^*\mathrm{Gr}_k^{\hat{W}}(\mathcal{H}) \to \mathrm{Gr}_k^{\hat{W}}(\mathcal{H}) \otimes \mathrm{Gr}_k^{\hat{W}}(\mathcal{H}), \tag{3.6}$$

the natural inclusion $\mathrm{Gr}_k^{\hat{W}}(\mathcal{H})^{\otimes 2} \hookrightarrow \mathrm{Gr}_k^{p_\lambda^* W} p_\lambda^* K(*\{0,\infty\}\times(M,0))^{\otimes 2}$, and the pull back $p_\lambda^* S_k$. By (3.1) and (3.3), P_k gives a morphism

$$P_k : \mathrm{Gr}_k^{\hat{W}}(\mathcal{H}) \otimes j_\lambda^*\mathrm{Gr}_k^{\hat{W}}(\mathcal{H}) \to \lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda\times(M,0)}.$$

This construction is known as Rees construction. We get the following.

Lemma 3.3. *The tuple* $\mathcal{T}(\mathscr{H}, S, U) := \big((\mathcal{H}, p_\lambda^*\nabla), (\hat{W}_k)_k, (P_k)_k\big)$ *defined above is a mixed trTLEP-structure on* $(M,0)$.

Proof. By (3.2), the adjunction map $p_{\lambda*}p_\lambda^*\mathrm{Gr}_k^W(\mathcal{H}) \to \mathrm{Gr}_k^W(\mathcal{H})$ is an isomorphism for every k. Since any extension of two trivial bundles on $\mathbb{P}^1$ is trivial, the adjunction map $p_{\lambda*}p_\lambda^*\mathcal{H} \to \mathcal{H}$ is also an isomorphism. Since U_ℓ is flat, the connection $p_\lambda^*\nabla$ is logarithmic along $\{\infty\}\times(M,0)$. The Griffith transversality implies that $p_\lambda^*\nabla$ is pole order 1 along $\{\infty\}\times(M,0)$. Since S_k is $(-1)^k$-symmetric and non-degenerate, P_k is $(-1)^k$-symmetric and non-degenerate. □

We recall the definition of H^2-generation condition in [14].

Definition 3.4 ([14, Definition 5.3]). Let $\mathscr{H} := (V_\mathbb{Q}, F, W)$ be a VMHS on a germ $(M,0)$ of a complex manifold. Put $K := V_\mathbb{Q}\otimes\mathcal{O}_{M,0}$, $\nabla := \mathrm{id}_{V_\mathbb{Q}}\otimes d$, and $w := \max\{l\in\mathbb{Z} \mid F^l \neq 0\}$. Let $\theta := \mathrm{Gr}_F(\nabla) : \mathrm{Gr}_F K \to \mathrm{Gr}_F K \otimes \Omega^1_{M,0}$ be the induced Higgs field. The **H^2-generation condition** for $\mathscr{H}$ is the following.

(i) The rank of F^w is 1, and the rank of $\mathrm{Gr}_F^{w-1}(K)$ is equal to the dimension of $(M,0)$,

(ii) The map $\mathrm{Sym}\,\Theta_{M,0}\otimes F^w \to \mathrm{Gr}_F K$ induced by θ is surjective.

Standard discussion on Rees construction shows the following.

Lemma 3.5. *Let $\mathscr{H} = (V_{\mathbb{Q}}, F, W)$ be a VMHS on a germ of complex manifold $(M, 0)$. Take the integer w as above and non-zero vector $\zeta_0 \in F^w|_0$. Fix a graded polarization S and an opposite filtration U.*

(a) *The vector ζ_0 satisfies $(\mathrm{EC})_{2w}$ with respect to $\mathcal{T}(\mathscr{H}, S, U)$.*

(b) *Assume moreover that the rank of F^w is 1. Then, the vector ζ_0 satisfies* (GC), (IC) *if and only if $\mathscr{H}$ satisfies H^2-generation condition.*

Proof. Let $(\mathcal{H}, p_\lambda^* \nabla)$ be the underlying trTLE-structure in $\mathcal{T}(\mathscr{H}, S, U)$ (defined by (3.4) and (3.5)). Let $(\nabla^r, \mathcal{C}, \mathcal{U}, \mathcal{V})$ be the associated Frobenius type structure (Lemma 2.7). Then, using the decomposition $K = \bigoplus_\ell F^\ell \cap U_\ell$, we have $\mathcal{V} = \ell \cdot \mathrm{id}$ on $F^\ell \cap U_\ell$. Since ζ_0 is in $F^w \cap U_w$, this proves (a). The Higgs field $\mathcal{C}$ corresponds to θ via the natural isomorphism $\mathcal{H}|_{\lambda=0} \simeq \mathrm{Gr}^F K$. We also remark that $\mathcal{U} = 0$. Hence the condition (ii) in Definition 3.4 is equivalent to (GC). If we assume the condition (ii), the morphism $\Theta_{M,0} \to \mathrm{Gr}^F_{w-1} K$ is surjective. Then the morphism is injective (this is equivalent to (IC)) if and only if the rank of $\mathrm{Gr}^F_{w-1} K$ is equal to the dimension of M. This proves (b). □

Hence, combining Corollary 2.28, we have the following.

Corollary 3.6. *Let $\mathscr{H} = (\mathbb{V}_{\mathbb{Q}}, F, W)$ be a VMHS on a germ $(M, 0)$ of a complex manifold with H^2-generation condition. Take the integer w as above and non-zero vector $\zeta_0 \in F^w|_0$. Fix a graded polarization S and an opposite filtration U. Then there exists a tuple $(\mathscr{F}, \iota, i)$ with following conditions up to isomorphisms.*

1. *$\mathscr{F}$ is a MFS of charge $2w$ on a germ of a complex manifold $(\widetilde{M}, 0)$.*
2. *$\iota : (M, 0) \hookrightarrow (\widetilde{M}, 0)$ is a closed embedding.*
3. *$i : \mathcal{T}(\mathscr{H}, U, S) \xrightarrow{\sim} \mathcal{T}(\mathscr{F})$ is an isomorphism of mixed* trTLEP*-structure with $i|_{(0,0)}(\zeta_0) = e|_0$ where e is the unit vector field of $\mathscr{F}$.* □

We give an example of VMHS which satisfies H^2-generation condition.

Definition 3.7. Let Y be a projective complex manifold and put $d := \dim Y$. Let D_i $(i = 0, 1)$ be hypersurfaces in Y such that D_0 is smooth and $D := D_0 \cup D_1$ is normal crossing. The triple (Y, D_0, D_1) is called an **open Calabi-Yau manifold with a divisor** if $\Omega^d_Y(D_1)$ is trivial.

Remark 3.8. Take an open Calabi-Yau manifold with divisor (Y, D_0, D_1). Put $d := \dim Y$. Let F be the Hodge filtration on $H^d(Y \setminus D_1, D_0 \setminus D_1)$.

1. By the degeneration of Hodge-to-de Rham spectral sequence, we have the following.

$$\mathrm{Gr}_F^p H^d(Y \setminus D_1, D_0 \setminus D_1) \simeq H^{d-p}\big(Y, \Omega^p(\log D)(-D_0)\big). \tag{3.7}$$

2. Since $\Omega_Y^d(D_1)$ is trivial, the dimension of $\mathrm{Gr}_F^d H^d(X \setminus D_1, D_0 \setminus D_1) \simeq H^0(Y, \Omega^d(D_1))$ is 1.

Definition 3.9. We say that an open Calabi-Yau manifold with a divisor (X, D_0, D_1) satisfies H^2**-generation condition** if the natural morphism

$$\mathrm{Sym}\Big(H^1\big(Y, \Theta_Y(\log D)\big)\Big) \otimes H^0\big(Y, \Omega_Y^d(D_1)\big) \to \mathrm{Gr}_F H^d(Y \setminus D_1, D_0 \setminus D_1) \tag{3.8}$$

is surjective.

Remark 3.10. If $d = 1$, then $H^1(Y, \Theta_Y(-D))$ is isomorphic to

$$H^1(Y, \mathcal{O}(-D_0)) \simeq \mathrm{Gr}_F^1$$

and hence H^2-generation condition is automatically satisfied.

We then consider a complete family of open Calabi-Yau manifold with a divisor. That is, we consider a smooth projective morphism $\pi : (\mathcal{Y}, Y) \to (M, 0)$ and divisors $(\mathcal{D}_i, D_i)$ $(i = 0, 1)$ with the following properties.

- $\mathcal{D}_0$ is smooth and $\mathcal{D} := \mathcal{D}_0 \cup \mathcal{D}_1$ is normal crossing in $\mathcal{Y}$.
- $\Omega_{\mathcal{Y}/M}^d(\mathcal{D}_1)$ is isomorphic to $\mathcal{O}_{\mathcal{Y}}$ where $d = \dim \mathcal{Y} - \dim M$.
- The Kodaira-Spencer morphism $\rho : \Theta_{M,0} \to R^1\pi_* \Theta_{\mathcal{Y}/M}(\log \mathcal{D})$ is an isomorphism.

Let $j^1 : \mathcal{Y} \setminus \mathcal{D} \hookrightarrow \mathcal{Y} \setminus \mathcal{D}_0$ and $j^2 : \mathcal{Y} \setminus \mathcal{D}_0 \hookrightarrow \mathcal{Y} \setminus$ be the inclusions. Then $R^d\pi_* j_!^2 j_*^1 \mathbb{Q}_{Y \setminus D}$ gives a VMHS on $(M, 0)$ which we denote by $\mathscr{H}$.

Lemma 3.11. *The VMHS $\mathscr{H}$ satisfies H^2-generation condition in the sense of Definition* 3.4 *if the open Calabi-Yau manifold with a divisor (Y, D_0, D_1) satisfies H^2-generation condition in the sense of Definition* 3.9.

Proof. By Remark 3.8 and since the Kodaira-Spencer map ρ is an isomorphism, the condition (i) in Definition 3.4 is satisfied. The natural pairing

$$\Theta_{\mathcal{Y}/\mathcal{M}}(\log \mathcal{D}) \otimes \Omega^p_{\mathcal{Y}/\mathcal{M}}(\log \mathcal{D}_1)(-\mathcal{D}_0) \to \Omega^{p-1}_{\mathcal{Y}/\mathcal{M}}(\log \mathcal{D}_1)(-\mathcal{D}_0)$$

induces the morphism

$$R^1\pi_*\Theta_{\mathcal{Y}/\mathcal{M}}(\log \mathcal{D}) \otimes R^q\pi_*\Omega^p_{\mathcal{Y}/\mathcal{M}}(\log \mathcal{D}_1)(-\mathcal{D}_0) \to R^{q+1}\pi_*\Omega^{p-1}_{\mathcal{Y}/\mathcal{M}}(\log \mathcal{D}_1)(-\mathcal{D}_0).$$

Using the Kodaira-Spencer morphism ρ and (3.7), this corresponds to

$$\mathrm{Gr}_F(\nabla) : \Theta_{M,0} \otimes F^d \to \mathrm{Gr}_F^d H^d(Y \setminus D_1, D_0)$$

at $0 \in (M, 0)$. Therefore the surjectivity of (3.8) implies the condition (ii) in Definition 3.4. □

Example 3.12. Put $Y := \mathbb{P}^1$, $D_1 := \{0, \infty\}$, and $D_0 := \{1, z_1, \dots z_m\}$ where $z_i \neq 0, 1, \infty$ $(i = 1, 2, \dots m)$ and $z_i \neq z_j$ $(i \neq j)$. Then (Y, D_0, D_1) is a open Calabi-Yau manifold with a divisor. As mentioned in Remark 3.10, this satisfies the H^2-generation condition and hence the complete family of (Y, D_0, D_1) gives rise to a mixed Frobenius manifold.

3.2. Combinatorial description of VMHS for local B-models

3.2.1. Settings for local B-models. Let N be a finitely generated free abelian group and d the rank of N. Let $N^\vee$ be the dual lattice of N and put $\overline{N^\vee} := N^\vee \oplus \mathbb{Z}$. Consider the group ring $\mathbb{C}[\overline{N^\vee}] = \mathbb{C}[t_0, t_0^{-1}] \otimes \mathbb{C}[N^\vee]$ as a graded ring by $\deg(t_0^k t^m) := k$ $(k \in \mathbb{Z}, m \in N^\vee)$. For an integral polyhedron $\Delta \subset N^\vee_{\mathbb{R}} := N^\vee \otimes \mathbb{R}$, let σ_Δ be the cone in $\overline{N^\vee_{\mathbb{R}}}$ generated by $\{1\} \times \Delta$. This defines a graded subring $S_\Delta := \mathbb{C}[\sigma_\Delta \cap N^\vee]$ in $\mathbb{C}[\overline{N^\vee}]$. $\mathbb{P}_\Delta := \mathrm{Proj}\ S_\Delta$ is a toric variety which contains an algebraic torus $T_N := \mathrm{Spec}\ \mathbb{C}[N^\vee]$ as an open dense subset. We also note that $D_\Delta := \mathbb{P}_\Delta \setminus T_N$ is a hypersurface.

Recall that the Newton polygon of a Laurent polynomial $f = \sum_{m \in N^\vee} a_m t^m \in \mathbb{C}[N^\vee]$ is the convex hull of the subset $\{m \in N^\vee \mid a_m \neq 0\}$ in $N^\vee_{\mathbb{R}}$. Put $A := A(\Delta) := \Delta \cap N^\vee$ and let $\mathbb{L}(\Delta)$ be the set of functions whose newton polygon is contained in Δ. Then $\mathbb{L}(\Delta)$ is naturally identified with $\mathbb{C}^A$.

Definition 3.13. Let Δ' be a face of Δ. For $f := \sum_{m \in A} a_m t^m \in \mathbb{L}(\Delta)$, we define the function $f^{\Delta'}$ by $f^{\Delta'} := \sum_{m \in \Delta' \cap N^\vee} a_m t^m$.

For a basis $u_1, u_2, \ldots, u_d$ of N, let $\theta_1, \theta_2, \ldots, \theta_d$ be the corresponding vector field on T_N. Each θ_i defines a differential operator on $\mathbb{C}[N^\vee]$ by $\theta_i(t^m) = \langle u_i, m\rangle t^m$.

Definition 3.14 ([18, Definition 3.1]). A Laurent polynomial $f \in \mathbb{C}[N^\vee]$ is called **Δ-regular** if the following conditions are satisfied.

1. The Newton polygon of f is Δ.
2. For each face Δ' of Δ, there is no point in T_N such that

$$f^{\Delta'} = \theta_1(f^{\Delta'}) = \cdots = \theta_d(f^{\Delta'}) = 0. \tag{3.9}$$

Let $\mathbb{L}_{\mathrm{reg}} := \mathbb{L}_{\mathrm{reg}}(\Delta)$ be the set of Δ-regular Laurent polynomials.

3.2.2. Mixed Hodge structure. For $f \in \mathbb{L}(\Delta)$, we define the differential operators $\mathcal{L}_f^i$ $(0 \le i \le d)$ on S_Δ by

$$\mathcal{L}_f^0 := t_0\partial_{t_0} + t_0 f,\ \mathcal{L}_f^i := \theta_i + t_0\theta_i f,\ (i = 1, 2, \ldots, d). \tag{3.10}$$

Definition 3.15. We define the vector space $\mathcal{R}_f$ by

$$\mathcal{R}_f := S_\Delta / \sum_{i=0}^{d} \mathcal{L}_f^i S_\Delta. \tag{3.11}$$

We define the decreasing filtration $\mathcal{E}$ on S_Δ by $\mathcal{E}^\ell := \bigoplus_{\ell \le k} S_\Delta^k$. We denote the induced filtration on $\mathcal{R}_f$ by the same letter. Denote by $\sigma_\Delta(\ell)$ the set of the co-dimension ℓ faces of σ_Δ. Put $|\sigma_\Delta(\ell)| := \bigcup_{\tau \in \sigma_\Delta(\ell)} \tau$ and $I(\ell) := (\sigma_\Delta \setminus |\sigma_\Delta(\ell)|) \cap \overline{N^\vee}$. Then an increasing filtration

$$\mathcal{I}_\ell := \bigoplus_{(k,m) \in I(\ell)} \mathbb{C} t_0^k t^m \tag{3.12}$$

on S_Δ is defined. We also denote the induced filtration on $\mathcal{R}_f$ by the same letter.

For $f \in \mathbb{L}_{\mathrm{reg}}(\Delta)$, put $V_f := f^{-1}(0)$. In [18], Konishi and Minabe constructed an isomorphism

$$\rho : \mathcal{R}_f \xrightarrow{\sim} H^d(T_N, V_f) \tag{3.13}$$

with the following properties.

(*a*) If $F = (F^\ell \mid \ell \in \mathbb{Z})$ is the Hodge filtration on $H^d(T_N, V_f)$, then $\rho(\mathcal{E}^{i-d}) = F^i$ for $0 \le i \le d$.

(*b*) If $W = (W_k \mid k \in \mathbb{Z})$ is the weight filtration on $H^d(T_N, V_f)$, then

$$\rho(\mathcal{I}_i) = W_{d-2+i},\ (0 < i \le d-1),$$
$$\rho(\mathcal{I}_{d+1}) = W_{2d-2} = W_{2d-1}, \quad H^d(T_N, V_f) = W_{2d}.$$

3.2.3. Gauss-Manin connection. Let $\mathcal{O}_{\mathbb{L}(\Delta)}$ be the sheaf of algebraic functions on $\mathbb{L}(\Delta)$. Since $\mathbb{L}(\Delta) = \mathbb{C}^A$, we have

$$\mathbb{C}[(a_m)_{m\in A}] = H^0(\mathbb{L}(\Delta), \mathcal{O}_{\mathbb{L}(\Delta)}).$$

For $i = 0, \dots, d$, let $\mathcal{L}^i$ be the differential operator on $S_\Delta \otimes \mathcal{O}_{\mathbb{L}(\Delta)}$ given by

$$\begin{aligned} \mathcal{L}^0 &:= t_0\partial_0 + t_0 \sum_{m\in A} a_m t^m, \\ \mathcal{L}^i &:= \theta_i + \sum_{m\in A} a_m \langle u_i, m\rangle t^m \ (1 \le i \le d). \end{aligned} \tag{3.14}$$

Remark that $\mathcal{L}^i = \mathcal{L}^i_f$ at $f \in \mathbb{L}(\Delta)$. We define the $\mathcal{O}_{\mathbb{L}(\Delta)}$-module $\mathcal{R}$ by

$$\mathcal{R} := S_\Delta \otimes \mathcal{O}_{\mathbb{L}(\Delta)} \Big/ \sum_{i=1}^{d} \mathcal{L}^i (S_\Delta \otimes \mathcal{O}_{\mathbb{L}(\Delta)}). \tag{3.15}$$

The restriction of $\mathcal{R}$ to $\mathbb{L}_{\mathrm{reg}}(\Delta)$ defines an algebraic vector bundle, which we denote by the same letter. We note that the fiber of $\mathcal{R}$ at $f \in \mathbb{L}_{\mathrm{reg}}(\Delta)$ is $\mathcal{R}_f$.

We define differential operators $\mathcal{D}_{a_m}$ $(m \in A)$ on $S_\Delta \otimes \mathcal{O}_{\mathbb{L}(\Delta)}$ by

$$\mathcal{D}_{a_m} := \frac{\partial}{\partial a_m} + t_0 t^m. \tag{3.16}$$

Let ∇ be the connection on $\mathcal{R}$ defined by $\nabla_{\partial_{a_m}} := \mathcal{D}_{a_m}$.

Put $\mathcal{X} := \mathbb{P}_\Delta \times \mathbb{L}_{\mathrm{reg}}(\Delta)$, $\widetilde{\mathcal{M}} := \mathbb{L}_{\mathrm{reg}}(\Delta)$ and let $\pi : \mathcal{X} \to \widetilde{\mathcal{M}}$ be the projection. Define the divisors D_0, D_1 and D by $D_0 := \{(p, f) \in \mathcal{X} \mid p \in V_f\}$, $D_1 := D_\Delta \times \mathbb{L}_{\mathrm{reg}}(\Delta)$, and $D := D_0 \cup D_1$. Let $j^1 : \mathcal{X} \setminus D_0 \hookrightarrow \mathcal{X}$, $j^2 : \mathcal{X} \setminus D \hookrightarrow \mathcal{X} \setminus D_0$ be the inclusions. The stalk of the sheaf $R^d\pi_* j^1_! j^2_* \mathbb{C}_{\mathcal{X}\setminus D}$ at $f \in \widetilde{\mathcal{M}}$ is $H^d(T_N, V_f)$.

Lemma 3.16 ([18, Lemma 4.1],[31, Section 6]). *The isomorphism* (3.13) *gives an isomorphism between the local system of flat section of the analytic flat bundle* $(\mathcal{R}, \nabla)^{\mathrm{an}}$ *and* $R^d\pi_* j^1_! j^2_* \mathbb{C}_{\mathcal{X}\setminus D}$. □

3.2.4. The moduli space. We recall the definition of the moduli space $\mathcal{M}(\Delta)$ of the affine hypersurfaces of T_N. Define the action of T_N on $\mathbb{L}(\Delta)$ by $(sf)(t) := f(st)$ where $f \in \mathbb{L}(\Delta)$ and $s, t \in T_N$. We put $\mathbb{C}[a_m] := \mathbb{C}[(a_m)_{m\in A}]$. We regard the invariant ring $\mathbb{C}[a_m]^{T_N}$ as a graded ring using the natural grading on $\mathbb{C}[a_m]$.

Definition 3.17 ([2, Definition 10.4]). We define the moduli space of affine hypersurface in T_N by $\mathcal{M}(\Delta) := \mathrm{Proj}(\mathbb{C}[a_m]^{T_N})$.

If we put $\mathbb{P}(A) := \mathrm{Proj}(\mathbb{C}[a_m])$, $\mathcal{M}(\Delta)$ is a GIT-quotient of $\mathbb{P}(A)$ by the action of T_N. The stability condition of this GIT-quotient is defined as follows.

Definition 3.18. For a point $x \in \mathbb{P}(A)$, take a non-zero vector $v \in \mathbb{C}^A$ which represents x. The point x is stable if the orbit $T_N \cdot v$ is closed and of d-dimensions.

Put $R_f^i := \mathrm{Gr}_{\mathcal{E}}^{-i}\mathcal{R}_f$, $R_f := \bigoplus_i R_f^i$. By the property (a) of the isomorphism (3.13), we have the isomorphism

$$R_f^i \simeq \mathrm{Gr}_F^{d-i} H^d(T_N, V_f),\ (0 \le i \le d). \tag{3.17}$$

Let J_f be a homogeneous ideal of S_Δ generated by $t_0 f, t_0\theta_1 f, \ldots, t_0\theta_d f$. Then we naturally have the isomorphism $R_f \simeq S_\Delta/J_f$ of graded rings.

Proposition 3.19 ([2, Proposition 11.2, Corollary 11.3]). *Consider the action of the torus $T_{\overline{N}}$ on $\mathbb{L}(\Delta)$*

$$T_{\overline{N}} \times \mathbb{L}(\Delta) \to \mathbb{L}(\Delta) : (t_0, t) \times f(s) \mapsto t_0 f(ts). \tag{3.18}$$

If we identify $L(\Delta)$ and S_Δ^1 by $f \mapsto t_0 f$, the tangent space to the orbit $T_{\overline{N}} f$ is isomorphic to the homogeneous component J_f^1. Moreover, if we assume that $f \in \mathbb{L}_{\mathrm{reg}}(\Delta)$, and the corresponding class $[f] \in \mathcal{M}(\Delta)$ is a smooth stable point. Then, the tangent space $\Theta_{\mathcal{M}(\Delta),[f]}$ is naturally isomorphic to $R_f^1 = S_\Delta^1/J_f^1$. □

Proposition 3.20. *Let f be a Δ-regular function in $\mathbb{L}(\Delta)$ such that corresponding class $[f]$ in $\mathcal{M}(\Delta)$ is smooth stable point. Then under the isomorphism* (3.17) *and the isomorphism $\Theta_{\mathcal{M}(\Delta),[f]} \simeq R_f^1$ in Proposition* 3.19*, the*

Higgs field

$$\mathrm{Gr}_F(\nabla)_{[f]} : \Theta_{\mathcal{M}(\Delta),[f]} \otimes \mathrm{Gr}_F H^d(T_N, V_f) \to \mathrm{Gr}_F H^d(T_N, V_f)$$

corresponds to the multiplication

$$R_f^1 \otimes R_f \to R_f.$$

Proof. By Lemma 3.16, the Gauss-Manin connection corresponds to the connection

$$\nabla_{\partial_{a_m}} = \frac{\partial}{\partial a_m} + t^0 t^m \quad (m \in A) \tag{3.19}$$

on $\mathcal{R}$ over $\mathbb{L}(\Delta)$. Since the filtration $\mathcal{E}$ is determined by the degree of t_0, under the identifications $\mathbb{L}(\Delta) \simeq S_\Delta^1$ and (3.17), the Higgs field

$$\mathrm{Gr}_F(\nabla) : \Theta_{\mathbb{L}(\Delta),f} \otimes \mathrm{Gr}_F H^d(T_N, V_f) \to \mathrm{Gr}_F H^d(T_N, V_f)$$

corresponds to the multiplication

$$S_\Delta^1 \otimes R_f \to R_f.$$

By Proposition 3.19, this implies the conclusion. □

3.3. H^2-generation condition and MFS for local B-models

Let $[f_0] \in \mathcal{M}(\Delta)$ be a smooth stable point and assume that f_0 is Δ-regular. And let $\mathscr{H}_\Delta$ be the variation of mixed Hodge structure on the germ of complex manifold $(\mathcal{M}(\Delta), [f_0])$ defined by $H^d(T_N, V_f), (f \in \mathcal{M}(\Delta))$. By proposition 3.20, $\mathscr{H}_\Delta$ satisfies the H^2-generation condition if and only if R_f is generated by R_f^1. In this section, we consider the following condition: S_Δ^1 generates S_Δ. If this condition is satisfied, then $\mathscr{H}_\Delta$ satisfies the H^2-generation condition and hence gives rise to a mixed Frobenius manifold.

Definition 3.21 ([2, Definition 12.3]). A polyhedron $\Delta \subset N_{\mathbb{R}}^\vee$ is called **reflexive** if it satisfies the following conditions.

1. Δ contains $0 \in N^\vee$.
2. For any codimension 1 face Δ', there exists a primitive element $u \in N$ such that
$$\Delta' = \{m \in N_{\mathbb{R}}^\vee \mid \langle m, u \rangle = -1\}.$$

In the following, we assume that Δ is reflexive.

Remark 3.22. Reflexive polyhedron Δ has following properties ([2, Theorem 12.2]).

1. Its dual polyhedron $\Delta^* := \{u \in N_{\mathbb{R}} \mid \langle \Delta, u \rangle \geq -1\}$ is also reflexive.
2. $D_\Delta = \mathbb{P}_\Delta \setminus T_N$ is an anti-canonical divisor of $\mathbb{P}_\Delta$.
3. $\mathbb{P}_\Delta$ is a Fano variety with Gorenstein singularities.

Lemma 3.23. *Let $\Delta \subset N^\vee_{\mathbb{R}}$ be a 2-dimensional reflexive polyhedron. Then, S_Δ is generated by S^1_Δ.*

Proof. Fix an isomorphism $N^\vee \simeq \mathbb{Z}^2$. Label the elements of $\Delta \cap N^\vee \setminus \{0\}$ anti-clockwise with $\{m_1, m_2, \ldots, m_l\}$. For each i, let τ_i be the cone generated by m_i and m_{i+1}. (Here, we put $m_{\ell+1} := m_1$). The cones $\{\tau_i\}_i$ define a complete fan, which we denote by $\Sigma(\Delta^*)$. It is known that the toric manifold corresponding to $\Sigma(\Delta^*)$ is smooth and weak Fano. Therefore, the pair $\{m_i, m_{i+1}\}$ is a basis of $N^\vee$ for every i. Put

$$\sigma_i := \mathrm{Cone}((1,0),(1,m_i),(1,m_{i+1})) \subset \mathbb{R} \times N^\vee_{\mathbb{R}}$$

and $S_i := \mathrm{Spec}(\mathbb{C}[\sigma_i \cap (\mathbb{Z} \times N^\vee)])$. Since $\{m_i, m_{i+1}\}$ is a basis of $N^\vee$, each S_i is generated by S^1_i. The equation $S_\Delta = \sum_i S_i$ shows the lemma. □

For higher dimensional case, we consider following condition.

Definition 3.24 ([2, Definition 12.5, Remark 12.6]). Let Δ be a reflexive polyhedron and Δ^* its dual. Then Δ is called **Fano polyhedron** if $\mathbb{P}_\Delta$ is smooth Fano variety.

Lemma 3.25 ([2, Lemma 12.9]). *If Δ is Fano polyhedron, then S_Δ is generated by S^1_Δ.* □

Now, we assume that Δ is 2-dimensional or Fano polyhedron. Fix a Laurent polynomial $f_0 \in \mathbb{L}_{\mathrm{reg}}(\Delta)$ such that corresponding $[f_0] \in \mathcal{M}(\Delta)$ is smooth stable point.

Corollary 3.26. *Let $\mathscr{H}_\Delta$ be the variation of mixed Hodge structure on $(\mathcal{M}(\Delta), [f_0])$ defined by $H^d(T_N, V_f)$, $([f] \in \mathcal{M}(\Delta))$. Fix a graded polarization S and an opposite filtration U. Fix a generator ζ_0 of $\mathrm{Gr}^d_F(H^d(T_N, V_{f_0}))$. There exists the tuple $\left((\widetilde{M}, 0)\mathscr{F}, \iota, i\right)$ with following properties uniquely up to isomorphisms.*

- $\mathscr{F} = (\boldsymbol{\nabla}, \circ, E, e, \mathcal{I}, g)$ *is a MFS of charge* $2d$ *on a germ* $(\widetilde{M}, 0)$ *of a complex manifold.*
- $\iota : (\mathcal{M}(\Delta), [f_0]) \hookrightarrow (\widetilde{M}, 0)$ *is a closed embedding.*
- $i : \mathcal{T}(\mathscr{H}_\Delta, S, U) \to \iota^*\mathcal{T}(\mathscr{F})$ *is an isomorphism of mixed trTLEP-structure with* $i|_{(0,0)}(\zeta_0) = e|_0$.

Proof. By Lemma 3.23 and Lemma 3.25, ζ_0 generates $\mathrm{Gr}_F H^d(T_N, V_{f_0})$ over $\Theta_{\mathcal{M}(\Delta),[f_0]}$. Hence, by Corollary 3.6, we have the conclusion. □

4. Application to local A-models

In this section, we give an application of the construction theorem (Corollary 2.28) to local A-models.

4.1. Limit mixed trTLEP-structure

4.1.1. Mixed trTLEP-structure defined by a nilpotent endomorphism. Let $(\mathcal{H}, \nabla, P)$ be a trTLEP(0)-structure on a complex manifold M. Let $p_\lambda : \mathbb{P}^1_\lambda \times M \to M$ be the projection. Assume that there is a nilpotent endomorphism $\mathfrak{N}$ on $\mathcal{H}$ with the following conditions;

$$[\nabla, \mathfrak{N}] = \mathfrak{N}\frac{\mathrm{d}\lambda}{\lambda}, \tag{4.1}$$

$$\mathfrak{N} = p_\lambda^*(\mathfrak{N}|_{\lambda=0}), \tag{4.2}$$

$$P(\mathfrak{N}a, b) = P(a, \mathfrak{N}b). \tag{4.3}$$

We obtain a mixed trTLEP-structure as follows. Let $\mathcal{G}$ be the cokernel of $\mathfrak{N}$. By (4.2), $\mathcal{G}$ is a vector bundle over $\mathbb{P}^1_\lambda \times M$ such that $p_{\lambda *}p_\lambda^*\mathcal{G} \to \mathcal{G}$ is an isomorphism. Condition (4.1) implies that ∇ induces a flat connection $\overline{\nabla}$ on $\mathcal{G}$. Let $W = (W_k \mid k \in \mathbb{Z})$ be a filtration on $\mathcal{G}$ defined by

$$W_k := \begin{cases} 0 & (k < 0) \\ \mathrm{Im}\big(\mathrm{Ker}(\mathfrak{N}^{k+1}) \to \mathcal{G}\big) & (k \geq 0). \end{cases} \tag{4.4}$$

The graded pairing $Q = (Q_k : \mathrm{Gr}^W_k(\mathcal{G}) \otimes j_\lambda^*\mathrm{Gr}^W_k(\mathcal{G}) \to \lambda^{-k}\mathcal{O}_{\mathbb{P}^1_\lambda \times M} \mid k \in \mathbb{Z})$ is given by

$$Q_k([a], [b]) := \begin{cases} 0 & (k < 0) \\ P(\lambda^{-k}\mathfrak{N}^k a, b) & (k \geq 0). \end{cases} \tag{4.5}$$

Here, a is a local section of $\mathrm{Ker}(\mathfrak{N}^{k+1})$ and $[a]$ is the corresponding class in $\mathrm{Gr}_k^W(\mathcal{G})$. Similarly, b is a local section of $j_\lambda^*\mathrm{Ker}(\mathfrak{N}^{k+1})$ and $[b]$ is the corresponding class in $j_\lambda^*\mathrm{Gr}_k^W(\mathcal{G})$. We get the following.

Lemma 4.1. *The tuple* $\mathcal{T}_{\mathfrak{N}} := (\mathcal{G}, \overline{\nabla}, W, Q)$ *is a mixed* trTLEP*-structure.*

Proof. By (4.2), the adjunction $p_{\lambda *}p_\lambda^*\mathrm{Gr}_k^W\mathcal{G} \to \mathrm{Gr}_k^W\mathcal{G}$ is an isomorphism for every k. By (4.1), we have $[\nabla, \mathfrak{N}^k] = k\mathfrak{N}^k \mathrm{d}\lambda/\lambda$. Therefore, for $a \in \mathrm{Ker}(\mathfrak{N}^{k+1})$, we have

$$\mathfrak{N}^{k+1}\nabla a = \nabla\mathfrak{N}^{k+1}a + (k+1)\mathfrak{N}^{k+1}a\frac{\mathrm{d}\lambda}{\lambda} = 0.$$

Hence $\nabla a \in \mathrm{Ker}\mathfrak{N}^{k+1}$. This implies that the subbundle W_k is $\overline{\nabla}$-flat.

Put $P_k(a,b) := P(\mathfrak{N}^k a, b)$. For a subbundle $\mathcal{J}$ of $\mathcal{H}$, put $\mathcal{J}^{\perp k} := \{a \in \mathcal{H} \mid P_k(a,b) = 0 \text{ for all } b \in j_\lambda^*\mathcal{J}\}$. Since P is non-degenerate, we have $\mathcal{H}^{\perp k} = \mathrm{Ker}\mathfrak{N}^K$ and $\mathrm{Im}\mathfrak{N}^{\perp k} = \mathrm{Ker}\mathfrak{N}^{k+1}$. Therefore, we have

$$\left(\mathrm{Ker}\mathfrak{N}^{k+1}\right)^{\perp k} = \mathrm{Im}\mathfrak{N} + \mathrm{Ker}\mathfrak{N}^k. \tag{4.6}$$

For $a \in \mathrm{Ker}(\mathfrak{N}^{k+1})$, and $b \in j^*\mathrm{Ker}(\mathfrak{N}^{k+1})$, Let $[a] \in \mathrm{Gr}_k^W(\mathcal{G}), [b] \in j^*\mathrm{Gr}_k^W(\mathcal{G}))$ be the corresponding classes. The relation $\lambda^k Q_k([a],[b]) = P_k(a,b)$ and (4.6) shows that Q_k is well defined and non-degenerate.

Let a, b and $[a], [b]$ as above. We have

$$\begin{aligned}
&\mathrm{d}Q_k([a],[b]) - Q_k(\overline{\nabla}[a],[b]) - Q_k([a],\overline{\nabla}[b]) \\
&= \frac{1}{\lambda^k}\Big\{(-k)P(\mathfrak{N}^k a, b)\frac{\mathrm{d}\lambda}{\lambda} + \mathrm{d}P(\mathfrak{N}^k a, b) - P(\mathfrak{N}^k\nabla a, b) - P(\mathfrak{N}^k a, \nabla b)\Big\} \\
&= \frac{1}{\lambda^k}\Big\{\big((-k)P(\mathfrak{N}^k a, b) + kP(\mathfrak{N}^k a, b)\big)\frac{\mathrm{d}\lambda}{\lambda} \\
&\qquad + \mathrm{d}P(\mathfrak{N}^k a, b) - P(\nabla\mathfrak{N}^k a, b) - P(\mathfrak{N}^k a, \nabla b)\Big\} \\
&= 0.
\end{aligned}$$

This proves the flatness of Q_k. □

4.1.2. Logarithmic trTLEP-structure and limit mixed trTLEP-structure. Let Z be a normal crossing hypersurface of a complex manifold M. Recall the definition of logarithmic trTLEP-structure.

Definition 4.2 ([23, Definition 1.8]). Let k be an integer. A **trTLEP**(k)**-structure** on M **logarithmic along** Z is a tuple $\mathcal{T} = (\mathcal{H}, \nabla, P)$ with the following properties.

- $\mathcal{H}$ is a holomorphic vector bundle over $\mathbb{P}^1 \times M$ such that the adjoint morphism $p^*p_*\mathcal{H} \to \mathcal{H}$ is an isomorphism.
- ∇ is a meromorphic flat connection on $\mathcal{H}$ such that

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega_{\mathbb{P}^1_\lambda \times M}(\log Z_0) \otimes \mathcal{O}_{\mathbb{P}^1_\lambda \times M}(\{0\} \times M) \tag{4.7}$$

  where $Z_0 := (\{0, \infty\} \times M) \cup (\mathbb{P}^1_\lambda \times Z)$.
- $P : \mathcal{H} \otimes j^*\mathcal{H} \to \lambda^k \mathcal{O}_{\mathbb{P}^1_\lambda \times M}$ is a $(-1)^k$-symmetric, non-degenerate, ∇-flat pairing.

We also call $\mathcal{T}$ a **logarithmic trTLEP(k)-structure (or log$Z$-trTLEP(k)-structure)** for short.

We also recall the notion of logarithmic Frobenius type structure.

Definition 4.3 ([23, Definition 1.6]). Let K be a holomorphic flat bundle on M. Let $\mathcal{U}$ and $\mathcal{V}$ be endomorphisms on K. A tuple $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$ is called Frobenius type structure on K with logarithmic pole along Z if

- ∇^{r} is a flat connection on K with logarithmic pole along Z,
- $\mathcal{C}$ is a Higgs field on K with logarithmic pole along Z,

and these data satisfy the relations (2.2) and (2.3). We also call the tuple $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$ a **logarithmic Frobenius type structure** for short.

Remark 4.4.
- This definition of logarithmic Frobenius type structure lacks the pairing.
- If we assume that Z is smooth, we have the residue endomorphisms $\mathrm{Res}_Z \nabla^{\mathrm{r}}$, and $\mathrm{Res}_Z \mathcal{C}$.

The following lemma is proved by the same way as Lemma 2.7.

Lemma 4.5 ([23, Proposition 1.10]). *Let $(\mathcal{H}, \nabla, P)$ be a logarithmic* trTLEP(0)*-structure. Then there is a unique logarithmic Frobenius type structure $(\nabla^r, \mathcal{C}, \mathcal{U}, \mathcal{V})$ on $\mathcal{H}|_{\lambda=0}$ such that*

$$\nabla = p_\lambda^* \nabla^{\mathrm{r}} + \frac{1}{\lambda} p_\lambda^* \mathcal{C} + \left(\frac{1}{\lambda} p_\lambda^* \mathcal{U} - p_\lambda^* \mathcal{V} \right) \frac{\mathrm{d}\lambda}{\lambda}. \tag{4.8}$$

□

In the following, we assume that Z is smooth. Let $\mathcal{T} = (\mathcal{H}, \nabla, P)$ be a trTLEP(0)-structure on M logarithmic along Z such that

$$\mathrm{Res}_{\mathbb{P}^1_\lambda \times Z}(\nabla) \mid_{\{\infty\}\times Z} = 0. \tag{4.9}$$

Fix a point z in Z and a defining function q of Z on a neighborhood of z. Then the residual connection ∇^q on $\mathcal{H} \mid_{\mathbb{P}^1_\lambda \times (Z,z)}$ is induced. It is easy to see that the tuple $\mathcal{T}^q := (\mathcal{H} \mid_{\mathbb{P}^1_\lambda \times (Z,z)}, \nabla^q, P \mid_{(Z,z)})$ is a trTLEP(0)-structure on the germ (Z, z) of a complex manifold.

Lemma 4.6. *The endomorphism* $\mathfrak{N} := \lambda \mathrm{Res}_{\mathbb{P}^1_\lambda \times Z}(\nabla)$ *is nilpotent and satisfies the conditions* (4.1)–(4.3) *with respect to the* trTLEP(0)*-structure* $\mathcal{T}^q = (\mathcal{H} \mid_{(Z,z)}, \nabla^q, P \mid_{(Z,z)})$.

Proof. First of all, we show that $\mathrm{Res}_{\mathbb{P}^1_\lambda \times Z}(\nabla)$ is nilpotent. Let $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$ be the logarithmic Frobenius type structure on $\mathcal{H}|_{\lambda=0}$ such that

$$\nabla = p_\lambda^* \nabla^{\mathrm{r}} + \frac{1}{\lambda} p_\lambda^* \mathcal{C} + \left(\frac{1}{\lambda} p_\lambda^* \mathcal{U} - p_\lambda^* \mathcal{V} \right) \frac{\mathrm{d}\lambda}{\lambda}. \tag{4.10}$$

The condition (4.9) is equivalent to $\mathrm{Res}_{Z,z}(\nabla^r) = 0$. Then we have

$$\mathrm{Res}_{\mathbb{P}^1_\lambda \times (Z,z)}(\nabla) = \lambda^{-1} \mathrm{Res}_{Z,z} \mathcal{C}.$$

Since the eigenvalues of $\mathrm{Res}_{\mathbb{P}^1_\lambda \times (Z,z)}(\nabla)$ and $\mathrm{Res}_{Z,z}\mathcal{C}$ are both constant along $\mathbb{P}^1_\lambda$, they are all zero. Therefore, the endomorphism $\mathrm{Res}_{\mathbb{P}^1_\lambda \times (Z,z)}(\nabla)$ is nilpotent.

The following shows (4.1):

$$\begin{aligned} [\nabla^q, \mathfrak{N}] &= \lambda [\nabla^q, \mathrm{Res}_{\mathbb{P}^1_\lambda \times (Z,z)}(\nabla)] + \lambda \mathrm{Res}_{\mathbb{P}^1_\lambda \times (Z,z)}(\nabla) \frac{\mathrm{d}\lambda}{\lambda} \\ &= \mathfrak{N} \frac{\mathrm{d}\lambda}{\lambda}. \end{aligned}$$

The condition (4.2) is clear by $\mathfrak{N} = \mathrm{Res}_{Z,z}\mathcal{C}$. The flatness of P implies (4.3). □

This lemma together with Lemma 4.1 defines mixed trTLEP-structure on (Z, z).

Definition 4.7. We call the mixed trTLEP-structure $\mathcal{T}_{Z,z} := \mathcal{T}^q_{\mathfrak{N}}$ a **limit mixed trTLEP structure**.

Remark 4.8. The mixed trTLEP-structure $\mathcal{T}_{Z,z}$ does not depend on the choice of the defining function q of Z around z.

4.2. Quantum D-modules on A-models

We recall the definition of quantum D-modules and their properties. Let X be a smooth projective toric variety and $\Lambda_X \subset H_2(X,\mathbb{C})$ a semi-subgroup consists of effective classes. For each $d \in \Lambda_X$ and $n \in \mathbb{Z}_{\geq 0}$, let $\overline{X}_{0,n,d}$ be the moduli space of genus 0 stable maps to X of degree d. We denote the i-th evaluation map by $ev_i : \overline{X}_{0,n,d} \to X$. Fix a Hermitian metric $\| * \|$ on $H^{\neq 2}(X,\mathbb{C})$

Theorem 4.9 ([16, Theorem 1.3], See also [24, Theorem 4.2]). *The Gromov-Witten potential*

$$\Phi_X(\tau) = \sum_{d\in\Lambda_X}\sum_{n\geq 0}\frac{1}{n!}\int_{[\overline{X}_{0,n,d}]^{\mathrm{vir}}}\prod_{i=1}^{n} ev_i^*(\tau) \tag{4.11}$$

$$= \sum_{d\in\Lambda_X}\sum_{n\geq 0}\frac{1}{n!}\int_{[\overline{X}_{0,n,d}]^{\mathrm{vir}}} ev_i^*(\tau')e^{\delta(d)}. \tag{4.12}$$

converges on a simply connected domain

$$U_X := \{\tau = \tau' + \delta \in H^*(X,\mathbb{C}) |\ \mathrm{Re}(\delta(d)) < -C,\ \|\tau'\| < e^{-C}\} \tag{4.13}$$

for $C \gg 0$. Hence $\Phi_X(\tau)$ defines a holomorphic function on U_X. □

Define $g : H^*(X,\mathbb{C}) \otimes H^*(X,\mathbb{C}) \to \mathbb{C}$ by $g(\alpha,\beta) := \int \alpha \cup \beta$, and a quantum cup product $\circ$ on $H^*(X,\mathbb{C})$ by

$$g(\alpha \circ_\tau \beta, \gamma) = \alpha\beta\gamma\Phi_X(\tau)\ (\alpha,\beta,\gamma \in H^*(X,\mathbb{C})). \tag{4.14}$$

Here, $\tau \in U_X$ and we regard α,β,γ on the right hand side as differential operators. Let E be a vector field on $H^*(X,\mathbb{C})$ defined as a sum of first Chern class $c_1(X)$ and fundamental vector field of the action of $\mathbb{C}^*$ defined by $t\cdot\alpha = t^{\frac{2-i}{2}}\alpha$ ($\alpha \in H^i(X,\mathbb{C})$). Then it is well known that $\mathscr{F}_X := (\circ, E, g)$ is a Frobenius structure on U_X of charge $\dim X$.

Definition 4.10 ([24, Definition-Lemma 4.3]). We call the trTLEP(0)-structure $\mathcal{T}(\mathscr{F}_X)$ on U_X a **big quantum D-module**. We also call its restriction to $U'_X := U_X \cap H^2(X,\mathbb{C})$ a **small quantum D-module**.

Remark 4.11. This definition is equivalent to the definition in [24]. The trTLEP($\dim X$)-structure $\mathcal{T}(\mathscr{F}_X)(\dim X/2)$ is considered there (see Remark 2.15).

Denote by V_X^0 (resp. $V_X'^0$) the quotient space of U_X (resp. U'_X) by the natural action of $2\pi\sqrt{-1}H^2(X,\mathbb{Z})$. The following lemma is trivial by construction.

Lemma 4.12. *The big quantum D-module $\mathcal{T}(\mathscr{F}_X)$ induces a* trTLEP(0)*-structure $\mathcal{T}_X^{\mathrm{big}}$ on V_X^0. The small quantum D-module also induces a* trTLEP(0)*-structure $\mathcal{T}_X^{\mathrm{small}}$ on $V_X'^0$.* □

Fix a nef basis $T_1, T_2, \dots, T_r$ of $H^2(X,\mathbb{Z})$. Then the embedding of V'_X into $\mathbb{C}^r$ is naturally defined. Let $q = (q_1, q_2, \dots, q_r)$ be the canonical coordinate on $\mathbb{C}^r$ and $|*|$ the canonical Hermitian metric. We define

$$V_X := \{(q,\tau') \in \mathbb{C}^r \times H^{\neq 2}(X,\mathbb{C}) \big| |q| < e^{-C}, \|\tau'\| < e^{-C}\} \tag{4.15}$$

$$V'_X := \{q \in \mathbb{C}^r \big| |q| < e^{-C}\}. \tag{4.16}$$

Then we have the following.

Proposition 4.13 ([24, Corollary 4.5]). *$\mathcal{T}_X^{\mathrm{big}}$ (resp. $\mathcal{T}_X^{\mathrm{small}}$) is extended to a logarithmic* trTLEP(0)*-structure on V_X (resp. V'_X).* □

4.3. MFS for local A-models

4.3.1. Construction of MFS for local A-models. Let S be a weak Fano toric surface, $\gamma_0 = 1 \in H^0(S,\mathbb{Z})$ a unit, and γ_{r+1} its Poincare dual. Fix a nef basis $\gamma_1, \gamma_2, \dots, \gamma_r \in H^2(S,\mathbb{Z})$. Then $\{\gamma_0, \gamma_1, \dots, \gamma_{r+1}\}$ is a basis of $H^*(S,\mathbb{C})$. Denote by Λ_S the semi-subgroup consists of effective classes in $H_2(S,\mathbb{Z})$.

Let X be the projective compactification of the canonical bundle K_S (i.e. $X := \mathbb{P}(K_S \oplus \mathcal{O}_S)$). Let $p : X \to S$ be the natural projection and $i : S \to X$ the embedding defined by the zero section of K_S. Put $\Gamma_i := p^*\gamma_i$ $(0 \le i \le r+1)$ and $\Delta_0 := c_1(\mathcal{O}_{X/S}(1)) \in H^2(X,\mathbb{Z})$. Put $\Delta_i := \Delta_0 \cup \Gamma_i$ $(0 \le i \le r+1)$. Then the classes Γ_i, Δ_j $(0 \le i,j \le r+1)$ form a basis of $H^*(X,\mathbb{C})$. This basis gives a coordinate

$$(t,s) = (t^0, \dots, t^i, \dots, t^{r+1}, s^0, \dots, s^j, \dots, s^{r+1})$$

on $H^*(X,\mathbb{C})$. Put $q_0 := e^{s_0}$ and $q_i := e^{t_i}$. Then, the quantum cup product on $H^*(X,\mathbb{C})$ is given as follows.

Lemma 4.14 ([19, Lemma 8.9, 8.10]).

$$\Delta_i \circ * = \Delta_i \cup * + O(q_0), \tag{4.17}$$

$$\Gamma_i \circ \Gamma_j = \Gamma_i \cup \Gamma_j + \sum_{k=1}^{r} \sum_{d \in i_* \Lambda_S \setminus \{0\}} (\Gamma_i(d)\Gamma_j(d)\Gamma_k(d) N_d q^d) \Gamma_k^\vee + O(q_0). \tag{4.18}$$

Here, $q^d := \prod_{i=1}^r q^{\langle d, \Gamma_i \rangle}$ *and* $N_d := \int_{[\overline{X}_{0,0,d}]^{\mathrm{vir}}} 1$. *The symbol* $O(q_0)$ *represents the higher order term with respect to* q_0. □

By Proposition 4.13, we have logarithmic trTLEP(0)-structure $\mathcal{T}_X^{\mathrm{small}}$ on V_X'. (Note that $q = (q_0, q_1, \ldots, q_r) \in \mathbb{C}^{r+1}$.) Put $M := V_X' \setminus \bigcup_{i>0} \{q_i = 0\}$ and denote the restriction of $\mathcal{T}_X^{\mathrm{small}}$ to M by the same latter. Let Z be a divisor of M defined by $q_0 = 0$. For each point z in Z, the restriction of $\mathcal{T}_X^{\mathrm{small}}$ to the germ (M, z) satisfies the condition (4.9) along (Z, z). Hence we get the limit mixed trTLEP-structure $(\mathcal{T}_X^{\mathrm{small}})_{Z,z}$. To compare with the results of Konishi-Minabe [19] later, we consider the Tate twist $(\mathcal{T}_X^{\mathrm{small}})_{Z,z}(-1/2)$ (See Remark 2.15).

Proposition 4.15. *Let* $(\mathcal{T}_X^{\mathrm{small}})_{Z,z}(-1/2) = (\mathcal{H}, W, P)$ *be the mixed trTLEP-structure on* (Z, z) *constructed above and* $(\nabla^{\mathrm{r}}, \mathcal{C}, \mathcal{U}, \mathcal{V})$ *the corresponding Frobenius type structure. (See Lemma* 2.7.*) Put* $\mathfrak{N} := \Delta \cup * : H^*(X) \to H^*(X)$. *Then we get the following.*

$$\mathcal{H}|_{\lambda=0} = \mathrm{Cok}(\mathfrak{N}) \times (Z, z) \simeq \left(\bigoplus_{i=0}^{r+1} \mathbb{C}\Gamma_i \right) \times (Z, z), \tag{4.19}$$

$$W_k|_{\lambda=0} = \begin{cases} 0 & (k \le 0) \\ \mathrm{Im}(\mathrm{Ker}(\mathfrak{N}^k) \to \mathcal{H}|_{\lambda=0}) & (k > 0), \end{cases} \tag{4.20}$$

$$\mathcal{C}_{q_i \partial_{q_i}}(\Gamma_j) = \begin{cases} \Gamma_i, & (j = 0) \\ \Gamma_i \cup \Gamma_j + \\ \sum_{k=1}^r \sum_{d \in i_* \Lambda_S \setminus \{0\}} (\Gamma_i(d)\Gamma_j(d)\Gamma_k(d) N_d q^d) \Gamma_k^\vee & (j > 0), \end{cases} \tag{4.21}$$

$$\mathcal{U} = 0, \tag{4.22}$$

$$\mathcal{V} = -\frac{\deg}{2} + 2, \tag{4.23}$$

where $i = 1, 2, \ldots, r$, q^d *and* N_d *in the equation* (4.21) *are defined as in Lemma* 4.14, *and the operator* deg *is defined by* $\deg(\Gamma_j) := m\Gamma_j$ *for* $\Gamma_j \in H^m(X)$.

Proof. The Lemma 4.14 implies (4.19) and (4.21). In particular, we have $\lambda \mathrm{Res}_{\mathbb{P}^1_\lambda \times (Z,z)} \nabla = \Delta \cup *$ for the connection ∇ underlying $\mathcal{T}_X^{\mathrm{small}}$. The equation 4.4 twisted by $(-1/2)$ gives (4.20). As is shown in [19], the Euler vector field E of $\mathscr{F}_X$ is given by

$$E = t^0 \frac{\partial}{\partial t^0} + 2\frac{\partial}{\partial s^0} - t^{r+1}\frac{\partial}{\partial t^{r+1}} - \sum_{i=1}^{r} s^i \frac{\partial}{\partial s^i} - 2s^{r+1}\frac{\partial}{\partial s^{r+1}}.$$

Hence $\mathcal{C}_E$ is 0 on $\mathrm{Cok}(\mathfrak{N})$ over Z. Since the charge of $\mathscr{F}_X$ is 3, considering the twist, we have $\mathcal{V} = \overline{\nabla E} - (2-3)/2 + 1/2 = -\deg/2 + 2$, where $\overline{\nabla E}$ is the endomorphism induced on $\mathrm{Cok}(\mathfrak{N})$ over Z by ∇E. □

Using this proposition,we get the following.

Theorem 4.16. *If $z \in Z$ is in a sufficiently small neighborhood of $0 \in V'_X$, there exits a tuple $\big((\widetilde{M}, 0), \mathscr{F}_A^{\mathrm{loc}}, \iota, i\big)$ with the following conditions uniquely up to isomorphisms.*

1. *$\mathscr{F}_A^{\mathrm{loc}}$ is a MFS of charge 4 on a germ of complex manifold $(\widetilde{M}, 0)$.*
2. *$\iota : (Z, z) \hookrightarrow (\widetilde{M}, 0)$ is a closed embedding.*
3. *$i : (\mathcal{T}_X^{\mathrm{small}})_{Z,z}(-1/2) \xrightarrow{\sim} \iota^* \mathcal{T}(\mathscr{F})$ is an isomorphism of mixed trTLEP-structure such that the restriction $i|_{(0,z)}$ sends Γ_0 to the unit vector field of $\mathscr{F}$.*

Proof. By (4.20), (4.21), and (4.23), Γ_0 satisfies (IC), (GC), and $(EC)_4$ when z is sufficiently small. Therefore, by Corollary 2.28, we have the conclusion. □

4.3.2. Comparison with the result of Konishi and Minabe. The mixed Frobenius manifold $\mathscr{F}_A^{\mathrm{loc}}$ constructed in Theorem 4.16 is isomorphic to the mixed Frobenius manifold constructed in [19] as follows. Let $\mathscr{F}_{\mathrm{KM}}$ be the mixed Frobenius structure on a open subset of $H^*(S, \mathbb{C})$ defined in [19, Theorem 8.7]. Regard Z as a subset of the quotient $H^2(S, \mathbb{C})/2\pi\sqrt{-1}H^2(S, \mathbb{Z})$ via the pull back p^*. It is easy to see that $\mathscr{F}_{\mathrm{KM}}$ induces MFS on $H^*(S, \mathbb{C})/2\pi\sqrt{-1}H^*(S, \mathbb{Z})$, which we denote by the same notation. We restrict the induced mixed trTLEP-structure $\mathcal{T}(\mathscr{F}_{\mathrm{KM}})$ to the germ (Z, z) and denote it by $\mathcal{T}(\mathscr{F}_{\mathrm{KM}})|_{(Z,z)}$. We have the following proposition.

Proposition 4.17. *We have a natural isomorphism*

$$\mathcal{T}(\mathscr{F}_{\mathrm{KM}})|_{(Z,z)} \simeq (\mathcal{T}_X^{\mathrm{small}})_{Z,z}(-1/2).$$

Proof. The isomorphism is given by $\gamma_i \mapsto \Gamma_i$ $(0 \le i \le r+1)$. Comparing the proof of [19, Theorem 8.7] with (4.5) and Proposition 4.15, we can check that this gives an isomorphism of mixed trTLEP-structure over (Z, z). □

This proposition together with the uniqueness in Theorem 4.16 shows the following.

Corollary 4.18. *We have an isomorphism of mixed Frobenius manifolds*

$$\big((\widetilde{M}, 0), \mathscr{F}_A^{\mathrm{loc}}\big) \simeq \big((H^*(S, \mathbb{C})/2\pi\sqrt{-1}H^*(S, \mathbb{Z}), z), \mathscr{F}_{\mathrm{KM}}\big).$$

□

This shows that we have constructed the mixed Frobenius manifold $\mathscr{F}_{\mathrm{KM}}$ by using the limit mixed trTLEP-structure and the unfolding theorem.

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Research Institute for Mathematical Sciences, Kyoto University
Kyoto 606-8502 Japan
E-mail address: shamoto@kurims.kyoto-u.ac.jp