ADV. THEOR. MATH. PHYS.
Volume 21, Number 1, 121–145, 2017

4d $\mathcal{N} = 2$ SCFT and singularity theory Part II: complete intersection

Bingyi Chen, Dan Xie, Shing-Tung Yau,
Stephen S.-T. Yau, and Huaiqing Zuo

We classify three dimensional isolated weighted homogeneous rational complete intersection singularities, which define many new four dimensional $\mathcal{N} = 2$ superconformal field theories. We also determine the mini-versal deformation of these singularities, and therefore solve the Coulomb branch spectrum and Seiberg-Witten solution.

1. Introduction

This is the second of a series of papers in which we try to classify four dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs) using classification of singularity. This program has several interesting features:

- The classification of field theory is reduced to the classification of singularities, which in many cases are much simpler than the classification using field theory tools.
- Many highly non-trivial physical questions such as Coulomb branch spectrum and the Seiberg-Witten solution [SW1, SW2] can be easily found by studying the mini-versal deformation of the singularity.

In [XY], we conjecture that any three dimensional rational Gorenstein graded isolated singularity should define a $\mathcal{N} = 2$ SCFT. A complete list of

The work of S.-T. Yau is supported by NSF grant DMS-1159412, NSF grant PHY- 0937443, and NSF grant DMS-0804454. The work of Stephen S.-T. Yau is supported by NSFC grant 11531007 and Tsinghua University startup fund. The work of Huaiqing Zuo is supported by NSFC (grant nos. 11531007, 11401335) and Tsinghua University Initiative Scientic Research Program. The work of Dan Xie is supported by Center for Mathematical Sciences and Applications at Harvard University, and in part by the Fundamental Laws Initiative of the Center for the Fundamental Laws of Nature, Harvard University.

hypersurface singularities was obtained in [YY, YY1], and this immediately gives us a large number of new four dimensional $\mathcal{N}=2$ SCFTs.

The natural next step is to classify three dimensional rational weighted homogeneous isolated complete intersection singularities (ICIS). To our surprise, the space of such singularities is also very rich, and we succeed in giving a complete classification. Let's summarize our major findings:

- The number of polynomials defining ICIS is two, i.e. the singularity is defined as $f_1 = f_2 = 0$.
- We find a total of 303 class of singularities, and some of them consist only finite number of models, but we do get many infinite sequences.

Our classification gives many new interesting 4d $\mathcal{N}=2$ SCFTs. Some of these singularities describe the familiar gauge theory, i.e. the singularity $(f_1, f_2) = (z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^{2N}, z_1^2 + 2z_2^2 + 3z_3^2 + 4z_4^2 + 5z_5^{2N})$ describes the affine D_5 quiver gauge theory with SU type gauge group, see Figure 1. The major purpose of this paper is to describe the classification, and more detailed study of the corresponding SCFTs will appear in a different publication.

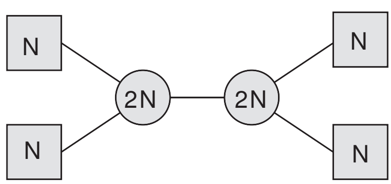

Figure 1: 4d $\mathcal{N}=2$ SCFT described by the singularity $(f_1, f_2) = (z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^{2N}, z_1^2 + 2z_2^2 + 3z_3^2 + 4z_4^2 + 5z_5^{2N})$. Here the gauge group is $SU(2N)$.

This paper is organized as follows: Section two reviews the connection between the physics of 4d $\mathcal{N}=2$ SCFT and the property of ICIS; Section III reviews some preliminary facts about the general property of singularities; Section IV proves that the maximal embedding dimension of 3d rational ICIS is five, and section V classifies the general weighted homogeneous ICIS and we also compute the Milnor number and the monomial basis of the mini-versal deformations.

2. ICIS and 4d $\mathcal{N} = 2$ SCFT

Four dimensional $\mathcal{N} = 2$ SCFT has an important $SU(2)_R \times U(1)_R$ R symmetry. These theories have half BPS scalar operators $\mathcal{E}_{r,(0,0)}$ which can get expectation value and parameterize the Coulomb branch, here r is the $U(1)_R$ charge and this operator is a singlet under $SU(2)_R$ symmetry. The scaling dimension of operator $\mathcal{E}_{r,(0,0)}$ is $\Delta = r$, and if $1 < r < 2$, one can turn on the relevant deformation

$$\delta S = \lambda \int d^4\theta \mathcal{E}_{r,(0,0)} + c.c. \tag{2.1}$$

The scaling dimension of the coupling constant is determined by the relation $[\lambda] + [\mathcal{E}_{r,(0,0)}] = 2$. The coupling constants do not lift the Coulomb branch, but will change the infrared physics, so we need to include those coupling constants besides the expectation values of $\mathcal{E}_{r,(0,0)}$ to parameterize the Coulomb branch. We also have the dimension one mass parameters m_i which can also change the IR physics, which should also be included as the parameters of Coulomb branch. To solve the Coulomb branch of a $\mathcal{N} = 2$ SCFT, we need to achieve the following two goals:

- Determine the set of rational numbers which include the scaling dimension of the coupling constants λ, Coulomb branch operators $\mathcal{E}_{r,(0,0)}$, and mass parameters m_i.

$$(r_1, \ldots, 1, \ldots, 1, \ldots r_\mu). \tag{2.2}$$

  Since the scaling dimension of the coupling constant is paired with that of Coulomb branch operator, this set is symmetric with respect to identity.

- Once we find out the parameters on the Coulomb branch, we want to write down a Seiberg-Witten curve which describes the low energy effective theory on the Coulomb branch

$$F(z_i, u) = 0. \tag{2.3}$$

  Here u includes all the parameters discussed above.

These two questions are central in understanding Coulomb branch of a $\mathcal{N} = 2$ SCFT, and in general are quite hard to answer.

If our 4d SCFT is engineered using 3-fold singularity, the above two questions can be found from the miniversal deformation of the singularity[1]. The formulae for the hypersurface case has been given in [SV, CNV]. In the following, we will review the relevant formulas for ICIS case, which is first derived in [XY].

Consider a three dimensional ICIS defined by two polynomials $f = (f_1, f_2)$, where$f$ is the map $f : (C^5, 0) \to (C^2, 0)$ [2]. We require the defining polynomials to have a manifest $\mathbb{C}^*$ action, which is proportional to the $U(1)_R$ symmetry of the field theory. We normalize the $\mathbb{C}^*$ action so that the weights of the coordinates $(z_1, \ldots, z_5)$ are $(w_1, w_2, \ldots, w_5)$, and the degree of f_1 is one, the degree of f_2 is d:

$$f_1(\lambda^{w_i} z_i) = \lambda f_1(z_i), \quad f_2(\lambda^{w_i} z_i) = \lambda^d f_2(z_i). \tag{2.4}$$

This singularity has a distinguished $(3, 0)$ form:

$$\Omega = \frac{dz_1 \wedge dz_2 \wedge \cdots \wedge dz_5}{df_1 \wedge df_2}, \tag{2.5}$$

which has charge $\sum w_i - 1 - d$ under the $\mathbb{C}^*$ action. To define a sensible 4d SCFT, we require this charge to be positive, which means that

$$\sum w_i > 1 + d. \tag{2.6}$$

We conjecture that this condition is necessary and sufficient to define a SCFT. Such singularity is called rational singularity, see section III for the definition. The SW solution is described by the mini-versal deformation of the singularity:

$$F(\lambda, z_i) = f(z_i) + \sum_{\alpha=1}^{\mu} \lambda_\alpha \phi_\alpha, \tag{2.7}$$

here ϕ_α is the monomial basis of the Jacobi module of f, and μ is the Milnor number. The coefficient λ_α is identified with the parameters on Coulomb branch. The scaling dimension of λ_α is determined by requiring Ω to have

[1] If the theory is engineered using M5 branes [Ga, GMN, NX, CDY, Xie, WX], these questions are solved by spectral curve of Hitchin system. If the theory is engineered using Kodaira singularity [ALLM1, ALLM2, ALLM3], these questions can also be solved by studying the deformations of the singularity.

[2] ICIS defined by two polynomials is enough for our purpose, see section IV.

dimension one as its integration over the middle homology cycle of Milnor fibration gives the mass of BPS particle, and we have

$$
\begin{aligned}
&\phi_\alpha = [\phi_\alpha, 0]: \quad [\lambda_\alpha] = \frac{1-Q_\alpha}{\sum w_i - 1 - d}, \\
&\phi_\alpha = [0, \phi_\alpha]: \quad [\lambda_\alpha] = \frac{d-Q_\alpha}{\sum w_i - 1 - d}.
\end{aligned} \tag{2.8}
$$

Here Q_α is the $\mathbb{C}^*$ charge of the monomial ϕ_α. The spectrum is classified into following categories:

- Coulomb branch operator $\mathcal{E}_{r,(0,0)}$ if $[\lambda_\alpha] > 1$.
- Mass parameters if $[\lambda_\alpha] = 1$.
- Coupling constants for relevant deformations if $0 < [\lambda_\alpha] < 1$.
- Exact marginal deformations if $[\lambda_\alpha] = 0$. These deformations are related to the moduli of the singularity.
- Irrelevant deformations if $[\lambda_\alpha] < 0$.

The spectrum is paired and is symmetric with respect to one, which is in perfect agreement with the field theory expectation.

Example. Consider the singularity $f = (f_1, f_2) = (z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2, z_1^2 + 2z_2^2 + 3z_3^2 + 4z_4^2 + 5z_5^2) = 0$. The weights and degrees of two polynomials are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1)$. The Jacobi module J_f has the basis

$$
\begin{aligned}
&\phi_1 = [0, z_5^2],\ \phi_2 = [0, z_4^2],\ \phi_3 = [0, z_5],\ \phi_4 = [0, z_4],\ \phi_5 = [0, z_3], \\
&\phi_6 = [0, z_2],\ \phi_7 = [0, z_1],\ \phi_8 = [0, 1],\ \phi_9 = [1, 0].
\end{aligned} \tag{2.9}
$$

Using the Formula (2.8), we find the scaling dimension of the coefficients λ_i in mini-versal deformation:

$$
[\lambda_1] = [\lambda_2] = 0,\ [\lambda_3] = [\lambda_4] = [\lambda_5] = [\lambda_6] = [\lambda_7] = 1,\ [\lambda_8] = [\lambda_9] = 2; \tag{2.10}
$$

So this theory has two exact marginal deformations, two Coulomb branch operators with dimension 2, and five mass parameters. The corresponding gauge theory is depicted in Figure 1 with $N = 1$.

In the following sections, we will classify all possible 3 dimensional weighted homogeneous ICIS, and describe the miniversal deformation. The Coulomb branch is then solved using Formulas 2.7 and 2.8.

3. Preliminaries

In this section, we recall some definitions and known results about the Gorenstein singularity.

Definition 3.1. For a commutative Noetherian local ring R, the depth of R (the maximum length of a regular sequence in the maximal ideal of R) is at most the Krull dimension of R. The ring R is called Cohen-Macaulay if its depth is equal to its dimension. More generally, a commutative ring is called Cohen-Macaulay if it is Noetherian and all of its localizations at prime ideals are Cohen-Macaulay. In geometric terms, a scheme is called Cohen-Macaulay if it is locally Noetherian and its local ring at every point is Cohen-Macaulay.

Definition 3.2. Let (X, x) be an isolated singularity of dimension n. (X, x) is said to be normal or Cohen-Macaulay if the local ring $\mathcal{O}_{X,x}$ has such a property.

Let (X, x) be an isolated singularity of dimension n. Then we have the following propositions.

Proposition 3.1. *(Corollary 3.10, [Ha]) (X, x) is called Cohen-Macaulay iff $H^1_{\{x\}}(\mathcal{O}_X) = \cdots = H^{n-1}_{\{x\}}(\mathcal{O}_X) = 0$. (X, x) is normal iff $H^1_{\{x\}}(\mathcal{O}_X) = 0$.*

Proposition 3.2. *Let (X, x) be an isolated singularity of dimension n and $\pi : (\widetilde{X}, E) \to (X, x)$ be a resolution of (X, x). Then $H^i(\widetilde{X}, \mathcal{O}) \cong H^i_\infty(\widetilde{X}, \mathcal{O}) \cong H^{i+1}_{\{x\}}(X, \mathcal{O}), 1 \le i \le n-2$.*

Proof. Following Laufer [La1], we consider the sheaf cohomology with support at infinity. The following sequence is exact:

$$\begin{aligned} 0 \to \Gamma(\widetilde{X}, \mathscr{O}) \to \Gamma_\infty(\widetilde{X}, \mathscr{O}) \to H^1_c(\widetilde{X}, \mathscr{O}) \\ \to H^1(\widetilde{X}, \mathscr{O}) \to H^1_\infty(\widetilde{X}, \mathscr{O}) \to H^2_c(\widetilde{X}, \mathscr{O}) \to \cdots \end{aligned}$$

By Serre duality,

$$H^i_c(\widetilde{X}, \mathscr{O}) \cong H^{n-i}(\widetilde{X}, \mathscr{O}(K)).$$

by Grauert-Riemenschneider Vanishing Theorem, we have

$$H^{n-i}(\widetilde{X}, \mathscr{O}(K)) = 0, \text{ for } i \le n-1.$$

It follows that

$$H^i(\widetilde{X}, \mathcal{O}) \cong H^i_\infty(\widetilde{X}, \mathcal{O}),\ 1 \le i \le n-2. \tag{3.1}$$

On the other hand, we also have the following exact sequence:

$$\begin{aligned}0 \to \Gamma(X, \mathscr{O}) \to \Gamma(X - \{x\}, \mathscr{O}) \to H^1_{\{x\}}(X, \mathscr{O}) \\ \to H^1(X, \mathscr{O}) \to H^1(X - \{x\}, \mathscr{O}) \to H^2_{\{x\}}(\widetilde{X}, \mathscr{O}) \to \cdots\end{aligned}$$

Since $H^i(X, \mathscr{O}) = 0$, $i \ge 1$. Thus we have

$$H^i(X - \{x\}, \mathscr{O}) \cong H^{i+1}_{\{x\}}(X, \mathcal{O}),\ i \ge 1. \tag{3.2}$$

Take a 1-convex exhaustion function ϕ on $\widetilde{X}$ such that $\phi \ge 0$ on $\widetilde{X}$ and $\phi(y) = 0$ if and only if $y \in E_i$ where E_i is the irreducible component of E. Put $\widetilde{X}_r = \{y \in \widetilde{X} : \phi(y) \le r\}$. Then by Laufer [La1],

$$\varinjlim_r H^q(\widetilde{X} - \widetilde{X}_r, \mathcal{O}) \cong H^q_\infty(\widetilde{X}, \mathcal{O}).$$

On the other hand, by Andreotti and Grauert (Théorème 15 of [An-Gr]), $H^q(\widetilde{X} - E, \mathcal{O})$ is isomorphic to $H^q(\widetilde{X} - \widetilde{X}_r, \mathcal{O})$ for $q \le n-2$ and $H^{n-1}(\widetilde{X} - E, \mathcal{O}) \to H^{n-1}(\widetilde{X} - \widetilde{X}_r, \mathcal{O})$ is injective. Thus we have

$$H^i_\infty(\widetilde{X}, \mathcal{O}) \cong H^i(\widetilde{X} - E, \mathcal{O}) \cong H^i(X - \{x\}, \mathscr{O}),\ 1 \le i \le n-2. \tag{3.3}$$

Combining with (3.1),(3.2) and (3.3) we have

$$H^i(\widetilde{X}, \mathcal{O}) \cong H^i_\infty(\widetilde{X}, \mathcal{O}) \cong H^{i+1}_{\{x\}}(X, \mathcal{O}), 1 \le i \le n-2.$$

This completes the proof. □

Corollary 3.1. *(1) (X, x) is Cohen-Macaulay $\Rightarrow H^i(\tilde{X}, \mathcal{O}) = 0, 1 \le i \le n-2$.*

(2) (X, x) is normal and $H^i(\tilde{X}, \mathcal{O}) = 0, 1 \le i \le n-2 \Rightarrow (X, x)$ is Cohen-Macaulay.

Definition 3.3. A normal variety X is called Gorenstein if it is Cohen-Macaulay and the sheaf $\omega_X := \mathcal{O}(K_X)$ is locally free.

Definition 3.4. A Gorenstein point $x \in X$ of an n-dimensional variety X is rational (respectively minimally elliptic) if for a resolution $f : Y \to X$

we have $f_*\omega_Y = \omega_X$(respectively $f_*\omega_Y = m_x\omega_X$, where m_x is the ideal of x). (This is equivalent via duality to the cohomological assertion $R^{n-1}f_*\mathcal{O}_Y = 0$ (respectively, is a 1-dimensional $\mathbb{C}$-vector space at x).

It is convenient to make intrinsic (and generalize slightly) the notion of a general hyperplane section through x:

Definition 3.5. Let $(\mathcal{O}_X, x, m_x)$ be the local ring of a point $x \in X$ of a $\mathbb{C}$- scheme, and let $V \subset m_x$ be a finite-dimensional $\mathbb{C}$-vector space which maps onto m_x/m_x^2 (equivalently, by Nakayamas lemma, V generates the $\mathcal{O}_{X,x}$-ideal m_x); by a general hyperplane section through x is mean the sub-scheme $H \subset X_0$ defined in a suitable neighborhood X_0 of x by the ideal $\mathcal{O}_{X,v}$, where $v \in V$ is a sufficiently general element (that is, v is a $\mathbb{C}$-point of a certain dense Zariski open $U \subset V$).

Theorem 3.1. *([Mi], Theorem 2.6) If $x \in X$ is a rational Gorenstein point (dim $X = n \geq 3$. Then for a general hyperplane section S through x, $x \in S$ is minimally elliptic or rational Gorenstein.*

Proof. Suppose that S runs through any linear system of sections $x \in S \subset X$ whose equations generate the maximal ideal m_x of $\mathcal{O}_{X,x}$. Then a general element S of this linear system is normal.

Let $f : Y \to X$ be any resolution of X which dominates the blow-up of the maximal ideal m_x; by definition of the blow-up, the scheme-theoretic fiber over x is an effective divisor E such that $m_x\mathcal{O}_Y = \mathcal{O}_Y(-E)$. Hence $f^*S = T + E$, where T runs through a free linear system on Y. By Bertini's theorem, $\phi = f\,|_T \colon T \to S$ is a resolution of S. Now we use the adjunction formula to compare K_T and φ^*K_S.

In the diagram (Figure 1), we have

$$\begin{tikzcd} Y \arrow[d, "f"] & \supset & T+E \arrow[d, "\varphi"] \\ X & \supset & S \end{tikzcd}$$

Figure 2.

$$K_Y = f^*K_X + \Delta, \text{ with } \Delta \geq 0$$

and

$$T = f^*S - E,$$

so that

$$K_Y + T = f^*(K_X + S) + \Delta - E$$

and

$$K_T = (K_Y + T) \mid_T = \varphi^* K_S + (\Delta - E) \mid_T .$$

This just means that any $s \in \omega_s$ has at worst $(\Delta - E) \mid_T$ as pole on T. On the other hand, since the maximal ideal $m_{S,x} \subset \mathcal{O}_{S,x}$ is the restriction to S of the maximal ideal $m_{X,x} \subset \mathcal{O}_X$, it follows that every element of $m_{S,x}$ vanishes along $E \cap T$. Hence every element of $m_{S,x}\omega_S$ is regular on T, that is

$$m_x \omega_S \subset \varphi_* w_T \subset \omega_S.$$

Thus $m_x \omega_S = \varphi_* w_T$ implies $x \in S$ is minimally elliptic and $\omega_S = \varphi_* w_T$ implies $x \in S$ is rational. □

Theorem 3.2. *[La2] Let x be a minimally elliptic singularity. Let $\pi \colon M \to V$ be a resolution of a Stein neighborhood V of x with x as its only singular point. Let m be the maximal ideal in $\mathcal{O}_{V,x}$. Let Z be the fundamental cycle on $E = \pi^{-1}(x)$.*

(1) *If $Z^2 \leq -2$, then $\mathcal{O}(-Z) = m\mathcal{O}$ on E.*

(2) *If $Z^2 = -1$, and π is the minimal resolution or the minimal resolution with non-singular E_i and normal crossings, $\mathcal{O}(-Z)/m\mathcal{O}$ is the structure sheaf for an embedded point.*

(3) *If $Z^2 = -1$ or -2, then x is a double point.*

(4) *If $Z^2 = -3$, then for all integers $n \geq 1$, $m^n \approx H^0\big(E, \mathcal{O}(-nZ)\big)$ and* $\dim m^n/m^{n+1} = -nZ^2$.

(5) *If $-3 \leq Z^2 \leq -1$, then x is a hypersurface singularity.*

(6) *If $Z^2 = -4$, then x is a complete intersection and in fact a tangential complete intersection.*

(7) *If $Z^2 \leq -5$, then x is not a complete intersection.*

3.1. Deformation of singularities

Let (X_0, x_0) be an isolated singularity with dimension n, a deformation of (X_0, x_0) will be simply a realization of (X_0, x_0) as the fiber of a map-germ between complex manifolds whose dimensions differ by n. To be precise, it consists of holomorphic map-germ $f:(X,x)\to(S,o)$ between complex manifold germs with $\dim(X,x)$-$\dim(S,o)=n$ and an isomorphism ι : of (X_0,x_0) onto the fiber (X_o,x) of f. A morphism from a deformation (ι',f') to another (ι,f) is a pair of map-germs $(\tilde g,g)$ such that the diagram

$$\begin{array}{ccc} (X',x') & \xrightarrow{\tilde g} & (X,x) \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ (S',o') & \xrightarrow{g} & (S,o) \end{array}$$

is Cartesian and $\tilde g\circ\iota'=\iota$. We say that a deformation (ι,f) of (X_0,x_0) is versal if for any deformation (ι',f') of (X_0,x_0) there exists a morphism $(\tilde g,g)$ from (ι',f') to (ι,f). Notice that we do not require this morphism to be unique in any sense. If, however, the derivative of g in o', $\partial g(o'):T_{o'}(S')\to T_o(S)$ is unique, then we say that (ι,f) is miniversal.

Proposition 3.3. *([AGLV] (2.10)) Let $f:(\mathbb{C}^{n+k},0)\to(\mathbb{C}^k,0)$ define an icis at the origin and has dimension n. A miniversal deformation of $f=0$ can be taken in the form*

$$F(z,\lambda)=f(z)+\lambda_1e_1(z)+\cdots+\lambda_\tau e_\tau(z), \tag{3.4}$$

where $e_i\in\mathcal{O}^k_{n+k}$ are the representative of a basis of the linear space:

$$T^1_f=\mathcal{O}^k_{n+k}/\{I\mathcal{O}^k_{f^{n+k}}+\mathcal{O}_{n+k}\langle\partial f/\partial z_1,\dots,\partial f/\partial z_{n+k}\rangle\}. \tag{3.5}$$

Here τ is the Tyurina number and is equal to the Milnor number μ if f is weighted homogeneous.

4. Homogeneous isolated complete intersection singularity

In this section we shall prove the following conjecture for homogeneous isolated complete intersection singularity (ICIS) in Theorem 4.1 and three dimensional isolated complete intersection singularity in Theorem 4.4. We shall also give a classification of three-dimensional rational homogeneous isolated complete intersection singularities in Theorems 4.2 and 4.3.

Conjecture. Let p be the dimension of rational isolated complete intersection singularity with $\mathbb{C}^*$-action. The the embedding dimension of the singularity is at most $2p-1$.

Definition 4.1. Let $(V,0) \subset (\mathbb{C}^N,0)$ be the analytic germ of an n-dimensional complex homogeneous isolated complete intersection singularity. Let $\pi : (M,E) \to (V,0)$ be a resolution of singularity of dimension n with exceptional set $E = \pi^{-1}(0)$. The geometric genus p_g of the singularity $(V,0)$ is the dimension of $H^{n-1}(M,\mathcal{O})$ and is independent of the resolution M.

We have the following proposition.

Proposition 4.1. *[KN] Let $(V,0) = \{f_1 = \cdots = f_r = 0\}$ be a homogeneous isolated complete intersection singularity of multidegree $(d_1,\dots,d_r)$ and dimension n, that is $degf_i = d_i$, then*

$$p_g = \sum_{\underline{k}\in K_{n,r}} \prod_{i=1}^{r} \binom{d_i}{k_i+1},$$

where $K_{n,r} := \{\underline{k} = (k_1,\dots,k_r) : k_i \geq 0 \text{ for all } i, and \sum_i k_i = n\}$.

We prove that the above conjecture is true in homogeneous case.

Theorem 4.1. *Let $(V,0) = \{f_1 = \cdots = f_r = 0\} \subset (\mathbb{C}^N,0)$ be a homogeneous rational isolated complete intersection singularity of multidegree $(d_1,\dots,d_r)$ and dimension n, then $r \leq n-1$ (i.e. $N \leq 2(N-r)-1$).*

Proof. Since $(V,0)$ is a homogeneous isolated rational complete intersection singularity, so by Proposition 4.1, we have

$$p_g = \sum_{\underline{k}\in K_{n,r}} \prod_{i=1}^{r} \binom{d_i}{k_i+1} = 0.$$

Thus for any $\underline{k} \in K_{n,r}$, we have $\prod_{i=1}^{r} \binom{d_i}{k_i+1} = 0$. If we assume the contrary, $r \geq n$, without loss of generality, we consider thel $\underline{k} = (k_1,\dots,k_r)$ with $k_1 = k_2 = \cdots = k_n = 1, k_{n+1} = \cdots = k_r = 0$. Then for this choice $\underline{k} = (1,\dots,1,0,\dots,0)$, we have $\prod_{i=1}^{r} \binom{d_i}{k_i+1} \geq 1$ since $d_i \geq 2$. This contradicts with $\prod_{i=1}^{r} \binom{d_i}{k_i+1} = 0$. Therefore we have $r \leq n-1$. □

We have the following two classification theorems for homogeneous case.

Theorem 4.2. *Let $(V,0) = \{f_1 = \cdots = f_r = 0\} \subset (\mathbb{C}^N,0)$ be a three dimensional homogeneous rational isolated complete intersection singularity of multidegree $(d_1,\dots,d_r)$ which is not a hypersurface singularity, then $r = 2, N = 5$ and $d_1 = d_2 = 2$.*

Proof. If follows from Theorem 4.1 that $r \leq 2$. Since $(V, 0)$ is not a hypersurface singularity, so $r = 2$. Let $\underline{k} = (k_1, k_2) \in K_{3,2}$, we have $(k_1, k_2) = (0, 3), (3, 0), (1, 2)$ or $(2, 1)$. For $(k_1, k_2) = (1, 2)$, by Proposition 4.1

$$p_g = \sum_{\underline{k} \in K_{3,2}} \prod_{i=1}^{2} \binom{d_i}{k_i + 1} = 0,$$

we have

$$\binom{d_1}{2}\binom{d_2}{3} = 0$$

which implies $d_2 = 2$ since d_1 and d_2 are at least 2. Similarly, $(k_1, k_2) = (2, 1)$ implies $d_1 = 2$. Thus $d_1 = d_2 = 2$. □

Theorem 4.3. *Let $(V, 0) = \{f = 0\} \subset (\mathbb{C}^4, 0)$ be a three dimensional homogeneous rational isolated hypersurface singularity of degree d, then $d = 2, 3$.*

Proof. By Proposition 4.1, we have

$$p_g = \binom{d}{4} = 0,$$

so we $d = 2$ or 3. □

We can also prove the conjecture for $p = 3$.

Lemma 4.1. *Let $(V, 0)$ be a n-dimensional isolated singularity in $\mathbb{C}^N$. If $2n - N > 0$, then $(V, 0)$ cannot have two components of dimension n.*

Proof. If V is union of two components V_1 and V_2, each of which is of dimension n, then V_1 and V_2 will intersect with at least dimension $2n - N > 0$. This is a contradiction since V is singular along the intersection. □

Theorem 4.4. *Let $(V, 0)$ be a three dimensional rational isolated complete intersection singularity. Then the embedding dimension of $(V, 0)$ is at most 5.*

Proof. Take a generic section $(H, 0)$ of $(V, 0)$. Then by Theorem 3.1, $(H, 0)$ is either a 2-dimensional rational Gorenstein singularity or minimally elliptic singularity. It is well-known that 2-dimensional rational Gorenstein singularity must be rational double points. So the embedding dimension of $(H, 0)$ is 3. On the other hand, by Theorem 3.2 asserts that minimally elliptic complete intersection isolated singularity has embedding dimension at most 4. So the embedding dimension of $(V, 0)$ is at most 5. □

An immediate corollary is as follows.

Corollary 4.1. *The three dimensional weighted homogeneous rational isolated complete intersection singularity which is not hypersurface singularity is defined by two weighted homogeneous polynomials in 5 variables.*

Lemma 4.2. *Both f_1 and f_2 in Corollary 4.1 are irreducible.*

Proof. Assume that $f_1 = f_{11} f_{12} \cdots f_{1k_1}$ and $f_2 = f_{21} f_{22} \cdots f_{2k_2}$,where $f_{1i}, 1 \leq i \leq k_1$ and $f_{2j}, 1 \leq j \leq k_2$ are irreducible. $V(f_{1i}, f_{2j})$ are $k_1 k_2$ irreducible components of dimension 3. Since $n = 3, N = 5$, so $2n - N = 1 > 0$. By Lemma 4.1, we have $k_1 k_2 = 1$, which implies $k_1 = k_2 = 1$, thus both f_1 and f_2 are irreducible. □

5. Classification of three dimensional weighted homogeneous rational isolated complete intersection singularity

In this section, we shall give an complete classification of three-dimensional rational weighted homogeneous complete intersection singularities. We first recall some definitions and then we prove some properties which are used in the proof of classification theorem.

Definition 5.1. Let $w = (w_1, \ldots, w_n; d)$ be an $(n+1)$-tuple of positive rational numbers. A polynomial $f(z_1, \ldots, z_n)$ is said to be a weighted homogeneous polynomial with weights w if each monomial $\alpha z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}$ of f satisfies $a_1 w_1 + \cdots + a_n w_n = d$. And we say a pair of polynomials (f_1, f_2) are weighted homogeneous of type $(w_1, \ldots, w_n; d_1, d_2)$ if f_1 is weighted homogeneous of type $(w_1, \ldots, w_n; d_1)$ and f_2 is weighted homogeneous of type $(w_1, \ldots, w_n; d_2)$.

By Corollary 4.1, in order to classify three dimensional weighted homogeneous rational isolated complete intersection singularity, we only need to study weighted homogeneous polynomials in 5 variables.

Theorem 5.1. *[Ma] Let $X = V(f_1, \ldots, f_k)$ be an weighted homogeneous ICIS of type*

$$(w_1, \ldots, w_n; d_1, \ldots, d_k).$$

Let

$$A(N) = \left\{ (a_1, \ldots, a_n) \in \mathbb{N}^n | a_i > 0 \text{ and } \sum_{i=1}^{k} a_i w_i \leq N \right\}$$

and

$$\ell(N) = \#A(N).$$

Then we have

$$p_g(X) = \ell(d_1 + \cdots + d_k) \\ - \sum_{i=1}^{k} \ell(d_1 + \cdots + \hat{d_i} + \cdots + d_k) + \cdots + (-1)^{k-1} \sum_{i=1}^{k} \ell(d_i).$$

The following observations plays key role in our proof.

Lemma 5.1. *Let $X = V(f_1, f_2)$ be a weighted homogeneous ICIS of type $(w_1, \ldots, w_5; d_1, d_2)$. Then X is rational if and only if*

$$w_1 + \cdots + w_5 > d_1 + d_2.$$

Proof. By Theorem 5.1,we have $p_g(X) = \ell(d_1 + d_2) - \ell(d_1) - \ell(d_2)$. Thus X is rational if and only if $\ell(d_1 + d_2) - \ell(d_1) - \ell(d_2) = 0$. It is easy to see that $w_1 + \cdots + w_5 > d_1 + d_2$ implies $\ell(d_1 + d_2) = \ell(d_1) = \ell(d_2) = 0$, it follows that X is rational. If X is rational, and without lose of generality, we assume that $\ell(d_1) > 0$, then $A(d_1)$ is not empty. Let $\mathbf{a}_{\max} \in A(d_1)$ such that $\mathbf{wa}_{\max} \geq \mathbf{wa}$ for any $\mathbf{a} \in A(d_1)$, where $\mathbf{wa} = \sum_{i=1}^{5} w_i a_i$ for $\mathbf{a} = (a_1, \ldots, a_5)$. Then let $B = \{\mathbf{a}_{\max} + \mathbf{b} \mid \mathbf{b} \in A(d_2)\}$ so we have $d_1 < \mathbf{wc} \leq d_1 + d_2$ for any $\mathbf{c} \in B$ and $\#B = \ell(d_2)$. It is easy to seen that there exist $i \in \{1, 2, 3, 4, 5\}$ such that $w_i \leq d_2$, because if not then $f_2 = 0$. Without lose of generality, we may assume that $w_1 \leq d_2$. Let $\mathbf{d} = \mathbf{a}_{\max} + (1, 0, 0, 0, 0)$, then $d_1 < \mathbf{wd} \leq d_1 + d_2$. Notice that $\mathbf{d} \in A(d_1 + d_2) \setminus (A(d_1) \cup B)$ and $A(d_1) \cap B = \emptyset$, thus we have $\ell(d_1 + d_2) \geq \ell(d_1) + \ell(d_2) + 1$. It follows that $p_g(X) \geq 1$. It contradicts with X is rational, so we conclude that $\ell(d_1) = 0$. Similarly we can prove that $\ell(d_2) = 0$. So $p_g(X) = \ell(d_1 + d_2) = 0$ which implies $w_1 + \cdots + w_5 > d_1 + d_2$. □

Lemma 5.2. *Let $X = V(f_1, f_2)$ be a three dimensional weighted homogeneous ICIS of type $(w_1, \ldots, w_5; d_1, d_2)$. Then we have*

(1) *for any $i \in \{1, 2, 3, 4, 5\}$, one of the following cases occurs:*
   (1a) *z_i^n appears in f_1 for some n,*
   (1b) *z_i^n appears in f_2 for some n,*
   (1c) *there exist $j, k \in \{1, 2, 3, 4, 5\} \setminus \{i\}$ $j \neq k$ such that $z_i^n z_j$ appears in f_1 for some n and $z_i^m z_k$ appears in f_2 for some m.*

(2) *for any* $l=1,2$ *and any* $\{i,j\}\subset\{1,2,3,4,5\}$, *one of the following cases occurs:*
  (2a) $z_i^a z_j^b$ *appears in* f_l, *for some non-negative integers* a,b,
  (2b) *there exists* $k\in\{1,2,3,4,5\}\setminus\{i,j\}$ *such that* $z_k z_i^a z_j^b$ *appears in* f_l, *for some non-negative integers* a,b.

(3) *for any* $\{i,j\}\subset\{1,2,3,4,5\}$, *one of the following cases occurs:*
  (3a) $z_i^a z_j^b$ *appears in* f_1 *for some non-negative integer* a,b,
  (3b) $z_i^a z_j^b$ *appears in* f_2 *for some non-negative integer* a,b,
  (3c) *there exist* $\{p_1,p_2\},\{s_1,s_2\}\subset\{1,2,3,4,5\}\setminus\{i,j\}$ *and* $\{p_1,p_2\}\neq\{s_1,s_2\}$ *such that* $z_{p_1}z_i^{a_1}z_j^{b_1}, z_{p_2}z_i^{a_2}z_j^{b_2}$ *appear in* f_1 *for some non-negative integers* a_1,a_2,b_1,b_2 *and* $z_{s_1}z_i^{c_1}z_j^{d_1}, z_{s_2}z_i^{c_2}z_j^{d_2}$ *appear in* f_2 *for some non-negative integers* c_1,c_2,d_1,d_2.

(4) *for any* $l=1,2$ *and any* $\{i,j,k\}\subset\{1,2,3,4,5\}$, *let* $\{p,s\}=\{1,2,3,4,5\}\setminus\{i,j,k\}$, *then one of the following cases occurs:*
  (4a) $z_i^a z_j^b z_k^c$ *appears in* f_l, *for some non-negative integers* a,b,c,
  (4b) $z_p z_i^{a_1} z_j^{b_1} z_k^{c_1}$ *and* $z_s z_i^{a_2} z_j^{b_2} z_k^{c_2}$ *appear in* f_l, *for some non-negative integers* a_1,b_1,c_1,a_2,b_2,c_2.

(5) *for any* $\{i,j,k\}\subset\{1,2,3,4,5\}$, *there exists* $l\in\{1,2\}$ *such that* $z_i^a z_j^b z_k^c$ *appears in* f_l *for some non-negative integers* a,b,c.

Proof. (1) Without lose of generality, we may assume that $i=1$. Assume on the contrary that neither of $(1a),(1b)$ and $(1c)$ occurs. Then z_1^n does not appear in f_l for any $l=1,2$ and integer n, so we have $f_1=f_2=\frac{\partial f_1}{\partial z_1}=\frac{\partial f_2}{\partial z_1}=0$ when $z_2=z_3=z_4=z_5=0$. And for any $\{j,k\}\subset\{2,3,4,5\}$, we have $z_1^a z_j$ doesn't appear in f_1 for any non-negative integer a or $z_1^b z_k$ doesn't appear in f_2 for any non-negative integer b. It follows that $\frac{\partial f_1}{\partial z_j}=0$ or $\frac{\partial f_2}{\partial z_k}=0$ when $z_2=z_3=z_4=z_5=0$. Similarly we have $\frac{\partial f_1}{\partial z_k}=0$ or $\frac{\partial f_2}{\partial z_j}=0$ when $z_2=z_3=z_4=z_5=0$. Thus we have

$$\frac{\partial f_1}{\partial z_j}\frac{\partial f_2}{\partial z_k}-\frac{\partial f_1}{\partial z_k}\frac{\partial f_2}{\partial z_j}=0,\ \forall\,\{j,k\}\subset\{2,3,4,5\}, z_2=z_3=z_4=z_5=0,$$

which implies $(\frac{\partial f_1}{\partial z_1},\dots,\frac{\partial f_1}{\partial z_5})$ and $(\frac{\partial f_2}{\partial z_1},\dots,\frac{\partial f_2}{\partial z_5})$ are linear dependent. Thus $V(z_2,z_3,z_4,z_5)$, which has dimension one, is contained in the singular locus of X. This contradicts with X has an isolated singularity.

(2) We may assume that $l=1$ and $i,j=1,2$. Assume on the contrary that neither of $(2a)$ and $(2b)$ occurs, then $z_1^a z_2^b$ does not appear in f_1, for any non-negative integers a,b and $z_k z_1^a z_2^b$ does not appear in f_1. And for

any $k \in \{3,4,5\}$ and for any non-negative integers a, b, we have $f_1 = 0$ and $(\frac{\partial f_1}{\partial z_1}, \ldots, \frac{\partial f_1}{\partial z_5}) = 0$ when $z_3 = z_4 = z_5 = 0$. Thus $V(f_2, z_3, z_4, z_5)$, which has dimension at least one, is contained in the singular locus of X. This contradicts with X has an isolated singularity.

(3) We may assume that $i, j = 1, 2$. Assume on the contrary that neither of $(3a), (3b)$ and $(3c)$ occurs, then one of the following two cases occurs:

subcase (a) $z_1^a z_2^b$ does not appear in f_q for any $q = 1, 2$ and any non-negative integer a, b, and there exist $l \in \{1, 2\}, s, p \in \{3, 4, 5\} (s \neq p)$ such that both $z_s z_1^{a_1} z_2^{b_1}$ and $z_p z_1^{a_2} z_2^{b_2}$ do not appear in f_l for any non-negative integer a_1, a_2, b_1, b_2.

subcase (b) $z_1^a z_2^b$ does not appear in f_q for any $q = 1, 2$ and any non-negative integer a, b, and there exists $k \in \{3, 4, 5\}$ such that $z_k z_1^a z_2^b$ does not appear in f_q for any $q = 1, 2$ and any non-negative integer a, b.

If **subcase (a)** occurs, without lose of generality, we may assume that $l = 1$ and $s, p = 3, 4$, then $f_1 = f_2 = \frac{\partial f_1}{\partial z_1} = \cdots = \frac{\partial f_1}{\partial z_4} = 0$ when $z_3 = z_4 = z_5 = 0$. Thus $V(z_3, z_4, z_5, \frac{\partial f_1}{\partial z_5})$, which has dimension at least one, is contained in the singular locus of X. This contradicts with X has an isolated singularity.

If **subcase (b)** occurs, without lose of generality, we may assume $k = 3$. Then when $z_3 = z_4 = z_5 = 0$, we have $f_1 = f_2 = 0$, $(\frac{\partial f_1}{\partial z_1}, \ldots, \frac{\partial f_1}{\partial z_5}) = (0, 0, 0, \frac{\partial f_1}{\partial z_4}, \frac{\partial f_1}{\partial z_5})$ and $(\frac{\partial f_2}{\partial z_1}, \ldots, \frac{\partial f_2}{\partial z_5}) = (0, 0, 0, \frac{\partial f_2}{\partial z_4}, \frac{\partial f_2}{\partial z_5})$. Thus $V(z_3, z_4, z_5, \frac{\partial f_1}{\partial z_4}\frac{\partial f_2}{\partial z_5} - \frac{\partial f_2}{\partial z_4}\frac{\partial f_1}{\partial z_5})$, which has dimension at least one, is contained in the singular locus of X. This contradicts with X has an isolated singularity.

(4) We may assume that $i, j, k = 1, 2, 3$ and $l = 1$. Assume on the contrary that neither of $(4a)$ and $(4b)$ occurs, thus $z_1^a z_2^b z_3^c$ does not appear in f_1 for any non-negative integers a, b, c and there exists $p \in \{4, 5\}$ such that $z_p z_1^a z_2^b z_3^c$ does not appear in f_1 for any non-negative integers a, b, c. Without lose of generality, we may assume that $p = 4$. Then $f_1 = \frac{\partial f_1}{\partial z_1} = \cdots = \frac{\partial f_1}{\partial z_4} = 0$ when $z_4 = z_5 = 0$. Thus $V(z_4, z_5, f_2, \frac{\partial f_1}{\partial z_5})$, which has dimension at least one, is contained in the singular locus of X. This contradicts with X has an isolated singularity.

(5) We may assume that $i, j, k = 1, 2, 3$. Assume on the contrary that $z_1^a z_2^b z_3^c$ does not appear in f_l for any $l = 1, 2$ and any non-negative integers a, b, c. Then when $z_4 = z_5 = 0$ we have $f_1 = f_2 = 0$ and $(\frac{\partial f_1}{\partial z_1}, \ldots, \frac{\partial f_1}{\partial z_5}) = (0, 0, 0, \frac{\partial f_1}{\partial z_4}, \frac{\partial f_1}{\partial z_5})$ and $(\frac{\partial f_2}{\partial z_1}, \ldots, \frac{\partial f_2}{\partial z_5}) = (0, 0, 0, \frac{\partial f_2}{\partial z_4}, \frac{\partial f_2}{\partial z_5})$. Thus $V(z_4, z_5, \frac{\partial f_1}{\partial z_4}\frac{\partial f_2}{\partial z_5} -$

$\frac{\partial f_2}{\partial z_4}\frac{\partial f_1}{\partial z_5}$), which has dimension at least two, is contained in the singular locus of X. This contradicts with X has an isolated singularity. $\square$

We define $N(a) = \{ka \mid k \text{ is non-negative integer}\}$, $N(a,b) = \{ka + sb \mid k, s \text{ are non-negative integers}\}$ and $N(a,b,c) = \{ka + sb + tc \mid k, s, t \text{ are non-negative integers}\}$.

Corollary 5.1. *Let $X = V(f_1, f_2)$ be a weighted homogeneous ICIS of type $(w_1, \ldots, w_5; d_1, d_2)$, then we have:*

(1) *for any $i \in \{1,2,3,4,5\}$, we have $d_1 \in N(w_i)$ or $d_2 \in N(w_i)$ or there exist $j, k \in \{1,2,3,4,5\} \setminus \{i\}$, $j \neq k$ such that $d_1 - w_j \in N(w_i)$ and $d_2 - w_k \in N(w_i)$.*

(2) *for any $\{i,j\} \subset \{1,2,3,4,5\}$ and any $l \in \{1,2\}$, we have $d_l \in N(w_i, w_j)$ or there exists $k \in \{1,2,3,4,5\} \setminus \{i,j\}$ such that $d_l - w_k \in N(w_i, w_j)$.*

(3) *for any $\{i,j\} \subset \{1,2,3,4,5\}$, if $d_1, d_2 \notin N(w_i, w_j)$, then there exist $\{p_1, p_2\}, \{s_1, s_2\} \subset \{1,2,3,4,5\} \setminus \{i,j\}$ and $\{p_1, p_2\} \neq \{s_1, s_2\}$ such that $d_1 - w_{p_1}, d_1 - w_{p_2} \in N(w_i, w_j)$ and $d_2 - w_{s_1}, d_2 - w_{s_2} \in N(w_i, w_j)$.*

(4) *for any $\{i,j,k\} \subset \{1,2,3,4,5\}$ and any $l \in \{1,2\}$, let $\{p,s\} = \{1,2,3,4,5\} \setminus \{i,j,k\}$, if $d_l \notin N(w_i, w_j, w_k)$, then we have $d_l - w_p, d_l - w_s \in N(w_i, w_j, w_k)$.*

(5) *for any $\{i,j,k\} \subset \{1,2,3,4,5\}$, we have $d_1 \in N(w_i, w_j, w_k)$ or $d_2 \in N(w_i, w_j, w_k)$.*

Theorem 5.2. *Let $X = V(f_1, f_2)$ be a weighted homogeneous ICIS of type $(w_1, \ldots, w_5; 1, d)$, with $d \geq 1$. Then (f_1, f_2) has the same weight type as one of the following weight homogeneous singularities in the list up to permutation of coordinates. (there are total 303 classes in the list, we only list part of these classes in order to save place. The complete list can be found at* `https://arxiv.org/abs/1604.07843`*)*

Remark 5.1. *We also list the Milnor number μ and the vector basis of the miniversal deformation of the singularities in the list. In order to save space, we only list the set of maximum elements (i.e. **mini**) of the vector basis of the corresponding singularity. That is, $\{[a,0] \mid a \in \boldsymbol{m},\ \exists\ [b,0] \in \boldsymbol{mini} \text{ s.t. } b \geq a\} \cup \{[0,a] \mid a \in \boldsymbol{m},\ \exists\ [0,b] \in \boldsymbol{mini} \text{ s.t. } b \geq a\}$ form a basis of miniversal deformation where $\boldsymbol{m} =$ ideal $(z_1, \ldots, z_5)$. A monomial $z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4} z_5^{a_5} \leq z_1^{b_1} z_2^{b_2} z_3^{b_3} z_4^{b_4} z_5^{b_5}$ if and only if $a_i \leq b_i$ for $i = 1, \ldots, 5$. For example, if $\boldsymbol{mini} =$*

$\{[z_1^3,0],[z_1z_2^2,0],[0,z_3z_4]\}$, *then*

$$\{[0,1],[1,0],[z_1,0],[z_1^2,0],[z_1^3,0],\\ [z_2,0],[z_2^2,0],[z_1z_2,0],[z_1z_2^2,0],[0,z_3],[0,z_4],[0,z_3z_4]\}$$

form a vector basis of miniversal deformation.

(1) $\begin{cases} z_1^2+z_2^2+z_3^2+z_4^2+z_5^n=0 \\ z_1^2+2z_2^2+3z_3^2+4z_4^2+5z_5^n=0 \end{cases}$
$n\geq 2$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{n};1,1)$
$\mu=-7+8n$

$mini=\{[z_5^{-2+2n},0],[z_4^2z_5^{-2+n},0],[z_3z_5^{-2+n},0],[z_2z_5^{-2+n},0],$
$[z_1z_5^{-2+n},0],[0,z_5^{-2+n}]\}$

(2) $\begin{cases} z_1^2+z_2^2+z_3^2+z_4^3+z_5^3=0 \\ z_1^2+2z_2^2+3z_3^2+4z_4^3+5z_5^3=0 \end{cases}$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3};1,1)$
$\mu=32$

$mini=\{[z_4z_5^4,0],[z_4^4z_5,0],[z_3z_4z_5,0],[z_2z_4z_5,0],[z_1z_4z_5,0],[0,z_4z_5]\}$

(3) $\begin{cases} z_1^2+z_2^2+z_3^2+z_4^3+z_5^4=0 \\ z_1^2+2z_2^2+3z_3^2+4z_4^3+5z_5^4=0 \end{cases}$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{4};1,1)$
$\mu=47$

$mini=\{[z_4z_5^6,0],[z_4^4z_5^2,0],[z_3z_4z_5^2,0],[z_2z_4z_5^2,0],[z_1z_4z_5^2,0],[0,z_4z_5^2]\}$

(4) $\begin{cases} z_1^2+z_2^2+z_3^2+z_4^3+z_5^5=0 \\ z_1^2+2z_2^2+3z_3^2+4z_4^3+5z_5^5=0 \end{cases}$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{5};1,1)$
$\mu=62$

$mini=\{[z_4z_5^8,0],[z_4^4z_5^3,0],[z_3z_4z_5^3,0],[z_2z_4z_5^3,0],[z_1z_4z_5^3,0],[0,z_4z_5^3]\}$

$$(5)\ \begin{cases} z_1z_3 + z_2^2 + z_4^2 = 0 \\ z_1^2 + z_3^4 + z_5^3 + z_2^2z_3 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{4}{9}; 1, \frac{4}{3})$
$\mu = 45$

$$mini = \{[z_5^2, 0], [z_3^2z_5, 0], [0, z_3^2z_4^2z_5], [0, z_3^3z_4z_5], [0, z_2z_4z_5], [0, z_3^6z_5], [0, z_2z_3^3z_5]\}$$

$\vdots$

$$(55)\ \begin{cases} z_1z_2 + z_3^2 + z_4^2 + z_5^2 = 0 \\ z_1z_3 + 2z_2^5 + z_2z_4^2 = 0 \end{cases}$$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{5}{4})$
$\mu = 31$

$$mini = \{[z_3, 0], [0, z_2^3z_5^2], [0, z_4z_5], [0, z_2^4z_5], [0, z_4^2], [0, z_2^4z_4], [0, z_2^3z_3], [0, z_2^8]\}$$

$\vdots$

$$(59)\ \begin{cases} z_1z_2 + z_3^2 + z_4^2 + z_5^n = 0 \\ z_1z_5 + 2z_3^2 + z_4^2 + 3z_2^n = 0 \end{cases}$$
$n \geq 3$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{-1+n}{n}, \frac{1}{n}, \frac{1}{2}, \frac{1}{2}, \frac{1}{n}; 1, 1)$
$\mu = -3 + 4n + n^2$

$$mini = \{[z_5^{-2+2n}, 0], [z_4z_5^{-2+n}, 0], [z_3z_5^{-2+n}, 0], [0, z_5^{-2+2n}], [0, z_2^{-2+n}z_5^{-1+n}], [0, z_2^{-1+n}]\}$$

$$(60)\ \begin{cases} z_1z_2 + z_3^2 + z_4^2 + z_5^n = 0 \\ z_1z_5 + 3z_2^{1+2n} + 2z_2z_3^2 + z_2z_4^2 = 0 \end{cases}$$
$n \geq 3$
$(w_1, w_2, w_3, w_4, w_5; 1, d) = (\frac{-1+2n}{2n}, \frac{1}{2n}, \frac{1}{2}, \frac{1}{2}, \frac{1}{n}; 1, \frac{1+2n}{2n})$
$\mu = 5 + 9n + 2n^2$

$$mini = \{[z_5^{-1+n}, 0], [0, z_2z_5^n], [0, z_2^{-1+2n}z_5^{-1+n}], [0, z_2^{-3+4n}z_5], [0, z_4^2], [0, z_3z_4], [0, z_2^{2n}z_4], [0, z_2^{2n}z_3], [0, z_2^{4n}]\}$$

$\vdots$

$$(303)\ \begin{cases} z_1z_2+z_3z_4=0 \\ z_1z_5+z_2z_3^2+z_2z_4^5+z_5^3+z_2^{51}=0 \end{cases}$$
$(w_1,w_2,w_3,w_4,w_5;1,d)=(\frac{34}{35},\frac{1}{35},\frac{5}{7},\frac{2}{7},\frac{17}{35};1,\frac{51}{35})$
$\mu=463$

$mini=\{[z_5^2,0],[0,z_2^{50}z_5^2],[0,z_4^5],[0,z_2^{50}z_4^4],[0,z_3^2],[0,z_2^{50}z_3],[0,z_2^{99}]\}$

Proof. It is easy to check that each singularities defined by pairs of polynomials in the list above are three dimensional isolated rational complete intersection singularities. By Lemma 5.2 (1), we know that for any $i\in\{1,2,3,4,5\}$, one of the following cases occurs:

(1a) $z_i^{n_i}$ appears in f_1 for some n_i,

(1b) $z_i^{n_i}$ appears in f_2 for some n_i,

(1c) there exist $j_i,k_i\in\{1,2,3,4,5\}\setminus\{i\}$ *and* $j_i\neq k_i$ such that $z_i^{n_i}z_{j_i}$ appears in f_1 for some n_i and $z_i^{m_i}z_{k_i}$ appears in f_2 for some m_i.

For each $i\in\{1,2,3,4,5\}$, if one of (1a), (1b) and (1c) occurs, then there are $3^5=243$ cases. If (1a) or (1b) occurs, one monomial which appear in f_1 and f_2 can be determined. And if (1c) occurs, then two monomials which appear in f_1 and f_2 can be determined. Now we consider the following two cases:

(I) There exists $i\in\{1,2,3,4,5\}$ such that (1c) occurs. Therefore more than 6 monomials in f_1 and f_2 are determined. Thus we get more than 6 equations of $w_1,\ldots,w_5,d$ (for instance, if we have $z_1^{n_1}z_2$ appears in f_2, then we have $n_1w_1+w_2=d$). So $(w_1,\ldots,w_5,d)$ is uniquely determined by solving these 6 linear equations. And we have checked that each weight type $(w_1,\ldots,w_5;1,d)$ obtained by this way, which satisfies the rational condition $w_1+\cdots+w_5>1+d$ and the conditions listed in Corollary 5.1, is the same as one of the weight types of the singularities in the list up to permutation of coordinates.

More explicitly, for example, we treat the case that $z_1^{n_1},\ldots,z_4^{n_4},z_5^{n_5}z_4$ appear in f_1 and $z_5^{m_5}z_3$ appears in f_2. Then we can get $w_1=\frac{1}{n_1},\ldots,w_4=\frac{1}{n_4},w_5=\frac{n_4-1}{n_4n_5},d=\frac{m_5(n_4-1)}{n_4n_5}+\frac{1}{n_3}$ by solving the 6 corresponding linear equations. Without lose of generality, we may assume that $w_1\geq w_2$. Since we have $w_1+\cdots+w_5>1+d$ and $d\geq 1$, so we conclude that $(n_1,\ldots,n_5,m_5)$ can only be one of the following cases:

(1) $(2,2,u,v,1,1),\ 2\leq u\leq v$

(2) $(2,3,u,v,1,1),\ 2\le u\le v\le 5$

(3) $(2,4,u,v,1,1),\ 2\le u\le v\le 3$

(4) $(2,5,u,v,1,1),\ 2\le u\le v\le 3$

(5) $(2,u,2,2,1,1),\ u\ge 6$

(6) $(3,u,2,2,1,1),\ 3\le u\le 5$

(7) $(2,2,2,2,u,u),\ u\ge 2$

(8) $(2,2,2,u,2,1),\ u\ge 2$

(9) $(2,3,2,2,2,2)$.

Then we only consider the infinite cases (1), (5) and (7). The other finite cases can be checked easily.

For infinite case (1), we have

$$w_1=w_2=\frac{1}{2},\ w_3=\frac{1}{u},\ w_4=\frac{1}{v},\ w_5=1-\frac{1}{v},\ d=1-\frac{1}{v}+\frac{1}{u},\ 2\le u\le v.$$

By Corollary 5.1 (2), we have $d\in N(w_1,w_2)$ or there exists $k\in\{3,4,5\}$ such that $d-w_k\in N(w_1,w_2)$, it follows that one of following cases occurs:

(i) $d\in N(\frac{1}{2},\frac{1}{2})$

(ii) $d-\frac{1}{u}\in N(\frac{1}{2},\frac{1}{2})$

(iii) $d-\frac{1}{v}\in N(\frac{1}{2},\frac{1}{2})$

(iv) $d+\frac{1}{v}-1\in N(\frac{1}{2},\frac{1}{2})$.

If (i) $d\in N(\frac{1}{2},\frac{1}{2})$ occurs, since $v\ge u\ge 2$, so we have $1\le d=1-\frac{1}{v}+\frac{1}{u}<\frac{3}{2}$, thus $d=1$ and $u=v$.

If (ii) $d-\frac{1}{u}\in N(\frac{1}{2},\frac{1}{2})$ occurs, since $v\ge 2$, we have $\frac{1}{2}\le d-\frac{1}{u}=1-\frac{1}{v}<1$, thus $1-\frac{1}{v}=\frac{1}{2}$. Hence we have $v=2$. Since $2\le u\le v$, so we have $u=2$.

If (iv) $d+\frac{1}{v}-1\in N(\frac{1}{2},\frac{1}{2})$ occurs, i.e. $\frac{1}{u}=d+\frac{1}{v}-1\in N(\frac{1}{2},\frac{1}{2})$. Since $0<\frac{1}{u}\le\frac{1}{2}$, we have $\frac{1}{u}=\frac{1}{2}$. Therefore we have $u=2$.

If (iii) $d-\frac{1}{v}\in N(\frac{1}{2},\frac{1}{2})$ occurs, since $2\le u\le v$, we have $0<d-\frac{1}{v}=1-\frac{2}{v}+\frac{1}{u}<\frac{3}{2}$. Thus we have $d-\frac{1}{v}=1-\frac{2}{v}+\frac{1}{u}=\frac{1}{2}$ or 1. If $d-\frac{1}{v}=\frac{1}{2}$, notice that $d\ge 1$ and $2\le u\le v$, thus we have $v=u=2$. If $d-\frac{1}{v}=1-\frac{2}{v}+\frac{1}{u}=1$, we have $2u=v$.

In conclusion, we have $2\le u=v$ or $2=u\le v$ or $4\le 2u=v$. So $(w_1,\ldots,w_5;1,d)$ is same as one of the following cases:

case (1) $2\le u=v\Rightarrow(w_1,\ldots,w_5;1,d)=(\frac{1}{2},\frac{1}{2},\frac{1}{n},\frac{1}{n},\frac{n-1}{n};1,1),\ n\ge 2$

case (2) $2 = u \leq v \Rightarrow (w_1, \ldots, w_5; 1, d) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{n}, \frac{1}{2}; 1, 1),\ n \geq 2$
case (3) $2u = v \Rightarrow (w_1, \ldots, w_5; 1, d) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{n}, \frac{1}{2n}, \frac{2n-1}{2n}; 1, \frac{2n+1}{2n}),\ n \geq 2.$
And we can see that case (1); case (2) and case (3) corresponds to the 1st ($n = 2$), 59th; 1st; and 55th, 60th singularities in the list respectively.

For infinite case (5), we have

$$w_1 = w_3 = w_4 = w_5 = \frac{1}{2},\ w_2 = \frac{1}{u},\ d = 1.$$

For infinite case (7), we have

$$w_1 = w_2 = w_3 = w_4 = \frac{1}{2},\ w_5 = \frac{1}{2u},\ d = 1.$$

it is easy to seen that $(w_1, \ldots, w_5; 1, d)$ of infinite cases (5) and (7) is included in the weight types of the 1st singularities in the list up to permutation of coordinates.

(II) For each $i \in \{1, 2, 3, 4, 5\}$, (1a) or (1b) occurs. Then there are only 5 monomials in f_1 and f_2 can be determined. In order to determine $(w_1, \ldots, w_5; d)$, we need at least one more monomial included in f_1 and f_2. Since 5 monomials in f_1 and f_2 are known, so it is easy to seen that one of following will occurs:

(a) there exists $\{i_1, i_2\} \subset \{1, 2, 3, 4, 5\}$ such that (1a) occurs when $i = i_1, i_2$

(b) there exists $\{j_1, j_2\} \subset \{1, 2, 3, 4, 5\}$ such that (1b) occurs when $i = j_1, j_2$.

If (a) occurs, then by Lemma 5.2 (2), we have $z_{i_1}^a z_{i_2}^b$ appears in f_2 for some non-negative integer a, b or there exist $k \in \{1, 2, 3, 4, 5\} \setminus \{i_1, i_2\}$ such that $z_k z_{i_1}^a z_{i_2}^b$ appears in f_2 for some non-negative integer a, b. Thus we have 6 monomials in f_1 and f_2 are determined now.

If (b) occurs, then by Lemma 5.2 (2), we have $z_{j_1}^a z_{j_2}^b$ appears in f_1 for some non-negative integer a, b or there exist $k \in \{1, 2, 3, 4, 5\} \setminus \{j_1, j_2\}$ such that $z_k z_{j_1}^a z_{j_2}^b$ appears in f_1 for some non-negative integer a, b. Thus there are 6 monomials in f_1 and f_2 are determined now.

More explicitly, let us consider the example that $z_1^{n_1}, z_2^{n_2}, z_3^{n_3}$ appear in f_1, and $z_4^{n_4}, z_5^{n_5}$ appear in f_2. Then by Lemma 5.2 (2) we have $z_4^a z_5^b$ appears in f_1 for some non-negative integers a, b or there exist $k \in \{1, 2, 3\}$ such that $z_k z_4^a z_5^b$ appears in f_1 for some non-negative integer a, b. Thus there are 6 monomials in f_1 and f_2 are determined. It follows that $(w_1, \ldots, w_5; d)$ is determined as above. And we have checked that each weight type $(w_1, \ldots, w_5; 1, d)$ gotten by this way, which satisfies the rational condition $w_1 + \cdots +$

$w_5 > 1 + d$ and the conditions listed in Corollary 5.1, is the same as one of the weight types of the singularities in the above list up to permutation of coordinates. □

References

[ALLM1] P. C. Argyres, M. Lotito, Y. Lü, and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs I: physical constraints on relevant deformations*, preprint, `arXiv:1505.04814`.

[ALLM2] P. C. Argyres, M. Lotito, Y. Lü, and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs II: Construction of special Kahler geometries and RG flows*, preprint, `arXiv:1601.00011`.

[ALLM3] P. C. Argyres, M. Lotito, and Y. Lü, and M. Martone, *Expanding the landscape of $\mathcal{N} = 2$ rank 1 SCFTs*, preprint, `arXiv:1602.02764`.

[AGLV] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vasilev, *Singularity Theory II: Classification and Applications*, Springer, Berlin-Heidelberg, 1993.

[An-Gr] A. Andreotti and H. Grauert *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

[CDY] O. Chacaltana, J. Distler, and Y. Tachikawa, *Nilpotent orbits and codimension-2 defects of 6d $N = (2, 0)$ theories*, IJMA **28** (2013).

[CNV] S. Cecotti, A. Neitzke, and C. Vafa, *R-twisting and 4d/2d correspondences*, preprint, `arXiv:1006.3435`.

[Ga] D. Gaiotto, *$N = 2$ dualities*, JHEP (2012), no. 8, 1–58.

[GMN] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, preprint, `arXiv:0907.3987`.

[Ha] R. Hartshorne, *Local Cohomology*, Lecture Notes in Mathematics, 1967.

[KN] D. Kerner and A. Némethi, *The ‘corrected Durfee’s inequality’ for homogeneous complete intersections*, Math. Z. **274** (2013), no. 3, 1385–1400.

[La1] H. Laufer, *On rational singularities*, Amer. J. Math. **94** (1972), 597–608.

[La2] H. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99** (1977), 1257–1295.

[Ma] M. Morales, *Fonctions de Hilbert, genre géométrique d'une singularité quasi-homogène Cohen-Macaulay*, C. R. Acad. Sc. Pairs, t.301, Série I **14** (1985), 699–702.

[Mi] R. Miles, *Canonical 3-folds*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Alphen aan den Rijn: Sijthoff and Noordhoff, 273–310.

[NX] D. Nanopoulos and D. Xie, *Hitchin equation, singularity, and $N = 2$ superconformal field theories*, JHEP (2010), no. 3, 1–43.

[SV] A. D. Shapere and C. Vafa, *BPS structure of Argyres-Douglas superconformal theories*, preprint, `arXiv:hep-th/9910182`.

[SW1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **426** (1994), no. 1, 19–52.

[SW2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. B **431** (1994), no. 3, 484–550.

[WX] Y. Wang, and D. Xie, *Classification of Argyres-Douglas theories from M5 branes*, preprint, `arXiv:1509.00847`.

[Xie] D. Xie, *General Argyres-Douglas theory*, JHEP (2013), no. 1, 1–53.

[XY] D. Xie and S.-T. Yau, *4d $\mathcal{N} = 2$ SCFT and singularity theory Part I: classification*, preprint, `arXiv:math/1510.01324`, (2015).

[YY] S. S.-T. Yau and Y. Yu, *Algebraic classification of rational CR structures on topological 5-sphere with transversal holomorphic S^1 action in $\mathbb{C}^4$*, Math. Nachr. **246** (2002), no. 1, 207–233.

[YY1] S. S.-T. Yau and Y. Yu, *Classification of 3-dimensional isolated rational hypersurface singularities with $\mathbb{C}^*$-action*, preprint, `arXiv:math/0303302`, (2003).

Department of Mathematical Sciences, Tsinghua University
Beijing 100084, P. R. China
E-mail address: chenby16@mails.tsinghua.edu.cn

Center of Mathematical Sciences and Applications
Jefferson Physical Laboratory, Harvard University
Cambridge, 02138, USA
E-mail address: dxie@cmsa.fas.harvard.edu

Department of Mathematics
Center of Mathematical Sciences and Applications
Jefferson Physical Laboratory, Harvard University
Cambridge, 02138, USA
E-mail address: yau@math.harvard.edu

Department of Mathematical Sciences, Tsinghua University
Beijing 100084, P. R. China
E-mail address: yau@uic.edu

Yau Mathematical Sciences Center, Tsinghua University
Beijing, 100084, P. R. China
E-mail address: hqzuo@math.tsinghua.edu.cn