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On the precanonical structure of the Schrödinger wave functional

Igor V. Kanatchikov

(*In loving memory of Nina Efimova*)

We show that the Schrödinger wave functional may be obtained as the product integral of precanonical wave functions on the space of field and space-time variables. The functional derivative Schrödinger equation underlying the canonical field quantization is derived from the partial derivative covariant analogue of the Schrödinger equation, which appears in the precanonical field quantization based on the De Donder-Weyl generalization of the Hamiltonian formalism for field theory. The representation of precanonical quantum operators typically contains an ultraviolet parameter $\varkappa$ of the dimension of the inverse spatial volume. The transition from the precanonical description of quantum fields in terms of Clifford-valued wave functions and partial derivative operators to the standard functional Schrödinger representation obtained from canonical quantization is accomplished if $\frac{1}{\varkappa} \to 0$ and $\frac{1}{\varkappa}\gamma_0$ is mapped to the infinitesimal spatial volume element $\mathrm{d}\mathbf{x}$. Thus the standard QFT obtained via canonical quantization corresponds to the quantum theory of fields derived from the precanonical quantization in the limiting case of an infinitesimal value of the parameter $\frac{1}{\varkappa}$.

1. Introduction

Field theories are commonly considered as systems with an infinite number of degrees of freedom. This notion originates in the canonical Hamiltonian treatment of field theory and it is transferred to quantum field theory by the procedure of canonical quantization. The resulting version of quantum field theory has evolved into a very successful framework in contemporary theoretical physics whose applications range from condensed matter physics to quantum cosmology. In this framework, even the divergences, viewed as pathologies in the earlier days of QFT, have turned into triumphs for the concepts of renormalization and effective field theory.

However, there remain conceptual tensions between quantum theory and relativity, which we face in the context of discussions of foundational issues, and particularly in quantum field theory in curved space-times, quantum gravity, and unification of all interactions, where our current understanding of QFT is pushed to the limits of its applicability due to the distinguished role of time in the formalism of quantum theory on the one side and the generally covariant, geometric and nonlinear nature of general relativity on the other.

In approaching those issues, we draw attention to the fact that the progress of QFT has essentially overlooked developments in the calculus of variations of multiple integral problems, where the extension of the Hamiltonian formulation from mechanics to field theoryis known to be far from unique [1]. Moreover, as opposed to the canonical Hamiltonian formalism, the approaches to its generalization developed in the setting of the calculus of variations have no need of a distinguished "time variable" in the set of space-time variables (i.e. the independent variables of the variational problem which defines a field theory). Nor do they necessarily imply the picture of fields as infinite-dimensional systems evolving in time (which would fail on non-globally hyperbolic space-times).

Thus, the question arises whether a formulation of quantum field theory could be built based on these alternative space-time symmetric Hamiltonizations of field theory, and if the inherent features of the latter, such as manifest respect for the space-time symmetries and the finite dimensionality of the corresponding analogue of the configuration space (i.e. the bundle of field variables over space-time, whose sections are field configurations appearing in the standard formulations) can help in clarifying fundamental issues of QFT at the frontiers of current research, e.g. in the context of quantum gravity.

Furthermore, the existence of the Hamilton-Jacobi formulations of field theories associated with each of these alternative Hamiltonizations [1–5] naturally leads to the question whether alternative formulations of quantum field theories exist which would reproduce the corresponding Hamilton-Jacobi equations in the classical limit, and what would be their physical significance.

To be more specific, let us recall that in a field theory given by the first order Lagrangian density $L = L(y^a, y^a_\mu, x^\mu)$, where y^a denote the field variables of any nature, y^a_μ are (the first jet space coordinates of) their first derivatives, and x^μ are the space-time variables, the simplest of the above mentioned alternative Hamiltonizations is the so-called De Donder-Weyl (DW) theory (see e.g. [1, 2]). It is based on the following Hamiltonian-like covariant reformulation of the Euler-Lagrange field equations:

$$\partial_\mu p^\mu_a = -\frac{\partial H}{\partial y^a}, \quad \partial_\mu y^a = \frac{\partial H}{\partial p^\mu_a}, \tag{1.1}$$

which uses the following covariant Legendre transformation to define new variables: $p^\mu_a := \partial L/\partial y^a_\mu$ (*polymomenta*) and $H(y^a, p^\mu_a, x) := p^\mu_a \partial_\mu y^a - L$ (*DW Hamiltonian function*).

The DW Hamiltonian equations (1.1) can be compared with the standard Hamilton equations in the canonical formalism:

$$\partial_t p^0_a(\mathbf{x}) = -\frac{\delta \mathbf{H}}{\delta y^a(\mathbf{x})}, \quad \partial_t y^a(\mathbf{x}) = \frac{\delta \mathbf{H}}{\delta p^0_a(\mathbf{x})}, \tag{1.2}$$

where the canonical Hamiltonian functional is introduced:

$$\mathbf{H}([y(\mathbf{x}), p^0(\mathbf{x})]) := \int \mathrm{d}\mathbf{x} \Big(\partial_t y^a(\mathbf{x}) p^0_a(\mathbf{x}) - L \Big), \tag{1.3}$$

and a decomposition into the space and time is performed, so that $x^\mu := (\mathbf{x}, t)$. Here and in what follows the capital bold letters denote functionals.

When both formulations are regular the equivalence between (1.1) and (1.2) can be established by noticing that

$$\mathbf{H} = \int \mathrm{d}\mathbf{x} \Big(H - \partial_i y^a(\mathbf{x}) p^i_a(\mathbf{x}) \Big). \tag{1.4}$$

Then it is easy to check that the canonical Hamilton equations can be derived from the (precanonical) DW Hamiltonian equations (1.1).

Whereas the field quantization based on the canonical Hamiltonization is well elaborated and underlies QFT as we know it, an approach to quantization of fields based on the De Donder-Weyl (DW) generalization of Hamiltonian mechanics to field theory was put forward only recently [6–8] (c.f. also discussions of similar ideas by other authors in [10–14]). In the context of quantization of gravity [15–17] the approach was later given the name of *precanonical quantization.*

While the connection between the canonical and DW Hamiltonizations is sufficiently clear on the classical level (see e.g. [18]), the relation between the respective quantizations has been rather problematic for a long time (see [6, 19] for earlier discussions). In the recent paper [20] we found a formula connecting the Schrödinger wave functional with the precanonical wave functions in the case of scalar field theory. However, the derivation was based on an *ad hoc* Ansatz, so that it remained unclear how general the result is. In this paper we establish a connection between QFT based on canonical quantization in the functional Schrödinger representation and a formulation based on precanonical quantization without any *a priori* assumptions regarding the form of this relation, except a very general possible relation between the Schrödinger wave functional and precanonical wave function (c.f. Eq. (3.6) below).

In Sect. 2 we present a comparative outline of the elements of canonical and precanonical quantization, which are essential for our purposes. In Sect. 3 we derive the functional derivative Schrödinger equation for quantum scalar field theory from the corresponding Dirac-like partial derivative precanonical analogue of the Schrödinger equation. This consideration leads to a relation between the Schrödinger wave functional known from the canonical quantization and the Clifford-valued wave function appearing in precanonical quantization. As an application of our result, we construct the vacuum state functional of the free scalar field theory from the precanonical ground state wave functions. A concluding discussion is found in Sect. 4.

2. Canonical and precanonical quantization

Let us present a brief comparative overview of the elements of canonical and precanonical quantization, which are relevant for the following discussion.

Canonical quantization (in the Schrödinger picture [21]) is known to lead to the description of quantum fields in terms of the Schrödinger wave functional $\mathbf{\Psi}([y(\mathbf{x})], t)$ on the infinite-dimensional configuration space of field configurations $y(\mathbf{x})$ at time t. Precanonical quantization [6–8] leads to the description in terms of Clifford algebra-valued wave functions $\Psi(y, x)$ on the finite dimensional "covariant configuration space" (in the terminology of [18]) of field variables y and space-time variables x.

While the Schrödinger wave functional $\mathbf{\Psi}$ fulfils the Schrödinger equation

$$i\hbar\partial_t \mathbf{\Psi} = \widehat{\mathbf{H}}\mathbf{\Psi}, \tag{2.1}$$

where $\widehat{\mathbf{H}}$ stands for the functional derivative operator of the canonical Hamiltonian, the precanonical wave function $\Psi(y, x)$ satisfies the following covariant generalization of the Schrödinger equation [6–8]

$$i\hbar\varkappa\gamma^\mu\partial_\mu\Psi = \widehat{H}\Psi, \tag{2.2}$$

where $\widehat{H}$ is the partial derivative operator of the De Donder-Weyl Hamiltonian function, γ^μ are Dirac matrices of n-dimensional space-time, and $\varkappa$ is a "very large" constant of dimension $\ell^{-(n-1)}$. The latter routinely appears on dimensional grounds in the expressions of precanonical quantum operators, which follow from quantization of the Poisson-Gerstenhaber brackets of differential forms representing the dynamical variables in field theory. These brackets were found in our earlier work on the mathematical structure of the DW Hamiltonian formulation [22–24] (for further generalizations see e.g. [25–30]). Their geometric prequantization, which can be viewed as an intermediate step toward precanonical quantization, was considered in [31]. It is on the level of geometric prequantization we can justify the appearance of Clifford algebra-valued wave functions and operators in precanonical quantization.

More specifically, let us consider the theory of interacting scalar fields, which is given by

$$L = \frac{1}{2}\partial_\mu y^a \partial^\mu y^a - V(y), \tag{2.3}$$

where the potential term $V(y)$ also includes the mass terms like $\frac{1}{2}m^2y^2$ (henceforth we set $\hbar = 1$ and use the metric signature $+ - \cdots -$).

In this case the operator of the canonical conjugate momentum of $y^a(\mathbf{x})$:

$$p_a^0(\mathbf{x}) := \frac{\partial L}{\partial \partial_t y^a(\mathbf{x})},$$

in the Schrödinger $y(\mathbf{x})$-representation is given by

$$\hat{p}_a^0(\mathbf{x}) = -i\frac{\delta}{\delta y^a(\mathbf{x})}. \tag{2.4}$$

This representation follows from quantization of the equal-time Poisson bracket

$$\{p_a^0(\mathbf{x}), y^b(\mathbf{x}')\} = \delta_a^b \delta(\mathbf{x} - \mathbf{x}'). \tag{2.5}$$

In precanonical quantization, the representation of the operators of polymomenta:

$$\hat{p}_a^\mu = -i\varkappa\gamma^\mu\frac{\partial}{\partial y^a}, \tag{2.6}$$

follows from quantization of the Heisenberg subalgebra of the above mentioned Poisson-Gerstenhaber algebra of forms (c.f. [6–9]):

$$\begin{aligned} \{\![p^{\mu}_{a}\omega_{\mu}, y^{b}]\!\} &= \delta^{b}_{a}, \\ \{\![p^{\mu}_{a}\omega_{\mu}, y^{b}\omega_{\nu}]\!\} &= \delta^{b}_{a}\omega_{\nu}, \\ \{\![p^{\mu}_{a}, y^{b}\omega_{\nu}]\!\} &= \delta^{b}_{a}\delta^{\mu}_{\nu}, \end{aligned} \tag{2.7}$$

where $\omega_{\mu} := \partial_{\mu} \rfloor \omega$ are the contractions of the volume n-form on the space-time $\omega := \mathrm{d}x^{0} \wedge \mathrm{d}x^{1} \wedge \cdots \wedge \mathrm{d}x^{n-1}$ with the basis vectors ∂_{μ} of its tangent space. This quantization also implies the existence of a map q from the co-exterior forms on the classical level[1] to the Clifford algebra elements (Dirac matrices) on the quantum level:

$$\omega_{\mu} \overset{q}{\longmapsto} \frac{1}{\varkappa}\gamma_{\mu}, \tag{2.8}$$

which is similar to the “Chevalley map” [32], or “quantization map” [33], known in the theory of Clifford algebras. The constant $\varkappa$ appears here on dimensional grounds. From the association of ω_{0}, which represents the infinitesimal spatial volume element $\mathrm{d}\mathbf{x}$, with γ_{0}, which is dimensionless, it is evident that $\frac{1}{\varkappa}$ corresponds to a “very small” volume and has the meaning of a physically infinitesimal or elementary volume.

Furthermore, while the canonical Hamiltonian operator of the quantum scalar field theory:

$$\widehat{\mathbf{H}} = \int \mathrm{d}\mathbf{x} \left\{ -\frac{1}{2}\frac{\delta^{2}}{\delta y(\mathbf{x})^{2}} + \frac{1}{2}(\nabla y(\mathbf{x}))^{2} + V(y(\mathbf{x})) \right\}, \tag{2.9}$$

is formulated in terms of functional derivative operators, the DW Hamiltonian operator in this case (see [6–8]):

$$\widehat{H} = -\frac{1}{2}\varkappa^{2}\partial_{yy} + V(y) \tag{2.10}$$

is expressed in terms of the partial derivative operators with respect to the field variables.

The question naturally arises, how those two descriptions, which seem to be so different both physically and mathematically, can be related: how the Schrödinger wave functional is related to the Clifford algebra-valued

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[1]We have explained in our earlier papers [23] that the natural multiplication of forms here is given by the *co-exterior* product: $\alpha \bullet \beta := *^{-1}(*\alpha \wedge *\beta)$, where $*$ is the Hodge duality operator.

precanonical wave function and how the functional derivative canonical Schrödinger equation (2.1) is related to the precanonical Schrödinger equation (2.2).

3. Schrödinger wave functional from precanonical wave function

To make the above mentioned relation less mysterious, let us first recall our earlier observation [19] that the functional derivative Hamilton-Jacobi equation of the canonical Hamiltonian formalism:

$$\partial_t \mathbf{S} + \mathbf{H}\left(y^a(\mathbf{x}), p^0_a(\mathbf{x}) = \frac{\delta \mathbf{S}}{\delta y(\mathbf{x})}, t\right) = 0, \tag{3.1}$$

can be derived from the partial differential Hamilton-Jacobi equation of precanonical De Donder-Weyl theory [1, 2]:

$$\partial_\mu S^\mu + H\left(y^a, p^\mu_a = \frac{\partial S^\mu}{\partial y^a}, x^\mu\right) = 0, \tag{3.2}$$

if the canonical HJ functional $\mathbf{S}([y(\mathbf{x}), t])$ is constructed in terms of the DW-HJ functions $S^\mu(y, x)$ as follows:

$$\mathbf{S} = \int_\Sigma S^\mu \omega_\mu. \tag{3.3}$$

Here Σ: $(y{=}y(\mathbf{x}), t{=}const)$ is the subspace in the covariant configuration space, which represents the field configuration $y(\mathbf{x})$ at the moment of time t.

This result of [19] demonstrates that the transition from an object of DW (precanonical) theory, such us $S^\mu(y, x)$, to an object of canonical theory, such as $\mathbf{S}([y(\mathbf{x}), t])$, involves a restriction of the former to the subspace Σ and subsequent integration over it. In this way the functionals of field configurations are constructed from the functions on the covariant finite-dimensional configuration space, and the functional derivative equations of the canonical formalism are derived from their precanonical partial derivative counterparts.

A similar relationship exists also between the forms representing dynamical variables and their Poisson-Gerstenhaber brackets in the DW Hamiltonian formulation and the "observables" represented by functionals and their Poisson brackets in the canonical formalism [22, 34].

Having obtained this result on the classical level, we can also expect a similar relation between the wave functional and precanonical wave function on the quantum level, because both the Schrödinger wave functional $\mathbf{\Psi}$ and the precanonical wave function Ψ are related to the exponentials of, respectively, the HJ functional and DW-HJ functions, viz.,

$$\mathbf{\Psi} \sim e^{i\mathbf{S}} \quad \text{and} \quad \Psi \sim e^{\frac{i}{\varkappa} S^\mu \gamma_\mu} \tag{3.4}$$

(see [8], where the second expression is used to argue that in the classical limit the DW-HJ equation follows from the precanonical Schrödinger equation). Using the fact that $\mathbf{S}$ is the spatial integral of DW-HJ functions, we can anticipate that $\mathbf{\Psi}$ is related to the *product integral* [36] of precanonical wave functions restricted to the subspace Σ:

$$\begin{aligned} \mathbf{\Psi}([y(\mathbf{x})]) \sim e^{i\mathbf{S}} = e^{i\int_\Sigma S^\mu \omega_\mu} &= \prod_{\mathbf{x}\in\Sigma} e^{iS^\mu(y=y(\mathbf{x}),\mathbf{x},t)\omega_\mu} \\ &\sim \prod_{\mathbf{x}\in\Sigma} \Psi(y=y(\mathbf{x}),\mathbf{x},t)|_{\frac{1}{\varkappa}\gamma_\mu\to\omega_\mu}, \end{aligned} \tag{3.5}$$

where the last step implies the inverse of the "quantization map" in Eq. (2.8). The consideration below will make this idea more precise.

Now, let us assume that the Schrödinger wave functional $\mathbf{\Psi}$ can be expressed in terms of the precanonical wave function $\Psi(y,x)$ restricted to the subspace Σ: $\Psi(y,x)|_\Sigma := \Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)$, i.e.

$$\mathbf{\Psi}([y(\mathbf{x})],t) = \mathbf{\Psi}([\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)],[y^a(\mathbf{x})]). \tag{3.6}$$

Then the time evolution of the Schrödinger wave functional is determined by the time evolution of precanonical wave function. By applying the chain rule to the composite functional (3.6) we obtain:

$$i\partial_t \mathbf{\Psi} = \int \mathrm{d}\mathbf{x}\ \mathsf{Tr}\left\{ \frac{\delta \mathbf{\Psi}}{\delta \Psi^T_\Sigma(y^a(\mathbf{x}),\mathbf{x},t)} i\partial_t \Psi_\Sigma(y^a(\mathbf{x}),\mathbf{x},t) \right\}. \tag{3.7}$$

Note that the additional dependence of $\mathbf{\Psi}$ from $y^a(\mathbf{x})$, which is not incorporated in $\Psi|_\Sigma$, is supposed to take into account the fact that the amplitudes $\Psi|_\Sigma$ in space-like separated points in general are not independent — one of the manifestations of quantum nonlocality.

The equation of time evolution of $\Psi_\Sigma(\mathbf{x})$ is given by the space-time decomposed precanonical Schrödinger equation, Eq. (2.2), restricted to Σ, viz.,

$$i\partial_t \Psi_\Sigma(\mathbf{x}) = -i\alpha^i \frac{\mathrm{d}}{\mathrm{d}x^i}\Psi_\Sigma(\mathbf{x}) + i\alpha^i \partial_i y(\mathbf{x})\partial_y \Psi_\Sigma(\mathbf{x}) + \frac{1}{\varkappa}\beta(\widehat{H}\Psi)_\Sigma(\mathbf{x}), \tag{3.8}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x^i} := \partial_i + \partial_i y(\mathbf{x})\partial_y + \partial_{ij} y(\mathbf{x})\partial_{y_j} + \cdots \tag{3.9}$$

is the total derivative along the subspace Σ, $\beta := \gamma^0$, $\beta^2 = 1$, and $\alpha^i := \beta\gamma^i$. Here we have also introduced the shorthand notation $\Psi_\Sigma(\mathbf{x}) := \Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)$ to be used henceforth.

Hence, the time evolution of the wave functional of the quantum scalar field is given by:

$$\begin{aligned} i\partial_t \mathbf{\Psi} = \int \mathrm{d}\mathbf{x}\ \mathsf{Tr}\Bigg\{ \frac{\delta \mathbf{\Psi}}{\delta \Psi^T_\Sigma(\mathbf{x},t)} \Bigg[& -i\alpha^i \frac{\mathrm{d}}{\mathrm{d}x^i}\Psi_\Sigma(\mathbf{x}) + i\alpha^i \partial_i y(\mathbf{x})\partial_y \Psi_\Sigma(\mathbf{x}) \\ & - \frac{1}{2}\varkappa\beta\partial_{yy}\Psi_\Sigma(\mathbf{x}) + \frac{1}{\varkappa}\beta V(y(\mathbf{x}))\Psi_\Sigma(\mathbf{x}) \Bigg]\Bigg\}. \end{aligned} \tag{3.10}$$

Eq. (3.10) is obtained by inserting the explicit expression of the DW Hamiltonian operator of the nonlinear scalar field given by Eq. (2.10) into Eq. (3.8).

From Eq. (3.6) we obtain the expressions for the first and the second total functional derivatives of $\mathbf{\Psi}$ with respect to $y(\mathbf{x})$, viz.,

$$\frac{\delta \mathbf{\Psi}}{\delta y(\mathbf{x})} = \mathsf{Tr}\left\{ \frac{\delta \mathbf{\Psi}}{\delta \Psi^T_\Sigma(\mathbf{x},t)} \partial_y \Psi_\Sigma(\mathbf{x}) \right\} + \frac{\bar{\delta}\mathbf{\Psi}}{\bar{\delta} y(\mathbf{x})}, \tag{3.11}$$

$$\begin{aligned} \frac{\delta^2 \mathbf{\Psi}}{\delta y(\mathbf{x})^2} = {} & \mathsf{Tr}\left\{ \frac{\delta \mathbf{\Psi}}{\delta \Psi^T_\Sigma(\mathbf{x},t)}\ \delta(\mathbf{0})\partial_{yy}\Psi_\Sigma(\mathbf{x}) \right\} \\ & + \mathsf{Tr}\,\mathsf{Tr}\left\{ \frac{\delta^2 \mathbf{\Psi}}{\delta \Psi^T_\Sigma(\mathbf{x}) \otimes \delta \Psi^T_\Sigma(\mathbf{x})}\ \partial_y \Psi_\Sigma(\mathbf{x}) \otimes \partial_y \Psi_\Sigma(\mathbf{x}) \right\} \\ & + 2\ \mathsf{Tr}\left\{ \frac{\delta\bar{\delta}\mathbf{\Psi}}{\delta \Psi^T_\Sigma(\mathbf{x})\ \bar{\delta} y(\mathbf{x})}\ \partial_y \Psi_\Sigma(\mathbf{x}) \right\} + \frac{\bar{\delta}^2 \mathbf{\Psi}}{\bar{\delta} y(\mathbf{x})^2}\ . \end{aligned} \tag{3.12}$$

Here and in what follows $\bar{\delta}$ denotes the partial functional derivative with respect to $y(\mathbf{x})$, and $\delta(\mathbf{0})$ is the result of functional differentiation of a function with respect to itself at the same spatial point.

Our **first** observation is that the potential energy term in the canonical functional derivative Schrödinger equation for the scalar field, see Eqs. (2.1) and (2.9), should be obtained from the potential energy term in the precanonical equation, Eq. (3.10), i.e.,

$$\int \mathrm{d}\mathbf{x}\ \mathsf{Tr}\left\{\frac{\delta\boldsymbol{\Psi}}{\delta\Psi^T_\Sigma(\mathbf{x})}\ \frac{1}{\varkappa}\beta V(y(\mathbf{x}))\Psi_\Sigma(\mathbf{x}))\right\} \mapsto \int \mathrm{d}\mathbf{x}\ V(y(\mathbf{x}))\ \boldsymbol{\Psi}. \tag{3.13}$$

It can be accomplished if

$$\mathsf{Tr}\left\{\frac{\delta\boldsymbol{\Psi}}{\delta\Psi^T_\Sigma(\mathbf{x})}\ \frac{1}{\varkappa}\beta\Psi_\Sigma(\mathbf{x})\right\} \mapsto \boldsymbol{\Psi} \tag{3.14}$$

in any point $\mathbf{x}$. The precise meaning of the operation $\mapsto$ will be established below.

The **second** observation is obtained by functionally differentiating both sides of Eq. (3.14) with respect to $\Psi^T_\Sigma(\mathbf{x})$:

$$\mathsf{Tr}\left\{\frac{\delta^2\boldsymbol{\Psi}}{\delta\Psi^T_\Sigma(\mathbf{x})\otimes\delta\Psi^T_\Sigma(\mathbf{x})}\frac{1}{\varkappa}\beta\Psi_\Sigma(\mathbf{x})\right\} + \frac{\delta\boldsymbol{\Psi}}{\delta\Psi^T_\Sigma(\mathbf{x})}\frac{1}{\varkappa}\beta\delta(\mathbf{0}) \mapsto \frac{\delta\boldsymbol{\Psi}}{\delta\Psi^T_\Sigma(\mathbf{x})}, \tag{3.15}$$

where $\delta(\mathbf{0}) = \delta\Psi_\Sigma(\mathbf{x})/\delta\Psi^T_\Sigma(\mathbf{x})$. We conclude, that the second term in (3.12), which has no counterparts in the familiar functional Schrödinger equation, vanishes, provided

$$\frac{1}{\varkappa}\beta\delta(\mathbf{0}) \mapsto 1. \tag{3.16}$$

Similarly, our **third** observation is that the term $\varkappa\beta\partial_{yy}\Psi_\Sigma$ in (3.10) reproduces the first term in Eq. (3.12) and therefore, in the functional Schrödinger equation with $\hat{\mathbf{H}}$ given by Eq.(2.9), if

$$\beta\varkappa \mapsto \delta(\mathbf{0}). \tag{3.17}$$

We see that this condition is consistent with Eq. (3.16) in the sense that (3.16) is fulfilled provided (3.17) is also satisfied.

Now, if we recall the origin of Dirac matrices in precanonical quantization as the quantum representations of differential forms, we can readily recognize the conditions (3.16) and (3.17) as the inverse quantization map q in Eq. (2.8) in the limit of the infinitesimal elementary volume $\frac{1}{\varkappa} \to 0$, i.e.

Eq. (3.17) is understood as follows:

$$\beta\varkappa \overset{q^{-1}}{\longmapsto} \delta(\mathbf{0}). \tag{3.18}$$

Note, that one may think of the mapping in Eq. (3.18) as the "Wick rotation" in the hyperbolic complex plane $(1, \beta)$ combined with the limit $\varkappa \to \infty$.

Fourth, if Eq. (3.10) is supposed to lead to a description in terms of the wave functional $\mathbf{\Psi}$ alone, then the third term in (3.12), which is proportional to $\partial_y \Psi_\Sigma(\mathbf{x})$, should cancel the second term in (3.10), which is also proportional to $\partial_y \Psi_\Sigma(\mathbf{x})$. This requirement leads to a condition which further restricts the dependence of $\mathbf{\Psi}$ on $\Psi_\Sigma(\mathbf{x})$ and $y(\mathbf{x})$, viz.,

$$\frac{\delta \mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x})} i\beta\gamma^i \partial_i y(\mathbf{x}) \mapsto -\frac{\delta\bar{\delta}\mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x})\bar{\delta} y(\mathbf{x})}. \tag{3.19}$$

By introducing the notation

$$\mathbf{\Phi}(\mathbf{x}) := \frac{\delta \mathbf{\Psi}}{\delta \Psi_\Sigma^T(\mathbf{x})}, \tag{3.20}$$

and taking into account that $\partial_i \delta(\mathbf{0}) = 0$, the solution of Eq. (3.19) can be found in the form

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{\Xi}([\Psi_\Sigma]; \breve{\mathbf{x}}) e^{-iy(\mathbf{x})\gamma^i \partial_i y(\mathbf{x})/\varkappa}, \tag{3.21}$$

where $\mathbf{\Xi}([\Psi_\Sigma]; \breve{\mathbf{x}})$ denotes a functional of $\Psi_\Sigma(\mathbf{x}')$ at $\mathbf{x}' \neq \mathbf{x}$ and the exponential function of the hypercomplex argument is understood as an eigenfunction of the Dirac operator $\gamma^i \partial_i$ with the condition $e^0 = 1$ (see e.g. [35]). Consequently,

$$\frac{\delta \mathbf{\Phi}(\mathbf{x})}{\delta \Psi_\Sigma^T(\mathbf{x})} = 0 \tag{3.22}$$

or equivalently,

$$\frac{\delta^2 \mathbf{\Psi}}{\delta \Psi_\Sigma(\mathbf{x}) \otimes \delta \Psi_\Sigma(\mathbf{x})} = 0. \tag{3.23}$$

We note that the latter equality is consistent with Eqs. (3.15) and (3.16).

Now, Eqs. (3.20) and (3.21) lead to the following solution:

$$\boldsymbol{\Psi} = \mathsf{Tr}\left\{ \boldsymbol{\Xi}([\Psi_\Sigma]; \check{\mathbf{x}})\ e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}\ \frac{1}{\varkappa}\beta\Psi_\Sigma(\mathbf{x}) \right\}\Big|_{\beta\varkappa \xrightarrow{q^{-1}} \delta(\mathbf{0})}, \tag{3.24}$$

which is valid for any $\mathbf{x}$ and in combination with the inverse quantization map (3.18). It is easy to check that Eq. (3.24) is consistent with Eq. (3.14).

The **fifth** observation is that the last term in (3.12), evaluated on the solution (3.24), yields:

$$\frac{\bar{\delta}^2\boldsymbol{\Psi}}{\bar{\delta}y(\mathbf{x})^2} \mapsto (\nabla y(\mathbf{x}))^2\boldsymbol{\Psi}. \tag{3.25}$$

Hence, it correctly reproduces the $\frac{1}{2}(\nabla y(\mathbf{x}))^2$ term in the functional Schrödinger equation (2.1) with $\hat{\mathbf{H}}$ given by Eq. (2.9). This calculation thus indicates that those are the twisting phase factors $e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}$ in front of the precanonical wave functions in (3.24) which account for the non-ultralocality (in Klauder's terminology [37]) of relativistic scalar field theory.

Our **sixth** observation concerns the first term in the right hand side of Eq. (3.10), which contains the total derivative. Namely, by integration by parts it takes the form

$$\int \mathrm{d}\mathbf{x}\ \mathsf{Tr}\left\{ \left(i\frac{\mathrm{d}}{\mathrm{d}x^i}\boldsymbol{\Phi}(\mathbf{x}) \right) \gamma^i\Psi_\Sigma(\mathbf{x}) \right\}. \tag{3.26}$$

Then, by taking the total derivative $\frac{\mathrm{d}}{\mathrm{d}x^i}$ of the explicit expression of $\boldsymbol{\Phi}$ in Eq. (3.21):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x^i}\boldsymbol{\Phi}(\mathbf{x}) = -\frac{i}{\varkappa}\boldsymbol{\Xi}([\Psi_\Sigma]; \check{\mathbf{x}})e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}(&\gamma^k\partial_k y(\mathbf{x})\partial_i y(\mathbf{x}) \\ &+ y(\mathbf{x})\gamma^k\partial_{ik}y(\mathbf{x})),\end{aligned} \tag{3.27}$$

and using the expression of $\boldsymbol{\Psi}$ in Eq. (3.24), we transform Eq. (3.26) to the following form:

$$-i\boldsymbol{\Psi}\int \mathrm{d}\mathbf{x}\ (\gamma^k\partial_k y(\mathbf{x})\partial_i y(\mathbf{x}) + y(\mathbf{x})\gamma^k\partial_{ik}y(\mathbf{x}))\gamma^i, \tag{3.28}$$

which obviously vanishes upon integration by parts. Hence, the first (total derivative) term in the right hand side of (3.10) does not contribute to the functional derivative equation describing the time evolution of $\boldsymbol{\Psi}$.

Finally, the functional $\Xi([\Psi_\Sigma(\mathbf{x})])$ in (3.24) is specified by combining all the above observations together and noticing that the formula Eq. (3.24) is valid for *any* $\mathbf{x}$. It can be accomplished only if the functional $\boldsymbol{\Psi}$ has the structure of the continuous product of identical terms at all points $\mathbf{x}$, viz.,

$$\boldsymbol{\Psi} = \mathsf{Tr}\left\{ \prod_{\mathbf{x}} e^{-iy(\mathbf{x})\gamma^i \partial_i y(\mathbf{x})/\varkappa} \Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t) \right\}\Big|_{\beta\varkappa \xmapsto{q^{-1}} \delta(\mathbf{0})} . \tag{3.29}$$

Thus we have obtained the expression of the Schrödinger wave functional in terms of precanonical wave functions. The equality in the above expression implies the inverse of the Clifford algebraic quantization map q and the limit of the infinitesimal elementary volume element $\frac{1}{\varkappa} \to 0$. Moreover, the preceding consideration also derives term by term the functional Schrödinger equation for the wave functional $\boldsymbol{\Psi}$ from the precanonical Schrödinger equation for the wave function Ψ restricted to the subspace Σ.

As we have already mentioned in the previous paper [20], the inverse quantization map in the limit of infinitesimal $\frac{1}{\varkappa}$ means that

$$\frac{1}{\varkappa}\,\beta \xmapsto{q^{-1}} \mathrm{d}\mathbf{x}. \tag{3.30}$$

Therefore the expression of the wave functional in Eq. (3.29) can be written in the form of the multidimensional product integral (c.f. [36]):

$$\boldsymbol{\Psi} = \mathsf{Tr}\left\{ \prod_{\mathbf{x}} e^{-iy(\mathbf{x})\alpha^i \partial_i y(\mathbf{x})\mathrm{d}\mathbf{x}} \Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)\big|_{\frac{1}{\varkappa}\beta\mapsto \mathrm{d}\mathbf{x}} \right\}, \tag{3.31}$$

which may be more practical to use than Eq. (3.29).

Further, let us recall that Ψ_Σ obeys Eq. (3.8). According to Eqs. (3.26)–(3.28) the total derivative term does not contribute to the functional derivative equation for $\boldsymbol{\Psi}$. In the case of scalar field theory, Eq. (3.8) without the total derivative term can be cast in the form

$$\begin{aligned} i\partial_t \Psi_\Sigma = {} & \frac{1}{2\varkappa}\beta\Big(i\varkappa\partial_y + \gamma^i\partial_i y(\mathbf{x})\Big)^2 \Psi_\Sigma \\ & + \frac{1}{\varkappa}\beta\left(V(y(\mathbf{x}) + \frac{1}{2}(\nabla y(\mathbf{x}))^2\right)\Psi_\Sigma =: \beta\mathcal{E}\Psi_\Sigma. \end{aligned} \tag{3.32}$$

The structure of the operator $\mathcal{E}$ in the right hand side of Eq. (3.32):

$$\mathcal{E} = \frac{1}{2\varkappa}\Big(i\varkappa\partial_y + \gamma^i\partial_i y(\mathbf{x})\Big)^2 + \frac{1}{\varkappa}\left(V(y(\mathbf{x}) + \frac{1}{2}(\nabla y(\mathbf{x}))^2\right), \tag{3.33}$$

resembles the structure of the magnetic Schrödinger operator in y-space with the "matrix magnetic potential" $\gamma^i\partial_i y(\mathbf{x})$ and the "electric potential" $V(y(\mathbf{x})) + \frac{1}{2}(\nabla y(\mathbf{x}))^2$.

The "magnetic" term in Eq. (3.33) is pure gauge (in y-space), so that it does not change the eigenvalues of $\mathcal{E}$ in comparison with $\hat{H}$. Its influence reduces to the multiplication of the eigenstates of $\hat{H}$ by the hypercomplex phase factor $e^{i\gamma^i y(\mathbf{x})\partial_i y(\mathbf{x})/\varkappa}$. Note that Eq. (3.32) is valid in the fibers of field variables and their first jets over each point $\mathbf{x}$, with $\mathbf{x}$-s here just labeling the fiber in which Eq. (3.32) is written.

The addition $\frac{1}{2}(\nabla y(\mathbf{x}))^2$ to the "electric potential" term in Eq. (3.33) modifies the mass term in the potential term of the DW Hamiltonian operator. Namely, by substituting it into Eq. (3.7) and integrating by parts using the property (3.14), we conclude that under the restriction to Σ the mass term $\frac{1}{2}m^2y^2$ in $V(y)$ is replaced by

$$\frac{1}{2}y(\mathbf{x})(m^2 - \nabla^2)y(\mathbf{x}). \tag{3.34}$$

Correspondingly, the parameter m in the expressions of precanonical wave functions is formally replaced by $\sqrt{m^2 - \nabla^2}$, when they are restricted to Σ.

For example, in the case of free massive scalar field theory the ground state of the DW Hamiltonian operator, $\hat{H} = -\frac{1}{2}\varkappa^2\partial_{yy} + \frac{1}{2}m^2y^2$, is given, up to the normalization factor, by $\Psi_0 \sim e^{-\frac{m}{2\varkappa}y^2}$, and its eigenvalue is $\frac{1}{2}m\varkappa$ [6–8]. Then the eigenstates of $\beta\hat{H}$ corresponding to the positive eigenvalues are given by $\sim(1+\beta)e^{-\frac{m}{2\varkappa}y^2}$. Therefore, the corresponding ground state wave function restricted to Σ: $\Psi_{0\Sigma}$, will take the form

$$\Psi_{0\Sigma} \sim e^{iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}(1+\beta)e^{-\frac{1}{2\varkappa}y(\mathbf{x})\sqrt{m^2-\nabla^2}\,y(\mathbf{x})}. \tag{3.35}$$

By substituting the last expression into (3.31) we see that the "magnetic" phase factors in Eq. (3.35) and Eq. (3.29) will cancel each other, so that finally we obtain

$$\mathbf{\Psi} \sim \mathsf{Tr}\prod_{\mathbf{x}}(1+\beta)e^{-\frac{1}{2}\beta y(\mathbf{x})\sqrt{m^2-\nabla^2}y(\mathbf{x})\mathrm{d}\mathbf{x}} \sim e^{-\frac{1}{2}\int y(\mathbf{x})\sqrt{m^2-\nabla^2}\,y(\mathbf{x})\mathrm{d}\mathbf{x}}, \tag{3.36}$$

where the identity $\beta(1+\beta) = (1+\beta)$ satisfied by β-matrix is used (c.f. our earlier treatment in [19]).

The right hand side of Eq. (3.36) reproduces the vacuum state solution of the functional derivative Schrödinger equation for the free scalar field (see e.g. [21]). Usually it corresponds to the picture of the vacuum as the continuum of harmonic oscillators with the zero-point energy $\frac{1}{2}\sqrt{m^2+\mathbf{k}^2}$

at every point of $\mathbf{k}$-space. Here the vacuum state of free quantum scalar field is obtained as the product of the ground state wave functions of the DW Hamiltonian operator (which in this case corresponds to the harmonic oscillator in y-space) over all points $\mathbf{x}$ of space.

4. Conclusion

Precanonical quantization, which is based on the space-time symmetric generalization of the Hamiltonian formalism to field theory (the De Donder-Weyl theory), leads to the description of quantum fields in terms of Clifford-valued wave functions on the bundle of field variables over space-time. These wave functions obey a Dirac-like generalization of the Schrödinger equation with the mass term replaced by the DW Hamiltonian operator. The formulation introduces a small parameter $\frac{1}{\varkappa}$ of the dimension of spatial volume, which appears on dimensional grounds in the representation of precanonical quantum operators and has the meaning of the minimal resolution of a spatial volume.

A proper understanding of the connection between precanonical quantization and the standard methods of quantization in field theory is important for the physical interpretation of the results of precanonical quantization. In this paper, we discuss how the results of canonical quantization in the functional Schrödinger representation are related to the precanonical quantization and improve the arguments of the previous discussions in [19, 20].

Summarizing the considerations in Sect. 3 and those in the preceding paper [20], we have proven that

Proposition. *If* $\Psi(y, x)$ *is a precanonical wave function obeying the precanonical covariant analogue of the Schrödinger equation* (2.2)*, and* $\Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)$ *is its restriction to the subspace* Σ *representing a field configuration* $y(\mathbf{x})$ *at time* t*, then in the limiting case* $\beta\varkappa \mapsto \delta(\mathbf{0})$ *or equivalently,* $\frac{1}{\varkappa}\beta \mapsto \mathrm{d}\mathbf{x}$*, there exists a composite functional* $\boldsymbol{\Psi}$ *of* Ψ_Σ *and* $y(\mathbf{x})$*, whose time evolution is governed by the standard functional derivative Schrödinger equation obtained from the canonical quantization. The time evolution of* $\boldsymbol{\Psi}$ *is completely determined by the time evolution of* $\Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)$ *determined by the precanonical Schrödinger equation restricted to* Σ*. The expression of the Schrödinger wave functional* $\boldsymbol{\Psi}$ *in terms of precanonical wave functions* Ψ *is given by the product integral formula, Eq.* (3.31)*, which is the* necessary and sufficient *condition for the functional* $\boldsymbol{\Psi}$ *to satisfy the canonical Schrödinger equation if* Ψ *satisfies the precanonical Schrödinger equation.*

This result leads to the conclusion that the canonical QFT in the functional Schrödinger representation is the limiting case of the theory obtained from precanonical quantization corresponding to $\beta\varkappa \mapsto \delta(\mathbf{0})$ or $\beta/\varkappa \mapsto \mathrm{d}\mathbf{x}$, i.e. to an infinitesimal value of $\frac{1}{\varkappa}$.

It is interesting to note that the introduction of the ultraviolet scale $\varkappa$ in precanonical quantization does not modify the relativistic space-time at small distances. It rather defines the scale of "very small" distances for the specific field theory under consideration. It is tempting, however, to interpret $\varkappa$ as the universal fundamental ultraviolet scale similar to the Planck scale, where the idea of the space-time continuum is supposed to break down due to quantum gravity effects. In this case, precanonical quantization might be able to provide new insights into the Planck scale physics.

Note in conclusion that the manifest respect for space-time symmetry within the precanonical quantization approach together with the nonperturbative nature of the construction of interacting quantum field theories, potentially make it a suitable framework for the exploration of quantum gravity and quantum gauge theories (c.f. [38, 39] for the recent discussions).

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National Quantum Information Centre in Gdańsk (KCIK)
81-824 Sopot, Poland
& Research Center of Einstein Physics, Free University of Berlin
14195 Berlin, Germany
E-mail address: kanattsi@gmail.com