ADV. THEOR. MATH. PHYS.
Volume 21, Number 1, 243–287, 2017

# Duality for toric Landau-Ginzburg models

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We introduce a duality construction for toric Landau-Ginzburg models, applicable to complete intersections in toric varieties via the sigma model / Landau-Ginzburg model correspondence. This construction is shown to reconstruct those of Batyrev-Borisov, Berglund-Hübsch, Givental, and Hori-Vafa. It can be done in more general situations, and provides partial resolutions when the above constructions give a singular mirror. An extended example is given: the Landau-Ginzburg models dual to elliptic curves in $(\mathbb{P}^1)^2$.

## Introduction

Motivated by the mirror involution on $N = 2$ superconformal theories and the fact that one can define from a Calabi-Yau such a theory, string theorists believe that Calabi-Yaus exist in "mirror pairs". In other words, to any Calabi-Yau, $Z$, there exists another, $\check{Z}$, such that the superconformal theories they define are mirror to each other.

Mathematically, such a pair would have mirror symmetric Hodge diamonds,

$$h^{p,q}(Z) = h^{n-p,q}(\check{Z}), \tag{1}$$

it would be possible to compute the Gromov-Witten invariants of $Z$ from the periods of $\check{Z}$, and, following the homological mirror symmetry conjecture of Kontsevich [Kon95a], the derived category of coherent sheaves on $Z$, $D^b(coh\ Z)$, would be equivalent to the derived Fukaya category of $\check{Z}$, $DFuk(\check{Z})$. Broadly speaking, mirror symmetry transforms invariants of the

*2000 Mathematics Subject Classification:* Primary 14J32; Secondary 14M10, 14M25, 14J81.
*Key words and phrases:* Landau-Ginzburg, mirror symmetry, complete intersection, toric variety
The author was supported in part by NSF Award #0703643.

symplectic topology of $Z$ into invariants of the complex structure of $\check{Z}$, and vice versa.

For general Kähler manifolds $Z$ it is possible to define Hodge numbers and $D^b(coh\ Z)$, and often (e.g. if $Z$ is Fano) Gromov-Witten invariants and $DFuk(Z)$ make sense. This leads naturally to the question of mirror pairs for general Kähler manifolds, rather than just Calabi-Yaus. One immediately obvious hurdle is the existence of a Kähler manifold, $\check{Z}$, satisfying the mirror symmetric Hodge diamond relation above is only possible in the Calabi-Yau case since

$$h^{n,0}(Z) = h^{0,0}(\check{Z}) = 1. \tag{2}$$

A promising solution to problems such as this is the use of Landau-Ginzburg models as mirrors. A *Landau-Ginzburg model* is a superconformal theory defined by a Kähler manifold $\check{X}$ equipped with a holomorphic function $\check{W}: \check{X} \to \mathbb{C}$. The function $\check{W}$ is referred to as the *superpotential*. The Hodge numbers are then replaced by dimensions of graded components of a certain "chiral" ring associated with $Z$ or the pair $(\check{X}, \check{W})$. There also exists a version of the derived category of coherent sheaves for Landau-Ginzburg models, $DB(\check{X}, \check{W})$, introduced by Orlov [Orl04] (generalizing the category of matrix factorizations), and a version of the derived Fukaya category, $DFS(\check{X}, \check{W})$, due to Seidel [Sei01].

There are four general predictive methods for computing the mirror of a complete intersection in a toric variety. The first to appear in the literature is that of Berglund and Hübsch [BH92]. Their construction produces a mirror candidate to a Calabi-Yau hypersurface in weighted projective space. Their mirror is also a Calabi-Yau hypersurface in (a different) weighted projective space.

Shortly after this, Batyrev [Bat94] gave a construction for Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties. This was subsequently generalized by Borisov [Bor93] to Calabi-Yau complete intersections that arise from "nef-partitions" of the anti-canonical divisor. Since a weighted projective space is never Gorenstein unless it is projective space itself, Berglund-Hübsch and Batyrev-Borisov address distinct situations.

The combined efforts of Batyrev [Bat], Batyrev-Borisov [BB], and Kontsevich [Kon95b] ultimately led to the proof that $n$-dimensional Batyrev-Borsiov pairs, $Z$ and $\check{Z}$, have mirror symmetric (stringy) Hodge diamonds. Thus giving one of the first general and rigorous results in mirror symmetry.

The same year, Givental published his "mirror theorem" [Giv98b] that made rigorous and generalized the approach to computing Gromov-Witten invariants pioneered by physicists in [CdlOGP91]. Given a Fano, complete intersection $Z$ in a smooth projective toric variety, Givental constructs a Landau-Ginzburg model mirror candidate $(\check{X}, \check{W})$. He then shows that structure constants of the quantum cohomology of $Z$ can be found by considering certain integrals over cycles in $\check{X}$ related to the Morse theory of $\check{W}$. It is worth mentioning that the recipe given by Givental for the mirror Landau-Ginzburg model can be done for arbitrary complete intersections in toric varieties, even though he only considers the case of Fano manifolds.

The most recent algorithm to compute a mirror candidate is given by Hori and Vafa [HV00]. Using physical arguments, from a complete intersection in a smooth toric variety they obtain a mirror Landau-Ginzburg model.

This paper puts forth a new method for computing mirror candidates for complete intersections in toric varieties. Given an $n$-dimensional toric variety $X$, an element $\omega \in$ the Chow group $A_{n-1}(X)_{\mathbb{C}/\mathbb{Z}}$ (coefficients in $\mathbb{C}/\mathbb{Z}$), and a morphism $W\colon X \to \mathbb{C}$, we produce a $n$-dimensional toric variety $X'$, a Chow group element $\omega' \in A_{n-1}(X')_{\mathbb{C}/\mathbb{Z}}$, and a morphism $W'\colon X' \to \mathbb{C}$ (strictly speaking the most natural objects to consider are toric Deligne-Mumford stacks, but we will not need this and the generalization is obvious). We call the new Landau-Ginzburg model *dual* to the original.

Using an idea from physics called the *sigma model / Landau-Ginzburg model correspondence*, this process can be applied to generate a mirror candidate for a complete intersection in a toric variety. This correspondence goes as follows. Assume $Z$ is the zero locus of a global section $\mathrm{w} \in \Gamma(Y, \mathcal{V})$ of some vector bundle $\mathcal{V}$ over a Kähler manifold $Y$. The identification $\mathrm{Hom}(\mathcal{V}^\vee, \mathcal{O}_Y) \cong \Gamma(Y, \mathcal{V})$ allows one to use w to define a morphism $W\colon X \to \mathbb{C}$ on the total space $X = \mathrm{tot}(\mathcal{V}^\vee)$. Physically, the superconformal theories defined by $Z$ and $(X, W)$ are the same [GS08]. Based on this, one would expect that the Hodge numbers of $Z$ give the graded component dimensions of the chiral ring of $(X, W)$, etc. Landau-Ginzburg model mirror candidates are then formed by the composition:

$$Z \overset{\Sigma/LG}{\longmapsto} (X, W) \overset{dual}{\longmapsto} (X', W'). \tag{3}$$

If $(X', W')$ has the form of a vector bundle paired with a section of its dual, we can run the correspondence backwards, $(X', W') \overset{LG/\Sigma}{\longmapsto} Z'$, to obtain a Kähler manifold mirror candidate.

After initial definitions and describing the construction of the dual Landau-Ginzburg model, we compare the mirror candidate obtained using four methods above with the candidate given by the dual. Ultimately, all methods are shown to be special cases of the duality defined here.

The Givental and Hori-Vafa mirrors are considered together. By a simple computation, we show they both produce the same mirror. The starting point for their construction is a split vector bundle $\mathcal{V}$ over toric variety $Y$. The complete intersection is given by a global section of $\mathcal{V}$. Note that in their formulation, the complex structure of $Z$ is ignored. Concretely, this means that $Z$ is the zero locus of some global section $\mathrm{w} \in \Gamma(Y, \mathcal{V})$, but the choice of w is not important. This is justified by the fact that whenever integral, smooth subvarieties $Z_1$ and $Z_2$ are given by global sections $\mathrm{w}_1$ and $\mathrm{w}_2$ respectively, they are symplectomorphic. With this in mind, we prove the following theorem.

**Theorem.** *For a specific choice* $\mathrm{w}_{GHV} \in \Gamma(Y, \mathcal{V})$*, the mirror Landau-Ginzburg model of Givental-Hori-Vafa is the dual Landau-Ginzburg model,* $(X', W')$*, to* $(X = \mathrm{Tot}(\mathcal{V}^\vee), W_{GHV})$*. Where* $W_{GHV}$ *is defined by* $\mathrm{w}_{GHV}$*.*

It is nice to note that the dual to $(X', W')$ is $(X, W^-)$, where $X$ is the original space and $W^-$ is closely related to the original superpotential $W$. This gives a nice resolution to the apparent lack of symmetry in the generalization of mirror symmetry to non-Calabi-Yaus bemoaned by Givental in [Wit93]. This is explained in Remark 6.11.

The methods of Berglund-Hübsch, and Batyrev-Borisov produce mirror families. This is to say that starting from a family of Calabi-Yaus $(Z_t)_t$ they give a new family $(\check{Z}_s)_s$ without specifying which $\check{Z}_s$ is mirror to a given $Z_t$. This suffices for the comparison of Hodge numbers since these are constant in families.

The Berglund and Hübsch construction is easily treated and a simple calculation gives the following theorem.

**Theorem.** *Let* $(X, W)$ *be obtained from the sigma model / Landau-Ginzburg model correspondence applied to a Calabi-Yau hypersurface* $Z$ *in weighted projective space. Then the dual Landau-Ginzburg model equals the Landau-Ginzburg model corresponding to a member* $\check{Z}$ *of the Berglund-Hübsch mirror family.*

The last method we analyze is that of Batyrev and Borisov. The starting point for their construction is a split bundle obtained from a nef-partition over a Gorenstein toric Fano variety. After some technical results concerning rational convex polyhedral subsets, we arrive at the following theorem.

**Theorem.** *Let* $(X, W)$ *be obtained from the sigma model / Landau-Ginzburg model correspondence applied to a certain Calabi-Yau* $Z_{BB}$ *given by a global* *section of a split bundle defined by a nef-partition over a toric Fano variety. Then the dual Landau-Ginzburg model equals the Landau-Ginzburg model corresponding to a member of the Batyrev-Borisov mirror family.*

One nice aspect about the dual Landau-Ginzburg model is that varying the original superpotential, $W$, causes the symplectic form $\omega' \in A_{n-1}(X')_{\mathbb{C}/\mathbb{Z}}$ to vary. When $X$ is obtained via the sigma model / Landau-Ginzburg correspondence, varying the superpotential is the same as varying the complex structure of $Z$. This identification of complex moduli with symplectic moduli is expected between mirror pairs.

This can be used to avoid potential difficulties that arise when the mirror candidate is singular. For instance, every element of the Batryev-Borisov mirror family may be singular since the ambient toric variety may be singular. However, varying $Z$ away from $Z_{BB}$ leads to a partial resolution of the mirror, thus taking some of the arbitrariness out of the choice of resolution.

There is a small example in section 6, and we conclude the paper in with an extended example that makes up section 10. Here, the case of elliptic curves in $(\mathbb{P}^1)^2$ is treated from the point of view of Givental and Batyrev-Borisov, and it is shown how our point of view allows for symplectic a resolution of these singular mirrors.

**Acknowledgements.** This work would not have been possible without the support and insight of Ludmil Katzarkov. We would also like thank Elizabeth Gasparim for her close reading of the text, and David Cox for pointing out several references. Thanks to Eric Sharpe for showing me the Berglund-Hübsch construction and explaining several ideas from physics to me. Finally, thanks to Lev Borisov for pointing out a way to modify the construction given here. This modification is discussed at the end of the paper.

## 1. Definitions

### Rational convex polyhedral sets

The theory of quasi-projective toric varieties is essentially the same as the theory of rational convex polyhedral sets. We will use this fact extensively throughout the paper. Here we review the main points we need.

Given a finite rank free abelian group $M$, we write $M_G$ for the tensor product $M \otimes_{\mathbb{Z}} G$ with an abelian group $G$. Denote the dual, $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, by $N$.

A rational convex polyhedral set $P \subseteq M_{\mathbb{R}}$ is the set of solutions to a set of linear inequalities:

$$\{\xi \in M_{\mathbb{R}} \mid \nu_j(\xi) + \alpha_j \geq 0,\ j = 1, \dots, r\}, \tag{4}$$

where $\nu_j \in \{\nu \otimes 1 \colon M_{\mathbb{R}} \to \mathbb{R} \mid \nu \in N\}$ and $\alpha_j \in \mathbb{R}$.

The inequalities can be packaged together into a homomorphism $A = (\nu_1, \dots, \nu_r) \colon M \to \mathbb{Z}^r$, and an element $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r = (\mathbb{Z}^r)_{\mathbb{R}}$. With these we have $P = \{\xi \in M_{\mathbb{R}} \mid A(\xi) + \alpha \geq 0\}$.

A face of $P$ is either the intersection of $P$ with the boundary of an affine half-space containing $P$, or $P$ itself. The dimension of a face is the dimension of the real vector space given by the span of the elements of the face. A facet of $P$ is a face whose dimension is one less that the dimension of $P$.

If $C$ is an arbitrary subset of $N_{\mathbb{R}}$, the dual of $C$ is the set

$$\check{C} := \{\xi \in M_{\mathbb{R}} \mid \nu(\xi) \geq 0,\ \forall\ \nu \in C\}. \tag{5}$$

Defined similarly to the dual, is the polar of $C \subseteq N_{\mathbb{R}}$

$$C^{\circ} := \{\xi \in M_{\mathbb{R}} \mid \nu(\xi) + 1 \geq 0,\ \forall\ \nu \in C\}. \tag{6}$$

Associated to a convex rational polyhedral set with non-empty interior is an *inward normal fan*. This is made up of rational convex polyhedral cones. A rational convex polyhedral cone is rational convex polyhedral set closed under multiplication by $\mathbb{R}_{\geq 0}$, and it is called strongly convex if 0 is a vertex (i.e. $\{0\}$ is a 0 dimensional face).

A fan, $\Sigma$, is a non-empty finite collection of strongly convex rational polyhedral cones $\subset N_{\mathbb{R}}$ such that

1) if $\sigma \in \Sigma$ then all faces of $\sigma$ are in $\Sigma$, and
2) the intersection of any two cones in $\Sigma$ is also in $\Sigma$.

The inward normal fan $\Sigma_P$ to $P$ is defined to be the collection of cones

$$\sigma_f := \{\nu \in N_{\mathbb{R}} \mid \nu(\xi) = \min(\nu|_P),\ \forall\ \xi \in f\} \tag{7}$$

for each face $f$ of $P$.

**Lemma 1.1.** *(see for instance [Ful93]) If $P$ has non-empty interior, then $\Sigma_P$ is a fan.*

We finish this discussion with some small, but useful results about the polar of a convex set.

**Lemma 1.2.** *If $C$ is convex and $0 \in \mathrm{int}(C)$, then $(C^\circ)^\circ = C$.*

*Proof.* $C$ is determined by the affine half-spaces containing it. These half-spaces contain 0 in their interior, so they have a defining inequality with constant part = 1. So we have $C = \cap_{\xi \in C^\circ}\{\nu \in N_\mathbb{R} \mid \nu(\xi) + 1 \geq 0\}$, and the result is clear. □

**Corollary 1.3.** *Assume $0 \in \mathrm{int}(P)$, then $P^\circ = \mathrm{conv}(\{\nu_j/\alpha_j\}_{j=0}^r \cup \{0\})$.*

*Proof.* Since $\alpha_j > 0$, $P = \mathrm{conv}(\{\nu_j/\alpha_j\}_{j=0}^r \cup \{0\})^\circ$ . Taking polars of convex sets reverses inclusions of convex sets, and preserves strictness for convex sets containing 0. □

**Corollary 1.4.** *Assume $0 \in \mathrm{int}(P)$. Given $\nu_0 \in M_\mathbb{R}$ and $\alpha_0 \in \mathbb{R}_{>0}$. $P$ is contained in the affine half-space $\{\xi \in M_\mathbb{R} \mid \nu_0(\xi) + \alpha_0 \geq 0\}$ if and only if $\nu_0/\alpha_0 \in \mathrm{conv}(\{\nu_j/\alpha_j\}_{j=0}^r \cup \{0\})$.*

## Toric varieties

A toric variety $X$ is a normal irreducible complex algebraic variety on which an algebraic torus $T \cong (\mathbb{C}^\times)^n$ acts and $\exists\ x \in X$ such that $t \mapsto t \cdot x$ defines an open immersion

$$\iota_x \colon T \hookrightarrow X. \tag{8}$$

Some standard references for toric varieties are [Oda88, Ful93, Aud04].

The open immersion $\iota_x$ identifies characters $\xi : T \to \mathbb{C}^\times$ with rational functions on $X$. Since the divisor of a character is supported on the union of $T$-invariant subvarieties, this gives the *character-to-divisor map*

$$\mathrm{div} \colon M \to \mathbb{Z}^R, \tag{9}$$

where $M$ is the group of characters of $T$, and $R := \{\rho_1, \dots, \rho_r\}$ is the set of components of $X \setminus \iota_x(T)$. This means $\mathbb{Z}^R$ is the set of $T$-invariant divisors of $X$.

The cokernel of div is the Chow group $A_{n-1}(X)$. This group can often be identified with the second integral cohomology group as the following theorem and corollary state.

**Theorem 1.5.** *([Ful93, pp. 63–64]) If $Y$ is a complete toric variety, $A_{n-1}(Y) = \mathrm{H}^2(Y;\mathbb{Z})$ and is torsion free.*

**Corollary 1.6.** *If $X$ is the total space of a split bundle of rank $c$ over a complete toric variety, $Y$, then $X$ is toric, $A_{n-1}(X) = \mathrm{H}^2(X, \mathbb{Z}) = A_{n-1-c}(Y) = \mathrm{H}^2(Y; \mathbb{Z})$, and these groups are torsion free.*

Denote the cokernel of div by

$$[-]\colon \mathbb{Z}^R \to A_{n-1}(X), \tag{10}$$

and write the image of and element $d \in \mathbb{Z}^R$ by $[d] \in A_{n-1}(X)$.

Consequently, in the case of Corollary 1.6 we have the sequence

$$0 \to M \to \mathbb{Z}^R \to \mathrm{H}^2(X; \mathbb{Z}) \to 0, \tag{11}$$

which is exact.

For toric varieties, the $T$-invariant divisor

$$-\kappa_X := 1\rho_1 + \cdots + 1\rho_r \tag{12}$$

gives a canonical choice of representative for the anticanonical divisor.

The group $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^R, \mathbb{C}^\times)$ acts diagonally on $\mathbb{C}^R$, and thus so does $\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^\times)$ via the group homomorphism

$$(- \circ [-])\colon \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^\times) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^R, \mathbb{C}^\times) \tag{13}$$

defined by $h \mapsto h \circ [-]$.

If we restrict the induced map on the cotangent space of the identity,

$$T^*_{Id}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^R, \mathbb{C}^\times) \to T^*_{Id}\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^\times) \tag{14}$$

to the lattice dual to the exponential kernel, we get $fr \circ [-]$. Here $fr$ is the projection of $A_{n-1}(X)$ onto $A_{n-1}(X)/\text{torsion}$. For the main applications considered here, $A_{n-1}(X)$ is torsion free, and so we can interpret $[-]$ as the pullback map on covectors at the identity.

Using the action of $\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^\times)$ on $\mathbb{C}^R$, the quotient construction for projective space generalizes to toric varieties.

**Theorem 1.7.** *(Cox [Cox95]) A toric variety, $X$, can be obtained as a (GIT) quotient of $\mathbb{C}^R \setminus \mathcal{P}$ by $\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^\times)$ [Cox95]. $\mathcal{P}$ is a collection of coordinate subspaces determined by the intersection theory on $X$.*

The ring of regular functions on $\mathbb{C}^R$ is called the *Cox homogeneous coordinate ring* of $X$, and is generated by $\{X_\rho\}_{\rho \in R}$. The degree of $X_\rho$ is $[1\rho] \in A_{n-1}(X)$.

A more classical way to define a toric variety is from a fan of rational strongly convex polyhedral cones. The construction of a toric variety goes as follows. Fix a torus $T$ and let $M$ be its character group. Denote $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ as before. Let $\Sigma$ be a fan of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$. Then define the toric variety $X(\Sigma)$ acted on by $T$, to be the union of affine charts $U_\sigma$ for each $\sigma \in \Sigma$. Where $U_\sigma$ is the spectrum of the subring of regular functions $R_\sigma$ on $T$ that is generated by characters $\xi$ in the dual cone $\check{\sigma}$.

**Theorem 1.8.** *(Sumihiro [Sum74, Sum75]) Every toric variety can be obtained via the fan construction in a unique way.*

### Functions on a toric variety

As indicated by the character-to-divisor map ((9) above), the function theory of $X$ is understood in terms of the function theory of the torus $T$. The identification of rational function on $X$ with those on $T$ depends on the choice of $x$ defining the open immersion $\iota_x$.

If $W$ is a rational function on $X$,

$$\iota_x^* W = \sum_j q_j \xi_j \tag{15}$$

is the pullback to $T$, where $\xi_j \in M$.

If one makes another choice, $x'$, there is a unique element $t \in T$ such that $x' = t \cdot x$. For the same function $W$,

$$\iota_{x'}^* W = \iota_{(t\cdot x)}^* W = \sum_j q_j \xi_j(t) \xi_j. \tag{16}$$

Denote by $(\mathbb{C}^\times)^\Xi$ the space of functions on $T$ with terms $\Xi := \{\xi_1, \dots, \xi_s\}$. The action $T \circlearrowright (\mathbb{C}^\times)^\Xi$ coming from the different choices of $x \in X$ is given by

$$t \cdot (q_1\xi_1 + \cdots + q_s\xi_s) = q_1\xi_1(t)\xi_1 + \cdots + q_s\xi_s(t)\xi_s. \tag{17}$$

It is then natural to eliminate the dependence on $x$ of the expression of $\iota_x^* W$ by thinking of $W$ as an element the quotient

$$W \in (\mathbb{C}^\times)^\Xi / T. \tag{18}$$

## 2. Linear data

Associated to a toric variety is the character-to-divisor map, div. It is a remarkable fact that this map is almost enough information to recover the original toric variety. For instance, if the toric variety is projective and we are also given a (very) ample divisor class, $a$, in the Chow group ($=$ coker(div)), the variety can be recovered.

We will prove this, and a more general result concerning total spaces of split bundles over certain toric varieties. Motivated by this result we call the pair $(\mathrm{div}, a)$ *the linear data* associated to $X$. Here $a$ is an arbitrary element of $A_{n-1}(X)_{\mathbb{R}}$.

**Definition 2.1.** Precisely, we define for an abelian group $H$, linear $H$-data. This the following information:

1) a finite rank free abelian group $G$,
2) a homomorphism $C\colon G \to \mathbb{Z}^t$, and
3) an element $c \in \mathrm{coker}(C)_H$.

A key fact, that we will exploit in our construction of the dual Landau-Ginzburg model, is that we can define linear $\mathbb{C}/\mathbb{Z}$-data associated to a function $W$ on a toric variety. Analogous to the case of projective toric varieties, the linear data associated to $W$ recovers $W$.

### Linear data associated to a toric variety

In section 1, we defined a polyhedral set from a homomorphism $A\colon M \to \mathbb{Z}^r$, and an element $\alpha \in (\mathbb{Z}^r)_{\mathbb{R}}$. If a polyhedral set had non-empty interior, we defined its inward normal fan. Finally, from a fan we defined a toric variety.

Instead, we would like to use as our starting point the linear data

$$(A, a). \tag{19}$$

**Definition 2.2.** Assume there exists $\alpha \in (\mathbb{Z}^r)_{\mathbb{R}}$ such that $\alpha$ maps to $a$ in $\mathrm{coker}(A)$, and $(A, \alpha)$ defines a rational convex polyhedral with non-empty interior. Define $X(A, a)$ to be the toric variety defined by $A$ and $\alpha$.

The lemma below shows that $X(A, a)$ is independent of the choice of $\alpha$. Specifically, different choices of $\alpha$ correspond to translation of the rational convex polyhedral set by an element of $M_{\mathbb{R}}$, and thus give the same inward normal fan.

The following notation for our polyhedral sets will be used throughout the paper.

$$P_\alpha = \{\xi \in M_{\mathbb{R}} \mid A(\xi) + \alpha \geq 0\}. \tag{20}$$

**Lemma 2.3.** *If $\alpha - \alpha' = A(\xi_0)$ then $P_{\alpha'} = \xi_0 + P_\alpha$.*

*Proof.*

$$\begin{aligned} \xi \in P_\alpha &\iff A(\xi) + \alpha \geq 0 \iff A(\xi) + A(\xi_0) + \alpha' \geq 0 \\ &\iff A(\xi + \xi_0) + \alpha' \geq 0 \iff \xi + \xi_0 \in P_{\alpha'}. \end{aligned}$$

□

**Theorem 2.4.** *If $Y$ is projective and $a \in A_{n-1}(Y)$ corresponds to a very ample line bundle then $Y = X(\mathrm{div}, a)$.*

*Proof.* $P_\alpha$ is the polytope of $T$-linearized global sections of the very ample line bundle corresponding to $a$. The result is then the standard fact that the fan of $Y$ is the normal cone fan of this polytope. □

The following lemma helps us to get a concrete handle on the linear data associated with a line bundle over a toric variety. Note that we identify vectors with one-column matrices. Therefore homomorphisms are written as matrices that multiply vectors from the left.

**Lemma 2.5.** *If $D$ is a $T$-invariant Cartier divisor and $E$ is the total space of a line bundle $\mathcal{O}_Y(-D)$ over a toric variety $Y$, then the character group of $E$ is*

$$M_E = M_Y \oplus \mathbb{Z}\xi, \tag{21}$$

*where $\xi$ is a rational section of $\mathcal{O}_Y(D)$ whose divisor is $D$. The $T$-invariant Weil divisors of $E$ are the preimages under $p$ of the $T$-invariant Weil divisors of $Y$ as well as the image of the zero section $\sigma_0\colon Y \to E$, and so*

$$\mathrm{div}_E = \left[\begin{array}{c|c} \mathrm{div}_Y & D \\ 0 & \sigma_0(Y) \end{array}\right] \tag{22}$$

*with respect to the decomposition of $M_E$ above and $\mathbb{Z}^{R_E} = \mathbb{Z}^{R_Y} \oplus \mathbb{Z}\sigma_0(Y)$.*

*Proof.* The function $\xi$ is a section of a line bundle, so it vanishes both over $D$ when $\xi \equiv 0$ and along the zero section $\sigma_0(Y)$. Other characters are pullbacks, so they vanish as they did on $Y$. □

**Theorem 2.6.** *Assume $Y = X(\mathrm{div}_Y, a)$, and $E = \mathrm{Tot}\ \Gamma(Y, \mathcal{O}_Y(-D_1) \oplus \cdots \oplus \mathcal{O}_Y(-D_c))$, and the $D_j$ are $T$-invariant Cartier divisors with $|D_j|$ base-point free, then $E = X(\mathrm{div}_E, p^*a)$. Where $p\colon E \to Y$ is the projection.*

*Proof.* The pullback of a base-point free linear system is base-point free, so by induction, it suffices to check this for when $c = 1$.

In this case $E = \mathrm{Tot}(\mathcal{O}_Y(-D))$. The pullback map on divisors sending $\rho \mapsto p^{-1}(\rho)$ induces the pullback map $p^*\colon A_{n-1}(Y) \to A_n(E)$, and so if $a = [\alpha]$ then $p^*a = [(\alpha, 0)]$.

The polyhedral set $P_{(\alpha,0)}$ is defined by the inequality $\mathrm{div}_E(\xi \oplus s\xi) + (\alpha, 0) = (\mathrm{div}_Y(m) + sD + \alpha) \oplus s\sigma_0(Y) \geq 0$. Immediate observations we make are $s \geq 0$ and $P_{(\alpha,0)} \cap \{s = s_0\} = P_{\alpha+s_0 D} \oplus s_0$. The second observation becomes important in light of the fact that $|D|$ base-point free implies that $|s_0 D|$ is base-point free for $s_0 \in \mathbb{Z}_{>0}$, and very-ample plus base-point free implies very-ample. So $P_{\alpha+s_0D}$ has the same face structure as $P_\alpha$ and consequently the polytope $P_{p^*a}$ has the face structure of $P_\alpha \times [0, \infty)$.

The 1-cones in the fan of $E$ and $X(\mathrm{div}_E, p^*a)$ are the same, and the agreement of face structures implies $X(\mathrm{div}_E, p^*a)$ is covered by charts centered at torus fixed points of $Y$ just like $E$. This implies the $n$-cones agree as well, and thus the full fans. □

## Linear data associated to rational functions on toric varieties

In Section 1 we introduced the homomorphism $T \to (\mathbb{C}^\times)^\Xi$, given by $t \mapsto \xi_1(t)\xi_1 + \cdots + \xi_s(t)\xi_s$. This was used to write the space of rational functions on a toric variety with terms $\Xi = \{\xi_1, \ldots, \xi_s\}$ as a quotient $(\mathbb{C}^\times)^\Xi/T$.

The homomorphism induces a map between Lie algebras that is integral with respect to the kernels of the exponential maps. The kernel of the exponential map on $T$ is typically denoted $N$, and is naturally identified with the dual space to the characters, $\mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$, and the one parameter subgroups $\mathbb{C} \to T$ (see for instance [Ful93]). If we write $\mathbb{Z}^\Xi$ for the kernel of the exponential map on $(\mathbb{C}^\times)^\Xi$, the homomorphism between these lattices is

$$\mathrm{mon}\colon N \to \mathbb{Z}^\Xi, \tag{23}$$

where mon is short for "infinitesimal action on monomials".

The isomorphism $\exp(2\pi i -)\colon \mathbb{C}/\mathbb{Z} \to \mathbb{C}^\times$ induces an isomorphism between the exact sequences

$$N_{\mathbb{C}/\mathbb{Z}} \overset{\text{mon}}{\to} (\mathbb{Z}^\Xi)_{\mathbb{C}/\mathbb{Z}} \to \operatorname{coker}(\text{mon})_{\mathbb{C}/\mathbb{Z}} \to 0 \tag{24}$$

and

$$T \to (\mathbb{C}^\times)^\Xi \to (\mathbb{C}^\times)^\Xi/T \to 1. \tag{25}$$

This uses the standard identification of $T$ with $N_{\mathbb{C}^\times}$.

**Definition 2.7.** Let $W$ be a rational function on a toric variety, $X$, with terms $\Xi$. Denote by $L \in \operatorname{coker}(\text{mon})_{\mathbb{C}/\mathbb{Z}}$ the point corresponding to $W$. We define the linear $\mathbb{C}/\mathbb{Z}$-data of $W$ to be the pair $(\text{mon}, L)$.

**Lemma 2.8.** *Let $\lambda \in \mathbb{C}^\Xi$ such that $\lambda$ maps to $L$ in* $\operatorname{coker}(\text{mon})_{\mathbb{C}/\mathbb{Z}}$. *Let $W(\text{mon}, \lambda)$ be the function on $T$ determined by* mon *and $\lambda$, then there is a unique $x \in X$ such that $\iota^* W = W(\text{mon}, \lambda)$.*

*Proof.* This is an immediate consequence of the definition of mon, or equivalently the definition of the homomorphism $T \to (\mathbb{C}^\times)^\Xi$. ☐

It is worth pointing out that $W(\text{mon}, \lambda) = \sum_j \exp(2\pi i \lambda_j)\xi_j$.

**Remark 2.9.** Despite the fact that the linear data of a rational function only determines the function up to the action of $T$, we will refer to the "rational function" $W(\text{mon}, L)$. This is because the action of $T$ will be viewed as a coordinate choice and not intrinsic to the situation.

## 3. Toric Landau-Ginzburg models

In the introduction, a Landau-Ginzburg model was a superconformal theory defined by Kähler manifold, $X$, equipped with a holomorphic function $W\colon X \to \mathbb{C}$, where $W$ is called the superpotential. It is common to add an additional piece of information called the *B-field*. This is simply a cohomology class $b \in \mathrm{H}^2(X; \mathbb{R}/\mathbb{Z})$.

We will use the term *toric Landau-Ginzburg model* to mean the following information:

1) a toric variety $X$,
2) an element $K \in A_{n-1}(X)_{\mathbb{C}/\mathbb{Z}}$, and

3) a regular function $W\colon X \to \mathbb{C}$.

The class $K = b + ia$ is thought of as the B-field, $b$, and the Kähler class, $a$, packaged together into a single "complexified Kähler class".

In the cases where Corollary 1.6 applies, this is the same as a Landau-Ginzburg model except that we have retained only the class of the Kähler form and forgotten the form itself.

### Toric Landau-Ginzburg models defined by linear data

Associated to a toric Landau-Ginzburg model is its *linear data*:

1) the linear $\mathbb{C}/\mathbb{Z}$-data $(\mathrm{div}, K)$, and
2) the linear $\mathbb{C}/\mathbb{Z}$-data $(\mathrm{mon}, L)$,

One can also start from two sets of linear $\mathbb{C}/\mathbb{Z}$-data, $(A, K), (B, L)$, and, provided the polyhedral set defined by $(A, \Im(K))$ has non-empty interior, define a toric variety $X(A, \Im(K))$ and rational function $W(B, L)$. Here $\Im(K)$ denotes the imaginary part of $K$. This is not quite a toric Landau-Ginzburg model, because $K$ may not define a complexified Kähler class, or $W$ may not be regular.

**Definition 3.1.** To avoid problematic situations, we will define linear $\mathbb{R}-$data $(C, c)$ to be *kopasetic* if

1) the polyhedral set defined by $(C, c)$ has non-empty interior, and
2) there exists a surjection $k : \mathbb{Z}^r \to \mathbb{Z}^{R_{X(C,c)}}$ that sends standard generators to standard generators or zero, and the diagram

$$\begin{array}{ccc} M & \xrightarrow{C} & \mathbb{Z}^r \\ & \searrow^{\mathrm{div}_{X(C,c)}} & \big\downarrow k \\ & & \mathbb{Z}^{R_{X(C,c)}} \end{array} \tag{26}$$

commutes.

We will also denote by $k$ the induced map on the cokernels.

Note that $k$ is essentially unique in the sense that it is constructed, by eliminating unnecessary inequalities from the family $A(\xi) + \gamma \geq 0$, where $\gamma$

is a lift of $c$. Non-uniqueness arises if two of the inequalities are identical, in which case we drop one of them.

To address the regularity of $W(B, L)$ we first make the simple observation that a rational function $W$ is regular if and only if all its monomials are regular, and a monomial $\xi \in M_X$ is regular if and only if

$$\operatorname{div}(\xi) \geq 0. \tag{27}$$

One can easily use these facts to check the statement:

**Lemma 3.2.** *$W$ is regular* $\iff \operatorname{div} \circ \operatorname{mon}^\tau \geq 0.$

**Definition 3.3.** With this in mind we define a pair of linear $\mathbb{C}/\mathbb{Z}$-data, $(A, K)$, and $(B, L)$, to be *kopasetic* if

1) $(A, \Im(K))$ is kopasetic, and
2) $A \circ B^\tau \geq 0$.

**Definition 3.4.** Given a pair of kopasetic linear $\mathbb{C}/\mathbb{Z}$-data, $(A, K)$, and $(B, L)$, we define a toric Landau-Ginzburg model with

1) the toric variety $X(A, \Im(K))$,
2) the regular function $W(B, L)$, and
3) the complexified Kähler class $k(K)$.

### The dual toric Landau-Ginzburg model

So far we have seen how one can extract linear data from a toric Landau-Ginzburg data, and under kopasetic conditions define a toric Landau-Ginzburg model from a pair of linear $\mathbb{C}/\mathbb{Z}$-data.

On the level of linear data, there is a simple involution defined by simply interchanging the roles of $(\operatorname{div}, K), (\operatorname{mon}, L)$.

**Definition 3.5.** Let $(A, K), (B, L)$ be a pair of linear $\mathbb{C}/\mathbb{Z}$-data. Define the *dual* pair $(A', K'), (B', L')$, by

$$(A', K') = (B, L), \text{ and, } (B', L') = (A, K). \tag{28}$$

**Definition 3.6.** Given a toric Landau-Ginzburg model, $(X, W, K)$, assume that the dual to its linear data is kopasetic. Define the *dual toric Landau-Ginzburg model* to be the toric Landau-Ginzburg model defined by the dual linear data. Denote the dual by $(X', W', K')$.

**Remark 3.7.** To check that the dual data is kopasetic, one only needs to verify that $(A', \Im(K'))$ is kopasetic since $A' \circ B'^{\tau} = (A \circ B^{\tau})^{\tau}$, and $W$ is regular so $(A \circ B^{\tau}) \geq 0$.

## 4. The sigma model / Landau-Ginzburg model correspondence

In physics, a *sigma model* is a superconformal theory defined by Kähler manifold $Z$ equipped with a B-field $b \in \mathrm{H}^2(Z, \mathbb{R}/\mathbb{Z})$. We will take a sigma model to be an algebraic variety with a choice of complexified Kähler class in its Chow group of divisors. This class will be a pullback from such a class on an ambient space.

One can produce interesting toric Landau-Ginzburg models from the sigma model / Landau-Ginzburg model correspondence applied to a complete intersection in a toric variety.

As mentioned in the introduction, when a subvariety $Z$ of a variety $Y$ is the zero locus of a global section $\mathrm{w}$ of a vector bundle $\mathcal{V}$, we can define a Landau-Ginzburg model. Strictly speaking, this information defines a morphism $W \colon X \to \mathbb{C}$ from the total space of the dual bundle $\mathcal{V}^{\vee}$, $X = \mathrm{Tot}(\mathcal{V}^{\vee})$, to $\mathbb{C}$. The morphism is defined by the pairing $\mathcal{V} \otimes \mathcal{V}^{\vee} \to \mathcal{O}_X$ that identifies sections of $\mathcal{V}$ with functions on $X$.

If $Y$ is equipped with complexified divisor class $K \in A_{n-1}(Y)_{\mathbb{C}/\mathbb{Z}}$, then we can pull it back to define a complexified divisor class on $X$. Also if $Y$ is toric and $\mathcal{V}$ is split, then it is easy to see that $X$ is a toric variety. So we make the following definition.

**Definition 4.1.** Let $\mathcal{V}$ be a rank $c$ split bundle over an $n$-dimensional toric variety $Y$. Let $\mathrm{w}$ be a global section of $\mathcal{V}$, and let $K \in A_{n-1}(Y)_{\mathbb{C}/\mathbb{Z}}$. Finally let $Z$ be the zero locus of $\mathrm{w}$. We define *toric Landau-Ginzburg model corresponding to* $(Z, K)$ to be $(X, W, K)$, where $X = \mathrm{Tot}(\mathcal{V}^{\vee})$, $W$ is the function defined by $\mathrm{w}$, and $K$ is the pullback to $A_{n+c-1}(X)$ of $K \in A_{n-1}(Y)$.

Note that the resulting toric Landau-Ginzburg model actually depends on $Y$, $\mathrm{w} \in \Gamma(Y, \mathcal{V})$, and $K$, rather than $(Z, K)$. Also notice $A_{n+c-1}(X) = A_{n-1}(Y)$.

**Remark 4.2.** As explained to us [Sha]: in the literature, when studying a sigma model $Z$ it is a common trick to move to a "Landau-Ginzburg point" of the gauged linear sigma model Kähler moduli space of $Z$ to obtain a Landau-Ginzburg model whose $B$-twist is the same as that of $Z$. We are not using this Landau-Ginzburg model. The Landau-Ginzburg model $(X, W, K)$ and the sigma model $(Z, K)$ lie in the same universality class and so truly define the same superconformal theory. See [GS08] for a detailed treatment.

### Linear data associated to $(X, W, K)$

An immediate consequence of Lemma 2.5 is the following formula for the map $\operatorname{div}_X$.

**Corollary 4.3.** *(of Lemma 2.5) If $D_1, \dots, D_c$ are $T$-invariant Cartier divisors and $X$ is the total space of the split bundle $\mathcal{O}_Y(-D_1) \oplus \cdots \oplus \mathcal{O}_Y(-D_c)$ over a toric variety $Y$, then the character group of $X$ is*

$$M_X = M_Y \oplus \mathbb{Z}\sigma_1 \oplus \cdots \oplus \mathbb{Z}\sigma_c, \tag{29}$$

*where $\sigma_j$ is a rational section of $\mathcal{O}_Y(D_j)$ whose divisor is $D_j$. The $T$-invariant Weil divisors of $X$ are the preimages under $p$ of the $T$-invariant Weil divisors of $Y$ as well as the total spaces $X_j$ of the $c$ subbundles $\mathcal{V}_j^\vee$, where $\mathcal{V}_j^\vee$ is the dual bundle to $\ker(\pi_j \colon \mathcal{V} \to \mathcal{O}(D_j))$. Furthermore,*

$$\operatorname{div}_X = \left[\begin{array}{c|c|c|c} \operatorname{div}_Y & D_1 & \cdots & D_c \\ 0 & \multicolumn{3}{c}{\mathrm{Id}} \end{array}\right] \tag{30}$$

*with respect to the decomposition of $M_X$ above and $\mathbb{Z}^{R_X} = \mathbb{Z}^{R_Y} \oplus \mathbb{Z}X_1 \oplus \cdots \oplus \mathbb{Z}X_c$.*

*Proof.* This formula is obtained by repeated application of Lemma 2.5. □

The linear data corresponding to the superpotential $W$ is easily obtained via the following lemma and the standard practice of identification of $T$-linearized global sections of $\mathcal{O}_Y(D)$ with integral points of the polytope $P_D$.

Rather than $\operatorname{mon}\colon N \to \mathbb{Z}^{\Xi}$, it is easier to write down the transpose, $\operatorname{mon}^\intercal$. Using the usual identification of characters and covectors,

$$\xi \leftrightarrow d\xi|_{Id}, \tag{31}$$

and of the transpose and the pullback, we get the lemma below.

**Lemma 4.4.** *Let $X$ be a toric variety with character group $M$, and $W$ a rational function on $X$ with terms $\Xi$.* $\mathrm{mon}^\tau : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Xi}, \mathbb{Z}) \to M$*, takes the basis element dual to $1\xi_j \in \mathbb{Z}^{\Xi}$ to $\xi_j \in M$.*

**Definition 4.5.** The set of terms of $W$, $\Xi_W$, is naturally identified with a subset of $\coprod_j (P_{D_j} \cap M_Y)$. Where $P_{D_j}$ is the polytope corresponding to $D_j \in \mathbb{Z}^{R_Y}$. If $\Xi_W$ is in bijection with this set, we say w is *generic*.

**Lemma 4.6.** *In the generic case the transpose map,* $\mathrm{mon}^\tau$*, is given by the matrix*

$$\begin{bmatrix} P_{D_1} \cap M_Y & | & \cdots & | & P_{D_c} \cap M_Y \\ \xi_1 & | & \cdots & | & \xi_c \end{bmatrix} \tag{32}$$

*with respect to the decomposition of $M_X$ above and the identification* $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Xi_W}, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(P_{D_1} \cap M)}, \mathbb{Z}) \oplus \cdots \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(P_{D_c} \cap M)}, \mathbb{Z})$*.*

*Proof.* A one parameter subgroup acts on a monomial coefficient by multiplication of the coefficient by the subgroup plugged into the monomial itself. Therefore the transpose simply picks out the appropriate monomial. □

**Corollary 4.7.** *If we write $P^{\times}_{D_j}$ for $P_{D_j} \setminus \{0\}$ and $0_j$ for the $j^{th}$ zero in $\coprod_j (P_{D_j} \cap M_Y)$, the matrix* $\mathrm{mon}^\tau$ *takes the form*

$$\begin{bmatrix} P^{\times}_{D_1} \cap M_Y & | & \cdots & | & P^{\times}_{D_c} \cap M_Y & | & 0 \\ \sigma_1 & | & \cdots & | & \sigma_c & | & \mathrm{Id} \end{bmatrix}, \tag{33}$$

*where now*

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Xi_W}, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(P^{\times}_{D_1} \cap M)}, \mathbb{Z}) \oplus \cdots \oplus \mathrm{Hom}(\mathbb{Z}^{(P^{\times}_{D_c} \cap M)}, \mathbb{Z}) \\ \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}0_1, \mathbb{Z}) \oplus \cdots \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}0_c, \mathbb{Z}).$$

*Proof.* This is just the matrix of Lemma 4.6 with the columns permuted. □

**Definition 4.8.** Following the decompositions above, we write $\Xi^{\times} := \coprod_j (P^{\times}_{D_j} \cap M_Y)$, and so $\mathbb{Z}^{\Xi_W} = \mathbb{Z}^{\Xi^{\times}} \oplus \mathbb{Z}0_1 \oplus \cdots \oplus \mathbb{Z}0_c$.

**Definition 4.9.** $N_X$ also has a natural decomposition in terms of the one parameter subgroups associated to $Y$ and the summands of $\mathcal{V}^{\vee}$ :

$$N_X = N_Y \oplus \mathbb{Z}\phi_1 \oplus \cdots \oplus \mathbb{Z}\phi_c. \tag{34}$$

Here $\phi_j$ is just multiplication by $\mathbb{C}^{\times}$ along the $j^{\mathrm{th}}$ summand of $\mathcal{V}^{\vee}$ and the elements of $N_Y$ are constant in the fiber direction.

**Remark 4.10.** In this decomposition $\mathbb{Z}\sigma_1 \oplus \cdots \oplus \mathbb{Z}\sigma_c \subset N_Y^{\perp}$, and thus the pairing between $M_X$ and $N_X$ is the sum of the pairing for $Y$ and pairing between $\mathbb{Z}\sigma_1 \oplus \cdots \oplus \mathbb{Z}\sigma_c$ and $\mathbb{Z}\phi_1 \oplus \cdots \oplus \mathbb{Z}\phi_c$ where $\langle \sigma_i, \phi_j \rangle = \delta(i-j)$.

## 5. Toric Landau-Ginzburg models dual to sigma models: existance

Corollaries 4.3 and 4.7, give a very concrete picture of the linear data of a toric Landau-Ginzburg model $(X, W, K)$ obtained from a complete intersection $Z$ determined by a global section w of a bundle $\mathcal{V}$ over a toric variety $Y$.

In this section, we show the dual data, $(A', K')$, and $(B', L')$ is kopasetic provided $(A', \Im(K'))$ is kopasetic, as mentioned in Remark 3.7. This involves checking two things

1) the polytope defined by $(A', \Im(K'))$ has non-empty interior, and
2) there exists a surjection $k$ as in Equation (26).

**Lemma 5.1.** *Assuming the codimension, $c$, of $Z$ is positive, $(A', \Im(K'))$ is kopasetic for all values of $\Im(K')$.*

*Proof.* The existence of $k$ is immediate, since the columns of $\mathrm{mon}_W^{\tau}$ distinct an primitive. Now let $\alpha'$ denote a lift of $\Im(K')$, and consider the polyhedral set $P_{\alpha'}$. The inward normals of facets of $P_{\alpha'}$ form a subset of the columns of $\mathrm{mon}_W^{\tau}$. These normals all live in the interior of the half-space of $N'_{\mathbb{R}} = (M_X)_{\mathbb{R}}$, $H = \{\xi \in (M_X)_{\mathbb{R}} \mid \sum_j \phi_j(\xi) \geq 0\}$, where the $\phi_j$ are as in Equation (34). The lemma now follows from the following statement: *The intersection of affine half-spaces, such that the non-negative $\mathbb{R}$-span of their normals does not contain a half-space, is non-empty.*

Define $C$ to be the cone dual to the cone defined by the non-negative span of the columns of $\mathrm{mon}_W^{\tau}$. $C$ has 0 as an apex since its dual is contained in a half-space, and more importantly it has a non-empty interior since its properly contained in a half-space. If $p \in P_\alpha$ is is clear that $p + C \subset P_\alpha$ since the affine half-spaces defining $P_\alpha$ can only be translates away from $p$. On the other hand $P_\alpha$ can be obtained by intersection translates of half-planes containing $C$, so it is clear that $P_\alpha \neq \varnothing$. □

**Corollary 5.2.** *A toric Landau-Ginzburg model $(X, W, K)$ obtained from a complete intersection $Z \neq Y$ determined by a global section* w *of a bundle $\mathcal{V}$ over a toric variety $Y$ has a dual toric Landau-Ginzburg model $(X', W', K')$.*

## 6. Toric Landau-Ginzburg models dual to sigma models: structure

The toric variety $X'$ is very closely related to the total space $E'$ of some vector bundle $(\mathcal{V}')^\vee$ over a toric variety $Y'$. We will define these objects below.

We proceed to give sufficient conditions under which $X' = E'$, first in terms of the columns of $\mathrm{mon}_W^\tau$, or equivalently the rows of $A'$. Finally, if $X' = E'$ then we give conditions under which $W'$ comes from a global section of $\mathcal{V}'$. In other words, we give conditions under which we can construct a dual sigma model $(Z', K')$ to the sigma model $(Z, K)$ corresponding to $(X, W, K)$.

### The toric varieties $Y'$ and $E'$.

Corollary 4.7, describing $\mathrm{mon}_W^\tau$, implies that since $A' = \mathrm{mon}_W$ it has the block form

$$A' = \left[ \begin{array}{c|c} d' & D' \\ 0 & \mathrm{Id} \end{array} \right]. \tag{35}$$

This is with respect to the decompositions $N_X = N_Y \oplus \mathbb{Z}\phi_1 \oplus \cdots \oplus \mathbb{Z}\phi_c$ and $\mathbb{Z}^\Xi = \mathbb{Z}^{\Xi^\times} \oplus \mathbb{Z}0_1 \oplus \cdots \oplus \mathbb{Z}0_c$.

The identity matrix in the lower right block guarantees $\mathrm{coker}(A') = \mathrm{coker}(d')$. Thus we can find a lift of $\Im(K')$ of the form $(\alpha', 0)$, where $\alpha' \in \mathbb{Z}^{\Xi^\times}$. We would like to make the definition $Y' = X(d', \Im(K'))$, but unfortunately, there is no guarantee that $(d', \Im(K'))$ is kopasetic.

We will make the assumption that $(d', \Im(K'))$ is kopasetic. This is a comfortable assumption to make in light of the following lemma.

**Lemma 6.1.** *There is a non-empty open cone $U_{kopa}$ of* $\mathrm{coker}(d')$ *such that $(d', a')$ is kopasetic if and only if $a' \in U_{kopa}$.*

*Proof.* An affine half-space can be translated to contain the origin within its interior. Choose such an arrangement for all the affine half-spaces with inward normals coming from the row of $d'$. This corresponds to a point $a'_0 \in \mathrm{coker}(d')$. The origin remains in the interior of the intersection the of the half-spaces for small deformations of $a'$ such that $(d', a')$ is kopasetic. □

Note that it is standard to interpret moving from one value $a'_0$ to another $a'$ as a deformation of the symplectic structure of the toric variety $X(d', a'_0)$. This may involve birational transformations corresponding to when the volumes of curves (or higher dimensional subvarieties) shrink to 0.

**Definition 6.2.** Define the toric variety

$$Y' = X(d', \Im(K')). \tag{36}$$

Denote the map of Equation (26) by $k_{Y'}$ and the columns of $D'$ by $D'_j$. These are elements of $\mathbb{Z}^{\Xi^\times}$. We now have the vector bundle

$$\mathcal{V}' = \mathcal{O}_{Y'}(k_{Y'}(D'_1)) \oplus \cdots \oplus \mathcal{O}_{Y'}(k_{Y'}(D'_c)), \tag{37}$$

the toric variety

$$E' = \mathrm{Tot}(\mathrm{Hom}_{\mathcal{O}_{Y'}}(\mathcal{V}', \mathcal{O}_{Y'})), \tag{38}$$

and a function $W'$ defined on it by $B'$ and $L'$.

### Conditions for $X' = E'$

**Definition 6.3.** We will spend the rest of the section discussing the "rows" of the homomorphisms $A'\colon M_{X'} \to \mathbb{Z}^{\Xi}$ and $d'\colon M_{Y'} \to \mathbb{Z}^{\Xi^\times}$. By this we mean the images under the transpose of the standard basis vectors. The rows of $d'$ may be a multiset.

Keep in mind that we will be using the follow identifications.

$$\begin{aligned} M_Y &= N_{Y'}\ , \\ N_Y &= M_{Y'}\ , \\ M_X &= N_{E'}\ , \text{ and} \\ N_X &= M_{E'}\ . \end{aligned} \tag{39}$$

The strategy for comparing $X'$ and $E'$ is based on comparing $\mathrm{div}_{X'}$ and $\mathrm{div}_{E'}$. Both of these toric varieties are defined from rational convex polyhedral sets. The defining inequalities for $X'$ come directly from the matrix $A' = \mathrm{mon}_W$, and the element $a' \in \mathrm{coker}(A')_{\mathbb{C}/\mathbb{Z}}$.

On the other hand $E'$ is formed by first using the upper left block, $d'$, of $A'$ to define the toric variety $Y'$. Then a certain submatrix of the upper right block $D'$, of the $A'$ are treated as divisors and we get the family of inequalities coming from the formula in Equation (4.3).

We will compare

$$\mathrm{div}_{X'} = k_{X'} \circ A'$$

and

$$\mathrm{div}_{E'} = \begin{bmatrix} k_{Y'} \circ d' & | & k_{Y'} \circ D' \\ 0 & | & \mathrm{Id} \end{bmatrix}.$$

Post-composition with $k$ results in deleting rows whose affine half-space is not necessary for defining the polytope. We need some conditions under which the rows deleted from $A'$ using $k_{X'}$ are the same of those deleted using $k_{Y'}$. Recall, the rows of $A'$ are of the form $(P_{D_j} \cap M_Y) \times \{\sigma_j\}$.

The only case when such a comparison make sense is when $a'$ lifts to $(\alpha', 0) \in \mathbb{Z}^{\Xi^\times} \oplus \mathbb{Z}0_1 \oplus \cdots \oplus \mathbb{Z}0_c$, such that $\alpha' > 0$, as pointed out in the lemma below.

**Lemma 6.4.** *If $E'$ is defined (i.e. $(d', a')$ is kopasetic), then $a'$ has a lift as above.*

*Proof.* Choose $\alpha'$ so that 0 lies in the interior of the polyhedral set defining $Y'$. □

**Definition 6.5.** Let $\alpha' = (\alpha'_1, \ldots, \alpha'_c)$ with respect to the decomposition $\Xi^\times = \coprod_j (\Gamma(Y, \mathcal{O}_Y(D_j)) \cap \Xi^\times)$. Furthermore, we can write $\alpha'_j$ in components as $\alpha'_j = (\alpha'_{(\nu,\sigma_j)})_{\nu \in P^\times_{D_j}}$.

**Lemma 6.6.** *The facets of the polytope of $X'$ correspond to the non-zero vertices of the convex hull $C_j$ points*

$$\{(\nu, \sigma_j)/\alpha'_{(\nu,\sigma_j)} \mid \nu \in P^\times_{D_j} \cap M_Y\} \cup \{\lambda(0, \sigma_j) \mid 0 \le \lambda \in \mathbb{R}\}, \tag{40}$$

*including the vertex "$\infty(0, \sigma_j)$".*

*Proof.* If we deform $(\alpha', 0)$ to $(\alpha', \epsilon)$, we can use Corollary 1.4 to write the dual polytope to the one defined by $A'$ can be expressed as the convex hull of the points $(\nu, \sigma_j)/\alpha'_{(\nu,\sigma_j)}$ and $(0, \sigma_j)/\epsilon$, $j = 1, \ldots, c$.

If a convex polyhedral set $C$ in a vector space $V \oplus W$ defined by points that lie in either $V \oplus \{0\}$ or $\{0\} \oplus W$. Has vertices that are exactly the vertices of convex hull of the points in $V \oplus \{0\}$ and the vertices of the convex hull of the points in $C \cap \{0\} \oplus W$.

Putting these facts together and taking the limit $\epsilon \to 0$ gives the result. □

**Definition 6.7.** Denote the projection $(M_{Y'})_{\mathbb{R}} \oplus \mathbb{R}\sigma_j \to (M_{Y'})_{\mathbb{R}}$, by $\pi_j$, the vertices of $C_j$ by $V_j$, and the set $V_j \setminus \{(0,0),(0,\sigma_j)\}$ by $V_j^{\times}$.

**Theorem 6.8.** *A row* $\nu \in P^{\times}_{D_j}$ *appears in* $\mathrm{div}_{Y'}$ *iff* $\pi_j((\nu,\sigma_j)/\alpha_\nu)$ *defines a non-zero vertex of* $\mathrm{conv}(\{0\} \cup \bigcup_j \pi_j(V_j))$*.*

*Proof.* The projection takes $(\nu,\sigma_j)/\alpha'_{(\nu,\sigma_j)}$ to $\nu/\alpha'_{(\nu,\sigma_j)}$, and it is the convex hull of $\{0\}$ and these points is exactly the dual polytope to the on defined by $(d',\alpha')$. □

**Theorem 6.9.** *(Assuming $a'$ has a lift as above) $X' = E'$ if and only if, for all $j$, every element in $V_j^{\times}$ defines a vertex of* $\mathrm{conv}(\{0\} \cup \bigcup_j \pi_j(V_j))$*.*

*Proof.* This simply states that the rows are the same, which is exactly what we need. □

### The dual superpotential

Finally, we discuss the superpotential $W'$ on the dual. First we discuss arbitrary functions on the total space, $E'$, of a split vector bundle over a toric variety $Y'$.

Let $Y'$ be a toric variety, $\{D'_1,\ldots,D'_c\}$ a set of $T$-invariant divisors, and $f$ a function on $E' := \mathrm{Tot}(\mathcal{O}_Y(-D'_1) \oplus \cdots \oplus \mathcal{O}_Y(-D'_c))$. We want to know when $f$ comes from a global section of $\mathcal{O}_Y(D'_1) \oplus \cdots \oplus \mathcal{O}_Y(D'_c)$.

The character group of $E'$ is given by $M_{Y'} \oplus \mathbb{Z}\xi'_1 \oplus \cdots \oplus \mathbb{Z}\xi'_c$. Meromorphic global sections have the form $f = f_1\xi'_1 + \cdots + f_c\xi'_c$, where the $f_j$'s have terms in $M_{Y'}$. The key here is that $f$ is a linear form in the $\xi'$'s. As we mentioned before in Equation (27), $\mu \in M_{E'}$ is regular iff $\mathrm{div}_{E'}(\mu) \geq 0$.

Now, assume that $X = \mathrm{Tot}(\mathcal{O}_Y(-D_1) \oplus \cdots \oplus \mathcal{O}_Y(-D_c))$ over a toric variety $Y$. Also assume the dual toric variety $X(A',a')$ equals $E'$.

**Lemma 6.10.** *$W'$ comes from a global section exactly when $\exists$ effective $T$-invariant divisors on $X$ such that $\tilde{D}_1,\ldots,\tilde{D}_c$ such that $D_j \sim \tilde{D}_j$, and $\tilde{D}_1 + \cdots + \tilde{D}_c = -\kappa_X$. Where $-\kappa_X$ is the canonical choice of anticanonical divisor representative of Equation (12).*

*Proof.* The terms of $W'$ are the rows of $A = \mathrm{div}_X$. The lemma is then clear from the matrix of $A^{\tau}$, see Theorem 4.3. □

**Remark 6.11.** The dual to $(X',W',K')$ is computed from the dual of its linear data. This data is the same as that of $(X,W,K)$ except some rows of

$\text{mon}_W$ may have been deleted. This means that the double dual of $(X, W, K)$ is $(X, W^-, K)$, where $X$ and $K$ are the same and $W^-$ is obtained from $W$ by deleting some terms.

If $(X, W, K)$ is obtained from a sigma model then $(X, W^-, K)$ comes from a sigma model in the same ambient space and the same bundle. However the sigma models might differ by a complex deformation. It is possible that this unfortunate discrepancy can be addressed with a slight modification of the dualization process. We will explain this in the discussion section the end of this paper.

## An example: three points on $\mathbb{P}^1$

A configuration on three points on $\mathbb{P}^1$ corresponds to a Landau-Ginzburg model $X = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-3))$, $W$, and $K$. If we identify $X$ with the blow up of $\mathbb{C}^2/\mathbb{Z}_3$ at the fixed point, $W$ is the pullback of a degree three homogeneous polynomial on $\mathbb{C}^2$ pushed forward to a function on the quotient.

The character group is generated by elements that correspond to $u/v$ and $u^2v$ on $\mathbb{C}^2$. With respect to these coordinates we have

$$\text{div}_X = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \text{mon}_W = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -2 & 1 \\ 0 & 1 \end{bmatrix}. \tag{41}$$

If we write $x$ and $y$ for the coordinates dual to these on $X'$, there are four possibilities for $X'$ depending on the choice of $W$. These are indicated by the polyhedral sets in Figure 1. If $\Lambda$ is a lift of $L$, examples of values that give these are $\Im(\Lambda) = (\alpha', 0) = (0, 2, 5, 0), (0, 3, 5, 0), (-1, -1, 0, 0)$, and $(-1, 0, -1, 0)$ respectively.

The inward normals come from the rows of $\text{mon}_W$. The superpotential $W'$ above, has monomials whose exponent vectors are the rows of $\text{div}_X$. $q$ is determined $K$ by choosing a lift of $K$ of the form $(\beta + i\alpha, 0, 0)$ and setting $q = \exp(-2\pi(\alpha + i\beta))$. $W'$ is not linear in $y$, so it will not come from a global section of $E'$.

$$d' = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \tag{42}$$

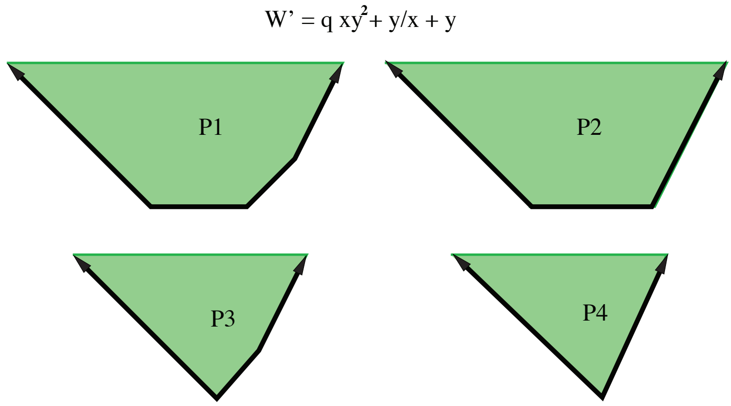


Figure 1: Mirror possibilities for three points on $\mathbb{P}^1$.

The toric variety $Y'$ is given by the polyhedral set in $\mathbb{R}$ defined by

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \xi + \alpha' \geq 0 \; . \tag{43}$$

For the polyhedral set P1, $(d', a')$ is kopasetic. The other polyhedral sets can give kopasetic data at specific values. For example P2 for $\alpha' = (0, 1, 2, 0)$, P3 for $\alpha' = (-1, -1, 0, -1)$, and P4 for $\alpha' = (0, 0, 0, 0)$. One can easily check this by noting that the rows $d'$ that define facets, must be primitive for the map $k$ to exist. However, we will only look at P1 since these other cases do not shed more light on the situation.

In this case

$$\mathrm{div}_{Y'} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} , \tag{44}$$

and $k$ projects onto the first two basis vectors, so $k((0, 2, 5, 0)) = (0, 2)$.

For the divisors we have

$$D' = D'_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \quad k(D'_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{45}$$

This means that $Y' = \mathbb{P}^1$, and $E'$ is the total space of $\mathcal{O}(-2)$. The polytope corresponding to this toric variety is below.

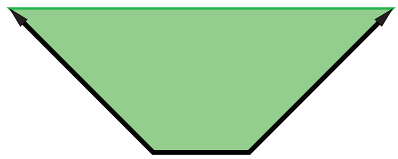

Figure 2: Tot $\mathcal{O}(-2)$ over $\mathbb{P}^1$.

This figure does not agree with P1. This is predicted by Theorem 6.9 since all facets are present, but for $Y'$ not all are needed. One interesting thing that is apparent in this example and happens in general is that the elements of $V_j^\times$ that are not vertices of $\operatorname{conv}(\pi_j(V_j^\times))$ serve to partially compactify $E'$. Furthermore, both $E'$ and $X'(A', \Im(K'))$ are always local Calabi-Yau (i.e. have trivial canonical class).

## 7. Comparison: Givental and Hori-Vafa

### 7.1. Givental

Let $Y$ be a $n$-dimensional smooth complete toric variety with $T$-invariant Cartier divisors $R_Y = \{\rho_1, \ldots, \rho_r\}$. Recall,

$$\operatorname{coker}(\operatorname{div}_Y) = \mathrm{H}^2(Y; \mathbb{Z}) \tag{46}$$

and is torsion free.

Let $D_1, \ldots, D_c$ be effective $T$-invariant Cartier divisors. Set $\mathcal{V} := \mathcal{O}_Y(D_1) \oplus \cdots \oplus \mathcal{O}_Y(D_c)$, as usual, and write $X$ for the total space of the dual. Assume that there is no point in $Y$ at which all global sections of $\mathcal{V}$ vanish.

Let $p_1, \ldots, p_{r-n}$ be a positive basis for $\mathrm{H}^2(Y; \mathbb{Z})$, and $\omega$ a symplectic form on $Y$ with cohomology class

$$[\omega] = \sum_i t_i p_i. \tag{47}$$

Denote $\exp(t_i)$ by $Q_i$. Set

$$[\rho_v] = \sum_i m_{iv} p_i \quad \text{and} \quad [D_j] = \sum_i d_{ij} p_i. \tag{48}$$

So a representative for $t$ is $\hat{t}$ with $\sum \hat{t}_v m_{iv} = t_i$.

In this notation, $[-]\colon \mathbb{Z}^{R_X} \to \mathrm{H}^2(Y;\mathbb{Z})$ is given by

$$\left[\ \mathcal{M} \quad | \quad -D\ \right]. \tag{49}$$

Define

$$F(x,y) := x_1 + \cdots + x_r + y_1 + \cdots + y_c \tag{50}$$

as a function on

$$H' := \left\{ (x,y) \mid \prod_{v=1}^{r} x_v^{m_{iv}} = Q_i \prod_{j=1}^{c} y_j^{d_{ij}} \ ,\ i = 1,\ldots,r-n \right\} \subseteq (\mathbb{C})^{r+c}_{x,y} \times (\mathbb{C}^\times)^{r-n}_{Q}. \tag{51}$$

According to [Giv98a, pg. 45], relations in the quantum cohomology of the zero locus $(\mathrm{w})_0$ of a generic section w of $\mathcal{V}$ can be described by differential operators annihilating integrals of the form

$$\int \exp(F/\hbar) \prod_{v=1}^{r} x_v^{\lambda_v/\hbar} \prod_{j=1}^{c} y_j^{-\lambda'_j/\hbar} \quad \mathrm{dlog}(x) \wedge \mathrm{dlog}(y)/\mathrm{dlog}(Q) \tag{52}$$

along certain cycles in the fiber over $Q$, $H'_Q$. Here $\mathrm{dlog}(x) = \mathrm{dlog}(x_1) \wedge \cdots \wedge \mathrm{dlog}(x_r)$, $\mathrm{dlog}(y) = \mathrm{dlog}(y_1) \wedge \cdots \wedge \mathrm{dlog}(y_c)$, and $\mathrm{dlog}(Q) = \mathrm{dlog}(Q_1) \wedge \cdots \wedge \mathrm{dlog}(Q_{r-n})$.

Note that the quotient of differential forms is unambiguous since any form annihilated when multiplied by $\mathrm{dlog}(Q)$ restricts to zero along $Q =$ constant.

Let $(X, W, K)$ be defined by w and $K = -it/2\pi \in \mathrm{H}^2(Y;\mathbb{C}/\mathbb{Z})$. Denote the dual Landau-Ginzburg model by $(X', W', K')$. We will compare $(X', W', K')$ to $(H_Q, F, 0)$. The main step is the theorem below. We restate our assumptions about $Y$ and $\mathcal{V}$ for clarity.

**Theorem 7.1.** *Assume $Y$ is smooth and there is no point that lies on $(\sigma)_0$ for all $\sigma \in \Gamma(Y,\mathcal{V})$. Then $M^+_{X'}$ is generated by the terms $\xi' \in \Xi'$ of $W'$, and there is an isomorphism for any $Q$,*

$$\mathrm{Spec}(\mathbb{C}[M^+_{X'}]) \xrightarrow{\sim} H'_Q \subset \mathbb{C}^{r+c}\ . \tag{53}$$

*If enumerate the terms of $\Xi'$ in the same way as the divisors of $X$, $R_X = \{\rho_v\}_{v=1}^{r} \cup \{X_j\}_{j=1}^{c}$, the map is given by $x_v \mapsto \hat{q}_v \xi'_v$ and $y_j \mapsto \xi_j$. Here $\hat{q}_v := \exp(\hat{t}_v)$.*

*Proof.* Since $Y$ is complete, applying $\mathrm{Hom}(-,\mathbb{Z})$ to Equation (11) yields the exact sequence

$$0 \to \mathrm{H}^{2n-2}(Y;\mathbb{Z})^\vee \to \mathrm{Hom}(\mathbb{Z}^{\Xi'},\mathbb{Z}) \to M_{X'} \to 0. \tag{54}$$

The image of the standard basis vectors satisfy relations coming from the rows of

$$\left[\ \mathcal{M} \quad \middle| \quad -D\ \right]. \tag{55}$$

Therefore, $H' = \mathrm{Spec}(\mathbb{C}[\ \xi' \mid \xi' \in \Xi'\ ])$.

It remains to check that the terms of $W'$ generate the semigroup $M^+_{X'}$. All of the terms are global, so they lie in $M^+_{X'}$.

The terms of $W'$ are the same as the inward normals of the polytope defining the original toric variety $X$, and their $\mathbb{R}_{\geq 0}$-span intersected with $M_{X'}$ is $M^+_{X'}$. One way to see this is to consider a 1-parameter subgroup $\nu$ of $T$ that does not lie in the fan of $X$. Consider the fan of the projective bundle compactifying $X$ to a complete toric variety. Then it is clear that $\lim_{\lambda\to 0}\nu(\lambda)$ lies in on the divisor at infinity. All of the monomials defining $W$ have a pole along this divisor and there is no chance that all the terms of $W$ have $\nu(0)$ on their divisor of zeros since this would mean that the sections they correspond to would all vanish at the point $\nu(0)$ projects to in $Y$. Finally, this means that there is at least one term for which the limit at $\nu(0)$ is infinity, and so $n$ is not in $M^+_{X'}$.

It remains to check positive integer multiples of these normals pick up every integer point in the cone they define. The cone they generate is naturally decomposed into the fan of $X$. Since the fan is contained in a half-space, we pick up all integral points with integral linear combinations of the normals if and only if $X$ is smooth.

The last observation needed to guarantee that we land in $H'_Q$ under the isomorphism above is that $Q_i = \prod_v \hat{q}_v^{m_{iv}}$. □

We see from the proof that if there is a point at which all global sections vanish, then $\mathrm{Spec}(\mathbb{C}[M^+_{X'}])$ is an affine open set of $H'_Q$. If $Y$ is singular, then there is an étale map $\mathrm{Spec}(\mathbb{C}[M^+_{X'}]) \to H'_Q$. In any case we have a map $\mathrm{Spec}(\mathbb{C}[M^+_{X'}]) \to H'_Q$.

The last thing to check is that $F$ pulls back to $W'$, but this is obvious from the definition of the isomorphism.

So we get the following theorem.

**Theorem 7.2.** *There is a morphism $X' \to \mathbb{C}^{r+c}$ such the image is $H_Q$ and the function $F(x,y) = x_1 + \cdots + x_r + y_1 + \cdots + y_c$ pulls back to $W'$. This morphism is naturally interpreted as the result of the Kähler degeneration of $X'$ under which $K' \rightsquigarrow 0$.*

**Remark 7.3.** If $L$ already equals zero, then $X' = H_Q$. The section corresponding to $L = 0$ of $\mathcal{V}$ is the $\mathrm{w}_{GHV}$ mentioned in the introduction.

### 7.2. Hori and Vafa

In this section we show that the mirror used by Hori and Vafa is the same as the one used by Givental.

To get started, observe the integral above in Equation (52) used by Givental can be manipulated according to the following rule for forms with delta functions as coefficients:

$$\delta(z)\varphi = (\varphi/dz)|_{z=0}. \tag{56}$$

With repeated application of this rule we can write the integral of Givental on $(n+c)$-cycles in $H'_Q$ as

$$\int \exp(F/\hbar) \prod_{v=1}^{r} x_v^{\lambda_v/\hbar} \prod_{j=1}^{c} y_j^{-\lambda'_j/\hbar} \prod_{i=1}^{r-n} \delta(\log(Q_i) - t_i) \mathrm{dlog}(x) \wedge \mathrm{dlog}(y) \tag{57}$$

over $(r+c)$-cycles in $\mathbb{C}^{r+c}$.

Now we consider the construction of Hori and Vafa. As before $Y$ is an $n$-dimensional toric variety, This time it is obtained as a quotient $\mathbb{C}^r//(\mathbb{C}^\times)^{r-n}$. This can always be done as shown by Cox [Cox95]. Let $\mathfrak{m}_{iv}$ be the weight of the action of $i^{\text{th}}$ $\mathbb{C}^\times$ on the $v^{\text{th}}$ $\mathbb{C}$. Let $G_1, \ldots, G_c$ be multi-homogeneous polynomials on $\mathbb{C}^r$ where $\mathfrak{d}_{ij}$ is the degree of the $j^{\text{th}}$ polynomial with respect to the $i^{\text{th}}$ $\mathbb{C}^\times$.

Set $\mathfrak{W}' = \sum_v \exp(-\mathfrak{Z}_v) + \sum_j \exp(-Y_j)$. In [HV00] the following integrals are considered:

$$\int \prod_{v=1}^{r} \mathrm{d}\mathfrak{Z}_v \prod_{j=1}^{c} \mathrm{d}Y_j \prod_{j=1}^{c} \exp(-Y_j) \times \prod_{i=1}^{r-n} \delta\left(\sum_{v=1}^{r} \mathfrak{m}_{iv}\mathfrak{Z}_v - \sum_{j=1}^{c} \mathfrak{d}_{ij} Y_j - \mathfrak{t}_i\right) \exp(-\mathfrak{W}'). \tag{58}$$

**Remark 7.4.** The integral above is a slight modification of [HV00] Equation (7.78) which is supposed to be a generalization of Equation (7.32) in that paper. However, Equation (7.78) does not have Equation (7.32) as a special case. On the other hand, the integral here does.

Set for $i = 1, \ldots, r-n$

$$\log(\mathfrak{Q}_i) := -\sum_{\upsilon=1}^{r} \mathfrak{m}_{i\upsilon}\mathfrak{Z}_\upsilon + \sum_{j=1}^{c} \mathfrak{d}_{ij} Y_j. \tag{59}$$

Writing $-\mathfrak{Z}_\upsilon = \log(x_\upsilon)$ and $-Y_j = \log(y_j)$ we obtain

$$(-1)^{(r+c)} \int \prod_{\upsilon=1}^{r} \mathrm{dlog}(x_\upsilon) \prod_{j=1}^{c} \mathrm{dlog}(y_j) \times \prod_{\upsilon=1}^{r} x_\upsilon^{-1} \prod_{j=1}^{c} y_j^0 \prod_{i=1}^{r-n} \delta(\log(\mathfrak{Q}_{\mathfrak{i}}) + \mathfrak{t}_i) \exp(-F(x,y)). \tag{60}$$

**Remark 7.5.** Observe that when setting $\hbar = -1$, $\lambda_\upsilon = 1$, and $\lambda'_j = 0$ in Givental's integrals we have

$$\int \exp(-F(x,y)) \times \prod_{\upsilon=1}^{r} x_\upsilon^{\lambda_\upsilon/\hbar} \prod_{j=1}^{c} y_j^{-\lambda'_j/\hbar} \prod_{i=1}^{r-n} \delta(\log(Q_i) - t_i) \;\; \mathrm{dlog}(x) \wedge \mathrm{dlog}(y). \tag{61}$$

The following theorem verifies that the integrals considered by Hori and Vafa are exactly those of Givental specialized as in the above remark.

**Theorem 7.6.** *$\mathfrak{m}_{i\upsilon} = m_{i\upsilon}$, $\mathfrak{d}_{ij} = d_{ij}$, $\mathfrak{t}_i = -t_i$, and $\mathfrak{Q}_{\mathfrak{i}} = Q_i$*

*Proof.* The weight matrix, $(\mathfrak{m})_{i\upsilon}$, is simply the map on Lie algebras $\mathrm{d}_{\mathrm{id}}(- \circ [-]) = (fr \circ [-])^\tau = [-]^\tau$ . □

## 8. Comparison: Berglund-Hübsch

Let $Y$ be the $n$-dimensional weighted projective space $\mathbb{P}(l_0, \ldots, l_n)$ and $X$ a Calabi–Yau hypersurface in $Y$. If we set $d := l_0 + \cdots + l_n$, $X$ is defined by a weighted homogeneous polynomial $G$ of degree $d$ in the variables $x_0, \ldots, x_n$, where the degree of $x_i$ is $l_i$. In [BH92] Berglund and Hübsch consider the situation in which all degree $d$ monomials except $x_0 \cdots x_n$ appear in the expansion of $G$.

They define $P$ to be the matrix whose columns are the exponent vectors of the terms of $G$. To be clear,

$$(62) \qquad \begin{bmatrix} l_0 & \cdots & l_n & -d \end{bmatrix} \cdot \begin{bmatrix} & P & & 1 \\ & & & \vdots \\ 1 & \cdots & & 1 \end{bmatrix} = 0 \,.$$

They then define $\hat{l}_0, \ldots, \hat{l}_n$, and $\hat{d}$ by

$$(63) \qquad \begin{bmatrix} \hat{l}_0 & \cdots & \hat{l}_n & -\hat{d} \end{bmatrix} \cdot \begin{bmatrix} & \hat{P} & & 1 \\ & & & \vdots \\ 1 & \cdots & & 1 \end{bmatrix} = 0 \,,$$

where $\hat{P} := P^\tau$.

They then obtain the mirror $\hat{X} \subset \mathbb{P}(\hat{l}_0, \ldots, \hat{l}_n)$ defined by a homogeneous degree $\hat{d}$ polynomial $\hat{G}$ in the variables $\hat{x}_0, \ldots, \hat{x}_n$, where now the degree of $\hat{x}_i$ is $\hat{l}_i$. All degree $\hat{d}$ terms appear in $\hat{G}$ except $\hat{x}_0 \cdots \hat{x}_n$. $\hat{P}$ is the matrix whose columns are the exponent vectors of the terms of $\hat{G}$. Note that the Berglund and Hübsch set all coefficients to one.

Observe that

$$(64) \qquad \begin{bmatrix} & P & & 1 \\ & & & \vdots \\ 1 & \cdots & & 1 \end{bmatrix}$$

factors as $A \cdot B^\tau$, where $A$ and $B$ are obtained from the sigma model / Landau–Ginzburg model correspondence applied to a degree $d$ hypersurface $X_\epsilon$ in $\mathbb{P}(l_0, \ldots, l_n)$ whose equation uses all degree $d$ monomials.

Similarly,

$$(65) \qquad \begin{bmatrix} & \hat{P} & & 1 \\ & & & \vdots \\ 1 & \cdots & & 1 \end{bmatrix}$$

factors as $\hat{A} \cdot \hat{B}^\tau$

The key point is that $\hat{A}$ and $\hat{B}$ are obtained from the sigma model / Landau–Ginzburg model correspondence applied to a degree $\hat{d}$ hypersurface $\hat{X}_{\hat{\epsilon}}$ in $\mathbb{P}(\hat{l}_0, \ldots, \hat{l}_n)$ whose equation uses all degree $\hat{d}$ monomials. Furthermore, since $\left[\begin{array}{cccc} \hat{l}_0 & \cdots & \hat{l}_n & -\hat{d} \end{array}\right]$ is the cokernel of $B$ it is immediate that $\hat{A} = B$ and $\hat{B} = A$. Thus $X_\epsilon$ and $\hat{X}_{\hat{\epsilon}}$ are in mirror families.

Finally, these families contain the hypersurfaces considered by Berglund and Hübsch, and in fact one can simply apply (non-toric) automorphisms to $\mathbb{P}(l_0, \dots l_n)$ and $\mathbb{P}(\hat{l}_0, \dots, \hat{l}_n)$ to bring $X$ and $\hat{X}$ to the form $X_\epsilon$ and $\hat{X}_{\hat{\epsilon}}$. The coefficients can be set to one for appropriate choices of $L$ and $L'$.

## 9. Comparison: Batyrev-Borisov

In order to describe precisely the construction of Batyrev and Borisov, we require several more definitions.

**Definition 9.1.** [Bat94] A lattice polytope $P \subset M_{\mathbb{R}}$ with $0 \in \mathrm{int}(P)$ is called *reflexive* if $P^\circ$ also a lattice polytope. It is clear that $P^\circ$ is also reflexive.

**Remark 9.2.** In light of Corollary 1.4, reflexivity simply means that $P = \{\mu \in M_{\mathbb{R}} \mid \nu(\mu) + 1 \geq 0,$ where $\nu$ is a primitive inward normal$\}$. This is the same as saying that $P$ is the anticanonical polytope of a toric variety. In particular, such a $P$ is be the anticanonical polytope of the toric variety it defines.

**Definition 9.3.** [Bor93] If $P$ is a reflexive polytope, and $\Sigma_P$ its inward normal fan, a *nef-partition* of $\mathrm{vert}(P)$ is a partition $\{\mathbb{E}_1, \dots, \mathbb{E}_c\}$ of $\mathrm{vert}(P)$ such that there exist integral convex $\Sigma_P$-piecewise linear functions $\phi_1, \dots, \phi_c$ on $M_{\mathbb{R}}$ satisfying $\phi_i(e_j) = \delta(i - j)$ for $e_j \in \mathbb{E}_j$.

**Remark 9.4.** It is standard procedure in the theory of toric varieties to equate an integral convex function $\phi_j$ with a $T$-invariant Cartier divisor $D_j$ on $X(\Sigma)$. In our case, $D_j$ is the sum of divisors whose corresponding facet of $P$ has its inward normal in $\mathbb{E}_j$.

If $Y$ is a toric variety defined by a reflexive polytope $P$, it will be more natural to consider nef-partitions of $P^\circ$.

**Definition 9.5.** [Bor93] Let $P^\circ$ be a reflexive polytope and $\{\mathbb{E}_1, \dots, \mathbb{E}_c\}$ a nef partition of $\mathrm{vert}(P^\circ)$. Define $\nabla_j := \{\mu \in M_{\mathbb{R}} \ |\nu(\mu) \geq -\phi_j(\nu), \ \forall \nu \in N_{\mathbb{R}}\}$, and $(P^*)^\circ := \mathrm{conv}(\nabla_1 \cup \cdots \cup \nabla_c)$. Set $\nabla_j^\times = \mathrm{conv}(\nabla_j \setminus \{0\})$, and $\mathbb{E}_j^* = \mathrm{vert}(\nabla_j^\times)$. Then $(P^*)^\circ$ is reflexive, and $\{\mathbb{E}_1^*, \dots, \mathbb{E}_c^*\}$ is a nef partition of $\mathrm{vert}((P^*)^\circ)$ [Bor93, Prop. 3.4]. $\{\mathbb{E}_1^*, \dots, \mathbb{E}_c^*\}$ is called the *dual nef partition* to $\{\mathbb{E}_1, \dots, \mathbb{E}_c\}$.

**Definition 9.6.** We will write $P^*$ for $((P^*)^\circ)^\circ$.

**Definition 9.7.** Let $Y$ be a toric variety defined by the reflexive polytope $P$, and let $\{\mathbb{E}_1, \ldots, \mathbb{E}_c\}$ a nef partition of $\mathrm{vert}(P^\circ)$ with corresponding divisors $\{D_1, \ldots, D_c\}$. From this define $Y^*$ to be the toric variety defined by $P^*$, $\{\mathbb{E}_1^*, \ldots, \mathbb{E}_c^*\}$ the dual nef partition, and $\{D_1^*, \ldots, D_c^*\}$ the corresponding divisors on $Y^*$. Denote $\mathcal{V} := \mathcal{O}_Y(D_1) \oplus \cdots \oplus \mathcal{O}_Y(D_c)$ and $\mathcal{V}^* := \mathcal{O}_{Y^*}(D_1^*) \oplus \cdots \oplus \mathcal{O}_{Y^*}(D_c^*)$. The complete intersections in $Y$ given by the vanishing of global sections of $\mathcal{V}$ are *Batyrev-Borisov mirror* to the complete intersections in $Y^*$ given by vanishing of global sections of $\mathcal{V}^*$.

## Application of Section 6

To make a comparison between our duality with this construction, we revisit the results of Section 6, in particular Theorem 6.9. Now let $Y$ be an arbitrary toric variety.

**Definition 9.8.** A collection $\{D_1, \ldots, D_c\} \subseteq \mathbb{Z}^R$ is called *Givental* if $D_j > 0$ for all $j$ and $-\kappa_Y - \sum_j D_j \geq 0$. If $[-\kappa_Y] = [\sum_j D_j]$, we say the collection is *Calabi–Yau.*

**Definition 9.9.** Denote $\mathrm{conv}(P_{D_j}^\times \cap M_Y)$ by $C_j$, and $\mathrm{conv}(\cup_j(P_{D_j}^\times \cap M_Y))$ by $C$.

It is always true that $\mathrm{vert}(C) \subseteq \cup_j \mathrm{vert}(C_j)$, and this leads to the following defintion.

**Definition 9.10.** A *nef sub-partition* is a Givental collection such that $\mathrm{vert}(C) = \cup_j \mathrm{vert}(C_j)$, and $C_j \neq \varnothing$ for all $j$.

The following theorem allows us to express Borisov's notion of nef partition in these terms.

**Theorem 9.11.** *If $Y$ defined by a reflexive polytope $P = P_{-\kappa_Y}$, a Calabi–Yau nef sub-partiton is the same as a nef partition of* $\mathrm{vert}((P_{-\kappa_Y})^\circ)$.

*Proof.* Given a Calabi–Yau nef sub-partition, define $\mathbb{E}_j$ = set of primitive inward normals to the affine half-spaces corresponding the the toric divisors appearing as summands of $D_j$. Now define $\phi_j(\nu) = -\inf_{\mu \in P_{D_j}}(\nu(\mu))$. Then the definition of $P_{D_j}$ guarantees the $\phi_i(e_j) = \delta(i-j)$ as needed. So $\{\mathbb{E}_1, \ldots, \mathbb{E}_c\}$ is a nef-partition (Definition 9.3).

Now assume we have a nef-partition of $\mathrm{vert}((P_{-\kappa_Y})^\circ)$. Since $Y$ is Fano, these vertices are primitive inward normals to the affine hypersurfaces that correspond to toric divisors of $Y$. Define $D_j :=$ sum of toric divisors whose primitive inward normal is in $\mathbb{E}_j$. This trivially gives a Givental, Calabi–Yau collection. It remains to check the nef sub-partition condition. This is proved in [Bor93, Prop. 3.4]. □

**Definition 9.12.** Let $Y$ be a Fano toric variety, and $\{D_1, \dots, D_c\}$ a Calabi-Yau nef sub-partition and $\mathcal{V} = \mathcal{O}_Y(D_1) \oplus \cdots \oplus \mathcal{O}_Y(D_c)$. Identify integral points $P_{D_j}$ with a basis for global sections of $\mathcal{O}_Y(D_j)$. Define $\mathrm{w}_{BB}$ to be the global section of $\mathcal{V}$ given by

$$\mathrm{w}_{BB} := \sum_j \sum_{0 \neq \sigma \in P_{D_j} \cap M_Y} \exp(-2\pi)\sigma \ . \tag{66}$$

Finally, the following theorem shows that the construction of Batryev and Borisov is a special case of our duality (refer to Definition 9.7 for the definition of $\mathcal{V}^*$.)

**Theorem 9.13.** *Let $Y$ be an $n$ dimensional Fano toric variety, $\{D_1, \dots, D_c\}$ a Calabi-Yau nef sub-partition, and $\mathcal{V} = \mathcal{O}_Y(D_1) \oplus \cdots \oplus \mathcal{O}_Y(D_c)$. Let $X$ be the total space of the dual, $\mathcal{V}^\vee$, and $W_{BB}\colon X \to \mathbb{C}$ the superpotential defined by $\mathrm{w}_{BB}$. For arbitrary $K \in A_{n-1}(Y)$, the Landau-Ginzburg model $(X', W', K')$, dual to $(X, W, K)$, has $X' = \mathrm{Tot}((\mathcal{V}^*)^\vee)$ and $W'$ comes from a global section of $\mathcal{V}^*$.*

*Proof.* The choice of $W_{BB}$ means that on the dual $K' = (\alpha', 0)$, with $\alpha'_{(\nu,\sigma_j)} = 1$ for all $\nu \in P^\times_{D_j} \cap M_Y$. For this value $(d', a')$ is easily seen to be kopasetic. Now the set $V^\times_j$ in Definition 6.7 is $(P^\times_{D_j} \cap M_Y) \times \{\sigma_j\}$. The projection is $(P^\times_{D_j} \cap M_Y) \subset (M_Y)_\mathbb{R}$. The nef sub-partition condition and Theorem 6.9 guarantee that $X' = E' = \mathrm{Tot}((\mathcal{V}')^\vee)$ over $Y'$.

If we denote the polytope defining $Y$ by $P$, it remains to show that $Y'$ is defined by $P^*$, and $D'_j = D^*_j$. The first fact follows from checking that $P_{D'_j} = \Delta_j$, where $\Delta_j = \mathrm{conv}(\{0\} \cup \mathbb{E}_j)$. Since $P' = P_{-k_{Y'}} = \sum_j P_{D'_j}$ from the standard theory of toric varieties, and $P^* = \sum_j \Delta_j$ by [Bor93, Prop. 3.2]. On the other hand, the second fact follows from $P_{D_j} = \nabla_j$ which implies $\mathrm{vert}(\nabla^\times_j) = \mathrm{vert}(C_j)$ and so $\mathbb{E}^*_j$ is made up of exactly the vertices of $(P')^\circ$ defining $D'_j$.

$P_{D_j} = \nabla_j$: $\mu \in \nabla_j \iff \nu(\mu) + \phi_j(\nu) \geq 0,\ \forall \nu \in (N_Y)_\mathbb{R}$. Since $Y$ is complete, we can find $\lambda \geq 0$ such that $\nu = \sum_{e \in \mathrm{vert}(P^\circ)} c_e e$. Plugging this in above gives $\sum_e c_e(e(\mu) + \phi_j(e)) \geq 0$. Evaluating $\phi_j$ gives $\sum_{e \notin \mathbb{E}_j} c_e e(\mu) + \sum_{e \in \mathbb{E}_j} c_e(e(\mu) + 1) \geq 0$. This equation must hold for all choices $c_e \geq 0$, so we find $e(\mu) \geq 0$ for all $e \in \operatorname{vert}(P^\circ) \setminus \mathbb{E}_j$, and $(e(\mu) + 1) \geq 0$ for all $e \in \mathbb{E}_j$. This is exactly the condition that $\mu \in P_{D_j}$.

$P_{D'_j} = \Delta_j$: Using the equality of $P_{D_j}$ and $\nabla_j$ above and the fact we only need to check non-zero vertices, we can write $P_{D'_j}$ to be the $\nu$ satisfying $\nu(e^*) + 1 \geq 0$ for all $e^* \in \mathbb{E}^*_j$ and $\nu(e^*) \geq 0$ for all $e^* \in \mathbb{E}^*_i$ when $i \neq j$. $\Delta_j$ can be defined by $\{\nu \in N_{\mathbb{R}} \; | \nu(\mu) \geq -\phi^*_j(\mu), \;\; \forall \mu \in M_{\mathbb{R}}\}$, where $\phi^*_j(\mu) := -\inf_{\nu \in \nabla_j}(\nu(\mu))$ [Bor93, Cor 2.12 ]. Now if we unravel this definition as we did in the previous paragraph, we arrive at the same conditions defining $P_{D'_j}$.

Finally, we check that the superpotential comes from a section by simply applying Lemma 6.10. □

## 10. An example: elliptic curves in $(\mathbb{P}^1)^2$

We consider elliptic curves on $(\mathbb{P}^1)^2$. This is a nice example since many of the features of our duality are exhibited and the dimension is low enough so that we can actually draw the polytopes involved. Also, since they are Calabi–Yau, duality can be compared to the Batyrev–Borisov, Givental, and Hori–Vafa constructions.

Let $Z$ be an effective divisor with $Z \sim D$, where $D$ is a $T$-invariant $(2, 2)$ divisor on $Y = (\mathbb{P}^1)^2$. $Z$ is an elliptic curve. This can be seen using the adjunction formula, $\kappa_Z = (\kappa_Y + Z)|_Z$, and the fact $\kappa_Y = (-2, -2)$.

In order to write down the Landau-Ginzburg model corresponding to $Z$ as in Section 4, we will use the inclusion $(\mathbb{C}^\times)^2_{x,y} \hookrightarrow Y$ defined by the point $([1 : 1], [1 : 1]) \in Y$ and the action $(x, y) \cdot ([a : b], [c : d]) = ([a : xb], [c : yd])$. This gives the character group

$$M_Y = \mathbb{Z}x \oplus \mathbb{Z}y, \tag{67}$$

the group of $T$-invariant divisors

$$\mathbb{Z}^{R_Y} = \mathbb{Z}\rho^x_0 \oplus \mathbb{Z}\rho^y_0 \oplus \mathbb{Z}\rho^x_\infty \oplus \mathbb{Z}\rho^y_\infty, \tag{68}$$

the $1^{\text{st}}$ Chow group

$$A_1(Y) = \mathbb{Z}[\rho^x_0] \oplus \mathbb{Z}[\rho^y_0], \tag{69}$$

the character-to-divisor map

$$\mathrm{div}_Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{70}$$

and the cokernel of $\mathrm{div}_Y$,

$$[-]_Y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{71}$$

Now for $E = \mathrm{Tot}(\mathcal{O}_Y(-2,-2))$ we use the formulas provided in Section 4. First the (character-to-divisor) $A$-side:

$$M = M_Y \oplus \mathbb{Z}\xi, \tag{72}$$

and

$$\mathbb{Z}^R = \mathbb{Z}^{R_Y} \oplus \mathbb{Z}D. \tag{73}$$

Recall

$$A_2(E) = A_1(Y) = \mathrm{H}^2(Y;\mathbb{Z}), \tag{74}$$

and we choose

$$D = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{75}$$

Note that this is the canonical choice for $-\kappa_Y$. A quick check shows

$$[D] = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{76}$$

as needed.

Putting this together gives

$$\text{div}_E = \begin{bmatrix} d_Y & D \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ 0 \quad 0 & 1 \end{array}\right]. \tag{77}$$

Since $A_2(E)$ is torsion free the cokernel of $\text{div}_E$, $[-]$, is given by the matrix

$$[-] = \left[\; [-]_Y \;\middle|\; [-D] \;\right] = \left[ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right]. \tag{78}$$

At this point we will not commit ourselves to a choice of $K$.

Now the (superpotential) B-side: $N_Y = \mathbb{Z}\nu_1 \oplus \mathbb{Z}\nu_2$ with the obvious pairing. So $N_E = N_Y \oplus \mathbb{Z}\phi$. $P_D$ has integral points equal to the characters with $\text{div}_Y(\mu) + D \geq 0$. This means $-1 \leq x, y \leq 1$. Thus

$$\Xi = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}. \tag{79}$$

We can rewrite these vectors as functions on $E$:

$$\Xi = \{y\xi, xy\xi, x\xi, x\xi/y, \xi/y, \xi/xy, \xi/x, y\xi/x, \xi\}. \tag{80}$$

Now we can obtain mon by transposing the elements of $\Xi$ as mentioned after Equation (30):

$$\text{mon} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{81}$$

Applying duality we have $A = B' = \text{div}_E$ and $B = A' = \text{mon}$.

### $K'$ = "Givental choice"

At this point we make a choice for $K' = L$ so that we can investigate the geometry of $X(A', a')$. First choose $K' = 0$. This leads to the Givental mirror and we know it is described by reading the rows off the cokernel of $A = \mathrm{div}_{X=E}$ which appears in Equation (78):

$$\{(x_1, x_2, x_3, x_4, y_1) \in \mathbb{C}^5 \mid x_1x_3 = Q_1y_1^2, \text{and } x_2x_4 = Q_2y_1^2\}, \tag{82}$$

with the superpotential

$$W' = x_1 + x_2 + x_3 + x_4 + y_1 \ . \tag{83}$$

Indicating coordinates dual to $x, y$ and $\xi$ with primes, the identification of the Givental mirror with $X(A', 0)$ sets

$$x_1 = x'\xi', \quad x_2 = y'\xi', \quad x_3 = Q_1\xi'/x', \quad x_4 = Q_2\xi'/y', \quad \text{and} \quad y_1 = \xi' \ . \tag{84}$$

The superpotential becomes

$$W' = (x' + y' + Q_1/x' + Q_2/y' + 1)\xi' \tag{85}$$

on $X(A', 0) = \mathrm{Spec}(\mathbb{C}[x'\xi', y'\xi', \xi'/x', \xi'/y', \xi'])$.

### $K'$ = "Batyrev-Borisov choice"

Now observe what happens for the "Batyrev–Borisov" choice $a' = (-\kappa', 0)$. In terms of the basis for $\mathbb{Z}^{\Xi}$,

$$L_{BB} = ia' = i[y\xi + xy\xi + x\xi + x\xi/y + \xi/y + \xi/xy + \xi/x + y\xi/x]. \tag{86}$$

So $(\alpha', 0) = (y\xi + xy\xi + x\xi + x\xi/y + \xi/y + \xi/xy + \xi/x + y\xi/x, 0)$ gives a representative for $a'$ with $\alpha' = -\kappa \in \mathbb{Z}^{\Xi^\times}$ as desired.

The polytope of $X(A', a')$ is the set of solutions $\begin{bmatrix} n_1 \\ n_2 \\ p \end{bmatrix} \in N_E = \mathbb{Z}\nu_1 \oplus \mathbb{Z}\nu_2 \oplus \mathbb{Z}\phi$ to

$$\begin{bmatrix} n_1 \\ n_2 \\ p \end{bmatrix} + \begin{bmatrix} \alpha' \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} n_1 + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} n_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} p + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \geq 0. \tag{87}$$

A careful check shows that inequalities corresponding to rows 1, 3, 5, and 7 above are unnecessary. Removing them yields the system of inequalities

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} n_1 + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} n_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} p + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \geq 0. \tag{88}$$

It follows

$$\operatorname{div}_{X(A',a')} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{89}$$

and $k(\alpha') = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ as in (as in Equation (26)).

Comparing $\operatorname{div}_{X(A',a')}$ and $k(\alpha')$ to the formula in Equation (22), or more generally Equation (4.3), we see that $X(A', a')$ is the total space, $E'$, of the canonical bundle over a variety $Y'$ with

$$\text{div}_{Y'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{90}$$

Furthermore, $-\kappa_{Y'}$ pulls back to $k(\alpha')$.

After looking at the equation $\text{div}_{Y'} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} + \kappa_{Y'} \geq 0$, one finds $Y'$ is determined by the reflexive polytope $P_{-\kappa_{Y'}}$ that is given by $\pm n_1 \pm n_2 + 1 \geq 0 \subseteq N_Y = \mathbb{Z}\nu_1 \oplus \mathbb{Z}\nu_2$:

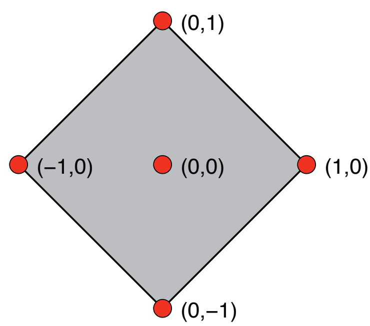


Figure 3: Polytope for $Y'$ under the anticanonical choice for $K'$.

Note that this is dual to the polytope of $P_{-\kappa_Y}$ ( $P_{-\kappa_Y}$ is a square whose sides have length $= 2$ and is centered at the origin of $M_{\mathbb{R}}$).

This is predicted by Theorem 9.13. To check that a function $W'$ on $X(A', a')$ defined by any choice of $L'$ comes from a section of $\mathcal{O}_{Y'}(-\kappa_{Y'})$, one only needs to look at $\text{div}_E$ in Equation (77) and notice that each row is the transpose of an element of $P_{-\kappa_{Y'}} \times \{\sigma\} \subset N_E$. Note the toric variety associated to this polytope is not smooth.

Also note that one can easily check that the $X(A', K'_{BB})$ can be obtained from Givental's mirror by blowing up the scheme defined by $\xi' = 0$.

**$K'$ = "very ample"**

If we make a different choice for $K'$, for instance

(91) $$K' = i[\begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 0 \end{bmatrix}]$$

A similar check shows that with this choice $A' = \mathrm{div}_{X(A',a')}$, and $X(A', a')$ is the total space of the canonical bundle of the toric variety associated with the stop-sign polytope:

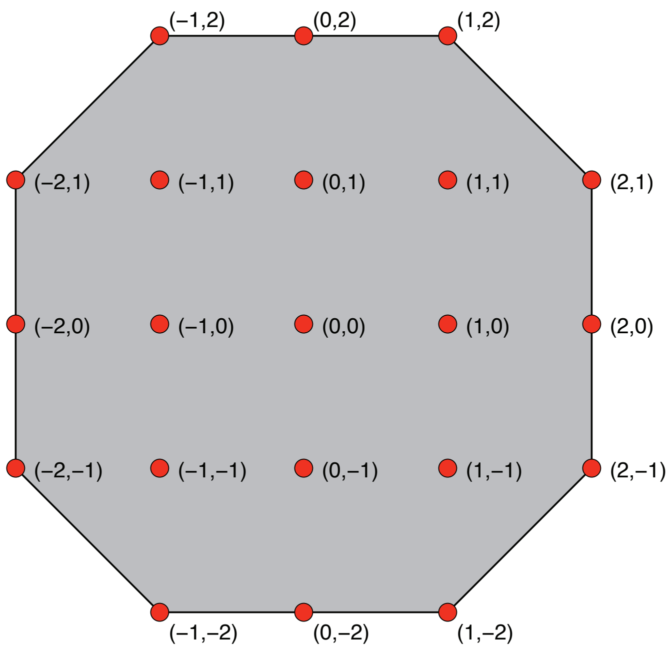


Figure 4: Polytope for $Y'$ under a very ample choice for $K'$.

This toric variety is smooth, and is in fact a crepant resolution of the variety given by the diamond polytope. However, although $W'$ comes from a global section of the anticanonical bundle, it is not generic.

It is helpful to think of all three of these varieties as both living in the family of Kähler deformations

$$K'_{(s,t)} = it(s\left[\begin{array}{c}2\\3\\2\\3\\2\\3\\2\\3\\0\end{array}\right] + i(1-s)\left[\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\0\end{array}\right]). \tag{92}$$

## 11. Discussion

The duality introduced here provides a simple framework under which all existing mirror candidate constructions for complete intersections fall. It is also holds some promise for use in homological mirror symmetry since it often provides partial resolutions when the mirror according to other constructions is singular.

The main shortcoming in the construction is the apparent asymmetry that sometimes happens under double dualization. This was mentioned in Remark 6.11. It was pointed out to us by Lev Borisov that a possible fix would be to include in the dualization procedure a pair of (formal) automorphisms of $F_A\colon \mathrm{coker}(A)_{\mathbb{C}/\mathbb{Z}} \to \mathrm{coker}(B')_{\mathbb{C}^\times}$, and $F_B\colon \mathrm{coker}(B)_{\mathbb{C}^\times} \to \mathrm{coker}(A')_{\mathbb{C}/\mathbb{Z}}$.

In the construction as defined here we are simply using the maps $F_A = \exp(2\pi i -)$ and $F_B = \frac{1}{2\pi i}\log(-)$. These would take $K$ to $\exp(2\pi i L')$ and $\exp(2\pi i L)$ to $K'$. We could replace these with arbitrary maps, so long as $F_{B'} \circ F_A = \mathrm{Id}$, and $F_B \circ F_{A'} = \mathrm{Id}$, and all the results in the paper would still hold.

The "correct" maps $F_A$ and $F_B$ should be closely related to the mirror map relating the moduli space of complex structures of a sigma model $Z$, and the moduli space of complexified Kähler structures of the mirror sigma model $Z'$. In the case when both our Landau-Ginzburg models correspond to sigma models, then $\mathrm{coker}(B)_{\mathbb{C}/\mathbb{Z}}$ should map to the moduli $\mathcal{M}_K$ of complexified Kähler structures of $Z$ and $\mathrm{coker}(A')_{\mathbb{C}^\times}$ to the moduli, $\mathcal{M}_{L'}$ of complex structures of $Z'$. This should lead to a commutative diagram

$$\begin{array}{ccc} \operatorname{coker}(B)_{\mathbb{C}/\mathbb{Z}} & \stackrel{F_B}{\to} & \operatorname{coker}(A')_{\mathbb{C}^\times} \\ \downarrow & & \downarrow \\ \mathcal{M}_K & \stackrel{\text{mirror}}{\to} & \mathcal{M}_{L'} \end{array} . \tag{93}$$

Nailing down the exact maps $F_A$ and $F_B$ in general requires more than what is known about the mirror map since we hope that homological mirror symmetry could be made to work for more than just Calabi-Yau manifolds. It is the subject of further research to understand mirror symmetry and the mirror map in this more general setting.

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