

BPS/CFT correspondence II: Instantons at crossroads, Moduli and Compactness Theorem

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Gieseker-Nakajima moduli spaces $\mathcal{M}_k(n)$ parametrize the charge k noncommutative $U(n)$ instantons on \mathbb{R}^4 and framed rank n torsion free sheaves \mathcal{E} on \mathbb{CP}^2 with $\text{ch}_2(\mathcal{E}) = k$. They also serve as local models of the moduli spaces of instantons on general four-manifolds. We study the generalization of gauge theory in which the four dimensional spacetime is a stratified space X immersed into a Calabi-Yau fourfold Z . The local model $\mathfrak{M}_k(\vec{n})$ of the corresponding instanton moduli space is the moduli space of charge k (noncommutative) instantons on origami spacetimes. There, X is modelled on a union of (up to six) coordinate complex planes \mathbb{C}^2 intersecting in Z modelled on \mathbb{C}^4 . The instantons are shared by the collection of four dimensional gauge theories sewn along two dimensional defect surfaces and defect points. We also define several quiver versions $\mathfrak{M}_k^\gamma(\vec{n})$ of $\mathfrak{M}_k(\vec{n})$, motivated by the considerations of sewn gauge theories on orbifolds \mathbb{C}^4/Γ .

The geometry of the spaces $\mathfrak{M}_k^\gamma(\vec{n})$, more specifically the compactness of the set of torus-fixed points, for various tori, underlies the non-perturbative Dyson-Schwinger identities recently found to be satisfied by the correlation functions of qq -characters viewed as local gauge invariant operators in the $\mathcal{N} = 2$ quiver gauge theories.

The cohomological and K-theoretic operations defined using $\mathfrak{M}_k(\vec{n})$ and their quiver versions as correspondences provide the geometric counterpart of the qq -characters, line and surface defects.

1. Introduction

Recently we introduced a set of observables in quiver $\mathcal{N} = 2$ supersymmetric gauge theories which are useful in organizing the non-perturbative Dyson-Schwinger equations, relating contributions of different instanton sectors to the expectation values of gauge invariant chiral ring observables. In this paper

we shall provide the natural geometric setting for these observables. We also explain the gauge and string theory motivations for these considerations.

Notations. We are going to explore the moduli spaces, which parametrize, roughly speaking, the sheaves \mathcal{E} supported on a union of coordinate complex two-planes $\approx \mathbb{C}^2$ inside \mathbb{C}^4 . The \mathbb{C}^4 with a set of \mathbb{C}^2 's inside is a local model (Z^{loc}, X^{loc}) of a pair (Z, X) consisting of a Calabi-Yau fourfold Z which contains a possibly singular complex surface $X \subset Z$:

$$(1) \quad Z^{loc} = \mathbb{C}^4, \quad X^{loc} = \bigcup_{A \in \underline{\mathbf{6}}} \mathbb{C}_A^2, \quad \text{supp}(\mathcal{E}) = \bigcup_{A \in \underline{\mathbf{6}}} n_A \mathbb{C}_A^2$$

We denote by $\underline{\mathbf{4}}$ the set of complex coordinates in \mathbb{C}^4 :

$$(2) \quad \underline{\mathbf{4}} = \{1, 2, 3, 4\}, \quad a \in \underline{\mathbf{4}} \leftrightarrow z_a \in \mathbb{C}$$

and by

$$(3) \quad \underline{\mathbf{6}} = \binom{\underline{\mathbf{4}}}{2} = \left\{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \right\}$$

the set of two-element subsets of $\underline{\mathbf{4}}$, i.e. the set of coordinate two-planes in \mathbb{C}^4 .

We shall sometimes denote the elements of $\underline{\mathbf{6}}$ by the pairs $ab = ba \leftrightarrow \{a, b\} \in \underline{\mathbf{6}}$. We also define, for $A \in \underline{\mathbf{6}}$, $\bar{A} = \underline{\mathbf{4}} \setminus A$, and

$$(4) \quad \epsilon(A) = \epsilon_{abcd}, \quad A = \{a, b\}, \quad \bar{A} = \{c, d\}, \quad a < b, \quad c < d$$

so that, e.g. $\overline{12} = 34$, $\epsilon(23) = \epsilon_{2314} = 1$, $\epsilon(24) = \epsilon_{2413} = -1$.

The two-plane $\mathbb{C}_A^2 \subset \mathbb{C}^4$ corresponding to $A \in \underline{\mathbf{6}}$ is defined by the equations: $z_{\bar{a}} = 0$, for all $\bar{a} \in \bar{A}$.

We denote by $\underline{\mathbf{3}}$ the quotient $\underline{\mathbf{6}}/\mathbb{Z}_2$ where \mathbb{Z}_2 acts by the involution $A \mapsto \bar{A}$. The elements $a \in \underline{\mathbf{3}}$ are the unordered pairs (A, \bar{A}) .

We can visualize the sets $\underline{\mathbf{3}}$, $\underline{\mathbf{4}}$, $\underline{\mathbf{6}}$, using the tetrahedron:

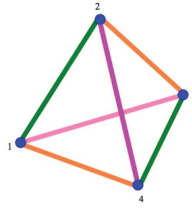


Figure 1: Tetrahedron with the sets $\underline{4}$ and $\underline{6}$ of vertices and edges, and the set $\underline{3} = \{\text{red, green, orange}\}$ of crossed edges.

Our story will involve four complex parameters $q_a \in \mathbb{C}^\times$, $a \in \underline{4}$, obeying

$$(5) \quad \prod_{a \in \underline{4}} q_a = 1$$

We shall also use the additive variables $\mathbf{e}_a \in \mathbb{C}$, $a \in \underline{4}$, obeying

$$(6) \quad \sum_{a \in \underline{4}} \mathbf{e}_a = 0$$

Define the lattice

$$(7) \quad Z_{\underline{\mathbf{e}}} = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2 + \mathbb{Z}\mathbf{e}_3 \subset \mathbb{C}$$

which is the image of the projection:

$$(8) \quad \mathbb{Z}^4 \rightarrow \mathbb{C}, \quad (i, j, k, l) \mapsto \mathbf{e}_1 i + \mathbf{e}_2 j + \mathbf{e}_3 k + \mathbf{e}_4 l$$

We shall use the following functions on $\underline{6}$:

$$(9) \quad \begin{aligned} p_A &= \prod_{a \in A} (1 - q_a) , \\ q_A &= \prod_{a \in A} q_a , \\ \mathbf{e}_A &= \sum_{a \in A} \mathbf{e}_a = -\mathbf{e}_{\bar{A}} . \end{aligned}$$

In what follows we denote by $[n]$, for $n \in \mathbb{Z}_{>0}$, the set $\{1, 2, 3, \dots, n\} \subset \mathbb{Z}_{>0}$.

Let S be a finite set, and $(V_s)_{s \in S}$ a collection of vector spaces. We use the notation

$$(10) \quad \sum_{s \in S} V_s$$

for the vector space which consists of all linear combinations

$$(11) \quad \sum_{s \in S} \psi_s, \quad \psi_s \in V_s.$$

1.1. Organization of the paper

We review the gauge and string theory motivation in the Section **2**. The moduli space $\mathfrak{M}_k(\vec{n})$ of spiked instantons is introduced in the Section **3**. The symmetries of spiked instantons are studied in the Section **4**. The moduli space of ordinary $U(n)$ instantons on (noncommutative) \mathbb{R}^4 is reviewed in Section **5**. The Section **6** discusses in more detail two particular cases of spiked instantons, the crossed instantons and the folded instantons. The crossed instantons live on two four-dimensional manifolds transversely intersecting in the eight-dimensional ambient manifold (a Calabi-Yau fourfold), the folded ones live on two four-dimensional manifolds intersecting transversely in the six dimensional ambient manifold. The Section **7** constructs the spiked instantons out of the ordinary ones, and studies the toric spiked instantons in some detail. The Section **8** is the main result of this paper: the compactness theorem. In Section **9** we enter the theory of integration over the spiked and crossed instantons, and relate the analyticity of the partition functions to the compactness theorem. The Section **10** discusses the *ADE*-quiver generalizations of crossed instantons. The Section **11** describes the spiked instantons on cyclic orbifolds, and the associated compactness theorem. The Section **12** is devoted to future directions and open questions.

1.2. Acknowledgements

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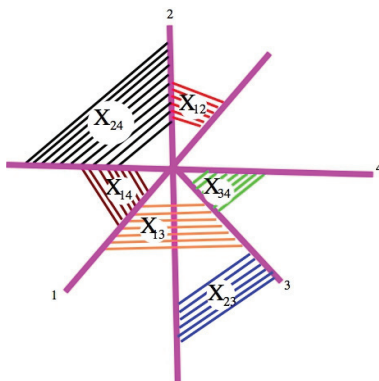


Figure 2: The origami worldvolume $X = \bigcup_A X_A$.

stonysbrook.edu/video_portal/video.php?id=2202, at the Institute for Advanced Studies at Hebrew University <https://www.youtube.com/watch?v=vGNfXQ3-Rjg>, at the Center for Mathematical Sciences and Applications at Harvard University <http://cmsa.fas.harvard.edu/nikita-nekrasov-crossed-instantons-qq-character/> and at the String-Math-2015 conference in Sanya, China.

2. Gauge and string theory motivations

2.1. Generalized gauge theory

We study the moduli spaces $\mathfrak{M}_{X,G}$ of what might be called supersymmetric gauge fields in the generalized gauge theories, whose space-time X contains several, possibly intersecting, components: see Fig. 2. We call such X the *origami worldvolume*. The gauge groups $G|_{X_A} = G_A$ on different components may be different. The intersections $X_A \cup X_B$ lead to the bi-fundamental matter fields charged under $G_A \times G_B$. The arrangement is motivated by the string theory considerations, where the open string Hilbert space, in the presence of several D -branes, splits into sectors labelled by the boundary conditions. It is well-known [10, 34] that some features of the open string theory are captured by the noncommutative gauge theory. In fact, the theories we shall study descend from the maximally supersymmetric Yang-Mills theory, which is twisted and deformed. One can view the fields of this theory as describing the deformations of the four dimensional stratified manifolds $\mathbf{X} = (X_A, n_A)$, i.e. singular, in general, spaces, which can be represented as

unions $X = \cup_A X_A$ of manifolds with certain conditions on closures and intersections, endowed with multiplicities, i.e. the strata X_A are allowed to have different multiplicity n_A . The local gauge group G_A is simply $U(n_A)$. The particular twist of the super-Yang-Mills theory we study corresponds to $X \hookrightarrow Z \times E$, where E is a two torus \mathbb{T}^2 , a cylinder $\mathbb{R}^1 \times \mathbb{S}^1$, or a plane \mathbb{R}^2 , while Z is a special holonomy eight dimensional manifold, e.g. the Calabi-Yau fourfold.

2.2. Gauge origami

Now suppose Z has non-trivial isometries (it ought to be non-compact). It is natural, in this case, to deform the problem to take into account the symmetries of Z . The partition function of the theory of stratified multiple \mathbf{X} 's localizes onto the set of fixed points, which are the configurations $\mathbf{X} = (X_A, n_A)$ where X_A 's are invariant under the isometries of Z . For example, when Z is toric, with the three dimensional torus \mathbf{T} acting by isometries, preserving the holomorphic top form, then at each vertex $z \in Z^{\mathbf{T}}$ pass at most six strata X_A , $A \in \underline{\mathbf{6}}$.

We are interested in integrals over the moduli spaces $\mathfrak{M}_{\mathbf{X},G}$. We shall view $\mathfrak{M}_{\mathbf{X},G}$ as the ‘‘space, defined by some equations modulo symmetry’’. More formally, $\mathfrak{M}_{\mathbf{X},G}$ is the quotient of a set of zeroes of some $\mathbf{G}_{\mathbf{g}}$ -equivariant section $s : M \rightarrow V$ of $\mathbf{G}_{\mathbf{g}}$ -equivariant vector bundle $V \rightarrow M$ over some smooth space (vector space in our case) with $\mathbf{G}_{\mathbf{g}}$ -action, with some Lie group $\mathbf{G}_{\mathbf{g}}$. If M is compact the integral over $\mathfrak{M}_{\mathbf{X},G}$ of a closed differential form can be represented by the $\mathbf{G}_{\mathbf{g}}$ -equivariant integral over M of the pull-back of the corresponding form times the Euler class of V . In the non-compact case one uses equivariant cohomology (mass deformation, in the physics language) with respect to both $\mathbf{G}_{\mathbf{g}}$ and some global symmetry group \mathbf{H} , and Mathai-Quillen representatives of the Euler class.

The resulting partition functions

$$(12) \quad \mathcal{Z}_{\mathbf{X},G}(\xi) \sim \int_{\mathfrak{M}_{\mathbf{X},G}}^{\mathbf{H}_{\text{equiv}}} 1 \sim \frac{1}{\text{Vol}(\mathbf{G}_{\mathbf{g}})} \int_{\text{Lie}(\mathbf{G}_{\mathbf{g}})} \int_M^{(\mathbf{G}_{\mathbf{g}} \times \mathbf{H})_{\text{equiv}}} \text{Euler}(V)$$

are functions on the Lie algebra of $\mathbf{H}_{\mathbb{C}}$, $\xi \in \text{Lie}(\mathbf{H}_{\mathbb{C}})$. The analytic properties of $\mathcal{Z}_{\mathbf{X},G}(\xi)$ reflect some of the geometric and topological features of $\mathfrak{M}_{\mathbf{X},G}$. They are the main focus of this paper.

The equivariant localization expresses $\mathcal{Z}_{\mathbf{X},G}$ as the sum over the fixed points of \mathbf{H} -action, which are typically labelled by multiple partitions, i.e. collections of Young diagrams. The resulting statistical mechanical model is

called the gauge origami and is studied in detail in the companion paper [29].

2.3. Symmetries, twisting, equivariance

The partition functions $\mathcal{Z}_{X,G}(\xi)$ are analytic functions of $\xi \in \text{Lie}(\mathbf{H}_{\mathbb{C}})$, with possible singularities. Given $\xi \in \text{Lie}(\mathbf{H}_{\mathbb{C}})$, the closure of the subgroup $\exp t\xi$, $t \in \mathbb{C}$ defines a torus \mathbb{T}_{ξ} . The partition function $\mathcal{Z}_{X,G}(\xi)$ can be computed, by Atiyah-Bott fixed point formula, as a sum over the \mathbb{T}_{ξ} -fixed points. Even though the moduli space $\mathfrak{M}_{X,G}$ may be noncompact (it is noncompact for noncompact X), the fixed point set, for suitable ξ , may still be compact, so that the integrals over $\mathfrak{M}_{X,G}$ of the equivariant differential forms converge. The set $\mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}}$ of \mathbb{T}_{ξ} -fixed points may have several connected components:

$$(13) \quad \mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}} = \bigcup_f \left(\mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}} \right)_f$$

The contributions \mathcal{Z}_f of $\left(\mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}} \right)_f$ are rational functions on $\text{Lie}(\mathbf{H}_{\mathbb{C}})$, they have poles. In the nice situations the component $\left(\mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}} \right)_f$ has a normal bundle in $\mathfrak{M}_{X,G}$ (or in the ambient smooth variety, as in the case of the obstructed theory), N_f , which inherits an action of \mathbb{T}_{ξ} , and decomposes into the sum of complex line bundles (real rank two bundles) $L_{f,w}$, with w going through the set of \mathbb{T}_{ξ} -weights. The fixed point formula states

$$(14) \quad \mathcal{Z}_f = \int_{\left(\mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}} \right)_f} \frac{\text{Euler}_{\xi}(\text{Obs}_f)}{\prod_w (w(\xi) + c_1(L_{f,w}))}$$

The poles in \mathcal{Z}_f occur when the Lie algebra element ξ crosses the hyperplane $w(\xi) = 0$ for some w occurring in the decomposition of N_f . Geometrically this means that the ξ belongs to a subalgebra of $\text{Lie}\mathbb{T} \subset \text{Lie}(\mathbf{H}_{\mathbb{C}})$ which fixes not only $\left(\mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}} \right)_f$, but also (at least infinitesimally, at the linearized order) a two-dimensional surface passing through f , in the direction of $L_{f,w}$.

We shall be interested in the analytic properties of $\mathcal{Z}_{X,G}$ and one of the questions we shall be concerned with is whether the poles in \mathcal{Z}_f are cancelled by the poles in the contribution of some other component $\left(\mathfrak{M}_{X,G}^{\mathbb{T}_{\xi}} \right)_{f'}$ of the fixed point set. More precisely, once $\xi \rightarrow \xi_c$ where ξ_c belongs to the hyperplane $w(\xi) = 0$ defined relative to the weight decomposition of N_f , the

component of the fixed point set may enhance,

$$(15) \quad \left(\mathfrak{M}_{X,G}^{\tau_\xi} \right)_f \subset \left(\mathfrak{M}_{X,G}^{\tau_{\xi_c}} \right)_{f''},$$

reaching out to the other component $\left(\mathfrak{M}_{X,G}^{\tau_\xi} \right)_{f'}$,

$$(16) \quad \left(\mathfrak{M}_{X,G}^{\tau_\xi} \right)_{f'} \cap \left(\mathfrak{M}_{X,G}^{\tau_{\xi_c}} \right)_{f''} \neq \emptyset$$

If the enhanced component $\left(\mathfrak{M}_{X,G}^{\tau_{\xi_c}} \right)_{f''}$ is compact, then the pole at $\xi = \xi_c$ in \mathcal{Z}_f will be cancelled by the pole in $\mathcal{Z}_{f'}$.

So the issue in question is the compactness of the fixed point set for the torus generated by the non-generic infinitesimal symmetries ξ_c .

In our case we shall choose a class of subgroups $\mathfrak{H} \subset \mathbf{H}$. We shall show that the set of \mathfrak{H} -fixed points is compact. It means that for generic choice of $\widehat{\xi} \in \text{Lie}(\mathfrak{H}_{\mathbb{C}})$ the partition function $\mathcal{Z}_{X,G}(\widehat{\xi} + \mathbf{x})$ as a function of $\mathbf{x} \in \text{Lie}(\mathfrak{H}_{\mathbb{C}})^\perp \subset \text{Lie}(\mathbf{H})$ has no singularities.

The procedure of restricting the symmetry group of the physical system to a subgroup is well-known to physicists under the name of twisting [38]. It is used in the context of topological field theories, which are obtained from the supersymmetric field theories having an R -symmetry group \mathbf{H}_R such that the group of rotations \mathbf{G}_{rot} of flat spacetime can be embedded nontrivially into the direct product

$$(17) \quad \mathbf{G}_{\text{rot}} \longrightarrow \mathbf{G}_{\text{rot}} \times \mathbf{H}_R .$$

We shall encounter a lot of instances of the procedure analogous to (17) in what follows.

2.4. Gauge theories on stacks of D-branes

The maximally supersymmetric Yang-Mills theory in $p + 1$ -dimensions models [39] the low energy behavior of a stack of parallel Dp -branes. This description can be made p -blind by turning on a background constant B -field. In the strong B -field the “non-abelian Born-Infeld/Yang-Mills” theory description of the low energy physics of the open strings connecting the Dp -branes crosses over to the noncommutative Yang-Mills description [34]. In this paper we shall use the noncommutative Yang-Mills to study the dynamics of intersecting stacks of Dp -branes.

2.4.1. The Matrix models. Recall the dimensional reductions of the maximally supersymmetric Yang-Mills theory down to 0 + 0, 0 + 1 and 1 + 1 dimensions [4], [18], [9]. We take the gauge group to be $G_{\mathbf{g}} = U(\mathbf{N})$ for some large \mathbf{N} . Following [23] we shall view the model of [18]

$$(18) \quad \frac{1}{\text{Vol}G_{\mathbf{g}}} \int_{\mathbb{R}^{10|16} \otimes \text{Lie}G_{\mathbf{g}}} DX^m D\theta^\alpha \exp \left(-\frac{1}{4} \sum_{m < n} \text{Tr} [X^m, X^n]^2 - \frac{1}{2} \sum_{m, \alpha, \beta} \Gamma_{\alpha\beta}^m \text{Tr} \theta^\alpha [X^m, \theta^\beta] \right)$$

with the adjoint bosons X^m and the adjoint fermions θ^α transforming in the representations $\mathbf{10}$ and $\mathbf{16}$ of $\text{Spin}(10)$, respectively, as the cohomological field theory in 0 dimensions, while [4] and [9] are obtained by the lift procedure of [5].

The approach to the noncommutative gauge theory in which the gauge field A_m is traded for the (infinite) matrix X^m is used in the background-independent (p -uniform) formalism of [33].

2.4.2. (0, 1)-formalism, Spin(7)-instantons. Let us start in the 1 + 1-dimensional case. Let Σ be the worldsheet of our theory, with the local complex coordinates z, \bar{z} . The theory has a gauge field $A = A_z dz + A_{\bar{z}} d\bar{z}$, 8 Hermitian adjoint scalars X^m , and 16 adjoint fermions, which split into 8 right ψ_+^a , and 8 left $\chi_-^{\dot{a}}$ ones. Here m, a and \dot{a} are the indices of the $\mathbf{8}_v, \mathbf{8}_s$ and $\mathbf{8}_c$ representations of the global symmetry group $\text{Spin}(8)$, respectively. The formalism that we shall adopt singles out a particular spinor among $\mathbf{8}_c$. The isotropy subgroup $\text{Spin}(7)$ of that spinor has the following significance. The representation $\mathbf{8}_c$, under $\text{Spin}(7)$, decomposes as $\mathbf{7} \oplus \mathbf{1}$, the $\mathbf{1}$ being the invariant subspace. Accordingly, we split $(\chi_-^{\dot{a}})_{\dot{a} \in \mathbf{8}_c} \rightarrow (\chi_-^i)_{i \in \mathbf{7}} \oplus \eta_-$. The representations $\mathbf{8}_s$ and $\mathbf{8}_v$ become the spinor $\mathbf{8}$ of $\text{Spin}(7)$. We shall also need the auxiliary fields h^i , which are the worldsheet scalars, transform in the adjoint of the gauge group, and in the representation $\mathbf{7}$ of $\text{Spin}(7)$. The theory, in this formalism, has one supercharge δ_+ , which squares to the chiral translation on the worldsheet

$$(19) \quad \delta_+^2 = D_{++} = \bar{\partial}_{\bar{z}} + A_{\bar{z}}$$

(the conjugate derivative $D_{--} = \partial_z + A_z$) which acts on the fields of the model as follows:

$$(20) \quad \begin{aligned} \delta_+ X^m &= \psi_+^m, & \delta_+ \psi_+^m &= D_{++} X^m = \bar{\partial}_{\bar{z}} X^m + [A_{\bar{z}}, X^m] \\ \delta_+ \chi_-^i &= h^i, & \delta_+ h^i &= D_{++} \chi_-^i = \bar{\partial}_{\bar{z}} \chi_-^i + [A_{\bar{z}}, \chi_-^i] \\ \delta_+ A_z &= \eta_-, & \delta_+ \eta_- &= F_{z\bar{z}} \end{aligned}$$

The Lagrangian

$$(21) \quad L = \delta_+ \int_{\Sigma} \text{Tr} (\psi_+^m D_{--} X^m + \chi_-^i (i\varphi_{mn}^i [X^m, X^n] - h^i) + \eta_- F_{z\bar{z}})$$

becomes that of the standard $\mathcal{N} = 8$ supersymmetric Yang-Mills once the auxiliary fields h^i are eliminated by their equations of motion. Here φ_{mn}^i is the matrix of the projection $\wp_7 : \Lambda^2 \mathfrak{so}_8 = \text{LieSpin}(8) \rightarrow \mathfrak{so}_7$ onto the orthogonal complement to $\text{LieSpin}(7) \subset \text{LieSpin}(8) = \text{LieSpin}(7) \oplus \mathfrak{so}_7$.

Upon the dimensional reduction to 0 dimensions, the gauge field A becomes a complex scalar $\sigma = A_z$ and its conjugate $\bar{\sigma} = A_{\bar{z}}$.

2.4.3. (0, 2)-formalism, SU(4)-instantons. In this formalism we have two supercharges Q_+, \bar{Q}_+ , obeying

$$(22) \quad Q_+^2 = \bar{Q}_+^2 = 0, \quad Q_+ \bar{Q}_+ + \bar{Q}_+ Q_+ = D_{++}$$

so that $\delta_+ = Q_+ + \bar{Q}_+$. We split 8 Hermitian adjoint scalars X^m into 4 complex adjoint scalars Z^a , $a \in \mathbf{4}$, and their conjugates $\bar{Z}^{\bar{a}}$, and the same for the fermions $\psi_+^m \rightarrow \psi_+^a, \bar{\psi}_+^{\bar{a}}$. The \mathfrak{so}_7 χ_-^i 's split as $\mathfrak{so}_6 \oplus \mathfrak{so}_1$: $(\chi_-^i)_{i \in \mathfrak{so}_7} \rightarrow (\chi_{A,-} = \varepsilon_{AA\bar{A}\bar{A},-})_{A \in \mathfrak{so}_6} \oplus \chi_-$. This splitting breaks the symmetry group $\text{Spin}(8) \times \text{Spin}(2) \subset \text{Spin}(10)$ of (18) down to $\text{SU}(4)$.

The Lagrangian (21), in this formalism, reads as follows:

$$(23) \quad \begin{aligned} L &= \delta_+ \Psi, \\ \Psi &= \int_{\Sigma} \text{Tr} (\psi_+^a D_{--} \bar{Z}^{\bar{a}} + \bar{\psi}_+^{\bar{a}} D_{--} Z^a + \eta_- F_{z\bar{z}}) \\ &\quad + i \int_{\Sigma} \text{Tr} \left(\chi_{ab,-} \left([Z^a, Z^b] + \frac{1}{2} \varepsilon^{abcd} [\bar{Z}^{\bar{c}}, \bar{Z}^{\bar{d}}] \right) + \chi_- \mu \right) \\ &\quad - \int_{\Sigma} \text{Tr} (\chi_- h + \chi_{ab,-} h^{ab}) \end{aligned}$$

where

$$(24) \quad \mu = \sum_{a \in \mathbf{4}} [Z^a, \bar{Z}^{\bar{a}}]$$

The supersymmetric (for flat Σ) solutions of (23) are the covariantly holomorphic matrices, solving the equations

$$(25) \quad D_{\bar{z}}Z^a = 0, \quad D_{\bar{z}}\bar{Z}^{\bar{a}} = 0, \quad a \in \underline{4}, \quad \mu = 0$$

and

$$(26) \quad [Z^a, Z^b] + \frac{1}{2}\varepsilon^{abcd}[\bar{Z}^{\bar{c}}, \bar{Z}^{\bar{d}}] = 0, \quad \{a, b\} \in \underline{6}$$

For finite dimensional $\mathbb{C}^{\mathbf{N}}$ these equations imply that all matrices commute and can be simultaneously diagonalized.

2.4.4. Noncommutative gauge theory. We now wish to consider a generalization of the model [18] in which the finite dimensional vector space $\mathbb{C}^{\mathbf{N}}$ is replaced by a Hilbert space \mathcal{H} . In order to keep the action (21) finite the combination

$$(27) \quad \wp_{mn}^i[X^m, X^n]$$

could be deformed to

$$(28) \quad \wp_{mn}^i[X^m, X^n] - i\vartheta^i \cdot \mathbf{1}_{\mathcal{H}}$$

for some constants ϑ^i . One possibility to have a finite action configuration (after h^i 's are integrated out) is to have the operators X^m obey the Heisenberg algebra:

$$(29) \quad [X^m, X^n] = i\vartheta^{mn} \cdot \mathbf{1}_{\mathcal{H}}$$

with the c -number valued matrix $\vartheta^{mn} = -\vartheta^{nm}$ obeying

$$(30) \quad \vartheta^i = \sum_{m < n} \wp_{mn}^i \vartheta^{mn} .$$

Since there are too many choices of ϑ^{mn} given ϑ^i , the modification (28) is not what we need. A more sensible modification is to define the action (with the auxiliary fields eliminated) to have the bosonic potential:

$$(31) \quad \sum_{m < n} \text{Tr}_{\mathcal{H}} \left([X^m, X^n] - i\vartheta^{mn} \cdot \mathbf{1}_{\mathcal{H}} \right)^2$$

whose absolute minima are given by the representation of the Heisenberg algebra (29) in \mathcal{H} . These are classified, for the non-degenerate ϑ^{mn} , modulo

the gauge group $G_{\mathfrak{g}} = U(\mathcal{H})$, by the Stone-von Neumann theorem. Fix a non-negative integer N , and a standard oscillator representation H of the Heisenberg algebra $[\hat{x}^m, \hat{x}^n] = i\vartheta^{mn} \cdot \mathbf{1}_H$. Then $H = L^2(\mathbb{R}^4)$, $\mathcal{H} = \mathbb{C}^N \otimes H$. For example, let us choose a block diagonal basis for ϑ^{mn} , in which

$$(32) \quad \hat{z}^a = \hat{x}^{2a-1} + i\hat{x}^{2a}, \quad \hat{z}^{a\dagger} = \hat{x}^{2a-1} - i\hat{x}^{2a}, \quad a \in \underline{4}$$

obey

$$(33) \quad \begin{aligned} [\hat{z}^a, \hat{z}^b] &= 0, & [\hat{z}^{a\dagger}, \hat{z}^{b\dagger}] &= 0, & \{a, b\} &\in \underline{6} \\ [\hat{z}^a, \hat{z}^{b\dagger}] &= -\theta^a \delta^{ab} \end{aligned}$$

The supersymmetric solution of the matrix model, the 0-dimensional reduction of (21) is given by the operators

$$(34) \quad \begin{aligned} X^m &= \mathbf{1}_N \otimes \hat{x}^m, & m &= 1, \dots, 8 \\ \sigma &= \text{diag}(\sigma_1, \dots, \sigma_N) \otimes \mathbf{1}_H \end{aligned}$$

This solution, for $\theta_a \neq 0$ for all $a \in \underline{4}$ describes a stack of N $D7$ branes whose worldvolume extends in the $1, \dots, 8$ directions. They are localized in the remaining two dimensions, parametrized by the eigenvalues of the complex matrix σ .

Now let us assume all θ_a equal to $\zeta > 0$. Take $H = L^2(\mathbb{R}^2)$ to be the Fock space representation of the algebra

$$(35) \quad \begin{aligned} [c_1, c_2] &= [c_1^\dagger, c_2^\dagger] = 0 \\ [c_i, c_j^\dagger] &= \zeta \delta_{ij}, & i, j &= 1, 2 \end{aligned}$$

Define:

$$(36) \quad \begin{aligned} \hat{z}^1 &= \frac{1}{\sqrt{2}} \left(\mathbf{1}_{N_{12}} \otimes c_1^\dagger + \mathbf{1}_{N_{13}} \otimes c_1^\dagger + \mathbf{1}_{N_{14}} \otimes c_1^\dagger \right) \\ \hat{z}^2 &= \frac{1}{\sqrt{2}} \left(\mathbf{1}_{N_{12}} \otimes c_2^\dagger + \mathbf{1}_{N_{23}} \otimes c_1^\dagger + \mathbf{1}_{N_{24}} \otimes c_1^\dagger \right) \\ \hat{z}^3 &= \frac{1}{\sqrt{2}} \left(\mathbf{1}_{N_{13}} \otimes c_2^\dagger + \mathbf{1}_{N_{23}} \otimes c_2^\dagger + \mathbf{1}_{N_{34}} \otimes c_1^\dagger \right) \\ \hat{z}^4 &= \frac{1}{\sqrt{2}} \left(\mathbf{1}_{N_{14}} \otimes c_2^\dagger + \mathbf{1}_{N_{24}} \otimes c_2^\dagger + \mathbf{1}_{N_{34}} \otimes c_2^\dagger \right) \end{aligned}$$

These operators obey:

$$(37) \quad \begin{aligned} [\widehat{z}^a, \widehat{z}^b] &= [\widehat{z}^{a\dagger}, \widehat{z}^{b\dagger}] = 0 \\ \sum_{a \in \underline{4}} [\widehat{z}^a, \widehat{z}^{a\dagger}] &= -\zeta \mathbf{1}_{N \otimes H} \end{aligned}$$

where

$$(38) \quad N = \bigoplus_{A \in \underline{6}} N_A$$

The solution (37) describes six stacks of $D3$ -branes spanning the coordinate two-planes $\mathbb{C}^2 \subset \mathbb{C}^4$, with $n_A = \dim N_A$ branes spanning the two-plane \mathbb{C}^2_A . This is a generalization of the “piercing string” and “fluxon” solutions of [13, 14].

We can easily produce more general solutions of the BPS equations. Take six solutions $\widehat{C}_A^1, \widehat{C}_A^2$ of non-commutative instanton equations in \mathbb{R}^4 , viewed as operators in $N_A \otimes H$, obeying:

$$(39) \quad [\widehat{C}_A^1, \widehat{C}_A^2] = 0, \quad [\widehat{C}_A^1, \widehat{C}_A^{1\dagger}] + [\widehat{C}_A^2, \widehat{C}_A^{2\dagger}] = \zeta$$

Define operators in $\mathcal{H} = \bigoplus_{A \in \underline{6}} N_A \otimes H$:

$$(40) \quad \widehat{Z}^a = \frac{1}{\sqrt{2}} \bigoplus_{A \ni a} \widehat{C}_A^{h_A(a)\dagger}$$

where $h_{\{a,b\}}(a) = 1$ for $a < b$, and $h_{\{a,b\}}(a) = 2$ for $a > b$. These operators satisfy the higher dimensional analogues of the noncommutative instanton equations (26), (25), [28]:

$$(41) \quad \begin{aligned} +\frac{1}{2} \sum_{c,d} \varepsilon_{abcd} [\widehat{Z}^{d\dagger}, \widehat{Z}^{c\dagger}] &= 0, \quad a, b, c, d \in \underline{4} \\ \sum_{a \in \underline{4}} [\widehat{Z}^a, \widehat{Z}^{a\dagger}] &= -\zeta \mathbf{1}_{N \otimes H} \end{aligned}$$

The aim of the next section is to produce the (almost) finite-dimensional model of the moduli space of finite action solutions to (41). Some of these solutions are of the form (40).

...

Recently the field theory description of two stacks of intersecting $D3$ branes in IIB string theory sharing a common $1 + 1$ -dimensional worldvolume was explored in [6, 22]. The theories exhibit unusual holographic and renormalization properties.

...

The string theory of two stacks of transversely intersecting $D3$ branes in IIB theory has been recently studied in [35, 36], albeit in the $\zeta = 0$ case. None of the beauty (to the trained eye) of the picture presented below seem to survive in this limit.

3. Spiked instantons

We are going to work with the collections of vector spaces and linear maps between them. The vector spaces will be labelled by the coordinate complex two-planes in the four dimensional complex vector space \mathbb{C}^4 .

3.1. Generalized ADHM equations

We start by fixing seven Hermitian vector spaces: K and N_A , $A \in \underline{\mathbf{6}}$. Let $k = \dim_{\mathbb{C}}(K)$, $n_A = \dim_{\mathbb{C}}(N_A)$. Consider the vector space $\mathcal{A}_k(\vec{n})$ of linear maps $(\mathbf{B}, \mathbf{I}, \mathbf{J})$

$$\begin{aligned}
 (42) \quad \mathbf{B} &= (B_a)_{a \in \underline{\mathbf{4}}}, & B_a &: K \rightarrow K, \\
 \mathbf{I} &= (I_A)_{A \in \underline{\mathbf{6}}}, & I_A &: N_A \rightarrow K, \\
 \mathbf{J} &= (J_A)_{A \in \underline{\mathbf{6}}}, & J_A &: K \rightarrow N_A.
 \end{aligned}$$

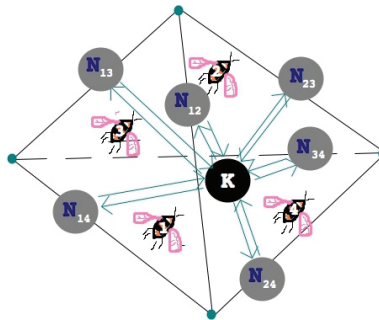


Figure 3: Seven vector spaces and maps between them.

The vector spaces and the maps are conveniently summarized by the tetrahedron diagram on Fig. 3. The choice of the matrices can be motivated by the string theory considerations. Namely, consider k $D(-1)$ -branes in the vicinity of the six stacks of $D3$ -branes (some of these stacks could be $\overline{D3}$ -branes) spanning the coordinate two-planes $\mathbb{C}_A^2 \subset \mathbb{C}^4$. The number of branes spanning \mathbb{C}_A^2 is n_A .

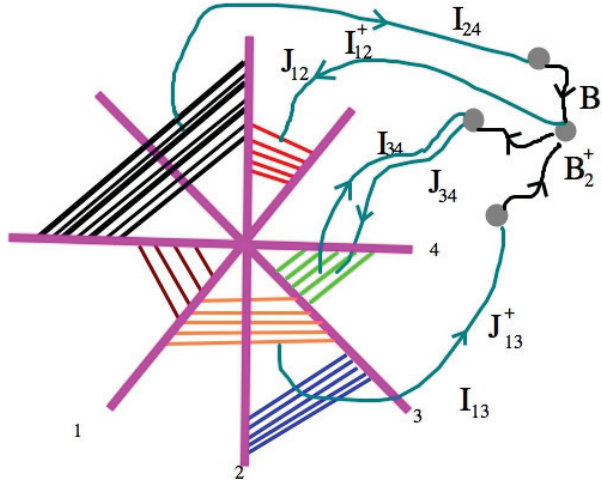


Figure 4: Open string sectors: fields \mathbf{B} , \mathbf{I} , \mathbf{J} .

Then the open strings stretched between the $D(-1)$ and $D(-1)$'s produce, upon quantization, the matrices B_a, B_a^\dagger , together with their superpartners, and some auxiliary fields, which enter the effective Lagrangian in such a way so as to impose the following

3.1.1. KK equations. Define, for $A = \{a, b\}$, $a < b$,

$$(43) \quad \mu_A = [B_a, B_b] + I_A J_A,$$

and

$$(44) \quad s_A = \mu_A + \epsilon(A) \mu_{\bar{A}}^\dagger : K \rightarrow K, \quad A \in \underline{\mathbf{6}}$$

obeying

$$(45) \quad s_A^\dagger = \epsilon(A) s_{\bar{A}}$$

Define the *real moment map*

$$(46) \quad \mu = \sum_{a \in \mathbf{4}} [B_a, B_a^\dagger] + \sum_{A \in \mathbf{6}} (I_A I_A^\dagger - J_A^\dagger J_A)$$

The symmetry (45) allows to view the collection $\vec{s} = (s_A)_{A \in \mathbf{6}} \oplus \mu$ as the $U(K)$ -equivariant map

$$(47) \quad \vec{s} : \mathcal{A}_k(\vec{n}) \longrightarrow \text{Lie}U(K)^* \otimes \mathbb{R}^7,$$

as a sort of an octonionic version of the hyperkähler moment map [15].

...

Likewise, the open strings stretched between the $D(-1)$ and $D3$'s produce, upon quantization, the matrices $I_A, J_A, I_A^\dagger, J_A^\dagger$, together with their superpartners, and some auxiliary fields,

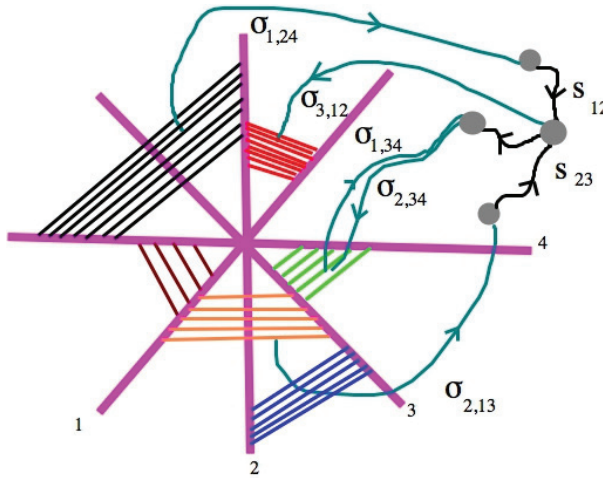


Figure 5: Open string sectors: mutiplets of the equations $\sigma_{\bar{a}A}, s_A$.

which enter the effective Lagrangian in such a way so as to impose the following

3.1.2. KN equations. For each pair (\bar{a}, A) , where $A \in \mathbf{6}$, and $\bar{a} \in \bar{A}$, define

$$(48) \quad \sigma_{\bar{a}A} = B_{\bar{a}} I_A + \varepsilon_{\bar{a}\bar{b}A} B_{\bar{b}}^\dagger J_A^\dagger : N_A \rightarrow K$$

where $\bar{b} \in \bar{A}$, and $\bar{b} \neq \bar{a}$.

3.1.3. NN equations. Now, for each $A \in \underline{\mathbf{6}}$ define

$$(49) \quad \Upsilon_A = J_{\bar{A}} I_A - \epsilon(A) I_{\bar{A}}^\dagger J_A^\dagger : N_A \rightarrow N_{\bar{A}}$$

which obey

$$(50) \quad \Upsilon_A^\dagger = -\Upsilon_{\bar{A}}.$$

Because of the symmetry (50) the collection of the maps $(\Upsilon_A)_{A \in \underline{\mathbf{6}}}$ takes values in the real vector space of dimension

$$(51) \quad \sum_{A \in \underline{\mathbf{6}}} n_A n_{\bar{A}}$$

The equations (50) result from integrating out the open strings connecting the two stacks of $D3$ -branes which intersect only at a point, the origin in \mathbb{C}^4 .

...

For each pair $A', A'' \in \underline{\mathbf{6}}$, such that $A' \cap A'' = \{a\}$, and $i \geq 1$, define

$$(52) \quad \Upsilon_{A', A''; i} = J_{A'} B_a^{i-1} I_{A''}.$$

These equations result (conjecturally) from integrating out the $3 - 3$ strings connecting the neighbouring stacks $\mathbb{C}_{A'}^2$ and $\mathbb{C}_{A''}^2$, intersecting along a real two-dimensional plane \mathbb{C}_a^1 .

...

Finally, for each $A = \{a', a''\} \in \underline{\mathbf{6}}$, e.g. $a' < a''$, and $i, j \geq 1$, define

$$(53) \quad \Upsilon_{A; i, j} = J_A B_{a'}^{i-1} B_{a''}^{j-1} I_A$$

3.1.4. A very useful identity. Let us compute

$$\begin{aligned}
(54) \quad & \sum_{A \in \underline{\mathfrak{6}}} \text{Tr } s_A s_A^\dagger + \sum_{A \in \underline{\mathfrak{6}}, \bar{a} \in \bar{A}} \text{Tr } \sigma_{\bar{a}A} \sigma_{\bar{a}A}^\dagger + \sum_{A \in \underline{\mathfrak{6}}} \text{Tr } \Upsilon_A \Upsilon_A^\dagger \\
&= 2 \sum_{A \in \underline{\mathfrak{6}}} \text{Tr } \mu_A \mu_A^\dagger + \sum_{A \in \underline{\mathfrak{6}}} \varepsilon(A) \left(\text{Tr } \mu_A \mu_{\bar{A}} + \text{Tr } \mu_A^\dagger \mu_{\bar{A}}^\dagger \right) \\
&+ \sum_{A \in \underline{\mathfrak{6}}, a \in \bar{A}} \text{Tr} \left(B_a^\dagger B_a \Pi_A^I + B_a B_a^\dagger \Pi_A^J \right) + 2 \sum_{A \in \underline{\mathfrak{6}}} \text{Tr } \Pi_A^J \Pi_A^I \\
&- 2 \sum_{A \in \underline{\mathfrak{6}}, \bar{A} = \{\bar{a}, \bar{b}\}} \varepsilon(A) \text{Tr} \left([B_{\bar{a}}, B_{\bar{b}}] I_A J_A + c.c. \right) \\
&- \sum_{A \in \underline{\mathfrak{6}}} \varepsilon(A) \text{Tr} \left(I_A J_A I_{\bar{A}} J_{\bar{A}} + c.c. \right) \\
&= 2 \sum_{A \in \underline{\mathfrak{6}}} \left(\|\mu_A\|^2 + \|J_{\bar{A}} I_A\|^2 \right) + \sum_{A \in \underline{\mathfrak{6}}, a \in \bar{A}} \|B_a I_A\|^2 + \|J_A B_a\|^2
\end{aligned}$$

where

$$(55) \quad \Pi_A^J = J_A^\dagger J_A, \quad \Pi_A^I = I_A I_A^\dagger$$

3.2. Holomorphic equations

Using the identity (54) it is easy to show that the equations

$$(56) \quad \begin{aligned} s_A &= 0, & \Upsilon_A &= 0, & A &\in \underline{\mathfrak{6}} \\ \sigma_{\bar{a}A} &= 0, & \bar{a} &\in \bar{A}, \end{aligned}$$

which are not holomorphic in the variables $\mathbf{B}, \mathbf{I}, \mathbf{J}$, imply stronger holomorphic equations: for each $A \in \underline{\mathfrak{6}}$,

$$(57) \quad \begin{aligned} \mu_A &= 0, & J_{\bar{A}} I_A &= 0 \\ B_{\bar{a}} I_A &= 0, & J_A B_{\bar{a}} &= 0, & \bar{a} &\in \bar{A}, \end{aligned}$$

3.3. The moduli spaces $\mathfrak{M}_k^*(\vec{n})$

Define $\mathfrak{M}_k^i(\vec{n})$ to be the $U(k)$ -quotient of the space of solutions to (56) (which imply, by the above argument, (57)), the additional equations

$$(58) \quad \Upsilon_{A', A''; j} = 0, \quad 1 \leq j \leq i,$$

for all $A', A'' \in \underline{\mathbf{6}}$ with $\#A' \cap A'' = 1$, and the “moment map” equation

$$(59) \quad \mu = \zeta \cdot \mathbf{1}_K$$

The group $U(k)$ acts by:

$$(60) \quad (\mathbf{B}, \mathbf{I}, \mathbf{J}) \mapsto (g^{-1}\mathbf{B}g, g^{-1}\mathbf{I}, \mathbf{J}g), \quad g \in U(k)$$

It is clear that, as a set

$$(61) \quad \mathfrak{M}_k^\infty(\vec{n}) \subset \dots \subset \mathfrak{M}_k^i(\vec{n}) \subset \mathfrak{M}_k^{i-1}(\vec{n}) \subset \dots \subset \mathfrak{M}_k^1(\vec{n}) \subset \mathfrak{M}_k^0(\vec{n})$$

and that the sequence stabilizes at $i \geq k$ (use the fact that a $k \times k$ matrix obeys the degree k polynomial equation).

3.4. Stability

Imposing (59) with $\zeta > 0$ and dividing by $U(k)$ is equivalent to imposing the stability condition and dividing by the action (60) with $g \in GL(k) \equiv GL(k, \mathbb{C})$. Note that we deal with the equations (57) when talking about the $GL(k)$ symmetry. The stability condition reads:

<p>Any subspace $K' \subset K$, such that</p> <p style="color: red;">$I_A(N_A) \subset K'$, for all $A \in \underline{\mathbf{6}}$</p> <p>and</p> <p style="color: blue;">$B_a(K') \subset K'$, for all $a \in \underline{\mathbf{4}}$</p> <p>coincides with all of K, $K' = K$</p> <p>in other words, $\sum_{A \in \underline{\mathbf{6}}} \mathbb{C}[B_1, B_2, B_3, B_4] I_A(N_A) = K$</p>
--

(62)

The proof is standard. In one direction, let us prove (62) holds given that the $GL(k)$ -orbit of the tuple $(\mathbf{B}, \mathbf{I}, \mathbf{J})$ of matrices crosses the locus $\mu = \zeta \mathbf{1}_K$. Indeed, assume there is K' which is \mathbf{B} -invariant, and contains the image of I_A 's. Let K'' be the orthogonal complement $K'' = (K')^\perp$. Let P', P'' be the

orthogonal projections onto K' , K'' , respectively:

$$(63) \quad \begin{aligned} \mathbf{1}_K &= P' + P'', & P'P'' &= P''P' = 0, \\ (P')^2 &= (P')^\dagger = P', & (P'')^2 &= (P'')^\dagger = P'' \end{aligned}$$

Since the images of \mathbf{I} 's are in K' , we have:

$$(64) \quad P''I_A = 0, \quad A \in \underline{\mathbf{6}}$$

Since \mathbf{B} preserve K' , we have:

$$(65) \quad P''B_aP' = 0, \quad a \in \underline{\mathbf{4}}$$

Define

$$(66) \quad \begin{aligned} b_a &= P''B_aP'', & b_a^\dagger &= P''B_a^\dagger P'', & a &\in \underline{\mathbf{4}}, \\ j_A &= J_AP'', & P''J_A^\dagger &= j_A^\dagger, & A &\in \underline{\mathbf{6}} \end{aligned}$$

Thus:

$$(67) \quad \zeta P'' = P''\mu P'' = \sum_{a \in \underline{\mathbf{4}}} [b_a, b_a^\dagger] - \sum_{a \in \underline{\mathbf{4}}} P''B_a^\dagger P' B_a P'' - \sum_{A \in \underline{\mathbf{6}}} j_A^\dagger j_A$$

Now, taking the trace of both sides of (67) we arrive at the conclusion $K'' = 0$:

$$(68) \quad 0 \leq \zeta \dim K'' = - \sum_{a \in \underline{\mathbf{4}}} \|P'B_aP''\|^2 - \sum_{A \in \underline{\mathbf{6}}} \|j_A\|^2 \leq 0 \implies \dim K'' = 0$$

Conversely, assume (62) holds. Let

$$(69) \quad f = \frac{1}{2} \text{Tr} (\mu - \zeta \mathbf{1}_K)^2$$

Consider the gradient flow, generated by f with respect to the flat Kähler metric

$$(70) \quad ds^2 = \|d\mathbf{B}\|^2 + \|d\mathbf{I}\|^2 + \|d\mathbf{J}\|^2$$

The function f decreases along the gradient trajectory. Moreover, the trajectory belongs to the $GL(k)$ -orbit. Eventually, the trajectory stops at a critical

point of f . Either it is the absolute minimum, i.e. the solution to (59), or the higher critical point, where

$$(71) \quad \langle \xi, \nabla \mu \rangle = 0, \quad \xi = \mu - \zeta \mathbf{1}_K \neq 0$$

The one-parametric subgroup $(\exp t\xi)_{t \in \mathbb{C}} \subset GL(K)$, preserves $(\mathbf{B}, \mathbf{I}, \mathbf{J})$,

$$(72) \quad \begin{aligned} [B_a, \xi] &= 0, & a \in \underline{\mathbf{4}}, \\ \xi I_A &= 0, & J_A \xi = 0, & A \in \underline{\mathbf{6}} \end{aligned}$$

Define $K' = \ker \xi$. The Eq. (72) implies K' is \mathbf{B} -invariant, and contains the image of \mathbf{I} . Therefore, by (62), $K' = K$, $\xi \equiv 0$, i.e. (59) is satisfied.

Notation. We denote by $[\mathbf{B}, \mathbf{I}, \mathbf{J}]$ the $GL(k)$ -orbit

$$(g^{-1} B_a g, g^{-1} I_A, J_A g)_{a \in \underline{\mathbf{4}}, A \in \underline{\mathbf{6}}, g \in GL(k)}.$$

4. The symmetries of spiked instantons

The moduli spaces $\mathfrak{M}_k^*(\vec{n})$ are acted on by a group $\mathbf{H} = \mathbf{H}_{\vec{n}}$ of symmetries, defined below. The symmetry of $\mathfrak{M}_k^*(\vec{n})$ will be used in several ways. First, we shall be studying \mathbf{H} -equivariant integration theory of the spiked instanton moduli, in cohomology and equivariant K -theory. Second, we shall define new moduli spaces by studying the Γ -fixed loci $(\mathfrak{M}_k^*(\vec{n}))^\Gamma$ in $\mathfrak{M}_k^*(\vec{n})$, for subgroups $\Gamma \subset \mathbf{H}$. These moduli spaces have the commutant $C_\Gamma(\mathbf{H})$ as the symmetry group. Finally, the connected components $\mathfrak{M}_{\mathbf{k}}^{*,\gamma}(\vec{\mathbf{n}}) \subset (\mathfrak{M}_k^*(\vec{n}))^\Gamma$ can be defined using only the quiver of Γ , not the group Γ . The definition can be then generalized to define more general quiver spiked instantons. Their symmetry \mathbf{H}_γ generalizes the commutant $C_\Gamma(\mathbf{H})$.

4.1. Framing and spatial rotations

First of all, we can act by a collection $\underline{\mathbf{h}} = (h_A)_{A \in \underline{\mathbf{6}}}$ of unitary matrices $h_A \in U(n_A)$, defined up to an overall $U(1)$ multiple:

$$(73) \quad \underline{\mathbf{h}} \cdot [B_a, I_A, J_A] = [B_a, I_A h_A, h_A^{-1} J_A]$$

We call the symmetry (73) the framing rotation.

Secondly, we can multiply the matrices B_a by the phases $B_a \mapsto q_a B_a$, as long as their product is equal to 1:

$$(74) \quad \prod_{a \in \underline{4}} q_a = 1$$

and we supplement this transformation with the transformation $J_A \mapsto q_A J_A$:

$$(75) \quad \underline{\mathfrak{q}} \cdot [B_a, I_A, J_A] = [q_a B_a, I_A, q_A J_A]$$

We can view $\underline{\mathfrak{q}}$ as the diagonal matrix

$$(76) \quad \underline{\mathfrak{q}} = \text{diag}(q_1, q_2, q_3, q_4) \in U(1)_{\underline{\mathfrak{e}}}^3 \subset SU(4)$$

which belongs to the maximal torus $U(1)_{\underline{\mathfrak{e}}}^3$ of the group $SU(4)$ of rotations of \mathbb{C}^4 preserving some supersymmetry. We call (75) the spatial rotations.

The group

$$(77) \quad \mathbf{H} = \text{P} \left(\prod_{A \in \underline{\mathfrak{6}}} U(n_A) \right) \times U(1)_{\underline{\mathfrak{e}}}^3$$

is the symmetry of the moduli space of spiked instantons for generic ζ and \vec{n} . The complexification $\mathbf{H}_{\mathbb{C}}$ preserves the holomorphic equations (57) and the stability condition (62).

The center $Z_{\mathbf{H}}$ of \mathbf{H} is the eight dimensional torus

$$(78) \quad Z_{\mathbf{H}} = U(1)_{\underline{\mathfrak{x}}}^5 \times U(1)_{\underline{\mathfrak{e}}}^3$$

The maximal torus $T_{\mathbf{H}}$ of \mathbf{H} is the torus

$$(79) \quad T_{\mathbf{H}} = \left(\left(\times_{A \in \underline{\mathfrak{6}}} T_A \right) / U(1) \right) \times U(1)_{\underline{\mathfrak{e}}}^3$$

where

$$(80) \quad T_A \subset U(n_A)$$

is the group of diagonal $n_A \times n_A$ unitary matrices, the maximal torus of $U(n_A)$, $T_A \approx U(1)^{n_A}$. In the Eq. (79) we divide by the $U(1)$ embedded diagonally into the product of all $\sum_A n_A U(1)$'s.

4.1.1. Coulomb parameters. Let $(\underline{\mathfrak{a}}, \underline{\mathfrak{e}}) \in \text{Lie}(T_{\mathbb{H}}) \otimes \mathbb{C}$,

$$(81) \quad \begin{aligned} \underline{\mathfrak{e}} &= (\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4), & \mathfrak{e}_a &\in \mathbb{C}, & \sum_{a \in \underline{4}} \mathfrak{e}_a &= 0, \\ \underline{\mathfrak{a}} &= (\mathfrak{a}_A)_{A \in \underline{6}}, & \mathfrak{a}_A &= \text{diag}(\mathfrak{a}_{A,1}, \dots, \mathfrak{a}_{A,n_A}) \in \text{Lie}(T_A) \otimes \mathbb{C} \end{aligned}$$

The eigenvalues $\mathfrak{a}_{A,\alpha} \in \mathbb{C}$ are defined modulo the overall shift $\mathfrak{a}_{A,\alpha} \mapsto \mathfrak{a}_{A,\alpha} + \mathbf{x}$, $\mathbf{x} \in \mathbb{C}$.

The integrals (12) which we define below are meromorphic functions of $(\underline{\mathfrak{e}}, \underline{\mathfrak{a}})$.

4.1.2. Symmetry enhancements. Sometimes the symmetry of the spiked ADHM equations enhances. First of all, if all $\mathbf{I} = \mathbf{J} = 0$ (for $N_A = 0$, for all A), then the \mathfrak{q} -transformations can be generalized to the action of the full $\text{SU}(4) = \text{Spin}(6)$:

$$(82) \quad B_a \mapsto \sum_{c \in \underline{4}} g_{ac} B_c, \quad gg^\dagger = 1, \quad \det(g) = 1$$

In the case of less punitive restrictions on N_A 's, e.g. in the crossed instanton case, the symmetry enhances to $\text{SU}(2) \times U(1) \times \text{SU}(2)$, and, if $\zeta = 0$, to $\text{SU}(2)^3$. Let us assume, for definiteness, that only N_{12} and N_{34} are non-zero. Then the transformations:

$$(83) \quad \begin{aligned} &(B_1, B_2, B_3, B_4, I_{12}, J_{12}, I_{34}, J_{34}) \\ &\mapsto (uaB_1 + ubB_2, -u\bar{b}B_1 + u\bar{a}B_2, \bar{u}cB_3 + \bar{u}dB_4, \\ &\quad -\bar{u}d\bar{B}_3 + \bar{u}c\bar{B}_4, uI_{12}, uJ_{12}, \bar{u}I_{34}, \bar{u}J_{34}), \\ &\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \text{SU}(2)_{12}, \quad \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} \in \text{SU}(2)_{34}, \\ &\begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} \in U(1)_\Delta \subset \text{SU}(2)_\Delta \\ &a\bar{a} + b\bar{b} = c\bar{c} + d\bar{d} = u\bar{u} = 1 \end{aligned}$$

preserve the crossed instanton equations (56). When $\zeta = 0$ the $U(1)_\Delta$ symmetry enhances to the full $\text{SU}(2)_\Delta$, acting by:

$$\begin{aligned}
 & (B_1, B_2, B_3, B_4), (I_{12}, J_{12}, I_{34}, J_{34}) \\
 \mapsto & \left(uB_1 - vB_2^\dagger, vB_1^\dagger + uB_2, \bar{u}B_3 - \bar{v}B_4^\dagger, \bar{v}B_3^\dagger + \bar{u}B_4 \right), \\
 (84) \quad & \left(uI_{12} - vJ_{12}^\dagger, uJ_{12} + vI_{12}^\dagger, \bar{u}I_{34} - \bar{v}J_{34}^\dagger, \bar{u}J_{34} + \bar{v}I_{34}^\dagger \right), \\
 & \begin{pmatrix} u & \bar{v} \\ -v & \bar{u} \end{pmatrix} \in \text{SU}(2)_\Delta, \quad u\bar{u} + v\bar{v} = 1
 \end{aligned}$$

The equation $\Upsilon_{12} \equiv J_{34}I_{12} - I_{34}^\dagger J_{12}^\dagger$, the equations $s_{13} = -s_{24}^\dagger, s_{14} = s_{23}^\dagger$, the equations $\sigma_{3,12}, \sigma_{4,12}$ as well as the equations $\sigma_{1,34}, \sigma_{2,34}$ are $\text{SU}(2)_\Delta$ -invariant, while the equations $s_{12}, \mu, s_{34} = s_{12}^\dagger$ form a doublet.

4.2. Subtori

In what follows we shall encounter the arrangement of hyperplanes \mathfrak{H}_l in $\text{Lie}(T_{\mathbb{H}}) \otimes \mathbb{C}$ defined by the system of linear equations:

$$(85) \quad L_i(\underline{\mathbf{a}}, \underline{\boldsymbol{\epsilon}}) = \sum_{A \in \underline{\mathfrak{6}}} \sum_{\alpha \in [n_A]} \varpi_{i;A,\alpha} \mathbf{a}_{A,\alpha} + \sum_{a \in \underline{\mathfrak{4}}} n_{i;a} \boldsymbol{\epsilon}_a = 0$$

with $\varpi_{i;A,\alpha} \in \{-1, 0, +1\}$, $n_{i;a} \in \mathbb{Z}$ and the matrix $\varpi_{i;A,\alpha}$ of maximal rank. Such equations (85) can be interpreted as defining a subtorus $\mathfrak{H} = T_L \subset T_{\mathbb{H}}$: simply solve (85) for the subset of $\mathbf{a}_{A,\alpha}$'s for which the matrix $\varpi_{i;A,\alpha}$ is invertible. We shall not worry about the integrality of the inverse matrix in this paper, by using the covering tori, if necessary.

One of the reasons we need to look at the subtori T_L is the following construction.

4.3. Orbifolds, quivers, defects

In this section the global symmetry group \mathbb{H} is equal to

$$\mathbb{H} = P \left(\prod_{A \in \underline{\mathfrak{6}}} U(n_A) \right) \times \mathbb{G}_{\text{rot}}$$

where

1)

$$G_{\text{rot}} = U(1)_{\underline{\mathfrak{g}}}^3$$

if there are at least two $A' \neq A'' \in \underline{\mathfrak{6}}$ with non-empty intersection with $n_{A'} n_{A''} \neq 0$, and to

2)

$$G_{\text{rot}} = \text{SU}(2)_A \times U(1)_{\Delta} \times \text{SU}(2)_{\bar{A}}$$

otherwise, i.e. there is at most one pair A, \bar{A} with $n_A n_{\bar{A}} \neq 0$.

In all cases

$$G_{\text{rot}} \subset \text{SU}(4),$$

so that to every $\gamma \in \Gamma$ one associates a unitary 4×4 matrix $\underline{\mathfrak{q}} = \|q_a^b(\gamma)\|_{a,b \in \underline{\mathfrak{4}}}$ with unit determinant. In the first case this matrix is diagonal, in the second case it is a 2×2 block-diagonal matrix with unitary 2×2 blocks of inverse determinants.

The symmetry of $\mathfrak{M}_k^*(\vec{n})$ can be used to define new moduli spaces. Suppose $\Gamma \subset \mathfrak{H}$ is a discrete subgroup. Let $\mathfrak{H}^{\Gamma} \subset \mathfrak{H}$ be the maximal subgroup commuting with Γ , the centralizer of Γ . Let Γ^{\vee} be the set of irreducible unitary representations $(\mathcal{R}_{\omega})_{\omega \in \Gamma^{\vee}}$ of Γ , and $\vec{k} \in \mathbb{Z}_{\geq 0}^{\Gamma^{\vee}}$. The representations $N_A, A \in \underline{\mathfrak{6}}$ of \mathfrak{H} decompose as representations of Γ

$$(86) \quad N_A = \bigoplus_{\omega \in \Gamma^{\vee}} N_{A,\omega} \otimes \mathcal{R}_{\omega}$$

Let \vec{n} now denote the collection $(n_{A,\omega})_{A \in \underline{\mathfrak{6}}, \omega \in \Gamma^{\vee}}$ of dimensions

$$(87) \quad n_{A,\omega} = \dim N_{A,\omega}$$

of multiplicity spaces. The vector $\mathbf{k} = (k_{\omega})_{\omega \in \Gamma^{\vee}}$ defines a representation of Γ :

$$(88) \quad \gamma \in \Gamma \mapsto g_{\gamma} \in U(K), \quad K = \bigoplus_{\omega \in \Gamma^{\vee}} K_{\omega} \otimes \mathcal{R}_{\omega}, \quad k_{\omega} = \dim K_{\omega}$$

We call the components k_{ω} of the vector \mathbf{k} fractional instanton charges. The moduli space of Γ -folded spiked instantons of charge \vec{k} is the component $\mathfrak{M}_{\mathbf{k}}^{*,\Gamma}(\vec{n})$ set of Γ -fixed points $(\mathfrak{M}_k^*(\vec{n}))^{\Gamma}$. The representation (88) enters the

realization of the Γ -fixed locus in the space of matrices $(\mathbf{B}, \mathbf{I}, \mathbf{J})$:

$$(89) \quad \gamma \cdot (\mathbf{B}, \mathbf{I}, \mathbf{J}) = (g_\gamma \mathbf{B} g_\gamma^{-1}, g_\gamma \mathbf{I}, \mathbf{J} g_\gamma^{-1}) , \quad g_\gamma \in U(K)$$

where

$$(90) \quad \begin{aligned} \gamma \cdot (\mathbf{B}, \mathbf{I}, \mathbf{J}) &\equiv \left(q_a^b(\gamma) B_b, I_A h_A(\gamma)^{-1}, q_A(\gamma) h_A(\gamma) J_A \right) , \\ \gamma \in \Gamma &\mapsto (h_A(\gamma))_{A \in \mathfrak{G}} \times \|q_a^b(\gamma)\|_{a,b \in \mathfrak{A}} \in \mathbf{H} \end{aligned}$$

is the defining representation of Γ , with $q_A(\gamma)$ given by (9) in the case (1), and by the projection to $U(1)_\Delta$ in the case (2). The equations (89) are invariant under the subgroup

$$(91) \quad \prod_{\omega \in \Gamma^\vee} U(K_\omega) \subset U(K)$$

of unitary transformations of K commuting with Γ . The holomorphic equations (57) restricted onto the locus of Γ -equivariant i.e. obeying (89) matrices $\mathbf{B}, \mathbf{I}, \mathbf{J}$ become the holomorphic equations defining $\mathfrak{M}_k^\Gamma(\underline{\mathbf{n}})$. The stability condition (62) can be further refined, analogously to the refinement of the real moment map equation $\mu = \sum_{\omega \in \Gamma^\vee} \zeta_\omega \mathbf{1}_{K_\omega} \otimes \mathbf{1}_{\mathcal{R}_\omega}$. We shall work in the chamber where all $\zeta_\omega > 0$.

The moduli spaces $\mathfrak{M}_k^\Gamma(\underline{\mathbf{n}})$ in the case (1) parametrize the spiked instantons in the presence of $U(1)_\xi^3$ -invariant surface operators, while in the case (2) they parametrize the instantons in supersymmetric quiver gauge theories on the ALE spaces, with additional defect.

The commutant \mathbf{H}^Γ acts on $\mathfrak{M}_k^\Gamma(\underline{\mathbf{n}})$, so that the partition functions we study are meromorphic functions on $\text{Lie}(\mathbf{H}^\Gamma) \otimes \mathbb{C}$.

Note that if Γ has trivial projection to \mathbf{G}_{rot} then the moduli space of Γ -folded instantons is simply the product of the moduli spaces of spiked instantons for N_ω 's. In what follows we assume the projection to \mathbf{G}_{rot} to be non-trivial.

4.3.1. Subtori for Γ -folds. Let us now describe the maximal torus $T_{\mathbf{H}^\Gamma}$ of the Γ -commutant as T_L . In other words, the choice of a discrete subgroup $\Gamma \subset \mathbf{H}$ defines the hyperplanes $L_i(\mathbf{a}, \underline{\mathbf{e}}) = 0$ in $\text{Lie}(T_{\mathbf{H}})$.

In the case (1) the \mathbf{G}_{rot} -part of Γ is abelian, i.e. it is a product of cyclic groups (if Γ is finite) or it is a torus itself. In either case there is no restriction on the $\underline{\mathbf{e}}$ -parameters. The framing part of Γ reduces $P(\times_A U(n_A))$ to $P(\times_{A,\omega} U(n_{A,\omega}))$ which means that some of the eigenvalues $\mathbf{a}_{A,\alpha}$, viewed as

the generators of $\text{Lie}U(n_A)$, must coincide, more precisely to be of multiplicity $\dim \mathcal{R}_\omega$. The minimal case, when Γ is abelian, imposes no restrictions on $(\mathfrak{a}, \mathfrak{e})$, so that $T_{\text{Hr}} = T_{\text{H}}$.

In the case (2) the \mathbf{G}_{rot} -part of Γ need not be abelian. Let us assume, for definiteness, that $A = 12, \bar{A} = 34$. If the image of Γ in $SU(2)_{12}$ is non-abelian, then $\mathfrak{e}_1 = \mathfrak{e}_2$. Likewise if the image of Γ in $SU(2)_{34}$ is non-abelian then $\mathfrak{e}_3 = \mathfrak{e}_4$. The non-abelian discrete subgroups of $SU(2)$ have irreducible representations of dimensions 2 and higher, up to 6. Thus the corresponding $\alpha_{A,\alpha}$ eigenvalues will have the multiplicity up to 6.

4.3.2. Subtori for sewing. Let us specify the integral data for the subtori, i.e. the explicit solutions to the constraints (85). Let $\mathfrak{e} = (e_a)_{a \in \mathbf{4}}, e_a \in \mathbb{Z}_{\neq 0}$, be a 4-tuple of non-zero integers, with no common divisors except for ± 1 , which sum up to zero:

$$(92) \quad \sum_{a \in \mathbf{4}} e_a = 0 .$$

Such a collection \mathfrak{e} defines a split $\mathbf{6} = \mathbf{6}^+ \amalg \mathbf{6}^-$, where $\mathbf{6}^\pm$ being the set of $A = \{a, b\}$ such that $\pm e_a e_b > 0$.

For $A = \{a, b\} \in \mathbf{6}^-$, i.e. $e_a e_b < 0$, let $p_A = \text{gcd}(|e_a|, |e_b|) > 0$. Let us also fix for such $A \in \mathbf{6}^-$ a partition $\nu_A = (\nu_{A,\iota})$ of size n_A , whose parts do not exceed e_A : $1 \leq \nu_{A,\iota} \leq p_A$. Let $\ell_A = \ell(\nu_A)$ be its length.

Given ν_A we partition the set $[n_A]$ as the union of nonintersecting subsets

$$(93) \quad [n_A] = \bigcup_{\iota} [n_A]_{\iota} , \quad \#[n_A]_{\iota} = \nu_{A,\iota} ,$$

$[n_A]_{\iota'} \cap [n_A]_{\iota''} = \emptyset$ for $\iota' \neq \iota''$. Fix a map $c_{A,\iota} : [n_A]_{\iota} \rightarrow \mathbb{Z}$ obeying, for any $a', a'' \in [n_A]_{\iota}, a' \neq a''$:

$$(94) \quad c_{A,\iota}(a') - c_{A,\iota}(a'') \neq 0 \pmod{p_A}$$

When $p_A = 1$ the condition (94) is empty.

For $A = \{a, b\} \in \mathbf{6}^+$, let us fix a map $c_A : [n_A] \rightarrow \mathbb{Z}$, obeying, for any $a', a'' \in [n_A]$,

$$(95) \quad c_A(a') - c_A(a'') \notin \mathbb{Z}_{>0} e_a + \mathbb{Z}_{>0} e_b$$

Note that (95) does not forbid the situation where $c_A(a') = c_A(a'')$ for some $a', a'' \in [n_A]$. To make the notation uniform we assign to such $A, \nu_A = (n_A), \ell_A = 1$.

The final piece of data is the choice $\iota_A \in [\ell_A]$ for each $A \in \underline{\mathfrak{6}}^-$. Define the set $\lambda_A = [\ell_A] \setminus \{\iota_A\}$, of cardinality $\ell_A - 1$.

Now, we associate to the data

$$(96) \quad \mathfrak{L} = (\nu, \lambda, c, \underline{\mathfrak{e}}) ,$$

the torus

$$(97) \quad \mathfrak{H} = T_{\mathfrak{L}} = \left(\prod_{A \in \underline{\mathfrak{6}}} U(1)^{\ell_A - 1} \right) \times U(1)$$

Note that only $A \in \underline{\mathfrak{6}}^-$ contribute to the product in (97). This torus is embedded into $T_{\mathfrak{H}}$ as follows: the element

$$(98) \quad (e^{i\xi}, e^{it}) \equiv \prod_{A \in \underline{\mathfrak{6}}^-} \left(e^{i\xi_{A,i}} \right)_{i \in \lambda_A} \times e^{it} \in T_{\mathfrak{L}}$$

is mapped to

$$(99) \quad \left[\prod_{A \in \underline{\mathfrak{6}}^-} \text{diag}_A^- \times \prod_{A \in \underline{\mathfrak{6}}^+} \text{diag}_A^+ \right] \times \text{diag} (e^{ie_a t})_{a \in \underline{\mathfrak{4}}} \\ \in \mathbb{P} \left(\prod_{A \in \underline{\mathfrak{6}}} U(n_A) \right) \times \text{SU}(4)$$

where $\text{diag}_A^{\pm} \in U(n_A)$ are the diagonal matrices with the eigenvalues

$$(100) \quad \text{Eigen} (\text{diag}_A^-) = \left\{ e^{ic_{A,\iota_A}(\alpha)t} \mid \alpha \in [n_A]_{\iota_A} \right\} \\ \cup \left\{ e^{i(\xi_{A,\iota} + c_{A,i}(\alpha)t)} \mid \iota \in \lambda_A, \alpha \in [n_A]_{\iota} \right\}$$

$$(101) \quad \text{Eigen} (\text{diag}_A^+) = \left\{ e^{ic_A(\alpha)t} \mid \alpha \in [n_A] \right\}$$

Thus, the torus $T_{\mathfrak{L}}$ corresponds to the solution of the Eqs. (85) with

$$(102) \quad \mathfrak{e}_a = e_a u, \quad a \in \underline{\mathfrak{4}}, \\ \mathfrak{a}_{A,\alpha} = c_A(\alpha)u, \quad A \in \underline{\mathfrak{6}}^+, \alpha \in [n_A] \\ \mathfrak{a}_{A,\alpha} = c_{A,\iota_A}(\alpha)u, \quad A \in \underline{\mathfrak{6}}^-, \alpha \in [n_A]_{\iota_A} \\ \mathfrak{a}_{A,\alpha} = \xi_{A,\iota} + c_{A,\iota}(\alpha)u, \quad A \in \underline{\mathfrak{6}}^-, \alpha \in [n_A]_{\iota}, \iota \in \lambda_A$$

In other words, the Ω -background parameters are maximally rationally dependent (the worst way to insult the rotational parameters), the framing of the spaces N_A , $A \in \underline{\mathfrak{6}}^+$ is completely locked with space rotations (spin-color locking), while the framing of the spaces N_A , $A \in \underline{\mathfrak{6}}^-$ is locked partially.

4.4. Our goal: compactness theorem

Our goal is to establish the compactness of the fixed point sets $\mathfrak{M}_k(\vec{n})^{T\mathfrak{e}}$ and $(\mathfrak{M}_k^\Gamma(\vec{n}))^{T\text{hr}}$. Before we attack this problem we shall discuss a little bit the ordinary instantons, then look at a few examples of the particular types of spiked instantons: the crossed and the folded instantons, and then proceed with the analysis of the general case. The reader interested only in the compactness theorem can skip the next two sections at the first reading.

5. Ordinary instantons

In this section we discuss the relations between the ordinary four dimensional $U(n)$ instantons and the spiked instantons.

5.1. ADHM construction and its fine print

In the simplest case only one of six vector spaces is non-zero, e.g.

$$(103) \quad N_A = 0, \quad A \neq \{1, 2\}.$$

Let $n = n_{12}$. We shall now show that, set theoretically, the moduli space of spiked instantons in this case is $\mathcal{M}_k(n)$, the ADHM moduli space (more precisely, its Gieseker-Nakajima generalization).

Recall the ADHM construction of the $U(n)$ framed instantons of charge k on (noncommutative) \mathbb{R}^4 [3, 24, 27]. It starts by fixing Hermitian vector spaces N and K of dimensions n and k , respectively. Consider the space of quadruples (B_1, B_2, I, J) ,

$$(104) \quad I : N \rightarrow K, \quad J : K \rightarrow N, \quad B_\alpha : K \rightarrow K, \quad \alpha = 1, 2$$

obeying

$$(105) \quad \vec{\mu}_{12} \equiv \left(2\mu_{12}^{\mathbb{R}}, \mu_{12}^{\mathbb{C}} + \mu_{12}^{\mathbb{C}\dagger}, i \left(\mu_{12}^{\mathbb{C}} - \mu_{12}^{\mathbb{C}\dagger} \right) \right) = (\zeta, 0, 0) \cdot \mathbf{1}_K,$$

where

$$(106) \quad \mu_{12}^{\mathbb{C}} = [B_1, B_2] + IJ, \quad 2\mu_{12}^{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J$$

Note that the number of equations (105) plus the number of symmetries is less than the number of variables. The moduli space $\mathcal{M}_k(n)$ of solutions to (105) modulo the $U(k)$ action

$$(107) \quad (B_1, B_2, I, J) \mapsto (g^{-1}B_1g, g^{-1}B_2g, g^{-1}I, Jg), \quad g \in U(k)$$

has the positive dimension

$$(108) \quad \dim_{\mathbb{R}}\mathcal{M}_k(n) = 4k(n + k) - 3k^2 - k^2 = 4nk$$

Again, the $\mu_{12}^{\mathbb{R}}$ -equation, with $\zeta > 0$, can be replaced by the stability condition, and the $GL(k)$ -symmetry:

Any subspace $K' \subset K$, such that $I(N) \subset K'$,
 and $B_\alpha(K') \subset K'$, for all $\alpha = 1, 2$
 coincides with all of K , $K' = K$
 in other words, $\mathbb{C}[B_1, B_2]I(N) = K$

(109)

Notation. We denote by $[B_1, B_2, I, J]$ the $GL(k)$ -orbit

$$(g^{-1}B_1g, g^{-1}B_2g, g^{-1}I, Jg)_{g \in GL(k)}.$$

5.2. Ordinary instantons from spiked instantons

Now, to show that the spiked instantons reduce to the ordinary instantons when (103) is obeyed, we need to show that $B_3 = B_4 = 0$ on the solutions of our equations (57). This is easy:

$$(110) \quad B_3f(B_1, B_2)I = f(B_1, B_2)B_3I = 0$$

where we used $[B_1, B_3] = \mu_{13} = 0$, $[B_2, B_3] = \mu_{23} = 0$, and $B_3I = 0$. Therefore B_3 acts by zero on all of K . The same argument proves the vanishing of B_4 .

5.3. One-instanton example

Let $k = 1$. We can solve the equations (106) explicitly. The matrices B_1, B_2 are just complex numbers, e.g. $b_1, b_2 \in \mathbb{C}$. The pair I, J obeys $IJ = 0$, $\|I\|^2 - \|J\|^2 = \zeta$. Assuming $\zeta > 0$ define the vectors $w_1 = J(K) \in N$, $w_2 = \frac{1}{\sqrt{\zeta + \|J\|^2}} I^\dagger(K) \in N$. They obey $\langle w_2, w_1 \rangle = 0$, $\langle w_2, w_2 \rangle = 1$. Dividing by the $U(1) = U(K)$ symmetry we arrive at the conclusion:

$$(111) \quad \mathcal{M}_1(n) = \mathbb{C}^2 \times T^*\mathbb{C}\mathbb{P}^{n-1}$$

The first factor parametrizes (b_1, b_2) , the base $\mathbb{C}\mathbb{P}^{n-1}$ of the second factor is the space of w_2 's obeying $\|w_2\|^2 = 1$ modulo $U(1)$ symmetry.

5.4. U versus PU

In describing the action of H_{12} in (73) specified to the case of ordinary instantons we use an element h of the group $U(n)$ yet it is the group $PU(n) = U(n)/U(1) = SU(n)/\mathbb{Z}_n$ which acts faithfully on $\mathcal{M}_k(n)$. Indeed, multiplying h by a scalar matrix

$$(112) \quad h \rightarrow h\tilde{u}, \quad \tilde{u} \in U(1)$$

does not change the effect of the transformation (73) since it can be undone by the $U(k)$ -transformation (107) with $g = \tilde{u}^{-1}\mathbf{1}_K \in U(k)$.

5.5. Tangent space

Let $m \in \mathcal{M}_k(n)$. Let (B_1, B_2, I, J) be the representative of $m = [B_1, B_2, I, J]$. Consider the nearby quadruple

$$(113) \quad (B_1 + \delta B_1, B_2 + \delta B_2, I + \delta I, J + \delta J)$$

Assuming it solves the ADHM equations to the linear order, the variations $\delta B_1, \delta B_2, \delta I, \delta J$ are subject to the linearized $\mu_{12}^{\mathbb{C}}$ equation:

$$(114) \quad d_2(\delta B_1, \delta B_2, \delta I, \delta J) := [B_1, \delta B_2] + [\delta B_1, B_1] + (\delta I)J + I(\delta J) = 0$$

and we identify the variations which differ by an infinitesimal $GL(K)$ -transformation of (B_1, B_2, I, J) :

$$(115) \quad (\delta B_1, \delta B_2, \delta I, \delta J) \sim (\delta B_1, \delta B_2, \delta I, \delta J) + d_1(\delta\sigma),$$

$$d_1(\delta\sigma) := ([B_1, \delta\sigma], [B_2, \delta\sigma], -\delta\sigma \cdot I, J \cdot \delta\sigma)$$

Since $d_2 \circ d_1 = 0$, the tangent space is the degree 1 cohomology, $T_m\mathcal{M}_k(n) = \ker d_2 / \text{im} d_1 = H^1\mathcal{T}_m\mathcal{M}_k(n)$, of the complex

$$(116) \quad \mathcal{T}_m\mathcal{M}_k(n) = [0 \rightarrow \text{End}(K) \xrightarrow{d_1} \text{End}(K) \otimes \mathbb{C}_{12}^2 \oplus \text{Hom}(N, K) \oplus \text{Hom}(K, N) \xrightarrow{d_2} \text{End}(K) \otimes \wedge^2 \mathbb{C}_{12}^2 \rightarrow 0]$$

5.6. Fixed locus

In applications we will be interested in the fixed point set $\mathcal{M}_k(n)^{\mathbf{T}_{12}}$ with $\mathbf{T}_{12} \subset \mathbf{H}_{12}$ a commutative subgroup. The maximal torus $\mathbf{T}_{12} \subset \mathbf{H}_{12}$ is the product of the maximal torus $\mathbf{T}_n \subset PU(n)$ and the two dimensional torus $U(1)_{12} \times U(1)'_{12} \subset SU(2)_{12} \times U(1)'_{12}$. Let

$$(117) \quad \mathfrak{a} = i \text{diag}(\mathfrak{a}_1, \dots, \mathfrak{a}_n), \quad \mathfrak{a}_\alpha \in \mathbb{R}$$

be the generic element of $\text{Lie}(\mathbf{T}_n)$. It means that the numbers \mathfrak{a}_α are defined up to the simultaneous shift

$$(118) \quad \mathfrak{a}_\alpha \sim \mathfrak{a}_\alpha + a, \quad a \in \mathbb{R}$$

and we assume $\mathfrak{a}_\alpha \neq \mathfrak{a}_\beta$, for $\alpha \neq \beta$. Let

$$(119) \quad \mathfrak{e}_{12} = \frac{1}{2}(\mathfrak{e}_1 - \mathfrak{e}_2, \mathfrak{e}_1 + \mathfrak{e}_2)$$

be the generic element of $\text{Lie}(U(1)_{12} \times U(1)'_{12})$. The pair $(\mathfrak{a}, \mathfrak{e}_{12})$ generates an infinitesimal transformation (73), (75) of the quadruple (B_1, B_2, I, J) :

$$(120) \quad \delta_{\mathfrak{a}, \mathfrak{e}_{12}}(B_1, B_2, I, J) = (\mathfrak{e}_1 B_1, \mathfrak{e}_2 B_2, I\mathfrak{a}, (\mathfrak{e}_1 + \mathfrak{e}_2 - \mathfrak{a})J)$$

For the $U(k)$ -equivalence class $f = [B_1, B_2, I, J]$ to be fixed under the infinitesimal transformation generated by the generic pair $(\mathfrak{a}, \mathfrak{e})$ there must

exist an infinitesimal $U(k)$ -transformation (107)

$$(121) \quad \delta_\sigma(B_1, B_2, I, J) = ([B_1, \sigma], [B_2, \sigma], -\sigma I, J\sigma)$$

undoing it: $\delta_\sigma(B_1, B_2, I, J) + \delta_{\mathbf{a}, \mathbf{e}_{12}}(B_1, B_2, I, J) = 0$. In other words, there must exist the operators $\widehat{d}_\alpha, \widehat{d}_1, \widehat{d}_2 \in \text{End}(K)$, such that:

$$(122) \quad \begin{aligned} \sigma &= \sum_{\alpha \in [n]} \mathbf{a}_\alpha \widehat{d}_\alpha + \mathbf{e}_1 \widehat{d}_1 + \mathbf{e}_2 \widehat{d}_2, \\ \sum_{\alpha \in [n]} \widehat{d}_\alpha &= \mathbf{1}_K \end{aligned}$$

obeys:

$$(123) \quad \begin{aligned} \mathbf{e}_a B_a &= [\sigma, B_a], \quad a \in \{1, 2\} \\ \mathbf{I} \mathbf{a} &= \sigma I, \quad -(\mathbf{e}_1 + \mathbf{e}_2) J + \mathbf{a} J = J\sigma, \end{aligned}$$

or, in the group form:

$$(124) \quad q_a B_a = g_t B_a g_t^{-1}, \quad q_1 q_2 h_t^{-1} J = J g_t^{-1}, \quad I h_t = g_t I$$

where $q_a = e^{t\mathbf{e}_a}, g_t = e^{t\sigma}, h_t = e^{t\mathbf{a}}$. Here t is an arbitrary complex number, the map $t \mapsto g_t$ defines the representation $\mathbf{T}_{12} \rightarrow GL(k)$. The Eqs. (124) imply:

$$(125) \quad g_t \left(B_1^{i-1} B_2^{j-1} I \right) = q_1^{i-1} q_2^{j-1} \left(B_1^{i-1} B_2^{j-1} I \right) h_t$$

The Eqs. (123) for generic (\mathbf{a}, \mathbf{e}) imply:

$$(126) \quad K = \bigoplus_{\alpha \in [n]} K_\alpha, \quad \widehat{d}_\alpha|_{K_\beta} = \delta_{\alpha, \beta}$$

The eigenspace K_α is generated by $I_\alpha = I(N_\alpha)$, where $N_\alpha \subset N$ is the eigenline of \mathbf{a} with the eigenvalue \mathbf{a}_α :

$$(127) \quad K_\alpha = \sum_{i, j \geq 1} B_1^{i-1} B_2^{j-1} I_\alpha.$$

The subspace $I_\alpha \subset K$ (it is one-dimensional for generic \mathbf{a}) obeys:

$$(128) \quad \widehat{d}_\beta I_\alpha = \delta_{\alpha, \beta}, \quad \widehat{d}_1 I_\alpha = \widehat{d}_2 I_\alpha = 0$$

On K_α the operators $\widehat{d}_1, \widehat{d}_2$ have non-negative spectrum:

$$(129) \quad \sigma \left(B_1^{i-1} B_2^{j-1} I_\alpha \right) = (\mathbf{a}_\alpha + \mathbf{e}_1(i-1) + \mathbf{e}_2(j-1)) \left(B_1^{i-1} B_2^{j-1} I_\alpha \right)$$

Therefore, as long as $\mathbf{a}_\beta - \mathbf{a}_\alpha \notin \mathbf{e}_1 \mathbb{Z}_{>0} + \mathbf{e}_2 \mathbb{Z}_{>0}$,

$$(130) \quad J \left(B_1^{i-1} B_2^{j-1} I_\alpha \right) = 0,$$

as follows from the last Eq. in (123).

Thus, we have shown that

$$(131) \quad J = 0, \quad [B_1, B_2] = 0$$

Define an ideal $\mathcal{I}^{(\alpha)} \subset \mathbb{C}[x, y]$ in the ring of polynomials in two variables by:

$$(132) \quad P(x, y) \in \mathcal{I}^{(\alpha)} \Leftrightarrow P(B_1, B_2) I_\alpha = 0$$

Define the partition $\lambda^{(\alpha)} = \left(\lambda_1^{(\alpha)} \geq \lambda_2^{(\alpha)} \geq \dots \geq \lambda_{\ell_{\lambda^{(\alpha)}}}^{(\alpha)} \right)$ by

$$(133) \quad \lambda_i^{(\alpha)} = \min \{ j \mid B_1^{i-1} B_2^j I_\alpha = 0 \}$$

Thus, $K_\alpha = \mathbb{C}[z_1, z_2] / \mathcal{I}_{\lambda^{(\alpha)}}$. Here we denote by $\mathcal{I}_\lambda \subset \mathbb{C}[x, y]$ the ideal generated by the monomials $x^{i-1} y^{\lambda_i}$, $i = 1, 2, \dots, \ell_\lambda$.

Conversely, given the monomial ideal $\mathcal{I}_{\lambda^{(\alpha)}}$ define the vector $I_\alpha \subset K_\alpha$ to be the image of the polynomial 1 in the quotient $\mathbb{C}[x, y] / \mathcal{I}_{\lambda^{(\alpha)}}$. The operators B_1, B_2 act by multiplication by the coordinates x, y , respectively. Furthermore,

$$(134) \quad K_\alpha = \bigoplus_{i=1}^{\ell_\lambda} \bigoplus_{j=1}^{\lambda_i} K_{\alpha; i, j}$$

where

$$(135) \quad \widehat{d}_1|_{K_{\alpha; i, j}} = i - 1, \quad \widehat{d}_2|_{K_{\alpha; i, j}} = j - 1$$

The map $(\mathbf{a}, \mathbf{e}_{12}) \mapsto \sigma \in \text{End}(K)$ makes the space K a \mathbf{T}_{12} -representation. Its character is easy to compute:

$$(136) \quad K_\chi := \text{Tr}_K g t = \sum_{\alpha \in [n]} e^{t \mathbf{a}_\alpha} \sum_{i=1}^{\ell_{\lambda^{(\alpha)}}} q_1^{i-1} \sum_{j=1}^{\lambda_i^{(\alpha)}} q_2^{j-1}$$

The set of eigenvalues of σ is a union of n collections of centers of boxes of Young diagrams $\lambda^{(1)}, \dots, \lambda^{(n)}$

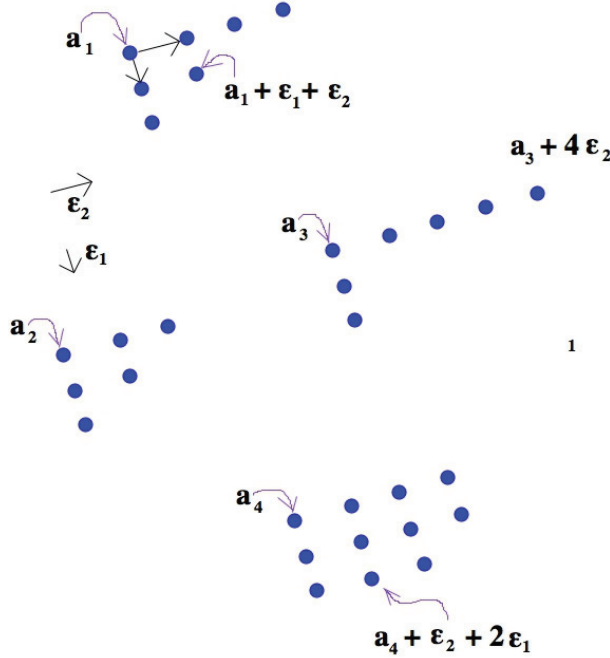


Figure 6: Eigenvalues of σ .

The space N is a \mathbf{T}_{12} -representation by definition:

$$(137) \quad N_\chi := \text{Tr}_N h_t = \sum_{\alpha \in [n]} e^{t\alpha}$$

We also define:

$$(138) \quad K_\chi^* = \text{Tr}_K g_t^{-1}, \quad N_\chi^* = \text{Tr}_N h_t^{-1}$$

5.7. Tangent space at the fixed point

Finally, the tangent space $T_f \mathcal{M}_k(n)$ to the moduli space at the fixed point f is also a \mathbf{T}_{12} -representation. Let us compute its character. Let $f = [B_1, B_2, I, J]$. The quadruple (B_1, B_2, I, J) is fixed by the composition of the \mathbf{T}_{12} transformation $(e^{t\alpha}, e^{t\epsilon_{12}}) \in \mathbf{T}_{12}^{\mathbb{C}}$ and the $GL(k)$ -transformation $e^{t\sigma} \in GL(k)$, for

any complex number t , cf. (123). Now take the nearby quadruple

$$\left(\tilde{B}_1 = B_1 + \delta B_1, \tilde{B}_2 = B_2 + \delta B_2, \tilde{I} = I + \delta I, \tilde{J} = J + \delta J\right)$$

and act on it by the combination of the $\mathbf{T}_{12}^{\mathbb{C}}$ transformation $(e^{t\mathbf{a}}, e^{t\mathbf{e}_{12}})$:

$$\left(\tilde{B}_1, \tilde{B}_2, \tilde{I}, \tilde{J}\right) \mapsto \left(q_1 \tilde{B}_1, q_2 \tilde{B}_2, \tilde{I}, q_1 q_2 \tilde{J}\right)$$

and the $GL(k)$ -transformation g_t :

$$\left(\tilde{B}_1, \tilde{B}_2, \tilde{I}, \tilde{J}\right) \mapsto \left(g_t^{-1} \tilde{B}_1 g_t, g_t^{-1} \tilde{B}_2 g_t, g_t^{-1} \tilde{I}, \tilde{J} g_t\right),$$

defining the \mathbf{T}_{12} -action on $(\delta B_1, \delta B_2, \delta I, \delta J)$:

$$(139) \quad \begin{aligned} & e^t \cdot [\delta B_1, \delta B_2, \delta I, \delta J] \\ &= [q_1 g_t^{-1} \delta B_1 g_t, q_2 g_t^{-1} \delta B_2 g_t, g_t^{-1} \delta I h_t, q_1 q_2 h_t^{-1} \delta J g_t]. \end{aligned}$$

So the space $\mathcal{T}_f^1 \mathcal{M}_k(n)$ of variations $(\delta B_1, \delta B_2, \delta I, \delta J)$ is a \mathbf{T}_{12} representation, with the character:

$$(140) \quad \text{Tr}_{\mathcal{T}_f^1 \mathcal{M}_k(n)}(h, q) = (q_1 + q_2) K_\chi K_\chi^* + N_\chi K_\chi^* + N_\chi^* K_\chi q_1 q_2$$

The first two terms on the right hand side account for $\delta B_1, \delta B_2$, the third term corresponds to the δI variations, and the last term accounts for the δJ variations.

Now, the tangent space $T_f \mathcal{M}_k(n)$ is the degree 1 cohomology $H^1 \mathcal{T}_f \mathcal{M}_k(n)$ of the complex (116), which has no H^0 or H^2 cohomology (for $\zeta > 0$). The character of $T_f \mathcal{M}_k(n)$ can be therefore computed by taking the alternating sum of the characters of $\mathcal{T}_f^0 \mathcal{M}_k(n)$, $\mathcal{T}_f^1 \mathcal{M}_k(n)$ and $\mathcal{T}_f^2 \mathcal{M}_k(n)$, giving:

$$(141) \quad \text{Tr}_{T_f \mathcal{M}_k(n)}(h, q) = N_\chi K_\chi^* + q_{12} N_\chi^* K_\chi - p_{12} K_\chi K_\chi^*$$

Thus,

$$(142) \quad \text{Tr}_{T_f \mathcal{M}_k(n)}(h, q) = \sum_{\alpha, \beta \in [n]} e^{t(\mathbf{a}_\alpha - \mathbf{a}_\beta)} T_\chi(\lambda^{(\alpha)}, \lambda^{(\beta)})$$

where

$$(143) \quad T_\chi(\mu, \lambda) = \sum_{(i,j) \in \lambda} q_1^{i-\lambda_j^t} q_2^{\mu_i+1-j} + \sum_{(i,j) \in \mu} q_1^{\mu_j^t+1-i} q_2^{j-\lambda_i}$$

We see that, as long as there is no rational relation between \mathbf{e}_1 and \mathbf{e}_2 , and $\mathbf{a}_\alpha - \mathbf{a}_\beta \notin \mathbf{e}_1\mathbb{Z} + \mathbf{e}_2\mathbb{Z}$ the weights which appear in the character of the tangent space are non-zero. In other words, the tangent space $T_f\mathcal{M}_k(n)$ does not contain trivial representations of \mathbf{T}_{12} , i.e. f is an isolated fixed point.

5.8. Smaller tori

Let (Γ^\vee, d) be a pair consisting of a finite or a countable set Γ^\vee (the meaning of the notation will become clear later), and a function $d : \Gamma^\vee \rightarrow \mathbb{N}$, which we shall call the dimension. We assign to each $\omega \in \Gamma^\vee$ a vector space

$$(144) \quad \mathcal{R}_\omega = \mathbb{C}^{d(\omega)}$$

of the corresponding dimension.

Let \underline{n} be a d -partition of n ,

$$(145) \quad n = \sum_{\omega \in \Gamma^\vee} n_\omega d(\omega), \quad n_\omega \geq 0$$

with only a finite number of $n_\omega > 0$. Let

$$(146) \quad \ell_{\underline{n}} = \#\{\omega \mid n_\omega > 0\}$$

We associate to \underline{n} a decomposition of N into the direct sum of tensor products:

$$(147) \quad N = \bigoplus_{\omega \in \Gamma^\vee} N_\omega \otimes \mathcal{R}_\omega$$

with n_ω -dimensional complex vector spaces N_ω .

Define, for the d -partition \underline{n} and a pair (e_1, e_2) of non-zero integers, the sub-torus

$$(148) \quad \mathbf{T}_{\underline{n}; \underline{e}} \approx \mathbf{T}_{\underline{n}} \times U(1)_{\underline{e}} \subset PU(N) \times \text{Spin}(4)_{12}.$$

Here $U(1)_{\underline{e}}$ is embedded into $U(1)_{12} \times U(1)'_{12} \subset U(2)_{12} \subset \text{Spin}(4)_{12}$ by

$$(149) \quad U(1)_{\underline{e}} : e^{i\vartheta} \mapsto \left(e^{\frac{i}{2}(e_1+e_2)\vartheta}, e^{\frac{i}{2}(e_1-e_2)\vartheta} \right),$$

in other words, it acts on \mathbb{C}_{12}^2 by:

$$(150) \quad (z_1, z_2) \mapsto \left(e^{ie_1\vartheta} z_1, e^{ie_2\vartheta} z_2 \right).$$

The torus $\mathbf{T}_{\mathbf{n}} \subset \mathbf{T}_n$ is defined to be a quotient of the product of the maximal tori of $U(n_\omega)$ by the overall center $U(1)$:

$$(151) \quad \begin{aligned} h &= \bigoplus_{\omega \in \Gamma^\vee} h_\omega \otimes \mathbf{1}_{\mathcal{R}_\omega} \in U(N), \\ (h_1 : \dots : h_{\ell_{\mathbf{n}}}) &= (h_1 u : \dots : h_{\ell_{\mathbf{n}}} u) \in \mathbf{T}_{\mathbf{n}}, \end{aligned}$$

where $h_\omega h_\omega^\dagger = 1$, $\omega \in \Gamma^\vee$, $h_\omega = \text{diag}(u_{\omega,1}, \dots, u_{\omega,n_\omega})$, and $|u|^2 = 1$.

5.9. Fixed points of smaller tori

Let us start with $n = 1$, so that $\mathbf{T}_{12} = \mathbf{T}_{\mathbf{e}}$. The $\mathbf{T}_{\mathbf{e}}$ -fixed points on $\mathcal{M}_k(1) = \text{Hilb}^{[k]}(\mathbb{C}^2)$ are isolated for $e_1 e_2 < 0$ and non-isolated for $e_1 e_2 > 0$, as we see from the $T_\chi(\lambda, \lambda)$ character (143). Indeed, as soon as the partition λ has a box $\square = (i, j)$ whose arm plus one-to-leg, or leg plus one-to-arm ratio is equal to $e_1 : e_2$,

$$(152) \quad \begin{aligned} e_1(i - \lambda_j^t) + e_2(\lambda_i + 1 - j) &= 0, \\ \text{or } e_1(\lambda_j^t + 1 - i) + e_2(j - \lambda_i) &= 0 \end{aligned}$$

then $T_\lambda \mathcal{M}_k(1)$ contains trivial $\mathbf{T}_{\mathbf{e}}$ -representations, i.e. λ is not an isolated fixed point. Geometrically, the fixed points of the $\mathbf{T}_{\mathbf{e}}$ -action for $e_1 e_2 > 0$ are the (e_1, e_2) -graded ideals \mathcal{I} in $\mathbb{C}[x, y]$, i.e. the ideals which are invariant under the \mathbb{C}^\times -action:

$$(153) \quad (x, y) \rightarrow (t^{e_1} x, t^{e_2} y)$$

For such an ideal \mathcal{I} the quotient $K = \mathbb{C}[x, y]/\mathcal{I}$ is also a graded vector space:

$$(154) \quad K = \bigoplus_{s=0}^{d_K} K_s$$

For general Γ^\vee and the general partition \mathbf{n} the $\mathbf{T}_{\mathbf{n}, \mathbf{e}}$ -fixed point set is a finite union of finite product

$$(155) \quad \mathcal{M}_k(n)^{\mathbf{T}_{\mathbf{n}, \mathbf{e}}} = \sum_{\omega \in \Gamma^\vee, \alpha \in [n_\omega]} \bigcup_{k_{\alpha, \omega} = k} \prod_{\omega \in \Gamma^\vee} \prod_{\alpha \in [n_\omega]} \mathcal{M}_{k_{\alpha, \omega}}(d(\omega))^{\mathbf{T}_{\mathbf{e}}}$$

of the $\mathbf{T}_{\mathbf{e}}$ -fixed point sets on the moduli spaces $\mathcal{M}_{k'}(n')$. This is easy to show using the same methods as we employed so far. It suffices then to analyze

the structure of $\mathcal{M}_k(n)^{\mathbf{T}_{\mathbf{e}}}$ where the torus $\mathbf{T}_{\mathbf{e}} \approx U(1)$ acts on the matrices (B_1, B_2, I, J) via:

$$(156) \quad e^{it} \cdot [B_1, B_2, I, J] = [e^{ie_1 t} B_1, e^{ie_2 t} B_2, I, e^{i(e_1+e_2)t} J]$$

As usual, the $GL(K)$ -equivalence class of the quadruple $[B_1, B_2, I, J]$ is $\mathbf{T}_{\mathbf{e}}$ -invariant if for every $e^t \in \mathbb{C}^\times$ there is an operator $g_t \in GL(K)$ which undoes (156), i.e.

$$(157) \quad \begin{aligned} e^{ie_1 t} g_t^{-1} B_1 g_t &= B_1, & e^{ie_2 t} g_t^{-1} B_2 g_t &= B_2 \\ g_t^{-1} I &= I, & e^{i(e_1+e_2)t} J g_t &= J \end{aligned}$$

The correspondence $e^t \mapsto g_t$ splits K as the sum of irreducible representations of $\mathbf{T}_{\mathbf{e}}$

$$(158) \quad K = \bigoplus_{\mathfrak{s} \geq 0} K_{\mathfrak{s}} \otimes \mathcal{R}_{\mathfrak{s}}$$

with $K_{\mathfrak{s}}$ being the multiplicity space of the charge \mathfrak{s} representation $\mathcal{R}_{\mathfrak{s}}: e^t \mapsto e^{t\mathfrak{s}}$. Let

$$(159) \quad k_{\mathfrak{s}} = \dim K_{\mathfrak{s}} .$$

We have:

$$(160) \quad \sum_{\mathfrak{s}} k_{\mathfrak{s}} = k$$

The grade 0 component is 1-dimensional:

$$(161) \quad K_0 = I(N) .$$

The operators B_1, B_2 raise the grading by e_1 and e_2 , respectively:

$$(162) \quad B_a : K_{\mathfrak{s}} \rightarrow K_{\mathfrak{s}+e_a}, \quad a = 1, 2$$

The complex \mathcal{S}_0 and its cohomology P_{12}^{\pm} are also graded:

$$(163) \quad P_{12}^{\pm} = \bigoplus_{\mathfrak{s}} P_{12, \mathfrak{s}}^{\pm} .$$

The dimensions k_s are constant throughout the connected component of the set of (e_1, e_2) -homogeneous ideals. In fact, for $e_1 = e_2$ the component is a smooth projective variety, [17]. See also [16], [20].

5.10. Compactness of the fixed point set

The topology of the fixed point set $\mathcal{M}_k(n)^{\mathbf{T}_{12}}$ depends on the choice of the torus \mathbf{T}_{12} . In other words, it depends on how non-generic the choice of $(\underline{\mathbf{a}}, \underline{\mathbf{e}})$ is.

If there is no rational relation between $\underline{\mathbf{a}}$ and $\mathbf{e}_1, \mathbf{e}_2$, more precisely, if for any $\alpha, \beta \in [n]$ and $p, q \in \mathbb{Z}$

$$(164) \quad \mathbf{a}_\alpha - \mathbf{a}_\beta + \mathbf{e}_1 p + \mathbf{e}_2 q = 0 \quad \implies \quad \alpha = \beta, \quad p = q = 0$$

then the fixed points f are isolated, $f \leftrightarrow (\lambda^{(\alpha)})_{\alpha \in [n]}$. Their total number, for fixed k is finite, therefore the set of fixed points is compact.

What if there is a rational relation between $\mathbf{a}_\alpha - \mathbf{a}_\beta$ and $\mathbf{e}_1, \mathbf{e}_2$? That is for some non-trivial $\alpha, \beta \in [n]$ and $p, q \in \mathbb{Z}$,

$$(165) \quad \mathbf{a}_\alpha - \mathbf{a}_\beta + \mathbf{e}_1 p + \mathbf{e}_2 q = 0.$$

We shall assume all the rest of the parameters $\mathbf{a}_\gamma, \mathbf{e}_1, \mathbf{e}_2$ generic. In particular we assume both $\mathbf{e}_1, \mathbf{e}_2$ non-zero. There are three cases to consider:

- 1) $\alpha \neq \beta$ and $pq > 0$;
- 2) $\alpha = \beta$ and $pq > 0$;
- 3) $pq < 0$ and no restriction on α, β ;

In the case (1) the fixed locus is non-compact. It is parametrized by the value of the invariant

$$(166) \quad J_\beta B_1^{p-1} B_2^{q-1} I_\alpha$$

We therefore must make sure, in what follows, that the eigenvalues $(\mathbf{a}_\alpha)_{\alpha \in [n]}$ of the infinitesimal framing rotations and the parameters $(\mathbf{e}_1, \mathbf{e}_2)$ of the spatial rotation do not land on the hyperplanes:

$$(167) \quad \mathbf{a}_\alpha - \mathbf{a}_\beta + \mathbf{e}_1 p + \mathbf{e}_2 q \neq 0,$$

for all $\alpha \neq \beta$, and integer $p, q \geq 1$.

In the case (2) the fixed points corresponding to the monomial ideals are isolated, since the weights in (143) have the form $\mathbf{e}_1 p' + \mathbf{e}_2 q'$ with $p'q' \leq 0$. We shall show below that

the $U(1)$ -fixed points in the case $pq > 0$ correspond to the monomial ideals. In other words they are $U(1) \times U(1)$ -invariant

For fixed k the sizes of the Young diagrams $\lambda^{(\alpha)}$ are bounded above, since

$$(168) \quad \sum_{\alpha=1}^n |\lambda^{(\alpha)}| = k$$

Since the number of collections of Young diagrams which obey (168) is finite, the set of points fixed by the action of the maximal torus \mathbf{T} is compact. This set, as we just showed, is in one-to-one correspondence with the collections

$$(169) \quad f \leftrightarrow \left(\lambda^{(\alpha)} \right)_{\alpha \in [n]}$$

obeying (168).

In the case (3) the fixed points are not isolated, but the fixed point set is compact. Let us show the $\mathbf{T}_{\mathbf{n};\mathbf{e}}$ -fixed point set is compact. There are two cases:

- 1) $e_1 e_2 > 0$. In this case the minimal torus corresponds to $\Gamma^\vee = \{0\}$, $d(0) = n$, $n_0 = 1$, i.e. for $\mathbf{T}_{\mathbf{n}} = 1$, $\mathbf{T}_{\mathbf{n};\mathbf{e}} = \mathbf{T}_{\mathbf{e}}$. The corresponding Coulomb parameter vanishes, $\mathbf{a} = 0$.

We are going to demonstrate that for all $\mathbf{T}_{\mathbf{n};\mathbf{e}}$ -fixed points on $\mathcal{M}_k(n)$ the L^2 -norm of (B_1, B_2, I, J) is bounded above by a constant which depends only on n, k , and ζ . We use the real moment map equation (46):

$$(170) \quad \begin{aligned} k\zeta &= \text{Tr}_K(\mu) = \|I\|^2 - \|J\|^2 \\ \zeta \text{Tr}_K \sigma &= \text{Tr}_K(\sigma\mu) = e_1 \|B_1\|^2 + e_2 \|B_2\|^2 + (e_1 + e_2) \|J\|^2 \end{aligned}$$

where we used the Eqs. (123) with the specialization $\mathbf{e}_1 = e_1, \mathbf{e}_2 = e_2$:

$$(171) \quad \begin{aligned} e_a B_a &= [\sigma, B_a], \quad e_a B_a^\dagger = [B_a^\dagger, \sigma], \quad a = 1, 2, \\ 0 &= \sigma I, \quad 0 = I^\dagger \sigma, \quad (e_1 + e_2)J = -J\sigma, \quad J^\dagger(e_1 + e_2) = -\sigma J^\dagger, \end{aligned}$$

The Eqs. (171) imply, by the same arguments as before, that the spectrum of σ has the form:

$$(172) \quad \mathfrak{s} = e_1(i - 1) + e_2(j - 1), \quad (i, j) \in \Sigma$$

for a finite set Σ of pairs (i, j) of positive integers, and that J maps the eigenvectors of σ in K to zero, unless the eigenvalue is equal to $-e_1 - e_2$. Now, the eigenvalues of σ are of the form (172), which are never equal to $-e_1 - e_2$. Thus, $J|_K = 0$, therefore B_1 and B_2 commute on K .

Now, $\sigma|_{I(N)} = 0$, i.e. $m_0 = \dim(\text{im} I) \leq n$. Now, the vector spaces

$$(173) \quad K_{i,j} = \mathbb{C} \cdot B_1^{i-1} B_2^{j-1} I(N)$$

if non-zero, contribute $\dim K_{i,j} \leq n$ to $k_{\mathfrak{s}}$ with $\mathfrak{s} = e_1(i - 1) + e_2(j - 1)$. It is clear that

$$(174) \quad \begin{aligned} k_{\mathfrak{s}} &= \dim \sum_{e_1(i-1)+e_2(j-1)=\mathfrak{s}} \mathbb{C} \left(B_1^{i-1} B_2^{j-1} I(N) \right) \\ &\leq \sum_{e_1(i-1)+e_2(j-1)=\mathfrak{s}} \dim \mathbb{C} \left(B_1^{i-1} B_2^{j-1} I(N) \right) \\ &\leq n \text{Coeff}_{t^{\mathfrak{s}}} \frac{(1 - t^{ke_1})(1 - t^{ke_2})}{(1 - t^{e_1})(1 - t^{e_2})}, \end{aligned}$$

since both i and j cannot be greater than k . The trace $\text{Tr}_K \sigma$ can be estimated by

$$(175) \quad \begin{aligned} \text{Tr}_K \sigma &= \sum_{\mathfrak{s} \in \Sigma} \mathfrak{s} k_{\mathfrak{s}} \leq n \sum_{\mathfrak{s}=0}^{\infty} \mathfrak{s} \text{Coeff}_{t^{\mathfrak{s}}} \frac{(1 - t^{ke_1})(1 - t^{ke_2})}{(1 - t^{e_1})(1 - t^{e_2})} \\ &= n t \frac{d}{dt} \Big|_{t=1} \frac{(1 - t^{ke_1})(1 - t^{ke_2})}{(1 - t^{e_1})(1 - t^{e_2})} \\ &= \frac{1}{2} k^2 (k - 1) n (e_1 + e_2) \end{aligned}$$

Thus, $J = 0$, the norms $\|B_{1,2}\|^2$ of the operators $B_{1,2}$ are bounded above, while the norm of the operator I is fixed:

$$(176) \quad \|I\|^2 = \zeta k, \quad \frac{e_1}{e_1 + e_2} \|B_1\|^2 + \frac{e_2}{e_1 + e_2} \|B_2\|^2 \leq \frac{\zeta}{2} k^2 (k - 1) n$$

2) $e_1 e_2 < 0$. In this case we take $\Gamma^\vee = [n]$, $d(\omega) = 1$ for all $\omega \in [n]$. The Coulomb parameters are the generic n complex numbers $\mathbf{a}_\alpha \in \mathbb{C}$, $\alpha \in [n]$, defined up to an overall shift. Below we further restrict the parameters \mathbf{a}_α to be real, so that they belong to the Lie algebra of the compact torus $\mathbf{T}_{\mathbf{n}; \mathbf{e}}$. The equations (171) generalize to:

$$(177) \quad \begin{aligned} e_a B_a &= [\sigma, B_a], & e_a B_a^\dagger &= [B_a^\dagger, \sigma], & a &= 1, 2, \\ I\mathbf{a} &= \sigma I, & \mathbf{a}I^\dagger &= I^\dagger \sigma, & (e_1 + e_2 - \mathbf{a})J &= -J\sigma, \\ J^\dagger(e_1 + e_2 - \mathbf{a}) &= -\sigma J^\dagger, \end{aligned}$$

The fixed point set $\mathcal{M}_k(n)^{\mathbf{T}_{\mathbf{n}; \mathbf{e}}}$ splits:

$$(178) \quad \mathcal{M}_k(n)^{\mathbf{T}_{\mathbf{n}; \mathbf{e}}} = \bigcup_{k_1 + \dots + k_n = k} \prod_{\alpha \in [n]} \mathcal{M}_{k_\alpha}(1)^{\mathbf{T}_{\mathbf{e}}}$$

The fixed points are isolated, these are our friends $(\lambda^{(\alpha)})_{\alpha \in [n]}$, the n -tuples of partitions with the total size equal to k . Since it is a finite set, it is compact.

Note that we couldn't restrict the torus $\mathbf{T}_{\mathbf{n}; \mathbf{e}}$ any further in this case. Indeed, the crucial ingredient in arriving at (178) is vanishing of the J matrix for the $\mathbf{T}_{\mathbf{n}; \mathbf{e}}$ -invariant solutions of the ADHM equations. The argument below the Eq. (172) we used before would not work for $e_1 e_2 < 0$, since $\mathbf{a}_\alpha - (e_1 + e_2)$ may be equal to $\mathbf{a}_\beta + e_1(i - 1) + e_2(j - 1)$ for some $\alpha, \beta \in [n]$, $i, j \geq 1$. In this case J may have a non-trivial matrix element, giving rise to a non-compact fixed locus. Now, insisting on the $\mathbf{T}_{\mathbf{n}; \mathbf{e}}$ -invariance with $\mathbf{T}_{\mathbf{n}} = U(1)^{n-1}$ means \mathbf{a}_α 's in (177) are completely generic, in particular, for $\alpha \neq \beta$, $\mathbf{a}_\alpha - \mathbf{a}_\beta \notin \mathbb{Z}$. This still leaves the case $\alpha = \beta$ as a potential source of noncompactness. But this is the case of the $\mathbf{T}_{\mathbf{e}}$ -action on $\mathcal{M}_k(1) = \text{Hilb}^{[k]}(\mathbb{C}^2)$. In this case J vanishes not because of the toric symmetry, but rather because of the stability condition [25]:

$$(179) \quad \begin{aligned} JI &= \text{Tr} IJ = \text{Tr} [B_2, B_1] = 0, \\ J(xB_1 + yB_2)^l I &= \text{Tr} (xB_1 + yB_2)^l [B_2, B_1] \\ &= \text{Tr} (xB_1 + yB_2)^l [xB_1 + yB_2, x'B_1 + y'B_2] = 0, \end{aligned}$$

for any x, y, x', y' , s.t. $x'y - xy' = 1$,

$$\begin{aligned}
 & Jf(B_1, B_2)B_1B_2g(B_1, B_2)I \\
 &= Jf(B_1, B_2)B_2B_1g(B_1, B_2)I \\
 &+ (Jf(B_1, B_2)I)(Jg(B_1, B_2)I) = 0, \quad \text{by induction} \\
 \implies & J\mathbb{C}[B_1, B_2]I = 0 \implies J = 0, \quad \text{by stability}
 \end{aligned}$$

The compactness of $\mathcal{M}_k(n)^{\mathbf{T}^{n:e}}$ is thus established.

5.11. Ordinary instantons as the fixed set

Let us now consider the particular $\mathbf{T}_x = U(1)^5$ symmetry of the spiked instanton equations,

$$(180) \quad (I_A, J_A) \longrightarrow \left(e^{i\vartheta_A} I_A, e^{-i\vartheta_A} J_A \right)$$

where $\vec{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6) \sim (\vartheta_1 + \vartheta, \vartheta_2 + \vartheta, \vartheta_3 + \vartheta, \vartheta_4 + \vartheta, \vartheta_5 + \vartheta, \vartheta_6 + \vartheta)$ for any ϑ . The \mathbf{T}_x -invariant configuration $[\mathbf{B}, \mathbf{I}, \mathbf{J}]$ defines a homomorphism of the covering torus $\tilde{\mathbf{T}}_x \approx U(1)^6 \rightarrow U(k)$, via the compensating $U(k)$ -transformation $g(\vec{\vartheta})$ obeying:

$$(181) \quad e^{i\vartheta_A} I_A = g(\vec{\vartheta}) I_A, \quad e^{-i\vartheta_A} J_A = J_A g(\vec{\vartheta})^{-1}, \quad g(\vec{\vartheta}) B_a g(\vec{\vartheta})^{-1} = B_a$$

The space K splits as the orthogonal direct sum

$$\begin{aligned}
 (182) \quad K &= \bigoplus_{A \in \mathfrak{6}} K_A, \quad g(\vec{\vartheta})|_{K_A} = e^{i\vartheta_A}, \\
 K_A &= \mathbb{C}[B_a, B_b] I_A(N_A), \quad \text{for } A = \{a, b\}
 \end{aligned}$$

This decomposition is preserved by the matrices $\mathbf{B}, \mathbf{I}, \mathbf{J}$. Thus the solution is the direct sum of the solutions of ADHM equations:

$$(183) \quad \mathfrak{M}_k^*(\vec{n})^{\mathbf{T}_x} = \bigcup_{\sum_{A \in \mathfrak{6}} k_A = k} \prod_{A \in \mathfrak{6}} \mathcal{M}_{k_A}(n_A)$$

6. Crossed and folded instantons

Distorted shadows fell
 Upon the lighted ceiling:
 Shadows of crossed arms, of crossed legs-
 Of crossed destiny.[†]

[†] Winter night, from “Dr. Zhivago”, B. Pasternak, English translation by Bernard G. Guerney.

The next special case is where only two e.g. $N_{A'}$ and $N_{A''}$ out of six vector spaces N_A are non-zero. There are two basic cases.

6.1. Crossed instantons

Suppose $A' \cup A'' = \emptyset$, e.g. $A' = \{1, 2\}$ and $A'' = \{3, 4\}$. In this case we define

$$(184) \quad \mathfrak{M}_k^+(n, w) = \mathfrak{M}_k^0(n_{12} = n, 0, 0, 0, n_{34} = w)$$

We call the space $\mathfrak{M}_k^+(n, w)$ the space of (n, w) -crossed instantons.

The virtual dimension of the space $\mathfrak{M}_k^+(n, w)$ is independent of k , it is equal to $-2nw$. As a set, $\mathfrak{M}_k^+(n, w)$ is stratified

$$(185) \quad \mathfrak{M}_k^+(n, w) = \bigcup_{k'+k'' \geq k} \mathfrak{M}_{k',k'';k}^+(n, w) .$$

The stratum

$$(186) \quad \mathfrak{M}_{k',k'';k}^+(n, w) = \left\{ [\mathbf{B}, \mathbf{I}, \mathbf{J}] \mid \dim K_{12} = k', \dim K_{34} = k'' \right\}$$

parametrizes the crossed instantons, whose ordinary instanton components have the charges k' and k'' , respectively: the crossed instanton $[\mathbf{B}, \mathbf{I}, \mathbf{J}]$ defines two ordinary instantons, $[B_1, B_2, I_{12}, J_{12}]$ on \mathbb{C}_{12}^2 and $[B_3, B_4, I_{34}, J_{34}]$ on \mathbb{C}_{34}^2 , of the charges

$$(187) \quad k' = \dim K_{12}, \quad k'' = \dim K_{34}$$

6.2. One-instanton crossed example

When $k = 1$ the matrices B_a are just complex numbers $b_a \in \mathbb{C}$, $a \in \underline{4}$. The equations $b_1 I_{34} = b_2 I_{34} = 0$ and $b_3 I_{12} = b_4 I_{12} = 0$ imply that if $(b_1, b_2) \neq 0$, then $I_{34} = 0$, $I_{12} \neq 0$, $(b_3, b_4) = (0, 0)$, $K = K_{12}$ and the rest of the matrices define the ordinary charge 1 $U(n)$ instanton, parametrized by the space (111). Likewise, if $(b_3, b_4) \neq 0$, then $I_{12} = 0$, $I_{34} \neq 0$, $(b_1, b_2) = (0, 0)$, $K = K_{34}$ and the rest of the matrices define the ordinary charge 1 $U(w)$ instanton. Finally, if $(b_1, b_2, b_3, b_4) = 0$, then both I_{12}, I_{34} need not vanish. If, indeed, both I_{12}, I_{34} do not vanish, then J_{12} and J_{34} vanish, by the v -equations,

while I_{12}, I_{34} obey

$$(188) \quad \|I_{12}\|^2 + \|I_{34}\|^2 = \zeta,$$

which, modulo $U(1) = U(K)$ symmetry, define a subset in $\mathbb{C}\mathbb{P}^{n+w-1}$, the complement to the pair of “linked” projective spaces, $\mathbb{C}\mathbb{P}^{n-1}$ and $\mathbb{C}\mathbb{P}^{w-1}$, corresponding to the vanishing of I_{34} and I_{12} , respectively. The result is, then

$$(189) \quad \mathfrak{M}_1^+(n, w) = \mathbb{C}^2 \times T^*\mathbb{C}\mathbb{P}^{n-1} \cup \mathbb{C}\mathbb{P}^{n+w-1} \cup \mathbb{C}^2 \times T^*\mathbb{C}\mathbb{P}^{w-1},$$

the first and second components intersect along $(0, 0) \times \mathbb{C}\mathbb{P}^{w-1}$ the second and the third components intersect along $(0, 0) \times \mathbb{C}\mathbb{P}^{n-1}$, where $\mathbb{C}\mathbb{P}^{n-1} \cup \mathbb{C}\mathbb{P}^{w-1} \subset \mathbb{C}\mathbb{P}^{n+w-1}$ are non-intersecting $\mathbb{C}\mathbb{P}^{n-1} \cap \mathbb{C}\mathbb{P}^{w-1} = \emptyset$ projective subspaces.

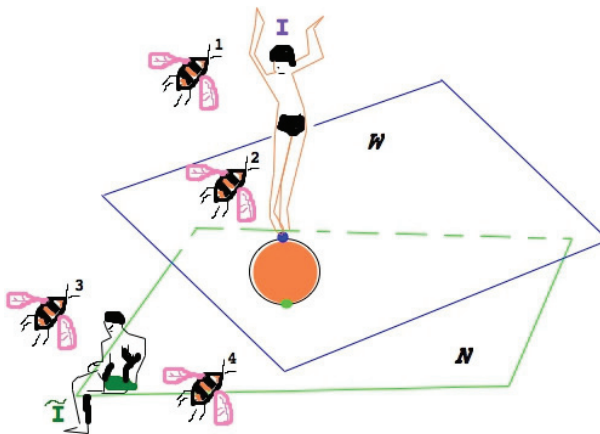


Figure 7: Charge one crossed instanton moduli space: the planes represent the complex plane factors \mathbb{C}^2_{12} and \mathbb{C}^2_{34} , the girl represents the $T^*\mathbb{C}\mathbb{P}^{w-1}$ factor, the man represents the $T^*\mathbb{C}\mathbb{P}^{n-1}$ factor, the orange ball is the $\mathbb{C}\mathbb{P}^{n+w-1}$ component, the blue and green dots are the $\mathbb{C}\mathbb{P}^{n-1}$ and $\mathbb{C}\mathbb{P}^{w-1}$ loci of the intersections of components.

6.3. Folded instantons

In this case $A' \cap A'' = \{a\}$, e.g. $A' = \{1, 2\}$, $A'' = \{1, 3\}$, $a = 1$.

We define:

$$(190) \quad \mathfrak{M}_k^\perp(n, w) = \mathfrak{M}_k^k(n_{12} = n, n_{13} = w, 0, 0, 0, 0)$$

We call the space $\mathfrak{M}_k^\perp(n, w)$ the space of (n, w) -folded instantons.

There exists an analogue of the stratification (185) for the folded instantons. Again, the folded instanton data $[B_1, B_2, B_3, B_4, I_{12}, I_{13}, J_{12}, J_{13}]$ defines two ordinary noncommutative instantons on \mathbb{R}^4 , one on $\mathbb{C}_{12}^2, [B_1, B_2, I_{12}, J_{12}]$, another on $\mathbb{C}_{13}^2, [B_1, B_3, I_{13}, J_{13}]$. The stability implies that B_4 vanishes. The spaces $K_{12} = \mathbb{C}[B_1, B_2]I_{12}(N_{12})$ and $K_{13} = \mathbb{C}[B_1, B_3]I_{13}(N_{13})$ generate all of K ,

$$(191) \quad K = K_{12} + K_{13}.$$

6.4. One-instanton folded example

When $k = 1$, as before, the matrices B_a are the complex numbers $b_a, a \in \underline{4}$, except that b_4 vanishes. Now, the equation $b_2 I_{13} = 0$ implies that if $b_2 \neq 0$ then $I_{13} = 0$, and we have the charge one ordinary $U(n)$ instanton on \mathbb{C}_{12}^2 . Likewise the equation $b_3 I_{12} = 0$ implies that if $b_3 \neq 0$ then $I_{12} = 0$ and we have the charge one ordinary $U(w)$ instaton on \mathbb{C}_{13}^2 . Finally, when both $b_2 = b_3 = 0$, the remaining equations $J_{12} I_{13} = J_{13} I_{12} = J_{12} I_{12} = J_{13} I_{13} = 0$, and $\|I_{12}\|^2 + \|I_{13}\|^2 - \|J_{12}\|^2 - \|J_{13}\|^2 = \zeta$, define the variety which is a product of a copy of \mathbb{C}^1 (parametrized by b_1) and our friend the union of three pieces: $\mathbb{C}\mathbb{P}^{n+w-1}$ (this is the locus where $J_{12} = J_{13} = 0$), $T^*\mathbb{C}\mathbb{P}^{n-1}$ (the locus where $I_{13} = J_{13} = 0$) and $T^*\mathbb{C}\mathbb{P}^{w-1}$ (the locus where $I_{12} = J_{12} = 0$):

$$(192) \quad \mathfrak{M}_1^\perp(n, w) = \mathbb{C}^2 \times T^*\mathbb{C}\mathbb{P}^{w-1} \cup \mathbb{C}^1 \times \mathbb{C}\mathbb{P}^{n+w-1} \cup \mathbb{C}^2 \times T^*\mathbb{C}\mathbb{P}^{n-1},$$

the first and second components intersect along $\mathbb{C}^1 \times \mathbb{C}\mathbb{P}^{w-1}$ the second and the third components intersect along $\mathbb{C}^1 \times \mathbb{C}\mathbb{P}^{n-1}$, where $\mathbb{C}\mathbb{P}^{n-1} \cup \mathbb{C}\mathbb{P}^{w-1} \subset \mathbb{C}\mathbb{P}^{n+w-1}$ are non-intersecting $\mathbb{C}\mathbb{P}^{n-1} \cap \mathbb{C}\mathbb{P}^{w-1} = \emptyset$ projective subspaces.

6.5. Fixed point sets: butterflies and zippers

Let us now discuss the fixed point sets of toric symmetries of the crossed and folded instantons. The torus $\mathbf{T}_{n,w} \times U(1)_\mathfrak{c}^3$ acts on $\mathfrak{M}_k^+(n, w)$ and $\mathfrak{M}_k^\perp(n, w)$:

$$(193) \quad (B_a, I_A, J_A) \mapsto (e^{it\epsilon_a} B_a, I_A e^{-ita_A}, e^{it\epsilon_A} e^{ita_A} J_A)$$

Here $\mathfrak{a}_A = \text{diag}(\mathfrak{a}_{A,1}, \dots, \mathfrak{a}_{A,n_A})$ where the complex numbers $\mathfrak{a}_{A,\alpha}$, $\alpha \in [n_A]$ are defined up to the overall shift

$$(194) \quad \mathfrak{a}_{A,\alpha} \sim \mathfrak{a}_{A,\alpha} + y,$$

with $y \in \mathbb{C}$. Let $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$,

$$(195) \quad \sum_{a \in \underline{4}} \epsilon_a = 0$$

We assume $\epsilon_a \neq 0$ for each $a \in \underline{4}$.

6.5.1. Toric crossed instantons. The fixed point set $\mathfrak{M}_k^+(n, w)^{\mathbf{T}_{n,w} \times U(1)_\epsilon^3}$ is easy to describe. The infinitesimal transformation generated by $(\mathfrak{a}, \underline{\epsilon})$ is compensated by the infinitesimal $GL(k)$ transformation, generated by $\sigma \in \text{End}(K)$. As in the previous section this makes K a representation of $\mathbf{T}_{n,w} \times U(1)_\epsilon^3$. The space K contains two subspaces, K_{12} and K_{34} , whose intersection $K_{12,34} = K_{12} \cap K_{34}$ belongs to both P_{12} and P_{34} :

$$(196) \quad K_{12,34} \subset P_{12} \cap P_{34}$$

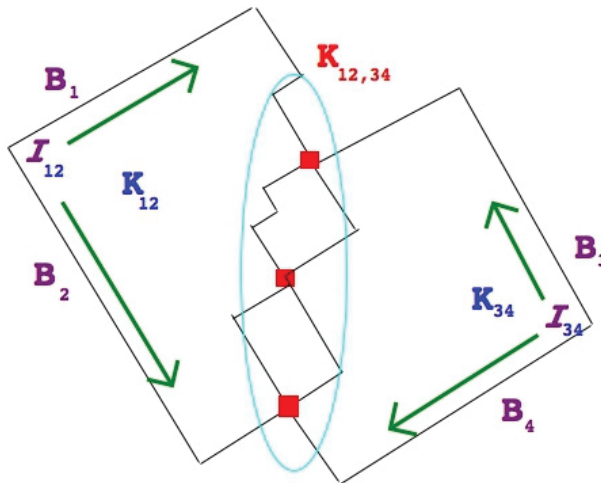


Figure 8: The butterfly.

The eigenvalues of σ on K_{12} have the form:

$$(197) \quad \text{Eigen}(\sigma|_{K_{12}}) = \left\{ \mathbf{a}_{12,\alpha} + \mathbf{e}_1(i-1) + \mathbf{e}_2(j-1) \mid \alpha \in [n], (i, j) \in \lambda^{(12,\alpha)} \right\}$$

The eigenvalues of σ on K_{34} have the form:

$$(198) \quad \text{Eigen}(\sigma|_{K_{34}}) = \left\{ \mathbf{a}_{34,\beta} + \mathbf{e}_3(i-1) + \mathbf{e}_4(j-1) \mid \beta \in [w], (i, j) \in \lambda^{(34,\beta)} \right\}$$

These two sets do not overlap when all the parameters $\mathbf{a}_{A,\alpha}, \mathbf{e}_a$ are generic. Therefore $K_{12} \perp K_{34}$ and the $\mathbf{T}_{n,w} \times U(1)_{\mathbf{e}}^3$ -fixed points are isolated. These fixed points are, therefore, in one-to-one correspondence with the pairs

$$(199) \quad (\lambda^{12}; \lambda^{34})$$

consisting of n - and w -tuples

$$\lambda^{12} = \left(\lambda^{(12,1)}, \dots, \lambda^{(12,n)} \right); \quad \lambda^{34} = \left(\lambda^{(34,1)}, \dots, \lambda^{(34,w)} \right),$$

of partitions, obeying

$$(200) \quad \sum_{\alpha \in [n]} \left| \lambda^{(12,\alpha)} \right| + \sum_{\beta \in [w]} \left| \lambda^{(34,\alpha)} \right| = k$$

Their number is finite, therefore the set $\mathfrak{M}_k^+(\mathbf{n})^{\mathbf{T}_{n,w} \times U(1)_{\mathbf{e}}^3}$ of fixed points is compact.

Now let us try to choose a sub-torus $\mathbf{T}' \subset \mathbf{T}_{n,w} \times U(1)_{\mathbf{e}}^3$, restricted only by the condition that $J_{12} = J_{34} = 0$ for the \mathbf{T}' -invariant solutions of (57). We wish to prove that the set of \mathbf{T}' -fixed points is compact in this case as well. In the next sections we shall describe such tori in more detail.

We start by the observation that if $K_{12,34} \neq 0$ then the two sets (197) and (198) of σ -eigenvalues must overlap. Therefore, for some $(\alpha, \beta) \in [n] \times [w]$, and for some $(i', j') \in \lambda^{(12,\alpha)}, (i'', j'') \in \lambda^{(34,\beta)}$

$$(201) \quad \mathbf{a}_{12,\alpha} + \mathbf{e}_1(i' - 1) + \mathbf{e}_2(j' - 1) = \mathbf{a}_{34,\beta} + \mathbf{e}_3(i'' - 1) + \mathbf{e}_4(j'' - 1)$$

Note that (201) is invariant under the shifts (194). Moreover, if (cf. (7))

$$(202) \quad \begin{aligned} \mathbf{a}_{12,\alpha'} - \mathbf{a}_{12,\alpha''} &\notin Z_{\mathbf{e}}, & \alpha' &\neq \alpha'' \\ \mathbf{a}_{34,\beta'} - \mathbf{a}_{34,\beta''} &\notin Z_{\mathbf{e}}, & \beta' &\neq \beta'' \end{aligned}$$

and $0 \notin Z_{\mathbf{e}}$, then the condition (201) determines (α, β) and $i'j'$ and $i''j''$ uniquely, up to the shifts $(i', j', i'', j'') \mapsto (i' + k, j' + k, i'' - k, j'' - k), k \in$

\mathbb{Z} . The relation (201) defines the codimension 1 subtorus $\mathbf{T}' \subset \mathbf{T}_{n,w} \times U(1)_{\mathfrak{e}}^3$. Let us describe its fixed locus.

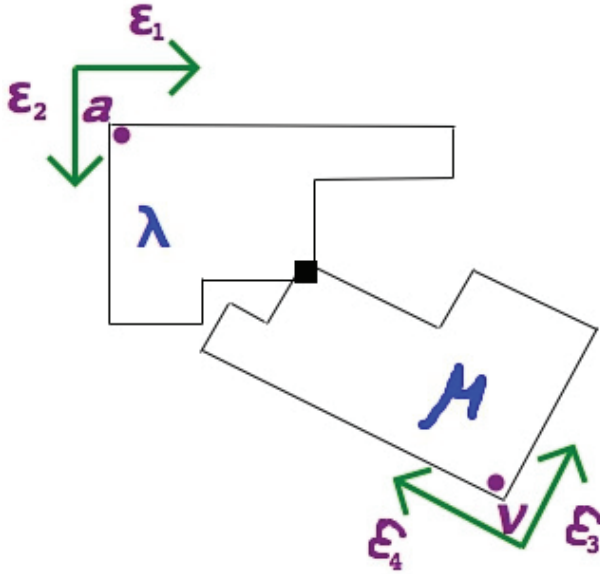
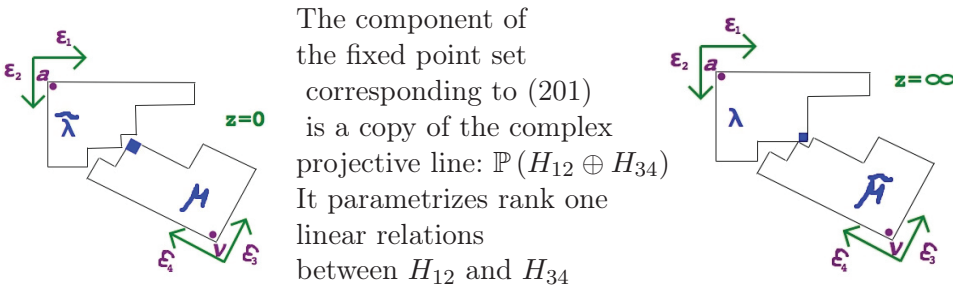


Figure 9: $\blacksquare = K_{12,34}$, $\mathbf{a} = \mathbf{a}_{12,\alpha}$, $\mathbf{v} = \mathbf{a}_{34,\beta}$, $\lambda = \lambda^{(12,\alpha)}$, $\mu = \lambda^{(34,\beta)}$.



The component of the fixed point set corresponding to (201) is a copy of the complex projective line: $\mathbb{P}(H_{12} \oplus H_{34})$. It parametrizes rank one linear relations between H_{12} and H_{34} .

Figure 10: The component $\mathbb{P}(H_{12} \oplus H_{34})$.

If the condition (201) on \mathbf{a}, \mathfrak{e} is obeyed, it does not imply that $K_{12,34} \neq 0$. However, if in addition to (201) also the condition (202) is obeyed, then the intersection $K_{12,34}$ is not more than one-dimensional. Let $H_{12} = P_{12;\alpha;i'j'} \subset P_{12}$, $H_{34} = P_{34;\beta;i''j''} \subset P_{34}$ be the one-dimensional eigenspaces of σ corresponding to the eigenvalue (201). If an eigenbasis of N_{12} for \mathbf{a}_{12} and the eigenbasis of N_{34} for \mathbf{a}_{34} are chosen, then H_{12} and H_{34} are endowed with the

basis vectors as well (act on the eigenvector of \mathfrak{a}_{12} corresponding to $\mathfrak{a}_{12,\alpha}$ by $B_1^{i'-1}B_2^{j'-1}I_{12}$ to get the basis vector of H_{12}).

Let z be the coordinate on $\mathbb{P}(H_{12} \oplus H_{34})$ such that $z = \infty$ corresponds to the line H_{12} while $z = 0$ corresponds to the line H_{34} . For $z \neq 0, \infty$ the linear spaces H_{12} and H_{34} coincide, z being the isomorphism. When $z \rightarrow 0$ the image $(p_1(z=0), p_2(z=0))$ is the pair $(\tilde{\lambda}^{12}, \lambda^{34})$, the image $(p_1(z=\infty), p_2(z=\infty))$ is the pair $(\lambda^{12}, \tilde{\lambda}^{34})$. Here $\tilde{\lambda}^{12}$ is the n -tuple of partitions which differs from λ^{12} in that the Young diagram of $\tilde{\lambda}^{(12,\alpha)}$ ($\tilde{\lambda}$ on the Fig. 10) is obtained by removing the (i', j') square from the Young diagram of $\lambda^{(12,\alpha)}$. Similarly, the w -tuple $\tilde{\lambda}^{34}$ ($\tilde{\mu}$ on the Fig. 10) is obtained by modifying $\lambda^{(34,\beta)}$ by removing the box (i'', j'') .

In the next chapters we shall relax the condition (202). In other words, we shall consider a subtorus in $\mathbf{T}_{n,w} \times U(1)_{\mathfrak{e}}^3$.

6.5.2. Toric folded instantons. Now let us explore the folded instantons invariant under the action of the maximal torus $\mathbf{T}_{n,w} \times U(1)_{\mathfrak{e}}^3$. It is easy to see that these are again the pairs $(\lambda^{12}, \lambda^{13})$, with $\lambda^{12} = (\lambda^{(12,1)}, \dots, \lambda^{(12,n)})$, $\lambda^{13} = (\lambda^{(13,1)}, \dots, \lambda^{(13,w)})$. The spaces K_{12} and K_{13} do not intersect, $K = K_{12} \oplus K_{13}$. In other words, the only $\mathbf{T}_{n,w} \times U(1)_{\mathfrak{e}}^3$ -invariant folded instantons are the superpositions of the ordinary instantons on \mathbb{C}_{12}^2 and \mathbb{C}_{13}^2 , of the charges k_{12} and k_{13} , respectively, with $k = k_{12} + k_{13}$.

Now let us consider the non-generic case, such that $K_{12,13} \neq \emptyset$. We call the corresponding fixed point “the zipper”, see the Fig. 11. The codimension one subtorus for which this is possible corresponds to the relation $\mathfrak{a}_{12,\alpha} - \mathfrak{a}_{13,\beta} \notin Z_{\mathfrak{e}}$ between the parameters of the infinitesimal torus transformation.

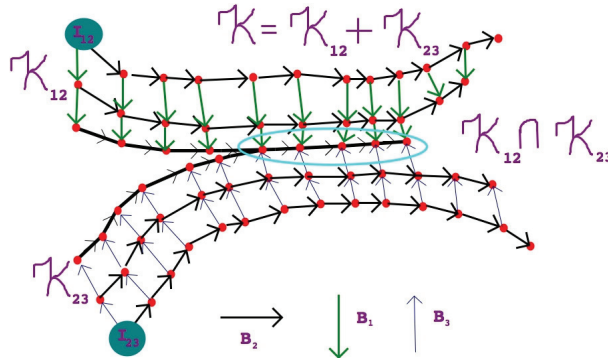


Figure 11: The zipper.

The non-empty overlap $K_{12} \cap K_{13}$ implies the sets of eigenvalues of σ on K_{12} and K_{13} overlap, leading to

$$(203) \quad \mathbf{a}_{12,\alpha} + \mathbf{e}_1(i' - 1) + \mathbf{e}_2(j' - 1) = \mathbf{a}_{13,\beta} + \mathbf{e}_1(i'' - 1) + \mathbf{e}_3(j'' - 1)$$

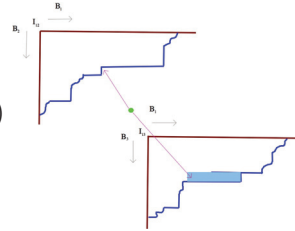
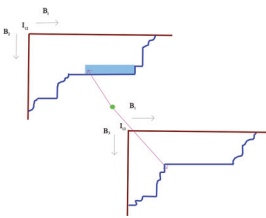
for some $\alpha \in [n], \beta \in [w], i', j', i'', j''$. Unlike the Eq. (201) the Eq. (203) the integers i', i'' are not uniquely determined. Since the left hand side of (203) is the eigenvalue of $\sigma|_{L_{1,2}}$, while the right hand side is the eigenvalue of $\sigma|_{L_{1,3}}$, we conclude:

$$(204) \quad \begin{aligned} (i', j') &\in \lambda^{(12,\alpha)}, & (i'', j'') &\in \lambda^{(13,\beta)}, \\ (i', j' + 1) &\notin \lambda^{(12,\alpha)}, & (i'', j'' + 1) &\notin \lambda^{(13,\beta)} \end{aligned}$$

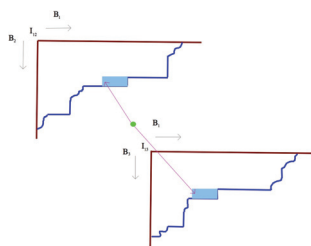
i.e. $j' = \lambda_{i'}^{(12,\alpha)t}, j'' = \lambda_{i''}^{(13,\beta)t}$. The change $(i', i'') \mapsto (i' \pm 1, i'' \pm 1)$ maps the solution of (203) to another solution of (203). Let $l \geq 0$ be the maximal integer such that $j' = \lambda_{i'-l}^{(12,\alpha)}, j'' = \lambda_{i''-l}^{(13,\beta)}$, and let $l \geq m \geq 0$ be the maximal integer such that $j' = \lambda_{i'-m}^{(12,\alpha)}, j'' = \lambda_{i''-m}^{(13,\beta)}$, and $e_{12} = B_1^{i'-m-1} B_2^{j'-1} I_{12,\alpha} \in K_{12,13}, e_{13} = B_1^{i''-m-1} B_3^{j''-1} I_{13,\beta} \in K_{12,13}$. In other words the vectors e_{12} and e_{13} are linearly dependent, $e_{12} = z e_{13}$. Consequently, the arm-lengths $a_{i'-m,j'} = \lambda_{j'}^{(12,\alpha)t} - i' + m, a_{i''-m,j''} = \lambda_{j''}^{(13,\beta)t} - i'' + m$ must be equal:

$$(205) \quad \lambda_{j'}^{(12,\alpha)t} - i' + m = a_{i'-m,j'} = \lambda_{j''}^{(13,\beta)t} - i'' + m$$

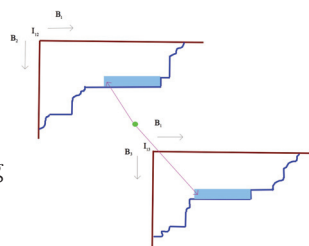
The component of the fixed point set corresponding to (203) is a copy of the complex projective line: $\mathbb{P}(\mathbb{C}e_{12} \oplus \mathbb{C}e_{13})$. It parametrizes rank one linear relations between e_{12} and e_{13}



Let z be the coordinate on $\mathbb{P}(\mathbb{C}e_{12} \oplus \mathbb{C}e_{13})$ such that $z = \infty$ corresponds to the line $\mathbb{C}e_{12}$ while $z = 0$ corresponds to the line $\mathbb{C}e_{13}$. Then the image $(p_1(z = 0), p_2(z = 0))$ is the pair $(\tilde{\lambda}^{(12,\alpha)}, \lambda^{(13,\beta)})$, the image $(p_1(z = \infty), p_2(z = \infty))$ is the pair $(\lambda^{(12,\alpha)}, \tilde{\lambda}^{(13,\beta)})$. Here we defined $\tilde{\lambda}^{(12,\alpha)}$ to be the partition whose Young diagram is obtained by removing the block of squares $(i' - m, j') \cdots (\lambda_{j'}^{(12,\alpha)t}, j')$ from the Young diagram of $\lambda^{(12,\alpha)}$. Similarly, the Young diagram of $\tilde{\lambda}^{(13,\beta)}$ is obtained by removing the block of squares $(i'' - m, j'') \cdots (\lambda_{j''}^{(13,\beta)t}, j'')$ from the Young diagram of $\lambda^{(13,\beta)}$.



Note that the pair of Young diagrams $\lambda^{(12,\alpha)}$, $\lambda^{(34,\beta)}$ gives rise to several components of the fixed point set, isomorphic to $\mathbb{C}\mathbb{P}^1$, e.g. the ones corresponding to the blocks of horizontal boxes of different length,



see the pictures above on the left and on the right. But they actually belong to moduli spaces of folded instantons of different charges (in computing the charge k we subtract the length of the block from the sum of the sizes of Young diagrams). So despite the similarity in graphic design, these are pieces of different architectures.

7. Reconstructing spiked instantons

In this section we describe the sewing procedure, which produces a spiked instanton out of six ordinary noncommutative instantons. We then use the stitching to describe the spiked instantons invariant under the toric symmetry, i.e. the $T_{\mathcal{G}}$ -fixed locus.

7.1. The local K-spaces

For $A = \{a, b\} \in \underline{\mathbf{6}}$ we define:

$$(206) \quad K_A = \mathbb{C}[B_a, B_b] \text{im}(I_A) \subset K$$

By definition, this is the minimal B_a, B_b -invariant subspace of K , containing the image $\text{im}(I_A) = I_A(N_A)$ of N_A .

The equations (57), (58) for $i \geq k$ imply that

$$(207) \quad J_B(K_A) = 0, \quad B \neq A$$

and

$$(208) \quad B_b(K_A) = 0, \quad b \notin A$$

7.2. Toric spiked instantons

Now let us describe the spiked instantons, invariant under the torus action. The tori in question are the subgroups of \mathbf{H} , the global symmetry group. We consider first the maximal torus $T_{\mathbf{H}}$ (cf. (79)) and then its subtori $T_{\mathcal{L}}$ for various choices of the \mathcal{L} data.

7.2.1. Maximal torus. First of all, let us consider the $T_{\mathbf{H}}$ -fixed points. Let $\underline{\mathfrak{a}} = (\mathfrak{a}_A)_{A \in \underline{\mathfrak{6}}}$ be the collection of diagonal matrices $\mathfrak{a}_A = \text{diag}(\mathfrak{a}_{A,1}, \dots, \mathfrak{a}_{A,n_A}) \in \text{Lie}U(n_A)$. The spiked instanton $[\mathbf{B}, \mathbf{I}, \mathbf{J}]$ is $T_{\mathbf{H}}$ -invariant iff for any $\underline{\mathfrak{a}}$ and $\underline{\mathfrak{e}}$ there exists an operator $\sigma \in \text{End}(K)$, such that:

$$(209) \quad \begin{aligned} \mathfrak{e}_a B_a &= [\sigma, B_a], & a \in \underline{\mathfrak{4}} \\ (\mathfrak{e}_A - \mathfrak{a}_A) J_A &= -J_A \sigma, \\ I_A \mathfrak{a}_A &= \sigma I_A, & A \in \underline{\mathfrak{6}} \end{aligned}$$

Let $N_{A,\alpha}$, $\alpha \in [n_A]$, be the eigenspace of \mathfrak{a}_A with the eigenvalue $\mathfrak{a}_{A,\alpha}$. Let $I_{A,\alpha} = I_A(N_{A,\alpha})$. We have (for $A = \{a, b\}$, $a < b$):

$$(210) \quad K_A = \sum_{\alpha \in [n_A], i, j \geq 1} K_{A,\alpha}^{i,j}, \quad K_{A,\alpha}^{i,j} = B_a^{i-1} B_b^{j-1} I_{A,\alpha}$$

The eigenvalue of σ on $K_{A,\alpha}^{i,j}$ is equal to

$$(211) \quad \sigma|_{K_{A,\alpha}^{i,j}} = \mathfrak{a}_{A,\alpha} + \mathfrak{e}_a(i-1) + \mathfrak{e}_b(j-1)$$

On the other hand, Eq. (209) implies that the vector

$$(212) \quad \psi = J_A(K_{A,\alpha}^{i,j}) \in N_A$$

is an eigen-vector of \mathfrak{a}_A with the eigen-value:

$$(213) \quad \mathfrak{a}_A \psi = (\mathfrak{a}_{A,\alpha} + \mathfrak{e}_a i + \mathfrak{e}_b j) \psi$$

The $T_{\mathbf{H}}$ -invariance means we are free to choose the parameters $\mathfrak{a}_{A,\alpha}$, \mathfrak{e}_a in an arbitrary fashion. It means, that $\mathfrak{a}_{A,\alpha} + \mathfrak{e}_a i + \mathfrak{e}_b j \neq \mathfrak{a}_{A,\beta}$ for $i, j \geq 1$, $\alpha, \beta \in [n_A]$. Therefore J_A vanishes on all $K_{A,\alpha}^{i,j}$ subspaces, and therefore on all of K_A . Therefore, all B_a 's commute with each other. Also, the eigenvalues (211) are different for different $(A, \alpha; i, j)$. Therefore, the spaces $K_{A,\alpha}^{i,j}$ are orthogonal to each other.

Define, for $A \in \underline{\mathbf{6}}, \alpha \in [n_A]$ the partition $\lambda^{(A,\alpha)}$, by:

$$(214) \quad \lambda_i^{(A,\alpha)} = \sup \left\{ j \mid j \geq 1, K_{A,\alpha}^{i,j} \neq 0 \right\}$$

We have:

$$(215) \quad k_A = \sum_{\alpha \in [n_A]} |\lambda^{(A,\alpha)}|$$

The T_H -fixed points are, therefore, in one-to-one correspondence with the collections

$$(216) \quad \mathbf{\Lambda} = \left(\lambda^{(A,\alpha)} \right)_{A \in \underline{\mathbf{6}}, \alpha \in [n_A]}$$

of

$$(217) \quad \mathbf{n} = \sum_{A \in \underline{\mathbf{6}}} n_A$$

Young diagrams. In the companion paper [29] we shall be studying the statistical mechanical model, where the random variables are the collections $\mathbf{\Lambda}$, while the complexified Boltzman weights are the contributions of $\mathbf{\Lambda}$ to the gauge partition function, defined below.

7.2.2. Subtori. Now fix the data \mathcal{L} and consider the $T_{\mathcal{L}}$ -invariant spiked instantons $[\mathbf{B}, \mathbf{I}, \mathbf{J}]$. As usual, these come with the homomorphism $T_{\mathcal{L}} \rightarrow U(K)$ which associates the compensating $U(K)$ -transformation $g_{t,\xi}$ for every $(e^{i\xi}, e^{it}) \in T_{\mathcal{L}}$. Since K decomposes into the direct sum of weight subspaces

$$(218) \quad K = \bigoplus_{w,n} K^{w,n}$$

where

$$(219) \quad g_{t,\xi}|_{K^{w,n}} = e^{i\langle w,\xi \rangle + int}$$

where $n \in \mathbb{Z}$, while w belongs to the weight lattice of $\prod_{A \in \underline{\mathbf{6}}} U(1)^{\ell_A - 1}$.

The relations:

$$\begin{aligned}
 & e^{ie_a t} B_a = g_{t,\xi} B_a g_{t,\xi}^{-1}, \\
 (220) \quad & I_A(N_{A,\alpha}) e^{ic_A(\alpha)t} = g_{t,\xi} I_A(N_{A,\alpha}), \quad A \in \underline{\mathfrak{6}}^+, \alpha \in [n_A] \\
 & I_A(N_{A,\iota,\alpha}) e^{ic_{A,\iota}(\alpha)t} = g_{t,\xi} I_A(N_{A,\iota,\alpha}), \quad A \in \underline{\mathfrak{6}}^-, \alpha \in [n_A]_\iota
 \end{aligned}$$

imply:

$$\begin{aligned}
 & I_A(N_{A,\alpha}) \in K^{0,c_A(\alpha)}, \quad \alpha \in [n_A], \quad A \in \underline{\mathfrak{6}}^+, \\
 (221) \quad & I_A(N_{A,\iota_A,\alpha}) \in K^{0,c_{A,\iota_A}(\alpha)}, \quad \alpha \in [n_A]_{\iota_A}, \quad A \in \underline{\mathfrak{6}}^-, \\
 & I_A(N_{A,\iota,\alpha}) \in K^{\varpi_{A,\iota},c_{A,\iota}(\alpha)}, \quad \alpha \in [n_A]_\iota, \quad \iota \in \lambda_A, \quad A \in \underline{\mathfrak{6}}^- \\
 & B_a(K^{w,n}) \subset K^{w,n+e_a}, \quad a \in \underline{\mathfrak{4}}
 \end{aligned}$$

where $\varpi_{A,\iota}$ is the fundamental weight, $\langle \varpi_{A,\iota}, \xi \rangle = \xi_{A,\iota}$.

Finally, the $T_{\mathfrak{L}}$ -invariance translates to

$$\begin{aligned}
 (222) \quad & J_A(K^{w,n}) \in N_{A,\alpha}, \quad A \in \underline{\mathfrak{6}}^+, \alpha \in [n_A] \Leftrightarrow n = c_A(\alpha) - e_A, \quad w = 0, \\
 & J_A(K^{w,n}) \in N_{A,\iota_A,\alpha}, \quad A \in \underline{\mathfrak{6}}^-, \alpha \in [n_A]_{\iota_A} \Leftrightarrow n = c_{A,\iota_A}(\alpha) - e_A, \quad w = 0, \\
 & J_A(K^{w,n}) \in N_{A,\iota,\alpha}, \quad A \in \underline{\mathfrak{6}}^-, \alpha \in [n_A]_\iota \Leftrightarrow n = c_{A,\iota}(\alpha) - e_A, \quad w = \varpi_{A,\iota}.
 \end{aligned}$$

which imply, with our choice of $T_{\mathfrak{L}}$, that $J_A = 0$. This is shown using the same arguments as we used around the Eq. (213).

7.2.3. K -spaces for toric instantons. Let $A \in \underline{\mathfrak{6}}^+$. The local space K_A is $g_{t,\xi}$ -invariant, and decomposes as

$$(223) \quad K_A = \bigoplus_n K_A^n$$

with integer n , via

$$(224) \quad g_{t,\xi}|_{K_A^n} = e^{int}$$

where $n \geq c_A^- = \inf_{\alpha \in [n_A]} c_A(\alpha)$, when $e_a, e_b > 0$ and $n \leq c_A^+ = \sup_{\alpha \in [n_A]} c_A(\alpha)$ when $e_a, e_b < 0$. For $e_a, e_b > 0$ both operators $B_a, a \in A$ raise the grading. For $e_a, e_b < 0$ both operators $B_a, a \in A$ lower the grading. Let $k_{A,n} = \dim K_A^n$. Since K_A is finite dimensional, $k_{A,n}$ vanish for $|n| > C_A$ for some some constant $C_A \leq k$.

Let $A \in \underline{\mathfrak{6}}^-$. The local space K_A is $g_{t,\xi}$ -invariant, and decomposes as

$$(225) \quad K_A = \bigoplus_n K_A^n \oplus \bigoplus_{\iota \in \lambda_{A,n}} K_{A,\iota}^n$$

with

$$(226) \quad g_{t,\xi}|_{K_{A,\iota}^n} = e^{int+i\xi_{A,\iota}}$$

for $i \in \lambda_A$, and

$$(227) \quad g_{t,\xi}|_{K_A^n} = e^{int}$$

Since the eigenvalues of $g_{t,\xi}$ on $K_{A,\iota}^n$ for $\iota \in \lambda_A$ differ from each other and from those on $K_B^{n'}$ for all $B \in \underline{\mathfrak{6}}$, $n' \in \mathbb{Z}$, the spaces $K_{A,\iota}^n$ are orthogonal to $K_B^{n'}$ and to each other:

$$(228) \quad \begin{aligned} K_{A,\iota'}^{n'} &\perp K_{A,\iota''}^{n''}, & \iota' \neq \iota'' \\ K_{A,\iota}^{n'} &\perp K_A^{n''} \end{aligned}$$

The action of \mathbf{B}, \mathbf{I} -operators respects the orthogonal decomposition (228).

We now prove that the spaces K_A^n and $K_{A,i}^n$ have an additional $U(1)$ -action. Indeed, let f_a, f_b be the two positive mutually prime integers, such that

$$(229) \quad e_a f_a + e_b f_b = 0,$$

so that $e_a = p_{ab} f_b$, $e_b = -p_{ab} f_a$ (assuming $e_a > 0 > e_b$). Then the operator

$$(230) \quad \mathcal{E} = B_a^{f_a} B_b^{f_b}$$

commutes with $g_{t,\xi}$, thanks to (220). Since all the eigenvalues of B_a and B_b vanish (again, thanks to (220)), the operators B_a, B_b , and \mathcal{E} are nilpotent. By Jacobson-Morozov theorem, \mathcal{E} can be included into the sl_2 -triple, i.e. for each $K_A^n, K_{A,i}^n$ there are operators $\mathcal{H}, \mathcal{E}^\vee$, such that

$$(231) \quad [\mathcal{E}, \mathcal{E}^\vee] = \mathcal{H}, \quad [\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, \quad [\mathcal{H}, \mathcal{E}^\vee] = -2\mathcal{E}^\vee$$

so that

$$(232) \quad K_A^n = \bigoplus_h K_A^{n,h}, \quad K_{A,\iota}^n = \bigoplus_h K_{A,\iota}^{n,h}$$

with h standing for the eigenvalue of \mathcal{H} . Now, it is not difficult to prove that the (n, h) -grading is equivalent to the (i, j) -grading, with $i, j \geq 1$:

$$\begin{aligned}
 (233) \quad & i = i' + (h - h'_A(\alpha))f_a, \\
 & j = j' + (h - h'_A(\alpha))f_b, \\
 & n = c_A(\alpha) + p_A(f_b(i' - 1) - f_a(j' - 1)), \\
 & i', j' \geq 1, i' \leq f_a \text{ and/or } j' \leq f_b \\
 & h'_A(\alpha) = \inf \text{Spec}(\mathcal{H}|_{K_A^n})
 \end{aligned}$$

and α is uniquely determined by $n \bmod p_A = c_A(\alpha) \bmod p_A$. Thus,

$$(234) \quad K_A = \bigoplus_{i,j \geq 1} K_A^{i,j}, \quad K_{A,\ell} = \bigoplus_{i,j \geq 1} K_{A,\ell}^{i,j}$$

with

$$(235) \quad B_a(K_A^{i,j}) \subset K_A^{i+1,j}, \quad B_b(K_A^{i,j}) \subset K_A^{i,j+1}$$

Now we are ready for the final push:

8. The compactness theorem

We now prove the compactness theorem which establishes the analyticity of the partition function defined in the next chapter. To this end we estimate the norm of $(\mathbf{B}, \mathbf{I}, \mathbf{J})$ whose $U(k)$ -orbit is invariant with respect to the action of any minimal torus $T_{\mathbb{Q}}$.

Since J_A 's vanish, the real moment map equation reads as follows:

$$(236) \quad \sum_{a \in \mathbf{4}} [B_a, B_a^\dagger] + \sum_{A \in \mathbf{6}} I_A I_A^\dagger = \zeta \mathbf{1}_K$$

The trace of this equation gives the norm of I_A 's:

$$(237) \quad \sum_{A \in \mathbf{6}} \|I_A\|^2 = \zeta k$$

But we need to estimate the norms $\|B_a\|^2$ which drop out of trace. However, it is not too difficult to chase them down. We have:

$$\begin{aligned}
 (238) \quad \sum_{a \in \underline{4}} \|B_a\|^2 + \sum_{A \in \underline{6}} \|I_A\|^2 &= \sum_{a \in \underline{4}} \text{Tr}_K B_a B_a^\dagger + \sum_{A \in \underline{6}} \text{Tr}_K I_A I_A^\dagger \\
 &\leq \sum_{A \in \underline{6}} \text{Tr}_{K_A} \left(\sum_{a \in \underline{4}} B_a B_a^\dagger + \sum_{C \in \underline{6}} I_C I_C^\dagger \right) \\
 &= \sum_{A \in \underline{6}} \left(\zeta k_A + \text{Tr}_{K_A} \sum_{a \in \underline{4}} B_a^\dagger B_a \right) \\
 &= \zeta \sum_{A \in \underline{6}} k_A + \sum_{A \in \underline{6}, a \in A} \text{Tr}_{K_A} B_a^\dagger B_a
 \end{aligned}$$

where we used the moment map equation (59), projected onto K_A , and the Eq. (208). Define:

$$\begin{aligned}
 (239) \quad \delta_{A,n} &= \frac{1}{\zeta} \text{Tr}_{K_A^n} \left(\sum_{a \in \underline{4}} B_a B_a^\dagger + \sum_{A' \in \underline{6}} I_{A'} I_{A'}^\dagger \right) \\
 \delta_A &= \frac{1}{\zeta} \text{Tr}_{K_A} \left(\sum_{a \in \underline{4}} B_a B_a^\dagger + \sum_{A' \in \underline{6}} I_{A'} I_{A'}^\dagger \right) = \sum_n \delta_{A,n}
 \end{aligned}$$

Now for $A \in \underline{6}^+$ use the decomposition (223), and (208) to show, that for $e_a \neq e_b$:

$$(240) \quad \delta_{A,n} = k_{A,n} + \frac{1}{\zeta} \sum_{a \in A} \text{Tr}_{K_A^{n+e_a}} (B_a B_a^\dagger) \leq k_{A,n} + \sum_{a \in A} \delta_{A,n+e_a}$$

where we very conservatively estimated:

$$(241) \quad \text{Tr}_{K_A^n} (B_a B_a^\dagger) \leq \delta_{A,n},$$

for any $a \in \underline{4}$. This very conservative inequality can be used to show the boundness of $\delta_{A,n}$.

From now on let us assume $e_a > e_b > 0$. The case of negative e_a, e_b is treated analogously. First of all, let introduce the sequence of generalized

Fibonacci numbers $F_n^{p,q}$, $n \in \mathbb{Z}$, for positive integers $p > q > 0$, by:

$$\begin{aligned}
 (242) \quad & F_n^{p,q} = 0, & 1 - p \leq n \leq 0 \\
 & F_1^{p,q} = 1, \\
 & F_n^{p,q} = F_{n-p}^{p,q} + F_{n-q}^{p,q}, \quad n > 0
 \end{aligned}$$

It is easy to write the formula for $F_n^{p,q}$ in terms of the roots λ_ω , $\omega = 0, \dots, p - 1$, of the characteristic equation

$$(243) \quad \lambda_\omega^p = \lambda_\omega^{p-q} + 1,$$

$$(244) \quad F_n^{p,q} = \sum_\omega f_\omega \lambda_\omega^n$$

where the coefficients f_ω are to be found from the linear equations $F_n^{p,q} = \delta_{n,1}$, $1 - p \leq n \leq 1$.

Now, we can estimate $\delta_{A,n}$ by induction in n :

$$(245) \quad \delta_{A,n} \leq \sum_{n' \geq n} k_{A,n'} F_{n'+1-n}^{e_a, e_b}$$

This leads to the following, also very conservative, bound on δ_A :

$$(246) \quad \delta_A < k_A F_{k_A}^{e_a, e_b}$$

When $e_a = e_b = e_A/2 \geq 1$ we can make a better estimate:

$$(247) \quad \delta_{A,n} = k_{A,n} + \frac{1}{\zeta} \sum_{a \in A} \text{Tr}_{K_A^{n+e_a}} (B_a B_a^\dagger) \leq k_{A,n} + \delta_{A,n+e_A/2}$$

which, by iteration, implies:

$$(248) \quad \delta_{A,n} \leq k_{A,n} + k_{A,n+e_A/2} + k_{A,n+e_A} + \dots$$

which in turns implies the upper bound on

$$(249) \quad \delta_A \leq \frac{1}{e_A} k_A (k_A + 1)$$

It remains to estimate δ_A for $A \in \underline{\mathfrak{6}}^-$. This is easy to do using the (i, j) -grading (234). Define:

$$(250) \quad \delta_{A,n} = \frac{1}{\zeta} \sum_{i+j=n+2} \text{Tr}_{K_A^{i,j}} \left(\sum_{a \in \underline{\mathfrak{4}}} B_a B_a^\dagger + \sum_{A' \in \underline{\mathfrak{6}}} I_{A'} I_{A'}^\dagger \right)$$

Then, using (59), projected onto $K_A^{i,j}$, and (235), we derive the estimate:

$$(251) \quad \delta_{A,n} \leq \delta_{A,n+1} + \sum_{i+j=n+2} \dim K_A^{i,j}$$

from which we get the estimate:

$$(252) \quad \delta_A \leq \frac{1}{2} k_A (k_A + 1)$$

9. Integration over the spiked instantons

The moduli spaces $\mathfrak{M}_k^i(\vec{n})$ are not your favorite smooth varieties. They can be stratified by smooth varieties of various dimensions. Over these smooth components the obstruction bundles keep track of the non-genericity of the equations we used to define the spaces $\mathfrak{M}_k^i(\vec{n})$.

In applications we need to compute the integrals over the spaces $\mathfrak{M}_k^\infty(\vec{n})$, as well as to define and compute the equivariant indices of various twisted Dirac operators (for five dimensional theories compactified on a circle).

Mathematically one can take the so-called virtual approach [12], where the fundamental cycle $[\mathfrak{M}_k^\infty(\vec{n})]$ is replaced by the virtual fundamental cycle $[\mathfrak{M}_k^\infty(\vec{n})]^{\text{vir}}$, which is defined as the Euler class of the bundle of equations over the smooth variety of the original variables $(\mathbf{B}, \mathbf{I}, \mathbf{J})$. There are two difficulties with this definition: *i*) the space of $(\mathbf{B}, \mathbf{I}, \mathbf{J})$, being a vector space, is non-compact; *ii*) the bundle of equations is infinite dimensional, unless we are in the situation with only the crossed or ordinary instantons.

The problem *i*) is solved by passing to the equivariant cohomology. The problem *ii*) is cured by working with $\mathfrak{M}_k^i(\vec{n})$ for large but finite i , and then regularize the limit $i \rightarrow \infty$ by using the Γ -functions.

Physically, the problem is solved by the considerations of the matrix integral (matrix quantum mechanics) of the system of k $D(-1)$ -branes (k $D0$ -branes whose worldlines wrap S^1) in the vicinity of six stacks of $D3$ branes ($D4$ branes) wrapping coordinate two-planes \mathbb{C}_A^2 (times a circle S^1) in the *IIB* (*IIA*) background $\mathbb{R}^2 \times \mathbb{C}^4$ ($\mathbb{R}^1 \times S^1 \times \mathbb{C}^4$).

One can also define the elliptic genus by the study of the two-dimensional gauge theory corresponding to the stack of k $D1$ -strings wrapping a \mathbf{T}^2 shared by six stacks of $D5$ branes in *IIB* string theory, wrapping $\mathbf{T}^2 \times \mathbb{C}_A^2$.

9.1. Cohomological field theory

Let us briefly recall the physical approach. For every variable, i.e. for every matrix element of the matrices B_a, I_A, J_A , we introduce the fermionic variables $\Psi_a^{\mathbf{B}}, \Psi_A^{\mathbf{I}}, \Psi_A^{\mathbf{J}}$ with the same $\mathbf{H} \times U(K)$ transformation properties.

For every equation $s_A, \mu, \sigma_{\bar{a}A}, \Upsilon_A, \Upsilon_{A,B;i}$ we introduce a pair of fermion-boson variables valued in the dual space: $(\chi_A, h_A), (\chi, \mu), (\chi_{\bar{a}A}, h_{\bar{a}A}), (\xi_A, r_A), (\varpi_{A',A'';i}, y_{A',A'';i})$.

Finally, we need a triplet of variables $(\sigma, \bar{\sigma}, \eta)$ (two bosons and a fermion), valued in the Lie algebra of $U(K)$.

Our model has the fermionic symmetry acting by:

$$(253) \quad \begin{aligned} \delta B_a &= \Psi_a^{\mathbf{B}}, & \delta \Psi_a^{\mathbf{B}} &= -[\sigma, B_a] + \mathbf{e}_a B_a \\ \delta I_A &= \Psi_A^{\mathbf{I}}, & \delta \Psi_A^{\mathbf{I}} &= -\sigma I_A + I_A \mathbf{a}_A \\ \delta J_A &= \Psi_A^{\mathbf{J}}, & \delta \Psi_A^{\mathbf{J}} &= -\mathbf{a}_A J_A + J_A \sigma + \mathbf{e}_A J_A \end{aligned}$$

(cf. (209)) and

$$(254) \quad \begin{aligned} \delta \chi_A &= h_A, & \delta h_A &= -[\sigma, \chi_A] + \mathbf{e}_A \chi_A, \\ \delta \chi &= h, & \delta h &= -[\sigma, \chi], \\ \delta \chi_{\bar{a}A} &= h_{\bar{a}A}, & \delta h_{\bar{a}A} &= -\mathbf{a}_A \chi_{\bar{a}A} + \chi_{\bar{a}A} \sigma - \mathbf{e}_{\bar{a}} \chi_{\bar{a}A}, \\ \delta \xi_A &= r_A, & \delta r_A &= \mathbf{a}_{\bar{A}} \xi_A - \xi_A \mathbf{a}_A + \mathbf{e}_A \xi_A, \\ \delta \varpi_{A',A'';i} &= y_{A',A'';i}, \\ \delta y_{A',A'';i} &= -\mathbf{a}_{A'} \varpi_{A',A'';i} + \varpi_{A',A'';i} \mathbf{a}_{A''} + (i-1) \mathbf{e}_a \varpi_{A',A'';i}, \\ \text{whenever } & A' \cap A'' = \{a\} \end{aligned}$$

and

$$(255) \quad \delta \bar{\sigma} = \eta, \quad \delta \eta = [\sigma, \bar{\sigma}], \quad \delta \sigma = 0$$

Now we can define the partition function

$$(256) \quad \mathcal{Z}_k^i(\vec{n}; \underline{\mathbf{e}}, \underline{\mathbf{a}}) = \int e^{-\mathbf{S}_i} \frac{D\sigma}{\text{Vol}(U(k))} D\mathbf{B} D\mathbf{B}^\dagger D\Psi^{\mathbf{B}} D\Psi^{\mathbf{B}\dagger} \dots D\chi D h \dots D\bar{\sigma} D\eta \\ \times \prod_{j=1}^{i-1} D\chi_{A,B;j} D h_{A,B;j} D\bar{\chi}_{A,B;j} D\bar{h}_{A,B;j}$$

where

$$\begin{aligned}
 \mathbf{S}_i &= \delta \Xi_i, \\
 \Xi_i &= \Xi^s + \Xi^f + \Xi^K + \sum_A \Xi_A^N + \sum_{A', A''; \#A' \cap A'' = 1} \sum_{j=1}^{i-1} \Xi_{A', A'', j}^N, \\
 \Xi^s &= \text{Tr}_K \eta[\sigma, \bar{\sigma}], \\
 \Xi^f &= \text{Tr}_K \sum_a \left(\Psi_a^{\mathbf{B}} \left(-[\bar{\sigma}, B_a^\dagger] + \bar{\varepsilon}_a B_a^\dagger \right) + c.c. \right) \\
 &\quad + \text{Tr}_K \sum_A \left((-\bar{\sigma} I_A + I_A \bar{\mathbf{a}}_A) \Psi_A^{\mathbf{I}^\dagger} + c.c. \right) \\
 (257) \quad &\quad + \text{Tr}_K \sum_A \left(\Psi_A^{\mathbf{J}^\dagger} (J_A \bar{\sigma} - \bar{\mathbf{a}}_A J_A + \bar{\varepsilon}_A J_A) + c.c. \right), \\
 \Xi^K &= \text{Tr}_K \chi(-i(\mu - \zeta \mathbf{1}_K) + h) \\
 &\quad + \sum_A \left(\chi_{\bar{A}}(-i s_A + h_A) + \sum_{\bar{a} \in \bar{A}} \chi_{\bar{a}A}^\dagger(-i \sigma_{\bar{a}A} + h_{\bar{a}A}) \right). \\
 \Xi_A^N &= \text{Tr}_{N_A} \left(\xi_A^\dagger(-i \Upsilon_A + r_A) + c.c. \right), \\
 \Xi_{A', A'', j}^N &= \text{Tr}_{N_A} \left(\varpi_{A', A'', j}^\dagger(-i \Upsilon_{A', A'', j} + y_{A', A'', j}) + c.c. \right)
 \end{aligned}$$

Here $\bar{\varepsilon}_a, \bar{\mathbf{a}}_A$ are auxiliary elements of the Lie algebra of \mathbf{H} , which can be chosen arbitrary, as long as the integral (256) converges.

9.2. Localization and analyticity

The usual manipulations with the integral (256), for generic $(\underline{\mathfrak{e}}, \underline{\mathfrak{a}})$, express it as a sum over the fixed points, which we enumerated in the Eq. (216). Each fixed point contributes a homogeneous (degree zero) rational function of $\mathfrak{a}_{A, \alpha}$'s, $1 \leq \alpha \leq n_A$ and \mathfrak{e}_a , times the product

$$\begin{aligned}
 (258) \quad & \prod_{A \in \underline{\mathfrak{G}}, 4 \in \bar{A}} \prod_{\alpha=1}^{n_A} \prod_{\beta=1}^{n_{\bar{A}}} (\mathfrak{a}_{A, \alpha} - \mathfrak{a}_{\bar{A}, \beta} + \mathfrak{e}_{\bar{A}}) \\
 & \times \prod_{A, B \in \underline{\mathfrak{G}}, A \cap B = \{c\}} \prod_{\alpha=1}^{n_A} \prod_{\beta=1}^{n_B} \prod_{j=1}^{i-1} (\mathfrak{a}_{A, \alpha} - \mathfrak{a}_{B, \beta} + \mathfrak{e}_c(j-1))
 \end{aligned}$$

The compactness theorem of the previous chapter implies, among other things, that the partition functions $\mathcal{Z}_k^i(\vec{n}; \underline{\mathfrak{e}}, \underline{\mathfrak{a}})$, for $i > k$, have no singularities

in

$$(259) \quad \mathbf{x}_A = \frac{1}{n_A} \sum_{\alpha=1}^{n_A} \mathbf{a}_{A,\alpha}$$

with fixed \mathbf{e}_a 's and $\tilde{\mathbf{a}}_{A,\alpha} = \mathbf{a}_{A,\alpha} - \mathbf{x}_A$. In other words, they are polynomials of \mathbf{x}_A .

10. Quiver crossed instantons

10.1. Crossed quivers

For oriented graph γ let V_γ denote the set of its vertices, and E_γ the set of edges, with $s, t : E_\gamma \rightarrow V_\gamma$ the source and the target maps. Sometimes we write $\gamma = (V_\gamma, E_\gamma, s, t)$. The crossed quiver is the data $\mathfrak{X} = (\gamma^+, \gamma^-, p)$, where γ^\pm are two oriented graphs, and $p \in \mathbb{Z}_{\geq 0}$ is a non-negative integer. Let $\Xi_p = \mathbb{Z}/p\mathbb{Z}$ be the additive group with p elements, for $p > 0$ and \mathbb{Z} for $p = 0$. Define $\mathfrak{V}_\mathfrak{X} = V_{\gamma^+} \times V_{\gamma^-} \times \Xi_p$. The group Ξ_p acts on $\mathfrak{V}_\mathfrak{X}$ by translations of the third factor. The generator of Ξ_p acts by $\omega = (v^+, v^-, n) \mapsto \omega + 1 \equiv (v^+, v^-, n + 1)$, with $v^\pm \in V_{\gamma^\pm}$. We also define ${}^\pm \mathfrak{E}_\mathfrak{X} = E_{\gamma^\pm} \times V_{\gamma^\mp} \times \Xi_p$ and the natural maps $s, t : {}^\pm \mathfrak{E}_\mathfrak{X} \rightarrow \mathfrak{V}_\mathfrak{X}$, e.g. $s(e, u, n) = (s(e), u, n)$ for $e \in E_{\gamma^+}, u \in V_{\gamma^-}$, $t(e, v, n) = (v, t(e), n)$ for $e \in E_{\gamma^-}, v \in V_{\gamma^+}$ etc. The group Ξ_p also acts on ${}^\pm \mathfrak{E}_\mathfrak{X}$, so we shall write: $\eta = (e, u, n) \mapsto \eta \pm 1 \equiv (e, u, n \pm 1)$. The source and target maps are Ξ_p -equivariant, i.e. $s(\eta \pm 1) = s(\eta) \pm 1$.

10.1.1. Paths and integrals. We shall use the notion of a path. Define the path $\mathbf{p}_{\omega', \omega''}^\pm$ of length ℓ to be a sequence of pairs:

$$(260) \quad \mathbf{p}_{\omega', \omega''}^\pm = (\eta_1, \text{or}_1), (\eta_2, \text{or}_2), \dots, (\eta_\ell, \text{or}_\ell)$$

with $\eta_j \in {}^\pm \mathfrak{E}_\mathfrak{X}, \text{or}_j \in \{-1, +1\}$, (we call or_j the orientation of the edge η_j relative to $\mathbf{p}_{\omega', \omega''}^\pm$) such that for any $j = 1, \dots, \ell - 1$ either

$$(261) \quad \begin{aligned} s(\eta_{j+1}) &= t(\eta_j \pm 1), & \text{or}_j &= \text{or}_{j+1} = 1, \\ \text{or } t(\eta_{j+1}) &= t(\eta_j), & \text{or}_j &= -\text{or}_{j+1} = 1, \\ \text{or } s(\eta_{j+1}) &= s(\eta_j), & -\text{or}_j &= \text{or}_{j+1} = 1, \\ \text{or } t(\eta_{j+1}) &= s(\eta_j \mp 1), & -\text{or}_j &= -\text{or}_{j+1} = 1, \end{aligned}$$

and also either $s(\eta_1) = \omega'$ ($\text{or}_1 = 1$) or $t(\eta_1 \pm 1) = \omega'$ ($\text{or}_1 = -1$) and also either $s(\eta_\ell) = \omega''$ ($\text{or}_\ell = -1$) or $t(\eta_\ell \pm 1) = \omega''$ ($\text{or}_\ell = 1$).

For a function $b : \pm \mathfrak{E}_x \rightarrow \mathbb{R}$ (a 1-chain) and a path $p_{\omega', \omega''}^\pm$ define the integral

$$(262) \quad \int_{p_{\omega', \omega''}^\pm} b = \sum_{j=1}^{\ell} \text{or}_j b(\eta_j)$$

The function $b : \pm \mathfrak{E}_x \rightarrow \mathbb{R}$ is a coboundary, $b = \delta c$, of a 0-chain $c : \mathfrak{V}_x \rightarrow \mathbb{R}$, iff $b(\eta) = c(t(\eta \pm 1)) - c(s(\eta))$. The integral of a coboundary obeys Stokes formula:

$$(263) \quad \int_{p_{\omega', \omega''}^\pm} \delta c = \sum_{j=1}^{\ell} \text{or}_j (c(t(\eta_j \pm 1)) - c(s(\eta_j))) = c(\omega'') - c(\omega')$$

10.1.2. Representations of crossed quivers. Fix four dimension vectors $\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-, \mathbf{m} : \mathfrak{V}_x \rightarrow \mathbb{Z}_{\geq 0}$. Let $(K_\omega, N_\omega^\pm, M_\omega)_{\omega \in \mathfrak{V}_x}$ be a collection of complex vector spaces whose dimensions are given by the components of the corresponding dimension vectors, e.g. $\dim K_\omega = k_\omega \equiv \mathbf{k}(\omega)$. We view the spaces N_ω^\pm, M_ω as fixed, e.g. with some fixed basis, while the spaces K_ω are varying, i.e. defined up to the automorphisms. We also fix a decomposition $M_\omega = M'_\omega \oplus M''_\omega$ as an additional refinement of our structure.

10.1.3. Weight assignment for crossed quivers. For the crossed quiver γ and its representation let us fix the integral data: $(\mathbf{n}_\omega^\pm, \mathbf{m}'_\omega, \mathbf{m}''_\omega)_{\omega \in \mathfrak{V}_\gamma}$, $\mathbf{t} = (t(\omega))_{\omega \in \mathfrak{V}_x}$ and $\mathbf{e} = (e(\eta))_{\eta \in \mathfrak{E}_x}$, with integers $t(\omega), e(\eta) \in \mathbb{Z}$, obeying $t(\omega + 1) = t(\omega)$, $e(\eta + 1) = e(\eta)$, and the integral vectors $\mathbf{n}_\omega^\pm = (n_{\omega, \alpha}^\pm)_{\alpha=1}^{n_\omega^\pm} \in \mathbb{Z}^{n_\omega^\pm}$ etc. The data $(\mathbf{n}^\pm, \mathbf{e}, \mathbf{m}', \mathbf{m}'')$ is defined up to the action of the lattice $\mathbb{Z}^{\mathfrak{V}_\gamma}$: a function $f : \mathfrak{V}_\gamma \rightarrow \mathbb{Z}$ shifts the data (289) by:

$$(264) \quad n_{\omega, \alpha}^\pm \mapsto n_{\omega, \alpha}^\pm - f(\omega), \quad e \mapsto e + \delta f$$

10.1.4. Crossed quiver instantons. Consider the vector superspace $\mathcal{A}_{\mathbf{k}}^\gamma(\mathbf{n}^\pm, \mathbf{m})$ of linear maps $(\mathbf{B}, \mathbf{I}, \mathbf{J}, \Theta)$

$$(265) \quad \begin{aligned} \text{bosons : } \mathbf{B} &= (B_\eta^\pm, \tilde{B}_\eta^\pm)_{\eta \in \pm \mathfrak{E}_x}, \quad \mathbf{I} = (I_\omega^+, I_\omega^-)_{\omega \in \mathfrak{V}_x}, \\ \mathbf{J} &= (J_\omega^+, J_\omega^-)_{\omega \in \mathfrak{V}_x}, \\ B_\eta^\pm &: K_{s(\eta)} \rightarrow K_{t(\eta) \pm 1}, \quad \tilde{B}_\eta^\pm : K_{t(\eta)} \rightarrow K_{s(\eta) \pm 1}, \quad \eta \in \pm \mathfrak{E}_x \\ I_\omega^\pm &: N_\omega^\pm \rightarrow K_{\omega \pm 1}, \quad J_\omega^\pm : K_\omega \rightarrow N_{\omega \pm 1}^\pm, \quad \omega \in \mathfrak{V}_x \\ \text{fermions : } \Theta &= (\Theta'_\omega, \Theta''_\omega)_{\omega \in \mathfrak{V}_x}, \\ \Theta'_\omega &\in \text{PIHom}(K_\omega, M'_\omega), \quad \Theta''_\omega \in \text{PIHom}(M''_{\omega-1}, K_\omega) \end{aligned}$$

Let $\mathcal{G}_{\underline{\mathbf{k}}}^\gamma, \mathcal{G}_{\underline{\mathbf{k}}}^{\mathbb{C}\gamma}$ be the groups:

$$(266) \quad \mathcal{G}_{\underline{\mathbf{k}}}^\gamma = \prod_{\omega \in \mathfrak{X}} U(k_\omega), \quad \mathcal{G}_{\underline{\mathbf{k}}}^{\mathbb{C}\gamma} = \prod_{\omega \in \mathfrak{X}} GL(k_\omega, \mathbb{C}),$$

which act on $\mathcal{A}_{\underline{\mathbf{k}}}^\gamma(\underline{\mathbf{n}}^\pm, \underline{\mathbf{m}})$ via:

$$(267) \quad (g_\omega)_{\omega \in \mathfrak{X}} \cdot (\mathbf{B}, \mathbf{I}, \mathbf{J}, \Theta) = \left(g_{t(\eta) \pm 1} B_\eta^\pm g_{s(\eta)}^{-1}, g_{s(\eta) \pm 1} \tilde{B}_\eta^\pm g_{t(\eta)}^{-1}; g_{\omega \pm 1} I_\omega^\pm, J_\omega^\pm g_\omega^{-1}; \Theta'_\omega g_\omega^{-1}, g_\omega \Theta''_\omega \right)$$

We impose the following analogues of the Eqs. (56):

$$(268) \quad \begin{aligned} \mu_\omega &= \zeta_\omega \mathbf{1}_{K_\omega}, & \omega \in \mathfrak{X}, & \quad \zeta_\omega > 0 \\ s_{e^+, e^-; n} &= 0, & e^\pm \in \mathbf{E}_{\gamma^\pm}, & \quad n \in \Xi_p \\ \tilde{s}_{e^+, e^-; n} &= 0, & e^\pm \in \mathbf{E}_{\gamma^\pm}, & \quad n \in \Xi_p \\ s_\omega &= 0, & \omega \in \mathfrak{X} \\ \Sigma_\eta &= 0, & \eta \in \mathfrak{E}_\mathfrak{X}, \\ \tilde{\Sigma}_\eta &= 0, & \eta \in \mathfrak{E}_\mathfrak{X}, \\ \Upsilon_\omega &= 0, & \omega \in \mathfrak{X}, \end{aligned}$$

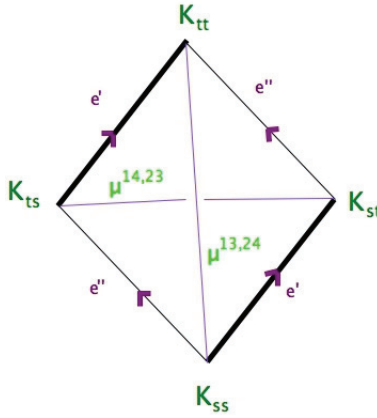
where

$$(269) \quad \begin{aligned} \Sigma. &= \sigma. + \hat{\sigma}.^\dagger, \\ \tilde{\Sigma}. &= \tilde{\sigma}. - \hat{\tilde{\sigma}}.^\dagger, \\ s_{...} &= \mu_{...}^{13} - (\mu_{...}^{24})^\dagger, \\ \tilde{s}_{...} &= \mu_{...}^{14} + (\mu_{...}^{23})^\dagger, \\ s_{...} &= \mu_{...}^{12} + (\mu_{...}^{34})^\dagger, \\ \Upsilon. &= v_{.-1}^+ - (v_{.+1}^-)^\dagger \end{aligned}$$

with the linear maps

$$(270) \quad \begin{aligned} \sigma_\eta &: N_{s(\eta)}^\pm \rightarrow K_{t(\eta)}, & \text{for } \eta \in {}^\mp \mathfrak{E}_\mathfrak{X} \\ \tilde{\sigma}_\eta &: N_{t(\eta)}^\pm \rightarrow K_{s(\eta)}, & \text{for } \eta \in {}^\mp \mathfrak{E}_\mathfrak{X} \\ v_{\omega \mp 1}^\pm &: N_\omega^\mp \rightarrow N_\omega^\pm, \end{aligned}$$

and



$$\begin{aligned}
 & \mu_\omega : K_\omega \rightarrow K_\omega , \\
 & \mu_\omega^{12} : K_\omega \rightarrow K_{\omega+2} , \\
 & \mu_\omega^{34} : K_\omega \rightarrow K_{\omega-2} , \\
 (271) \quad & \mu_{e^+,e^-;n}^{13} : K_{s(e^+),s(e^-),n} \rightarrow K_{t(e^+),t(e^-),n} , \\
 & \mu_{e^+,e^-;n}^{24} : K_{t(e^+),t(e^-),n} \rightarrow K_{s(e^+),s(e^-),n} , \\
 & \mu_{e^+,e^-;n}^{14} : K_{s(e^+),t(e^-),n} \rightarrow K_{t(e^+),s(e^-),n} , \\
 & \mu_{e^+,e^-;n}^{23} : K_{t(e^+),s(e^-),n} \rightarrow K_{s(e^+),t(e^-),n} .
 \end{aligned}$$

The maps (270), for $\eta \in \pm \mathfrak{E}_x$, are given by:

$$\begin{aligned}
 (272) \quad & \sigma_\eta = B_{\eta\pm 1}^\mp I_{s(\eta)}^\pm , \quad \widehat{\sigma}_\eta = J_{s(\eta)\pm 1}^\mp \widetilde{B}_\eta^\pm , \\
 & \widetilde{\sigma}_\eta = \widetilde{B}_{\eta\pm 1}^\mp I_{t(\eta)}^\pm , \quad \widehat{\widetilde{\sigma}}_\eta = J_{t(\eta)\pm 1}^\mp B_\eta^\pm ,
 \end{aligned}$$

and

$$(273) \quad v_{\omega\mp 1}^\pm = J_{\omega\mp 1}^\pm I_\omega^\mp$$

The crossed quiver analogues (271) of real and complex moment maps are given by: for $\omega \in \mathbb{V}_\gamma$,

$$\begin{aligned}
 (274) \quad \mu_\omega &= I_{\omega-1}^+ (I_{\omega-1}^+)^{\dagger} + I_{\omega+1}^- (I_{\omega+1}^-)^{\dagger} - (J_\omega^+)^{\dagger} J_\omega^+ - (J_\omega^-)^{\dagger} J_\omega^- \\
 &\quad - \sum_{\eta \in s^{-1}(\omega) \cap^+ \mathfrak{E}_x} (B_\eta^+)^{\dagger} B_\eta^+ - \sum_{\eta \in t^{-1}(\omega) \cap^+ \mathfrak{E}_x} (\tilde{B}_\eta^+)^{\dagger} \tilde{B}_\eta^+ \\
 &\quad - \sum_{\eta \in s^{-1}(\omega) \cap^- \mathfrak{E}_x} (B_\eta^-)^{\dagger} B_\eta^- - \sum_{\eta \in t^{-1}(\omega) \cap^- \mathfrak{E}_x} (\tilde{B}_\eta^-)^{\dagger} \tilde{B}_\eta^- \\
 &\quad + \sum_{\eta \in s^{-1}(\omega) \cap^+ \mathfrak{E}_x} \tilde{B}_{\eta-1}^+ (\tilde{B}_{\eta-1}^+)^{\dagger} + \sum_{\eta \in s^{-1}(\omega) \cap^- \mathfrak{E}_x} \tilde{B}_{\eta+1}^- (\tilde{B}_{\eta+1}^-)^{\dagger} \\
 &\quad + \sum_{\eta \in s^{-1}(\omega) \cap^+ \mathfrak{E}_x} B_{\eta-1}^+ (B_{\eta-1}^+)^{\dagger} + \sum_{\eta \in s^{-1}(\omega) \cap^- \mathfrak{E}_x} B_{\eta+1}^- (B_{\eta+1}^-)^{\dagger},
 \end{aligned}$$

and

$$\begin{aligned}
 (275) \quad \mu_\omega^{12} &= \begin{array}{c} \text{---} \nearrow \text{---} \searrow \text{---} \\ \text{---} \nwarrow \text{---} \nearrow \text{---} \end{array} - \begin{array}{c} \text{---} \nwarrow \text{---} \nearrow \text{---} \\ \text{---} \nearrow \text{---} \searrow \text{---} \end{array} \\
 &= I_{\omega+1}^+ J_\omega^+ + \sum_{\eta \in +\mathfrak{E}_x \cap t^{-1}(\omega)} B_{\eta+1}^+ \tilde{B}_\eta^+ - \sum_{\eta \in +\mathfrak{E}_x \cap s^{-1}(\omega)} \tilde{B}_{\eta+1}^+ B_\eta^+
 \end{aligned}$$

$$\begin{aligned}
 (276) \quad \mu_\omega^{34} &= \begin{array}{c} \text{---} \nwarrow \text{---} \nearrow \text{---} \\ \text{---} \nearrow \text{---} \searrow \text{---} \end{array} - \begin{array}{c} \text{---} \nearrow \text{---} \searrow \text{---} \\ \text{---} \nwarrow \text{---} \nearrow \text{---} \end{array} \\
 &= I_{\omega-1}^- J_\omega^- + \sum_{\eta \in -\mathfrak{E}_x \cap t^{-1}(\omega)} B_{\eta-1}^- \tilde{B}_\eta^- - \sum_{\eta \in -\mathfrak{E}_x \cap s^{-1}(\omega)} \tilde{B}_{\eta-1}^- B_\eta^-
 \end{aligned}$$

for $e^\pm \in \mathbb{E}_{\gamma^\pm}$, $n \in \Xi_p$,

$$\begin{aligned}
 (277) \quad \mu_{e^+, e^-; n}^{13} &= \begin{array}{c} \text{---} \nearrow \text{---} \searrow \text{---} \\ \text{---} \nwarrow \text{---} \nearrow \text{---} \end{array} - \begin{array}{c} \text{---} \nwarrow \text{---} \nearrow \text{---} \\ \text{---} \nearrow \text{---} \searrow \text{---} \end{array} \\
 &= B_{e^+, t(e^-), n-1}^+ B_{e^-, s(e^+), n}^- - B_{e^-, t(e^+), n+1}^- B_{e^+, s(e^-), n}^+
 \end{aligned}$$

$$(278) \quad \mu_{e^+,e^-;n}^{24} = \begin{array}{c} \begin{array}{ccc} & \nearrow & \searrow \\ + & & - \\ & \nwarrow & \swarrow \end{array} & - & \begin{array}{ccc} & \nearrow & \searrow \\ - & & + \\ & \nwarrow & \swarrow \end{array} \\ \\ \tilde{B}_{e^+,s(e^-),n-1}^+ \tilde{B}_{e^-,t(e^+),n}^- - \tilde{B}_{e^-,s(e^+),n+1}^- \tilde{B}_{e^+,t(e^-),n}^+ \end{array}$$

$$(279) \quad \mu_{e^+,e^-;n}^{23} = \begin{array}{c} \begin{array}{ccc} & \nearrow & \searrow \\ + & & - \\ & \nwarrow & \swarrow \end{array} & - & \begin{array}{ccc} & \nearrow & \searrow \\ - & & + \\ & \nwarrow & \swarrow \end{array} \\ \\ \tilde{B}_{e^+,t(e^-),n-1}^+ B_{e^-,t(e^+),n}^- - B_{e^-,s(e^+),n+1}^- \tilde{B}_{e^+,s(e^-),n}^+ \end{array}$$

$$(280) \quad \mu_{e^+,e^-;n}^{14} = \begin{array}{c} \begin{array}{ccc} & \nearrow & \searrow \\ + & & - \\ & \nwarrow & \swarrow \end{array} & - & \begin{array}{ccc} & \nearrow & \searrow \\ - & & + \\ & \nwarrow & \swarrow \end{array} \\ \\ B_{e^+,s(e^-),n-1}^+ \tilde{B}_{e^-,s(e^+),n}^- - \tilde{B}_{e^-,t(e^+),n+1}^- B_{e^+,t(e^-),n}^+ \end{array}$$

The moduli space of quiver crossed instantons $\mathfrak{M}_{\mathbf{k}}^\gamma(\mathbf{n}^\pm, \mathbf{m})$ is the space of solutions to (268) modulo the action (266) of $\mathcal{G}_{\mathbf{k}}^\gamma$.

The identity

$$(281) \quad \begin{aligned} & \sum_{\eta \in \mathbf{E}_\gamma} \text{Tr}_{K_{t(\eta)}} (\sigma_\eta \hat{\sigma}_\eta) - \text{Tr}_{K_{s(\eta)}} (\tilde{\sigma}_\eta \hat{\tilde{\sigma}}_\eta) \\ & + \sum_{\omega \in \mathbf{V}_\gamma} \text{Tr}_{N_\omega^+} (v_{\omega-1}^+ v_{\omega+1}^-) + \text{Tr}_{K_\omega} (\mu_{\omega-2}^{12} \mu_\omega^{34}) \\ & + \sum_{e^\pm \in \mathbf{E}_{\gamma^\pm}, n \in \Xi_p} \text{Tr}_{K_{t(e^+),s(e^-),n}} (\mu_{e^+,e^-;n}^{14} \mu_{e^+,e^-;n}^{23}) \\ & - \text{Tr}_{K_{t(e^+),t(e^-),n}} (\mu_{e^+,e^-;n}^{13} \mu_{e^+,e^-;n}^{24}) \\ & = 0 \end{aligned}$$

can be used to demonstrate, by the argument identical to that in (54), that the equations (268) imply the holomorphic equations

$$(282) \quad \begin{aligned} & \sigma_\eta = 0, \quad \tilde{\sigma}_\eta = 0, \quad \hat{\sigma}_\eta = 0, \quad \hat{\tilde{\sigma}}_\eta = 0, \\ & \mu_\omega^{12} = 0, \quad \mu_\omega^{34} = 0, \quad \mu_{e^+,e^-;n}^{13} = 0, \\ & \mu_{e^+,e^-;n}^{14} = 0, \quad \mu_{e^+,e^-;n}^{23} = 0, \quad \mu_{e^+,e^-;n}^{24} = 0, \\ & v_\omega^\pm = 0 \end{aligned}$$

Thus, $\mathfrak{M}_{\mathbf{k}}^\gamma(\underline{\mathbf{n}}^\pm, \underline{\mathbf{m}})$ is the space of stable solutions of (282) modulo the action (266) of $\mathcal{G}_{\mathbf{k}}^{\mathbb{C}\gamma}$. Here, the stability condition is formulated as follows:

Any collection $(K'_\omega)_{\omega \in \mathfrak{V}_x}$ of subspaces $K'_\omega \subset K_\omega$, such that

$$I_\omega^\pm(N_\omega^\pm) \subset K'_{\omega \pm 1}$$

and

$$B_\eta^\pm(K'_{s(\eta)}) \subset K'_{t(\eta) \pm 1}, \quad \text{for all } \eta \in \pm \mathfrak{E}_x$$

$$\tilde{B}_\eta^\pm(K'_{t(\eta)}) \subset K'_{s(\eta) \pm 1}, \quad \text{for all } \eta \in \pm \mathfrak{E}_x$$

coincides with all of K_ω , $K'_\omega = K_\omega$,

in other words,

$$K_\omega = K_\omega^+ + K_\omega^-$$

(283)

where $K_\omega^\pm \subset K_\omega$ is the subspace, generated by acting with arbitrary (non-commutative) polynomials in $B_\eta^\pm, \tilde{B}_\eta^\pm$'s with $\eta \in \pm \mathfrak{E}_x$ on the image $\sum_{\omega' \in \mathfrak{V}_x} I_{\omega'}^\pm(N_{\omega'}^\pm)$:

$$(284) \quad K_\omega^\pm = \sum_{\omega' \in \mathfrak{V}_x} \left(\mathbb{C} \left[B_\eta^\pm, \tilde{B}_\eta^\pm \right]_{\eta \in \pm \mathfrak{E}_x} I_{\omega'}^\pm(N_{\omega'}^\pm) \right) \cap K_\omega$$

The space $\mathfrak{M}_{\mathbf{k}}^\gamma(\underline{\mathbf{n}}^\pm, \underline{\mathbf{m}})$ is acted upon by the group

$$(285) \quad \mathbf{H}^\gamma = \left(\mathbf{H}_{\text{ff}}^\gamma \times \mathbf{H}_{\text{edg}}^\gamma \right) / \mathbf{H}_{\text{ver}}^\gamma \times U(1)_u,$$

where

$$(286) \quad \begin{aligned} \mathbf{H}_{\text{ff}}^\gamma &= \prod_{\omega \in \mathfrak{V}_\gamma} U(N_\omega^+) \times U(N_\omega^-) \times U(M'_\omega) \times U(M''_\omega), \\ \mathbf{H}_{\text{edg}}^\gamma &= \prod_{\eta \in \mathbf{E}_\gamma + \mathbb{I}\mathbf{E}_{\gamma-}} U(1), \quad \mathbf{H}_{\text{ver}}^\gamma = \prod_{\omega \in \mathfrak{V}_\gamma} U(1) \end{aligned}$$

and the embedding of $\mathbf{H}_{\text{ver}}^\gamma$ into $\mathbf{H}_{\text{ff}}^\gamma \times \mathbf{H}_{\text{edg}}^\gamma$ is given by:

$$(287) \quad \begin{aligned} (u_\omega)_{\omega \in \mathfrak{V}_\gamma} &\mapsto \left(u_\omega \cdot \mathbf{1}_{N_\omega^+}, u_\omega \cdot \mathbf{1}_{N_\omega^-}, u_\omega \cdot \mathbf{1}_{M'_\omega}, u_\omega \cdot \mathbf{1}_{M''_\omega} \right)_{\omega \in \mathfrak{V}_\gamma} \\ &\quad \times \left(u_{s(\eta) \pm 1}^{-1} u_{t(\eta)} \right)_{\eta \in \pm \mathfrak{E}_x} \end{aligned}$$

The group (285) acts on $\mathfrak{M}_{\underline{\mathbf{k}}}^\gamma(\underline{\mathbf{n}}^\pm, \underline{\mathbf{m}})$ in the following fashion:

$$(288) \quad \left((h_\omega^+, h_\omega^-, m'_\omega, m''_\omega)_{\omega \in V_\gamma} \times (u_\eta)_{\eta \in E_\gamma} \times u \right) \cdot [\mathbf{B}, \mathbf{I}, \mathbf{J}, \Theta] \\ = \left[u^{\pm 1} u_\eta g_{t(\eta) \pm 1} B_\eta^\pm g_{s(\eta)}^{-1}, u^{\pm 1} u_\eta^{-1} g_{s(\eta) \pm 1} \tilde{B}_\eta^\pm g_{t(\eta)}^{-1} \mid_{\eta \in E_\gamma} \right. \\ \left. u^{\pm 1} g_{\omega \pm 1} I_\omega^\pm (h_\omega^\pm)^{-1}, h_{\omega \pm 1}^\pm J_\omega^\pm g_{\omega}^{-1} u^{\pm 1}, m'_\omega \Theta'_\omega, \Theta''_\omega (m''_\omega)^{-1} \mid_{\omega \in V_\gamma} \right],$$

where we indicated the compensating $\mathcal{G}_{\underline{\mathbf{k}}}^\gamma$ -transformations. It is obvious from the Eq. (288) that the $\mathbf{H}_{\text{ff}}^\gamma \times \mathbf{H}_{\text{edg}}^\gamma$ -transformations which are in the image of $\mathbf{H}_{\text{Ver}}^\gamma$ can be undone by a $\mathcal{G}_{\underline{\mathbf{k}}}^\gamma$ -transformation.

10.1.5. Compactness theorem in the crossed quiver case. Let us demonstrate the compactness of the set $(\mathfrak{M}_{\underline{\mathbf{k}}}^\gamma(\underline{\mathbf{n}}^\pm, \underline{\mathbf{m}}))^{\text{T}_\gamma}$ of T_γ -fixed points in $\mathfrak{M}_{\underline{\mathbf{k}}}^\gamma(\underline{\mathbf{n}}^\pm, \underline{\mathbf{m}})$, where $\text{T}_\gamma \subset \mathbf{H}^\gamma$ is a subtorus of the global symmetry group. The choice of T_γ is restricted by the following requirement: it must contain a $U(1)$ -subgroup, to be denoted by $U(1)_v$, such that 1) the composition $p \circ i$, where $i : U(1)_v \hookrightarrow \mathbf{H}^\gamma$ is the embedding, and $p : \mathbf{H}^\gamma \rightarrow U(1)_u$ is the projection, is a non-trivial homomorphism, $v \mapsto v^k, k \neq 0$, 2) the embedding into \mathbf{H}^γ is parametrized by the collection

$$(289) \quad v \in U(1)_v \mapsto (v^{\underline{\mathbf{n}}^+}, v^{\underline{\mathbf{n}}^-}, v^{\underline{\mathbf{m}}'}, v^{\underline{\mathbf{m}}''})_{\omega \in V_\gamma} \times v^{\mathbf{e}} \times v^k$$

The symmetry (264) reflects the quotient by $\mathbf{H}_{\text{Ver}}^\gamma$ in (285).

We shall impose an additional requirement on the data $(\underline{\mathbf{n}}_{V_\gamma}^\pm, \underline{\mathbf{m}}'_{V_\gamma}, \underline{\mathbf{m}}''_{V_\gamma}, \mathbf{e}, k)$: for any $\omega', \omega'' \in V_\gamma$ and any $\alpha' \in [n_{\omega'}^\pm], \alpha'' \in [n_{\omega''}^\pm]$

$$(290) \quad n_{\omega', \alpha'}^\pm - n_{\omega'', \alpha''}^\pm \pm k(\ell + 1) + \int_{\text{path}_{\omega', \omega''}^\pm} e \neq 0$$

for any path $\text{p}_{\omega', \omega''}^\pm$. Note (290) is invariant under (264). The requirement (290) can be slightly weakened, namely one can allow (290) to fail for a single pair $(\omega', \alpha') = (\omega'', \alpha'')$. In what follows we insist on (290), though.

The proof goes as follows: Define the function δ on the Grassmanian of subspaces $V \subset \oplus_\omega K_\omega$:

$$(291) \quad \delta_V = \text{Tr}_V \left(\sum_{\eta \in E_{\gamma, \pm}} B_\eta^\pm (B_\eta^\pm)^\dagger + \tilde{B}_\eta^\pm (\tilde{B}_\eta^\pm)^\dagger + \sum_{\omega, \pm} I_\omega^\pm (I_\omega^\pm)^\dagger + J_\omega^\pm (J_\omega^\pm)^\dagger \right)$$

The function δ is monotone: $\delta_{V'} \leq \delta_{V''}$ for $V' \subset V''$. We have:

$$(292) \quad \|\mathbf{B}, \mathbf{I}, \mathbf{J}\|^2 = \sum_{\omega \in \mathbf{V}_\gamma} \delta_{K_\omega}$$

Now, the \mathbb{T}_γ -invariance implies, by the usual arguments, that the spaces K_ω^\pm, K_ω , for each $\omega \in \mathbf{V}_\gamma$ are \mathbb{T}_γ -representations,

$$(293) \quad K_\omega^\pm = \bigoplus_{w \in \mathbb{T}_\gamma^\vee} K_{\omega,w}^\pm \otimes \mathcal{R}_w$$

where \mathcal{R}_w are the irreps of \mathbb{T}_γ .

First, we need to prove that $J_\omega^\pm = 0$ for the \mathbb{T}_γ -invariant $(\mathbf{B}, \mathbf{I}, \mathbf{J})$. The equations (282) imply that $J_\omega^\pm(K_\omega^\mp) = 0$. Let us restrict (293) onto $U(1)_\vee$:

$$(294) \quad K_\omega^\pm = \bigoplus_{w \in \mathbb{Z}} K_{\omega,w}^\pm \otimes \mathcal{R}_w$$

where \mathcal{R}_w are the irreps of $U(1)_\vee: v \mapsto v^w$. We have:

$$(295) \quad I_\omega^\pm(N_\omega^\pm) = \bigoplus_{\alpha=1}^{n_\omega^\pm} N_{\omega,\alpha}^\pm \otimes \mathcal{R}_{n_{\omega,\alpha}^\pm}$$

and

$$(296) \quad B_\eta^\pm(K_{s(\eta),w}^\pm) \subset K_{t(\eta)+1,w+k+e(\eta)}^\pm, \quad \tilde{B}_\eta^\pm(K_{t(\eta),w}^\pm) \subset K_{s(\eta)+1,w+k-e(\eta)}^\pm$$

Thus the weights s which occur in the decomposition (294) have the form:

$$(297) \quad n_{\omega',\alpha'}^\pm \pm k\ell + \int_{\mathbf{p}_{\omega',\omega}^\pm} e$$

for some $\omega' \in \mathbf{V}_\gamma$ and some length ℓ path $\mathbf{p}_{\omega',\omega}^\pm$.

Now the $U(1)_\vee$ equivariance implies that $J_\omega^\pm(K_{\omega,w} \otimes \mathcal{R}_w)$ belongs to the eigenspace of $v \in U(1)_\vee$ in $N_{\omega\pm 1}^\pm$ with the eigenvalue $v^{w\mp k}$. Since the eigenvalues of $v \in U(1)_\vee$ on $N_{\omega\pm 1}^\pm$ are given by: $v^{n_{\omega\pm 1,\alpha}^\pm}$ the non-vanishing J_ω^\pm means that for some ω', α' the eigenvalue (297) coincides with $n_{\omega\pm 1,\alpha} \mp k$, which contradicts (290). Thus, $\mathbf{J} = 0$.

Now, use the real moment map equation to deduce:

$$\begin{aligned}
 (298) \quad \delta_{K_{\omega,w}^\pm} &= \zeta_\omega \dim(K_{\omega,w}^\pm) + \sum_{\eta \in s^{-1}(\omega) \cap \pm \mathfrak{E}_x} \text{Tr}_{K_{t(\eta)^\pm 1, w+k+e(\eta)}} B_\eta^\pm (B_\eta^\pm)^\dagger \\
 &\quad + \sum_{\eta \in t^{-1}(\omega) \cap \pm \mathfrak{E}_x} \text{Tr}_{K_{s(\eta)^\pm 1, w+k-e(\eta)}} \tilde{B}_\eta^\pm (\tilde{B}_\eta^\pm)^\dagger \\
 &\leq \zeta_\omega \dim(K_{\omega,w}^\pm) + \sum_{\eta \in s^{-1}(\omega) \cap \pm \mathfrak{E}_x} \delta_{K_{t(\eta)^\pm 1, w+k+e(\eta)}} \\
 &\quad + \sum_{\eta \in t^{-1}(\omega) \cap \pm \mathfrak{E}_x} \delta_{K_{s(\eta)^\pm 1, w+k-e(\eta)}}
 \end{aligned}$$

Now repeat the same estimate by pushing the arguments w' of the $\delta_{K_{\omega',w'}}$'s in the right hand side of (298) outside the domain where the corresponding $K_{\omega',w'}$ spaces are non-trivial (this is possible because the total dimension of the K space is finite). In this way we get an upper bound on $\delta_{K_{\omega',w'}}$'s and the norms of B 's, \mathbf{I} 's and \mathbf{J} 's, as promised.

10.2. Orbifolds and defects: ADE \times U(1) \times ADE

The construction above can be motivated by the following examples.

Recall that the moduli space $\mathcal{M}_k^+(n, w)$ of crossed instantons has an $SU(2) \times U(1) \times SU(2)$ symmetry.

10.2.1. Space action. Let Γ be a discrete subgroup of $SU(2)_{12} \times U(1)_\Delta \times SU(2)_{34}$,

$$(299) \quad \iota : \Gamma \longrightarrow \mathbf{G}_{\text{rot}} = SU(2)_{12} \times U(1)_\Delta \times SU(2)_{34},$$

10.2.2. Framing action. Now let us endow the spaces N_{12} and N_{34} with the structure of Γ -module:

$$\begin{aligned}
 (300) \quad N_{12} &= \bigoplus_{\omega \in \Gamma^\vee} N_\omega \otimes \mathcal{R}_\omega, \\
 N_{34} &= \bigoplus_{\omega \in \Gamma^\vee} W_\omega \otimes \mathcal{R}_\omega,
 \end{aligned}$$

in other words let us fix the homomorphisms

$$(301) \quad \rho_A : \Gamma \longrightarrow U(n_A)$$

Let us denote by \mathbf{n}, \mathbf{w} the vectors of dimensions $(\dim N_{12, \omega})_{\omega \in \Gamma^\vee}, (\dim N_{34, \omega})_{\omega \in \Gamma^\vee}$, respectively.

The data (300), (299) defines the embedding of the group Γ into \mathbf{H} , the symmetry group of $\mathcal{M}_k^+(n, w)$.

10.2.3. New moduli spaces. The set of Γ -fixed points in $\mathcal{M}_k^+(n, w)$ splits into components

$$(302) \quad \mathcal{M}_k^+(n, w)^\Gamma = \bigcup_{\mathbf{k}} \mathcal{M}_{\mathbf{k}}^{+, \gamma_\Gamma}(\mathbf{n}, \mathbf{w})$$

This is a particular case of the space $\mathfrak{M}_{\mathbf{k}}^\gamma(\mathbf{n}, \mathbf{w}, 0)$. Indeed, let $\mathbf{V}_\gamma = \Gamma^\vee$, while \mathbf{E}_γ^\pm are defined by decomposing

$$(303) \quad \mathcal{R}_\omega \otimes \mathbb{C}_{34}^2 = \bigoplus_{e \in s^{-1}(\omega) \cap \mathbf{E}_\gamma^\pm} \mathcal{R}_{t(e)} \oplus \bigoplus_{e \in t^{-1}(\omega) \cap \mathbf{E}_\gamma^\pm} \mathcal{R}_{s(e)}$$

The requirement that Γ preserves the $U(k)$ -orbit of $(\mathbf{B}, \mathbf{I}, \mathbf{J})$ translates to the fact that Γ is unitary represented in K , so that

$$(304) \quad \gamma \cdot (\mathbf{B}, \mathbf{I}, \mathbf{J}) = (g_\gamma \mathbf{B} g_\gamma^{-1}, g_\gamma \mathbf{I}, \mathbf{J} g_\gamma^{-1}), \quad \gamma \mapsto g_\gamma \in U(K)$$

Thus, we can decompose K into the irreps of Γ :

$$(305) \quad K = \bigoplus_{\omega \in \Gamma^\vee} K_\omega \otimes \mathcal{R}_\omega$$

The operators $\mathbf{B}, \mathbf{I}, \mathbf{J}$ then become linear maps between the spaces $K_{\omega'}$, $N_{A, \omega''}$, which can be easily classified by unraveling the equivariance conditions (304).

The components $\mathcal{M}_{\mathbf{k}}^{+, \gamma_\Gamma}(\mathbf{n}, \mathbf{w})$ can then be deformed by modifying the real moment map equation to

$$(306) \quad \mu = \sum_{\omega \in \Gamma^\vee} \zeta_\omega \mathbf{1}_{K_\omega}$$

In the particular case $\Gamma \subset \text{SU}(2)_{34}$ the orbifold produces the moduli spaces of supersymmetric gauge configurations in the

$$(307) \quad \mathbf{G}_g = \prod_{\omega \in \Gamma^\vee} U(n_\omega)$$

gauge theory in the presence of a point-like defect, the qq -character

$$(308) \quad \mathcal{X}_{(w_\omega)_{\omega \in \Gamma^\vee}}(x, \nu_{\omega, \beta})$$

The gauge theory in question is the affine ADE quiver gauge theory.

10.2.4. Odd dimensions and finite quivers. We can also obtain the moduli space of supersymmetric gauge field configurations in the quiver gauge theories built on finite quivers. The natural way to do that is to start with an affine quiver and send some of the gauge couplings to zero, i.e. by making some of dimensions k_ω vanishing.

Remarkably, this procedure produces the superspace, the odd variables originating in the multiplet of the cohomological field theory. Let us explain this in more detail. Let us consider, for simplicity, the group $\Gamma \subset SU(2)_{34}$, so that E_γ^+ has one element.

The linear algebra data

$$\left(B_{1,\omega}, B_{2,\omega}, I_{12,\omega}, J_{12,\omega}, I_{34,\omega}, J_{34,\omega}, B_e, \tilde{B}_e \right) \in \mathcal{A}_{\underline{\mathbf{k}}}^\gamma(\underline{\mathbf{n}}, \underline{\mathbf{w}}, 0)$$

is parametrized by the

$$(309) \quad 2 \sum_{\omega \in \Gamma^\vee} k_\omega (k_\omega + n_\omega + w_\omega) + 2 \sum_{e \in E} k_{t(e)} k_{s(e)}$$

complex dimensional space. The Eqs. (268) plus the $GL_\gamma(\underline{\mathbf{k}})$ -invariance remove

$$(310) \quad \sum_{\omega \in \Gamma^\vee} (n_\omega w_\omega + 2k_\omega (k_\omega + w_\omega)) + \sum_{e \in E} 2k_{s(e)} k_{t(e)} + k_{s(e)} n_{t(e)} + k_{t(e)} n_{s(e)}$$

dimensions (this is half the number of equations (282)). The result is $\underline{\mathbf{k}}$ -linear,

$$(311) \quad \text{virtual dim } \mathcal{M}_{\underline{\mathbf{k}}}^{+,\gamma}(\underline{\mathbf{n}}, \underline{\mathbf{w}}) = \sum_{\omega \in \Gamma^\vee} (k_\omega m_\omega - n_\omega w_\omega)$$

where

$$(312) \quad m_\omega = 2n_\omega - \sum_{e \in s^{-1}(\omega)} n_{t(e)} - \sum_{e \in t^{-1}(\omega)} n_{s(e)}$$

Now, if for all $\omega \in \Gamma^\vee$ the deficits m_ω are non-negative, and at least for one vertex the deficit is positive then the quiver is, in fact, a finite ADE Dynkin diagram. In this case we can add the odd variables taking values in the spaces $\text{Hom}(K_\omega, M_\omega)$ with the complex vector space M_ω of dimension m_ω , and define the moduli space to be the supermanifold which is the total space of the odd vector bundle $\text{Hom}(K_\omega, M_\omega)$ over the previously defined bosonic moduli space. In practice this means that the integration over the “true” moduli space is the integral over the coarse moduli space of the equivariant Euler

class of the vector bundle $\text{Hom}(K_\omega, M_\omega)$. This is what the cohomological field theory applied to the affine quiver case with the subsequent setting $k_{\omega'} = 0$ for some $\omega' \in \Gamma^\vee$ would amount to. With the “compensator” vector bundle in place the virtual dimension of the moduli space becomes \mathbf{k} -independent. This is the topological counterpart of the asymptotic conformal invariance of the gauge theory.

11. Spiked instantons on orbifolds and defects

Now let us go back to the general case of spiked instantons. Choose a discrete subgroup Γ of $U(1)_{\mathbf{e}}^3$, e.g. $\Gamma \approx \Xi_{p_1} \times \Xi_{p_2} \times \Xi_{p_3}$, $\Gamma^\vee \approx \Gamma$. Let $t_a : \Gamma \rightarrow U(1)$, $a \in \mathbf{4}$ be the homomorphisms corresponding to the embedding $U(1)_{\mathbf{e}}^3 \subset SU(4)$. We have:

$$(313) \quad \prod_{a \in \mathbf{4}} t_a(\gamma) = 1$$

Let $\mathcal{R}_a \in \Gamma^\vee, a \in \mathbf{4}$ be the corresponding one-dimensional representations of Γ , e.g $\mathcal{R}_1 = \mathcal{R}_{1,0,0}$, $\mathcal{R}_4 = \mathcal{R}_{-1,-1,-1}$. We shall use the additive notation, $\mathcal{R}_a = \mathcal{R}_{\vec{\alpha}_a}$ so that $\mathcal{R}_{\vec{\omega}} \otimes \mathcal{R}_a = \mathcal{R}_{\vec{\omega} + \vec{\alpha}_a}$. Fix the framing homomorphisms: $\rho_{A,\Gamma} \rightarrow U(n_A)$:

$$(314) \quad N_A = \bigoplus_{\vec{\omega} \in \Gamma^\vee} N_{A,\vec{\omega}} \otimes \mathcal{R}_{\vec{\omega}}$$

The set of Γ -fixed points in $\mathfrak{M}_k(\vec{n})$ splits into components

$$(315) \quad \mathfrak{M}_k(\vec{n})^\Gamma = \bigcup_{\mathbf{k}} \mathfrak{M}_{\mathbf{k}}^{\gamma_\Gamma}(\vec{\mathbf{n}})$$

It describes the moduli spaces of spiked instantons in the presence of additional surface and point-like conical defects. The compactness theorem holds in this case. The proof is a simple extension of the proof of Section 8 with the spaces K_A replaced by $K_{A,\vec{\omega}}$, where $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, $\omega_i \in \Xi_{p_i}$:

$$(316) \quad K_{A,\vec{\omega}} = \sum_{\vec{\omega}' \in \Gamma^\vee} \sum_{f \in \mathbb{C}[x,y]} f(B_a, B_b) I_A(N_{A,\vec{\omega}'})$$

where the sum is over polynomials obeying:

$$(317) \quad f(t_a(\gamma)x, t_b(\gamma)y) = \chi_{\mathcal{R}_{\vec{\omega}-\vec{\omega}'}}(\gamma)$$

for all $\gamma \in \Gamma$. As before $B_{c,\vec{\omega}'}(K_{A,\vec{\omega}}) = 0$ whenever $c \notin A$. The vector $\underline{\mathbf{k}}$ encodes the dimensions of the spaces

$$(318) \quad K_{\vec{\omega}} = \sum_{A \in \mathfrak{6}} K_{A,\vec{\omega}}$$

the operators B_a have the block form:

$$(319) \quad B_a(K_{\vec{\omega}}) \subset K_{\vec{\omega} + \vec{\omega}_a}$$

The norms $\|B_a\|^2$ are estimated with the help of the quantities

$$(320) \quad \delta_{A,\vec{\omega},n} = \text{Tr}_{K_{A,\vec{\omega}}} \left(\sum_a B_a B_a^\dagger + \sum_C I_C I_C^\dagger \right)$$

12. Conclusions and future directions

In this paper we introduced several moduli spaces: \mathfrak{M}^+ , \mathfrak{M}^\perp , \mathfrak{M}^* of matrices solving quadratic equations modulo symmetries. These moduli spaces generalize the Gieseker-Nakajima partial compactification \mathcal{M} of the ADHM moduli space of $U(n)$ instantons on \mathbb{R}^4 . We gave some motivations for these constructions and proved the compactness theorem which we shall use in the next papers to establish useful identities on the correlation functions of supersymmetric gauge theories in four dimensions.

In this concluding section we would like to make a few remarks.

First of all, one can motivate the crossed instanton construction by starting with the ordinary ADHM construction and adding the co-fields [7, 8] which mirror the embedding of the $\mathcal{N} = 2$ super-Yang-Mills vector multiplet into the $\mathcal{N} = 4$ super-Yang-Mills vector multiplet [37].

Secondly, we would like to find the crossed instanton analogue of the stable envelopes of [21].

Third, it would be nice to generalize the spiked instanton construction to allow more general orbifold groups $\Gamma \subset SU(4)$, and more general (Lagrangian?) subvarieties in \mathbb{C}^4/Γ .

Now, to the serious drawbacks of our constructions. The purpose of the ADHM construction, after all, is the construction of the solutions to the instanton equations

$$F_A^+ = 0$$

We didn't find the analogue of the ADHM construction for the spiked instantons. Conjecturally, the matrices $[\mathbf{B}, \mathbf{I}, \mathbf{J}]$ solving the Eqs. (56) are in one-to-one correspondence with the finite action solutions to Eqs. (41).

Finally, we have proposed a definition of quiver crossed instantons, which are defined for quivers more general than the products of ADE Dynkin diagrams. It would be interesting to find the precise restrictions on these quivers compatible with the compactness theorem.

In the forthcoming papers the compactness theorem will be used to derive the main statements of the theory of qq -characters [30]. While this paper was in preparation, the algebraic counterpart of our compactness theorem was studied in [19]. Various consequences of the compactness theorem will be studied in [29]. Some of them have already been observed in [1, 2, 11, 26, 31, 32].

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