ADV. THEOR. MATH. PHYS.
Volume 20, Number 6, 1473–1484, 2016

Automorphisms of the generalized quot schemes

Indranil Biswas and Sukhendu Mehrotra

Given a compact connected Riemann surface X of genus $g \geq 2$, and integers $r \geq 2$, $d_p > 0$ and $d_z > 0$, in [BDHW], a generalized quot scheme $\mathcal{Q}_X(r, d_p, d_z)$ was introduced. Our aim here is to compute the holomorphic automorphism group of $\mathcal{Q}_X(r, d_p, d_z)$. It is shown that the connected component of $\text{Aut}(\mathcal{Q}_X(r, d_p, d_z))$ containing the identity automorphism is $\text{PGL}(r, \mathbb{C})$. As an application of it, we prove that if the generalized quot schemes of two Riemann surfaces are holomorphically isomorphic, then the two Riemann surfaces themselves are isomorphic.

1. Introduction

In [BDHW], a generalized quot scheme was defined; we quickly recall the definition. Let X be a compact connected Riemann surface of genus $g \geq 2$, and $r \geq 2$, $d_p > 0$ and $d_z > 0$ are integers. Let $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ be the quot scheme that parametrizes the coherent subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank r and degree $-d_p$. This complex projective variety $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ is a moduli space of vortices [BDW], [Br], [BR], [Ba], [EINOS]. This moduli space is extensively studied (cf. [Bif], [BGL]). The universal vector bundle over $X \times \mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ will be denoted by $\mathcal{S}$. The generalized quot scheme $\mathcal{Q}_X(r, d_p, d_z)$ parametrizes torsionfree coherent sheaves F on X of rank r and degree $d_z - d_p$ such that some member of the family $\mathcal{S}$ is a subsheaf of F. In [BDHW], the fundamental group and the cohomology of $\mathcal{Q}_X(r, d_p, d_z)$ were computed.

2010 Mathematics Subject Classification: 14H60, 14D21, 14D23.
Key words and phrases: Generalized quot scheme, vector fields, automorphism group, symmetric product.

The natural action of $\mathrm{GL}(r, \mathbb{C})$ on the trivial vector bundle $\mathcal{O}_X^{\oplus r}$ produces a holomorphic action of $\mathrm{PGL}(r, \mathbb{C})$ on $\mathcal{Q}_X(r, d_p, d_z)$. The main result proved here says that $\mathrm{PGL}(r, \mathbb{C})$ is the connected component, containing the identity element, of the group of holomorphic automorphisms of $\mathcal{Q}_X(r, d_p, d_z)$; see Theorem 2.1.

Let X' be a compact connected Riemann surface of genus at least two. Fix positive integers $r' \geq 2$, d'_p and d'_z. Let $\mathcal{Q}'_X(r', d'_p.d'_z)$ be the corresponding generalized quot scheme. As an application of Theorem 2.1, we prove the following (see Proposition 3.2):

Proposition 1.1. *If the two varieties $\mathcal{Q}'_X(r', d'_p.d'_z)$ and $\mathcal{Q}_X(r, d_p, d_z)$ are isomorphic, then X is isomorphic to X'.*

2. Holomorphic vector fields on $\mathcal{Q}_X(r, d_p, d_z)$

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Fix positive integers $r \geq 2$, d_p and d_z. Let $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ denote the quot scheme that parametrizes all the torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree d_p. Therefore, elements of $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ represent subsheaves S of $\mathcal{O}_X^{\oplus r}$ such that $\mathrm{rank}(S) = r$ and $\mathrm{degree}(S) = -d_p$. Note that these two conditions on the subsheaf S are together equivalent to the condition that $\mathcal{O}_X^{\oplus r}/S$ is torsion of degree d_p. There is a universal short exact sequence of sheaves on $X \times \mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$

$$0 \longrightarrow \mathcal{S} \longrightarrow p_X^* \mathcal{O}_X^{\oplus r} = \mathcal{O}_{X \times \mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)}^{\oplus r} \longrightarrow \mathcal{T}_1 \longrightarrow 0, \tag{2.1}$$

where $p_X : X \times \mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p) \longrightarrow X$ is the natural projection.

Let

$$f : \mathcal{Q} = \mathcal{Q}_X(r, d_p, d_z) \longrightarrow \mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p) \tag{2.2}$$

be the relative quot scheme that parametrizes torsion quotients of $\mathcal{S}^*$ of degree d_z (see (2.1)). In other words, if z is the point of $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ representing a subsheaf $S \subset \mathcal{O}_X^{\oplus r}$, then the fiber $f^{-1}(z)$ is the the space of all subsheaves of S^* of rank r and degree $d_p - d_z$. Note that the degree of S^* is d_p. Therefore, elements of $\mathcal{Q}$ parametrize diagrams of the form

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & S & \longrightarrow & \mathcal{O}_X^{\oplus r} & \longrightarrow & T_1 & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & V & & & & & & \\
 & & \downarrow & & & & & & \\
 & & T_2 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\tag{2.3}
$$

where T_1 and T_2 are torsion sheaves of degree d_p and d_z respectively, and the subsheaf $S \subset \mathcal{O}_X^{\oplus r}$ corresponds to the image, under f, of the point of $\mathcal{Q}$ corresponding to V.

Let $\mathrm{Aut}(\mathcal{Q})$ denote the group of all holomorphic automorphisms of $\mathcal{Q}$. Since $\mathcal{Q}$ is a smooth complex projective variety, its holomorphic automorphisms are automatically algebraic. Thus $\mathrm{Aut}(\mathcal{Q})$ is a complex Lie group with Lie algebra $H^0(\mathcal{Q}, T\mathcal{Q})$, where $T\mathcal{Q}$ is the holomorphic tangent bundle of $\mathcal{Q}$; the Lie algebra structure is given by the Lie bracket operation of vector fields. Let

$$\mathrm{Aut}^0(\mathcal{Q}) \subset \mathrm{Aut}(\mathcal{Q})$$

be the connected component containing the identity element.

The standard action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$ produces an action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathcal{O}_X^{\oplus r}$, because the total space of $\mathcal{O}_X^{\oplus r}$ is identified with $X \times \mathbb{C}^r$. This action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathcal{O}_X^{\oplus r}$ defines an action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$. This action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ evidently lifts to an action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathcal{Q}$ (see (2.2)). Indeed, $\mathrm{GL}(r, \mathbb{C})$ acts on diagrams of type (2.3). Since $\mathrm{GL}(r, \mathbb{C})$ is connected, we get a homomorphism

$$\mathrm{GL}(r, \mathbb{C}) \longrightarrow \mathrm{Aut}^0(\mathcal{Q}).$$

The center $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$ of $\mathrm{GL}(r, \mathbb{C})$ acts trivially on $\mathcal{Q}$. Hence the above homomorphism produces a homomorphism

$$\varphi : \mathrm{PGL}(r, \mathbb{C}) \longrightarrow \mathrm{Aut}^0(\mathcal{Q}). \tag{2.4}$$

Theorem 2.1. *The homomorphism* φ *in* (2.4) *is an isomorphism.*

Proof. Let

$$p : \mathcal{Q} \longrightarrow \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X) \tag{2.5}$$

be the morphism that sends any $z \in \mathcal{Q}$ to the support of T_1 with multiplicity given by T_1 and the support of T_2 with multiplicity given by T_2, where T_1 and T_2 are the torsion sheaves in the diagram (2.3) corresponding to the point z.

The homomorphism φ is injective because the homomorphism

$$\mathrm{PGL}(r, \mathbb{C}) \longrightarrow \mathrm{Aut}(\mathbb{CP}^{r-1})$$

given by the standard action of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$ is injective. Indeed, for any

$$x = ((x_1, \ldots, x_{d_p}), (y_1, \ldots, y_{d_z})) \in \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X),$$

where all the above $d_p + d_z$ points are distinct, the fiber $p^{-1}(x)$ is $(\mathbb{CP}^{r-1})^{d_p} \times (CP^{r-1})^{d_z}$ (see (2.5)), and the action of $\mathrm{PGL}(r, \mathbb{C})$ on $p^{-1}(x)$ coincides with the diagonal action of $\mathrm{PGL}(r, \mathbb{C})$ on the factors in the above Cartesian product.

We need to prove that φ is surjective.

The Lie algebra of $\mathrm{PGL}(r, \mathbb{C})$ will be denoted by $\mathfrak{g}$; it is the Lie algebra structure on trace zero $r \times r$–matrices with complex entries given by commutator. Let

$$\theta : \mathfrak{g} \longrightarrow H^0(\mathcal{Q}, T\mathcal{Q}) \tag{2.6}$$

be the homomorphism of Lie algebras corresponding to the homomorphism φ in (2.4). To prove that φ is surjective, it suffices to show that θ is surjective.

The following lemma is a key step in the computation of $H^0(\mathcal{Q}, T\mathcal{Q})$.

Lemma 2.2. *The Lie algebra $H^0(\mathcal{Q}, T\mathcal{Q})$ has a natural injective homomorphism to $\mathfrak{g} \oplus \mathfrak{g}$.*

Proof of Lemma 2.2. For any positive integer k, let $P(k)$ denote the group of all permutations of $\{1, \ldots, k\}$. Consider the action of $P(d_p) \times P(d_z)$ on $X^{d_p + d_z}$ that permutes the first d_p factors and the last d_z factors. The quotient $X^{d_p+d_z}/(P(d_p) \times P(d_z))$ is $\mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X)$. Let

$$q : X^{d_p+d_z} \longrightarrow \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X)$$

be the corresponding quotient map.

Let

$$\widetilde{U} \subset X^{d_p+d_z} \tag{2.7}$$

be the complement of the big diagonal, so $\widetilde{U}$ parametrizes all possible distinct $d_p + d_z$ ordered points of X. The image $q(\widetilde{U}) \subset \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X)$ will be denoted by U. Since $p^{-1}(U)$ is a Zariski dense open subset of $\mathcal{Q}$, where p is defined in (2.5), we have

$$H^0(\mathcal{Q}, T\mathcal{Q}) \subset H^0(p^{-1}(U), T(p^{-1}(U))). \tag{2.8}$$

The Galois group $\Gamma := P(d_p) \times P(d_z)$ for the étale covering

$$q|_{\widetilde{U}} : \widetilde{U} \longrightarrow U,$$

where $\widetilde{U}$ is defined in (2.7), acts on the fiber product $\mathcal{Z} := p^{-1}(U) \times_U \widetilde{U}$. We have

$$H^0(p^{-1}(U), T\mathcal{Q}) = H^0(\mathcal{Z}, T\mathcal{Z})^{\Gamma}, \tag{2.9}$$

because the projection $\mathcal{Z} \longrightarrow p^{-1}(U)$ to the first factor of the fiber product is an étale Galois covering with Galois group Γ.

Now we have $\mathcal{Z} = \widetilde{U} \times (\mathbb{CP}^{r-1})^{d_p} \times (\mathbb{C}P^{r-1})^{d_z}$, where $\mathbb{C}P^{r-1}$ is the projective space parametrizing the lines in $\mathbb{C}^r$. Note that

$$H^0(\mathbb{CP}^{r-1}, T\mathbb{CP}^{r-1}) = \mathfrak{g} = H^0(\mathbb{C}P^{r-1}, T\mathbb{C}P^{r-1}).$$

It is known that $H^0(\widetilde{U}, T\widetilde{U}) = 0$ [BDH, p. 1452, Proposition 2.3]. Also, we have $H^0(\widetilde{U}, \mathcal{O}_{\widetilde{U}}) = 0$ [BDH, p. 1449, Lemma 2.2]. These together imply that

$$H^0(\mathcal{Z}, T\mathcal{Z}) = \mathfrak{g}^{\oplus d_p} \oplus \mathfrak{g}^{\oplus d_z}.$$

The action of $\Gamma = P(d_p) \times P(d_z)$ on $\mathfrak{g}^{\oplus d_p} \oplus \mathfrak{g}^{\oplus d_z} = H^0(\mathcal{Z}, T\mathcal{Z})$ (see (2.9)) is the one that permutes first d_p factors and the last d_z factors. Hence we have $H^0(\mathcal{Z}, T\mathcal{Z})^{\Gamma} = \mathfrak{g} \oplus \mathfrak{g}$. Therefore,

$$H^0(p^{-1}(U), T(p^{-1}(U))) = \mathfrak{g} \oplus \mathfrak{g} \tag{2.10}$$

by (2.9). Now the lemma follow from (2.8). □

Next we will need a property of the Hecke transformations.

Let Y be a smooth complex algebraic curve and $y_0 \in Y$ a point; the curve Y need not be projective. Fix a linear nonzero proper subspace $0 \neq S \subsetneq \mathbb{C}^r$. Consider the short exact sequence of sheaves on Y

$$0 \longrightarrow V \longrightarrow \mathcal{O}_Y^{\oplus r} \longrightarrow \mathcal{O}_{y_0}^{\oplus r}/S = \mathbb{C}^r/S \longrightarrow 0. \tag{2.11}$$

Let $P(V)$ denote the projective bundle over Y that parametrizes the lines in the fibers of V. Take any $A \in \mathrm{GL}(r, \mathbb{C})$. Let $\widehat{A}$ be the automorphism of $P(\mathcal{O}_Y^{\oplus r}) = Y \times CP^{r-1}$ given by A; this automorphism acts trivially on Y and has the standard action on CP^{r-1}. Since V and $\mathcal{O}_Y^{\oplus r}$ are identified over $Y \setminus \{y_0\}$, the above automorphism $\widehat{A}$ produces an automorphism of $P(V)|_{Y\setminus\{y_0\}}$. This automorphism of $P(V)|_{Y\setminus\{y_0\}}$ will be denoted by $\widehat{A}'$.

Lemma 2.3. *The above automorphism $\widehat{A}'$ of $P(V)|_{Y\setminus\{y_0\}}$ extends to a self-map of $P(V)$ if and only if $A(S) = S$.*

Proof of Lemma 2.3. Let $\mathrm{GL}(V)$ be the Zariski open subset of the total space of $\mathrm{End}(V) = V \otimes V^*$ parametrizing endomorphisms of fibers that are automorphisms. The quotient $\mathrm{PGL}(V) = \mathrm{GL}(V)/\mathbb{C}^*$ is a group–scheme over Y with fibers isomorphic to the group $\mathrm{PGL}(r, \mathbb{C})$. If an algebraic map of the total space

$$\tau : P(V) \longrightarrow P(V)$$

is an automorphism satisfies the condition that there is a nonempty Zariski open subset $U_\tau \subset Y$ such that τ restricts to an automorphism of $P(V)|_{U_\tau}$ over the identity map of U_τ, then τ is actually an automorphism over the identity map of Y. We note that the group of automorphisms of $P(V)$ over the identity map of Y is precisely the group of sections, over Y, of the group–scheme $\mathrm{PGL}(V)$.

Fix a subspace $S' \subset \mathbb{C}^r$ complementary to S, so $\mathbb{C}^r = S \oplus S'$. Let $E_S := Y \times S$ and $E_{S'} := Y \times S'$ be the trivial algebraic vector bundles over Y with fibers S and S' respectively. Then we have

$$\mathcal{O}_Y^{\oplus r} = E_S \oplus E_{S'} \quad \text{and } V = E_S \oplus (E_{S'} \otimes \mathcal{O}_Y(-y_0)). \tag{2.12}$$

From the above decompositions it follows immediately that if $A(S) = S$, then $\widehat{A}'$ extends to an automorphism of $P(V)$.

To prove the converse, assume that $\widehat{A}'$ extends to an automorphism of $P(V)$. It suffices to show that the subbundle $E_S \subset V$ in (2.12) is preserved by the section of $\mathrm{PGL}(V)$ corresponding to the automorphism of $P(V)$. Note that the restriction of this section of $\mathrm{PGL}(V)$ to $Y \setminus \{y_0\}$ is given by

A. There is no nonzero homomorphism from E_S to $E_{S'} \otimes \mathcal{O}_Y(-y_0)$ which is given by a constant homomorphism $B : S \longrightarrow S'$ on $Y \setminus \{y_0\}$ because such a homomorphism over $Y \setminus \{y_0\}$ extends to a homomorphism from $(E_S)_{y_0}$ to $(E_{S'})_{y_0}$ and this homomorphism $(E_S)_{y_0} \longrightarrow (E_{S'})_{y_0}$ coincides with B. On the other hand, the image of $(E_{S'} \otimes \mathcal{O}_Y(-y_0))_{y_0}$ in $(E_{S'})_{y_0}$ is the zero subspace. So if $B \neq 0$, then the homomorphism over $Y \setminus \{y_0\}$ does not extend to a homomorphism from E_S to $E_{S'} \otimes \mathcal{O}_Y(-y_0)$ over Y. Therefore, we conclude that $A(S) = S$. □

Fix distinct $d_p - 1$ points $x_1, \ldots, x_{d_p-1}$ on X. For each x_i, fix a hyperplane H_i in $(\mathcal{O}_X^{\oplus r})_{x_i} = \mathbb{C}^r$. Also, fix distinct $d_z - 1$ points $y_1, \ldots, y_{d_z-1}$ on X such that $x_i \neq y_j$ for all $1 \leq i \leq d_p - 1$ and $1 \leq j \leq d_z - 1$. Fix a line L_j in $(\mathcal{O}_X^{\oplus r})_{y_j} = \mathbb{C}^r$ for each j.

Now take the complement $Y = X \setminus \{x_1, \ldots, x_{d_p-1}, y_1, \ldots, y_{d_z-1}\}$. Take any nontrivial element

$$\mathrm{Id} \neq A \in \mathrm{PGL}(r, \mathbb{C}). \tag{2.13}$$

Fix a point $x_0 \in Y$ and also fix a hyperplane

$$S \subset (\mathcal{O}_X^{\oplus r})_{x_0} = \mathbb{C}^r$$

such that

$$A(S) \neq S; \tag{2.14}$$

since $A \neq \mathrm{Id}$, such a subspace exists. Consider the vector bundle V on Y constructed in (2.11) using S. As before, $P(V)$ denotes the projective bundle over Y parametrizing the lines in the fibers of V.

There is an embedding

$$\delta : P(V) \longrightarrow \mathcal{Q} \tag{2.15}$$

which we will now describe. For the map f in (2.2), the image $f \circ \delta(P(V))$ is the point given by the quotient

$$0 \longrightarrow \widehat{V} \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow (\oplus_{i=1}^{d_p-1} (\mathcal{O}_X^{\oplus r})_{x_i}/H_i) \oplus (\mathcal{O}_X^{\oplus r})_{x_0}/S) \longrightarrow 0,$$

where H_i are the hyperplanes fixed above; in particular, $f \circ \delta$ is a constant map. Note that $\widehat{V}$ is an extension of the vector bundle V to X. For any point $y \in Y$ and any point in the fiber $y' \in P(V)_y$, consider the short exact sequence on X

$$0 \longrightarrow E \longrightarrow \widehat{V}^* \longrightarrow (\oplus_{j=1}^{d_z-1}(\widehat{V}^*_{y_j}/L_j^\perp) \oplus (\widehat{V}^*_y/L(y')^\perp) \longrightarrow 0,$$

where $L(y') \subset V_y$ is the line in V_y corresponding to the above point y', and $L_j^\perp \subset \widehat{V}^*_{y_j}$ (respectively, $L(y')^\perp \subset \widehat{V}^*_y$) is the annihilator of L_j (respectively, $L(y')$); note that $\widehat{V}_{y_j}$ is identified with $\mathbb{C}^r$ and $\widehat{V}_y$ is identified with V_y. Therefore, we have

$$\widehat{V} \hookrightarrow E^*.$$

The map δ in (2.15) sends any y' to the above extension E^* of $\widehat{V}$ constructed from y'.

From (2.10) we know that $\mathrm{PGL}(r, \mathbb{C})$ is contained in $\mathrm{Aut}(p^{-1}(U))$ with

$$0 \oplus \mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g} = H^0(p^{-1}(U), T(p^{-1}(U)))$$

as its Lie algebra. This action of $\mathrm{PGL}(r, \mathbb{C})$ on $p^{-1}(U)$ clearly preserves the intersection $\delta(P(V)) \cap p^{-1}(U)$. Therefore, if the action of the element A in (2.13) extends to $\mathcal{Q}$, then the extended action must preserve the image $\delta(P(V))$.

On the other hand, from Lemma 2.3 we know that the action of A on $P(V)|_{Y\setminus\{y_0\}}$ does not extend to $P(V)$ because (2.14) holds. This completes the proof of the theorem. □

3. Holomorphic maps from a symmetric product

Proposition 3.1. *Let X and Y be compact connected Riemann surface with*

$$\mathrm{genus}(X) \geq \mathrm{genus}(Y) \geq 2.$$

If there is a nonconstant holomorphic map $\beta : \mathrm{Sym}^d(Y) \longrightarrow X$, then $d = 1$, and β is an isomorphism.

Proof. Let $\beta : \mathrm{Sym}^d(Y) \longrightarrow X$ be a nonconstant holomorphic map. Let

$$\beta^* : H^0(X, \Omega^1_X) \longrightarrow H^0(\mathrm{Sym}^d(Y), \Omega^1_{\mathrm{Sym}^d(Y)})$$

be the pull-back of 1–forms defined by $\omega \longmapsto \beta^*\omega$. This homomorphism β^* is injective, because β is surjective. Since

$$\dim H^0(\mathrm{Sym}^d(Y), \Omega^1_{\mathrm{Sym}^d(Y)}) = \mathrm{genus}(Y)$$

[Ma, p. 322, (4.3)], the injectivity of β^* implies that genus$(Y) \geq$ genus(X). Therefore, the given condition genus$(X) \geq$ genus(Y) implies that

- genus$(X) =$ genus(Y), and
- the above homomorphism β^* is an isomorphism.

If $d \geq 2$, the wedge product

$$\wedge^2 H^0(\mathrm{Sym}^d(Y), \Omega^1_{\mathrm{Sym}^d(Y)}) \longrightarrow H^0(\mathrm{Sym}^d(Y), \Omega^2_{\mathrm{Sym}^d(Y)})$$

is a nonzero homomorphism [Ma, p. 325, (6.3)]. On the other hand, the wedge product on $H^0(X, \Omega^1_X)$ is the zero homomorphism because $H^0(X, \Omega^2_X) = 0$. In other words, β^* is not compatible with the wedge product operation on holomorphic 1-forms if $d \geq 2$. So we conclude that $d = 1$.

Since genus$(X) =$ genus(Y), from Riemann–Hurwitz formula for Euler characteristic if follows that degree$(\beta) = 1$. In other words, β is an isomorphism. □

Let X' be a compact connected Riemann surface of genus at least two. Fix positive integers $r' \geq 2$, d'_p and d'_z. Let

$$\mathcal{Q}' = \mathcal{Q}'_X(r', d'_p.d'_z)$$

be the corresponding generalized quot scheme (see (2.2)).

Proposition 3.2. *If the two varieties $\mathcal{Q}'$ and $\mathcal{Q}$ (constructed in* (2.2)*) are isomorphic, then X is isomorphic to X'.*

Proof. Assume that $\mathcal{Q}'$ and $\mathcal{Q}$ are isomorphic. We will show that X and X' are isomorphic.

Let $\eta : \mathrm{Sym}^{d_p}(X) \times \mathrm{Sym}^{d_z}(X) \longrightarrow \mathrm{Pic}^{d_p}(X) \times \mathrm{Pic}^{d_z}(X)$ be the morphism defined by

$$((x_1, \ldots, x_{d_p}), (y_1, \ldots, y_{d_z})) \longmapsto (\mathcal{O}_X(x_1 + \cdots + x_{d_p}), \mathcal{O}_X(y_1 + \cdots + y_{d_z})).$$

Since the general fiber of the map p in (2.5) is a product of copies of projective spaces, the composition

$$\eta \circ p : \mathcal{Q} \longrightarrow \mathrm{Pic}^{d_p}(X) \times \mathrm{Pic}^{d_z}(X)$$

is the Albanese map for $\mathcal{Q}$, as there is no nonconstant holomorphic map from a projective space to an abelian variety. In particular, the Albanese variety of $\mathcal{Q}$ is of dimension $2g = 2 \cdot \text{genus}(X)$. Therefore, comparing the Albanese varieties of $\mathcal{Q}$ and $\mathcal{Q}'$ we conclude that $\text{genus}(X) = g = \text{genus}(X')$.

Fix a maximal torus T in $\text{Aut}^0(\mathcal{Q})$. In view of Theorem 2.1, this amounts to choosing a trivialization of $\mathcal{O}_X^{\oplus r}$, with two trivializations being identified if they differ by multiplication with a constant nonzero scalar. The fixed–point locus

$$\mathcal{Q}^T \subset \mathcal{Q},$$

for the action of T on $\mathcal{Q}$, is a disjoint union of copies of

$$(\text{Sym}^{a_1}(X) \times \cdots \times \text{Sym}^{a_r}(X)) \times (\text{Sym}^{b_1}(X) \times \cdots \times \text{Sym}^{b_r}(X))$$

with $\sum_{i=1}^r a_i = d_p$ and $\sum_{i=1}^r b_i = d_z$.

Take a component

$$Z = (\text{Sym}^{a_1}(X) \times \cdots \times \text{Sym}^{a_r}(X)) \times (\text{Sym}^{b_1}(X) \times \cdots \times \text{Sym}^{b_r}(X))$$

of $\mathcal{Q}^T$ such that at least one of the $2r$ integers $\{a_1, \ldots, a_r, b_1, \ldots, b_r\}$ is one.

We will first show that Z is not holomorphically isomorphic to $\text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y)$, where Y is compact connected Riemann surface of genus g and $c_j \geq 2$ for all $1 \leq j \leq n$. To prove this, assume that Z is isomorphic to $\text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y)$, where Y and c_i are as above. Consider the composition

$$\text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y) \xrightarrow{\sim} Z \xrightarrow{q} X,$$

where q is the projection to a factor of Z which is the first symmetric power of X (it is assumed that such a factor exists). Since all $c_j \geq 2$, from Proposition 3.1 it follows that there is no nonconstant map from $\text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y)$ to X. Therefore, we conclude that Z is not holomorphically isomorphic to $\text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y)$.

Fix a maximal torus $T' \subset \text{Aut}(\mathcal{Q}')^0$. Since $\mathcal{Q}'$ is isomorphic to $\mathcal{Q}$, there is a component

$$(\text{Sym}^{a'_1}(X') \times \cdots \times \text{Sym}^{a'_r}(X')) \times (\text{Sym}^{b'_1}(X') \times \cdots \times \text{Sym}^{b'_r}(X'))$$

of the fixed point locus $(\mathcal{Q}')^{T'} \subset \mathcal{Q}'$ which is isomorphic to Z. Now from Proposition 3.1 it follows that

- at least one of the $2r$ integers $\{a'_1, \ldots, a'_r, b'_1, \ldots, b'_r\}$ is one, and
- X is isomorphic to X'.

This completes the proof of the proposition. □

Acknowledgements

The first named author thanks Facultad de Matemáticas, PUC Chile, for hospitality. He received partial travel support from FONDECYT grant 1150404. He also acknowledges the support of a J. C. Bose Fellowship. The second named author was partially supported by FONDECYT grant 1150404 during the preparation of this article.

References

[Ba] J. M. Baptista, *On the L^2-metric of vortex moduli spaces*, Nucl. Phys. B **844** (2011), 308–333.

[BDW] A. Bertram, G. Daskalopoulos, and R. Wentworth, *Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians*, Jour. Amer. Math. Soc. **9** (1996), 529–571.

[Bif] E. Bifet, *Sur les points fixes schéma* $\mathrm{Quot}_{\mathcal{O}_X/X,k}$ *sous l'action du tore* $\mathbf{G}^r_{m,k}$, Com. Ren. Math. Acad. Sci. Paris **309** (1989), 609–612.

[BGL] E. Bifet, F. Ghione, and M. Letizia, *On the Abel-Jacobi map for divisors of higher rank on a curve*, Math. Ann. **299** (1994), 641–672.

[BDHW] I. Biswas, A. Dhillon, J. Hurtubise, and R. A. Wentworth, *A generalized Quot scheme and meromorphic vortices*, Adv. Theo. Math. Phys. **19** (2015), no. 4, `arXiv:1410.1182 [math.AG]`.

[BDH] I. Biswas, A. Dhillon, and J. Hurtubise, *Automorphisms of the Quot schemes associated to compact Riemann surfaces*, Int. Math. Res. Not. (2015), 1445–1460.

[BR] I. Biswas and N. M. Romão, *Moduli of vortices and Grassmann manifolds*, Comm. Math. Phy. **320** (2013), 1–20.

[Br] S. Bradlow, *Vortices in holomorphic line bundles over closed Kähler manifolds*, Commun. Math. Phys. **135** (1990), 1–17.

[EINOS] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, *Solitons in the Higgs phase: the moduli matrix approach*, J. Phys. A: Math. Gen. **39** (2006), 315–392.

[Ma] I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.

School of Mathematics, Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400005, India
E-mail address: indranil@math.tifr.res.in

Facultad de Matemáticas, PUC Chile
Av. Vicuña Mackenna 4860, Santiago, Chile
& Chennai Mathematical Institute, H1, SIPCOT IT Park
Siruseri, Kelambakkam 603103, India
E-mail address: smehrotra@mat.uc.cl