

Bogoliubov correction to the mean-field dynamics of interacting bosons

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We consider the dynamics of a large quantum system of N identical bosons in 3D interacting via a two-body potential of the form $N^{3\beta-1}w(N^\beta(x-y))$. For fixed $0 \leq \beta < 1/3$ and large N , we obtain a norm approximation to the many-body evolution in the N -particle Hilbert space. The leading order behaviour of the dynamics is determined by Hartree theory while the second order is given by Bogoliubov theory.

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1. Introduction

We consider a large system of N identical bosons living in \mathbb{R}^3 described by the Hamiltonian

$$(1) \quad H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N-1} \sum_{1 \leq j < k \leq N} w_N(x_j - x_k),$$

which acts on the symmetric space $\mathfrak{H}^N = \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^3)$. Here $x_j \in \mathbb{R}^3$ stands for the coordinate of the j -th particle. We assume that the interaction potential has the explicit form

$$(2) \quad w_N(x - y) = N^{3\beta} w(N^\beta(x - y))$$

where $\beta \geq 0$ is a fixed parameter and w a given nice function. For simplicity, we will assume that $w \in C_0^1(\mathbb{R}^3)$ is non-negative, spherically symmetric and decreasing.

Since the Hamiltonian H_N is bounded from below, and hence it can be defined as a self-adjoint operator on \mathfrak{H}^N by Friedrichs' method [51]. The coupling constant $1/(N - 1)$ in front of the interaction is to ensure that the kinetic energy and interaction energy are of the same order. We could choose $1/N$ instead of $1/(N - 1)$, but the latter will simplify some expressions.

Note that when $\beta > 0$, then w_N converges to the Dirac-delta interaction. In general, the larger the parameter β , the harder the analysis. In this paper we will focus on the mean-field regime $0 \leq \beta < 1/3$. In this case, the range of the interaction potential is much larger than the average distance between the particles and there are many but weak collisions. Therefore, to the leading order, the interaction potential experienced by each particle can be approximated by the mean-field potential $\rho * w_N$ where ρ is the density of the system. If $\beta > 1/3$, then the analysis is expected to be more complicated due to strong correlations between particles.

In the present paper, we are interested in the large N asymptotic behavior of the Schrödinger evolution

$$(3) \quad \Psi_N(t) = e^{-itH_N} \Psi_N(0)$$

generated by a special class of initial states $\Psi_N(0) \in \mathfrak{H}^N$. We are motivated by the physical picture that $\Psi_N(0)$ is a ground state (or an approximated ground state) of a trapped system described by the Hamiltonian

$$(4) \quad H_N^V = \sum_{j=1}^N (-\Delta_{x_j} + V(x_j)) + \frac{1}{N - 1} \sum_{1 \leq j < k \leq N} w_N(x_j - x_k)$$

with an external potential $V \in L_{\text{loc}}^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, and the time evolution $\Psi_N(t)$ in (3) is observed when the trapping potential $V(x)$ is turned off. This leads to certain properties of $\Psi_N(0)$ which will be described below.

1.1. Ground state properties

Bose-Einstein condensation. It is widely expected that ground states of trapped systems exhibit the (complete) Bose-Einstein condensation, namely $\Psi \approx u^{\otimes N}$ in an appropriate sense. In fact, when $0 \leq \beta < 1$ we have

$$(5) \quad \lim_{N \rightarrow \infty} \left(\inf_{\|\Psi\|_{\mathfrak{H}^N} = 1} \frac{\langle \Psi, H_N^V \Psi \rangle}{N} - \inf_{\|u\|_{\mathfrak{H}} = 1} \mathcal{E}_{\mathfrak{H}, N}^V(u) \right) = 0$$

where

$$(6) \quad \mathcal{E}_{\mathfrak{H}, N}^V(u) := \frac{1}{N} \langle u^{\otimes N}, H_N^V u^{\otimes N} \rangle = \int_{\mathbb{R}^3} |\nabla u|^2 + V|u|^2 + \frac{1}{2}|u|^2(w_N * |u|^2).$$

Moreover, if the Hartree energy functional $\mathcal{E}_{\mathfrak{H}, N}^V(u)$ has a unique minimizer $u_{\mathfrak{H}}$, then the ground state Ψ_N^V of H_N^V condensates on $u_{\mathfrak{H}}$ in the sense that

$$(7) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \langle u_{\mathfrak{H}}, \gamma_{\Psi_N^V} u_{\mathfrak{H}} \rangle = 1,$$

where $\gamma_{\Psi} : \mathfrak{H} \rightarrow \mathfrak{H}$ is the one-body density matrix of $\Psi \in \mathfrak{H}^N$ with kernel

$$(8) \quad \gamma_{\Psi}(x, y) = N \int \Psi(x, x_2, \dots, x_N) \overline{\Psi(y, x_2, \dots, x_N)} dx_2 \cdots dx_N.$$

The rigorous justifications for (5) and (7) in various specific cases has been given in [6, 20, 38, 42, 47, 50, 53, 54]. Recently, in a series of works [33–35], Lewin, Rougerie and the first author of the present paper provided proofs in a very general setting. Note that when $\beta = 1$ (the Gross-Pitaevskii regime), the Hartree functional has to be modified to capture the strong correlation between particles. This has been first done by Lieb, Seiringer and Yngvason in [39–41] (see also [44]).

Fluctuations around the condensation. The next order correction to the lower eigenvalues and eigenfunctions of H_N^V is predicted by Bogoliubov’s approximation [9]. This has been first derived rigorously by Seiringer in [53], and then extended in various directions in [15, 24, 37, 45].

Bogoliubov’s theory is formulated in the Fock space

$$\mathcal{F}(\mathfrak{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{H}^n = \mathbb{C} \oplus \mathfrak{H} \oplus \mathfrak{H}^2 \oplus \cdots,$$

where the excited particles are effectively described by a quadratic Hamiltonian \mathbb{H}^V acting on the subspace $\mathcal{F}(\{u_{\mathfrak{H}}\}^{\perp})$. In fact, \mathbb{H}^V is the second

quantization of (half) the Hessian of the Hartree functional $\mathcal{E}_H^V(u)$ at its minimizer u_H .

It was proved in [37, Theorem 2.2] by Lewin, Serfaty, Solovej and the first author of the present paper that if $\beta = 0$ and if the Hartree minimizer u_H is non-degenerate (in the sense that the Hessian of $\mathcal{E}_H^V(u)$ at u_H is bigger than a positive constant), then the ground state Ψ_N^V of H_N^V admits the norm approximation

$$(9) \quad \lim_{N \rightarrow \infty} \left\| \Psi_N^V - \sum_{n=0}^N u_H^{\otimes(N-n)} \otimes_s \psi_n \right\|_{\mathfrak{H}^N} = 0$$

where $\Phi^V = (\psi_n)_{n=0}^\infty \in \mathcal{F}(\{u\}^\perp)$ is the (unique) ground state of \mathbb{H}^V . The same is expected to be true for $\beta > 0$ small.

Note that the norm convergence (9) is much more precise than the convergence of density matrices in (7). In fact, if $w \neq 0$, then Φ^V is not the vacuum $\Omega := 1 \oplus 0 \oplus 0 \cdots$, and hence Ψ_N^V is *never* close to $u_H^{\otimes N}$ in norm.

Quasi-free states. The ground state Φ^V of the quadratic Hamiltonian \mathbb{H}^V is a quasi-free state (see [37, Theorem A.1]). Recall that a state Ψ in Fock space $\mathcal{F}(\mathfrak{H})$ is called a quasi-free state if it has finite particle number expectation and satisfies Wick’s Theorem:

$$(10) \quad \langle \Psi, a^\#(f_1)a^\#(f_2) \cdots a^\#(f_{2n-1})\Psi \rangle = 0,$$

$$(11) \quad \langle \Psi, a^\#(f_1)a^\#(f_2) \cdots a^\#(f_{2n})\Psi \rangle$$

$$= \sum_{\sigma \in P_{2n}} \prod_{j=1}^n \langle \Psi, a^\#(f_{\sigma(2j-1)})a^\#(f_{\sigma(2j)})\Psi \rangle$$

for all n and for all $f_1, \dots, f_n \in \mathfrak{H}$, where $a^\#$ is either the creation or annihilation operator (see Section 2) and P_{2n} is the set of pairings,

$$P_{2n} = \{ \sigma \in S(2n) \mid \sigma(2j-1) < \min\{\sigma(2j), \sigma(2j+1)\} \text{ for all } j \}.$$

It is clear that if Ψ is a quasi-free, then the projection $|\Psi\rangle\langle\Psi|$ is determined completely by its density matrices. Recall that for every state Ψ in $\mathcal{F}(\mathfrak{H})$, we define the density matrices $\gamma_\Psi : \mathfrak{H} \rightarrow \mathfrak{H}$ and $\alpha_\Psi : \overline{\mathfrak{H}} \rightarrow \overline{\mathfrak{H}}$ by

$$(12) \quad \langle f, \gamma_\Psi g \rangle = \langle \Psi, a^*(g)a(f)\Psi \rangle, \quad \langle f, \alpha_\Psi \bar{g} \rangle = \langle \Psi, a(g)a(f)\Psi \rangle$$

for all $f, g \in \mathfrak{H}$. If $\Psi \in \mathfrak{H}^N$, then γ_Ψ coincides with the density matrix defined in (8). In general, $\text{Tr}(\gamma_\Psi)$ is the particle number expectation of Ψ .

1.2. Time evolution

Let us recall the physical picture we have in mind. Let $\Psi_N(0)$ be a ground state of H_N^V (cf. (4)). When the external potential V is turned off, $\Psi_N(0)$ is no longer a ground state of the Hamiltonian H_N (cf. (1)) and the time evolution $\Psi_N(t) = e^{-itH_N}\Psi_N(0)$ is observed. The analysis of the behavior of $\Psi_N(t)$ when $N \rightarrow \infty$ is the main goal of our paper.

Leading order. It is a fundamental fact that the Bose-Einstein condensation is stable under the Schrödinger flow in the mean-field limit. To be precise, if $\Psi_N(0)$ condensates on a (one-body) state $u(0)$, in the sense of (7), then the time evolution $\Psi_N(t) = e^{-itH_N}\Psi_N(0)$ condensates on the state $u(t)$ determined by the Hartree equation

$$(13) \quad i\partial_t u(t) = (-\Delta + w_N * |u(t)|^2 - \mu_N(t))u(t)$$

and the initial datum $u(0)$.

Here $\mu_N(t) \in \mathbb{R}$ is a phase parameter which is free to choose. For the leading order, this phase plays no role as it does not alter the projection $|u(t)\rangle\langle u(t)|$. However, to simplify the second order expression discussed below, we will choose

$$(14) \quad \mu_N(t) := \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |u(t, x)|^2 w_N(x - y) |u(t, y)|^2 dx dy$$

which ensures the compatibility of the energies:

$$\begin{aligned} \langle u(t)^{\otimes N}, H_N(u(t)^{\otimes N}) \rangle &\approx \langle \Psi_N(t), H_N \Psi_N(t) \rangle \\ &= \langle \Psi_N(t), i\partial_t \Psi_N(t) \rangle \approx \langle u(t)^{\otimes N}, i\partial_t (u(t)^{\otimes N}) \rangle. \end{aligned}$$

Note that when $\beta > 0$, $w_N \rightharpoonup a_0\delta$ weakly with $a_0 = \int w$ and the solution to the Hartree equation (13) converges to that of the cubic nonlinear Schrödinger equation (NLS)

$$(15) \quad i\partial_t v(t) = (-\Delta + a_0|v(t)|^2 - \mu(t))v(t), \quad \mu(t) = \frac{a_0}{2} \int |v(t, x)|^4 dx,$$

with the same initial datum.

The rigorous derivation for the dynamics of the Bose-Einstein condensation has been the subject of a vast literature. For $\beta = 0$, the problem was studied by Hepp [29], Ginibre and Velo [22, 23] and Spohn [56]; see [1, 2, 4, 11, 19, 21, 31, 48, 52] for further results. For $0 < \beta \leq 1$, the problem

was solved by Erdős, Schlein and Yau [16–18] (see also [5, 12, 30, 49] for later developments for $\beta = 1$). Note that when $\beta = 1$, the strong correlation between particles yield a leading order correction to the effective dynamics and the NLS equation (15) has to be replaced by the Gross-Pitaevskii equation, i.e. a_0 has to be replaced by the scattering length of w .

Second order correction. In this paper, we are interested in the norm approximation for the time evolution $\Psi_N(t) = e^{-itH_N}\Psi_N(0)$.

For the norm approximation, a natural approach is to study the time evolution initiated by a *coherent state* in Fock space $\mathcal{F}(\mathfrak{H})$. Recall that a coherent state is obtained by applying Weyl’s unitary operator $W(f) = \exp(a^*(f) - a(f))$ to the vacuum:

$$W(f)\Omega = e^{-\|f\|^2/2} \sum_{n \geq 0} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

The N -body Hamiltonian H_N can be extended to the Fock space as

$$(16) \quad H_N = \int_{\mathbb{R}^3} a_x^*(-\Delta)a_x \, dx + \frac{1}{2(N-1)} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w(x-y)a_x^*a_y^*a_xa_y \, dx dy$$

where a_x^* and a_x are the operator-valued distributions (see Section 2). In the mean-field regime, the time evolution of a coherent state satisfies the norm approximation

$$(17) \quad \lim_{N \rightarrow \infty} \left\| \exp(-itH_N)W\left(\sqrt{N}u(0)\right)\Omega - W\left(\sqrt{N}u(t)\right)\Xi(t) \right\|_{\mathcal{F}} = 0$$

where $u(t)$ is the Hartree evolution (13) and $\Xi(t)$ is governed by a quadratic Hamiltonian on $\mathcal{F}(\mathfrak{H})$. The scale factor \sqrt{N} appears naturally as the particle number expectation of the coherent state $W(f)\Omega$ is $\|f\|^2$.

The convergence (17) has been justified rigorously by Hepp [29] and Ginibre and Velo [22, 23] for $\beta = 0$, and then by Grillakis and Machedon [25] for $0 \leq \beta < 1/3$, based on their previous works with Margetis [27, 28].

Note that by projecting the Fock-space approximation (17) onto the N -particle sector \mathfrak{H}^N , it is possible to derive a norm approximation for the time evolution initiated by a Hartree state $u(0)^{\otimes N}$ (see [36, Sec. 3]). This technique was first introduced by Rodnianski and Schlein in [52] to obtain the error estimate for the Hartree dynamics (see also [5, 11]). The coherent state approach, however, has two obvious drawbacks.

- First, projecting from Fock space to \mathfrak{H}^N makes certain estimates *weaker*. For example, it was shown in [25] that the coherent state

approximation (17) is valid for all $0 \leq \beta < 1/3$, but this only gives a meaningful approximation on \mathfrak{H}^N for $0 \leq \beta < 1/6$.

- Second, and more seriously, the initial state $u(0)^{\otimes N}$ is not really the physically relevant one. Recall that the ground state of H_N^V admits the approximation (9) and it is never close to a Hartree state $u(0)^{\otimes N}$ in norm (except when $w \equiv 0$).

Recently, a direct approach for N -particle initial states has been proposed in [36] by Lewin, Schlein and the first author of the present paper, based on ideas introduced in [37]. They considered the N -particle initial states of the form

$$(18) \quad \Psi_N(0) = \sum_{n=0}^N u(0)^{\otimes(N-n)} \otimes_s \psi_n(0)$$

where $(\psi_n(0))_{n=0}^\infty \in \mathcal{F}(\{u(0)\}^\perp)$. This form is motivated by the ground state property (9) of trapped systems. It was proved in [36] that when $\beta = 0$, the time evolution $\Psi_N(t) = e^{-itH_N} \Psi_N(0)$ satisfies the norm approximation

$$(19) \quad \lim_{N \rightarrow \infty} \left\| \Psi_N(t) - \sum_{n=0}^N u(t)^{\otimes(N-n)} \otimes_s \psi_n(t) \right\|_{\mathfrak{H}^N} = 0$$

where $u(t)$ is the Hartree evolution (13) and $\Phi(t) = (\psi(t))_{n=0}^\infty \in \mathcal{F}(\{u(t)\}^\perp)$ is generated by a quadratic Bogoliubov Hamiltonian (which is different from the effective Hamiltonian governing the evolution of $\Xi(t)$ in (17)).

In the present paper, we prove that the norm convergence (19) holds true for all $0 \leq \beta < 1/3$. This result can not be obtained by a straightforward modification of the proof in [36]. The main new ingredient is a different way to control the number of the particles outside of the condensate when $\beta > 0$. More precisely, we will show that the one-particle density matrices of the Bogoliubov dynamics $\Phi(t)$ satisfy a pair of Schrödinger-type linear equations, which then allow us to obtain the desired bound on the number of particles in the state $\Phi(t)$ by PDE techniques. Our approach is inspired by [25] where similar equations were used. However, our derivation of the equations is different and much simpler than that of [25].

The condition $0 \leq \beta < 1/3$ is typical for the mean-field regime. If $\beta > 1/3$, then the analysis becomes more complicated due to the strong correlation between particles. Very recently, independently to our work, new results for the Fock space norm approximation with $\beta > 1/3$ have been obtained by

Boccatto, Cenatiempo and Schlein [8], Grillakis and Machedon [26] and Kuz [32]. In [32], the author extends the result in [25] for $\beta < 1/2$. In [26], the authors prove a result similar to (17) for $\beta < 2/3$, but now the mean-field dynamics $u(t)$ and the quadratic generator have to be modified. In [8], the authors consider initial data of the form $W(\sqrt{N}u(0))\Phi(0)$ with a special quasi-free state $\Phi(0)$ and their result holds for $\beta < 1$.

Note that in our paper, we do not put any special assumptions on the initial states except the known properties of ground states of trapped systems. Therefore, the requirement $0 < \beta < 1/3$ seems to be reasonable for this large class of initial data. We expect that by using ideas from [8, 26], we can improve our result for larger β (and for more specified initial data), although the analysis in N -particle Hilbert space should be more complicated than that in Fock space. We hope to come back to this issue in the future.

In the most interesting case, $\beta = 1$, the norm approximation to the quantum dynamics is an open problem.

Finally, let us remark that our method is quite general and it can be applied to many different situations. For example, our result can be extended to $d = 1$ or $d = 2$ dimensions with attractive interaction potential (i.e. $w < 0$), with or without external potential, provided that $\beta < 1/d$ (cf. Remarks 3 and 4).

The precise statement of our main result will be given in the next section.

2. Main result

In this section we present our overall strategy and state our main theorem.

2.1. Fock space formalism

Let us quickly recall the Fock space formalism which is used throughout the paper. On the Fock space

$$\mathcal{F}(\mathfrak{H}) = \bigoplus_{n=0}^{\infty} \mathfrak{H}^n = \mathbb{C} \oplus \mathfrak{H} \oplus \mathfrak{H}^2 \oplus \dots$$

we can define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ for every $f \in \mathfrak{H}$ by

$$(a^*(f)\Psi)(x_1, \dots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} f(x_j)\Psi(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1})$$

$$(a(f)\Psi)(x_1, \dots, x_{n-1}) = \sqrt{n} \int \overline{f(x_n)}\Psi(x_1, \dots, x_n)dx_n$$

for all $\Psi \in \mathfrak{H}^n$ and for all n . These operators satisfy the canonical commutation relations (CCR)

$$(20) \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0, \quad [a(f), a^*(g)] = \langle f, g \rangle$$

for all $f, g \in \mathfrak{H}$. The creation and annihilation operators are widely used to represent many other operators on Fock space. The following result is well-known; see e.g. [7] or [55, Lemmas 7.8 and 7.12].

Lemma 1 (Second quantization). *Let H be a symmetric operator on \mathfrak{H} and let $\{f_n\}_{n \geq 1} \subset D(h)$ be an orthonormal basis for \mathfrak{H} . Then*

$$(21) \quad d\Gamma(H) := 0 \oplus \bigoplus_{N=1}^{\infty} \sum_{j=1}^N H_j = \sum_{m,n \geq 1} \langle f_m, H f_n \rangle a^*(f_m) a(f_n).$$

Let W be a symmetric operator on $\mathfrak{H} \otimes \mathfrak{H}$ and let $\{f_n\}_{n \geq 1}$ be an orthonormal basis for \mathfrak{H} such that $f_m \otimes f_n \in D(W)$ and

$$\langle f_m \otimes f_n, W f_p \otimes f_q \rangle = \langle f_n \otimes f_m, W f_p \otimes f_q \rangle$$

for all $m, n, p, q \geq 1$. Then

$$(22) \quad 0 \oplus 0 \oplus \bigoplus_{N=2}^{\infty} \sum_{1 \leq i < j \leq N} W_{ij}$$

$$= \frac{1}{2} \sum_{m,n,p,q \geq 1} \langle f_m \otimes f_n, W f_p \otimes f_q \rangle_{\mathfrak{H}^2} a^*(f_m) a^*(f_n) a(f_p) a(f_q).$$

If one does not want to work on a specific orthonormal basis, it is possible to use the operator-valued distributions a_x^* and a_x , with $x \in \mathbb{R}^3$, defined by

$$a^*(f) = \int_{\mathbb{R}^3} f(x) a_x^* dx \quad \text{and} \quad a(f) = \int_{\mathbb{R}^3} \overline{f(x)} a_x dx$$

for all $f \in \mathfrak{H}$. The canonical commutation relations (20) then imply that

$$(23) \quad [a_x^*, a_y^*] = [a_x, a_y] = 0 \quad \text{and} \quad [a_x, a_y^*] = \delta(x - y).$$

The second quantization formulas (21) and (22) can be rewritten as

$$(24) \quad d\Gamma(H) = \iint H(x, y) a_x^* a_y \, dx dy,$$

$$(25) \quad 0 \oplus 0 \oplus \bigoplus_{N=2}^{\infty} \sum_{1 \leq i < j \leq N} W_{ij} \\ = \frac{1}{2} \iiint\!\!\!\int W(x, y; x', y') a_x^* a_y^* a_{x'} a_{y'} \, dx dy dx' dy'$$

where $H(x, y)$ and $W(x, y; x', y')$ are the kernels of H and W , respectively.

For example, the particle number operator can be written as

$$\mathcal{N} := d\Gamma(1) = \bigoplus_{n=0}^n n \mathbf{1}_{\mathfrak{H}^n} = \int_{\mathbb{R}^3} a_x^* a_x \, dx$$

and the N -body Hamiltonian H_N can be extended to an operator on Fock space $\mathcal{F}(\mathfrak{H})$ as

$$(26) \quad H_N = d\Gamma(-\Delta) + \frac{1}{2(N-1)} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_N(x-y) a_x^* a_y^* a_x a_y \, dx dy.$$

2.2. Fluctuations around Hartree states

As discussed in the introduction, the starting point of our analysis is the Bose-Einstein condensation described by the Hartree equation. The following well-posedness of Hartree equation is taken from [25, Proposition 3.3].

Lemma 2 (Hartree evolution). *For every initial datum $u(0, \cdot) \in H^s(\mathbb{R}^3)$, $s \geq 1$, the Hartree equation (13) has a unique global solution $u(t, x)$ and*

$$\|u(t, \cdot)\|_{H^s(\mathbb{R}^3)} \leq C < \infty$$

for a constant C depending only on $\|u(0, \cdot)\|_{H^s(\mathbb{R}^3)}$ (independent of N and β). Moreover, if $u(0) \in W^{\ell,1}(\mathbb{R}^3)$ with ℓ sufficiently large, then

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C_1}{(1+t)^{3/2}}$$

for a constant C_1 depending only on $\|u(0)\|_{W^{\ell,1}(\mathbb{R}^3)}$.

A similar result for the cubic NLS has been proved in [10]. In the following, we will always denote by $u(t) = u(t, \cdot)$ the solution to the Hartree equation (13) with an initial datum $u(0) \in H^2(\mathbb{R}^3)$. In particular, by Sobolev's embedding $H^2(\mathbb{R}^3) \subset C(\mathbb{R}^3)$ we have the uniform bound $\|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq C$ for a constant C depending only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$.

To describe the particles outside of the condensate, we introduce

$$Q(t) := 1 - |u(t)\rangle\langle u(t)|, \quad \mathfrak{H}_+(t) := Q(t)\mathfrak{H} = \{u(t)\}^\perp$$

and the excited Fock space $\mathcal{F}_+(t) \subset \mathcal{F}(\mathfrak{H})$:

$$\mathcal{F}_+(t) := \mathcal{F}(\mathfrak{H}_+(t)) = \bigoplus_{n=0}^\infty \mathfrak{H}_+(t)^n = \bigoplus_{n=0}^\infty \bigotimes_{\text{sym}}^n \mathfrak{H}_+(t).$$

The corresponding particle number operator is

$$\mathcal{N}_+(t) := d\Gamma(Q) = \bigoplus_{n=0}^\infty n \mathbb{1}_{\mathfrak{H}_+^n(t)} = \mathcal{N} - a^*(u(t))a(u(t)).$$

As in [37, Sec. 2.3], we can decompose any function $\Psi \in \mathfrak{H}^N$ as

$$\Psi = \sum_{n=0}^N u(t)^{\otimes(N-n)} \otimes_s \psi_n = \sum_{n=0}^N \frac{(a^*(u(t)))^{N-n}}{\sqrt{(N-n)!}} \psi_n$$

with $\psi_n \in \mathfrak{H}_+(t)^n$, and this gives rises the *unitary* operator

$$(27) \quad U_N(t) : \mathfrak{H}^N \rightarrow \mathcal{F}_+^{\leq N}(t) := \bigoplus_{n=0}^N \mathfrak{H}_+(t)^n$$

$$\Psi \mapsto \psi_0 \oplus \psi_1 \oplus \dots \oplus \psi_N.$$

In our analysis, the unitary operator $U_N(t)$ plays the same role as Weyl's unitary operator $W(\sqrt{N}u(t))$, which has been used in [22, 23, 25, 27–29] to investigate the fluctuations around coherent states. However, the operator $U_N(t)$ is more suitable to play with on N -particle sector \mathfrak{H}^N .

2.3. Bogoliubov’s approximation

Following [36], we will consider

$$(28) \quad \Phi_N(t) := U_N(t)\Psi_N(t).$$

The vector $\Phi_N(t)$ belongs to $\mathcal{F}_+^{\leq N}(t)$ and it satisfies the equation

$$(29) \quad \begin{cases} i\partial_t\Phi_N(t) = \left[i(\partial_t U_N(t))U_N^*(t) + U_N(t)H_N U_N^*(t) \right] \Phi_N(t), \\ \Phi_N(0) = U_N(0)\Psi_N(0). \end{cases}$$

The first key ingredient in our approach is the following approximation

$$(30) \quad i(\partial_t U_N(t))U_N^*(t) + U_N(t)H_N U_N^*(t) \approx \mathbb{H}(t),$$

where $\mathbb{H}(t)$ is derived from Bogoliubov’s theory:

$$(31) \quad \mathbb{H}(t) := d\Gamma(-\Delta + |u(t)|^2 * w_N - \mu_N(t) + K_1(t)) + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(K_2(t, x, y)a^*(x)a^*(y) + \overline{K_2(t, x, y)}a(x)a(y) \right) dx dy.$$

Here $K_1(t) = Q(t)\tilde{K}_1(t)Q(t)$ where $\tilde{K}_1(t)$ is the operator on \mathfrak{H} with kernel $\tilde{K}_1(t, x, y) = u(t, x)w_N(x - y)\overline{u(t, y)}$, and $K_2(t, \cdot, \cdot) = Q(t) \otimes Q(t)\tilde{K}_2(t, \cdot, \cdot) \in \mathfrak{H}^2$ with $\tilde{K}_2(t, x, y) = u(t, x)w(x - y)u(t, y)$.

When $\beta = 0$, the approximation (30) in the meaning of quadratic forms has been justified in [36], inspired by ideas in [37]. To deal with the case $0 \leq \beta < 1/3$, we will need the following operator bound.

Proposition 3 (Bogoliubov’s approximation). *Let $\beta \geq 0$ and $N \in \mathbb{N}$ arbitrary. Let $u(t)$ be the Hartree evolution with initial datum $u(0) \in H^2(\mathbb{R}^3)$. Denote*

$$R(t) = \mathbf{1}_{\mathcal{F}_+^{\leq N}(t)} \left[i(\partial_t U_N(t))U_N^*(t) + U_N(t)H_N U_N^*(t) - \mathbb{H}(t) \right] \mathbf{1}_{\mathcal{F}_+^{\leq N}(t)}.$$

Then $R(t) = R^*(t)$ and

$$(32) \quad R^2(t) \leq C \left(N^{6\beta-2} \mathcal{N}_+^4(t) + N^{3\beta-1} \mathcal{N}_+^3(t) + N^{3\beta-2} \right)$$

on $\mathcal{F}(\mathfrak{H})$, for a constant C depending only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$.

A bound similar to (32) has been used in [45, Theorem 1] to study the collective excitation spectrum and stationary states of mean-field Bose gases. For the reader’s convenience, we will provide a full proof of Proposition 3 in Section 3.

Recall that we are interested in the evolution of the N -particle initial states of the form (18):

$$\Psi_N(0) = \sum_{n=0}^N u(0)^{\otimes(N-n)} \otimes_s \psi_n(0)$$

where $\Phi(0) := (\psi_n(0))_{n=0}^\infty \in \mathcal{F}_+(0)$. Under this choice,

$$\Phi_N(0) = U_N \Psi_N(0) = (\psi_n(0))_{n=0}^N$$

converges in norm to $\Phi(0)$. Combining with Bogoliubov’s approximation (30), we may expect that the evolution $\Phi_N(t)$ in (29) is close (in norm) to the solution of the effective Bogoliubov equation

$$(33) \quad \begin{cases} i\partial_t \Phi(t) = \mathbb{H}(t)\Phi(t), \\ \Phi(t = 0) = \Phi(0). \end{cases}$$

The existence and uniqueness of the solution of (33) in the quadratic form domain of $\mathbb{H}(t)$ have been proved in [36, Theorem 7]. Moreover, the proof in [36] also gives a bound on $\langle \Phi(t), \mathcal{N}\Phi(t) \rangle$ which, in particular, depends on $\|K_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}$. Indeed, a natural way to bound $\langle \Phi(t), \mathcal{N}\Phi(t) \rangle$ is to compute the derivative

$$\frac{d}{dt} \langle \Phi(t), \mathcal{N}\Phi(t) \rangle = -\langle \Phi(t), i[\mathcal{N}, \mathbb{H}]\Phi(t) \rangle$$

and then use Grönwall’s inequality. This requires a bound on the commutator $i[\mathcal{N}, \mathbb{H}]$ in terms of \mathcal{N} . To our knowledge, the best known bound of this type is

$$i[\mathcal{N}, \mathbb{H}] \leq C \|K_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} (\mathcal{N} + 1)$$

(see e.g. [36, Lemma 9]). Unfortunately, when $\beta > 0$,

$$\|K_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \sim \iint |u(t, x)|^2 |w_N(x - y)|^2 |u(t, y)|^2 dx dy \sim N^{3\beta},$$

and the Grönwall argument gives a bound on $\langle \Phi(t), \mathcal{N}\Phi(t) \rangle$ of the order $\exp(N^{3\beta/2})$, which is too big for our purposes.

The main new ingredient in our paper is a uniform bound on $\langle \Phi(t), \mathcal{N}\Phi(t) \rangle$, for any $\beta \geq 0$. More precisely, we have the following

Proposition 4 (Bogoliubov equation). *Let $\beta \geq 0$ and $N \in \mathbb{N}$ arbitrary. Let $u(t)$ be the Hartree evolution with initial datum $u(0) \in H^2(\mathbb{R}^3)$. Then for every initial state $\Phi(0) \in \mathcal{F}_+(0)$ satisfying $\langle \Phi(0), \mathcal{N}\Phi(0) \rangle < \infty$, the equation (33) has a unique global solution $\Phi(t)$. Moreover, $\Phi(t) \in \mathcal{F}_+(t)$ for all $t \geq 0$ and the following statements hold true.*

- (i) *The pair of density matrices $(\gamma(t), \alpha(t)) = (\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$ is the unique solution to the following system of one-body linear equations*

$$(34) \quad \begin{cases} i\partial_t \gamma = h\gamma - \gamma h + K_2 \alpha - \alpha^* K_2^*, \\ i\partial_t \alpha = h\alpha + \alpha h^T + K_2 + K_2 \gamma^T + \gamma K_2, \\ \gamma(t=0) = \gamma_{\Phi(0)}, \quad \alpha(t=0) = \alpha_{\Phi(0)}. \end{cases}$$

Here $h(t) = -\Delta + |u(t)|^2 * w_N - \mu_N(t) + K_1(t)$; $K_2(t) : \overline{\mathfrak{H}} \rightarrow \overline{\mathfrak{H}}$ is the operator with kernel $K_2(t, x, y)$; and $\gamma^T : \overline{\mathfrak{H}} \rightarrow \overline{\mathfrak{H}}$ is the operator with kernel $\gamma^T(t, x, y) = \gamma(t, y, x)$.

- (ii) *We have*

$$(35) \quad \|\alpha(t)\|_{\text{HS}}^2 + \|\gamma(t)\|_{\text{HS}}^2 \leq e^{Ct} (1 + \|\alpha(0)\|_{\text{HS}}^2 + \|\gamma(0)\|_{\text{HS}}^2).$$

- (iii) *If $\Phi(0)$ is a quasi-free state, then $\Phi(t)$ is a quasi-free state for all t and*

$$(36) \quad \langle \Phi(t), \mathcal{N}\Phi(t) \rangle \leq e^{Ct} \left(1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^2$$

for a constant C depending only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$. Moreover, if $u(0) \in W^{\ell,1}(\mathbb{R}^3)$ with ℓ sufficiently large, then

$$(37) \quad \langle \Phi(t), \mathcal{N}\Phi(t) \rangle \leq C_1 \left(\log(1+t) + 1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^2$$

for a constant C_1 depending only on $\|u(0)\|_{W^{\ell,1}(\mathbb{R}^3)}$.

Proposition 4 is a consequence of an abstract result proved in Section 4 on the evolution generated by a general quadratic Hamiltonian on Fock space.

The N -independent estimate (36) plays an essential role in our analysis and it can be derived directly from the equations (34).

Our derivation of (34) and (36) is inspired from the analysis in [25]. In fact, the bound (36) is similar to the pairing estimate in [25, Theorem 4.1] and the equations (34) is analogous to the pairing equations (17b)-(17c) in [25] (see also [27, 28] for earlier results). To be more precise, let us consider the case when $\Phi(t)$ is a quasi-free state. In this case, $\Phi(t) = T(t)\Omega$ for a unique Bogoliubov transformation on $\mathcal{F}(\mathfrak{H})$, and the equation (33) becomes

$$(38) \quad \left[T^*(t)(i\partial_t T(t)) - T^*(t)\mathbb{H}(t)T(t) \right] \Omega = 0.$$

In [25], the explicit form

$$T(t) = \exp \left(i\chi_N(t) + \iint \left[\overline{k(t, x, y)} a_x a_y - k(t, x, y) a_x^* a_y^* \right] dx dy \right)$$

has been taken, where $\chi_N(t) \in \mathbb{R}$ is a phase factor, and the pairing equations for $k(t, x, y)$ [25, Eqs. (17b)-(17c)] have been derived such that

$$T^*(t)(i\partial_t T(t)) - T^*(t)\mathbb{H}(t)T(t) = d\Gamma(\xi)$$

for some operator $\xi : \mathfrak{H} \rightarrow \mathfrak{H}$, which ensures that (38) holds true.

Our derivation of the linear equations (34) is different from and much shorter than the representation in [25, 27, 28]. In fact, (34) follows quickly from (33) by analyzing the dynamics of the two-point correlation functions $\langle \Psi_N(t), a_x^* a_y \Psi_N(t) \rangle$ and $\langle \Psi_N(t), a_x a_y^* \Psi_N(t) \rangle$.

The first statement in (iii) is a general fact that the set of quasi-free states is stable under the evolution generated by a time-dependent quadratic Hamiltonian. This interesting statement should be well-known but we could not localize a precise reference. As pointed out to us by Jan Dereziński (private communication), this statement follows from a similar statement for the evolution generated by a time-independent quadratic Hamiltonian and the closedness of the metaplectic group in Fock space. In the present paper, we will show that this statement is a direct consequence of the linear equations (34).

The last ingredient in our approach is the following

Lemma 5 (Fluctuations of quasi-free states). *For all $\ell \geq 1$, there exists a constant $C_\ell > 0$ such that for all quasi-free states Ψ in $\mathcal{F}(\mathfrak{H})$:*

$$\langle \Psi, \mathcal{N}^\ell \Psi \rangle \leq C_\ell (1 + \langle \Psi, \mathcal{N} \Psi \rangle)^\ell.$$

This result is well-known and a proof is provided in Section 5 for completeness. In our application, the case $\ell = 4$ is sufficient to control the error in Proposition 3.

2.4. Main result

Now we are able to state our main result.

Theorem 6 (Bogoliubov correction to mean-field dynamics). *Let $u(t)$ be the Hartree evolution in (13) with an initial state $u(0) \in H^2(\mathbb{R}^3)$. Let $\Phi(t) = (\psi_n(t))_{n=0}^\infty \in \mathcal{F}_+(t)$ be the Bogoliubov evolution in (33) with an initial quasi-free state $\Phi(0) = (\psi_n(0))_{n=0}^\infty \in \mathcal{F}_+(0)$. Then the Schrödinger evolution $\Psi_N(t) = e^{-itH_N}\Psi_N(0)$ with the initial state*

$$(39) \quad \Psi_N(0) = \sum_{n=0}^N u(0)^{\otimes(N-n)} \otimes_s \psi_n(0)$$

satisfies the following norm approximation:

$$(40) \quad \left\| \Psi_N(t) - \sum_{n=0}^N u(t)^{\otimes(N-n)} \otimes_s \psi_n(t) \right\|_{\mathfrak{H}^N} \leq C_0(t)N^{(3\beta-1)/2},$$

where

$$C_0(t) \leq e^{Ct} \left(1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^4$$

for a constant C depending only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$. Moreover, if $u(0) \in W^{\ell,1}(\mathbb{R}^3)$ with ℓ sufficiently large, then

$$C_0(t) \leq C_1(1+t) \left(1 + \log(1+t) + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^4$$

for a constant C_1 depending only on $\|u(0)\|_{W^{\ell,1}(\mathbb{R}^3)}$.

The proof of Theorem 6 will be provided in Section 5. Let us give some remarks on the result.

Remark 1. Since e^{-itH_N} is a unitary operator on \mathfrak{H}^N , the convergence

$$(41) \quad \lim_{N \rightarrow \infty} \left\| \Psi_N(t) - \sum_{n=0}^N u(t)^{\otimes(N-n)} \otimes_s \psi_n(t) \right\|_{\mathfrak{H}^N} = 0$$

still holds when (39) is replaced by the weaker assumption

$$\lim_{N \rightarrow \infty} \left\| \Psi_N(0) - \sum_{n=0}^N u(0)^{\otimes(N-n)} \otimes_s \psi_n(0) \right\|_{\mathfrak{H}^N} = 0.$$

Strictly speaking, the initial vector $\Phi_N(0)$ chosen in (39) is not normalized, but its norm converges to 1 and the renormalization is trivial.

We also note that the initial data $u(0)$ and $\Phi(0)$ in Theorem 6 can be chosen to be N -dependent, provided that the N -dependences of $\|u(0)\|_{H^s(\mathbb{R}^3)}$ and $\langle \Phi(0), \mathcal{N}\Phi(0) \rangle$ can be compensated by $N^{3\beta-1}$.

Remark 2. In many physical applications, one is often interested in the projection $|\Psi\rangle\langle\Psi|$ of a wave function instead of the wave function Ψ itself. From (41) we obtain

$$(42) \quad \lim_{N \rightarrow \infty} \text{Tr}_{\mathfrak{H}^N} \left| |\Psi_N(t)\rangle\langle\Psi_N(t)| - U_N^* |\Phi(t)\rangle\langle\Phi(t)| U_N(t) \right| = 0.$$

When $\Phi(t)$ is a quasi-free state, the projection $|\Phi(t)\rangle\langle\Phi(t)|$ is determined uniquely by its density matrices. Thus $|\Psi_N(t)\rangle\langle\Psi_N(t)|$ can be well approximated in trace norm using $(u(t), \gamma(t), \alpha(t))$ which, in principle, can be computed as accurate as we want using the one-body equations (33) and (34).

Moreover, since the one-particle density matrices can be obtained by taking the partial trace, namely

$$N^{-1}\gamma_{\Psi_N(t)} = \text{Tr}_{2 \rightarrow N} |\Psi_N(t)\rangle\langle\Psi_N(t)|,$$

the convergence (42) implies immediately the Bose-Einstein condensation

$$(43) \quad \lim_{N \rightarrow \infty} \text{Tr} \left| N^{-1}\gamma_{\Psi_N(t)} - |u(t)\rangle\langle u(t)| \right| = 0.$$

Note that, when $\beta > 0$, the Hartree dynamics $u(t)$ converges to the NLS dynamics $v(t)$ in (15) as $N \rightarrow \infty$. Therefore (43) is equivalent to

$$\lim_{N \rightarrow \infty} \text{Tr} \left| N^{-1}\gamma_{\Psi_N(t)} - |v(t)\rangle\langle v(t)| \right| = 0.$$

Remark 3. Our result is stated and proved in three dimensions, but it can be extended straightforwardly to one and two dimensions. More precisely, in $d \leq 3$ dimensions, we can consider the two-body interaction potential of the form $w_N(x - y) = N^{d\beta}w(N^\beta(x - y))$, and the result in Theorem (6) still holds true (on the right side of (40) the error now becomes $C_0(t)N^{(d\beta-1)/2}$).

Remark 4. Note that Lemma 2 is the only place where we need the assumption $w \geq 0$. The rest of our proof does not require the sign assumption on w (cf. Remark 5). In particular, our result can be extended to one or two dimensions with attractive interaction potential (i.e. $w < 0$), provided the well-posedness of the corresponding Hartree equation, as in Lemma 2, still holds. In particular, our method covers the derivation of the 1D and 2D focusing dynamics with a harmonic trap (see [13, 14] for results on the leading order).

3. Bogoliubov’s approximation

In this section we justify Bogoliubov’s approximation (30).

Proof of Proposition 3. Let us denote $\mathbb{1}_+^{\leq N} = \mathbb{1}_{\mathcal{F}_+^{\leq N}(t)} = \mathbb{1}(\mathcal{N}_+(t) \leq N)$ for short. Recall that from the calculations in [36, Eqs. (40)-(41)], we have

$$\begin{aligned}
 (44) \quad R(t) &= \mathbb{1}_+^{\leq N} \left[i(\partial_t U_N(t)) U_N^*(t) + U_N(t) H_N U_N^*(t) - \mathbb{H}(t) \right] \mathbb{1}_+^{\leq N} \\
 &= \frac{1}{2} \sum_{j=1}^5 \mathbb{1}_+^{\leq N} (R_j + R_j^*) \mathbb{1}_+^{\leq N}
 \end{aligned}$$

where

$$\begin{aligned}
 R_1 &= R_1^* = d\Gamma(Q(t)[w_N * |u(t)|^2 + K_1 - \mu_N(t)]Q(t)) \frac{1 - \mathcal{N}_+(t)}{N - 1}, \\
 R_2 &= -2 \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} a(Q(t)[w_N * |u(t)|^2]u(t)), \\
 R_3 &= \iint K_2(t, x, y) a_x^* a_y^* dx dy \left(\frac{\sqrt{(N - \mathcal{N}_+(t))(N - \mathcal{N}_+(t) - 1)}}{N - 1} - 1 \right), \\
 R_4 &= R_4^* = \frac{1}{2(N - 1)} \iiint (Q(t) \otimes Q(t) w_N Q(t) \otimes Q(t))(x, y; x', y') \\
 &\quad \times a_x^* a_y^* a_{x'} a_{y'} dx dy dx' dy', \\
 R_5 &= \frac{\sqrt{N - \mathcal{N}_+(t)}}{N - 1} \iiint (1 \otimes Q(t) w_N Q(t) \otimes Q(t))(x, y; x', y') \\
 &\quad \times \overline{u(t, x)} a_y^* a_{x'} a_{y'} dx dy dx' dy'.
 \end{aligned}$$

By the Cauchy-Schwarz inequality we have that

$$(45) \quad R^2(t) \leq 100 \sum_{j=1}^5 \mathbb{1}_+^{\leq N} (R_j \mathbb{1}_+^{\leq N} R_j^* + R_j^* \mathbb{1}_+^{\leq N} R_j) \mathbb{1}_+^{\leq N}.$$

Now we estimate all terms on the right side of (45). We will always denote by C a constant depending only on $\|u(0)\|_{H^s(\mathbb{R}^3)}$.

$\mathbf{j} = \mathbf{1}$. Using $\|w_N\|_{L^1} = \|w\|_{L^1}$ we get

$$(46) \quad \|w_N * |u(t)|^2\|_{L^\infty(\mathbb{R}^3)} \leq \|w\|_{L^1} \|u(t)\|_{L^\infty}^2 \leq C.$$

Similarly,

$$(47) \quad \begin{aligned} |\mu_N(t)| &= \frac{1}{2} \left| \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |u(t, x)|^2 w_N(x - y) |u(t, y)|^2 dx dy \right| \\ &\leq \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R}^3)}^2 \|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \|w_N\|_{L^1(\mathbb{R}^3)} \leq C. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| \langle f, \tilde{K}_1(t)g \rangle \right| &= \left| \iint \overline{f(x)} u(t, x) w_N(x - y) \overline{u(t, y)} g(y) dx dy \right| \\ &\leq \|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \left(\iint |f(x)|^2 |w_N(x - y)| dx dy \right)^{1/2} \\ &\quad \times \left(\iint |g(y)|^2 |w_N(x - y)| dx dy \right)^{1/2} \\ &\leq C \|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

for all $f, g \in L^2(\mathbb{R}^3)$. Therefore,

$$(48) \quad \|K_1(t)\| = \|Q(t)\tilde{K}_1(t)Q(t)\| \leq \|\tilde{K}_1(t)\| \leq C \|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \leq C.$$

Thus, in summary,

$$\pm d\Gamma(Q(t)[w_N * |u(t)|^2 + K_1 - \mu(t)]Q(t)) \leq C d\Gamma(Q(t)) = C\mathcal{N}_+(t).$$

Since $d\Gamma(Q(t)[w_N * |u(t)|^2 + K_1 - \mu(t)]Q(t))$ commutes with $\mathcal{N}_+(t)$, we find that

$$\begin{aligned} R_1^2 &= d\Gamma(Q(t)[w_N * |u(t)|^2 + K_1 - \mu(t)]Q(t))^2 \frac{(\mathcal{N}_+(t) - 1)^2}{(N - 1)^2} \\ &\leq C\mathcal{N}_+(t)^2 \frac{(\mathcal{N}_+(t) - 1)^2}{(N - 1)^2} \leq C \frac{\mathcal{N}_+^4(t)}{N^2}. \end{aligned}$$

Consequently,

$$(49) \quad R_1 \mathbf{1}_+^{\leq N} R_1 \leq R_1^2 \leq C \frac{\mathcal{N}_+^4(t)}{N^2}.$$

$\mathbf{j} = \mathbf{2}$. Note that $v := Q(t)[w_N * |u(t)|^2]u(t)$ satisfies

$$\begin{aligned} \|v\|_{L^2(\mathbb{R}^3)} &\leq \|w_N * |u(t)|^2 u(t)\|_{L^2(\mathbb{R}^3)} \\ &\leq \|w_N * |u(t)|^2\|_{L^\infty(\mathbb{R}^3)} \|u(t)\|_{L^2(\mathbb{R}^3)} \leq C. \end{aligned}$$

Using $a(v)a^*(v) = \|v\|^2 + a^*(v)a(v) \leq \|v\|^2(\mathcal{N}_+(t) + 1)$, we get

$$\begin{aligned} R_2 \mathbf{1}_+^{\leq N} R_2^* &= 4 \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} a(v) \mathbf{1}_+^{\leq N} a^*(v) \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} \\ &= 4 \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} \mathbf{1}^{\leq N-1} a(v) a^*(v) \mathbf{1}^{\leq N-1} \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} \\ &\leq C \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} \mathbf{1}^{\leq N-1} (\mathcal{N}_+ + 1) \mathbf{1}^{\leq N-1} \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} \\ &\leq C \frac{\mathcal{N}_+^3}{N}. \end{aligned}$$

Similarly, using

$$\begin{aligned} R_2^* &= -2a^*(v) \frac{\mathcal{N}_+(t) \sqrt{N - \mathcal{N}_+(t)}}{N - 1} \\ &= -2 \frac{(\mathcal{N}_+(t) - 1) \sqrt{N - \mathcal{N}_+(t) + 1}}{N - 1} a^*(v) \end{aligned}$$

we find that

$$\begin{aligned} R_2 \mathbf{1}_+^{\leq N} R_2^* &= 4 \frac{(\mathcal{N}_+(t) - 1) \sqrt{N - \mathcal{N}_+(t) + 1}}{N - 1} a^*(v) \mathbf{1}_+^{\leq N} a(v) \\ &\quad \times \frac{(\mathcal{N}_+(t) - 1) \sqrt{N - \mathcal{N}_+(t) + 1}}{N - 1} \\ &= 4 \frac{(\mathcal{N}_+(t) - 1) \sqrt{N - \mathcal{N}_+(t) + 1}}{N - 1} \mathbf{1}^{\leq N+1} a^*(v) a(v) \\ &\quad \times \mathbf{1}^{\leq N+1} \frac{(\mathcal{N}_+(t) - 1) \sqrt{N - \mathcal{N}_+(t) + 1}}{N - 1} \\ &\leq C \frac{(\mathcal{N}_+(t) - 1) \sqrt{N - \mathcal{N}_+(t) + 1}}{N - 1} \mathbf{1}^{\leq N+1} (\mathcal{N}_+(t) + 1) \\ &\quad \times \mathbf{1}^{\leq N+1} \frac{(\mathcal{N}_+(t) - 1) \sqrt{N - \mathcal{N}_+(t) + 1}}{N - 1} \\ &\leq C \frac{\mathcal{N}_+^3(t)}{N}. \end{aligned}$$

Thus

$$(50) \quad R_2^* \mathbf{1}_{\mp}^{\leq N} R_2 + R_2 \mathbf{1}_{\mp}^{\leq N} R_2^* \leq C \frac{\mathcal{N}_{\mp}^3(t)}{N}.$$

$\mathbf{j} = \mathbf{3}$. We can write $R_3 = \mathbb{K}_{\text{cr}} g(\mathcal{N}_+(t))$ where

$$(51) \quad \mathbb{K}_{\text{cr}} := \iint K_2(t, x, y) a_x^* a_y^* dx dy$$

and

$$g(\mathcal{N}_+(t)) := \frac{\sqrt{(N - \mathcal{N}_+(t))(N - \mathcal{N}_+(t) - 1)}}{N - 1} - 1.$$

Let us show that

$$(52) \quad \mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^* + \mathbb{K}_{\text{cr}}^* \mathbb{K}_{\text{cr}} \leq 2 \|K_2(t, \cdot, \cdot)\|_{L^2}^2 (\mathcal{N}_+(t) + 1)^2 \leq CN^{3\beta} (\mathcal{N}_+(t) + 1)^2.$$

Here we have used

$$(53) \quad \begin{aligned} \|K_2(t, \cdot, \cdot)\|_{L^2}^2 &= \|Q(t) \otimes Q(t) \tilde{K}_2(t, \cdot, \cdot)\|_{L^2}^2 \leq \|\tilde{K}_2(t, \cdot, \cdot)\|_{L^2}^2 \\ &= \iint |u(t, x)|^2 |w_N(x - y)|^2 |u(t, y)|^2 dx dy \\ &\leq \|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \|w_N\|_{L^2(\mathbb{R}^3)}^2 \|u(t)\|_{L^2(\mathbb{R}^3)}^2 \leq CN^{3\beta}. \end{aligned}$$

In fact, (52) is well-known (see e.g. [45, eq. (23) and (26)]), but we offer an alternative proof below because the proof strategy will be used later to control R_5 . First, using the Cauchy-Schwarz inequality $XY + Y^*X^* \leq XX^* + Y^*Y$ we get

$$(54) \quad \begin{aligned} \mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^* &= \iiint \iiint K_2(t, x, y) \overline{K_2(t, x', y')} a_x^* a_y^* a_{x'} a_{y'} dx dy dx' dy' \\ &= \frac{1}{2} \iiint \iiint \left(K_2(t, x, y) \overline{K_2(t, x', y')} a_x^* a_y^* a_{x'} a_{y'} + \text{h.c.} \right) dx dy dx' dy' \\ &\leq \frac{1}{2} \iiint \iiint \left(|K_2(t, x', y')|^2 a_x^* a_y^* a_{x'} a_{y'} \right. \\ &\quad \left. + |K_2(t, x, y)|^2 a_x^* a_y^* a_{x'} a_{y'} \right) dx dy dx' dy' \\ &= \|K_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \mathcal{N}_+(t) (\mathcal{N}_+(t) - 1). \end{aligned}$$

Here we have denoted $X + \text{h.c.} = X + X^*$ for short (h.c. stands for Hermitian conjugate). Moreover,

$$\begin{aligned} \mathbb{K}_{\text{cr}}^* \mathbb{K}_{\text{cr}} &= \iiint \overline{K_2(t, x, y)} K_2(t, x', y') a_x a_y a_x^* a_y^* dx dy dx' dy' \\ &= \iiint \overline{K_2(t, x, y)} K_2(t, x', y') a_{x'}^* a_{y'}^* a_x a_y dx dy dx' dy' \\ &\quad + \iiint \overline{K_2(t, x, y)} K_2(t, x', y') [a_x a_y, a_x^* a_y^*] dx dy dx' dy'. \end{aligned}$$

The first term of the right side of nothing but $\mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^*$ which has been already estimated. For the second term, using $K_2(t, x, y) = K_2(t, y, x)$ and

$$\begin{aligned} [a_x a_y, a_x^* a_y^*] &= \delta(x' - y) a_{y'}^* a_x + \delta(x - x') a_{y'}^* a_y + \delta(y - y') a_x^* a_x \\ &\quad + \delta(x - y') a_x^* a_y + \delta(x' - y) \delta(x - y') + \delta(x - x') \delta(y - y') \end{aligned}$$

we find that

$$\begin{aligned} &\iiint \overline{K_2(t, x, y)} K_2(t, x', y') [a_x a_y, a_x^* a_y^*] dx dy dx' dy' \\ &= 4 \iiint \overline{K_2(t, x, y)} K_2(t, y, y') a_{y'}^* a_x dx dy dy' + 2 \iint |K_2(t, x, y)|^2 dx dy \\ &= 4d\Gamma(K_2(t)K_2^*(t)) + 2\|K_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2, \end{aligned}$$

and hence

$$(55) \quad \mathbb{K}_{\text{cr}}^* \mathbb{K}_{\text{cr}} = \mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^* + 4d\Gamma(K_2(t)K_2^*(t)) + 2\|K_2(t, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2.$$

Here we have denoted by $K_2(t) : \mathfrak{H} \rightarrow \mathfrak{H}$ the operator with kernel $K_2(t, x, y)$. Putting differently, $K_2(t) = Q(t)\tilde{K}_2(t)\overline{Q(t)}$ with $\tilde{K}_2(t) : \mathfrak{H} \rightarrow \mathfrak{H}$ the operator with kernel $u(x)w_N(x - y)u(y)$. Similarly to (48) we have

$$(56) \quad \|K_2(t)\| = \|Q(t)\tilde{K}_2(t)\overline{Q(t)}\| \leq \|\tilde{K}_2(t)\| \leq C\|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \leq C.$$

Therefore, $d\Gamma(K_2(t)K_2^*(t)) \leq C\mathcal{N}_+(t)$. Thus (52) follows (55) and (54).

Now from $R_3 = \mathbb{K}_{\text{cr}}g(\mathcal{N}_+(t))$, using (52), (53) and the simple estimates

$$\mathbf{1}_+^{\leq N-2} g^2(\mathcal{N}_+(t)) + \mathbf{1}_+^{\leq N+2} g^2(\mathcal{N}_+(t) - 2) \leq 32 \frac{(\mathcal{N}_+(t) + 1)^2}{N^2} \mathbf{1}_+^{\leq N}$$

we conclude that

$$\begin{aligned}
 (57) \quad & R_3^* \mathbb{1}_+^{\leq N} R_3 + R_3 \mathbb{1}_+^{\leq N} R_3^* \\
 &= g(\mathcal{N}_+(t)) \mathbb{K}_{\text{cr}}^* \mathbb{1}_+^{\leq N} \mathbb{K}_{\text{cr}} g(\mathcal{N}_+(t)) + \mathbb{K}_{\text{cr}} g(\mathcal{N}_+(t)) \mathbb{1}_+^{\leq N} g(\mathcal{N}_+(t)) \mathbb{K}_{\text{cr}}^* \\
 &= g^2(\mathcal{N}_+(t)) \mathbb{1}_+^{\leq N-2} \mathbb{K}_{\text{cr}}^* \mathbb{K}_{\text{cr}} + g^2(\mathcal{N}_+(t) - 2) \mathbb{1}_+^{\leq N+2} \mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^* \\
 &\leq \left(g^2(\mathcal{N}_+(t)) \mathbb{1}_+^{\leq N-2} + g^2(\mathcal{N}_+(t) - 2) \mathbb{1}_+^{\leq N+2} \right) \left(\mathbb{K}_{\text{cr}}^* \mathbb{K}_{\text{cr}} + \mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^* \right) \\
 &\leq CN^{3\beta-2} (\mathcal{N}_+(t) + 1)^4.
 \end{aligned}$$

Here we have also used the fact that $\mathbb{K}_{\text{cr}}^* \mathbb{K}_{\text{cr}}$ and $\mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^*$ commute with $\mathcal{N}_+(t)$.

j = 4. By (25), for every two-body operator $W \geq 0$ one has

$$(58) \quad \frac{1}{2} \iiint \iiint W(x, y; x', y') a_x^* a_y^* a_{x'} a_{y'} \, dx dy dx' dy' \geq 0$$

where $W(x, y; x', y')$ is the kernel of W . Consequently,

$$\begin{aligned}
 \pm R_4 &= \pm \frac{1}{2(N-1)} \iiint \iiint (Q(t) \otimes Q(t) w_N Q(t) \otimes Q(t))(x, y; x', y') \\
 &\quad \times a_x^* a_y^* a_{x'} a_{y'} \, dx dy dx' dy', \\
 &\leq \frac{\|w_N\|_{L^\infty(\mathbb{R}^3)}}{2(N-1)} \iiint \iiint (Q(t) \otimes Q(t))(x, y; x', y') a_x^* a_y^* a_{x'} a_{y'} \, dx dy dx' dy' \\
 &\leq CN^{3\beta-1} \mathcal{N}_+^2(t).
 \end{aligned}$$

Here we have used the simple bound $\|w_N\|_{L^\infty(\mathbb{R}^3)} = N^{3\beta} \|w\|_{L^\infty(\mathbb{R}^3)}$ in the last estimate. Since R_4 commutes with \mathcal{N}_+ we find that

$$(59) \quad R_4 \mathbb{1}_+^{\leq N} R_4 \leq R_4^2 \leq CN^{6\beta-2} \mathcal{N}_+^4(t).$$

j = 5. This is the most complicated case. Recall that

$$R_5 = \frac{\sqrt{N - \mathcal{N}_+(t)}}{N - 1} R_6$$

with

$$R_6 := \iiint \iiint (1 \otimes Q(t) w_N Q(t) \otimes Q(t))(x, y; x', y') \overline{u(t, x)} a_y^* a_{x'} a_{y'} \, dx dy dx' dy'.$$

We will show that

$$(60) \quad R_6^* R_6 + R_6 R_6^* \leq CN^{3\beta} \mathcal{N}_+^3(t).$$

Remark 5. Note that in the following we use $w \geq 0$, but the proof can be adapted easily to cover any w without the sign assumption by decomposing $w = w_+ - w_-$ and treating each term w_{\pm} separately.

We will write $Q = Q(t)$ and $u = u(t)$ for short. We denote

$$Q(x, y) = \delta(x - y) - u(x)\overline{u(y)},$$

the kernel of Q , and introduce the operators

$$(61) \quad b_x := \int_{\mathbb{R}^3} Q(x, y)a_y dy, \quad B_x := \int_{\mathbb{R}^3} w_N(x - y)b_y^* b_y dy \geq 0.$$

The advantage of these notations is that using

$$(1 \otimes Q w_N Q \otimes Q)(x, y; x', y') = \int Q(y, y_1)w_N(x - y_1)Q(x, x')Q(y_1, y') dy_1$$

we can rewrite

$$(62) \quad \begin{aligned} R_6 &= \iiint\iiint Q(y, y_1)w_N(x - y_1) \\ &\quad \times Q(x, x')Q(y_1, y')\overline{u(x)}a_y^* a_{x'} a_{y'} dx dy dx' dy' \\ &= \iint \overline{u(x)}w_N(x - y_1)b_{y_1}^* b_{y_1} b_x dx dy_1 = \int \overline{u(x)}B_x b_x. \end{aligned}$$

Let us list some basic properties of b_x and B_x defined in (61). From the CCR (23) it is straightforward to see that

$$(63) \quad [b_x, b_y] = 0 = [b_x^*, b_y^*], \quad [b_x, b_y^*] = Q(x, y) = \delta(x - y) - u(x)\overline{u(y)}.$$

Moreover,

$$(64) \quad \begin{aligned} \int b_x^* b_x dx &= \iiint\iiint Q(z, x)Q(x, y)a_z^* a_y dx dy dz \\ &= \iint Q(z, y)a_z^* a_y dy dz = d\Gamma(Q) = \mathcal{N}_+(t) \end{aligned}$$

and consequently,

$$(65) \quad B_x \leq \|w_N\|_{L^\infty} \int b_y^* b_y dy \leq CN^{3\beta} \mathcal{N}_+(t),$$

$$(66) \quad \int B_x dx = \int \left(\int w_N(x-y) dx \right) b_y^* b_y dy \leq CN_+(t),$$

$$(67) \quad \int B_x^2 dx \leq CN^{3\beta} \mathcal{N}_+(t) \int B_x dx \leq CN^{3\beta} \mathcal{N}_+^2(t).$$

In the last estimate we have used the fact that B_x commutes with $\mathcal{N}_+(t)$.

Now using (62) we can write

$$(68) \quad \begin{aligned} R_6 R_6^* &= \iint \overline{u(x)} u(y) B_x b_x b_y^* B_y dx dy \\ &= \iint \overline{u(x)} u(y) B_x b_y^* b_x B_y dx dy \\ &\quad + \iint \overline{u(x)} u(y) B_x [b_x, b_y^*] B_y dx dy. \end{aligned}$$

The second term of (68) can be estimated easily using (63) and (67):

$$(69) \quad \begin{aligned} &\iint \overline{u(x)} u(y) B_x [b_x, b_y^*] B_y dx dy \\ &= \int |u(x)|^2 B_x^2 dx - \left(\int |u(x)|^2 B_x dx \right)^2 \\ &\leq \|u\|_{L^\infty(\mathbb{R}^3)}^2 \int B_x^2 dx \leq CN^{3\beta} \mathcal{N}_+^2(t). \end{aligned}$$

To estimate the first term of (68), we employ the Cauchy-Schwarz inequality $XY + Y^* X^* \leq XX^* + Y^* Y$ and obtain

$$(70) \quad \begin{aligned} &\iint \overline{u(x)} u(y) B_x b_y^* b_x B_y dx dy \\ &= \frac{1}{2} \iint \left(\overline{u(x)} u(y) B_x b_y^* b_x B_y + \text{h.c.} \right) dx dy \\ &\leq \frac{1}{2} \iint \left(|u(x)|^2 B_x b_y^* b_y B_x + |u(y)|^2 B_y b_x^* b_x B_y \right) dx dy \\ &\leq \|u\|_{L^\infty(\mathbb{R}^3)}^2 \iint B_x b_y^* b_y B_x dx dy \leq CN^{3\beta} \mathcal{N}_+^3(t). \end{aligned}$$

Here the last estimate follows from (64), (67) and the fact that B_x commutes with \mathcal{N}_+ . Thus, in summary, from (68)-(69)-(70) we get

$$(71) \quad R_6 R_6^* \leq CN^{3\beta} \mathcal{N}_+^3(t).$$

Now we estimate

$$(72) \quad \begin{aligned} R_6^* R_6 &= \iint u(x) \overline{u(y)} b_x^* B_x B_y b_y \, dx dy \\ &= \iint u(x) \overline{u(y)} b_x^* B_y B_x b_y \, dx dy \\ &\quad + \iint u(x) \overline{u(y)} b_x^* [B_x, B_y] b_y \, dx dy. \end{aligned}$$

The first term of (72) can be bounded similarly to the first term of (68). Indeed, by the Cauchy-Schwarz inequality $XY + Y^*X^* \leq XX^* + Y^*Y$ and (64), (67), we have

$$(73) \quad \begin{aligned} &\iint u(x) \overline{u(y)} b_x^* B_y B_x b_y \, dx dy \\ &= \frac{1}{2} \iint \left(u(x) \overline{u(y)} b_x^* B_y B_x b_y + \text{h.c.} \right) \, dx dy \\ &\leq \frac{1}{2} \iint \left(|u(x)|^2 b_x^* B_y^2 b_x + |u(y)|^2 b_y^* B_x^2 b_y \right) \, dx dy \\ &\leq \|u\|_{L^\infty(\mathbb{R}^3)}^2 \iint b_x^* B_y^2 b_x \, dx dy \\ &\leq CN^{3\beta} \int b_x^* \mathcal{N}_+^2(t) b_x \, dx \\ &= CN^{3\beta} \int b_x^* b_x (\mathcal{N}_+(t) + 1)^2 \, dx \\ &\leq CN^{3\beta} \mathcal{N}_+^3(t). \end{aligned}$$

To estimate the second term of (72), we use

$$\begin{aligned} [b_{x'}^*, b_{x'}, b_{y'}^*, b_{y'}] &= b_{x'}^* [b_{x'}, b_{y'}] b_{y'} - b_{y'}^* [b_{y'}, b_{x'}] b_{x'} \\ &= Q(x', y') b_{x'}^* b_{y'} - Q(y', x') b_{y'}^* b_{x'} \\ &= -u(x') \overline{u(y')} b_{x'}^* b_{y'} + u(y') \overline{u(x')} b_{y'}^* b_{x'} \end{aligned}$$

and write

$$\begin{aligned}
 & \iint u(x)\overline{u(y)}b_x^*[B_x, B_y]b_y dx dy \\
 &= \iint u(x)\overline{u(y)}b_x^* \left[\int w_N(x-x')b_{x'}^*b_{x'} dx', \int w_N(y-y')b_{y'}^*b_{y'} dy' \right] b_y dx dy \\
 &= \iiint \iint u(x)\overline{u(y)}w_N(x-x')w_N(y-y')b_x^*[b_{x'}^*b_{x'}, b_{y'}^*b_{y'}]b_y dx dy dx' dy' \\
 &= - \iiint \iint u(x)\overline{u(y)}w_N(x-x')w_N(y-y')u(x')\overline{u(y')}b_x^*b_{x'}^*b_{y'}b_y dx dy dx' dy' \\
 &\quad + \iiint \iint u(x)\overline{u(y)}w_N(x-x')w_N(y-y')u(y')\overline{u(x')}b_x^*b_{x'}^*b_{y'}b_y dx dy dx' dy'
 \end{aligned}$$

The term with the minus sign is negative because

$$\begin{aligned}
 & \iiint \iint u(x)\overline{u(y)}w_N(x-x')w_N(y-y')u(x')\overline{u(y')}b_x^*b_{x'}^*b_{y'}b_y dx dy dx' dy' \\
 &= \left(\iint u(x)w_N(x-x')u(x')b_x^*b_{x'}^* dx dx' \right) \times \\
 &\quad \times \left(\iint \overline{u(y)}w_N(y-y')\overline{u(y')}b_{y'}b_{y'} dy dy' \right) \\
 &= AA^* \geq 0
 \end{aligned}$$

with

$$A = \iint u(x)w_N(x-x')u(x')b_x^*b_{x'}^* dx dx'.$$

Thus

$$\begin{aligned}
 & \iint u(x)\overline{u(y)}b_x^*[B_x, B_y]b_y dx dy \\
 &\leq \iiint \iint u(x)\overline{u(y)}w_N(x-x')w_N(y-y')u(y')\overline{u(x')}b_x^*b_{x'}^*b_{y'}b_y dx dy dx' dy'.
 \end{aligned}$$

Next, we use

$$b_x^*b_{y'}^*b_{x'}b_y = b_x^*b_{x'}b_{y'}^*b_y - Q(x', y')b_x^*b_y.$$

For the term involving $b_x^*b_{x'}b_{y'}^*b_y$ we have

$$\iiint \iint u(x)\overline{u(y)}w_N(x-x')w_N(y-y')u(y')\overline{u(x')}b_x^*b_{x'}b_{y'}^*b_y dx dy dx' dy' = B^2$$

where

$$B := \iint u(x)w_N(x-x')\overline{u(x')}b_x^*b_{x'} dx dx'.$$

By the Cauchy-Schwarz inequality and (64), we can estimate

$$\begin{aligned} \pm B &\leq \frac{1}{2} \iint |w_N(x - x')| \left[|u(x)|^2 b_x^* b_x + |u(x')|^2 b_{x'}^* b_{x'} \right] dx dx' \\ &\leq C \|u(t)\|_{L^\infty}^2 \mathcal{N}_+. \end{aligned}$$

Moreover, since B commutes with \mathcal{N}_+ we thus obtain

$$B^2 \leq C \|u(t)\|_{L^\infty}^4 \mathcal{N}_+^2(t).$$

It remains to bound the term involving $Q(x', y') b_x^* b_y$:

$$\begin{aligned} &\iiint\iiint u(x) \overline{u(y)} w_N(x - x') w_N(y - y') u(y') \overline{u(x')} Q(x', y') b_x^* b_y dx dy dx' dy' \\ &= \iiint\iiint u(x) \overline{u(y)} w_N(x - y') w_N(y - y') |u(y')|^2 b_x^* b_y dx dy dy' \\ &\quad - \iiint\iiint u(x) \overline{u(y)} w_N(x - x') w_N(y - y') |u(y')|^2 |u(x')|^2 b_x^* b_y dx dy dx' dy' \end{aligned}$$

Now, by the Cauchy-Schwarz inequality and (64) again we have

$$\begin{aligned} &\pm \iiint\iiint u(x) \overline{u(y)} w_N(x - y') w_N(y - y') |u(y')|^2 b_x^* b_y dx dy dy' \\ &= \int \left(\pm \iint u(x) \overline{u(y)} w_N(x - y') w_N(y - y') b_x^* b_y dx dy \right) |u(y')|^2 dy' \\ &\leq \int \frac{1}{2} \left(\iint \left[|u(x)|^2 |w_N(y - y')|^2 b_x^* b_x + |u(y)|^2 |w_N(x - y')|^2 \right] b_y^* b_y dx dy \right) |u(y')|^2 dy' \\ &\leq \|u(t)\|_{L^\infty}^2 \|w_N\|_{L^2}^2 \mathcal{N}_+(t) \\ &\leq C N^{3\beta} \|u(t)\|_{L^\infty}^2 \mathcal{N}_+(t) \end{aligned}$$

and

$$\begin{aligned} &\pm \iiint\iiint u(x) \overline{u(y)} w_N(x - x') w_N(y - y') |u(y')|^2 |u(x')|^2 b_x^* b_y dx dy dx' dy' \\ &= \iint \left(\pm \iint u(x) \overline{u(y)} w_N(x - x') w_N(y - y') b_x^* b_y dx dy \right) |u(x')|^2 |u(y')|^2 dx' dy' \\ &\leq \iint \frac{1}{2} \left(\iint |w_N(x - x')| |w_N(y - y')| \left[|u(x)|^2 b_x^* b_x + |u(y)|^2 b_y^* b_y \right] dx dy \right) \\ &\quad \times |u(x')|^2 |u(y')|^2 dx' dy' \\ &\leq \|w_N\|_{L^1}^2 \|u(t)\|_{L^\infty}^6 \mathcal{N}_+^2(t). \end{aligned}$$

In summary, we have proved that

$$(74) \quad \iint u(x)\overline{u(y)}b_x^*[B_x, B_y]b_y dx dy \leq CN^{3\beta}\mathcal{N}_+^2(t).$$

From (72)-(73)-(74) we find that

$$(75) \quad R_6^*R_6 \leq CN^{3\beta}\mathcal{N}_+^3(t).$$

From (71) and (75), it follows that

$$(76) \quad \begin{aligned} & R_5\mathbb{1}_+^{\leq N}R_5^* + R_5^*\mathbb{1}_+^{\leq N}R_5 \\ &= \frac{\sqrt{N - \mathcal{N}_+(t)}}{N - 1}R_6\mathbb{1}_+^{\leq N}R_6^*\frac{\sqrt{N - \mathcal{N}_+(t)}}{N - 1} + R_6^*\frac{(N - \mathcal{N}_+(t))\mathbb{1}_+^{\leq N}}{(N - 1)^2}R_6 \\ &\leq \frac{\sqrt{N - \mathcal{N}_+(t)}}{N - 1}R_6R_6^*\frac{\sqrt{N - \mathcal{N}_+(t)}}{N - 1} + \frac{N}{(N - 1)^2}R_6^*R_6 \\ &\leq CN^{3\beta-1}\mathcal{N}_+^3(t). \end{aligned}$$

Conclusion. Substituting (49), (50), (57), (59) and (76) into (45) we get the desired bound

$$R^2(t) \leq C\left(N^{6\beta-2}\mathcal{N}_+^4(t) + N^{3\beta-1}\mathcal{N}_+^3(t) + N^{3\beta-2}\right).$$

□

4. Evolution generated by quadratic Hamiltonians

4.1. A general result

We have the following general result on the evolution generated by a quadratic Hamiltonian.

Proposition 7 (Evolution generated by quadratic Hamiltonians). *Let $\{\mathbb{H}(t)\}$, $t \in [0, 1]$, be a family of quadratic Hamiltonians on $\mathcal{F}(\mathfrak{H})$ of the form*

$$\mathbb{H}(t) := d\Gamma(h(t)) + \frac{1}{2} \iint \left(k(t, x, y)a_x^*a_y^* + \frac{1}{2}\overline{k(t, x, y)}a_x a_y \right) dx dy$$

where $h(t) = h_1 + h_2(t) : \mathfrak{H} \rightarrow \mathfrak{H}$, with $h_1 > 0$ time-independent and $h_2(t)$ bounded, and with $k(t) : \overline{\mathfrak{H}} \rightarrow \mathfrak{H}$ Hilbert-Schmidt with symmetric kernel

$k(t, x, y) = k(t, y, x)$. Assume that

$$\sup_{t \in [0,1]} \left(\|h_2(t)\| + \|\partial_t h_2(t)\| + \|k(t, \cdot, \cdot)\|_{\mathfrak{H}^2} + \|\partial_t k(t, \cdot, \cdot)\|_{\mathfrak{H}^2} \right) < \infty.$$

Then for every normalized vector $\Phi(0) \in \mathcal{F}(\mathfrak{H})$ satisfying $\langle \Phi(0), \mathcal{N}\Phi(0) \rangle < \infty$, the equation

$$(77) \quad \begin{cases} i\partial_t \Phi(t) = \mathbb{H}(t)\Phi(t), \\ \Phi(t=0) = \Phi(0) \end{cases}$$

has a unique solution $\Phi(t)$ with $t \in [0, 1]$ and the following statements hold true.

(i) The pair of density matrices $(\gamma(t), \alpha(t)) = (\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$ satisfies

$$(78) \quad \begin{cases} i\partial_t \gamma = h\gamma - \gamma h + k\alpha^* - \alpha k^*, \\ i\partial_t \alpha = h\alpha + \alpha h^\top + k + k\gamma^\top + \gamma k, \\ \gamma(t=0) = \gamma(0), \quad \alpha(t=0) = \alpha(0). \end{cases}$$

Moreover, $(\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$ is the unique solution to (78) under the constraints

$$(79) \quad \gamma = \gamma^*, \quad \alpha = \alpha^\top, \quad \sup_{t \in [0,1]} \left(\text{Tr}(\alpha(t)\alpha^*(t)) + \text{Tr}(\gamma^2(t)) \right) < \infty.$$

(ii) For every decomposition $k = k_1 + k_2$, we have

$$(80) \quad \|\alpha(t)\|_{\mathfrak{HS}}^2 + \|\gamma(t)\|_{\mathfrak{HS}}^2 \leq \Theta(t) + \int_0^t \exp\left(\int_s^t \xi(r)dr\right) \xi(s)\Theta(s)ds$$

for all $t \in [0, 1]$, where

$$\begin{aligned} \xi(t) &:= 6\left(\|k(t)\| + \|h_2(t)\|\right), \\ \Theta(t) &:= 2\|\alpha(0)\|_{\mathfrak{HS}}^2 + 2\|\gamma(0)\|_{\mathfrak{HS}}^2 \\ &\quad + 2\left(\|\mathcal{L}^{-1}k_1(t, \cdot, \cdot)\|_{L^2} + \|\mathcal{L}^{-1}k_1(0, \cdot, \cdot)\|_{L^2} \right. \\ &\quad \left. + \int_0^t (\|k_2(s, \cdot, \cdot)\|_{L^2} + \|\mathcal{L}^{-1}\partial_s k_1(s, \cdot, \cdot)\|_{L^2}) ds \right)^2, \\ \mathcal{L} &:= h_1 \otimes 1 + 1 \otimes h_1. \end{aligned}$$

(iii) If $\Phi(0)$ is a quasi-free state, then $\Phi(t)$ is a quasi-free state for all $t \in [0, 1]$. In this case,

$$(81) \quad \langle \Phi(t), \mathcal{N}\Phi(t) \rangle \leq \Theta_1(t) + \int_0^t \exp\left(\int_s^t \xi(r) dr\right) \xi(s) \Theta_1(s) ds$$

where

$$\Theta_1(t) := \Theta(t) - 2\|\alpha(0)\|_{\text{HS}}^2 - 2\|\gamma(0)\|_{\text{HS}}^2 + 4\left(1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle\right)^2.$$

Before proving Proposition 7, let us recall some well-known properties of density matrices.

Lemma 8 (Density matrices). *Let Ψ be a normalized vector in $\mathcal{F}(\mathfrak{H})$ with $\langle \Psi, \mathcal{N}\Psi \rangle < \infty$ and let $(\gamma_\Psi, \alpha_\Psi)$ be its density matrices. Then*

$$(82) \quad \gamma_\Psi \geq 0, \quad \text{Tr}(\gamma_\Psi) = \langle \Phi, \mathcal{N}\Phi \rangle, \quad \alpha = \alpha_\Psi^T, \quad \gamma_\Psi \geq \alpha_\Psi(1 + \gamma_\Psi^T)^{-1} \alpha_\Psi^*.$$

Moreover, Ψ is a quasi-free if and only if

$$(83) \quad \gamma\alpha = \alpha\gamma^T, \quad \alpha_\Psi\alpha_\Psi^* = \gamma_\Psi(1 + \gamma_\Psi).$$

Proof. The first three properties of (82) are obvious. The last inequality of (82) is often formulated as

$$\Gamma := \begin{pmatrix} \gamma_\Psi & \alpha_\Psi \\ \alpha_\Psi^* & 1 + \gamma_\Psi^T \end{pmatrix} \geq 0$$

on $\mathfrak{H} \oplus \overline{\mathfrak{H}}$, which can be seen immediately from the inequality

$$\langle \Psi, (a(f) + a^*(g))^*(a(f) + a^*(g))\Psi \rangle \geq 0$$

for all $f, g \in \mathfrak{H}$ and the CCR (20). The equivalence between $\Gamma \geq 0$ and the last inequality in (82) is straightforward, see e.g. [46, Lemma 1.1, p. 93] for a proof. It is also well-known, see e.g. [55, Theorem 10.4], that Ψ is a quasi-free if and only if

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma = -\Gamma.$$

A direct calculation shows that the latter equality is equivalent to (83). \square

Remark 6. In fact, the first condition $\gamma\alpha = \alpha\gamma^T$ of (83) is a consequence of the second condition $\alpha_\Psi\alpha_\Psi^* = \gamma_\Psi(1 + \gamma_\Psi)$, see [3, Proposition 4.5].

Now we are able to give

Proof of Proposition 7. The proof is divided into several steps.

Step 1 (Existence and uniqueness of solution to (77)). Note that from the estimate (52) and the operator monotonicity of the square root, we get

$$\pm \left(\iint \left(k(t, x, y) a_x^* a_y^* + \text{h.c.} \right) dx dy \right) \leq \sqrt{2} \|k(t, \cdot, \cdot)\|_{L^2} (\mathcal{N} + 1).$$

Recall that we have denoted $X + \text{h.c.} = X + X^*$ for short. Therefore, from the conditions on $h(t)$ and $k(t)$ we get

$$\begin{aligned} \pm(\mathbb{H}(t) - d\Gamma(h_1)) &= \pm \left(d\Gamma(h_2(t)) + \iint \left(k(t, x, y) a_x^* a_y^* + \text{h.c.} \right) dx dy \right) \\ &\leq C_2(\mathcal{N} + 1), \\ \pm \partial_t \mathbb{H}(t) &= \pm \left(d\Gamma(\partial_t h_2(t)) + \iint \left(\partial_t k(t, x, y) a_x^* a_y^* + \text{h.c.} \right) dx dy \right) \\ &\leq C_2(\mathcal{N} + 1), \\ \pm i[\mathbb{H}(t), \mathcal{N}] &= \pm \frac{i}{2} \iint \left[k(t, x, y) a_x^* a_y^* + \text{h.c.}, \mathcal{N} \right] dx dy \\ &= \pm \iint \left(-ik(t, x, y) a_x^* a_y^* + \text{h.c.} \right) dx dy \\ &\leq C_2(\mathcal{N} + 1) \end{aligned}$$

with

$$(84) \quad C_2 := \sup_{t \in [0,1]} \left(\|h_2(t)\| + \sqrt{2} \|k(t, \cdot, \cdot)\|_{L^2} \right) < \infty.$$

Thus by [36, Theorem 8], we know that for every $\Phi(0)$ in the quadratic form domain of $d\Gamma(h_1 + 1)$, the equation (77) has a unique solution $\Phi(t)$.

Note that for every time-independent operator \mathcal{O} on Fock space,

$$(85) \quad \begin{aligned} i\partial_t \langle \Phi(t), \mathcal{O}\Phi(t) \rangle &= -\langle i\partial_t \Phi(t), \mathcal{O}\Phi(t) \rangle + \langle \Phi(t), \mathcal{O}i\partial_t \Phi(t) \rangle \\ &= \langle \Phi(t), [\mathcal{O}, \mathbb{H}(t)]\Phi(t) \rangle. \end{aligned}$$

In particular, choosing $\mathcal{O} = 1$ we get $\|\Phi(t)\| = \|\Phi(0)\|$. Therefore, the propagator $\mathcal{U}(t) : \mathcal{F}(\mathfrak{H}) \rightarrow \mathcal{F}(\mathfrak{H})$ defined by $\mathcal{U}(t)\Phi(0) = \Phi(t)$ is a (partial) unitary, and hence it can be extended from the quadratic form domain of $d\Gamma(h_1 + 1)$ to the whole Fock space. In the following, we are interested in

the case when $\langle \Phi(0), \mathcal{N}\Phi(0) \rangle < \infty$. In this case, by (85) we have

$$\frac{d}{dt} \langle \Phi(t), (\mathcal{N} + 1)\Phi(t) \rangle = \langle \Phi(t), i[\mathbb{H}(t), \mathcal{N}]\Phi(t) \rangle \leq C_2 \langle \Phi(t), (\mathcal{N} + 1)\Phi(t) \rangle,$$

and by Grönwall's inequality,

$$(86) \quad \langle \Phi(t), (\mathcal{N} + 1)\Phi(t) \rangle \leq e^{C_2 t} \langle \Phi(0), (\mathcal{N} + 1)\Phi(0) \rangle.$$

Thus, in summary, for every $\Phi(0)$ with $\langle \Phi(0), \mathcal{N}\Phi(0) \rangle < \infty$, the equation (77) has a unique solution $\Phi(t)$ satisfying $\|\Phi(t)\| = \|\Phi(0)\|$ and (86).

Step 2 (Derivation of linear equations (78)). We will use (85) to compute the time-derivatives of the kernels

$$\gamma_{\Phi(t)}(x, y) = \langle \Phi(t), a_y^* a_x \Phi(t) \rangle, \quad \alpha_{\Phi(t)}(x, y) = \langle \Phi(t), a_x a_y \Phi(t) \rangle.$$

Let us use (85) with $\mathcal{O} = a_{y'}^* a_{x'}$. From the CCR (23) it follows that

$$\begin{aligned} [a_{y'}^* a_{x'}, \mathbb{H}(t)] &= \iint \left[a_{y'}^* a_{x'}, h(t, x, y) a_x^* a_y \right. \\ &\quad \left. + \frac{1}{2} k(t, x, y) a_x^* a_y^* + \frac{1}{2} k^*(t, x, y) a_x a_y \right] dx dy \\ &= \iint h(t, x, y) \left(\delta(x' - x) a_{y'}^* a_y - \delta(y' - y) a_x^* a_{x'} \right) dx dy \\ &\quad + \frac{1}{2} \iint k(t, x, y) \left(\delta(x' - x) a_{y'}^* a_y^* + \delta(x' - y) a_{y'}^* a_x^* \right) dx dy \\ &\quad - \frac{1}{2} \iint k^*(t, x, y) \left(\delta(y' - y) a_x a_{x'} + \delta(y' - x) a_y a_{x'} \right) dx dy. \end{aligned}$$

Therefore,

$$\begin{aligned} i\partial_t \gamma_{\Phi(t)}(x', y') &= i\partial_t \langle \Phi(t), a_{y'}^* a_x \Phi(t) \rangle = \langle \Phi(t), [a_{y'}^* a_x, \mathbb{H}(t)] \Phi(t) \rangle \\ &= \iint h(t, x, y) \left(\delta(x' - x) \gamma_{\Phi(t)}(y, y') - \delta(y' - y) \gamma_{\Phi(t)}(x', x) \right) dx dy \\ &\quad + \frac{1}{2} \iint k(t, x, y) \left(\delta(x' - x) \alpha_{\Phi(t)}^*(y, y') + \delta(x' - y) \alpha_{\Phi(t)}^*(y', x) \right) dx dy \\ &\quad - \frac{1}{2} \iint k^*(t, x, y) \left(\delta(y' - y) \alpha_{\Phi(t)}(x, x') + \delta(y' - x) \alpha_{\Phi(t)}(y, x') \right) dx dy \\ &= \left(h(t) \gamma_{\Phi(t)} - \gamma_{\Phi(t)} h(t) + k(t) \alpha_{\Phi(t)}^* - \alpha_{\Phi(t)} k^*(t) \right) (x', y'). \end{aligned}$$

Here we have used $k(t, x, y) = k(t, y, x)$ and $\alpha_{\Phi(t)}(x, y) = \alpha_{\Phi(t)}(y, x)$. Similarly, using (85) with $\mathcal{O} = a_{x'}a_{y'}$ and the identity

$$\begin{aligned} [a_{x'}a_{y'}, \mathbb{H}(t)] &= \iint [a_{x'}a_{y'}, h(t, x, y)a_x^*a_y \\ &\quad + \frac{1}{2}k(t, x, y)a_x^*a_y^* + \frac{1}{2}k^*(t, x, y)a_xa_y] dx dy \\ &= \iint h(t, x, y) (\delta(y' - x)a_{x'}a_y + \delta(x' - x)a_ya_{y'}) dx dy \\ &\quad + \frac{1}{2} \iint k(t, x, y) (\delta(y' - x)\delta(x' - y) + \delta(y' - x)a_y^*a_{x'} \\ &\quad \quad \quad + \delta(x' - x)\delta(y' - y) + \delta(x' - x)a_y^*a_{y'} \\ &\quad \quad \quad + \delta(y' - y)a_x^*a_{x'} + \delta(x' - y)a_x^*a_{y'}) dx dy, \end{aligned}$$

we find that

$$\begin{aligned} i\partial_t\alpha_{\Phi(t)}(x', y') &= i\partial_t\langle\Phi(t), a_{x'}^*a_{y'}^*\Phi(t)\rangle = \langle\Phi(t), [a_{x'}^*a_{y'}^*, \mathbb{H}(t)]\Phi(t)\rangle \\ &= \iint h(t, x, y) (\delta(y' - x)\alpha_{\Phi(t)}(x', y) + \delta(x' - x)\alpha_{\Phi(t)}(t, y, y')) dx dy \\ &\quad + \frac{1}{2} \iint k(t, x, y) (\delta(y' - x)\delta(x' - y) + \delta(y' - x)\gamma_{\Phi(t)}(x', y) \\ &\quad \quad \quad + \delta(x' - x)\delta(y' - y) + \delta(x' - x)\gamma_{\Phi(t)}(y', y) \\ &\quad \quad \quad + \delta(y' - y)\gamma_{\Phi(t)}(x', x) + \delta(x' - y)\gamma_{\Phi(t)}(y', x)) dx dy, \\ &= (\alpha_{\Phi(t)}h^\text{T}(t) + h(t)\alpha_{\Phi(t)} + k(t) + \gamma_{\Phi(t)}k(t) + k(t)\gamma_{\Phi(t)}^\text{T})(x', y'). \end{aligned}$$

Thus $(\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$ satisfies the couple of linear equations (78):

$$\begin{cases} i\partial_t\gamma = h\gamma - \gamma h + k\alpha^* - \alpha k^*, \\ i\partial_t\alpha = h\alpha + \alpha h^\text{T} + k + k\gamma^\text{T} + \gamma k, \end{cases}$$

with initial data $\gamma(0) = \gamma_{\Phi(0)}$, $\alpha(0) = \alpha_{\Phi(0)}$.

Step 3 (Uniqueness of solution to (78)). By the derivation in the previous step, we have proved that $(\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$ is a solution to (78). Moreover, from (86) and (82) we can see that if $\langle\Phi(0), \mathcal{N}\Phi(0)\rangle < \infty$, then $\text{Tr}(\gamma_{\Phi(t)})$ and $\text{Tr}(\alpha_{\Phi(t)}\alpha_{\Phi(t)}^*)$ are bounded uniformly in $t \in [0, 1]$.

Now we show that for every given initial condition $(\gamma(0), \alpha(0))$, the system (78) has at most one solution satisfying (79):

$$\gamma = \gamma^*, \quad \alpha = \alpha^T, \quad \sup_{t \in [0,1]} \left(\text{Tr}(\alpha(t)\alpha^*(t)) + \text{Tr}(\gamma^2(t)) \right) < \infty.$$

More precisely, we will show that if $(\gamma_j(t), \alpha_j(t))_{j=1,2}$ are two solutions to (78) with the same initial condition and they satisfy (79), then

$$X(t) := \gamma_1(t) - \gamma_2(t) \quad \text{and} \quad Y(t) := \alpha_1(t) - \alpha_2(t)$$

are 0 for all $t \in [0, 1]$. It is clear that X, Y satisfy

$$(87) \quad \begin{cases} i\partial_t X = hX - Xh + kY^* - Yk^*, \\ i\partial_t Y = hY + Yh^T + kX^T + Xk, \\ X(0) = 0, \quad Y(0) = 0 \end{cases}$$

and

$$(88) \quad X = X^*, \quad Y = Y^T, \quad \sup_{t \in [0,1]} \left(\text{Tr}(Y(t)Y^*(t)) + \text{Tr}(X^2(t)) \right) < \infty.$$

Note that the second equation of (87) is different from that of (78) because the inhomogeneous term k has been canceled.

From the first equation of (87) we have

$$(89) \quad \begin{aligned} i\partial_t X^2 &= (i\partial_t X)X + X(i\partial_t X) \\ &= (hX - Xh + kY^* - Yk^*)X + X(hX - Xh + kY^* - Yk^*) \\ &= hX^2 - X^2h + (kY^* - Yk^*)X + X(kY^* - Yk^*). \end{aligned}$$

We want to take the trace of (89) and use the cancellation $\text{Tr}(hX^2 - X^2h) = 0$ but it is a bit formal because hX^2 and X^2h might be not trace class. To make the argument rigorous, let us introduce the time-independent projection $\mathbb{1}^{\leq \Lambda} = \mathbb{1}(h_1 \leq \Lambda)$ with $0 < \Lambda < \infty$ and deduce from (89) that

$$(90) \quad i\partial_t(\mathbb{1}^{\leq \Lambda} X^2 \mathbb{1}^{\leq \Lambda}) = \mathbb{1}^{\leq \Lambda}(hX^2 - X^2h + \text{Er})\mathbb{1}^{\leq \Lambda}$$

where

$$\text{Er} := (kY^* - Yk^*)X + X(kY^* - Yk^*).$$

Now we can take the trace of both sides of (90) and then integrate over t . We will use the cyclicity of the trace

$$\text{Tr}(\mathfrak{X}_1 \mathfrak{X}_2) = \text{Tr}(\mathfrak{X}_2 \mathfrak{X}_1)$$

with \mathfrak{X}_1 bounded and \mathfrak{X}_2 trace class. In particular,

$$\begin{aligned} & \text{Tr}(\mathbb{1}^{\leq \Lambda} (h_1 X^2 - X^2 h_1) \mathbb{1}^{\leq \Lambda}) \\ &= \text{Tr}(\mathbb{1}^{\leq \Lambda} h_1 \cdot \mathbb{1}^{\leq \Lambda} X^2 \mathbb{1}^{\leq \Lambda}) - \text{Tr}(\mathbb{1}^{\leq \Lambda} X^2 \mathbb{1}^{\leq \Lambda} \cdot \mathbb{1}^{\leq \Lambda} h_1) = 0. \end{aligned}$$

Therefore, (90) gives us

$$(91) \quad \text{Tr}(\mathbb{1}^{\leq \Lambda} X^2(t) \mathbb{1}^{\leq \Lambda}) = -i \int_0^t \text{Tr} \left[\mathbb{1}^{\leq \Lambda} \left(h_2 X^2 - X^2 h_2 + \text{Er} \right) \mathbb{1}^{\leq \Lambda} \right] (s) ds.$$

Next, we pass $\Lambda \rightarrow +\infty$ and use the convergence

$$\lim_{\Lambda \rightarrow \infty} \text{Tr}(\mathbb{1}^{\leq \Lambda} \mathfrak{X} \mathbb{1}^{\leq \Lambda}) = \text{Tr}(\mathfrak{X})$$

which holds for every trace class operator \mathfrak{X} . Note that

$$\text{Tr} \left| h_2 X^2 - X^2 h_2 \right| \leq \|h_2\| \cdot \|X\|_{\text{HS}}^2, \quad \text{Tr} \left| \text{Er} \right| \leq 4\|k\| \cdot \|X\|_{\text{HS}} \cdot \|Y\|_{\text{HS}}$$

and $\|h_2(t)\|$, $\|k(t)\|$, $\|X(t)\|_{\text{HS}}$, $\|Y(t)\|_{\text{HS}}$ are bounded uniformly in $t \in [0, 1]$. Therefore, we can use Lebesgue's Dominated Convergence Theorem and deduce from (91) that

$$\|X(t)\|_{\text{HS}}^2 = -i \int_0^t \left[\text{Tr} \left(X^2 h_2 - h_2 X^2 + \text{Er} \right) \right] (s) ds.$$

Using again the cyclicity of the trace, $\text{Tr}(h_2 X^2) = \text{Tr}(X^2 h_2)$, we get

$$(92) \quad \|X(t)\|_{\text{HS}}^2 = -i \int_0^t \text{Er}(s) ds \leq 4 \int_0^t \|k(s)\| \cdot \|X(s)\|_{\text{HS}} \cdot \|Y(s)\|_{\text{HS}} ds.$$

Similarly, from the second equation of (87) we have

$$\begin{aligned} i\partial_t(Y Y^*) &= (i\partial_t Y) Y^* - Y (i\partial_t Y)^* \\ &= (hY + Y h^T + kX^T + Xk) Y^* - Y (hY + Y h^T + kX^T + Xk)^* \\ &= hY Y^* - Y Y^* h + (kX^T + Xk) Y^* - Y (kX^T + Xk)^*, \end{aligned}$$

and hence

$$(93) \quad \begin{aligned} \|Y(t)\|_{\text{HS}}^2 &= -i \int_0^t \left[(kX^T + Xk)Y^* - Y(kX^T + Xk)^* \right](s) ds \\ &\leq 4 \int_0^t \|k(s)\| \cdot \|X(s)\|_{\text{HS}} \cdot \|Y(s)\|_{\text{HS}} ds. \end{aligned}$$

Summing (92) and (93) we find that

$$\begin{aligned} \|X(t)\|_{\text{HS}}^2 + \|Y(t)\|_{\text{HS}}^2 &\leq 8 \int_0^1 \|k(s)\| \cdot \|X(s)\|_{\text{HS}} \cdot \|Y(s)\|_{\text{HS}} ds \\ &\leq 4 \left(\sup_{r \in [0,1]} \|k(r)\| \right) \int_0^t (\|X(s)\|_{\text{HS}}^2 + \|Y(s)\|_{\text{HS}}^2) \end{aligned}$$

for all $t \in [0, 1]$. Consequently, $X(t) \equiv 0$ and $Y(t) \equiv 0$ by Grönwall’s inequality. This finishes the proof of the uniqueness.

Thus $(\gamma(t), \alpha(t)) = (\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$ is the unique solution to the system (78) under conditions (79).

Step 4 (Improved bound on $\|\alpha\|_{\text{HS}}$) Recall that from (86) and (82) we already have upper bounds on $\text{Tr}(\gamma(t))$ and $\|\alpha\|_{\text{HS}}$. However, the constant C_2 defined in (84) depends on $\|k(t)\|_{\text{HS}} = \|k(t, \cdot, \cdot)\|_{L^2}$ which is large in our application. In the following we will derive another bound on $\|\alpha\|_{\text{HS}}$ which depends on the operator norm $\|k(t)\|$ instead of the Hilbert-Schmidt norm.

Inspired by [25, Proof of Theorem 4.1], we will decompose $\alpha(t) = Y_1(t) + Y_2(t)$ where $Y_1(t), Y_2(t) : \mathfrak{H} \rightarrow \mathfrak{H}$ satisfy

$$(94) \quad \begin{cases} i\partial_t Y_1 = h_1 Y_1 + Y_1 h_1^T + k, \\ Y_1(t=0) = 0, \end{cases}$$

and

$$(95) \quad \begin{cases} i\partial_t \gamma = h\gamma - \gamma h + k(Y_1 + Y_2)^* - (Y_1 + Y_2)k^*, \\ i\partial_t Y_2 = hY_2 + Y_2 h^T + h_2 Y_1 + Y_1 h_2^T + k\gamma^T + \gamma k, \\ Y_2(t=0) = \alpha(0), \quad \gamma(t=0) = \gamma(0). \end{cases}$$

Here (95) is derived from (94) and the system (78).

Estimation of $\|Y_1\|_{\text{HS}}$. Note that the kernel of the operator $h_1 Y_1 : \overline{\mathfrak{H}} \rightarrow \mathfrak{H}$ is the two-body function

$$\int h_1(x, z) Y_1(t, z, y) dz = \left[(h_1 \otimes 1) Y_1(t, \cdot, \cdot) \right](x, y).$$

The latter equality follows from a straightforward calculation:

$$\begin{aligned}
 & \iint \overline{f(x)g(y)} \left[(h_1 \otimes 1) Y_1(t, \cdot, \cdot) \right] (x, y) dx dy \\
 &= \left\langle f \otimes g, (h_1 \otimes 1) Y_1(t, \cdot, \cdot) \right\rangle_{L^2} = \left\langle (h_1 \otimes 1)(f \otimes g), Y_1(t, \cdot, \cdot) \right\rangle_{L^2} \\
 &= \iiint \overline{h_1(x, z) f(z) g(y)} Y_1(t, x, y) dx dy dz \\
 &= \iiint h_1(z, x) \overline{f(z) g(y)} Y_1(t, x, y) dx dy dz \\
 &= \iiint h_1(x, z) \overline{f(x) g(y)} Y_1(t, z, y) dx dy dz
 \end{aligned}$$

for all $f, g \in \mathfrak{H}$. Here we have used the fact that h_1 is self-adjoint, which implies that $h_1(x, y) = \overline{h_1(y, x)}$. Similarly, the kernel of $Y_1 h_1^\top = (h_1 Y_1)^\top$ is $\left[(1 \otimes h_1) Y_1(t, \cdot, \cdot) \right] (x, y)$. Therefore, the operator equation (94) can be rewritten as an equation of two-body functions:

$$(96) \quad \begin{cases} i\partial_t Y_1(t, x, y) = (\mathcal{L} Y_1)(t, x, y) + k(t, x, y), \\ Y_1(0, x, y) = 0. \end{cases}$$

where

$$\mathcal{L} := h_1 \otimes 1 + 1 \otimes h_1 > 0.$$

By Duhamel's formula and integration by parts, we can write

$$\begin{aligned}
 Y_1(t, x, y) &= \int_0^t e^{i(s-t)\mathcal{L}} k(s, x, y) ds \\
 &= \int_0^t e^{i(s-t)\mathcal{L}} k_2(s, x, y) ds - i \int_0^t \partial_s (e^{i(s-t)\mathcal{L}} \mathcal{L}^{-1}) k_1(s, x, y) ds \\
 &= -i \mathcal{L}^{-1} k_1(t, x, y) + i e^{-it\mathcal{L}} \mathcal{L}^{-1} k_1(0, x, y) \\
 &\quad + \int_0^t e^{i(s-t)\mathcal{L}} \left(k_2(s, x, y) + i \mathcal{L}^{-1} \partial_s k_1(s, x, y) \right) ds.
 \end{aligned}$$

Since $e^{-it\mathcal{L}}$ is a unitary operator on \mathfrak{H}^2 , by the triangle inequality we get

$$\begin{aligned}
 (97) \quad \|Y_1(t)\|_{\text{HS}} &= \|Y_1(t, \cdot, \cdot)\|_{L^2} \\
 &\leq \|\mathcal{L}^{-1} k_1(t, \cdot, \cdot)\|_{L^2} + \|\mathcal{L}^{-1} k_1(0, \cdot, \cdot)\|_{L^2} \\
 &\quad + \int_0^t \left(\|k_2(t, \cdot, \cdot)\|_{L^2} + \|\mathcal{L}^{-1} \partial_s k_1(s, \cdot, \cdot)\|_{L^2} \right) ds.
 \end{aligned}$$

Estimation of $\|\gamma\|_{\text{HS}}$ and $\|Y_2\|_{\text{HS}}$ from (95). We use again the argument of deriving (92) in Step 3. From the first equation of (95) we have

$$\begin{aligned} i\partial_t\gamma^2 &= (i\partial_t\gamma)\gamma + \gamma(i\partial_t\gamma) \\ &= \left(h\gamma - \gamma h + k(Y_1 + Y_2)^* - (Y_1 + Y_2)k^*\right)\gamma \\ &\quad + \gamma\left(h\gamma - \gamma h + k(Y_1 + Y_2)^* - (Y_1 + Y_2)k^*\right) \\ &= h\gamma^2 - \gamma^2 h + k(Y_1 + Y_2)^*\gamma - (Y_1 + Y_2)k^*\gamma \\ &\quad + \gamma k(Y_1 + Y_2)^* - \gamma(Y_1 + Y_2)k^*, \end{aligned}$$

and hence

$$\begin{aligned} (98) \quad & \|\gamma(t)\|_{\text{HS}}^2 - \|\gamma(0)\|_{\text{HS}}^2 \\ &= -i \int_0^t \text{Tr} \left(k(Y_1 + Y_2)^*\gamma - (Y_1 + Y_2)k^*\gamma \right. \\ &\quad \left. + \gamma k(Y_1 + Y_2)^* - \gamma(Y_1 + Y_2)k^* \right)(s) ds \\ &\leq 4 \int_0^t \left(\|k\| \cdot \|\gamma\|_{\text{HS}} \|Y_1\|_{\text{HS}} + \|k\| \cdot \|\gamma\|_{\text{HS}} \|Y_2\|_{\text{HS}} \right)(s) ds. \end{aligned}$$

From the second equation of (95) we have

$$\begin{aligned} i\partial_t(Y_2Y_2^*) &= (i\partial_tY_2)Y_2^* - Y_2(i\partial_tY_2)^* \\ &= (hY_2 + Y_2h^T + h_2Y_1 + Y_1h_2^T + k\gamma^T + \gamma k)Y_2^* \\ &\quad - Y_2(hY_2 + Y_2h^T + h_2Y_1 + Y_1h_2^T + k\gamma^T + \gamma k)^* \\ &= hY_2Y_2^* - Y_2Y_2^*h + (h_2Y_1 + Y_1h_2^T + k\gamma^T + \gamma k)Y_2^* \\ &\quad - Y_2(h_2Y_1 + Y_1h_2^T + k\gamma^T + \gamma k)^*, \end{aligned}$$

and hence

$$\begin{aligned} (99) \quad & \|Y_2(t)\|_{\text{HS}}^2 - \|\alpha(0)\|_{\text{HS}}^2 \\ &= -i \int_0^t \text{Tr} \left((h_2Y_1 + Y_1h_2^T + k\gamma^T + \gamma k)Y_2^* \right. \\ &\quad \left. + Y_2(h_2Y_1 + Y_1h_2^T + k\gamma^T + \gamma k)^* \right)(s) ds \\ &\leq 4 \int_0^t \left(\|h_2\| \cdot \|Y_1\|_{\text{HS}} \|Y_2\|_{\text{HS}} + \|k\| \cdot \|\gamma\|_{\text{HS}} \|Y_2\|_{\text{HS}} \right)(s) ds. \end{aligned}$$

Conclusion. Summing (97), (98) and (99), we get

$$\begin{aligned}
 (100) \quad & \|Y_1(t)\|_{\text{HS}}^2 + \|Y_2(t)\|_{\text{HS}}^2 + \|\gamma(t)\|_{\text{HS}}^2 \\
 & \leq \frac{1}{2}\Theta(t) + 4 \int_0^t \left(\|k\| \cdot \|\gamma\|_{\text{HS}} \|Y_1\|_{\text{HS}} \right. \\
 & \quad \left. + 2\|k\| \cdot \|\gamma\|_{\text{HS}} \|Y_2\|_{\text{HS}} + \|h_2\| \cdot \|Y_1\|_{\text{HS}} \|Y_2\|_{\text{HS}} \right) (s) ds \\
 & \leq \int_0^t 6 \left(\|k(s)\| + \|h_2(s)\| \right) \left(\|Y_1(s)\|_{\text{HS}}^2 + \|Y_2(s)\|_{\text{HS}}^2 + \|\gamma(s)\|_{\text{HS}}^2 \right) ds
 \end{aligned}$$

where

$$\begin{aligned}
 \Theta(t) := & 2\|\gamma(0)\|_{\text{HS}}^2 + 2\|\alpha(0)\|_{\text{HS}}^2 \\
 & + 2 \left(\|\mathcal{L}^{-1}k_1(t, \cdot, \cdot)\|_{L^2} + \|\mathcal{L}^{-1}k_1(0, \cdot, \cdot)\|_{L^2} \right. \\
 & \left. + \int_0^t \left(\|k_2(s, \cdot, \cdot)\|_{L^2} + \|\mathcal{L}^{-1}\partial_s k_1(s, \cdot, \cdot)\|_{L^2} \right) ds \right)^2.
 \end{aligned}$$

Note that if $f, g, \xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy

$$f(t) \leq g(t) + \int_0^t \xi(s)f(s) ds$$

for all t , then we have the Grönwall-type inequality

$$f(t) \leq g(t) + \int_0^t \exp\left(\int_s^t \xi(r) dr\right) \xi(s)g(s) ds.$$

Therefore, we can deduce from (100) that

$$\begin{aligned}
 (101) \quad & \|\alpha(t)\|_{\text{HS}}^2 + \|\gamma(t)\|_{\text{HS}}^2 \leq 2 \left(\|Y_1(t)\|_{\text{HS}}^2 + \|Y_2(t)\|_{\text{HS}}^2 + \|\gamma(t)\|_{\text{HS}}^2 \right) \\
 & \leq \Theta(t) + \int_0^t \exp\left(\int_s^t \xi(r) dr\right) \xi(s)\Theta(s) ds.
 \end{aligned}$$

where

$$\xi(t) := 6 \left(\|k(t)\| + \|h_2(t)\| \right).$$

Step 5 (Quasi-free states). In this final step we prove that if $\Phi(0)$ is a quasi-free state, then $\Phi(t)$ is a quasi-free for all $t \in [0, 1]$. By Lemma 8, it suffices

to show that the density matrices $(\gamma(t), \alpha(t))$ of $\Phi(t)$ satisfy

$$(102) \quad \gamma\alpha = \alpha\gamma^T, \quad \alpha\alpha^* = \gamma(1 + \gamma).$$

Indeed, using the equations (78), we can compute

$$\begin{aligned} i\partial_t(\gamma + \gamma^2 - \alpha\alpha^*) &= (i\partial_t\gamma)(1 + \gamma) + \gamma(i\partial_t\gamma) - (i\partial_t\alpha)\alpha^* + \alpha(i\partial_t\alpha)^* \\ &= (h\gamma - \gamma h + k\alpha^* - \alpha k^*)(1 + \gamma) + \gamma(h\gamma - \gamma h + k\alpha^* - \alpha k^*) \\ &\quad - (h\alpha + \alpha h^T + k + k\gamma^T + \gamma k)\alpha^* + \alpha(h\alpha + \alpha h^T + k + k\gamma^T + \gamma k)^* \\ &= h(\gamma + \gamma^2 - \alpha\alpha^*) - (\gamma + \gamma^2 - \alpha\alpha^*)h + k(\alpha^*\gamma - \gamma^T\alpha^*) - (\gamma\alpha - \alpha\gamma^T)k^* \end{aligned}$$

and

$$\begin{aligned} i\partial_t(\gamma\alpha - \alpha\gamma^T) &= (i\partial_t\gamma)\alpha + \gamma(i\partial_t\alpha) - (i\partial_t\alpha)\gamma^T - \alpha(i\partial_t\gamma)^T \\ &= (h\gamma - \gamma h + k\alpha^* - \alpha k^*)\alpha + \gamma(h\alpha + \alpha h^T + k + k\gamma^T + \gamma k) \\ &\quad - (h\alpha + \alpha h^T + k + k\gamma^T + \gamma k)\gamma^T - \alpha(h\gamma - \gamma h + k\alpha^* - \alpha k^*)^T \\ &= h(\gamma\alpha - \alpha\gamma^T) + (\gamma\alpha - \alpha\gamma^T)h - k(\gamma + \gamma^2 - \alpha\alpha^*)^T + (\gamma + \gamma^2 - \alpha\alpha^*)k. \end{aligned}$$

The latter two equations can be rewritten in the compact form

$$(103) \quad \begin{cases} i\partial_t Y_3 = hY_3 - Y_3h + kY_4^* - Y_4k^*, \\ i\partial_t Y_4 = hY_4 + Y_4h^T - kY_3^T + Y_3k, \\ Y_3(0) = 0, \quad Y_4(0) = 0, \end{cases}$$

where

$$Y_3 := \gamma + \gamma^2 - \alpha\alpha^*, \quad Y_4 := \gamma\alpha - \alpha\gamma^T.$$

Here the initial conditions $Y_3(0) = 0$ and $Y_4(0) = 0$ follow from Lemma 8 and the assumption that $\Phi(0)$ is a quasi-free state.

The system (103) is similar to (87) we have considered in Step 3, and by the same argument in the previous step we can show that $Y_3(t) = 0$ and $Y_4(t) = 0$ for all $t \in [0, 1]$. Thus $\Phi(t)$ is a quasi-free state for all $t \in [0, 1]$. Finally, since $\alpha\alpha^* = \gamma(1 + \gamma)$ we obtain $\langle \Phi(t), \mathcal{N}\Phi(t) \rangle = \text{Tr}(\gamma(t)) \leq \|\alpha\|_{\text{HS}}^2$. Therefore, (81) follows immediately from (101) and the simple estimate

$$\begin{aligned} \|\alpha(0)\|_{\text{HS}}^2 + \|\gamma(0)\|_{\text{HS}}^2 &= \text{Tr}(\gamma(0) + 2\gamma^2(0)) \\ &\leq 2(1 + \text{Tr}(\gamma(0)))^2 = 2(1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle)^2. \end{aligned}$$

□

4.2. Proof of Proposition 4

Now we apply Proposition 7 to the Bogoliubov Hamiltonian $\mathbb{H}(t)$ defined by (31), which corresponds to

$$h(t) = -\Delta + |u(t)|^2 * w_N - \mu_N(t) + K_1(t), \quad k(t) = K_2(t).$$

Recall that

- $K_1(t) = Q(t)\tilde{K}_1(t)Q(t) : \mathfrak{H} \rightarrow \mathfrak{H}$ where $\tilde{K}_1(t)$ is the operator on \mathfrak{H} with kernel $\tilde{K}_1(t, x, y) = u(t, x)w_N(x - y)\overline{u(t, y)}$;
- $K_2(t) = Q(t)\tilde{K}_2(t)\overline{Q(t)} : \overline{\mathfrak{H}} \rightarrow \overline{\mathfrak{H}}$ where $\tilde{K}_2(t) : \overline{\mathfrak{H}} \rightarrow \overline{\mathfrak{H}}$ is the operator with kernel $\tilde{K}_2(t, x, y) = u(t, x)w(x - y)u(t, y)$. Putting differently, $K_2(t, \cdot, \cdot) = Q(t) \otimes Q(t)\tilde{K}_2(t, \cdot, \cdot) \in \mathfrak{H}^2$.

We will decompose $h(t) = h_1 + h_2(t)$ and $K_2 = k_1 + k_2$ with

$$h_1 := -\Delta, \quad h_2(t) := |u(t)|^2 * w_N - \mu_N(t) + K_1(t)$$

and

$$k_1 := \tilde{K}_2, \quad k_2 := K_2 - \tilde{K}_2.$$

The conditions in Proposition 7 are verified in the following

Lemma 9. *The following bounds hold true for all $\beta \geq 0$, $N \in \mathbb{N}$ and $t \geq 0$:*

$$(104) \quad \|h_2(t)\| + \|K_2\| \leq C\|u(t)\|_{L^\infty(\mathbb{R}^3)}^2,$$

$$(105) \quad \|\partial_t h_2(t)\| + \|K_2(t, \cdot, \cdot)\|_{L^2} + \|\partial_t K_2(t, \cdot, \cdot)\|_{L^2} \leq CN^{3\beta},$$

$$(106) \quad \|K_2(t, \cdot, \cdot) - \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \leq C\|u(t)\|_{L^\infty(\mathbb{R}^3)}^2,$$

$$(107) \quad \begin{aligned} &\|(-\Delta_x - \Delta_y)^{-1}\tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \\ &+ \|(-\Delta_x - \Delta_y)^{-1}\partial_t\tilde{K}_2(t, \cdot, \cdot)\|_{L^2} \leq C\|u(t)\|_{L^\infty(\mathbb{R}^3)}^{2/3}. \end{aligned}$$

Here the constant C depends only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$.

Proof. The bound (104) follows from (46)-(47)-(48) and (56). Next, we consider (105). Recall that we have proved $\|K_2(t, \cdot, \cdot)\|_{L^2} \leq CN^{3\beta}$ in (53). Moreover,

$$(108) \quad \begin{aligned} \partial_t K_2(t, \cdot, \cdot) &= (\partial_t Q(t)) \otimes Q(t)\tilde{K}_2(t, \cdot, \cdot) \\ &+ Q(t) \otimes (\partial_t Q(t))\tilde{K}_2(t, \cdot, \cdot) \\ &+ Q(t) \otimes Q(t)(\partial_t \tilde{K}_2(t, \cdot, \cdot)). \end{aligned}$$

Note that $\partial_t Q(t) = -|\partial_t u(t)\rangle\langle u(t)| - |u(t)\rangle\langle \partial_t u(t)|$ is a rank-two operator and $\|\partial_t Q(t)\|_{\text{HS}} \leq C$ because

$$(109) \quad \|\partial_t u(t)\|_{L^2} = \|h(t)u(t)\|_{L^2} \leq \|\Delta u(t)\|_{L^2} + \|h_2(t)u(t)\|_{L^2} \leq C.$$

Consequently,

$$(110) \quad \begin{aligned} & \left\| (\partial_t Q(t)) \otimes Q(t) \tilde{K}_2(t, \cdot, \cdot) \right\|_{L^2} + \left\| Q(t) \otimes (\partial_t Q(t)) \tilde{K}_2(t, \cdot, \cdot) \right\|_{L^2} \\ &= \left\| (\partial_t Q(t)) \tilde{K}_2(t) \overline{Q(t)} \right\|_{\text{HS}} + \left\| Q(t) \tilde{K}_2(t) \overline{(\partial_t Q(t))} \right\|_{\text{HS}} \\ &\leq 2\|\partial_t Q(t)\|_{\text{HS}} \|\tilde{K}_2(t)\| \cdot \|Q(t)\| \leq \|\tilde{K}_2(t)\| \leq C\|u(t)\|_{L^\infty(\mathbb{R}^3)}^2. \end{aligned}$$

On the other hand,

$$(111) \quad \begin{aligned} & \|Q(t) \otimes Q(t) (\partial_t \tilde{K}_2(t, \cdot, \cdot))\|_{L^2}^2 \leq \|\partial_t \tilde{K}_2(t, \cdot, \cdot)\|_{L^2}^2 \\ &\leq 2 \iint |\partial_t u(t, x)|^2 |w_N(x - y)|^2 |u(t, y)|^2 dx dy \\ &\leq 2\|\partial_t u(t)\|_{L^2(\mathbb{R}^3)}^2 \|w_N\|_{L^2(\mathbb{R}^3)}^2 \|u(t)\|_{L^\infty(\mathbb{R}^3)}^2 \leq CN^{3\beta}. \end{aligned}$$

By the triangle inequality we deduce from (108), (110) and (111) that $\|\partial_t K_2(t, \cdot, \cdot)\|_{L^2} \leq CN^{3\beta}$. Similarly, we also have $\|\partial_t K_1(t)\|_{\text{HS}} \leq CN^{3\beta}$. Combining the latter inequality with

$$\begin{aligned} |\partial_t(|u(t)|^2 * w_N)(x)| &= \left| 2\Re \int \overline{(\partial_t u)(t, y)} u(t, y) w_N(x - y) dy \right| \\ &\leq 2\|\partial_t u(t)\|_{L^2(\mathbb{R}^3)} \|u(t)\|_{L^2(\mathbb{R}^3)} \|w_N\|_{L^\infty(\mathbb{R}^3)} \leq CN^{3\beta} \end{aligned}$$

we find that $\|\partial h_2(t)\| \leq CN^{3\beta}$. Thus (105) holds true.

The bound (106) can be proved using similarly as (110). More precisely, because $1 - Q = |u(t)\rangle\langle u(t)|$, we have

$$\begin{aligned} & \|K_2(t, \cdot, \cdot) - \tilde{K}_2(t, \cdot, \cdot)\|_{L^2} = \|Q(t) \tilde{K}_2(t) \overline{Q(t)} - \tilde{K}_2(t)\|_{\text{HS}} \\ &\leq \|(Q(t) - 1) \tilde{K}_2(t) \overline{Q(t)}\|_{\text{HS}} + \|\tilde{K}_2(t) (\overline{Q(t)} - 1)\|_{\text{HS}} \\ &\leq \|Q(t) - 1\|_{\text{HS}} \|\tilde{K}_2(t)\| \cdot \|\overline{Q(t)}\| + \|\tilde{K}_2(t)\| \cdot \|\overline{Q(t)} - 1\|_{\text{HS}} \\ &\leq C\|\tilde{K}_2(t)\| \leq C\|u(t)\|_{L^\infty(\mathbb{R}^3)}^2. \end{aligned}$$

The bound (107) is essentially [25, Lemma 4.3], but there is a technical modification that we will clarify below. It suffices to consider the most complicated term $f(t, x, y) := u(t, x)w_N(x - y)(\partial_t u)(t, y)$ which is derived from

$\partial_t \widetilde{K}_2(t, x, y)$. Following [25], we compute the Fourier transform :

$$\begin{aligned} \widehat{f}(t, p, q) &= \iint u(t, x) w_N(x - y) (\partial_t u)(t, y) e^{-i(p \cdot x + q \cdot y)} dx dy \\ &= \iint u(t, y + z) w_N(z) (\partial_t u)(t, y) e^{-i(p \cdot (y+z) + q \cdot y)} dz dy \\ &= \int w_N(z) (\widehat{u_z \partial_t u})(t, p + q) e^{-ip \cdot z} dz \end{aligned}$$

where $u_z(t, y) := u(t, y + z)$. Using the Cauchy-Schwarz inequality

$$\left| \widehat{f}(t, p, q) \right|^2 \leq \|w\|_{L^1} \int |w_N(z)| \cdot |(\widehat{u_z \partial_t u})(t, p + q)|^2 dz$$

and Plancherel’s Theorem, we can estimate

$$\begin{aligned} &\|(-\Delta_x - \Delta_y)^{-1} f(t, \cdot, \cdot)\|_{L^2}^2 \\ &= (2\pi)^6 \iint (|p|^2 + |q|^2)^{-2} \left| \widehat{f}(t, p, q) \right|^2 dp dq \\ &\leq C \iiint (|p|^2 + |q|^2)^{-2} |w_N(z)| \cdot |(\widehat{u_z \partial_t u})(t, p + q)|^2 dp dq dz \\ &= C \iiint (|p - q|^2 + |q|^2)^{-2} |w_N(z)| \cdot |(\widehat{u_z \partial_t u})(t, p)|^2 dp dq dz \\ &\leq C \iiint (|p|^2 + |q|^2)^{-2} |w_N(z)| \cdot |(\widehat{u_z \partial_t u})(t, p)|^2 dp dq dz \\ &= C \iint |w_N(z)| \cdot |p|^{-1} |(\widehat{u_z \partial_t u})(t, p)|^2 dp dz. \end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality we get

$$\begin{aligned} &\int |p|^{-1} |(\widehat{u_z \partial_t u})(t, p)|^2 dp \leq C \|u_z(t) \partial_t u(t)\|_{L^{3/2}}^2 \\ &\leq \|u_z(t)\|_{L^\infty}^{4/3} \|u_z(t)\|_{L^2}^{2/3} \|\partial_t u\|_{L^2}^2 \leq C \|u(t)\|_{L^\infty}^{4/3} \end{aligned}$$

and this gives the desired bound

$$\|(-\Delta_x - \Delta_y)^{-1} f(t, \cdot, \cdot)\|_{L^2}^2 \leq C \|u(t)\|_{L^\infty}^{4/3}.$$

Clarification. In [25, Lemma 4.3], the authors used $\|u_z(t) \partial_t u(t)\|_{L^{3/2}} \leq \|u(t)\|_{L^3} \|\partial_t u\|_{L^3}$, which we have avoided because we want to consider the case $u(0) \in H^2(\mathbb{R}^3)$ for which $\partial_t u(t)$ may be not in L^3 . □

Now we are able to give

Proof of Proposition 4. Recall that we will apply Proposition 7 to $h(t) = h_1 + h_2(t)$ and $k = K_2 = k_1 + k_2$ with

$$h_1 := -\Delta, \quad h_2(t) := |u(t)|^2 * w_N - \mu_N(t) + K_1(t)$$

and

$$k_1 := \tilde{K}_2, \quad k_2 := K_2 - \tilde{K}_2.$$

General results with $u(0) \in H^2(\mathbb{R}^3)$. Recall that if $u(0) \in H^2(\mathbb{R}^3)$, then we have the uniform bound

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq C_{\text{Sobolev}} \|u(t)\|_{H^2(\mathbb{R}^3)} \leq C$$

for a constant C depending only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$. Therefore, all relevant conditions in Proposition 7 have been verified by Lemma 9. Thus by Proposition 7, the Bogoliubov equation (33) has a unique global solution $\Phi(t)$ and the pair of density matrices $(\gamma(t), \alpha(t)) = (\gamma_{\Phi(t)}, \alpha_{\Phi(t)})$ is the unique solution to (34). The constraint $\Phi(t) \in \mathcal{F}_+(t)$ follows from the Hartree equation, see [36, Proof of Theorem 7] for an explanation.

Now we consider the bound (80). Since

$$\xi(t) := 6(\|h_2(t)\| + \|K_2(t)\|) \leq C$$

and

$$\begin{aligned} \Theta(t) &:= 2\|\alpha(0)\|_{\text{HS}}^2 + 2\|\gamma(0)\|_{\text{HS}}^2 \\ &\quad + 2\left(\|(-\Delta_x - \Delta_y)^{-1}\tilde{K}_2(t, \cdot, \cdot)\|_{L^2} + \|(-\Delta_x - \Delta_y)^{-1}\tilde{K}_2(0, \cdot, \cdot)\|_{L^2} \right. \\ &\quad \left. + \int_0^t (\|(K_2 - \tilde{K}_2)(s, \cdot, \cdot)\|_{L^2} + \|(-\Delta_x - \Delta_y)^{-1}\partial_s\tilde{K}_2(s, \cdot, \cdot)\|_{L^2}) ds\right)^2 \\ &\leq 2\|\alpha(0)\|_{\text{HS}}^2 + 2\|\gamma(0)\|_{\text{HS}}^2 + C(1 + t^2) \end{aligned}$$

we deduce from (80) that

$$\begin{aligned} \|\alpha(t)\|_{\text{HS}}^2 + \|\gamma(t)\|_{\text{HS}}^2 &\leq \Theta(t) + \int_0^t \exp\left(\int_s^t \xi(r) dr\right) \xi(s)\Theta(s) ds \\ &\leq e^{Ct}(1 + \|\alpha(0)\|_{\text{HS}}^2 + \|\gamma(0)\|_{\text{HS}}^2) \end{aligned}$$

for a constant C depending only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$.

Moreover, by Proposition 7, if $\Phi(0)$ is a quasi-free state, then $\Phi(t)$ is a quasi-free state for all $t \geq 0$, and from (81) we obtain

$$\langle \Phi(t), \mathcal{N}\Phi(t) \rangle \leq e^{Ct} \left(1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^2.$$

Improved bound with $u(0)$ smooth. Now we consider the case when the initial Hartree state is smooth, namely $u(0) \in W^{\ell,1}(\mathbb{R}^3)$ with ℓ sufficiently large. Recall that in this case

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C_1}{(1+t)^{3/2}}$$

for a constant C_1 depending only on $\|u(0)\|_{W^{\ell,1}(\mathbb{R}^3)}$. By Proposition 7, we have

$$\xi(t) = 6(\|h_2(t)\| + \|K_2(t)\|) \leq \frac{C_1}{(1+t)^3}$$

and

$$\begin{aligned} \Theta(t) &= 2\|\alpha(0)\|_{\text{HS}}^2 + 2\|\gamma(0)\|_{\text{HS}}^2 \\ &\quad + 2\left(\|(-\Delta_x - \Delta_y)^{-1}\tilde{K}_2(t, \cdot, \cdot)\|_{L^2} + \|(-\Delta_x - \Delta_y)^{-1}\tilde{K}_2(0, \cdot, \cdot)\|_{L^2} \right. \\ &\quad \left. + \int_0^t (\| (K_2 - \tilde{K}_2)(s, \cdot, \cdot) \|_{L^2} + \| (-\Delta_x - \Delta_y)^{-1} \partial_s \tilde{K}_2(s, \cdot, \cdot) \|_{L^2}) ds \right)^2, \\ &\leq 2\|\alpha(0)\|_{\text{HS}}^2 + 2\|\gamma(0)\|_{\text{HS}}^2 + C_1 \left(\frac{1}{1+t} + \int_0^t \left(\frac{1}{(1+t)^3} + \frac{1}{1+t} \right) ds \right)^2 \\ &\leq 2\|\alpha(0)\|_{\text{HS}}^2 + 2\|\gamma(0)\|_{\text{HS}}^2 + C_1 \log^2(1+t). \end{aligned}$$

Since $(1+t)^{-3}$ is integrable, the estimate (80) in Proposition 7 gives us

$$\begin{aligned} \|\alpha(t)\|_{\text{HS}}^2 + \|\gamma(t)\|_{\text{HS}}^2 &\leq \Theta(t) + \int_0^t \exp\left(\int_s^t \xi(r) dr\right) \xi(s) \Theta(s) ds \\ &\leq C_1(\log^2(1+t) + \|\alpha(0)\|_{\text{HS}}^2 + \|\gamma(0)\|_{\text{HS}}^2). \end{aligned}$$

Moreover, if $\Phi(0)$ is a quasi-free state, then from (81) we obtain

$$\langle \Phi(t), \mathcal{N}\Phi(t) \rangle \leq C_1 \left(\log(1+t) + 1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^2$$

for a constant C_1 depending only on $\|u(0)\|_{W^{\ell,1}(\mathbb{R}^3)}$. □

5. Proof of Main Theorem

Assuming Lemma 5 at the moment, we are ready to provide

Proof of Theorem 6. We will compare $\Phi_N(t) = U_N(t)\Psi_N(t)$ with the Bogoliubov evolution $\Phi(t)$. Using the equations (29) and (33), we can compute

$$\begin{aligned}
 (112) \quad & \partial_t \|\Phi_N(t) - \Phi(t)\|^2 \\
 &= 2\text{Im} \langle i\partial_t \Phi_N(t), \Phi(t) \rangle - 2\text{Im} \langle \Phi_N(t), i\partial_t \Phi(t) \rangle \\
 &= 2\text{Im} \langle (\mathbf{1}_+^{\leq N} \mathbb{H}(t) \mathbf{1}_+^{\leq N} + R(t)) \Phi_N(t), \Phi(t) \rangle - 2\text{Im} \langle \Phi_N(t), \mathbb{H} \Phi(t) \rangle \\
 &= 2\text{Im} \langle R(t) \Phi_N(t), \Phi(t) \rangle - 2\text{Im} \langle \Phi_N(t), \mathbf{1}_+^{\leq N} \mathbb{H}(1 - \mathbf{1}_+^{\leq N}) \Phi(t) \rangle
 \end{aligned}$$

where $\mathbf{1}_+^{\leq N} := \mathbb{1}_{\mathcal{F}_+^{\leq N}(t)}$ and

$$R(t) := \mathbf{1}_+^{\leq N} \left[i(\partial_t U_N(t)) U_N^*(t) + U_N(t) H_N U_N^*(t) - \mathbb{H}(t) \right] \mathbf{1}_+^{\leq N}.$$

Using the Cauchy-Schwarz inequality and Proposition 3, we can estimate

$$\begin{aligned}
 & \text{Im} \langle R(t) \Phi_N(t), \Phi(t) \rangle \\
 &= \text{Im} \langle \Phi_N(t) - \Phi(t), R(t) \Phi(t) \rangle \\
 &\leq \|R(t) \Phi(t)\| \cdot \|\Phi_N(t) - \Phi(t)\| \\
 &\leq CN^{(3\beta-1)/2} \left\langle \Phi(t), (\mathcal{N}_+ + 1)^4 \Phi(t) \right\rangle^{1/2} \|\Phi_N(t) - \Phi(t)\|.
 \end{aligned}$$

Moreover, by the Cauchy-Schwarz inequality again and (52) we get

$$\begin{aligned}
 & \text{Im} \left\langle \Phi_N(t), \mathbf{1}_+^{\leq N} \mathbb{H}(1 - \mathbf{1}_+^{\leq N}) \Phi(t) \right\rangle \\
 &\leq \|\mathbf{1}_+^{\leq N} \mathbb{H}(1 - \mathbf{1}_+^{\leq N}) \Phi(t)\| \\
 &= \|\mathbf{1}_+^{\leq N} \mathbb{K}_{\text{cr}}^* (1 - \mathbf{1}_+^{\leq N}) \Phi(t)\| \\
 &\leq \left\langle \Phi(t), \mathbb{K}_{\text{cr}} \mathbb{K}_{\text{cr}}^* (1 - \mathbf{1}_+^{\leq N}) \Phi(t) \right\rangle^{1/2} \\
 &\leq CN^{3\beta/2} \left\langle \Phi(t), (\mathcal{N}_+ + 1)^2 (1 - \mathbf{1}_+^{\leq N}) \Phi(t) \right\rangle^{1/2} \\
 &\leq CN^{3\beta/2-1} \left\langle \Phi(t), (\mathcal{N}_+ + 1)^4 \Phi(t) \right\rangle^{1/2}.
 \end{aligned}$$

Thus from (112) it follows that

$$\begin{aligned} \partial_t \|\Phi_N(t) - \Phi(t)\|^2 &\leq CN^{(3\beta-1)/2} \left\langle \Phi(t), (\mathcal{N}_+ + 1)^4 \Phi(t) \right\rangle^{1/2} \\ &\quad \times \left(\|\Phi_N(t) - \Phi(t)\| + N^{-1/2} \right). \end{aligned}$$

Consequently, the function $f(t) := \|\Phi_N(t) - \Phi(t)\|^2 + N^{-1}$ satisfies

$$\partial_t f(t) \leq CN^{(3\beta-1)/2} \left\langle \Phi(t), (\mathcal{N}_+ + 1)^4 \Phi(t) \right\rangle^{1/2} \sqrt{f(t)},$$

and hence

$$\partial_t \sqrt{f(t)} \leq CN^{(3\beta-1)/2} \left\langle \Phi(t), (\mathcal{N}_+ + 1)^4 \Phi(t) \right\rangle^{1/2}.$$

Taking the integral over t , we obtain

$$\begin{aligned} (113) \quad &\left(\|\Phi_N(t) - \Phi(t)\|^2 + N^{-1} \right)^{1/2} \\ &\leq \left(\|\Phi_N(0) - \Phi(0)\|^2 + N^{-1} \right)^{1/2} \\ &\quad + tCN^{(3\beta-1)/2} \sup_{s \in [0,t]} \left\langle \Phi(s), (\mathcal{N}_+ + 1)^4 \Phi(s) \right\rangle^{1/2}. \end{aligned}$$

Now we make a further estimate for every term in (113). First, since $U_N(t) : \mathfrak{H}^N \rightarrow \mathcal{F}_+^{\leq N}(t)$ is a unitary operator, we have

$$\begin{aligned} (114) \quad \|\Phi_N(t) - \Phi(t)\|^2 &= \|U_N(t)\Psi_N(t) - \mathbf{1}_+^{\leq N}\Phi(t)\|^2 + \|(1 - \mathbf{1}_+^{\leq N})\Phi(t)\|^2 \\ &\geq \|\Psi_N(t) - U_N(t)\mathbf{1}_+^{\leq N}\Phi(t)\|^2. \end{aligned}$$

On the other hand, the choice $\Phi_N(0) = \mathbf{1}_+^{\leq N}\Phi(0)$ implies that

$$(115) \quad \|\Phi_N(0) - \Phi(0)\|^2 = \left\langle \Phi(0), (\mathbf{1} - \mathbf{1}_+^{\leq N})\Phi(0) \right\rangle \leq \frac{1}{N} \left\langle \Phi(0), \mathcal{N}_+ \Phi(0) \right\rangle.$$

Moreover, recall that $\Phi(t)$ is a quasi-free state for all t due to Proposition 4 and the assumption that $\Phi(0)$ is a quasi-free state. Therefore, by Lemma 5,

$$(116) \quad \left\langle \Phi(t), (\mathcal{N} + 1)^4 \Phi(t) \right\rangle \leq C \left\langle \Phi(t), (\mathcal{N} + 1)\Phi(t) \right\rangle^4.$$

Inserting (114)-(115)-(116) into (113) we find that

$$\begin{aligned}
 (117) \quad & \|\Psi_N(t) - U_N(t)\mathbb{1}_\mp^{\leq N}\Phi(t)\| \\
 & \leq \frac{1}{\sqrt{N}} \left| \langle \Phi(0), (\mathcal{N} + 1)\Phi(0) \rangle \right|^{1/2} \\
 & \quad + tCN^{(3\beta-1)/2} \sup_{s \in [0,t]} \left\langle \Phi(s), (\mathcal{N} + 1)\Phi(s) \right\rangle^2.
 \end{aligned}$$

Finally, we derive (40) from (117) and the the upper bounds on $\langle \Phi(t), \mathcal{N}\Phi(t) \rangle$ in Proposition 4. To be precise, if we only know $u(0) \in H^2(\mathbb{R}^3)$, then from (117) and (36) we obtain

$$(118) \quad \|\Psi_N(t) - U_N(t)\mathbb{1}_\mp^{\leq N}\Phi(t)\| \leq N^{(3\beta-1)/2} e^{Ct} \left(1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^4$$

for a constant C depending only on $\|u(0)\|_{H^2(\mathbb{R}^3)}$. If we know that $u(0) \in W^{\ell,1}(\mathbb{R}^3)$ with ℓ sufficiently large, then from (117) and (37) we get the improved bound

$$\begin{aligned}
 (119) \quad & \|\Psi_N(t) - U_N(t)\mathbb{1}_\mp^{\leq N}\Phi(t)\| \\
 & \leq N^{(3\beta-1)/2} C_1 (1+t) \left(\log(1+t) + 1 + \langle \Phi(0), \mathcal{N}\Phi(0) \rangle \right)^4
 \end{aligned}$$

for a constant C_1 depending only on $\|u(0)\|_{W^{\ell,1}(\mathbb{R}^3)}$. □

For the completeness, let us provide

Proof of Lemma 5. Since the density matrices γ_Ψ, α_Ψ of Ψ satisfy the relations (83), we can diagonalize them simultaneously. More precisely (see also [43, Lemma 8]), we can find an orthonormal basis $\{u_n\}_{n=1}^\infty$ for \mathfrak{H} and non-negative numbers $\{\lambda_n\}_{n=1}^\infty$ such that

$$\gamma_\Psi = \sum_{n=1}^\infty \lambda_n |u_n\rangle\langle u_n|, \quad \alpha_\Psi u_n = \sqrt{\lambda_n + \lambda_n^2} (\bar{u}_n), \quad \forall n \in \mathbb{N}.$$

Let us denote $a_n = a(u_n)$ for short. By the definition of γ_Ψ and α_Ψ , we have

$$(120) \quad \langle \Psi, a_m^* a_n \Psi \rangle = \delta_{m,n} \lambda_n, \quad \langle \Psi, a_m a_n \Psi \rangle = \delta_{m,n} \sqrt{\lambda_n + \lambda_n^2}, \quad \forall n \in \mathbb{N}.$$

Moreover,

$$\sum_n \lambda_n = \text{Tr}(\gamma_\Psi) = \langle \Psi, \mathcal{N}\Psi \rangle.$$

Now we compute

$$\begin{aligned} & \langle \Psi, \mathcal{N}(\mathcal{N} - 1)(\mathcal{N} - 2)(\mathcal{N} - 3)\dots(\mathcal{N} - \ell + 1)\Psi \rangle \\ &= \sum_{n_1, n_2, \dots, n_\ell \geq 1} \langle \Psi, a_{n_1}^* \cdots a_{n_\ell}^* a_{n_1} \cdots a_{n_\ell} \Psi \rangle \\ &= \sum_{1 \leq s \leq \ell} \sum_{\substack{1 \leq n_1 < n_2 < \dots < n_s \\ (m_1, \dots, m_s) \in P(s, \ell)}} \langle \Psi, (a_{n_1}^*)^{m_1} (a_{n_1})^{m_1} \dots (a_{n_s}^*)^{m_s} (a_{n_s})^{m_s} \Psi \rangle, \end{aligned}$$

where $P(s, \ell)$ is the set of partitions:

$$P(s, \ell) = \{(m_1, \dots, m_s) \subset \mathbb{N} \mid m_1 + \dots + m_s = \ell\}.$$

For all $1 \leq n_1 < n_2 < \dots < n_s$ and $m_1, \dots, m_s \geq 1$, by using Wick's Theorem and (120) we find that

$$\langle \Psi, (a_{n_1}^*)^{m_1} (a_{n_1})^{m_1} \dots (a_{n_s}^*)^{m_s} (a_{n_s})^{m_s} \Psi \rangle = \prod_{j=1}^s \langle \Psi, (a_{n_j}^*)^{m_j} (a_{n_j})^{m_j} \Psi \rangle$$

and

$$\langle \Psi, (a_{n_j}^*)^{m_j} (a_{n_j})^{m_j} \Psi \rangle \leq |P(2m_j)| \lambda_{n_j} (1 + \lambda_{n_j})^{m_j - 1}.$$

Here recall that

$$P_{2n} = \{\sigma \in S(2n) \mid \sigma(2j - 1) < \min\{\sigma(2j), \sigma(2j + 1)\} \text{ for all } j\}.$$

is the set of pairing and we have denoted by $|P(2n)|$ the number of elements of P_{2n} . Thus

$$\begin{aligned} & \sum_{\substack{1 \leq n_1 < n_2 < \dots < n_s \\ (m_1, \dots, m_s) \in P(s, \ell)}} \langle \Psi, (a_{n_1}^*)^{m_1} (a_{n_1})^{m_1} \dots (a_{n_s}^*)^{m_s} (a_{n_s})^{m_s} \Psi \rangle \\ & \leq \sum_{\substack{1 \leq n_1 < n_2 < \dots < n_s \\ (m_1, \dots, m_s) \in P(s, \ell)}} \prod_{j=1}^s |P(2m_j)| \lambda_{n_j} (1 + \lambda_{n_j})^{m_j - 1} \\ & \leq \sum_{\substack{1 \leq n_1 < n_2 < \dots < n_s \\ (m_1, \dots, m_s) \in P(s, \ell)}} |P(2\ell)| (1 + \sup_p \lambda_p)^{\ell - s} \prod_{j=1}^s \lambda_{n_j} \end{aligned}$$

$$\begin{aligned}
 &\leq |P(s, \ell)| \cdot |P(2\ell)| \cdot \left(1 + \sup_p \lambda_p\right)^{\ell-s} \sum_{1 \leq n_1 < n_2 < \dots < n_s} \prod_{j=1}^s \lambda_{n_j} \\
 &\leq |P(s, \ell)| \cdot |P(2\ell)| \cdot \left(1 + \sup_p \lambda_p\right)^{\ell-s} \left(\sum_p \lambda_p\right)^s \\
 &\leq |P(s, \ell)| \cdot |P(2\ell)| \cdot \left(1 + \sum_p \lambda_p\right)^\ell \\
 &= |P(s, \ell)| \cdot |P(2\ell)| \cdot (1 + \langle \Psi, \mathcal{N} \Psi \rangle)^\ell
 \end{aligned}$$

Here in the second estimate we have used $\prod_{j=1}^s |P(2m_j)| \leq |P(2\ell)|$, which follows obviously from the definition of $P(2n)$ and the fact $\sum_{j=1}^s m_j = \ell$. Summing the latter estimate over $s = 1, 2, \dots, \ell$ we obtain

$$\begin{aligned}
 &\langle \Psi, \mathcal{N}(\mathcal{N} - 1)(\mathcal{N} - 2)(\mathcal{N} - 3) \dots (\mathcal{N} - \ell + 1) \Psi \rangle \\
 &= \sum_{1 \leq s \leq \ell} \sum_{\substack{1 \leq n_1 < n_2 < \dots < n_s \\ (m_1, \dots, m_s) \in P(s, \ell)}} \langle \Psi, (a_{n_1}^*)^{m_1} (a_{n_1})^{m_1} \dots (a_{n_s}^*)^{m_s} (a_{n_s})^{m_s} \Psi \rangle \\
 &\leq \left(\sum_{s=1}^{\ell} |P(s, \ell)| \right) \cdot |P(2\ell)| \cdot (1 + \langle \Psi, \mathcal{N} \Psi \rangle)^\ell.
 \end{aligned}$$

The desired result follows from a straightforward induction argument. □

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