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$4d$ $\mathcal{N} = 2$ SCFT from complete intersection singularity

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Detailed studies of four dimensional $\mathcal{N} = 2$ superconformal field theories (SCFT) defined by isolated complete intersection singularities are performed: we compute the Coulomb branch spectrum, Seiberg-Witten solutions and central charges. Most of our theories have exactly marginal deformations and we identify the weakly coupled gauge theory descriptions for many of them, which involve (affine) D and E shaped quiver gauge theories and theories formed from Argyres-Douglas matters. These investigations provide strong evidence for the singularity approach in classifying 4d $\mathcal{N} = 2$ SCFTs.

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1. Introduction

Six dimensional $(2, 0)$ theory has a remarkable ADE classification, which can actually be derived from the classification of two dimensional *isolated rational Gorenstein* singularities [45]. By compactifiying the $6d$ theory on torus [48], one can derive the classification of four dimensional $\mathcal{N} = 4$ superconformal field theories (SCFT)[1]. The natural next step is to classify four dimensional $\mathcal{N} = 2$ SCFTs [38, 39] and it transpires that the space of such theories is surprisingly large due to less supersymmetry and the possibility of non-Lagrangian theories [3, 5, 13, 24, 35, 36]. Those intrinsically strongly coupled theories make the classification much more difficult, and the traditional field theory tools become inadequate.

It turns out that the geometric tools are more suitable to implement classification: one can use wrapped M5 branes to engineer a large class of new $\mathcal{N} = 2$ SCFTs [25, 26, 44, 46, 50], and the classification of these theories is reduced to that of the punctures [14, 44, 47]; another seemingly much larger space of theories can be constructed using singularity theory [41, 51][2], and the classification of SCFTs boils down to the classification of three dimensional *isolated rational Gorenstein* singularities [51].[3] The isolated hypersurface singularities corresponding to $\mathcal{N} = 2$ SCFTs have been classified in [51, 55], and recently this program has been extended to the isolated complete intersection singularities (ICIS) [17]. The major result of [17] is that to obtain an $\mathcal{N} = 2$ SCFT, the ICIS must be defined by at most two polynomials (f_1, f_2), and there are in total 303 classes of singularities (many classes involve infinite sequences of singularities).

The purpose of this paper is to study properties of these new $\mathcal{N} = 2$ SCFTs engineered using ICISs in [17]. We will explore various physical properties of these theories such as the Coulomb branch spectrum, Seiberg-Witten solutions [38, 39], central charges [40], etc. using the data of the singularities. We also identify the weakly coupled gauge theory descriptions for some of these theories and compute various quantities from field theory techniques. The results are in complete agreements with those derived from singularity theory. Many of these theories take the form of (affine) D or E shaped quivers. We view these checks as compelling evidence for the power of our approach of employing singularity theory to classify four dimensional $\mathcal{N} = 2$ SCFTs.

The theory defined by an ICIS has some interesting and new features compared with the theory defined by a hypersurface singularity, for example, the multiplicity of the Coulomb branch operators (scalar chiral primaries) with the maximal scaling dimension can be larger than one; the number of A_1

[1]There are some subtleties involving non-local objects [1, 31, 42, 49].

[2]See [7–10] for an introduction to singularity theory.

[3]See recent work of [2, 2, 4] for classifications of rank 1 theories based on the Kodaira classification.

singularities (co-dimension one singularities on the Coulomb branch or the base of the Milnor fibration) is different from the Milnor number, etc. These models imply that some of the features of the SCFTs from hypersurface singularities may not be generic.

This paper is organized as follows: Section 2 explains how to derive physical properties of the SCFTs from the data of singularities; Section 3 describes the weakly coupled gauge theory descriptions for theories engineered using the singularities; we conclude with a short summary and discussion for future directions in Section 4.

2. $4d$ $\mathcal{N} = 2$ SCFT and ICIS

In this section, we will start by reviewing general properties of $4d$ $\mathcal{N} = 2$ SCFTs, and then explain the geometric constructions of a large class of such theories from ICISs. In particular, we will describe the necessary conditions on the ICISs to give rise to $4d$ $\mathcal{N} = 2$ SCFTs, and how to read off the Coulomb branch spectrum and conformal central charges of the $4d$ theory from the singularity.

2.1. Generality of $4d$ $\mathcal{N} = 2$ SCFT

The $\mathcal{N} = 2$ superconformal group $SU(2, 2|2)$ contains the bosonic conformal group $SO(4, 2)$ and $SU(2)_R \times U(1)_R$ R-symmetry. The unitary irreducible representations of $\mathcal{N} = 2$ superconformal algebra have been classified in [21], and a highest weight state is labeled as $|\Delta, R, r, j_1, j_2\rangle$, where Δ is the scaling dimension, R labels the representation of $SU(2)_R$ symmetry, r is the charge for $U(1)_R$ symmetry, and j_1, j_2 are the left and right spins. The important half-BPS operators (short multiplets) are $\mathcal{E}_{r,(0,0)}$ $(\Delta = r)$ and $\hat{B}_R$ $(\Delta = 2R)$, whose expectation values parametrize the Coulomb branch and the Higgs branch of the vacuum moduli space of the $\mathcal{N} = 2$ SCFT respectively. Moreover if $1 < \Delta[\mathcal{E}_{r,(0,0)}] \leq 2$, one can turn on the following relevant or exactly marginal deformations[4]

$$\delta S = \lambda \int d^4x\, \widetilde{Q}^4 \mathcal{E}_{r,(0,0)} + c.c \tag{1}$$

One can assign the scaling dimension to λ which satisfies the condition $\Delta[\lambda] + \Delta[\mathcal{E}_{r,(0,0)}] = 2$. If there is an operator $\hat{B}_1$ in the spectrum, one can

[4] We denote the $4 + 4$ supercharges by $Q, \widetilde{Q}$, suppressing the spacetime and R-symmetry indices.

also have the following relevant deformation

$$\delta S = m \int d^4x\, \widetilde{Q}^2 \hat{B}_1 + c.c \tag{2}$$

and m has scaling dimension one, which in Lagrangian theories gives the usual $\mathcal{N}=2$ mass deformation. The above deformations are all of the $\mathcal{N}=2$ preserving relevant or marginal deformations [2, 19]. The IR physics on the Coulomb branch depends on the parameters (m, λ, u) where $u \equiv \langle \mathcal{E}_{r,(0,0)} \rangle$ is the expectation value of the Coulomb branch operator. To solve the Coulomb branch, we need to achieve the following goals

- Determine the Coulomb branch spectrum, namely the scaling dimensions of the parameters (m, λ, u).
- Determine the Seiberg-Witten solution, which is often described by a family of geometric objects

$$F(z, m, \lambda, u) = 0. \tag{3}$$

  F can be a one-fold or a three-fold.

As for the Higgs branch, parametrized by the operators $\hat{B}_1$, the question is to determine the flavor symmetry group G and the corresponding affine ring, namely to identify the generators and relations for the Higgs branch.

2.2. Geometric engineering and 2d/4d correspondence

One can engineer a four dimensional $\mathcal{N}=2$ SCFT by starting with a three dimensional *graded rational Gorenstein* singularity [51]. Graded implies that the three-fold singularity should have a $\mathbb{C}^*$ action which is required by the $U(1)_R$ symmetry of $\mathcal{N}=2$ SCFTs. Gorenstein means that there is a distinguished top form Ω which will be identified with the Seiberg-Witten differential. The rational condition ensure that the Coulomb branch operators have the positive $U(1)_R$ charge, or the top form Ω has positive charge.

In the case of ICISs defined by $f = (f_1, f_2, \ldots, f_k) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ in $\mathbb{C}^n$ with $n = k+3 \geq 5$, the above conditions become

- Gorenstein: this is automatic for ICISs.
- $\mathbb{C}^*$ action: a set of positive weights w_a and d_i such that $f_i(\eta^{w_a} x_a) = \eta^{d_i} f_i(x_a)$
- Rationality: $\sum_{a=1}^n w_a > \sum_{i=1}^k d_i$

It was recently proved in [17] that the above conditions demand $k=2$.

Consider now graded ICISs which are defined by two polynomials $f = (f_1, f_2)$, which defines the map $f : (\mathbb{C}^5, 0) \to (\mathbb{C}^2, 0)$. We can denote the charge of the coordinates $x_a, a = 1, \ldots, 5$ and the degree of two polynomials by $(w_1, w_2, w_3, w_4, w_5; d_1, d_2)$ (up to an overall normalization). The rational condition is simply

$$\sum_{i=1}^{5} w_i > \sum_{i=1}^{2} d_i. \tag{4}$$

We can normalize the charges such that $d_1 = 1$ and $d_2 \le 1$ without losing any generality.

The above constraint can be interpreted from string theory. Considering type II string probing an ICIS. To decouple gravity, we send the string coupling $g_s \to 0$ while keeping the string scale ℓ_s fixed, this way we could end up with a non-gravitational $4d$ little string theory (LST) whose holographic dual is described by type II string theory in the background

$$\mathbb{R}^{3,1} \times \mathbb{R}_\phi \times (S^1 \times LG(W))/\Gamma \tag{5}$$

with a suitable GSO projection Γ that acts as an orbifold to ensure $4d$ $\mathcal{N}=2$ spacetime supersymmetry [28, 33, 37]. Here $\mathbb{R}_\phi$ denotes the $\mathcal{N}=1$ linear dilaton SCFT with dilaton profile $\varphi = -\frac{Q}{2}\phi$. For an ICIS, $LG(W)$ is the $\mathcal{N}=2$ Landau-Ginzburg (LG) theory with chiral superfields z_a and Λ.[5] Assuming $d_1 \ge d_2$ and normalizing the weights on z_a such that $d_1 = 1$, the superpotential is given by:

$$W = f_1(z_a) + \Lambda f_2(z_a), \tag{6}$$

where the $\mathbb{C}^*$ action on z_a is identified with the $U(1)_R$ charge of z_a in the LG model. The $U(1)_R$ charges of Λ is fixed to be $1-d_2$ so that W is quasi-homogeneous with charge 1. In the low energy limit ($\ell_s \to 0$), we expect to recover the $4d$ SCFT from the LST.

Now the $\mathcal{N}=1$ linear dilaton theory has central charge $3(1/2+Q^2)$, whereas the $\mathcal{N}=2$ LG model has central charge $3\hat{c}$. Consistency of the type II string theory on this background requires the worldsheet theory to have a total central charge of 15 which implies $Q^2 = 2-\hat{c}$ thus $\hat{c} < 2$. On the

[5] This has been considered for Calabi-Yau case in [29, 30] which extends naturally to the Fano case we consider here.

other hand, the central charge of the LG theory is determined by the $U(1)_R$ charges of the chiral superfields,

$$\hat{c} = \sum_{a=1}^{5}(1-2w_a) + (1-2(1-d_2)) = 4 - 2\sum_{a=1}^{5} w_a + 2d_2. \tag{7}$$

thus the condition $\hat{c} < 2$ is equivalent to

$$\sum_{a=1}^{5} w_a > 1 + d_2, \tag{8}$$

which is exactly the rationality condition for the singularity. Such $2d$ LG models are much less studied in the context of 2d $(2,2)$ SCFTs, in particular, the chiral ring structure is little explored. It is definitely useful to have a deeper understanding of these $2d$ theories so that the corresponding $4d$ theories can be understood better. In this paper, we will rely on the intrinsic properties of singularities to study 4d $\mathcal{N}=2$ theories.

2.3. Mini-versal deformations and Seiberg-Witten solution

The Seiberg-Witten (SW) solution of the $\mathcal{N}=2$ SCFT defined by an ICIS is identified with the mini-versal deformations of the singularity. Given a complete intersection singularity specified by polynomials $f = (f_2, f_2)$, the mini-versal deformations are captured by the Jacobi module

$$\mathcal{J} = \frac{\mathbb{C}^2[x_1, x_2, \ldots, x_5]}{\left(\frac{\partial f_i}{\partial x_a}\right)}. \tag{9}$$

We denote a monomial basis of the Jacobi module by ϕ_α which are 2×1 column vectors with only one non-zero entry. The mini-versal deformation of the ICIS is defined as

$$F(\lambda, z) = f(z) + \sum_{\alpha=1}^{\mu} \lambda_\alpha \phi_\alpha, \tag{10}$$

with the holomophic 3-form

$$\Omega = \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_5}{dF_1 \wedge dF_2}, \tag{11}$$

which describes the Milnor fibration of deformed 3-folds over the space of parameters λ_α. Here μ is the dimension of the Jacobi module and is also called Milnor number which counts the middle homology cycles (which are S^3 topologically) of the deformed 3-fold. The basis ϕ_α of the Jacobi module for ICIS can be computed using the software Singular [20], and the result is also listed in [17].

As explained in the previous subsection, type IIB string theory on this singular background gives rise to a $4d$ $\mathcal{N}=2$ SCFT. The coefficients λ_α in (10) are identified with the Coulomb branch parameters of $4d$ theory. The (assumed) $\mathbb{C}^*$ action on the singularity descends to the Jacobi module and is interpreted as (proportional to) the $U(1)_R$ symmetry of the $4d$ SCFT. The rank of the BPS charge lattice is given by μ. The BPS particles in the $4d$ theory comes from D3 branes wrapping special Lagrangian 3-cycles in the deformed 3-fold. Their masses are determined by the integral of Ω over the corresponding homology cycles.

Demanding the mass to have scaling dimension 1, we fix the relative normalization between the $\mathbb{C}^*$ and $U(1)_R$ charges . The scaling dimension ($U(1)_R$ charge) of a Coulomb branch parameter is then given by

$$\Delta[\lambda_\alpha] = \frac{d_j - Q[\phi_\alpha]}{\sum_{a=1}^5 w_a - d_1 - d_2}. \tag{12}$$

where $w_a > 0$ are the $\mathbb{C}^*$ charges of x_a. Here only the j-th entry of ϕ_α is nonzero. The Coulomb branch scaling dimensions $\Delta[\lambda_\alpha]$ are symmetric around 1 as shown in [43], which is in agreement with field theory result.

The Jacobi module of a graded ICIS will be captured by the following Poincare polynomial

$$P(z) = \sum z^\alpha \mathrm{dim} H_\alpha. \tag{13}$$

Here $\dim H_\alpha$ counts the number of basis elements in Jacobi module whose $\mathbb{C}^*$ charge is α. The Poincare polynomial has the following simple form [10] for the 3-fold ICISs here:

$$P(z) = z^c + z^c \frac{\prod_{i=1}^2 (1-z^{-d_i})}{\prod_{a=1}^5 (1-z^{-w_a})} \left[\left(\sum_{a=1}^5 z^{-w_a} - \sum_{i=1}^2 z^{-d_i} - 1 \right) - z^c \mathrm{res}_{t=0} \frac{1}{t^2(1+t)} \frac{\prod_{a=1}^5 (1+tz^{w_a})}{\prod_{i=1}^2 (1+tz^{d_i})} \right] \tag{14}$$

where $c \equiv \sum_{i=1}^2 d_i - \sum_{a=1}^5 w_a$.

The Milnor number can be computed from the Poincare polynomial by setting $z = 1$:

(15)
$$\mu = P(1) = \begin{cases} \prod_{a=1}^{5}\left(\frac{d}{w_a}-1\right)\left(4+\sum_{a=1}^{5}\frac{w_a}{d-w_a}\right) & d_1 = d_2 = d \\ \prod_{a=1}^{5}\left(\frac{d_1}{w_a}-1\right)\frac{d_2}{d_1-d_2}+\prod_{a=1}^{5}\left(\frac{d_2}{w_a}-1\right)\frac{d_1}{d_2-d_1} & d_1 \neq d_2 \end{cases}$$

2.4. Central charges a and c

Knowing the full spectrum of Coulomb branch parameters, we can compute the conformal central charges of the 4d $\mathcal{N} = 2$ SCFT using the following formula [40]:

$$a = \frac{R(A)}{4} + \frac{R(B)}{6} + \frac{5r}{24}, \quad c = \frac{R(B)}{3} + \frac{r}{6}, \tag{16}$$

where r is the rank of the Coulomb branch and we have used the fact that the generic fibre of the Milnor fibration (10) has only non-vanishing middle cohomology, thus there is no free massless hypermultiplets at a generic point on the Coulomb branch. $R(A)$ can be computed straightforwardly from the Coulomb branch spectrum which can be solved using the SW solution given above. The key point is to compute $R(B)$. In the hypersurface case, $R(B)$ takes an elegant form [51]:

$$R(B) = \frac{\mu u_{\max}}{4}; \tag{17}$$

Here μ is the Milnor number which is equal to the dimension of the charge lattice of our field theory, and u_{max} is the maximal scaling dimension of our theory.

For SCFTs defined by ICIS (f_1, f_2) with weights $(w_1, w_2, w_3, w_4, w_5; 1, d_2)$ and $d_2 \leq 1$, we propose that $R(A)$ and $R(B)$ are given by the following formula:

$$R(A) = \sum_{\Delta[u_i]>1}([u_i]-1), \quad R(B) = \frac{1}{4}\mu'\alpha. \tag{18}$$

Here μ' counts the "effective" number of A_1 singularities after generic deformations, which differs from the Milnor number μ, unlike in the case of hypersurface singularities, and α is the scaling dimension of one of the Coulomb branch operator which can be expressed as follows (with $d_2 \leq 1$):

$$\alpha = \frac{1}{\sum_{i=1}^{5} w_i - 1 - d_2}. \tag{19}$$

Notice that this is the scaling dimension of the operator associated with the deformation $(f_1 + t, f_2)$. The effective number of A_1 singularities is given by the following formula

$$\mu' = \mu + \mu_1, \tag{20}$$

where μ is the Milnor number of the ICIS, and μ_1 is the Milnor number for f_1 as an isolated hypersurface singularity in $\mathbb{C}^5$,

$$\mu_1 = \prod_{i=1}^{5} \left(\frac{1}{w_i} - 1 \right). \tag{21}$$

Let's explain how the effective number of A_1 singularities are computed. We have an ICIS (f_1, f_2) with weights $(w_1, w_2, w_3, w_4, w_5; 1, d_2)$ and $d_2 \leq 1$. We first deal with the situation where f_1 defines an isolated hypersurface singularity, and consider f_2 over the coordinates restricted on the variety $X_{f_1} = \{f_1(x_a) = 0\} \subset \mathbb{C}^5$. By deforming f_2 to $f_2' = f_2 + t$, the critical points of f_2' over X_{f_1} are Morse (the critical points are A_1 singularities). The number of such A_1 singularities is $\mu' = \mu + \mu_1$ [10], and μ_1 is the Milnor number of f_1. The scaling dimension of the coordinates (i.e Coulomb branch parameters, not to be confused with x_a) near the Morse critical points are determined by the scaling dimension of the polynomial f_1, which is given by

$$\alpha = \frac{1}{\sum_{i=1}^{5} w_i - 1 - d_2}; \tag{22}$$

This gives the $U(1)_R$ charge of the coordinates near the A_1 singularities, and explains the formula for $R(B)$ using the result in [40].

What is quite amazing is that even in the case that neither f_1 nor f_2 defines an isolated singularity, one can still use the above formula to compute the central charge although now $\mu^{'}$ is fractional, and the results match with the field theory expectations as we will see in the subsequent section.

3. Gauge theory descriptions

In the previous section, we have explained how to extract Coulomb branch data from the geometric information associated with the singularities (ICISs).

In this section, we will use these information to identify the gauge theory descriptions for a subclass of such constructions, thus providing nontrivial evidences for the general geometric constructions of $4d$ $\mathcal{N}=2$ SCFTs from ICISs.

3.1. Strategy of finding gauge theory descriptions

For some of the theories engineered using ICISs, the Coulomb branch spectrum contains operators with scaling dimension 2, which give rise to exactly marginal deformations. It appears that for $4d$ $\mathcal{N}=2$ SCFTs with such deformations, one can always find weakly coupled gauge theory descriptions.[6] We would like to find at least one weakly coupled gauge theory descriptions for our SCFTs with exactly marginal deformations. Unfortunately we do not have a systematic procedure at the moment and will take a brutal force approach: we compute the Coulomb branch spectrum from the ICIS and then try to guess a consistent quiver gauge theory such that the Coulomb branch spectrum matches (with additional consistency conditions such as central charges[7]). Even with this naive approach, we do find many interesting quiver gauge theories, and we compute the central charge from the quiver gauge theory which agrees with the result from that using singularity theory.

The essential strategy to identify gauge theory descriptions from the Coulomb branch spectrum consists of the following:

1) Identify candidate gauge groups G: among the integral scaling dimensions (greater than 1) in the CB spectrum, find the sequences that coincide with the set of fundamental degrees for various Lie groups listed in Table 1.

2) Identify candidate $\mathcal{N}=2$ matter: group the remaining scaling dimensions (those that are larger than 1) into isolated $\mathcal{N}=2$ SCFTs (many Argyres-Douglas type theories) which serve as non-Lagrangian matters with the necessary flavor symmetries to couple to the gauge groups.

3) Gluing the pieces: put together the gauge groups and matter by conformal gauging, check beta function

$$\beta_G = 2h^\vee(G) - \sum_\alpha \kappa_\alpha(G) \tag{1}$$

[6]It is interesting to prove or disprove this statement.

[7]We would like to emphasize here that there are examples of 4d $\mathcal{N}=2$ SCFTs with the same Coulomb branch spectrum and flavor symmetries yet different conformal central charges.

G	$\dim G$	$h^\vee$	$\{d_i\}_{i=1,\ldots,\text{rank}(G)}$
A_{n-1}	n^2-1	n	$2,3,\ldots,n$
B_n	$n(2n+1)$	$2n-1$	$2,4,\ldots,2n$
C_n	$n(2n+1)$	$n+1$	$2,4,\ldots,2n$
D_n	$n(2n-1)$	$2n-2$	$2,4,\ldots,2n-2;n$
E_6	78	12	$2,5,6,8,9,12$
E_7	133	18	$2,6,8,10,12,14,18$
E_8	248	30	$2,8,12,14,18,20,24,30$
F_4	52	9	$2,6,8,12$
G_2	14	4	$2,6$

Table 1: Relevant Lie algebra data: $h^\vee$ denotes the Coxeter number and $\{d_i\}$ are the degrees of the fundamental invariants.

vanishes for all gauge groups G. Here we use $\kappa_\alpha(G)$ to denote the flavor central charges of the matter theory for the symmetry G. We choose the normalization that for $G=A,B,C,D$, the flavor central charge $\kappa(G)=1$ for one fundamental hypermultiplet in the $G=A,C$ case and one half-hypermultiplet in the $G=B,D$ case. In the case when G is a subgroup of a larger non-abelian flavor symmetry J in the matter theory, the flavor central charges for G is related that of J by $\kappa(G)=I_{G\hookrightarrow J}\kappa(J)$ where $I_{G\hookrightarrow J}$ is known as the Dynkin index of embedding [6].[8]

4) Consistency check: make sure that the rank of remaining flavor symmetry matches with the the CB spectrum from singularity; in addition, if the quiver has a Lagrangian description, we compute the conformal

[8]Recall from [6] that the Dynkin index of embedding for $G\subset J$ is computed by

$$I_{G\hookrightarrow J}=\frac{\sum_i T(\mathbf{r}_i)}{T(\mathbf{r})} \tag{2}$$

where $\mathbf{r}$ denotes a representation of J which decomposes into $\oplus_i\mathbf{r}_i$ under G, and $T(\cdot)$ computes the quadratic index of the representation (which can be found for example in [34]).

central charges a and c from the gauge theory description using[9]

$$a = \frac{5n_v + n_h}{24}, \quad c = \frac{2n_v + n_h}{12} \tag{3}$$

and compare with that computed from (16). If the matter system includes the strongly coupled constituents, we also need to include the contributions from these subsectors.

We believe that these quiver gauge theory descriptions provide compelling evidence that our program of using singularity theory to classify $\mathcal{N} = 2$ SCFTs is correct. In the following subsection, we will list some of the interesting examples.

3.2. Matter system

The obvious matter system are free hypermultiplets. We also need to use strongly coupled matter systems such as theories defined by three punctured spheres [14, 26] and Argyres-Douglas matters [44, 50]. For later convenience, we will summarize some properties about these strongly coupled matter systems.

3.2.1. Gaiotto theory. We can have the strongly coupled matter defined by six dimensional (2, 0) type $J = ADE$ SCFTs on a sphere with regular punctures [26]. We are going to mainly use $J = A_{N-1}$ in which case the regular punctures are classified by Young Tableaux $Y = [n_s^{h_s}, \ldots, n_2^{h_2}, n_1^{h_1}]$ with the ordering $n_s > n_{s-1} > \cdots > n_1$. The flavor symmetry of a regular puncture is given by

$$G_Y = \left[\prod_{i=1}^{s} U(h_i)\right] / U(1). \tag{4}$$

The flavor central charge of a non-abelian factor $SU(h_i)$ is given by the following formula [14]:

$$k_{SU(h_i)} = \sum_{j \leq i} m_j s_j, \tag{5}$$

here s_i is defined by the transposed Young Tableaux $Y^T = [s_1^{m_1}, s_2^{m_2}, \ldots, s_{n_s}^{m_{n_s}}]$.

[9]For strongly coupled matter theories (e.g. Argyres-Douglas matters), we can think of n_h and n_v as counting the *effective* number of vector multiplets and hypermultiplets. To get a and c in those cases, the formula (16) is more useful.

The flavor symmetry of a theory defined by a three punctured sphere is $G = G_{Y_1} \times G_{Y_2} \times G_{Y_3}$, and it can be enhanced to a larger symmetry group using the $3d$ mirror as described in [11].

The Coulomb branch spectrum of these theories can be computed as follows: label the boxes of Y_j from $1, 2, \ldots, n$ row by row, and define the number (pole order associated with i-th fundamental invariant)

$$p_i^{(j)} = i - s_i \tag{6}$$

where s_i is the height of the ith box in Y_j. The number of degree (scaling dimension) i Coulomb branch operators are given by

$$d_i = \sum_{j=1}^{3} p_i^{(j)} - 2i + 1. \tag{7}$$

The conformal central charges a and c can be found (or rather n_v and n_h) using the puncture data [14] or equivelently the method described in [54].

3.2.2. Argyres-Douglas matters. The class of Argyres-Douglas (AD) matters (general $\mathcal{N} = 2$ SCFTs with large flavor symmetry) is huge but as we will see in the next subsection, many of the AD matters that show up in the gauge theory descriptions of our theories from ICISs, fall in the simple subclass of $D_p(G)$ theories[10] [12, 50] and its twisted analog [44].

One type of $D_p(G)$ matter theory that shows up often in our analysis is $D_2SU(N)$, and its basic property is summarized as follows (more details can be found in [12, 50]):

- When $N = 2n$, $D_2SU(N)$ represents the $SU(n)$ SQCD with $2n$ hypermultiplets in the fundamental representation, and we have the quiver description

$$SU(n) \text{ --- } \boxed{2n}$$

with flavor symmetry $SU(N)_{\frac{N}{2}} \times U(1)$.[11]

[10] This is denoted by $(G^{(b)}[k], F)$ in the more general classification of [44] with the restriction $b = h^\vee(G)$ and $p = k + h^\vee(G)$.

[11] For simplicity of notation, we will use subscript to denote the flavor central charge for the global symmetries.

- When $N = 2n+1$, $D_2SU(N)$ is an isolated SCFT with flavor symmetry $SU(N)_{\frac{N}{2}}$. The Coulomb branch spectrum is

$$\frac{N}{2}, \frac{N-2}{2}, \ldots, \frac{3}{2}, \underbrace{1, \ldots, 1}_{N-1}. \tag{8}$$

The conformal central charges are given by

$$a[D_2(SU(2n+1))] = \frac{7}{24}n(n+1), \quad c[D_2(SU(2n+1)] = \frac{1}{3}n(n+1). \tag{9}$$

We also use $D_nSU(n)$ to denote the following quiver tail

$$\boxed{1} \text{------} SU(2) \text{------} SU(3) \text{------} \ldots \text{------} SU(n-1) \text{------} \boxed{n}$$

with flavor symmetry $SU(n)_{n-1} \times U(1)^{n-1}$.

More general Argyres-Douglas matter can be engineered using 6d $J = ADE$ type $(2,0)$ theory on a sphere with one irregular puncture and one regular puncture [44]. These theories are denoted by $(J^{(b)}[k], Y)$ where $J^{(b)}[k]$ specifies the irregular puncture and Y labels the regular puncture (which are certain types of Young-Tableaux for classical Lie groups and Bala-Carter labels for exceptional groups [14, 18]). The simplest cases correspond to $Y = F$ by which we mean the regular puncture is of the full (maximal, principal) type thus enjoys a maximal J flavor symmetry but as we shall see, AD matters from degenerations of the full punctures also show up as building blocks of the $4d$ theories from ICISs.

In addition, we will make use of two matter theories from twisted D-type punctures in [44]. These theories are constructed by introducing a $\mathbb{Z}_2$ outer-automorphism twist in the compactification of D-type $(2,0)$ SCFT on a sphere with one irregular twisted puncture and one full regular twisted puncture. We denote such theories by $(D_n^{(b)}[k], \widetilde{F})$.

- The parameters b, k specifies the irregular singularity and satisfy **i.** $k \in \mathbb{Z}$ and $b = 2n-2$ or **ii.** $k \in \mathbb{Z} + 1/2$ and $b = n$ [44].
- $\widetilde{F}$ indicates that the regular twisted puncture is full (maximal) which gives rise to $USp(2n-2)$ flavor symmetry with central charge $\kappa = n - \frac{b}{2(b+k)}$.
- Moreover for $b = 2n-2$ and k odd, we have an additional $U(1)$ flavor symmetry.

3.3. Examples

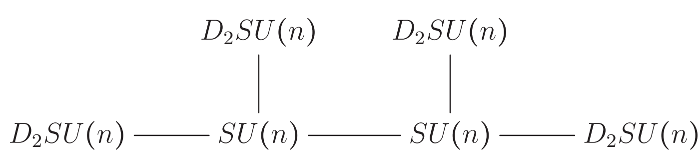
We adopt the labeling of [17] to keep track of the ICISs that we are going to study below. For each case, we list the quiver gauge theory description and compute the conformal central charges. The meaning of the symbols appearing in each example is

- **ICIS** (i) denotes the (i)-th ICIS appearing in [17], and we record the defining equations here.
- $(\boldsymbol{w_1, w_2, w_3, w_4, w_5; 1, d})$ are the weights of the coordinates and degrees of the polynomials. We do not normalize the weights here such that $d \leq 1$.
- $\boldsymbol{\mu}$ denotes the Milnor number of the ICIS, which is also the dimension of the charge lattice of the SCFT.
- $\boldsymbol{\mu_1 + \mu}$ gives the effective number of A_1 singularities.
- $\boldsymbol{\alpha}$ is the scaling dimension near the A_1 singularities.
- $\boldsymbol{r}$ is the dimension of Coulomb branch, namely counting the part of the spectrum with scaling dimension larger than one.
- $\boldsymbol{f}$ denotes the number of mass parameters.
- $\boldsymbol{a, c}$ denote the conformal central charges.

ICIS (1)

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^n = 0 \\ x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 5x_5^n = 0 \end{cases} \qquad n \geq 2$$

$$(w_1, w_2, w_3, w_4, w_5; 1, d) = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{n}; 1, 1\right)$$
$$\mu = -7 + 8n, \quad \mu_1 = n - 1, \quad \alpha = n,$$
$$r = \begin{cases} 4n - 6 & n \in 2\mathbb{Z} \\ 4n - 4 & n \in 2\mathbb{Z} + 1 \end{cases}, \quad f = \begin{cases} 5 & n \in 2\mathbb{Z} \\ 1 & n \in 2\mathbb{Z} + 1 \end{cases},$$
$$a = \begin{cases} \frac{3}{4}n^2 - \frac{5}{4} & n \in 2\mathbb{Z} \\ \frac{3}{4}n^2 - \frac{17}{24} & n \in 2\mathbb{Z} + 1 \end{cases}, \quad c = \begin{cases} \frac{3}{4}n^2 - 1 & n \in 2\mathbb{Z} \\ \frac{3}{4}n^2 - \frac{2}{3} & n \in 2\mathbb{Z} + 1 \end{cases}$$



For illustration, let's discuss how we identify the quiver. To start, we see from the singularity data (using the program Singular [20]) the Coulomb branch spectrum is given by (including positive scaling dimensions only)

$$\begin{aligned} n \in 2\mathbb{Z}+1 &: n, n, n-1, n-1, \ldots, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \\ &\quad \frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1 \\ n \in 2\mathbb{Z} &: n, n, n-1, n-1, \ldots, \frac{n+2}{2}, \frac{n+2}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}, \\ &\quad \frac{n-2}{2}, \frac{n-2}{2}, \ldots, 2, 2, 2, 2, 1, 1, 1, 1, 1 \end{aligned} \tag{10}$$

We see immediately that this is the affine D_5 theory when n is even (Figure 1)

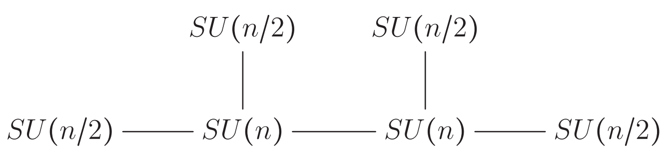
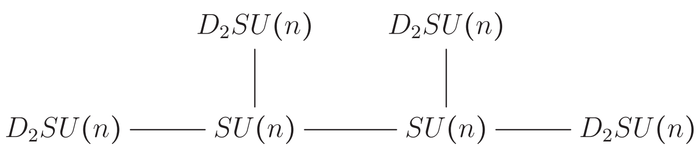


Figure 1: Affine D_5 quiver from ICIS (1) with $n \in 2\mathbb{Z}$

while more generally we have



We can compute the central charges from the spectrum (10) using (16)

$$a = \begin{cases} \frac{3}{4}n^2 - \frac{5}{4} & n \in 2\mathbb{Z} \\ \frac{3}{4}n^2 - \frac{17}{24} & n \in 2\mathbb{Z}+1 \end{cases}, \quad c = \begin{cases} \frac{3}{4}n^2 - 1 & n \in 2\mathbb{Z} \\ \frac{3}{4}n^2 - \frac{2}{3} & n \in 2\mathbb{Z}+1 \end{cases} \tag{11}$$

It's easy to check that they agree with those computed from the quiver description (using (9)).

For most of the examples below we will omit unnecessary details. The interested reader is welcome to follow the strategy outlined in the previous subsections to double check these gauge theory descriptions in comparison to the singularity theory point of view.

ICIS (2)

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^3 + x_5^3 = 0 \\ x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^3 + 5x_5^3 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}; 1, 1)$
$\mu = 32, \quad \mu_1 = 4, \quad \alpha = 6, \quad r = 12, \quad f = 6, \quad a = \frac{473}{24}, \quad c = \frac{121}{6}$

$$SU(2) \text{ —— } SU(4) \text{ —— } D_4\begin{pmatrix} [3,3,1,1] \\ [1,1,\ldots,1] \\ [1,1,\ldots,1]) \end{pmatrix} \text{ —— } SU(4) \text{ —— } SU(2)$$

The middle conformal matter theory $D_4([3,3,1,1],[1,1,\ldots,1]^2)$ is defined in the ordinary type D_4 class S construction, by two full $SO(8)$ punctures and one puncture of the type $[3,3,1,1]$ on S^2. This conformal matter supplies three CB operators $6,4,3$ and flavor symmetry $(SO(8)_6)^2 \times U(1)^2$. Note that the index of embedding $I_{SU(4)\hookrightarrow SO(8)} = 1$ from the decomposition $\mathbf{8} \to \mathbf{4} + \overline{\mathbf{4}}$. Therefore the conformal matter theory supply current central charge $\kappa_{SU(4)} = 6$ for both $SU(4)$ gauge groups, making them conformal

$$\beta_{SU(4)} = 8 - 6 - 2 = 0. \tag{12}$$

The left over flavor symmetry has rank 6 from the quiver which agrees with that from the singularity.

Furthermore, we can compute the conformal central charges from the quiver as follows. The conformal matter theory supplies $n_v = 41, n_h = 72$ (or $a = \frac{277}{24},\ c = \frac{77}{6}$). Including the contributions from the $SU(2)$ and $SU(4)$ vector and bifundamental multiplets we have

$$n_v = 41 + 2(3+15) = 77, \quad n_h = 72 + 2 \times 8 = 88 \tag{13}$$

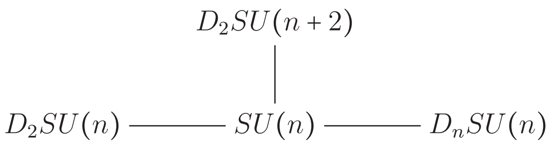
or

$$a = \frac{473}{24}, \quad c = \frac{121}{6}. \tag{14}$$

ICIS (7)

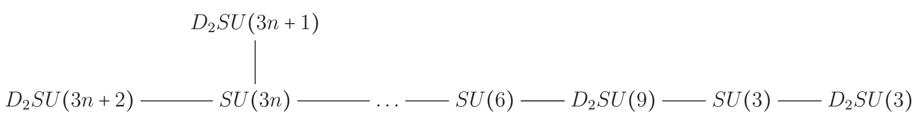
$$\begin{cases} x_1x_3 + x_4x_5 = 0 \\ x_1^2 + x_2^2 + x_3^n + x_4^2 + 2x_5^n = 0 \end{cases} \qquad n \geq 3$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{n}{2+n}, \frac{n}{2+n}, \frac{2}{2+n}, \frac{n}{2+n}, \frac{2}{2+n}; 1, \frac{2n}{2+n})$

$$\mu=(n+1)^2,\quad \mu_1=(n-1)^2,\quad \alpha=\tfrac{n-2}{2},$$
$$r=\begin{cases}\frac{n(n+1)-2}{2} & n\in 2\mathbb{Z}\\ \frac{n(n+1)}{2} & n\in 2\mathbb{Z}+1\end{cases},\quad f=\begin{cases}n+3 & n\in 2\mathbb{Z}\\ n+1 & n\in 2\mathbb{Z}+1\end{cases}$$
$$a=\begin{cases}\frac{(n+2)(2n-1)(2n+3)}{48} & n\in 2\mathbb{Z}\\ \frac{(n+2)(2n-1)(2n+3)+13}{48} & n\in 2\mathbb{Z}+1\end{cases},\quad c=\begin{cases}\frac{n(n+1)(n+2)}{12} & n\in 2\mathbb{Z}\\ \frac{n(n+1)(n+2)+2}{12} & n\in 2\mathbb{Z}+1\end{cases}$$



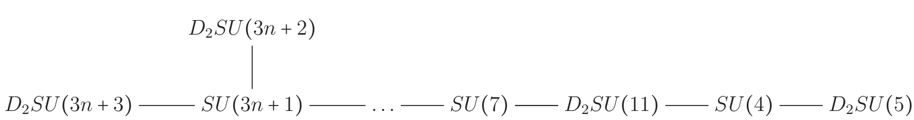
ICIS (8)

$$\begin{cases}x_1x_3+x_4x_5=0\\ x_1^2+x_2^2+x_3x_4^2+x_3^{3n}+2x_5^{2n}=0\end{cases}\qquad n\geq 2$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=\left(\tfrac{3n}{2+3n},\tfrac{3n}{2+3n},\tfrac{2}{2+3n},\tfrac{-1+3n}{2+3n},\tfrac{3}{2+3n};1,\tfrac{6n}{2+3n}\right)$$
$$\mu=3+7n+6n^2,\quad \mu_1=(3n+1)(2n-1),\quad \alpha=\tfrac{3n+2}{2},\quad r=3n(n+1),$$
$$f=n+3,\quad a=\tfrac{3}{8}n(4n^2+6n+3)+\tfrac{1}{12},\quad c=\tfrac{1}{4}n\left(6n^2+9n+5\right)+\tfrac{1}{6}$$



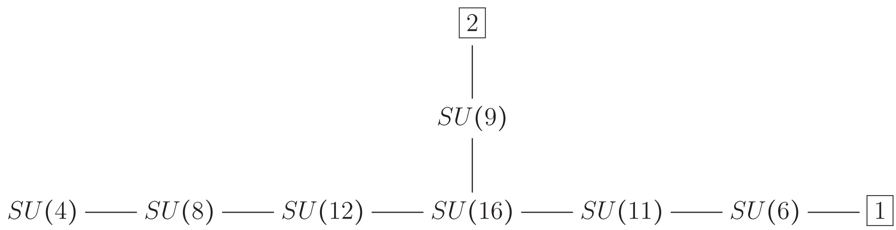
ICIS (9)

$$(9)\begin{cases}x_1x_3+x_4x_5=0\\ x_1^2+x_2^2+x_3x_4^2+x_3^{1+3n}+2x_3x_5^{2n}=0\end{cases}\qquad n\geq 1$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=\left(\tfrac{1+3n}{3(1+n)},\tfrac{1+3n}{3(1+n)},\tfrac{2}{3(1+n)},\tfrac{n}{1+n},\tfrac{1}{1+n};1,\tfrac{2(1+3n)}{3(1+n)}\right)$$
$$\mu=6+11n+6n^2,\quad \mu_1=6n^2+3n-\tfrac{2}{3},\quad \alpha=\tfrac{3(n+1)}{2},\quad r=n(3n+5)+1,$$
$$f=n+4,\quad a=\tfrac{1}{8}n(6n(2n+5)+25)+\tfrac{2}{3},\quad c=\tfrac{1}{4}n(3n(2n+5)+13)+\tfrac{5}{6}.$$



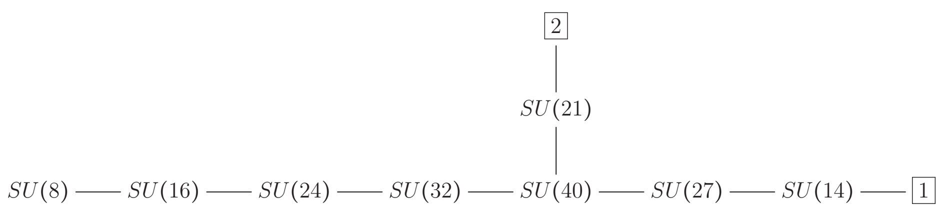
ICIS (10)

$$\begin{cases} x_1x_3+x_4x_5=0 \\ x_1^2+x_2^2+x_3x_4^3+x_3^{16}+x_5^4=0 \end{cases}$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=(\tfrac{8}{9},\tfrac{8}{9},\tfrac{1}{9},\tfrac{5}{9},\tfrac{4}{9};1,\tfrac{16}{9})$$
$$\mu=127,\quad \mu_1=99,\quad \alpha=9,\quad r=59,\quad f=9,\quad a=\tfrac{4285}{24},\quad c=\tfrac{538}{3}.$$



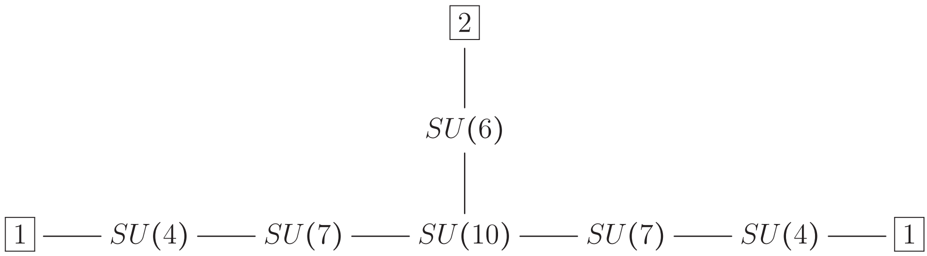
ICIS (11)

$$\begin{cases} x_1x_3+x_4x_5=0 \\ x_1^2+x_2^2+x_3x_4^3+x_3^{40}+x_5^5=0 \end{cases}$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=(\tfrac{20}{21},\tfrac{20}{21},\tfrac{1}{21},\tfrac{13}{21},\tfrac{8}{21};1,\tfrac{40}{21})$$
$$\mu=358,\quad \mu_1=324,\quad \alpha=21,\quad r=174,\quad f=10,\quad a=1221,\quad c=\tfrac{2445}{2}.$$



ICIS (12)

$$\begin{cases} x_1x_3+x_4x_5=0 \\ x_1^2+x_2^2+x_3x_4^3+x_3^{10}+x_3x_5^3=0 \end{cases}$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=(\tfrac{5}{6},\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{2},\tfrac{1}{2};1,\tfrac{5}{3})$$
$$\mu=73,\quad \mu_1=49,\quad \alpha=6,\quad r=32,\quad f=9,\quad a=\tfrac{197}{3},\quad c=\tfrac{199}{3}.$$

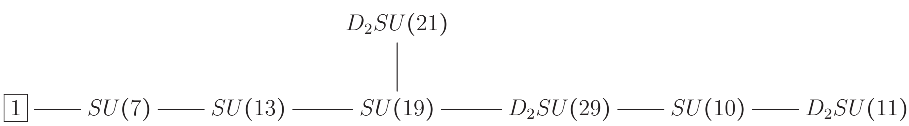


ICIS (13)

$$\begin{cases} x_1x_3 + x_4x_5 = 0 \\ x_1^2 + x_2^2 + x_3x_4^3 + x_3^{19} + x_3x_5^4 = 0 \end{cases}$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{19}{21}, \frac{19}{21}, \frac{2}{21}, \frac{4}{7}, \frac{3}{7}; 1, \frac{38}{21})$

$\mu = 155, \quad \mu_1 = \frac{377}{3}, \quad \alpha = \frac{21}{2}, \quad r = 74, \quad f = 7, \quad a = \frac{3085}{12}, \quad c = \frac{3095}{12}.$

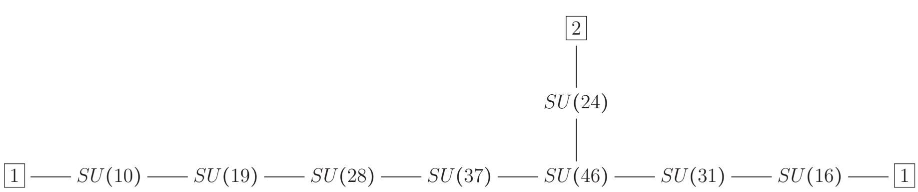


ICIS (14)

$$\begin{cases} x_1x_3 + x_4x_5 = 0 \\ x_1^2 + x_2^2 + x_3x_4^3 + x_3^{46} + x_3x_5^5 = 0 \end{cases}$$

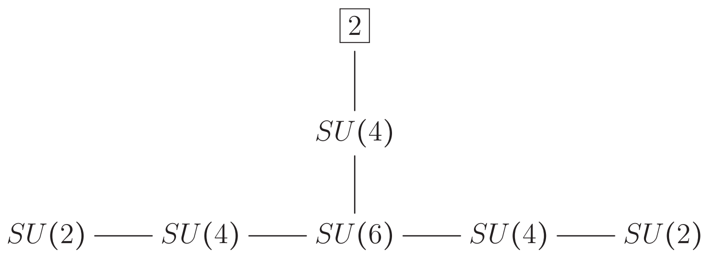
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{23}{24}, \frac{23}{24}, \frac{1}{24}, \frac{5}{8}, \frac{3}{8}; 1, \frac{23}{12})$

$\mu = 417, \quad \mu_1 = \frac{1147}{3}, \quad \alpha = 24, \quad r = 203, \quad f = 11, \quad a = \frac{13045}{8}, \quad c = \frac{3265}{2}.$



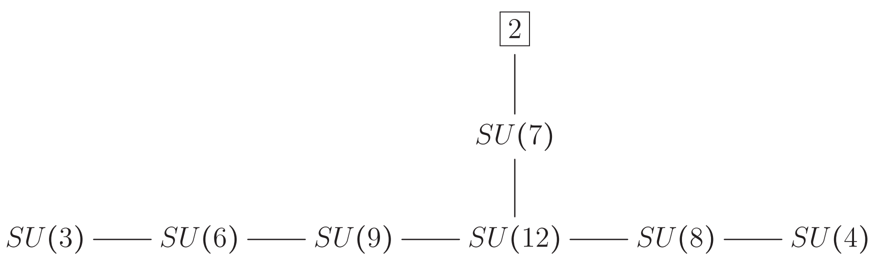
ICIS (15)

$$\begin{cases} x_1x_3 + x_4x_5 = 0 \\ x_1^2 + x_2^2 + x_4^3 + x_3^6 + x_5^3 = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}; 1, \tfrac{3}{2})$$
$$\mu = 39, \quad \mu_1 = 20, \quad \alpha = 4, \quad r = 16, \quad f = 7, \quad a = \tfrac{267}{12}, \quad c = \tfrac{67}{3}.$$



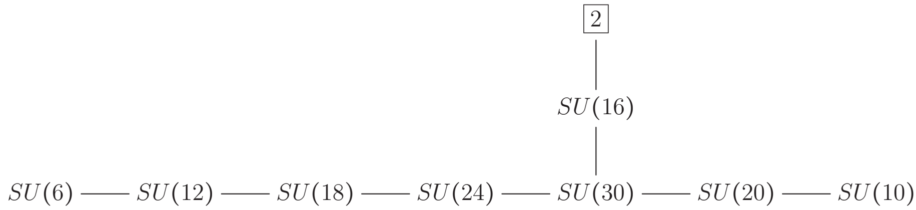
ICIS (16)

$$\begin{cases} x_1x_3 + x_4x_5 = 0 \\ x_1^2 + x_2^2 + x_4^3 + x_3^{12} + x_5^4 = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{6}{7}, \tfrac{6}{7}, \tfrac{1}{7}, \tfrac{4}{7}, \tfrac{3}{7}; 1, \tfrac{12}{7})$$
$$\mu = 92, \quad \mu_1 = 66, \quad \alpha = 7, \quad r = 42, \quad f = 8, \quad a = \tfrac{1183}{12}, \quad c = \tfrac{595}{6}.$$



ICIS (17)

$$\begin{cases} x_1x_3 + x_4x_5 = 0 \\ x_1^2 + x_2^2 + x_4^3 + x_3^{30} + x_5^5 = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{15}{16}, \tfrac{15}{16}, \tfrac{1}{16}, \tfrac{5}{8}, \tfrac{3}{8}; 1, \tfrac{15}{8})$$
$$\mu = 265, \quad \mu_1 = 232, \quad \alpha = 16, \quad r = 128, \quad f = 9, \quad a = 683, \quad c = 684.$$

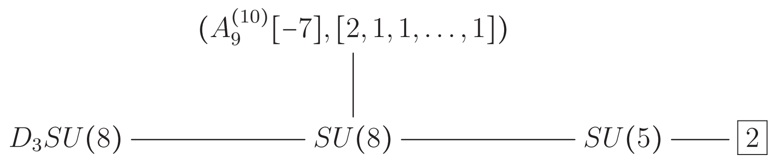


ICIS (18)

$$\begin{cases} x_1x_3 + x_4x_5 = 0 \\ x_1^2 + x_2^2 + x_4^3 + x_3^8 + x_3x_5^3 = 0 \end{cases}$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{8}{15}, \frac{7}{15}; 1, \frac{8}{5})$

$\mu = 56, \quad \mu_1 = 34, \quad \alpha = 5, \quad r = 26, \quad f = 4, \quad a = \frac{83}{2}, \quad c = \frac{251}{6}.$



The theory $(A_9^{(10)}[-7], [2, 1^6])$ is defined by six dimensional A_9 $(2, 0)$ theory on a sphere with irregular puncture $A_9^{(10)}[-7]$ and a regular puncture labeled by Young Tableaux $[2, 1, \ldots, 1]$. This theory has flavor symmetry $SU(8) \times U(1)$, and the flavor central charge of $SU(8)$ is $\frac{17}{3}$. The Coulomb branch spectrum is $(\frac{17}{3}, \frac{14}{3}, \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charges are $a = \frac{83}{12}$, $c = \frac{15}{2}$. The theory $D_3SU(8) = (A_7^{(8)}[-5], [1, 1, \ldots, 1])$ has flavor symmetry $SU(8)$, and the flavor central charge is $\frac{16}{3}$. The Coulomb branch spectrum of this theory is $(\frac{16}{3}, \frac{13}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$. Using the above data, one can check that the gauging of the $SU(8)$ gauge group is conformal,

$$\beta_{SU(8)} = 16 - \frac{16}{3} - \frac{17}{3} - 5 = 0 \tag{15}$$

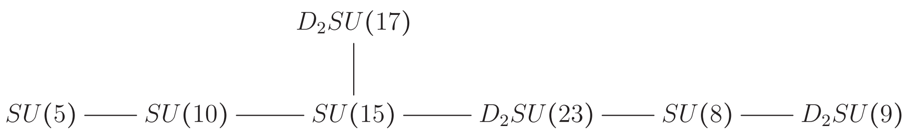
Futhermore, using (16), we find the central charges for $(A_9^{(10)}[-7], [2, 1^6])$ is $a = \frac{35}{3}$, $c = \frac{38}{3}$; while for $D_3SU(8)$ we have $a = \frac{77}{8}$, $c = \frac{21}{2}$, hence the central charges of the quiver is

$$(16)\qquad \begin{aligned} a &= \frac{35}{3}+\frac{77}{8}+\frac{5(24+63)+50}{24}=\frac{83}{2},\\ c &= \frac{38}{3}+\frac{21}{2}+\frac{2(24+63)+50}{12}=\frac{251}{6} \end{aligned}$$

which agrees with that from the singularity.

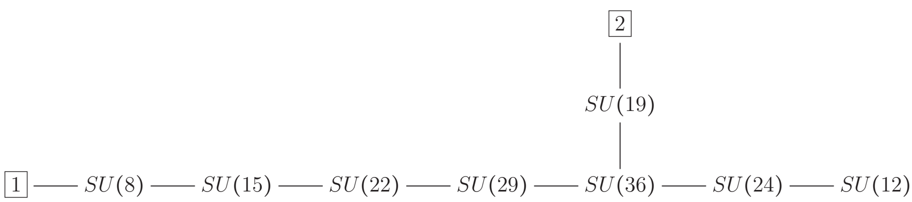
ICIS (19)

$$\begin{cases} x_1x_3+x_4x_5=0\\ x_1^2+x_2^2+x_4^3+x_3^{15}+x_3x_5^4=0 \end{cases}$$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{15}{17},\frac{15}{17},\frac{2}{17},\frac{10}{17},\frac{7}{17};1,\frac{30}{17})$
$\mu=120,\quad \mu_1=92,\quad \alpha=\frac{17}{2},\quad r=57,\quad f=6,\quad a=\frac{1909}{12},\quad c=\frac{479}{3}.$



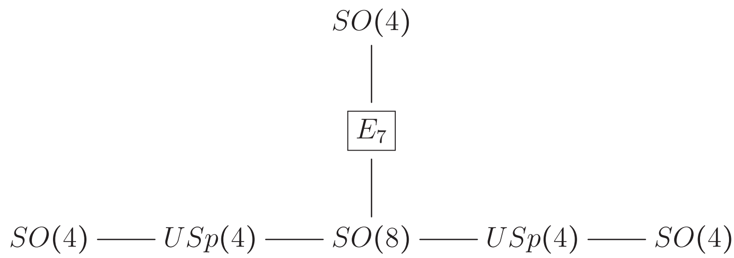
ICIS (20)

$$\begin{cases} x_1x_3+x_4x_5=0\\ x_1^2+x_2^2+x_4^3+x_3^{36}+x_3x_5^5=0 \end{cases}$$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{18}{19},\frac{18}{19},\frac{1}{19},\frac{12}{19},\frac{7}{19};1,\frac{36}{19})$
$\mu=324,\quad \mu_1=290,\quad \alpha=19,\quad r=157,\quad f=10,\quad a=\frac{23929}{24},\quad c=\frac{2995}{3}.$



ICIS (41)

$$\begin{cases} x_2^2+x_3^2+x_4^2+x_5^2=0\\ x_1^2+x_2^3+x_3^3+x_4^3=0 \end{cases}$$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{3}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2})$
$\mu=31,\quad \mu_1=16,\quad \alpha=4,\quad r=15,\quad f=1,\quad a=\frac{437}{24},\quad c=\frac{109}{6}.$



Here the conformal matter is provided by the E_7 Minahan-Nemeschansky theory with a single coulomb branch chiral primary of dimension 4 and $(E_7)_4$ flavor symmetry [36]. From the decomposition of $\mathbf{56}$ under $SO(8) \subset E_7$,

$$\mathbf{56} \to 8 \cdot \mathbf{1} + 2 \cdot \mathbf{8}_v + 2 \cdot \mathbf{8}_s + 2 \cdot \mathbf{8}_c \tag{17}$$

the embedding index

$$I_{SO(8) \hookrightarrow E_7} = \frac{2T(\mathbf{8}_v) + 2T(\mathbf{8}_s) + 2T(\mathbf{8}_c)}{T(\mathbf{56})} = \frac{6 \cdot 1}{6} = 1 \tag{18}$$

Hence the E_7 theory supplies $SO(8)_4$ for the $SO(8)$ gauge coupling, leading to conformal gauging, similarly for the $SO(4)$ gauge node.

The central charges can also be verified. The E_7 theory's central charges are characterized by $n_v = 7$ and $n_h = 24$. Together with the rest of the quiver, we have

$$n_v = 7 + 3 \cdot 6 + 2 \cdot 10 + 28 = 73, \quad n_h = 24 + \frac{2 \cdot 16 + 2 \cdot 32}{2} = 72 \tag{19}$$

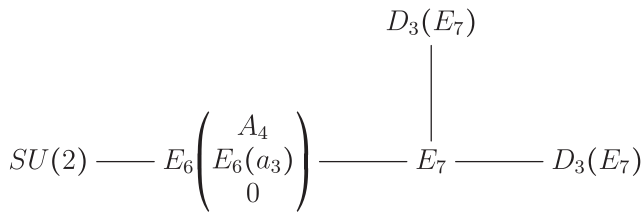
or

$$a = \frac{437}{24}, \quad c = \frac{109}{6} \tag{20}$$

in agreement with the singularity.

ICIS (42)

$$\begin{cases} x_2^2 + x_3^2 + x_4^2 + x_5^3 = 0 \\ x_1^2 + x_2^3 + x_3^3 + x_4^3 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}; 1, \frac{3}{2})$
$\mu = 54, \quad \mu_1 = 28, \quad \alpha = 12, \quad r = 27, \quad f = 0, \quad a = \frac{689}{8}, \quad c = \frac{173}{2}.$



Here the $\mathcal{N}=2$ SCFT in the weakly coupled frame is described by E_7 and $SU(2)$ $\mathcal{N}=2$ vector multiplets coupled conformally to matter of the $D_p(G)$ type and Gaiotto type.

The $E_6(A_4, E_6(a_3), 0)$ theory has Coulomb branch spectrum $(12, 8, 6, 4, 3)$ and flavor symmetry $(E_7)_{12} \times SU(2)_4$ [16]. It's conformal central charges are $a = \frac{139}{8}$, $c = \frac{39}{2}$ (or $n_v = 61,\ n_h = 112$).

The $D_3(E_7)$ theory has Coulomb branch $(12, 8, 6, 6, 4, 2, 2)$ and flavor symmetry $(E_7)_{12}$. It's conformal central charges are $a = \frac{485}{24}$, $c = \frac{133}{6}$ (or $n_v = 73,\ n_h = 120$).

We can check immediately that the gauge groups are conformal

$$\beta_{E_7} = 2 \cdot 18 - 2 \cdot 12 - 12 = 0 \tag{21}$$

similarly for the $SU(2)$ node.

The conformal central charges can also be verified. The E_7 theory's central charges are characterized by $n_v = 7$ and $n_h = 24$. Together with the rest of the quiver, we have

$$n_v = 3 + 61 + 133 + 2 \cdot 73 = 343, \quad n_h = 112 + 2 \cdot 120 = 352 \tag{22}$$

or

$$a = \frac{689}{8}, \quad c = \frac{173}{2}. \tag{23}$$

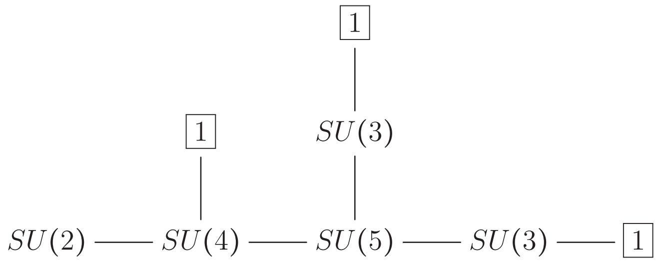
in agreement with the singularity.

ICIS (55)

$$\begin{cases} x_1x_2 + x_3^2 + x_4^2 + x_5^2 = 0 \\ x_1x_3 + 2x_2^5 + x_2x_4^2 = 0 \end{cases}$$

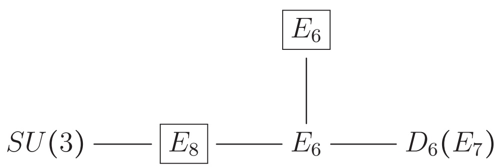
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{5}{4})$

$\mu = 31, \quad \mu_1 = 9, \quad \alpha = 4, \quad r = 12, \quad f = 7, \quad a = \frac{179}{12}, \quad c = \frac{46}{3}.$



ICIS (56)

$\begin{cases} x_1x_2 + x_3^2 + x_4^2 + x_5^3 = 0 \\ 2x_1x_3 + x_2^5 + x_2x_4^2 = 0 \end{cases}$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}; 1, \frac{5}{4})$
$\mu = 56, \quad \mu_1 = \frac{33}{2}, \quad \alpha = 12, \quad r = 27, \quad f = 2, \quad a = \frac{613}{8}, \quad c = 77.$



The conformal matter theories here are provided by E_6 and E_8 Minahan-Nemeschansky theories which have flavor symmetry $(E_6)_3$ and $(E_8)_6$ respectively [35, 36]., as well as an AD matter theory $D_6(E_7)$.

The $D_6(E_7)$ theory has Coulomb branch spectrum $(15, 12, 11, 9, 9, 8, 7, 6, 6, 5, 5, 3, 3)$ and flavor symmetry $(E_7)_{15} \times U(1)$. Its central charges are $a = \frac{1273}{24},\ c = \frac{166}{3}$ (or $n_v = 203,\ n_h = 258$).

Since index of embedding are $I_{E_6 \hookrightarrow E_8} = I_{E_6 \hookrightarrow E_7} = I_{SU(3) \hookrightarrow E_8} = 1$, it's easy to verify that the gauge couplings are finite. The central charges can also be matched with the singularity: together with E_6 theory $(n_v = 5,\ n_h = 16)$ and E_8 theory $(n_v = 11,\ n_h = 40)$,

$$n_v = 8 + 11 + 5 + 78 + 203 = 305, \quad n_h = 40 + 16 + 258 = 314 \tag{24}$$

or

$$a = \frac{613}{8}, \quad c = 77. \tag{25}$$

ICIS (57)

$$\begin{cases} x_1x_2 + x_3^2 + x_4^2 + x_5^4 = 0 \\ x_1x_3 + x_2^3 + x_2x_5^3 = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{5}{8}, \tfrac{3}{8}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}; 1, \tfrac{9}{8})$$
$$\mu = 43, \quad \mu_1 = \tfrac{35}{4}, \quad \alpha = 8, \quad r = 19, \quad f = 5, \quad a = \tfrac{893}{24}, \quad c = \tfrac{113}{3}.$$

$$\boxed{1} — SU(5) — A_8\begin{pmatrix} [1,1,\ldots,1] \\ [5,4] \\ [3,3,3] \end{pmatrix} — SU(7) — SU(5) — SU(3) — \boxed{1}$$

Here the conformal matter is a Gaiotto type theory $A_8([1,1,\ldots,1], [5,4],[3,3,3])$ with Coulomb branch spectrum $(9,8,6)$ and flavor symmetry $SU(12)_9$. Note that from the regular punctures, one can immediately read off the subgroup $SU(9)_9 \times SU(3)_9 \times U(1)$. The flavor symmetry enhancement can be seen from its $3d$ mirror in Figure 2.

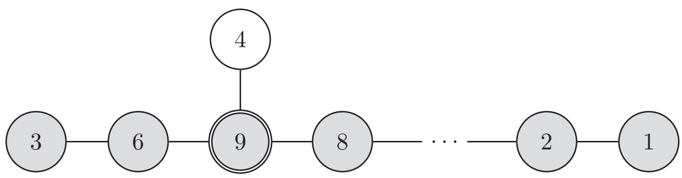


Figure 2: $3d\,\mathcal{N}=4$ mirror of $A_8([1,1,\ldots,1],[5,4],[3,3,3])$. The circle nodes denotes $U(n)$ gauge groups and the double circle node denotes an $SU(n)$ gauge group. The *balanced* nodes are shaded.

Recall that the enhancement is due to $3d$ monopole operators hitting the unitarity bound. This happens when the corresponding $3d$ quiver node is *balanced*, meaning $N_f - 2N_c = 0$ [27]. In the above $3d$ mirror quiver, we see that the balanced nodes form the Dynkin diagram of A_{11} thus the enhanced flavor symmetry.

ICIS (58)

$$\begin{cases} x_1x_2 + x_3^2 + x_4^2 + x_5^3 = 0 \\ x_1x_3 + x_2^2x_4 + x_4x_5^2 = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{3}; 1, \tfrac{7}{6})$$
$$\mu = 36, \quad \mu_1 = \tfrac{25}{3}, \quad \alpha = 6, \quad r = 15, \quad f = 6, \quad a = \tfrac{581}{24}, \quad c = \tfrac{74}{3}.$$

$$\boxed{1} — SU(3) — SU(5) — A_6\begin{pmatrix}[1,1,\ldots,1]\\ [4,3]\\ [2,2,2,1]\end{pmatrix} — SU(5) — SU(3) — \boxed{1}$$

Here the conformal matter is a Gaiotto type theory $A_6([1,1,\ldots,1],[4,3],[2,2,2,1])$ with Coulomb branch spectrum $(7,6,4)$ and (enhanced) flavor symmetry $SU(10)_7 \times U(1)$ which can be seen from the $3d$ mirror in Figure 3.

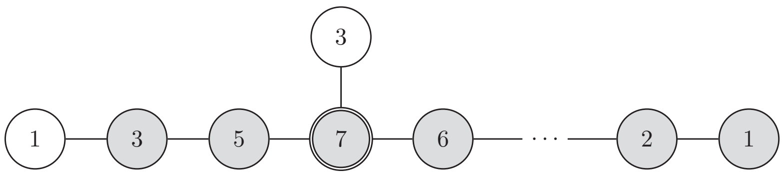


Figure 3: $3d$ $\mathcal{N}=4$ mirror of $A_6([1,1,\ldots,1],[4,3],[2,2,2,1])$. The *balanced* nodes are shaded.

ICIS (59)

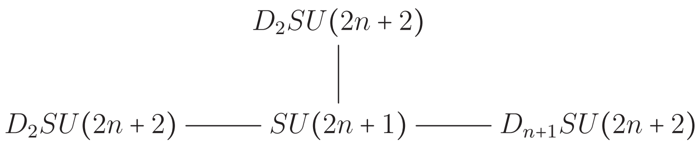
$$\begin{cases} x_1x_2 + x_3^2 + x_4^2 + x_5^n = 0 \\ x_1x_5 + 2x_3^2 + x_4^2 + 3x_2^n = 0 \end{cases} \quad n \geq 3$$
$$(w_1,w_2,w_3,w_4,w_5;1,d) = (\tfrac{-1+n}{n}, \tfrac{1}{n}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{n}; 1, 1)$$
$$\mu = -3+4n+n^2, \quad \mu_1 = n-1, \quad \alpha = n,$$
$$r = \begin{cases} \frac{1}{2}(n(n+3)-6) & n \in 2\mathbb{Z} \\ \frac{1}{2}(n-1)(n+4) & n \in 2\mathbb{Z}+1 \end{cases}, \quad f = \begin{cases} n+3 & n \in 2\mathbb{Z} \\ n+1 & n \in 2\mathbb{Z}+1 \end{cases},$$
$$a = \begin{cases} \frac{(n(4n(n+6)-7)-30)}{48} & n \in 2\mathbb{Z} \\ \frac{(n(4n(n+6)-7)-17)}{48} & n \in 2\mathbb{Z}+1 \end{cases}, \quad c = \begin{cases} \frac{(n-1)(n+1)(n+6)}{12} & n \in 2\mathbb{Z} \\ \frac{(n-1)(n+1)(n+6)+2}{12} & n \in 2\mathbb{Z}+1 \end{cases}$$

$$\begin{array}{ccc} & D_2SU(n) & \\ & | & \\ D_2SU(n) — & SU(n) & — D_{n+1}SU(n+1) \end{array}$$

ICIS (60)

$$\begin{cases} x_1x_2 + x_3^2 + x_4^2 + x_5^n = 0 \\ x_1x_5 + 3x_2^{1+2n} + 2x_2x_3^2 + x_2x_4^2 = 0 \end{cases} \quad n \geq 3$$
$$(w_1,w_2,w_3,w_4,w_5;1,d) = (\tfrac{-1+2n}{2n}, \tfrac{1}{2n}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{n}; 1, \tfrac{1+2n}{2n})$$

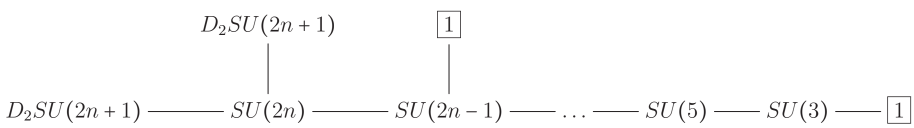
$\mu = 5 + 9n + 2n^2, \quad \mu_1 = \frac{2(n+1)^2}{n}, \quad \alpha = 2n, \quad r = n^2 + 4n, \quad f = n + 5,$
$a = \frac{n(8n(n+6)+49)+4}{24}, \quad c = \frac{n(2n(n+6)+13)+2}{6}.$



ICIS (74)

$$\begin{cases} x_1x_2 + x_3^2 + x_4^2 = 0 \\ x_1x_5 + x_2^{2n} + x_5^n + 2x_2x_3^2 + x_2x_4^2 = 0 \end{cases} \qquad n \geq 3$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{2(-1+n)}{-1+2n}, \frac{1}{-1+2n}, \frac{1}{2}, \frac{1}{2}, \frac{2}{-1+2n}; 1, \frac{2n}{-1+2n})$
$\mu = 1 + 7n + 2n^2, \quad \mu_1 = \frac{?2n+1)^2}{2n-1}, \quad \alpha = 2n - 1, \quad r = n(n+3) - 1,$
$f = n + 3, \quad a = \frac{n(4n(2n+9)+7)-5}{24}, \quad c = \frac{n(2n(2n+9)+5)-2}{12}.$



ICIS (113)

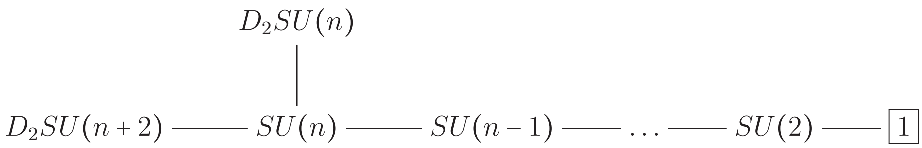
$$\begin{cases} x_1x_2 + x_4x_5 = 0 \\ x_1x_4 + x_3^2 + x_2^{n_2} + x_5^{n_5} = 0 \end{cases} \qquad n_2 \geq 3 \textit{ and } 2 \leq n_5 \leq n_2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \big(\frac{n_2-n_5+n_2n_5}{n_2+n_5+n_2n_5}, \frac{2n_5}{n_2+n_5+n_2n_5}, \frac{n_2n_5}{n_2+n_5+n_2n_5}, \frac{-n_2+n_5+n_2n_5}{n_2+n_5+n_2n_5},$
$\frac{2n_2}{n_2+n_5+n_2n_5}; 1, \frac{2n_2n_5}{n_2+n_5+n_2n_5}\big)$

For $\boldsymbol{n_2 = n_5 = n}$. $\mu = (n+1)^2, \quad \mu_1 = (n-1)^2, \quad \alpha = \frac{n+2}{2},$

$$r = \begin{cases} \frac{1}{2}(n^2 + n - 2) & n \in 2\mathbb{Z} \\ \frac{1}{2}n(n+1) & n \in 2\mathbb{Z}+1 \end{cases}, \quad f = \begin{cases} n+3 & n \in 2\mathbb{Z} \\ n+1 & n \in 2\mathbb{Z}+1 \end{cases},$$

$$a = \begin{cases} \frac{(n+2)(2n-1)(2n+3)}{48} & n \in 2\mathbb{Z} \\ \frac{n(2n+1)(2n+5)+7}{48} & n \in 2\mathbb{Z}+1 \end{cases}, \quad c = \begin{cases} \frac{n(n+1)(n+2)}{12} & n \in 2\mathbb{Z} \\ \frac{n(n+1)(n+2)+2}{12} & n \in 2\mathbb{Z}+1 \end{cases}.$$

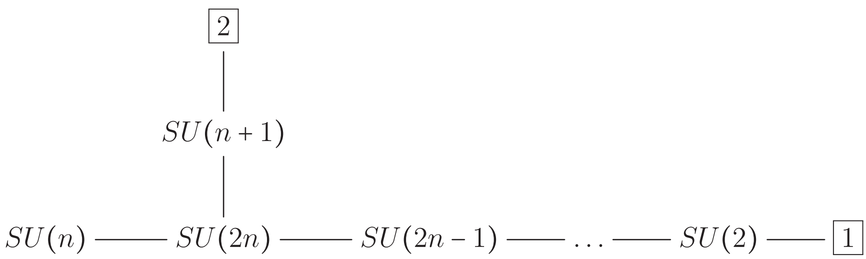


ICIS (114)

$$\begin{cases} x_1x_2 + x_4x_5 = 0 \\ x_1x_4 + x_3^2 + x_2^{n_2} + x_1x_5^{n_5} = 0 \end{cases} \qquad n_2 \geq 3 \textit{ and } 2 \leq 2n_5 \leq n_2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \left(\frac{n_2 - n_5 + n_2n_5}{(1+n_2)(1+n_5)}, \frac{1+2n_5}{(1+n_2)(1+n_5)}, \frac{n_2+2n_2n_5}{2(1+n_2)(1+n_5)}, \frac{n_5}{1+n_5}, \frac{1}{1+n_5};\right.$
$\left.1, \frac{n_2+2n_2n_5}{(1+n_2)(1+n_5)}\right)$.

For $2n_5 = n_2 = 2n$. $\mu = (1+2n)^2$, $\mu_1 = (2n-1)^2$, $\alpha = n+1$, $r = 2n^2 + n - 1$, $f = 2n+3$, $a = \frac{(n+1)(4n-1)(4n+3)}{24}$, $c = \frac{n(2n+1)(n+1)}{3}$.



Note that this is a special case of ICIS (113) for $n_2 = n_5 = 2n$.

ICIS (115)

$$\begin{cases} x_1x_2 + x_4x_5 = 0 \\ x_1x_4 + x_3^2 + x_2^{n_2}x_4 + x_5^{n_5} = 0 \end{cases} \qquad n_2 \geq 2 \textit{ and } 2 \leq n_5 \leq 2n_2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \left(\frac{n_2}{1+n_2}, \frac{1}{1+n_2}, \frac{n_5+2n_2n_5}{2(1+n_2)(1+n_5)}, \frac{-n_2+n_5+n_2n_5}{(1+n_2)(1+n_5)},\right.$
$\left.\frac{1+2n_2}{(1+n_2)(1+n_5)}; 1, \frac{n_5+2n_2n_5}{(1+n_2)(1+n_5)}\right)$, $\mu = 1 + 2(1+n_2)n_5$.

For $n_5 = 2n_2 = 2n$ this coincides with ICIS (114).

ICIS (116)

$$\begin{cases} x_1x_2 + x_4x_5 = 0 \\ x_1x_4 + 2x_3^2 + 3x_2^{n_2}x_4 + 4x_1x_5^{n_5} + 5x_2^{n_2}x_5^{n_5} = 0 \end{cases} \quad n_2 \geq 2 \textit{ and } 1 \leq n_5 \leq n_2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \big(\frac{n_2}{1+n_2}, \frac{1}{1+n_2}, \frac{n_2+n_5+2n_2n_5}{2(1+n_2)(1+n_5)}, \frac{n_5}{1+n_5}, \frac{1}{1+n_5};$ $1, \frac{n_2+n_5+2n_2n_5}{(1+n_2)(1+n_5)}\big)$, $\quad \mu = (1+2n_2)(1+2n_5)$.

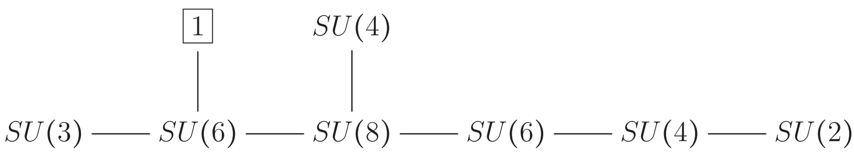
For $n_5 = n_2 = n$ this coincides with ICIS (114).

ICIS (118)

$$\begin{cases} x_1x_2 + x_4^2 + x_5^3 = 0 \\ x_1x_4 + x_3^2 + x_2^n = 0 \end{cases} \quad n \geq 3$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \big(\frac{-1+2n}{2(1+n)}, \frac{3}{2(1+n)}, \frac{3n}{4(1+n)}, \frac{1}{2}, \frac{1}{3}; 1, \frac{3n}{2(1+n)}\big)$

For $n = 8$. $\quad \mu = 59, \quad \mu_1 = 21, \quad \alpha = 6, \quad r = 26, \quad f = 7,$ $a = \frac{527}{12}, \quad c = \frac{133}{3}$.

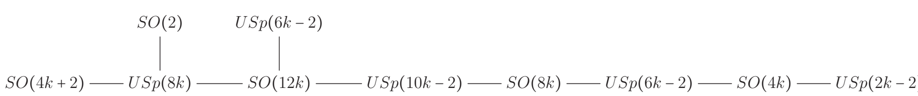


ICIS (119)

$$\begin{cases} x_1x_2 + x_4^2 + x_5^4 = 0 \\ x_1x_4 + x_3^2 + x_2^n = 0 \end{cases} \quad n \geq 3$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \big(\frac{-1+2n}{2(1+n)}, \frac{3}{2(1+n)}, \frac{3n}{4(1+n)}, \frac{1}{2}, \frac{1}{4}; 1, \frac{3n}{2(1+n)}\big)$

For $n = 6k - 1$. $\quad \mu = 60k - 5, \quad \mu_1 = 30k - 16 + \frac{2}{k}, \quad \alpha = \frac{6k-1}{4k},$ $r = 3(10k-3), \quad f = 1, \quad a = 60k^2 - \frac{37k}{4} + \frac{25}{24}, \quad c = 60k^2 - 9k + \frac{5}{6}$.



ICIS (143)

$$\begin{cases} x_1x_2 + x_4^2 + x_5^{n_5} = 0 \\ x_1x_5 + x_3^2 + x_2^{n_2} = 0 \end{cases} \quad n_2 \geq 3 \textit{ and } 3 \leq n_5 \leq n_2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \big(\frac{-1+n_2n_5}{(1+n_2)n_5}, \frac{1+n_5}{(1+n_2)n_5}, \frac{n_2+n_2n_5}{2(1+n_2)n_5}, \frac{1}{2}, \frac{1}{n_5}; 1, \frac{n_2+n_2n_5}{(1+n_2)n_5}\big)$

$\mu = -3 + 2n_5 + n_2(2+n_5)$

For $n_5 = n_2 = k$ this is the same as ICIS (59).

ICIS (144)

$$\begin{cases} x_1x_2 + x_4^2 + x_5^{n_5} = 0 \\ x_1x_5 + x_3^2 + x_2^{n_2}x_4 = 0 \end{cases} \qquad n_2 \geq 2 \textit{ and } 3 \leq n_5 \leq 2n_2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = \left(\frac{-2+n_5+2n_2n_5}{2(1+n_2)n_5}, \frac{2+n_5}{2(1+n_2)n_5}, \frac{2n_2+n_5+2n_2n_5}{4(1+n_2)n_5}, \frac{1}{2}, \frac{1}{n_5};\right.$ $\left.1, \frac{2n_2+n_5+2n_2n_5}{2(1+n_2)n_5}\right), \quad \mu = 3(-1+n_5) + 2n_2(1+n_5)$

For $n_5 = 2n_2 = 2k$ this is a subclass of ICIS (59) with $n = 2k$.

ICIS (152)

$$\begin{cases} x_1x_2 + x_4^3 + x_5^3 = 0 \\ x_1x_4 + x_3^2 + x_2^3 = 0 \end{cases} \qquad n = 3$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}; 1, 1)$

$\mu = 34, \quad \mu_1 = 4, \quad \alpha = 6, \quad r = 16, \quad f = 2, \quad a = \frac{259}{12}, \quad c = \frac{65}{3}$

$$\begin{array}{ccccccccccccc} & & & & USp(2) & & SO(2) & & & & & & \\ & & & & | & & | & & & & & & \\ SO(4) & - & USp(4) & - & SO(8) & - & USp(6) & - & SO(6) & - & USp(2) & - & SO(2) \end{array}$$

ICIS (154)

$$\begin{cases} x_1x_2 + x_4^3 + x_5^3 = 0 \\ x_1x_4 + x_3^2 + x_2^n x_5 = 0 \end{cases} \qquad n \geq 2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{n}{1+n}, \frac{1}{1+n}, \frac{1+4n}{6(1+n)}, \frac{1}{3}, \frac{1}{3}; 1, \frac{1+4n}{3(1+n)})$

For $n = 3k - 1$. $\mu = 2(18k - 1), \quad \mu_1 = 12k + \frac{2}{k} - 10, \quad \alpha = 6k,$ $r = 18k - 2 \quad f = 2, \quad a = 24k^2 - \frac{13}{4}k + \frac{5}{6}, \quad c = 24k^2 - 3k + \frac{2}{3}$

$$\begin{array}{ccccccccccccccc} & & & & & & USp(4k-2) & & SO(2) & & & & & & \\ & & & & & & | & & | & & & & & & \\ USp(2k-2) & - & SO(4k) & - & USp(6k-2) & - & SO(8k) & - & USp(6k) & - & SO(4k+2) & - & USp(2k) & - & SO(2) \end{array}$$

For general n. $\mu = 2(5 + 6n), \quad \mu_1 = \frac{2n(2n-1)}{n+1}, \quad \alpha = 2(n+1),$ $r = 6n + 4, \quad f = 2, \quad a = \frac{32n^2-77n+55}{12}, \quad c = \frac{8n^2-19n+13}{3}.$

$$\begin{array}{ccccccccc} & & & & & & SO(2) & & \\ & & & & & & | & & \\ (D_{n+1}^{(n+1)}[-n+\frac{1}{2}], \widetilde{F}) & - & USp(2n) & - & (D_{2n+2}^{(2n+2)}[-2n-\frac{1}{2}], \widetilde{F}) & - & USp(2n+2) & - & (D_{n+2}^{(2n+2)}[1-2n], \widetilde{F}) \end{array}$$

It's straightforward to check that the gauge groups are conformal. Explicitly, let's look at the $USp(2n)$ node. Recall from Subsection (3.2.2) that the $(D_{n+1}^{(n+1)}[k], \widetilde{F})$ has flavor symmetry $USp(2n)$ with flavor central charge $n+1-\frac{n+1}{2(n+1+k)}$, while $(D_{2n+2}^{(2n+2)}[k], \widetilde{F})$ has non-abelian flavor symmetry $USp(4n+2)$ with flavor central charge $2n+2-\frac{n+1}{2n+2+k}$ and the Dynkin index of embedding $I_{C_n \hookrightarrow C_{2n+1}} = 1$. Together the beta function is

$$\beta_{C_n} = 2(n+1) - \left(n+1-\frac{n+1}{2(n+1-n+1/2)}\right) - \left(2n+2-\frac{n+1}{2n+2-2n-1/2}\right) = 0 \tag{26}$$

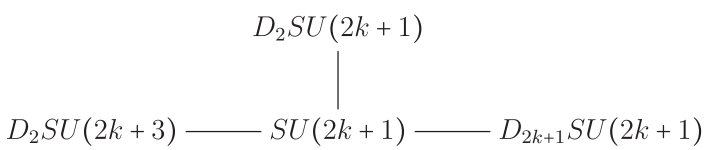
Similarly the other gauge node $USp(2n+2)$ is also conformal. One can also check that the left over flavor symmetry from the quiver has rank 2 which agrees with what we see from the singularity.

ICIS (182)

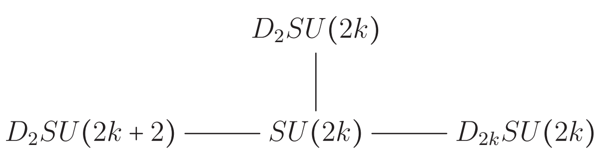
$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_3 + x_2^n + x_4^n + x_5^2 = 0 \end{cases}$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{n}{2+n}, \frac{2}{2+n}, \frac{n}{2+n}, \frac{2}{2+n}, \frac{n}{2+n}; 1, \frac{2n}{2+n})$

For $n = 2k+1$. $\mu = 4(k+1)^2$, $\mu_1 = 4k^2$, $\alpha = k+\frac{3}{2}$, $r = (k+1)(2k+1)$, $f = 2(k+1)$, $a = \frac{(k+1)(4k-1)(4k+3)}{24}$, $c = \frac{k(k+1)(2k+1)}{3}$.



For $n = 2k$. $\mu = (2k+1)^2$, $\mu_1 = (2k-1)^2$, $\alpha = k+1$, $r = 2k^2+k-1$, $f = 2(k+1)+1$, $a = \frac{k(16k(k+3)+41)+14}{24}$, $c = \frac{k(4k(k+3)+11)+4}{6}$.



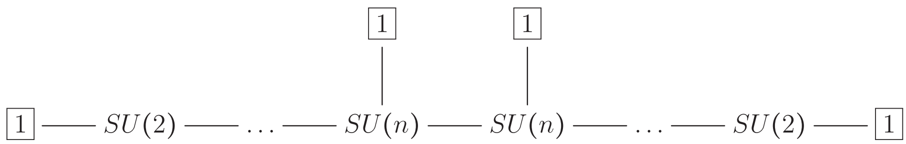
ICIS (234)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^n = 0 \\ x_1x_4 + x_3x_5 + x_2^n = 0 \end{cases} \qquad n_2 \geq 3 \ and \ 2 \leq n_5 \leq n_2$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{n-1}{n}, \frac{1}{n}, \frac{n-1}{n}, \frac{1}{n}, \frac{1}{n}; 1, 1)$

$\mu = 1 + 2n^2, \quad \mu_1 = n - 1, \quad \alpha = n, \quad r = n^2 - n, \quad f = 2n + 1,$

$a = \frac{n(4n^2+6n-7)}{24}, \quad c = \frac{n(n+2)(2n-1)}{12}$

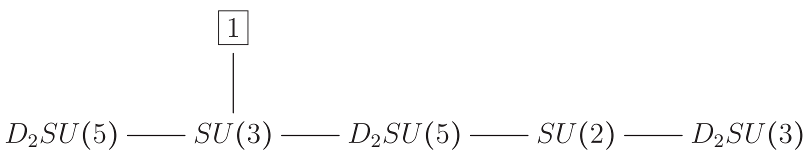


ICIS (253)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_4 + x_3^2 + x_2^4x_4 + x_5^2 = 0 \end{cases}$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{4}{5}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5}; 1, \frac{6}{5})$

$\mu = 21, \quad \mu_1 = 5, \quad \alpha = \frac{5}{2}, \quad r = 8, \quad f = 5, \quad a = \frac{13}{2}, \quad c = \frac{27}{4}$

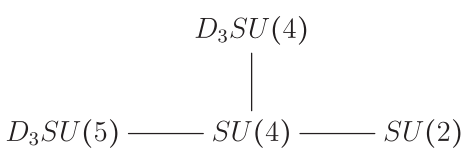


ICIS (261)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_3^2 + x_4^3 + x_2^4 = 0 \end{cases}$$

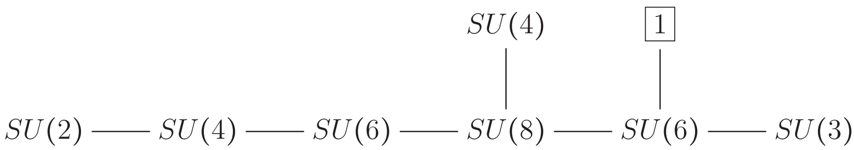
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{7}{10}, \frac{3}{10}, \frac{3}{5}, \frac{2}{5}, \frac{1}{2}; 1, \frac{6}{5})$

$\mu = 24, \quad \mu_1 = 6, \quad \alpha = \frac{10}{3}, \quad r = 11, \quad f = 2, \quad a = \frac{241}{24}, \quad c = \frac{61}{6}$



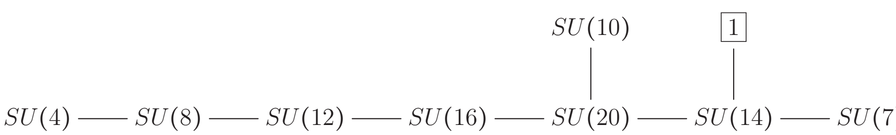
ICIS (262)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_3^2 + x_4^4 + x_2^8 = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{5}{6}, \tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{2}; 1, \tfrac{4}{3})$$
$$\mu = 59, \quad \mu_1 = 21, \quad \alpha = 6, \quad r = 26, \quad f = 7, \quad a = \tfrac{527}{12}, \quad c = \tfrac{133}{3}$$



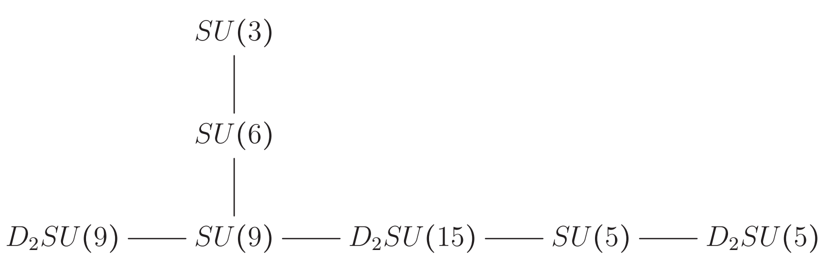
ICIS (263)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_3^2 + x_4^5 + x_2^{20} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{13}{14}, \tfrac{1}{14}, \tfrac{5}{7}, \tfrac{2}{7}, \tfrac{1}{2}; 1, \tfrac{10}{7})$$
$$\mu = 174, \quad \mu_1 = 76, \quad \alpha = 14, \quad r = 83, \quad f = 8, \quad a = \tfrac{2439}{8}, \quad c = \tfrac{611}{2}$$



ICIS (264)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^3 = 0 \\ x_1x_5 + x_3^2 + x_4^3 + x_2^9 = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{13}{15}, \tfrac{2}{15}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{3}; 1, \tfrac{6}{5})$$
$$\mu = 68, \quad \mu_1 = 32, \quad \alpha = \tfrac{15}{2}, \quad r = 32, \quad f = 4, \quad a = \tfrac{1381}{24}, \quad c = \tfrac{347}{6}$$

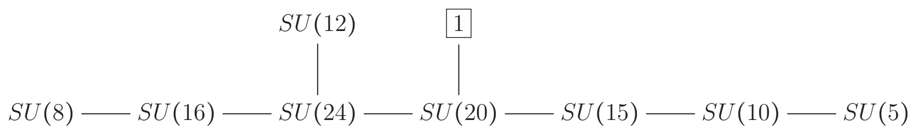


ICIS (265)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^4 = 0 \\ x_1x_5 + x_3^2 + x_4^3 + x_2^{24} = 0 \end{cases}$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{19}{20}, \frac{1}{20}, \frac{3}{5}, \frac{2}{5}, \frac{1}{4}; 1, \frac{6}{5})$

$\mu = 212, \quad \mu_1 = 46, \quad \alpha = 20, \quad r = 102, \quad f = 8, \quad a = \frac{1785}{4}, \quad c = 447$

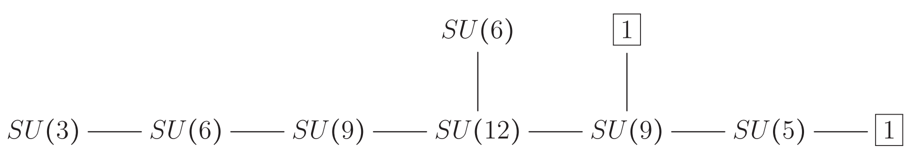


ICIS (266)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_4^4 + x_5^3 + x_2^{12} = 0 \end{cases}$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{8}{9}, \frac{1}{9}, \frac{2}{3}, \frac{1}{3}, \frac{4}{9}; 1, \frac{4}{3})$

$\mu = 94, \quad \mu_1 = 33, \quad \alpha = 9, \quad r = 43, \quad f = 8, \quad a = \frac{611}{6}, \quad c = \frac{1229}{12}$

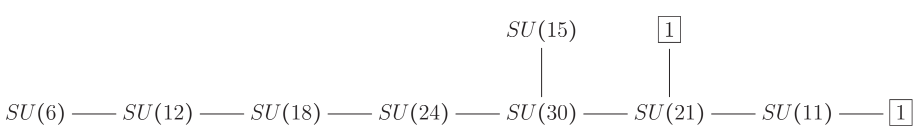


ICIS (267)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_4^5 + x_5^3 + x_2^{30} = 0 \end{cases}$$

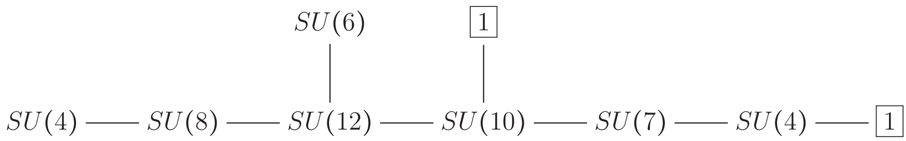
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{20}{21}, \frac{1}{21}, \frac{5}{7}, \frac{2}{7}, \frac{10}{21}; 1, \frac{10}{7})$

$\mu = 267, \quad \mu_1 = 116, \quad \alpha = 21, \quad r = 129, \quad f = 9, \quad a = \frac{2763}{4}, \quad c = \frac{2767}{4}$



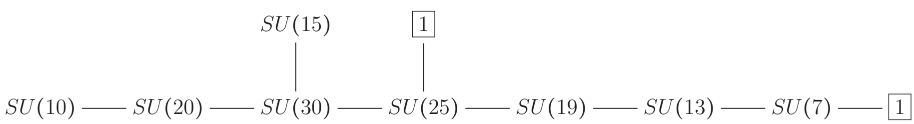
ICIS (268)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_4^3 + x_5^4 + x_2^{12} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{9}{10}, \tfrac{1}{10}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{3}{10}; 1, \tfrac{6}{5})$$
$$\mu = 96, \quad \mu_1 = 22, \quad \alpha = 10, \quad r = 44, \quad f = 8, \quad a = \tfrac{1261}{12}, \quad c = \tfrac{317}{3}$$



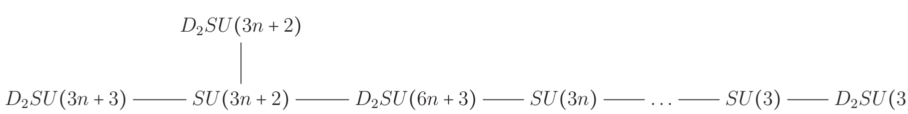
ICIS (269)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_4^3 + x_5^5 + x_2^{30} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{24}{25}, \tfrac{1}{25}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{6}{25}; 1, \tfrac{6}{5})$$
$$\mu = 271, \quad \mu_1 = 58, \quad \alpha = 25, \quad r = 131, \quad f = 9, \quad a = \tfrac{2825}{4}, \quad c = \tfrac{2829}{4}$$



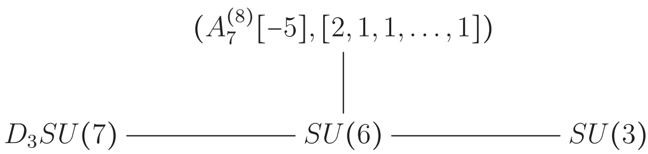
ICIS (271)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^{1+2n} = 0 \\ x_1x_5 + x_3^2 + x_2x_4^2 + x_2^{2+3n} = 0 \end{cases} \qquad n \geq 1$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{1+6n}{3(1+2n)}, \tfrac{2}{3(1+2n)}, \tfrac{2+3n}{3(1+2n)}, \tfrac{1+3n}{3(1+2n)}, \tfrac{1}{1+2n}; 1, \tfrac{2(2+3n)}{3(1+2n)})$$
$$\mu = 9 + 19n + 6n^2, \quad \mu_1 = 3(n+1), \quad \alpha = \tfrac{6n+3}{2}, \quad r = 3(n(n+3)+1),$$
$$f = 3, \quad a = \tfrac{3}{8}(n+1)(2n(2n+7)+5), \quad c = \tfrac{1}{4}(n+1)(3n(2n+7)+8).$$



ICIS (272)

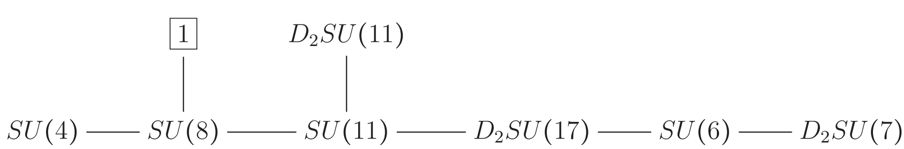
$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^3 + x_2^6 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{11}{14}, \frac{3}{14}, \frac{9}{14}, \frac{5}{14}, \frac{1}{2}; 1, \frac{9}{7})$
$\mu = 41, \quad \mu_1 = 13, \quad \alpha = \frac{14}{3}, \quad r = 19, \quad f = 3, \quad a = \frac{575}{24}, \quad c = \frac{145}{6}$



The AD matter theory $(A_7^{(8)}[-5], [2, 1^6])$ is defined by six dimensional A_7 $(2, 0)$ theory on a sphere with irregular puncture $A_7^{(8)}[-5]$ and a regular puncture labeled by Young Tableaux $[2, 1^6]$. This theory has flavor symmetry $SU(6) \times U(1)$, and the flavor central charge of $SU(6)$ is $\frac{13}{3}$. The Coulomb branch spectrum is $(\frac{13}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charges are $a = \frac{83}{12}, c = \frac{15}{2}$. The theory $D_3(SU(7))$ has flavor symmetry $SU(7)$, and the flavor central charge is $\frac{14}{3}$. The Coulomb branch spectrum of this theory is $(\frac{14}{3}, \frac{11}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$. Using the above data, one can check that the gauging for gauge group $SU(6)$ is conformal.

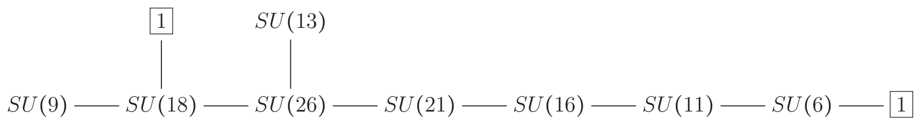
ICIS (273)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^4 + x_2^{11} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{7}{8}, \frac{1}{8}, \frac{11}{16}, \frac{5}{16}, \frac{1}{2}; 1, \frac{11}{8})$
$\mu = 87, \quad \mu_1 = 34, \quad \alpha = 8, \quad r = 87, \quad f = 5, \quad a = \frac{697}{8}, \quad c = \frac{175}{2}$



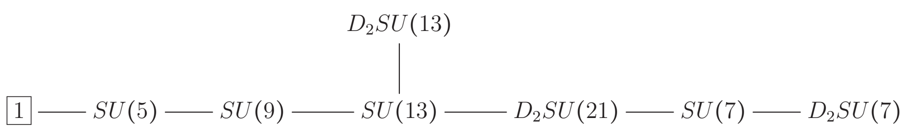
ICIS (274)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^5 + x_2^{26} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{17}{18}, \tfrac{1}{18}, \tfrac{13}{18}, \tfrac{5}{18}, \tfrac{1}{2}; 1, \tfrac{13}{9})$$
$$\mu = 233, \quad \mu_1 = 105, \quad \alpha = 18, \quad r = 112, \quad f = 9, \quad a = \tfrac{3149}{6}, \quad c = \tfrac{1577}{3}$$



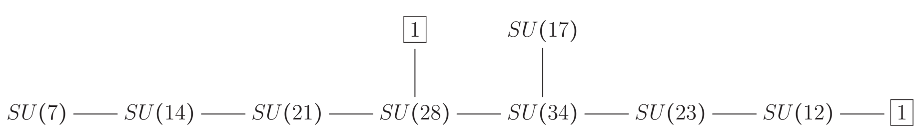
ICIS (275)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^3 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^3 + x_2^{13} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{19}{21}, \tfrac{2}{21}, \tfrac{13}{21}, \tfrac{8}{21}, \tfrac{1}{3}; 1, \tfrac{26}{21})$$
$$\mu = 103, \quad \mu_1 = 27, \quad \alpha = \tfrac{21}{2}, \quad r = 49, \quad f = 5, \quad a = \tfrac{2915}{24}, \quad c = \tfrac{1463}{12}$$



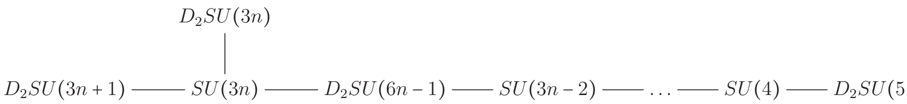
ICIS (276)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^4 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^3 + x_2^{34} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{27}{28}, \tfrac{1}{28}, \tfrac{17}{28}, \tfrac{11}{28}, \tfrac{1}{4}; 1, \tfrac{17}{14})$$
$$\mu = 305, \quad \mu_1 = 69, \quad \alpha = 28, \quad r = 148, \quad f = 9, \quad a = \tfrac{5377}{6}, \quad c = \tfrac{2692}{3}$$



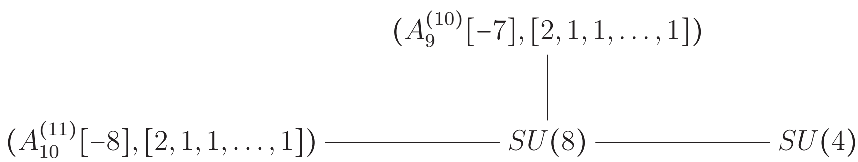
ICIS (278)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^2 + x_5^{2n} + x_2^{3n} = 0 \end{cases} \qquad n \geq 2$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{3(-1+2n)}{-1+6n}, \frac{2}{-1+6n}, \frac{3n}{-1+6n}, \frac{-1+3n}{-1+6n}, \frac{3}{-1+6n}; 1, \frac{6n}{-1+6n})$
$\mu = -1 + 11n + 6n^2, \quad \mu_1 = 1 + 3n, \quad \alpha = 3n - \frac{1}{2}, \quad r = (3n-1)(n+2),$
$f = n + 3, \quad a = \frac{1}{24}\left(36n^3 + 90n^2 + 3n - 10\right),$
$c = \frac{1}{12}\left(18n^3 + 45n^2 + 3n - 4\right)$



ICIS (279)

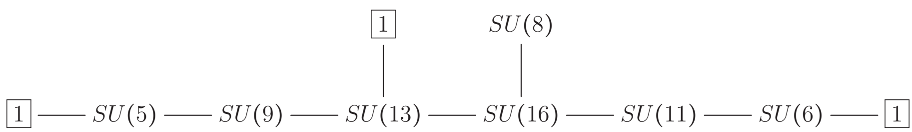
$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^3 + x_5^3 + x_2^8 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{16}{19}, \frac{3}{19}, \frac{12}{19}, \frac{7}{19}, \frac{8}{19}; 1, \frac{24}{19})$
$\mu = 58, \quad \mu_1 = 17, \quad \alpha = \frac{19}{3}, \quad r = 27, \quad f = 4, \quad a = \frac{175}{4}, \quad c = \frac{529}{12}$



The theory $(A_{10}^{(11)}[-8], [2, 1^9])$ is defined by six dimensional A_{10} $(2, 0)$ theory on a sphere with irregular puncture $A_{10}^{(11)}[-8]$ and a regular puncture labeled by Young Tableaux $[2, 1^9]$. This theory has flavor symmetry $SU(9) \times U(1)$, and the flavor central charge of $SU(9)$ is $\frac{19}{3}$. The Coulomb branch spectrum is $(\frac{19}{3}, \frac{16}{3}, \frac{13}{3}, \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charge is $a = \frac{29}{2}, c = \frac{63}{4}$. The theory $(A_9^{(10)}[-7], [2, 1^8])$ is defined by six dimensional A_{10} $(2, 0)$ theory on a sphere with irregular puncture $A_9^{(10)}[-7]$ and a regular puncture $[2, 1^8]$, and the flavor symmetry is $SU(8) \times U(1)$. The flavor central charge of $SU(8)$ is $\frac{17}{3}$. The Coulomb branch spectrum is $(\frac{17}{3}, \frac{14}{3}, \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charge is $a = \frac{35}{3}, c = \frac{38}{3}$. Using the above data, one can check that the gauging for gauge group $SU(8)$ is conformal.

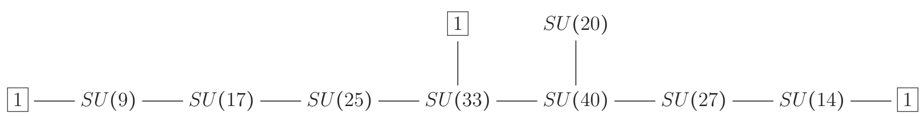
ICIS (280)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^3 + x_5^4 + x_2^{16} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{12}{13}, \tfrac{1}{13}, \tfrac{8}{13}, \tfrac{5}{13}, \tfrac{4}{13}; 1, \tfrac{16}{13})$$
$$\mu = 131, \quad \mu_1 = 33, \quad \alpha = 13, \quad r = 61, \quad f = 9, \quad a = \tfrac{4489}{24}, \quad c = \tfrac{1127}{6}$$



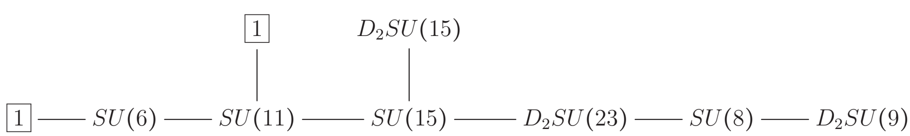
ICIS (281)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^3 + x_5^5 + x_2^{40} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{32}{33}, \tfrac{1}{33}, \tfrac{20}{33}, \tfrac{13}{33}, \tfrac{8}{33}; 1, \tfrac{40}{33})$$
$$\mu = 364, \quad \mu_1 = 81, \quad \alpha = 33, \quad r = 177, \quad f = 10, \quad a = \tfrac{5007}{4}, \quad c = \tfrac{5013}{4}$$



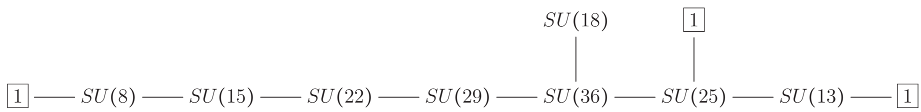
ICIS (282)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^4 + x_5^3 + x_2^{15} = 0 \end{cases}$$
$$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\tfrac{10}{11}, \tfrac{1}{11}, \tfrac{15}{22}, \tfrac{7}{22}, \tfrac{5}{11}; 1, \tfrac{15}{11})$$
$$\mu = 122, \quad \mu_1 = 46, \quad \alpha = 11, \quad r = 58, \quad f = 6, \quad a = \tfrac{1957}{12}, \quad c = \tfrac{491}{3}$$



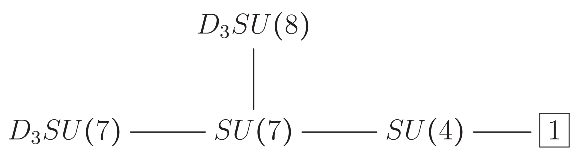
ICIS (283)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_3^2 + x_2x_4^5 + x_5^3 + x_2^{36} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{24}{25}, \frac{1}{25}, \frac{18}{25}, \frac{7}{25}, \frac{12}{25}; 1, \frac{36}{25})$
$\mu = 326, \quad \mu_1 = 145, \quad \alpha = 25, \quad r = 158, \quad f = 10, \quad a = \frac{24151}{24}, \quad c = \frac{12091}{12}$



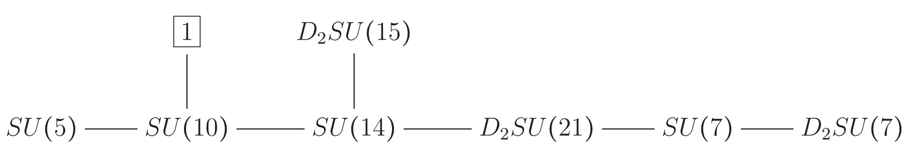
ICIS (284)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_2x_3^2 + x_4^3 + x_2^7 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{13}{16}, \frac{3}{16}, \frac{9}{16}, \frac{7}{16}, \frac{1}{2}; 1, \frac{21}{16})$
$\mu = 47, \quad \mu_1 = 16, \quad \alpha = \frac{16}{3}, \quad r = 22, \quad f = 3, \quad a = \frac{377}{12}, \quad c = \frac{95}{3}$



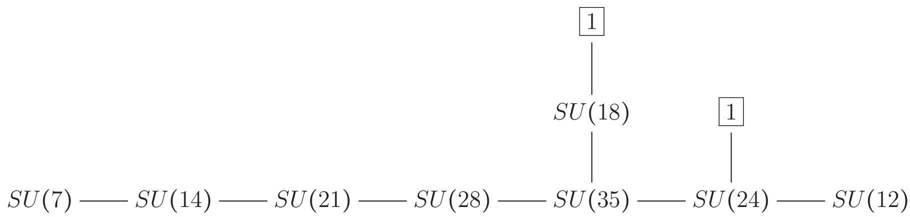
ICIS (285)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_2x_3^2 + x_4^4 + x_2^{14} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{9}{10}, \frac{1}{10}, \frac{13}{20}, \frac{7}{20}, \frac{1}{2}; 1, \frac{7}{5})$
$\mu = 109, \quad \mu_1 = 45, \quad \alpha = 10, \quad r = 52, \quad f = 5, \quad a = \frac{273}{2}, \quad c = 137$



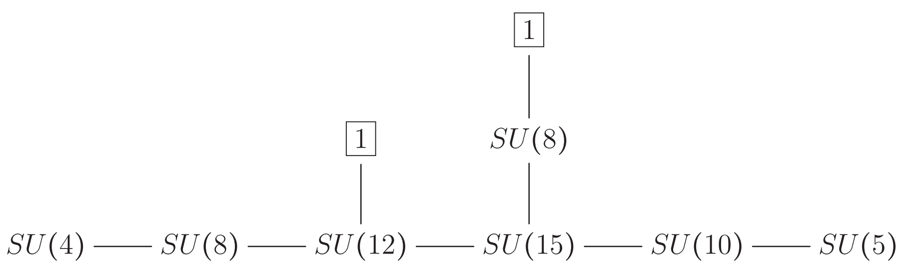
ICIS (286)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_2x_3^2 + x_4^5 + x_2^{35} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{23}{24}, \frac{1}{24}, \frac{17}{24}, \frac{7}{24}, \frac{1}{2}; 1, \frac{35}{24})$
$\mu = 311, \quad \mu_1 = 144, \quad \alpha = 24, \quad r = 151, \quad f = 9, \quad a = \frac{22415}{24}, \quad c = \frac{5611}{6}$



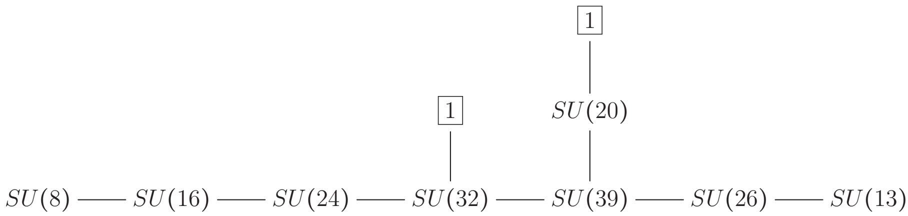
ICIS (287)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^3 = 0 \\ x_1x_5 + x_2x_3^2 + x_4^3 + x_2^{15} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{11}{12}, \frac{1}{12}, \frac{7}{12}, \frac{5}{12}, \frac{1}{3}; 1, \frac{5}{4})$
$\mu = 118, \quad \mu_1 = 32, \quad \alpha = 12, \quad r = 55, \quad f = 8, \quad a = \frac{3803}{24}, \quad c = \frac{955}{6}$



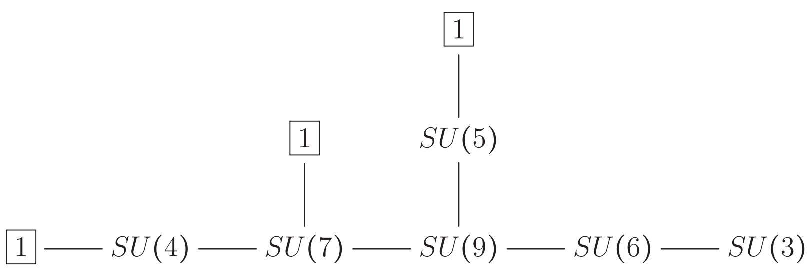
ICIS (288)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^4 = 0 \\ x_1x_5 + x_2x_3^2 + x_4^3 + x_2^{39} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{31}{32}, \frac{1}{32}, \frac{19}{32}, \frac{13}{32}, \frac{1}{4}; 1, \frac{39}{32})$
$\mu = 349, \quad \mu_1 = 80, \quad \alpha = 32, \quad r = 170, \quad f = 9, \quad a = \frac{14051}{12}, \quad c = \frac{3517}{3}$



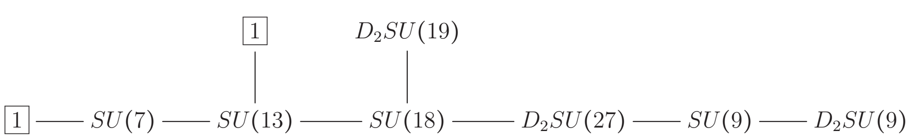
ICIS (289)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_2x_3^2 + x_4^3 + x_5^3 + x_2^9 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{6}{7}, \frac{1}{7}, \frac{4}{7}, \frac{3}{7}, \frac{3}{7}; 1, \frac{9}{7})$
$\mu = 64, \quad \mu_1 = 20, \quad \alpha = 7, \quad r = 28, \quad f = 8, \quad a = \frac{637}{12}, \quad c = \frac{161}{3}$



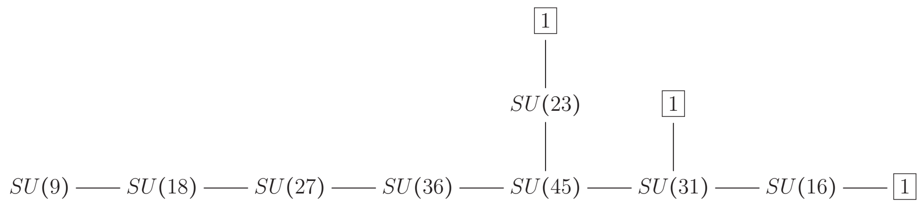
ICIS (290)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_2x_3^2 + x_4^4 + x_5^3 + x_2^{18} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{12}{13}, \frac{1}{13}, \frac{17}{26}, \frac{9}{26}, \frac{6}{13}; 1, \frac{18}{13})$
$\mu = 144, \quad \mu_1 = 57, \quad \alpha = 13, \quad r = 69, \quad f = 6, \quad a = \frac{457}{2}, \quad c = \frac{917}{4}$



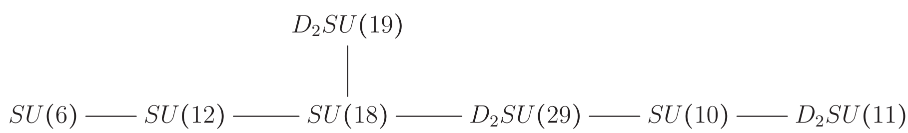
ICIS (291)

$$\begin{cases} x_1x_2+x_3x_4=0 \\ x_1x_5+x_2x_3^2+x_4^5+x_5^3+x_2^{45}=0 \end{cases}$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=(\tfrac{30}{31},\tfrac{1}{31},\tfrac{22}{31},\tfrac{9}{31},\tfrac{15}{31};1,\tfrac{45}{31})$$
$$\mu=404,\quad \mu_1=184,\quad \alpha=31,\quad r=197,\quad f=10,\quad a=\tfrac{37201}{24},\quad c=\tfrac{9311}{6}$$



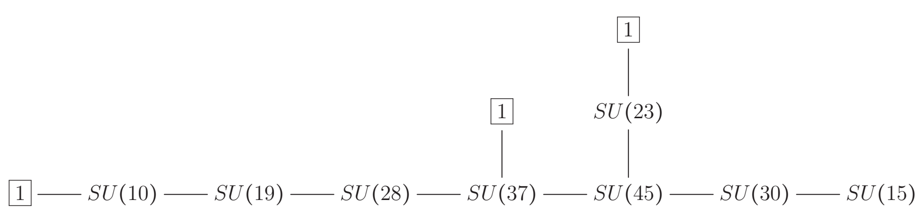
ICIS (292)

$$\begin{cases} x_1x_2+x_3x_4=0 \\ x_1x_5+x_2x_3^2+x_4^3+x_5^4+x_2^{18}=0 \end{cases}$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=(\tfrac{27}{29},\tfrac{2}{29},\tfrac{17}{29},\tfrac{12}{29},\tfrac{9}{29};1,\tfrac{36}{29})$$
$$\mu=146,\quad \mu_1=38,\quad \alpha=\tfrac{29}{2},\quad r=70,\quad f=6,\quad a=\tfrac{933}{4},\quad c=234$$



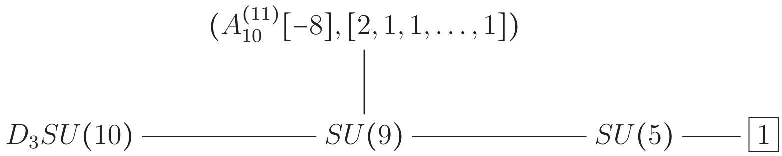
ICIS (293)

$$\begin{cases} x_1x_2+x_3x_4=0 \\ x_1x_5+x_2x_3^2+x_4^3+x_5^5+x_2^{45}=0 \end{cases}$$
$$(w_1,w_2,w_3,w_4,w_5;1,d)=(\tfrac{36}{37},\tfrac{1}{37},\tfrac{22}{37},\tfrac{15}{37},\tfrac{9}{37};1,\tfrac{45}{37})$$
$$\mu=408,\quad \mu_1=92,\quad \alpha=37,\quad r=199,\quad f=10,\quad a=\tfrac{37753}{24},\quad c=\tfrac{9449}{6}$$



ICIS (294)

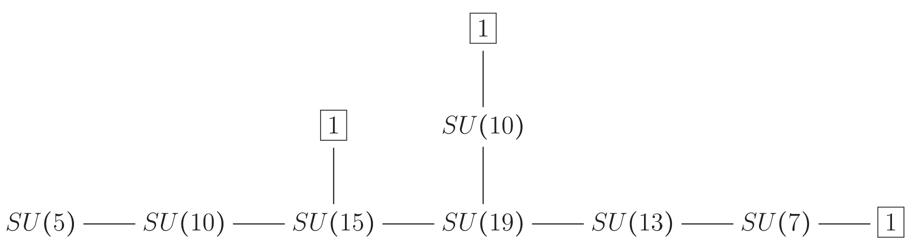
$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^3 + x_2^9 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{17}{20}, \frac{3}{20}, \frac{3}{5}, \frac{2}{5}, \frac{1}{2}; 1, \frac{27}{20})$
$\mu = 64, \quad \mu_1 = \frac{95}{4}, \quad \alpha = \frac{20}{3}, \quad r = 30, \quad f = 4, \quad a = \frac{427}{8}, \quad c = \frac{215}{4}$



Let's explain some details about AD matter $(A_{10}^{(11)}[-8], [2, 1^9])$, which is defined by six dimensional A_{10} $(2, 0)$ theory on a sphere with irregular puncture $A_{10}^{(11)}[-8]$ and the regular puncture is labeled by Young Tableaux $[2, 1^9]$. This theory has flavor symmetry $SU(9) \times U(1)$, and the flavor central charge of $SU(9)$ is $\frac{19}{3}$. The Coulomb branch spectrum of this theory is $(\frac{19}{3}, \frac{16}{3}, \frac{13}{3}, \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charges are $a = \frac{29}{2}, c = \frac{63}{4}$. The flavor central charge of $SU(10)$ flavor group of $D_3SU(10)$ theory is $\frac{20}{3}$. The Coulomb branch spectrum is $(\frac{20}{3}, \frac{17}{3}, \frac{14}{3}, \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charges are $a = \frac{121}{8}, c = \frac{33}{2}$. Using the above data, one can check that the gauging for gauge group $SU(9)$ is conformal.

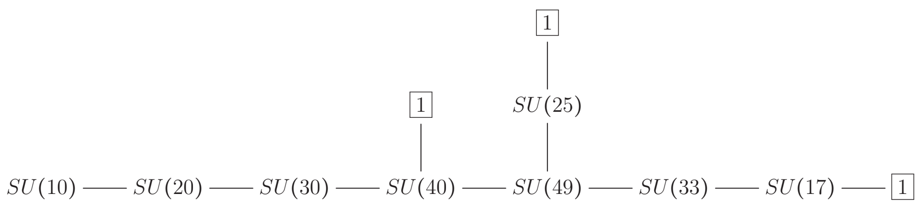
ICIS (295)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^3 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^3 + x_2^{19} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{14}{15}, \frac{1}{15}, \frac{3}{5}, \frac{2}{5}, \frac{1}{3}; 1, \frac{19}{15})$
$\mu = 153, \quad \mu_1 = \frac{130}{3}, \quad \alpha = 15, \quad r = 72, \quad f = 9, \quad a = \frac{6155}{24}, \quad c = \frac{3089}{12}$



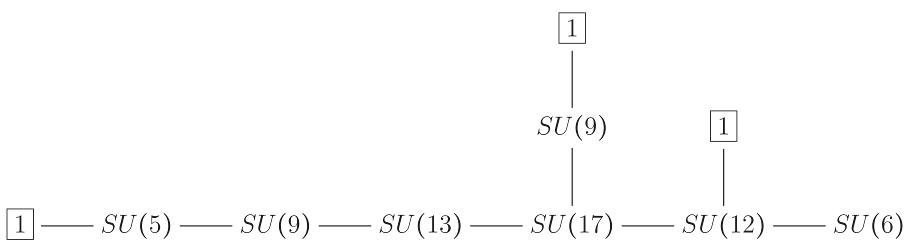
ICIS (296)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^4 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^3 + x_2^{49} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{39}{40}, \frac{1}{40}, \frac{3}{5}, \frac{2}{5}, \frac{1}{4}; 1, \frac{49}{40})$
$\mu = 442, \quad \mu_1 = \frac{825}{8}, \quad \alpha = 40, \quad r = 216, \quad f = 10, \quad a = \frac{44425}{24}, \quad c = \frac{22237}{12}$



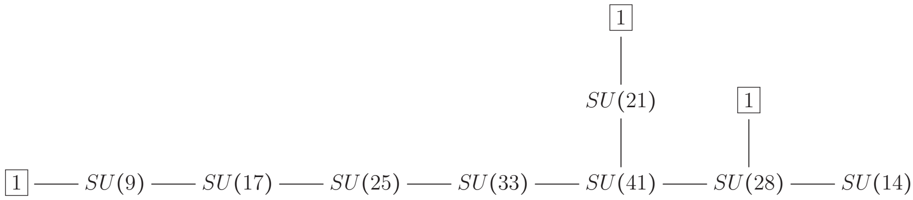
ICIS (297)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^4 + x_2^{17} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{11}{12}, \frac{1}{12}, \frac{2}{3}, \frac{1}{3}, \frac{1}{2}; 1, \frac{17}{12})$
$\mu = 137, \quad \mu_1 = \frac{117}{2}, \quad \alpha = 12, \quad r = 64, \quad f = 9, \quad a = \frac{616}{3}, \quad c = \frac{1237}{6}$



ICIS (298)

$$\begin{cases} x_1x_2 + x_3x_4 + x_5^2 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^5 + x_2^{41} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{27}{28}, \frac{1}{28}, \frac{5}{7}, \frac{2}{7}, \frac{1}{2}; 1, \frac{41}{28})$
$\mu = 81, \quad \mu_1 = \frac{693}{4}, \quad \alpha = 28, \quad r = 180, \quad f = 10, \quad a = \frac{31105}{24}, \quad c = \frac{15571}{12}$

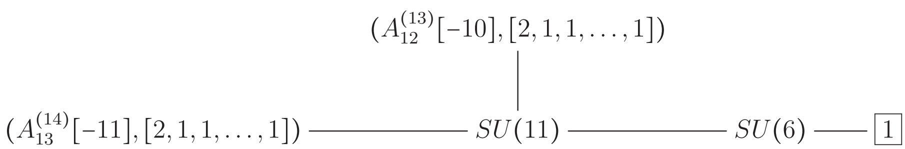


ICIS (299)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^3 + x_5^3 + x_2^{11} = 0 \end{cases}$$

$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{22}{25}, \frac{3}{25}, \frac{3}{5}, \frac{2}{5}, \frac{11}{25}; 1, \frac{33}{25})$

$\mu = 81, \quad \mu_1 = \frac{138}{5}, \quad \alpha = \frac{25}{3}, \quad r = 38, \quad f = 5, \quad a = \frac{1949}{24}, \quad c = \frac{327}{4}$



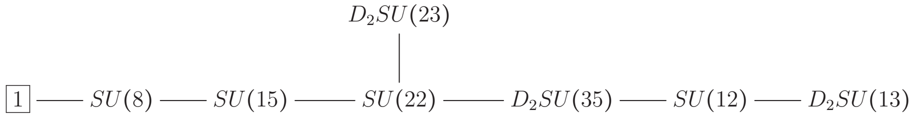
In above case, the AD matters are engineered using six dimensional A_{12} and A_{13} theory with the irregular puncture type $A_{12}^{(13)}[-10]$ and $A_{13}^{(14)}[-11]$, and the regular puncture has the the Young Tableaux type $[2, 1^{11}]$ and $[2, 1^{12}]$ respectively. The theory $(A_{12}^{(13)}[-10], [2, 1^{11}])$ has flavor symmetry $SU(11) \times U(1)$, and the flavor central charge for $SU(12)$ flavor group is $\frac{23}{3}$. The Coulomb branch spectrum is $(\frac{23}{3}, \frac{20}{3}, \frac{17}{3}, \frac{14}{3}, \frac{13}{3}, \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charge is $a = \frac{253}{12}$, $c = \frac{275}{12}$. The theory $(A_{13}^{(14)}[-11], [2, 1^{12}])$ has flavor symmetry $SU(12) \times U(1)$, and the flavor central charge for $SU(12)$ flavor group is $\frac{25}{3}$. The Coulomb branch spectrum is $(\frac{25}{3}, \frac{22}{3}, \frac{19}{3}, \frac{16}{3}, \frac{14}{3}, \frac{13}{3}, \frac{11}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{5}{3}, \frac{4}{3})$, and the central charge is $a = \frac{149}{6}$, $c = 27$. Using the above data, one can check that the gauging is conformal for $SU(11)$ gauge group.

ICIS (300)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^3 + x_5^4 + x_2^{22} = 0 \end{cases}$$

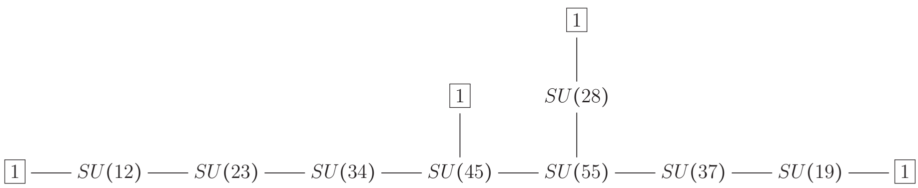
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{33}{35}, \frac{2}{35}, \frac{3}{5}, \frac{2}{5}, \frac{11}{35}; 1, \frac{44}{35})$

$\mu = 181, \quad \mu_1 = \frac{345}{7}, \quad \alpha = \frac{35}{2}, \quad r = 87, \quad f = 7, \quad a = \frac{8383}{24}, \quad c = \frac{1051}{3}$



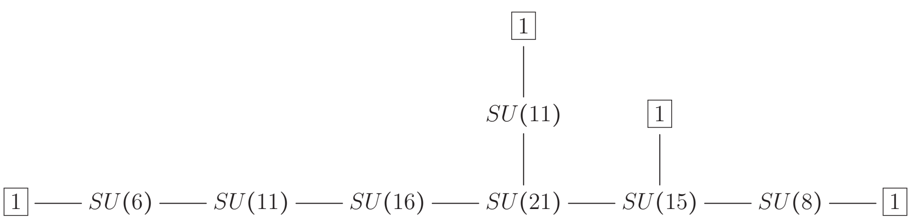
ICIS (301)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^3 + x_5^5 + x_2^{55} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{44}{45}, \frac{1}{45}, \frac{3}{5}, \frac{2}{5}, \frac{11}{45}; 1, \frac{11}{9})$
$\mu = 501, \quad \mu_1 = \frac{1036}{9}, \quad \alpha = 45, \quad r = 245, \quad f = 11, \quad a = \frac{9395}{4}, \quad c = \frac{9405}{4}$



ICIS (302)

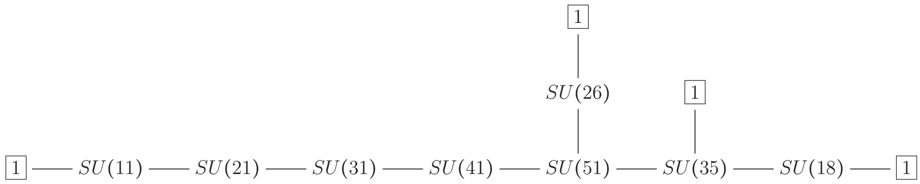
$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^4 + x_5^3 + x_2^{21} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{14}{15}, \frac{1}{15}, \frac{2}{3}, \frac{1}{3}, \frac{7}{15}; 1, \frac{7}{5})$
$\mu = 172, \quad \mu_1 = \frac{352}{5}, \quad \alpha = 15, \quad r = 81, \quad f = 10, \quad a = \frac{2523}{8}, \quad c = \frac{633}{2}$



ICIS (303)

$$\begin{cases} x_1x_2 + x_3x_4 = 0 \\ x_1x_5 + x_2x_3^2 + x_2x_4^5 + x_5^3 + x_2^{51} = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{34}{35}, \frac{1}{35}, \frac{5}{7}, \frac{2}{7}, \frac{17}{35}; 1, \frac{51}{35})$

$\mu = 463, \quad \mu_1 = \frac{1066}{5}, \quad \alpha = 35, \quad r = 226, \quad f = 11,$
$a = \frac{48185}{24}, \quad c = \frac{24119}{12}$



4. Discussion

In this paper, we studied physical properties of four dimensional $\mathcal{N} = 2$ SCFT defined by three-fold isolated complete intersection singularities with a $\mathbb{C}^*$ action which is classified in [17]. Let's summarize the major findings:

- The Seiberg-Witten solution is identified with the mini-versal deformation of the singularity, see formula (10), and the Seiberg-Witten differential is given by formula (11).
- The Coulomb branch spectrum (the scaling dimensions of Coulomb branch parameters) can be found from the basis of Jacobi module and the charges associated with the $\mathbb{C}^*$ action, see formula (12).
- The dimension of charge lattice is given by the weights and degrees of the defining equation of the singularity, see formula (15).
- The central charges a and c can be found using formula ((16), (18), (19), (20) and (21)).

We also provide a type IIB string theory realization of our SCFTs using the defining data of ICIS.

Many SCFTs studied in this paper have exactly marginal deformations, and it is expected that one can find the weakly coupled gauge theory descriptions. Using the Coulomb branch spectrum, we identify such descriptions for many SCFTs. Those quiver gauge theories often take the form of the (affine) D or E shape quivers (see [15, 32] for other descriptions of these theories). We compute various physical quantities such as the central charges which are in complete agreement with the results from singularity theory, which provides strong evidence for the correctness of the singularity approach.

There are some interesting questions that we would like to study in the future: a) We have identified the weakly coupled gauge theory descriptions by guessing, it is desirable to have a more systematic way of finding all the weakly coupled duality frames, perhaps the hypersurface examples studied in [52] can be useful. b): In the hypersurface case [51], we identify the BPS quiver as the intersection form of the vanishing cycles of the Milnor fibration; the naive generalization to ICIS case does not work so well as the number of the vanishing cycles is bigger than the dimension of the middle homology of the Milnor fibration [22], it is interesting to further investigate this issue. c): The renormalization flow of SCFTs defined by hypersurface singularities has a remarkable semicontinuity property [53], similar semicontinuity property of ICIS was also studied in [23], however, the details of ICIS case is much less understood, and it is interesting to further study RG flows along this line.

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