

Galois covers of $\mathcal{N} = 2$ BPS spectra and quantum monodromy

SERGIO CECOTTI AND MICHELE DEL ZOTTO

The BPS spectrum of many 4d $\mathcal{N} = 2$ theories may be seen as the (categorical) Galois cover of the BPS spectrum of a different 4d $\mathcal{N} = 2$ model. The Galois group \mathbb{G} acts as a physical symmetry of the covering $\mathcal{N} = 2$ model. The simplest instance is $SU(2)$ SQCD with $N_f = 2$ quarks, whose BPS spectrum is a \mathbb{Z}_2 -cover of the BPS spectrum of pure SYM. More generally, $\mathcal{N} = 2$ SYM with simply-laced gauge group G admits \mathbb{Z}_k -covers for all $k \in \mathbb{N}$; e.g. the \mathbb{Z}_2 -cover of $SO(8)$ SYM is $SO(8)$ SYM coupled to two copies of the E_6 Minahan–Nemeshanski SCFT. Galois covers simplify considerably the computation of the BPS spectrum at \mathbb{G} -symmetric points, in both finite and infinite chambers.

When the covering and quotient QFTs admit a geometric engineering, say for class \mathcal{S} models, the categorical spectral cover may be realized as a covering map in the geometry. A particularly nice instance is when the spectral Galois cover is induced by a modular cover of principal modular curves, $X(NM) \rightarrow X(M)$, or, more generally, by *regular* Grothendieck’s *dessins d’enfants*; the BPS spectra of the corresponding $\mathcal{N} = 2$ QFTs have magic properties.

The Galois covers allow to study effectively the action of the quantum (half)monodromy $\mathbb{K}(q)$ of 4d $\mathcal{N} = 2$ QFTs. We present several examples and applications of the spectral covering philosophy.

1	Introduction and outlook	1230
2	BPS spectra, GIM Lie algebras, and Galois covers	1235
2.1	Review of basic facts	1235
2.2	Triangular factors of Jacobian algebras	1238

2.3	Slodowy–GIM Lie algebras. Extended affine Lie algebras	1241
2.4	Galois covers of (locally) bounded \mathbb{C} –categories	1247
3	First examples and applications	1261
3.1	\mathbb{Z}_k covers in $SU(2)$ SQCD	1261
3.2	\mathbb{Z}_k covers of pure SYM with simply–laced gauge groups	1265
3.3	Abelian Galois covers of class $\mathcal{S}[A_1]$ models	1272
4	Non-Abelian \mathbb{G} : modular curves, <i>dessins d’enfants</i>	1273
4.1	Principal modular curves	1274
4.2	The modular quivers $Q(N)$ and their flat cousins $Q(N)^{\text{fl}}$	1276
4.3	Simple generalizations	1279
4.4	Generalization: regular Grothendieck’s <i>dessins d’enfants</i>	1280
4.5	Finite chambers of <i>dessin d’enfant</i> cover $\mathcal{N} = 2$ QFTs	1282
4.6	<i>Dessins d’enfants</i> models: general BPS chambers	1284
5	Galois covers and the quantum monodromy $\mathbb{M}(q)$	1284
5.1	A baby example: the covering $N_f = 2 \rightarrow N_f = 0$	1286
5.2	Galois coverings, cluster mutations, and finite BPS chambers	1289
5.3	\mathbb{Z}_k coverings of $SU(2)$ SQCD: friezes and SL_2 –tilings	1292
5.4	Covers of $\mathcal{N} = 2$ gauge theories with gauge group G	1298

$\mathcal{N} = 2$ BPS spectra and quantum monodromy	1229
6 BPS spectra of genus one class $\mathcal{S}[A_1]$ theories	1299
6.1 Categorical covers and duality frames	1301
6.2 Triangularity of the \mathbb{Z}_p -symmetric BPS spectra	1305
6.3 Physical interpretation: perturbing away from the \mathbb{Z}_p -point	1308
6.4 Relation with extended affine and Kac–Moody Lie algebras	1311
6.5 Irregular punctures	1314
Acknowledgements	1315
Appendix A Nullity κ and type \mathfrak{g}_r of class $\mathcal{S}[A_1]$ triangular factors	1315
Appendix B Quick review of strings and bands	1317
B.1 Indecomposable modules of string algebras	1317
B.2 The Assem–Bruštle–Charbonneau–Plamondon theorem	1319
B.3 Gentling class $\mathcal{S}[A_1]$ algebras	1320
Appendix C Proof of Fact 4	1323
C.1 The associated topological cover	1323
C.2 Exceptional cases	1326
Appendix D $g = 1$: no R_1^r summands at \mathbb{Z}_p -symmetric points	1326
D.1 Strings and bands	1327
D.2 No R_1^r	1328
References	1330

1. Introduction and outlook

The original elegant proof of the Kontsevich–Soibelman wall crossing formula (WCF) [1] for $\mathcal{N} = 2$ SYM with $G = SU(2)$ — due to Gaiotto, Moore, and Neitzke (GMN) [2] — was based on the observation that the BPS spectrum of $\mathcal{N} = 2$ SQCD with gauge group $SU(2)$ and two flavors is a ‘double cover’ of the BPS spectrum of pure SYM¹, so the WCF for SYM is obtained by ‘pushing down’ the simpler WCF for SQCD with $N_f = 2$.

There are several (related) senses in which $\mathcal{N} = 2$ SQCD with $N_c = N_f = 2$ is a ‘double cover’ of its $N_f = 0$ cousin. Geometrically, seeing both theories as class $\mathcal{S}[A_1]$ models [3, 4] defined by quadratic differentials $\phi_2^{(N_f)}$ on the sphere, we have $\phi_2^{(2)} = \xi^* \phi_2^{(0)}$ where $\xi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a certain double cover (a meromorphic function of degree 2). Thus the GMN spectral ‘double cover’ is induced from the Galois double cover of the field of rational functions $\mathbb{C}(z)$ by its algebraic extension $\mathbb{C}(\sqrt{z})$. From the point of view of the 2d/4d correspondence [5], the 2d (2,2) model associated to $N_f = 2$ SQCD is a double cover, in the sense of tt^* [6, 7], of the (2,2) theory corresponding to pure SYM.² All tt^* quantities (metric [6], CFIV index [9], and brane amplitudes [10]) for $N_f = 0$ SQCD are the tt^* push-down of the corresponding quantities for $N_f = 2$.

Since the 4d BPS spectrum may be computed geometrically [3, 11–14], the equality $\phi_2^{(2)} = \xi^* \phi_2^{(0)}$ implies that, in a sense, the BPS spectrum of $N_f = 2$ SQCD ‘doubly covers’ the one of pure SYM, as observed in [2]. On the other hand, the covering property of the corresponding 2d (2,2) models may be used to construct *new* 4d $\mathcal{N} = 2$ QFTs (generically without Lagrangian formulation) as multiple covers of known theories, as was done in [15, 16].

The purpose of this paper is to make more precise the notion of a $\mathcal{N} = 2$ QFT having a BPS spectrum which is a Galois cover of the BPS spectrum of another $\mathcal{N} = 2$ theory, introduce some technical tools to work effectively with such coverings, and discuss some generalization and new applications.

The basic observation is that the map between the BPS spectra of $N_f = 2$ and $N_f = 0$ $SU(2)$ SQCD is an elementary instance of a math gadget known as the *Bongartz–Gabriel push-down functor* [17, 18]. This functor is one of the most efficient tools in Representation Theory, and hence a powerful technique to compute BPS spectra of general 4d $\mathcal{N} = 2$ theories.

¹We stress that this holds in *all* BPS chambers of $SU(2)$ SYM.

²These 2d models are (2,2) Landau–Ginzburg theories with superpotentials $W_{N_f=2}(X) = e^{2X} + e^{-2X}$ and $W_{N_f=0}(Y) = e^Y + e^{-Y}$ [8].

The connection is as follows (see section 2 for details): to most 4d $\mathcal{N} = 2$ theories we may associate a bounded \mathbb{C} -category \mathcal{C} such that its BPS states are identified with continuous families of stable linear functors $X: \mathcal{C} \rightarrow \text{mod } \mathbb{C}$ (the category of finite-dimensional \mathbb{C} -spaces); we write $\text{mod } \mathcal{C}$ for the (Abelian) category of such functors. The BPS spectrum of a $\mathcal{N} = 2$ model is a ‘cover’ of the spectrum of another $\mathcal{N} = 2$ model when the category \mathcal{C}_1 of the first theory is a *Galois covering* of the category \mathcal{C}_2 of the second one.

$N_f = 2 \longrightarrow N_f = 0$ revisited. As an illustration of our strategy, in this introduction we rephrase the GMN computation in the categorical language, leaving the general story and more advanced applications for the main body of the paper. To SQCD with $N_c = N_f = 2$ it is associated the unique \mathbb{C} -category $\mathcal{C}^{(2)}$ on four objects \mathcal{O}_a , $a = 1, 2, 3, 4$, and morphism spaces such that

$$(1.1) \quad \dim \mathcal{C}^{(2)}(\mathcal{O}_a, \mathcal{O}_b) = \begin{cases} 1 & \text{if } a = b \\ 1 & \text{if } a \text{ even and } b \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

The category $\mathcal{C}^{(2)}$ has a \mathbb{Z}_2 group of automorphisms acting freely on the objects \mathcal{O}_a , i.e. $\mathcal{O}_1 \leftrightarrow \mathcal{O}_3$, $\mathcal{O}_2 \leftrightarrow \mathcal{O}_4$; then it makes sense to consider the *orbit category* $\mathcal{C}^{(0)} \equiv \mathcal{C}^{(2)}/\mathbb{Z}_2$. One says that the category $\mathcal{C}^{(2)}$ is the *Galois cover* of the category $\mathcal{C}^{(0)}$ with *Galois group* \mathbb{Z}_2 . In this example the orbit category $\mathcal{C}^{(0)}$ coincides with the linear category for pure $SU(2)$ SYM. Hence $N_f = 2$ SQCD is the \mathbb{Z}_2 Galois cover of pure SYM, as originally found by GMN. The canonical covering functor $F: \mathcal{C}^{(2)} \rightarrow \mathcal{C}^{(0)}$ induces two functors between the functor categories $\text{mod } \mathcal{C}^{(0)}$, $\text{mod } \mathcal{C}^{(2)}$: the *pull up* functor F^λ , and the *push down* functor F_λ

$$(1.2) \quad F^\lambda: \text{mod } \mathcal{C}^{(0)} \rightarrow \text{mod } \mathcal{C}^{(2)}, \quad F_\lambda: \text{mod } \mathcal{C}^{(2)} \rightarrow \text{mod } \mathcal{C}^{(0)}$$

which are each other adjoint. With respect to the Euler form,

$$(1.3) \quad \langle X, Y \rangle_{\mathcal{C}} = \sum_{k=0}^{\infty} (-1)^k \dim \text{Ext}_{\text{mod } \mathcal{C}}^k(X, Y), \quad X, Y \in \text{mod } \mathcal{C},$$

we have

$$(1.4) \quad \langle X, F^\lambda(Y) \rangle_{\mathcal{C}^{(2)}} = \langle F_\lambda(X), Y \rangle_{\mathcal{C}^{(0)}}, \quad X \in \text{mod } \mathcal{C}^{(2)}, Y \in \text{mod } \mathcal{C}^{(0)}.$$

The W boson of $N_f = 2$ SQCD is the pull-back of the pure SYM W boson. Then the adjoint formula (1.4) says that the push forward F_λ preserves the

magnetic charge of the states. F_λ restricted to the magnetically charged BPS states is then a two-to-one correspondence; this is the original GMN trick.

The cover $\mathcal{C}^{(2)} \rightarrow \mathcal{C}^{(0)}$ has a nice Lie algebraic interpretation. We know [8, 19–21] that the indecomposables of $\text{mod } \mathcal{C}^{(2)}$ are in correspondence with the positive roots of the affine Lie algebra $A_3^{(1)}$ which has a \mathbb{Z}_2 outer automorphism subgroup acting freely on the simple roots. We consider the \mathbb{Z}_2 -invariant Lie subalgebra $(A_3^{(1)})^{\mathbb{Z}_2} \cong A_1^{(1)}$. The indecomposables of $\text{mod } \mathcal{C}^{(0)}$ are in correspondence with the positive roots of the invariant Lie subalgebra $A_1^{(1)}$.

We may endow the category $\text{mod } \mathcal{C}^{(2)}$ with a stability function (i.e. $\mathcal{N} = 2$ central charge) which is the pull-back of the one for $\text{mod } \mathcal{C}^{(0)}$. In this case we may use the functors F^λ, F_λ to relate the stability of functors in $\text{mod } \mathcal{C}^{(2)}$ and $\text{mod } \mathcal{C}^{(0)}$, that is, to compare the BPS spectra of the two theories in corresponding chambers. The fact that the respective W bosons are related by pull back, implies that Yang–Mills weak (strong) coupling pulls back to weak (strong) YM coupling.

The Galois cover of BPS spectra leads to relations between the quantum monodromies [5] of the cover and quotient $\mathcal{N} = 2$ QFTs. In the simple case of $N_f = 2$ SQCD covering $N_f = 0$, this reduces to the functional identity of quantum operators (here $q = e^{2\pi i\tau}$ is a complex number in the unit disk)

$$(1.5) \quad \mathbb{M}^{(N_f=2)}(q) = \mathbb{M}^{(N_f=0)}(q^2),$$

where equality holds under a natural identification of the Hilbert spaces on which the two operators act.

A general strategy to compute (symmetric) BPS spectra. Going to the general case, the previous observation leads to a new strategy to compute the BPS spectrum of the covering theory from the one of the (usually simpler) quotient QFT. This strategy applies to all special points where the central charge Z_{cover} is the pull back of a central charge for the quotient theory, i.e. $Z_{\text{cover}} = F^\lambda Z_{\text{quot}}$. These special points are precisely the ones which are invariant under the Galois group³ \mathbb{G} : i.e. they are points of *enhanced symmetry* for the 4d QFT. The existence of an additional symmetry makes technically natural [22] to focus on such points (and chambers) in parameter space, even if they are highly non-generic.

³In the general case, the Galois group may be any group \mathbb{G} of automorphisms of the covering category \mathcal{C} which acts freely on the objects (and has finitely many orbits).

Non-generic points may be intersection points of walls of marginal stability. If this is the case, one must be careful with the physical interpretation of the mathematical results. In the examples, we find the following pattern which we expect to be rather general. Consider a generic point in parameter space at distance ϵ from the given \mathbb{G} -invariant point; for small ϵ , the BPS spectrum consists of two sectors:

- 1) the sector of particles one gets from the covering approach at the invariant point, which form complete \mathbb{G} -representations and have bounded \mathbb{G} -invariant masses (up to $O(\epsilon)$ corrections) as $\epsilon \rightarrow 0$;
- 2) if the \mathbb{G} -invariant point belongs to wall(s) of marginal stability, in addition we have BPS particles which do not form complete \mathbb{G} -representations, and have masses of order $O(1/\epsilon)$ unbounded as $\epsilon \rightarrow 0$. This sector of the spectrum depends on the particular perturbation away from the symmetric point (i.e. on which side of the wall(s) we are), and is also hard to compute.

The ‘hard’ sector 2) is physically irrelevant when we are very close to the \mathbb{G} -invariant point, and the ‘pull back’ spectrum 1) is asymptotically exact in the limit $\epsilon \rightarrow 0$.

A sample of results from the covering technique. There are many examples of Galois coverings between pairs of 4d $\mathcal{N} = 2$ QFTs; all of them may be used to compute BPS spectra at points in parameter space where the Galois group \mathbb{G} is realized as a physical symmetry. In this paper we focus on three large classes of examples. The first two are generalizations of $N_f = 2 \rightarrow N_f = 0$ in different directions.

A) Coverings between class $\mathcal{S}[A_1]$ models. We consider models defined by quadratic differentials on surfaces with several punctures (regular and irregular) [4, 8]. We describe the BPS spectrum in detail for class $\mathcal{S}[A_1]$ models associated to surfaces of genus zero and one. A large subclass of them covers $SU(2)$ SQCD or $SU(2)$ $\mathcal{N} = 2^*$. Since the BPS spectra of these quotient theories have an elegant description in terms of the root system of Lie algebras, the BPS spectra of the covering theories are also described by root systems of a class of Lie algebras known as *extended affine*. One may also characterize the allowed quantum numbers of the BPS states in terms of roots of Kac–Moody algebras, see section 6.

B) Coverings of pure SYM by SYM coupled to ‘matter’. $\mathcal{N} = 2$ SYM with simply-laced gauge group G admits, for all $k \in \mathbb{N}$, a \mathbb{Z}_k Galois

cover by SYM with the same gauge group coupled to two copies of a certain superconformal ‘matter’ system which depends on G and k (this reduces to two copies of the quark for $G = SU(2)$ and $k = 2$). A typical example is the double cover of pure SYM with gauge group $SO(8)$ by SYM with the same group coupled to two copies of the Minahan–Nemeshanski theory of type E_6 [23, 24]. Other examples involve in the ‘matter’ systems the Minahan–Nemeshanski theories of types E_7 , E_8 , and other strongly–coupled SCFTs with or without a Lagrangian formulation.

Despite the apparent complication of the coupled theory, the fact that — at \mathbb{Z}_k –symmetric points — it has a description as a Galois cover of pure SYM, allows to reduce the computations to the simpler ones for the uncoupled model. This simplification is particularly dramatic in the action of the monodromy.

C) Modular covers of $\mathcal{N} = 2$ theories, *dessins d’enfants*. When the Galois cover of the geometry (and hence of the linear category) arises from a *modular covering* — so that \mathbb{G} is a sub–quotient of the modular group $PSL(2, \mathbb{Z})$ — the covering $\mathcal{N} = 2$ QFTs are of extraordinary complexity: their gauge groups have a huge number of simple factors. For instance, the *minimal* example of §.4.3 has a gauge group of the form

$$(1.6) \quad G_{\text{gauge}} = \prod_{i=1}^{2304} G_i \quad G_i \text{ simple,}$$

and thousands of different matter sectors. Nevertheless the magic of the modular Galois covers allows to describe their BPS spectra (at \mathbb{G} –symmetric points) very easily.

Even more general examples may be constructed using regular Grothendieck’s *dessins d’enfants*, see section 4 for details.

Organization of the paper. Section 2 is a detailed catalogue of mathematical techniques we shall use in the subsequent sections to compute BPS spectra. §.2.1 is a review of basic facts mainly intended to establish the notation. §.2.2, §.2.3, and §.2.4 contain both reviews of mathematical tools not generally known in the physics community, and new materials which are specific to the algebras which appear in physics. Section 3 contains the first examples and applications. Section 4 describes the modular paradise and the relation with Grothendieck theory of the *dessins d’enfants*. Section 5 discusses the relation between Galois covers in the sense of §.2.4 and the (quantum) monodromy operator [5]. Section 6 contains some computations

of BPS spectra for class $\mathcal{S}[A_1]$ theories with genus one Gaiotto surfaces and describes their relations with the roots systems of the extended affine (GIM) Lie algebras. Several appendices contain additional examples, as well as technicalities and reviews of specific math tools.

2. BPS spectra, GIM Lie algebras, and Galois covers

In this section we collect the technical tools we shall use to study various $\mathcal{N} = 2$ models in later sections. §§.2.1, 2.2.1, 2.3.1, 2.3.2 and 2.4.1 are reviews of more or less known facts.

2.1. Review of basic facts

2.1.1. BPS states and representation theory. We quickly review the framework of [20, 21]. We consider the Coulomb branch of a 4d $\mathcal{N} = 2$ gauge theory. Its charges take value in a lattice Γ . Dirac quantization endows Γ with a skew-symmetric pairing $\langle \cdot, \cdot \rangle_D: \Gamma \times \Gamma \rightarrow \mathbb{Z}$. In a chosen Coulomb vacuum, the central charge of $\mathcal{N} = 2$ SUSY defines a linear map $Z: \Gamma \rightarrow \mathbb{C}$. Z encodes the stability condition of BPS states. Varying the moduli, the stability condition changes, and the BPS spectra jump discontinuously (wall-crossing). A BPS chamber is a region in parameter space with constant BPS spectrum; BPS spectra in contiguous chambers are related by the KS formula [1, 2, 25–27]. The $\mathcal{N} = 2$ model has a BPS–quiver if there exist stable BPS hypermultiplets of charge vectors $e_i \in \Gamma$ satisfying two conditions [8, 20]: $Z(e_i) \in H$ for some half-space $H \subset \mathbb{C}$, and all BPS states have charge vectors of the form $\pm \sum_i \mathbb{Z}_+ e_i$. Then the integral matrix⁴ $B_{ij} \equiv \langle e_i, e_j \rangle_D$ defines the model’s (BPS–)quiver Q : its nodes are in 1–to–1 correspondence with the simple charges e_i , and we draw B_{ij} oriented arrows from e_i to e_j . The BPS–quiver encodes a four–supercharges quiver quantum mechanics (SQM), with gauge group $\prod_{i=1}^D U(x_i)$ and bifundamental arrows, that captures the dynamics on the world-line of the BPS particle of charge $\mathbf{x} = \sum_{i=1}^D x_i e_i \in \Gamma$ [28]; its superpotential \mathcal{W} is a linear combination of oriented cycles in Q . The BPS states with charge $\mathbf{x} \in \Gamma$ correspond to the SUSY vacua of the quiver SQM given by the cohomology of its vacuum moduli. Standard GIT arguments allow to trade D –flatness for stability: then a (classical) vacuum configuration of the SQM describing particles of charge $\mathbf{x} \in \Gamma$ is identified

⁴ B_{ij} is the *exchange matrix* of the quiver. B_{ij} is the net number of arrows from e_i to e_j . For generic \mathcal{W} all 1 and 2-loops can be integrated out, and we remain with the 2-acyclic quiver specified by B .

with an isoclass of *stable* modules X of the Jacobian algebra $\mathcal{J} \equiv \mathbb{C}Q/(\partial\mathcal{W})$ (the path algebra $\mathbb{C}Q$ bounded by the bilateral ideal generated by $\partial\mathcal{W}$) with $\mathbf{dim} X = \mathbf{x}$. Rotating H to the upper-half plane, the stability condition defined by the central charge $Z: \Gamma \rightarrow \mathbb{C}$ becomes

$$(2.1) \quad X \text{ is stable} \iff \forall \text{ proper non-zero submodule } Y \text{ of } X: \\ \arg Z(Y) < \arg Z(X).$$

Isoclasses of stable modules appear in continuous families parametrized by Kählerian moduli spaces \mathcal{M}_X . The corresponding BPS $\mathcal{N} = 2$ supermultiplets have a Clifford vacuum in a $SU(2)_{\text{spin}} \times SU(2)_R$ representation determined by the Lefschetz/Hodge decompositions of $H^{p,q}(\mathcal{M}_X)$ [29], so the maximal spin occurring is $\dim_{\mathbb{C}} \mathcal{M}_X/2$. Hypermultiplets correspond to rigid representations with \mathcal{M}_X a point. A necessary condition for a \mathcal{J} -module X to be stable in *some* chamber is that X is a *brick*, namely $\text{End } X = \mathbb{C}$ [21].

The BPS spectral problem is then reduced to the Representation Theory (RT) of the Jacobian algebra \mathcal{J} . A $\mathcal{N} = 2$ model has different quiver descriptions: indeed there are many possible choices of the half-plane H . Different choices produce different, but physically equivalent, algebras \mathcal{J} [19, 20]. The corresponding SQMs are related by 1d Seiberg dualities a.k.a. quiver mutations [30].

2.1.2. Review of pure $SU(2)$ SYM. We shall need some results in the Representation Theory of the algebra associated to $SU(2)$ SYM, i.e. the Kronecker algebra $\text{Kr} \equiv$ the path algebra of the Kronecker quiver

$$(2.2) \quad \circ \begin{array}{c} \xrightarrow{A} \\ \xrightarrow{B} \end{array} \bullet$$

The Abelian category of Kr -modules has a separating decomposition into three linear categories (the *regular* category, \mathcal{T} , is an Abelian category in its own right) [31–33] (see also [21])

$$(2.3) \quad \text{mod Kr} = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q},$$

where the notation means that a Kr -module can be (uniquely) written as

$$(2.4) \quad p \oplus t \oplus q \quad \text{with } p \in \mathcal{P}, t \in \mathcal{T}, q \in \mathcal{Q}.$$

Separating means that in the RHS of (2.3) morphisms go from left to right, i.e.

$$(2.5) \quad \text{Hom}(\mathcal{Q}, \mathcal{T}) = \text{Hom}(\mathcal{Q}, \mathcal{P}) = \text{Hom}(\mathcal{T}, \mathcal{P}) = 0.$$

Objects of \mathcal{P} (resp. \mathcal{Q}) are called *preprojective modules* (resp. *preinjective modules*). Duality interchanges \mathcal{P} and \mathcal{Q} . Each object p, q , and t is, in turn, the direct sum of indecomposable modules in \mathcal{P} , \mathcal{Q} , and \mathcal{T} , respectively. They are characterized by the Ringel defect [31, 32]

$$(2.6) \quad m(X) = \dim X_{\circ} - \dim X_{\bullet} :$$

- a) An indecomposable Kr-module X is preprojective iff $m(X) < 0$. In this case the arrows A, B are injective non-iso. Indecomposable preprojective modules are rigid and have dimension $\mathbf{dim} P_n = (n - 1, n)$, $n \in \mathbb{N}$;
- b) An indecomposable Kr-module X is preinjective iff $m(X) > 0$. In this case the arrows A, B are surjective non-iso. Indecomposable injective modules are rigid and have dimension $\mathbf{dim} Q_n = (n, n - 1)$, $n \in \mathbb{N}$;
- c) An indecomposable Kr-module X is regular iff $m(X) = 0$. Regular indecomposable modules appear in families parametrized by \mathbb{P}^1 and have dimension $\mathbf{dim} R_n(\lambda) = (n, n)$, with $n \in \mathbb{N}$ and $\lambda \in \mathbb{P}^1$. The indecomposable $R_n(\lambda)$ has the form $\mathbb{C}^n \xrightarrow[\text{Id}]{J(\lambda, n)} \mathbb{C}^n$ with $J(\lambda, n)$ the Jordan block of size n and eigenvalue λ .

An indecomposable Kr-module X is a brick if either a) $m(X) \neq 0$, or b) $m(X) = 0$ and $\mathbf{dim} X = \delta \equiv (1, 1)$ [31]. We know that being a brick is a necessary condition for being stable in *some* chamber. In the Kr case, all bricks are simultaneously stable under a separating central charge, i.e. a central charge whose phase is ordered as the RHS of (2.3)

$$(2.7) \quad \arg Z(\mathcal{P}) < \arg Z(\mathcal{T}) < \arg Z(\mathcal{Q}).$$

The physical interpretation of these RT facts is [21]: a) the separating central charge gives the weakly coupled chamber; b) the Ringel defect is the magnetic charge; c) the weak-coupling BPS spectrum consists of a single vector multiplet of zero magnetic charge and electric charge 2, associated to the \mathbb{P}^1 family of regular bricks, together with two infinite towers of dyonic hypermultiplets, P_n of magnetic charge -1 and electric charge $2n$, and Q_n of magnetic charge $+1$ and electric charge $2n + 2$.

In terms of $Z_\bullet \equiv Z(e_\bullet)$, $Z_\circ \equiv Z(e_\circ)$, Eqn. (2.7) is just $\arg Z_\bullet < \arg Z_\circ$. We note that all regular modules $R \in \mathcal{T}$ have the same phase $\arg Z(R) = \theta_r \equiv \arg(Z_\circ + Z_\bullet)$ while

$$(2.8) \quad \arg Z(P_{n+1}) < \theta_r, \quad \arg Z(Q_{n+1}) > \theta_r.$$

2.2. Triangular factors of Jacobian algebras

2.2.1. Triangular factors and Tits forms. Let Q be a 2–acyclic quiver with (non degenerate, reduced) superpotential \mathcal{W} . We consider subquivers $\tilde{Q} \subset Q$ which are *acyclic* and not full; we take \tilde{Q} to be maximal in the sense that all nodes of Q are nodes of \tilde{Q} and there are no acyclic subquiver Q' with $\tilde{Q} \subsetneq Q' \subsetneq Q$. Then, given two nodes i and j , either \tilde{Q} contains all the arrows of Q between i and j , or does not contain any such arrow. Moreover, adding to \tilde{Q} any omitted arrow $\alpha: i \rightarrow j$ will close a cycle. We represent the pair (Q, \tilde{Q}) by a *bi-quiver* where the arrows of \tilde{Q} are solid and those of $Q \setminus \tilde{Q}$ are dashed. The underlying *bi-graph* is obtained by ignoring the orientation of the arrows but keeping the distinction dashed/solid.

We write **(dash)** for the two–sided ideal in the Jacobian algebra $\mathcal{J} = \mathbb{C}Q/(\partial\mathcal{W})$ generated by the dashed arrows. Let \mathcal{T} be the factor⁵ algebra $\mathcal{J}/(\mathbf{dash})$ of \mathcal{J} and $\pi: \mathcal{J} \rightarrow \mathcal{T}$ the canonical quotient. A module of \mathcal{T} is naturally a module of \mathcal{J} ; as a representation of the quiver, it is just a representation in which the dashed arrows happen to vanish.

We give an alternative description of \mathcal{T} . Let I be the (bilateral) ideal of $\mathbb{C}\tilde{Q}$

$$(2.9) \quad I := \left(\partial\mathcal{W} \Big| \right),$$

where $(\cdot) \Big|$ means that the dashed arrows in $Q \setminus \tilde{Q}$ are set to zero. Since \tilde{Q} is acyclic, while each dashed arrow α closes a cycle in Q , I is generated by the relations

$$(2.10) \quad \left\{ \partial_\alpha \mathcal{W} \Big| = 0, \mid \alpha \in Q \setminus \tilde{Q} \right\}.$$

The relations $\partial_\alpha \mathcal{W} \Big|$ are not necessarily independent; we choose a minimal set of generators of I , R , and write b_{ij} for the number of elements in R with

⁵Below we shall refer to \mathcal{T} simply as a *factor* of \mathcal{J} .

source i and target j . Obviously,

$$(2.11) \quad 0 \leq b_{ij} \leq a_{ij} \equiv \#\{\text{dashed arrows } j \rightarrow i\}.$$

One has $\mathcal{T} = \mathbb{C}\tilde{Q}/I$. We have a functor

$$(2.12) \quad F: \text{mod } \mathbb{C}\tilde{Q}/I \rightarrow \text{mod } \mathbb{C}Q/(\partial\mathcal{W})$$

whose image are the modules of $\mathcal{J} \equiv \mathbb{C}Q/(\partial\mathcal{W})$ having vanishing arrows in $Q \setminus \tilde{Q}$. F preserves indecomposables and isoclasses. The algebra $\mathcal{T} \equiv \mathbb{C}\tilde{Q}/I$ is *triangular*.⁶ Let n be the number of nodes of the quiver \tilde{Q} . With respect to this quiver, we define the three $n \times n$ symmetric matrices A_{ij} , B_{ij} , and C_{ij} as

$$(2.13) \quad \begin{aligned} A_{ij} &= C_{ij} + a_{ij} + a_{ji}, & B_{ij} &= C_{ij} + b_{ij} + b_{ji}, \\ C_{ij} &= 2\delta_{ij} - \#\{\text{arrows } i \rightarrow j\} - \#\{\text{arrows } j \rightarrow i\}. \end{aligned}$$

C is the Cartan matrix of \tilde{Q} in the sense of Kac [34–36]; B is the Brenner matrix of the triangular algebra $\mathbb{C}\tilde{Q}/I$ [37]. The corresponding integral quadratic forms $\mathbb{Z}^n \rightarrow \mathbb{Z}$ will be denoted as

$$(2.14) \quad q_A(\mathbf{x}) = \frac{1}{2} \mathbf{x}^t A \mathbf{x}, \quad q_B(\mathbf{x}) = \frac{1}{2} \mathbf{x}^t B \mathbf{x}, \quad q_C(\mathbf{x}) = \frac{1}{2} \mathbf{x}^t C \mathbf{x}.$$

$q_A(\cdot)$ is the quadratic form of the underlying bi-graph of (Q, \tilde{Q}) , $q_B(\cdot)$ is the *Tits form* of the triangular algebra $\mathbb{C}\tilde{Q}/I$, and $q_C(\cdot)$ is the *Tits form* of the hereditary algebra $\mathbb{C}Q$. In the examples of interest $q_A(\cdot) \equiv q_B(\cdot)$; we shall write the Tits form $q_A(\cdot)$ simply as $q(\cdot)$. Two Tits forms that differs by a linear isometry defined over \mathbb{Z} are said to be \mathbb{Z} -equivalent.

Counting dimensions, one sees that, if X is an indecomposable module of \mathcal{T} ,

$$(2.15) \quad \dim \mathcal{M}_X = 1 - q(\mathbf{dim} X).$$

In particular $q(\mathbf{dim} X) \leq 1$, that is, the dimension vector of an decomposable \mathcal{T} -module is a *root* of the Tits form $q(\cdot)$. *Real roots* \mathbf{x} , i.e. $\mathbf{x} \in \Gamma$ with $q(\mathbf{x}) = 1$ (and connected support in \tilde{Q}), correspond to dimension vectors

⁶An algebra is *triangular* if it has the form $\mathbb{C}\tilde{Q}/I$ with \tilde{Q} acyclic; \mathcal{T} was *constructed* to be triangular. Other authors use instead the adjective *directed* to denote the same class of algebras. We prefer to avoid this term to prevent confusion with representation-directed algebras.

of rigid indecomposables which, if stable, yield hypermultiplets. *Imaginary* roots of $q(\cdot)$, $q(\mathbf{x}) \leq 0$, correspond to higher spin BPS multiplets: the maximal spin of a BPS particle of charge vector $\mathbf{x} \in \Gamma$ is

$$(2.16) \quad \max \text{spin}(\mathbf{x}) = (1 - q(\mathbf{x}))/2.$$

The dimension \mathbf{x} of a stable module of a triangular algebra is not merely a root of $q(\cdot)$ which is just the dimension of an indecomposable X (i.e. $\text{End } X$ is a local ring). \mathbf{x} satisfies the stronger condition of being a *Schur root* i.e. the dimension of a *brick* X (i.e. $\text{End } X = \mathbb{C}$).

2.2.2. Triangular factors and BPS sectors. A BPS particle corresponds to a (family of) stable module of the Jacobian algebra $\mathcal{J} \equiv \mathbb{C}Q/(\partial\mathcal{W})$ of the quiver with superpotential. Any module of a factor \mathcal{J}/I is naturally a module of the original algebra \mathcal{J} ; so a module of \mathcal{J}/I , if stable, corresponds to a BPS particle. To each factor algebra \mathcal{J}/I is thus associated a *sector* of the BPS spectrum⁷. If the Representation Theory of the factor \mathcal{J}/I is elementary, the BPS particles in the corresponding sector are easily determined. This observation applies, in particular, to BPS sectors described by triangular factors \mathcal{T} of \mathcal{J} , which we call *triangular sectors*. The states of a triangular sector are controlled by the Tits form of the triangular algebra \mathcal{T} . Many previous computations of BPS spectra may be rephrased in the language of triangular sectors. The simplest instance is when there is a single triangular factor \mathcal{T} whose sector happens to be the *full* BPS spectrum; a slightly more general case is when there is a family of triangular factors \mathcal{T}_A such that the union of their sectors is the full BPS spectrum. When this happens we say that the BPS chamber is *triangular*. Obviously, all chambers are triangular when the algebra \mathcal{J} itself is triangular, as in *ADE* Argyres–Douglas systems or *SU(2)* SQCD with $N_f \leq 3$. Strong coupling finite chambers for $\mathcal{N} = 2$ SYM and SQCD with simply-laced gauge groups are also examples of triangular chambers [20], as are the ‘canonical’ finite chambers of (G, G') models [5], and Arnold models [38, 39]. A further example of a triangular chamber is the finite chamber of E_6 Minahan–Nemeshanski [40]. In all these cases the BPS spectrum is controlled by the Tits forms, and is described by the root systems of their associated Lie algebras as we are going to describe.

⁷The sectors overlap: the sectors of \mathcal{J}/I_1 and \mathcal{J}/I_2 both contain the sectors associated to the common factor $\mathcal{J}/(I_1, I_2)$ of \mathcal{J}/I_1 and \mathcal{J}/I_2 .

2.3. Slodowy–GIM Lie algebras. Extended affine Lie algebras

2.3.1. Slodowy GIM Lie algebras. The usual Kac–Moody construction [41] allows to associate to each graph (without loops) a Lie algebra. The construction may be generalized to bi–graphs [42, 43]. Let $A = \{a_{ij}\}$ be $n \times n$ matrix defined in §.2.2.1, namely, a_{ij} is the symmetric⁸ integral $n \times n$ matrix, with diagonal components $a_{ii} = 2$ for all $i = 1, 2, \dots, n$, where nodes i, j are connected by $|a_{ij}|$ undashed (resp. dashed) edges if $a_{ij} < 0$ (resp. $a_{ij} > 0$). To the matrix A we attach the GIM (Generalized Intersection Matrix) Lie algebra \mathfrak{L}_A [42, 43] generated by elements

$$(2.17) \quad \{e_i, f_i, h_i\}_{i=1,2,\dots,n}$$

and satisfying the generalized Serre relations

$$(2.18) \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j$$

$$(2.19) \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_i] = h_i$$

$$(2.20) \quad \text{if } i \neq j \text{ and } a_{ij} \leq 0 \quad \begin{cases} [e_i, f_j] = 0 \\ (\text{ad } e_i)^{1-a_{ij}} e_j = (\text{ad } f_i)^{1-a_{ij}} f_j = 0 \end{cases}$$

$$(2.21) \quad \text{if } i \neq j \text{ and } a_{ij} > 0 \quad \begin{cases} [e_i, e_j] = [f_i, f_j] = 0 \\ (\text{ad } e_i)^{1+a_{ij}} f_j = (\text{ad } f_i)^{1+a_{ij}} e_j = 0. \end{cases}$$

The corresponding bi–graph is called the *Dynkin* bi–graph of \mathfrak{L}_A . The graphical conventions are the same as in the theory of the quadratic integral (Tits) forms [37]; the relevant Tits form being $q(\mathbf{x}) \equiv \frac{1}{2} \mathbf{x}^t \mathbf{A} \mathbf{x}$. A given $\mathbf{x} \in \mathbb{Z}^n$ is a *root* of \mathfrak{L}_A iff $q(\mathbf{x}) \leq 1$. A given root \mathbf{x} is *Schur* (for a chosen orientation of the bi–graph) iff it is the dimension vector of a brick of $\mathbb{C}\tilde{Q}/I$. Two Tits forms which differ by a linear isometry defined over \mathbb{Z} are said to be \mathbb{Z} –equivalent: \mathbb{Z} –equivalent Tits forms correspond to the same Slodowy–GIM Lie algebra. In particular, an algebra \mathfrak{L}_A is Kac–Moody if and only if the Dynkin bi–graph is \mathbb{Z} –equivalent to one with no dashed edges.

In this paper we are interested in the GIM Lie algebras specified by the bi–graph underlying the bi–quiver (Q, \tilde{Q}) of a triangular factor of the Jacobian algebra of a $\mathcal{N} = 2$ model. Not all GIM Lie algebras may arise from a $\mathcal{N} = 2$ QFT. A (conjectural) necessary condition in order for a GIM Lie algebra to arise from a QFT is provided by the $2d/4d$ correspondence

⁸Our definitions are not the most general ones; they correspond to the symmetric (as contrasted to symmetrizable) case.

[5, 8]. A Slodowy GIM Lie algebra \mathfrak{L}_A satisfies this condition iff there is an unipotent integral matrix S with

$$(2.22) \quad A = S + S^t,$$

such that the following two properties hold:

- i.) the quiver Q_S with incidence matrix $B = S - S^t$ and the bi-graph G_A associated to the symmetric matrix A have the same underlying graph, and the set of arrows Q_S which correspond to solid edges of G_A form an acyclic subquiver \tilde{Q}_S ;
- ii.) the *spectral radius* of the matrix $M \equiv -(S^{-1})^t S$ is 1, and the Jordan blocks of M have at most size⁹ 2, and no size 2 block corresponds to the eigenvalue -1 .

This condition just says that S is a solution to the Diophantine equations of the 2d (2, 2) classification program [7] with $\hat{c} < 2$. A GIM Lie algebra \mathfrak{L}_A which satisfies the condition is called *2d/4d admissible*. The arguments of [7] give

Fact 1. *If the Tits form q is positive semi-definite then \mathfrak{L}_A is 2d/4d admissible.*

Proof. Choose a decomposition of the matrix A satisfying i.). Let¹⁰ $v \in \text{rad } q$; then

$$(2.23) \quad 0 = -(S^{-1})^t A v = (M - 1)v$$

so $\text{rad } q$ is invariant under M . M descends to a linear map \overline{M} on $\mathbb{Q}^n / \text{rad } q \otimes \mathbb{Q}$. The quadratic form q descends to a positive-definite form \overline{q} on $\mathbb{Q}^n / \text{rad } q \otimes \mathbb{Q}$. Now [7]

$$(2.24) \quad M^t A M = S^t S^{-1} (S + S^t) (S^{-1})^t S = S + S^t = A,$$

so \overline{M} is an orthogonal transformation with respect to the positive definite inner product corresponding to the form \overline{q} . Then \overline{M} has spectral radius 1 acting on $\mathbb{Q}^n / \text{rad } q \otimes \mathbb{Q}$; hence M has spectral radius 1 acting on \mathbb{Q}^n .

⁹For complete theories [5], the blocks of size 2 can be associated only to the eigenvalue $+1$.

¹⁰We recall that the *radical* of a quadratic form $q: \Gamma \rightarrow \mathbb{Z}$ is the sublattice $\text{rad } q \subset \Gamma$ of elements $v \in \Gamma$ annihilated by the matrix A of q , i.e. such that $Av = 0$.

Moreover, \overline{M} is semisimple, so M may have Jordan blocks only associated to the eigenvalue $+1$ and the size of the Jordan blocks is at most 2. \square

2.3.2. Extended affine Lie algebras. Slodowy GIM Lie algebras with semi-definite Tits form $q: \mathbb{Z}^n \rightarrow \mathbb{Z}$ have special interest. We already saw that they are automatically 2d/4d admissible. A GIM Lie algebra \mathfrak{L}_A having the Tits form semi-definite with a radical of rank κ , is called an *extended affine Lie algebra of nullity κ* [44, 45].

Consider the quadratic form \bar{q} induced by q on the lattice $\mathbb{Z}^n/\text{rad } q$; it is positive definite. A well-known theorem [32] says that a (connected) positive-definite form is \mathbb{Z} -equivalent to an *ADE* Tits form. We say that \mathfrak{L}_A is an *extended affine Lie algebra of type \mathfrak{g}_r* , if the corresponding induced form \bar{q} is equivalent to the Tits form of the finite-dimensional (simply-laced) Lie algebra \mathfrak{g}_r .

The extended affine Lie algebras of nullity κ and type \mathfrak{g}_r are constructed recursively from the same type Lie algebras of nullity $\kappa - 1$ by ‘*affinization*’, that is, as a central extension of their loop algebra. At nullity zero we have the finite-type Lie algebra \mathfrak{g}_r , and at nullity 1 we have its affinization $\mathfrak{g}_r^{(1)}$, which is the usual affine Kac–Moody algebra [41]. At nullity 2 we have the affinization of the Kac–Moody algebra, $\mathfrak{g}_r^{(1,1)} \equiv (\mathfrak{g}_r^{(1)})^{(1)}$, which is a *toroidal Lie algebra* [46]. In general

$$(2.25) \quad \mathfrak{g}_r^{\overbrace{(1,1,\dots,1)}^{\kappa \text{ times}}} \equiv \left(\mathfrak{g}_r^{\overbrace{(1,1,\dots,1)}^{\kappa-1 \text{ times}}} \right)^{(1)}.$$

Equivalently, an extended affine Lie algebra of nullity κ and type \mathfrak{g}_r is a central extension of the Lie algebra of maps $(S^1)^\kappa \rightarrow \mathfrak{g}_r$. The central extension is not unique [45] and we have a large family of inequivalent Lie algebras. However, the root system does not depend on the particular central extension but only on κ and \mathfrak{g}_r , that is, only on the Tits form q . If $v \in \mathbb{Z}^n$ is a *root* of an extended affine Lie algebra we say that it is *real* iff $q(v) = 1$, and *imaginary* iff $q(v) = 0$. It is clear that the set of real roots is given by $\{\alpha + \delta\}$ where α is a root of the finite-type Lie algebra of type \mathfrak{g}_r and δ is an element of the lattice of imaginary roots $\text{rad } q$. Equivalently, we may choose a generator $\delta \in \text{rad } q$ and write the roots in the form $\{\hat{\alpha} + \rho\}$ where $\hat{\alpha} = \alpha + n\delta$ is a root of the affine Lie algebra $\mathfrak{g}_r^{(1)}$ and $\rho \in \text{rad } q/\mathbb{Z}\delta$. We shall find convenient to represent (non-canonically) the root lattice of an extended Lie algebra as the direct sum of the root lattice of \mathfrak{g}_r (resp. $\mathfrak{g}_r^{(1)}$), which we shall call the *reduced charge lattice* $\Gamma_{\text{red.}}$, and the lattice $\text{rad } q$ (resp. $\text{rad } q/\mathbb{Z}\delta$).

The extended affine Lie algebra of nullity κ and type \mathfrak{g}_r will be written simply as $\mathfrak{g}_r^{[\kappa]}$.

2.3.3. Triangular factors of class $\mathcal{S}[A_1]$ algebras and their Lie algebras. The quiver with superpotential of a theory of class $\mathcal{S}[A_1]$ is determined by an ideal triangulation of the corresponding ultraviolet curve [47–49], ideal arc flips being in correspondence with quiver mutations [47]. To make the correspondence complete, it is necessary to allow ideal triangulations with self-folded triangles [47] (even if all surfaces admit triangulations without self-folded triangles).

In this subsection we describe the structure of the Tits form of the triangular factors of the Jacobian algebras for class $\mathcal{S}[A_1]$ models (*class $\mathcal{S}[A_1]$ algebras* for short). In view of §.2.3.1, we adopt the Lie theoretical language.

Let Q be the quiver arising from an ideal triangulation of a surface with punctures and marked points on the boundaries, and let \tilde{Q} be a maximal acyclic subquiver of Q . We write n for the number of nodes of Q . Since the algebras of a $\mathcal{S}[A_1]$ model are tame [50], for all the associated bi-quivers (Q, \tilde{Q}) their Tits form $q: \mathbb{Z}^n \rightarrow \mathbb{Z}$ is *weakly* non-negative. The bi-quivers (Q, \tilde{Q}) such that q is non-negative *definite* will be called *good*. It is easy to check that for all surfaces (admitting an ideal triangulation) good bi-quivers always exist.

Fact 2. Σ a surface of genus g with p punctures and b boundaries, with $(p + b) > 0$, the i -th boundary carrying $\ell_i \geq 1$ marked points. Σ has Euler characteristic $\chi = 2(1 - g) - p - b$, while the number of arcs in any ideal triangulation of Σ is $n = -3\chi + \sum_i \ell_i$. Let Q be a triangulation quiver of Σ , and q the Tits form of an associated good bi-quiver. Then

- 1) if Q corresponds to an ideal triangulation of Σ without self-folded triangles, q is the Tits form of an extended affine Lie algebra which is either

$$(2.26) \quad A_{n+\chi-1}^{[-\chi+1]} \quad \text{or} \quad D_{n+\chi}^{[-\chi]},$$

- 2) if Q corresponds to an ideal triangulation with at least one self-folded triangle, q is the Tits form of an extended affine Lie algebra which is either

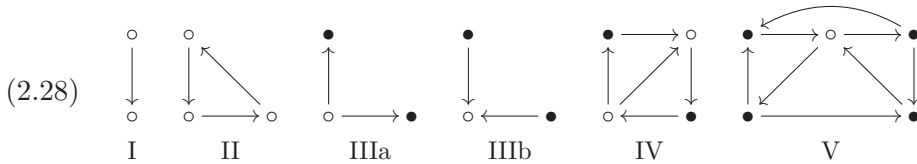
$$(2.27) \quad A_{n+\chi}^{[-\chi]} \quad \text{or} \quad D_{n+\chi}^{[-\chi]}.$$

Table 1: The geometric meaning of block Types.

block Type	n -gon
0	triangle with two sides on the boundary
I	triangle with one side on the boundary
II	triangle
III	punctured 2-gon with one side on the boundary
IV	punctured 2-gon
V	doubly punctured 1-gon

The maximal nullity κ vanishes only for $g = 0, b = 1, p = 0$, corresponding to the A_n AD models, and $g = 0, b = 1, p = 1$ which corresponds to the D_n AD models.

We sketch the proof of the above claims leaving the details to Appendix A. A quiver Q is a triangulation quiver of some Σ if and only if it can be decomposed into the following blocks of Types I – V (ref.[47] see also [8])



which have two kinds of nodes *black* \bullet and *white* \circ . All triangulation quivers are constructed by taking a finite collection of blocks of various types and gluing them together by identifying pairs of white nodes; gluing nodes belonging to the same block is forbidden. Some white node may remain unpaired. For our present purposes, it is convenient to introduce an additional block, of Type 0, which consists of a single white node \circ ; we reinterpret the unpaired white nodes as being paired with Type 0 blocks, so that in our conventions *all* white nodes of the building blocks of Types 0–V should be identified in pairs. Finally, all opposite pairs of arrows \rightleftarrows (between the same nodes), produced in the gluing process, should be deleted. We recall the geometric meaning of the block decomposition [47]: each block Type corresponds to an ideally triangulated, possibly punctured, oriented n -gon as in Table 1. White nodes correspond to the sides of the n -gon which are not boundary segments, thus internal ideal arcs along which two distinct n -gons are glued.

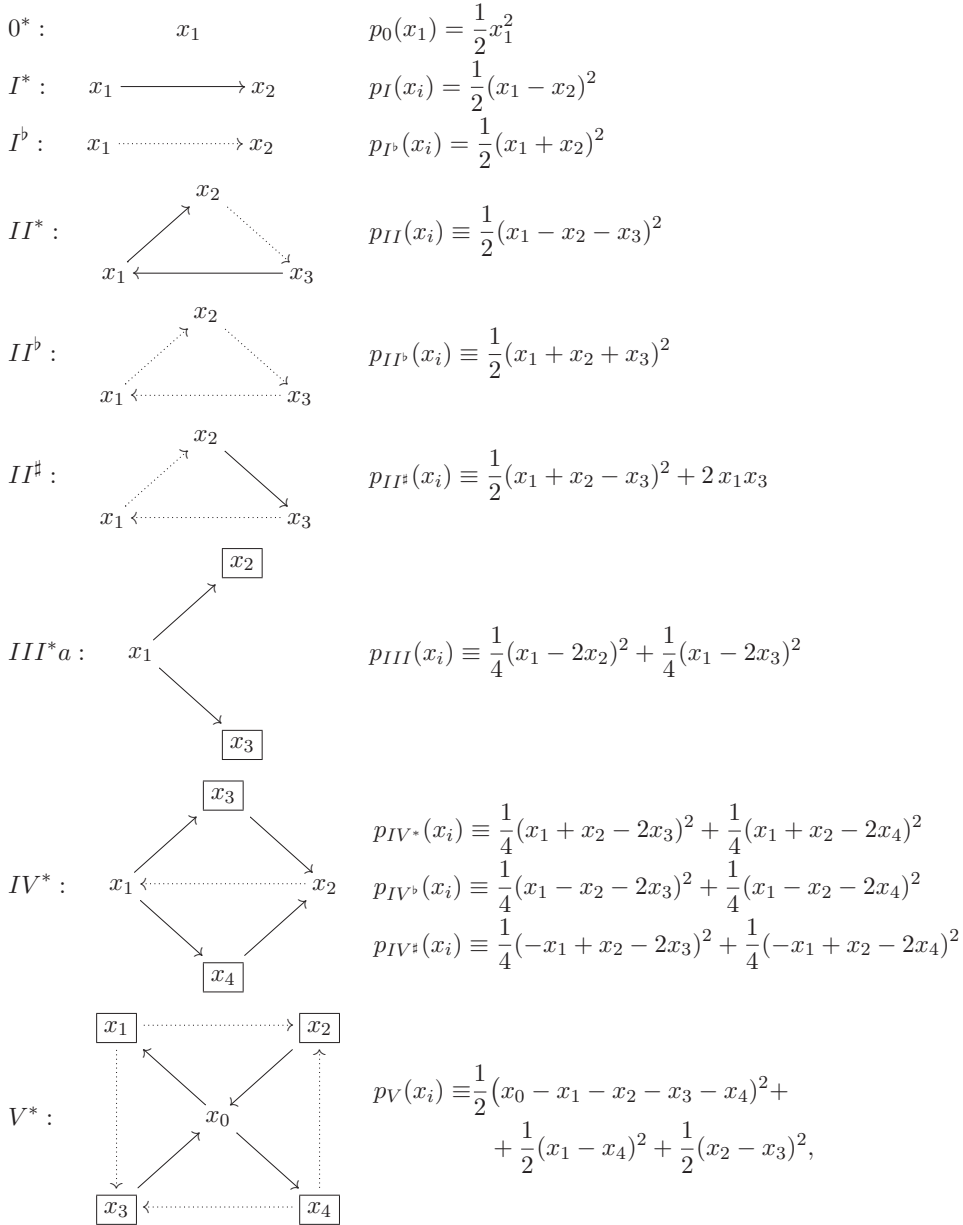


Figure 1: The collection \mathfrak{B} of bi-blocks of triangulation bi-quivers. Boxed labels correspond to the black nodes of blocks. The bi-block IV^b (IV^\sharp) is obtained from IV^* by inverting *dashed* \leftrightarrow *solid* all the arrows incident node x_2 (x_1). The bi-block of type V has other forms, we reduce to form V^* by flipping *dashed* \leftrightarrow *solid* all arrows incident to some black node. The bi-block III^*a has a second form III^*b with the arrows in the opposite direction.

A given block–decomposition of Q induces a decomposition of all its associated bi–quivers (Q, \tilde{Q}) in a finite collection \mathfrak{B} of bi–block Types, see Figure 1. By convention, in the bi–block decomposition the deleted pairs of opposite arrows \rightleftarrows are taken to be one solid and one dashed.

Attaching to the i –th node of Q an integer–valued variable x_i , the Tits form $q(\mathbf{x})$ of (Q, \tilde{Q}) is given by the sum over the polynomials $P_{\mathfrak{b}}(x_i)$ associated to the several bi–blocks

$$(2.29) \quad q(\mathbf{x}) = \sum_{\mathfrak{b} \in \mathfrak{B}} \sum_{\substack{\text{bi–blocks} \\ \text{of type } \mathfrak{b}}} p_{\mathfrak{b}}(x_i).$$

With the single exception of $p_{\text{II}^\sharp}(x_i)$, all these polynomials are sums of squares. Since $p_{\text{II}^\sharp}(x_i)$ is weakly non–negative, $q(\mathbf{x})$ is automatically weakly non–negative, and it is semi–definite iff no Type II^\sharp bi–block is present. By flipping *dashed* \leftrightarrow *solid* all arrows incident to a node of each offending II^\sharp bi–block, we get a *good* bi–quiver. For the computation of the nullity κ and type \mathfrak{g}_r of the Tits form $q(\mathbf{x})$, see Appendix A.

2.4. Galois covers of (locally) bounded \mathbb{C} –categories

2.4.1. Covering functors. We start by recalling some standard definitions [17, 18, 51–53] (for a nice survey see [54]).

A category \mathcal{A} is called \mathbb{C} –*linear* if, for all ordered pairs of objects, X, Y , the set of morphisms $\mathcal{A}(X, Y)$ has a \mathbb{C} –vector–space structure and their composition law \circ is bilinear. The category is *connected* if it is not empty nor the disjoint union of two non–empty categories. A connected \mathbb{C} –category \mathcal{A} is *locally bounded* if the following three conditions are satisfied:

- (a) distinct objects of \mathcal{A} are non–isomorphic;
- (b) for all object X , $\mathcal{A}(X, X)$ is a local algebra;
- (c) for all $X \in \mathcal{A}$, $\sum_{Y \in \mathcal{A}} (\dim_{\mathbb{C}} \mathcal{A}(X, Y) + \dim_{\mathbb{C}} \mathcal{A}(Y, X)) < \infty$.

A *bounded* category is a locally bounded category with finitely many objects.

A (locally) bounded category \mathcal{A} may always be written (uniquely) in the form $\mathbb{C}Q/I$, where Q is a (locally) finite quiver and I an admissible ideal in its path category [17, 18]. The objects of \mathcal{A} are identified with the nodes of Q . The identity morphism of the i –th object $i \in \mathcal{A}$ is identified with the minimal idempotent e_i of the algebra $\mathbb{C}Q/I$ given by the lazy path at the

i -th node of Q . One has

$$(2.30) \quad \mathcal{A}(i, j) = e_j \mathbb{C}Q/I e_i.$$

The bounded category \mathcal{A} is then canonically identified with the basic associative algebra $\mathbb{C}Q/I$, and we shall use the two languages interchangeably. Viewing \mathcal{A} as a category, however, allows to perform general categorical operations on it, in particular to construct its *covering* and *orbit* categories. In the categorical language, a (finite dimensional) representation X of the (opposite) quiver bounded by I , which is the same as a right module of the algebra $\mathbb{C}Q/I$, is seen as a *linear functor* from the \mathbb{C} -category \mathcal{A} to the \mathbb{C} -category of finitely-dimensional \mathbb{C} -vector-spaces

$$(2.31) \quad X: \mathcal{A} \rightarrow \text{mod } \mathbb{C}.$$

Such a functor is called a *module* of the bounded category \mathcal{A} , and the (Abelian) category of such functors is denoted as $\text{mod } \mathcal{A}$.

Let \mathcal{A} be a locally bounded \mathbb{C} -category and \mathbb{G} a group of \mathbb{C} -linear automorphisms of \mathcal{A} . The group \mathbb{G} acts on $\text{mod } \mathcal{A}$ by composition of functors

$$(2.32) \quad X \longmapsto X^g \equiv X \circ g.$$

This action is an autoequivalence and so:

- (a) preserves indecomposable, brick, simple, projective, and injective modules;
- (b) commutes with the Auslander–Reiten translations τ, τ^- .

To each $X \in \text{mod } \mathcal{A}$ we associate its *isotropy subgroup* $\mathbb{G}_X \subset \mathbb{G}$

$$(2.33) \quad \mathbb{G}_X = \left\{ g \in \mathbb{G} \mid X^g \cong X \right\}.$$

Let $\mathbb{H} \subseteq \mathbb{G}$ be a subgroup; we write $\text{mod}^{\mathbb{H}} \mathcal{A}$ for the full subcategory of \mathbb{H} -invariant modules.

An automorphism group \mathbb{G} is said to be *admissible* if it acts freely on the objects i of \mathcal{A} (i.e. on the nodes of its quiver) and has finitely many orbits. If \mathbb{G} is admissible for \mathcal{A} , we may consider its *orbit category* \mathcal{A}/\mathbb{G} which is a bounded \mathbb{C} -category. The objects of the category $\mathcal{B} \equiv \mathcal{A}/\mathbb{G}$ are the \mathbb{G} -orbits

$\mathbb{G}i$ of objects $i \in \mathcal{A}$, while the morphism spaces are

$$(2.34) \quad \mathcal{B}(\mathbb{G}i, \mathbb{G}j) = \bigoplus_{g \in \mathbb{G}} \mathcal{A}(i, gj).$$

Then we have a canonical *Galois covering* functor

$$(2.35) \quad F: \mathcal{A} \longrightarrow \mathcal{A}/\mathbb{G} \quad \text{given on objects by } i \longmapsto \mathbb{G}i.$$

Since \mathcal{A}/\mathbb{G} is again a bounded \mathbb{C} -category, there is a finite quiver \overline{Q} (whose nodes are the \mathbb{G} -orbits of nodes of Q) and an admissible ideal \overline{I} such that $\mathcal{A}/\mathbb{G} = \mathbb{C}\overline{Q}/\overline{I}$.

The process of going from the category \mathcal{A} and the freely acting automorphism group \mathbb{G} to the orbit category \mathcal{A}/\mathbb{G} may be reversed. Indeed, the definition (2.34) shows that the morphism spaces of \mathcal{A}/\mathbb{G} are vector spaces graded by elements $g \in \mathbb{G}$, the grading being compatible with compositions. By definition, this means that \mathcal{A}/\mathbb{G} is a \mathbb{G} -graded \mathbb{C} -category. Given a \mathbb{G} -graded \mathbb{C} -category \mathcal{B} we may construct the *smash product* \mathbb{C} -category, $\mathcal{B}\#\mathbb{G}$, whose objects are pairs (a, g) , $a \in \mathcal{B}$, $g \in \mathbb{G}$, and morphism spaces [54]

$$(2.36) \quad \mathcal{B}\#\mathbb{G}((a, g), (b, h)) = \mathcal{B}(a, b) \Big|_{\substack{\text{degree } h^{-1}g \\ \text{component}}}.$$

It is easy to see that, if \mathbb{G} acts freely on \mathcal{A} ,

$$(2.37) \quad (\mathcal{A}/\mathbb{G})\#\mathbb{G} \cong \mathcal{A}.$$

The theory of Galois covers of \mathbb{C} -categories has much the same flavor as the classical theory of Galois field extensions. A field k (say of zero characteristic) has an universal extension, namely its algebraic closure k^{al} , and universal Galois group $\text{Gal}(k^{\text{al}}/k)$. The fundamental theorem of the classical theory says that there is an inclusion reversing correspondence

$$(2.38) \quad \left\{ \text{normal subgroups } N \triangleleft \text{Gal}(k^{\text{al}}/k) \text{ of finite index} \right\} \longleftrightarrow \left\{ \text{finite degree Galois extensions of } k \right\}.$$

In the same fashion, a (locally bounded) \mathbb{C} -category \mathcal{A} admits an *universal Galois cover* $\tilde{\mathcal{A}}$ and an universal covering group $\Pi(\mathcal{A})$ called the *fundamental group* of the category \mathcal{A} . The fundamental group of \mathcal{A} is explicitly

constructed using the *walks* in the quiver Q of \mathcal{A} ; we shall not use this construction and refer to [51] for details. All locally bounded \mathbb{C} -category \mathcal{A} is automatically $\Pi(\mathcal{A})$ -graded. Eqn.(2.37) then gives an explicit formula for its universal cover category

$$(2.39) \quad \tilde{\mathcal{A}} = \mathcal{A} \# \Pi(\mathcal{A}).$$

The Galois correspondence then becomes

$$(2.40) \quad \{ \text{normal subgroups } \mathbb{H} \triangleleft \Pi(\mathcal{A}) \} \longleftrightarrow \{ \text{Galois coverings } \mathcal{C} \rightarrow \mathcal{A} \},$$

where the Galois group of $\mathcal{C} \rightarrow \mathcal{A}$ is $\mathbb{G} \equiv \Pi(\mathcal{A})/\mathbb{H}$, and $\mathcal{C} = \mathcal{A} \# (\Pi(\mathcal{A})/\mathbb{H})$. An algebra \mathcal{A} with $\Pi(\mathcal{A}) = (1)$ has no non-trivial covers; such an algebra, if connected, is said to be *simply-connected*.

We are not interested in the Galois cover of \mathbb{C} -categories $F: \mathcal{A} \rightarrow \mathcal{B}$ *per se*, but rather in the functors it induces between the module categories $\text{mod } \mathcal{A}$ and $\text{mod } \mathcal{B}$. For simplicity, we assume all categories to be bounded (so \mathbb{G} is a finite group of order $|\mathbb{G}|$) and all modules to be finite-dimensional; more general statements may be found in the literature [17, 18, 51–54]. A Galois cover $F: \mathcal{A} \rightarrow \mathcal{B} \equiv \mathcal{A}/\mathbb{G}$ induces two natural functors between the module categories:

- the *pull up* functor $F^\lambda: \text{mod } \mathcal{B} \rightarrow \text{mod } \mathcal{A}$ defined by composition of functors

$$(2.41) \quad F^\lambda: X \longmapsto F^\lambda X \equiv X \circ F;$$

- the *push down* functor $F_\lambda: \text{mod } \mathcal{A} \rightarrow \text{mod } \mathcal{B}$ is the map which associates to the functor $Y: \mathcal{A} \rightarrow \text{mod } \mathbb{C}$ the functor $F_\lambda Y: \mathcal{B} \rightarrow \text{mod } \mathbb{C}$ acting as follows

(2.42)

$$\diamond \text{ on objects } \mathbb{G}i: \quad \mathbb{G}i \longmapsto F_\lambda Y(\mathbb{G}i) = \bigoplus_{g \in \mathbb{G}} Y(gi)$$

(2.43)

$$\diamond \text{ on morphisms } \mathbb{G}i \xrightarrow{f} \mathbb{G}j: \quad \text{Eqn. (2.34)} \Rightarrow f = \sum_{g \in \mathbb{G}} f_g \text{ with } f_g \in \mathcal{A}(i, gj)$$

$$\text{then } F_\lambda Y(f) = \sum_{g \in \mathbb{G}} Y(f_g).$$

Properties [54]:

- 1) the categories $\text{mod}^{\mathbb{G}}\mathcal{A}$ and $\text{mod}\mathcal{B}$ are equivalent;
- 2) for all $X \in \text{mod}\mathcal{A}$ and all $g \in \mathbb{G}$ we have $F_\lambda X^g \cong X$ and $F^\lambda F_\lambda X \cong \bigoplus_{g \in \mathbb{G}} X^g$;
- 3) F_λ and F^λ are each other right- and left-adjoints:

$$\begin{aligned} \mathcal{A}(X, F^\lambda Y) &\cong \mathcal{B}(F_\lambda X, Y), \\ \mathcal{A}(F^\lambda Y, X) &\cong \mathcal{B}(Y, F_\lambda X) \quad \forall X \in \text{mod}\mathcal{A}, Y \in \text{mod}\mathcal{B}. \end{aligned}$$

The crucial property for our applications is

Fact 3 [54]. *\mathbb{G} an admissible group of automorphisms of \mathcal{A} . Suppose the \mathcal{A} -module X is indecomposable and $\mathbb{G}_X = (1)$. Then $F_\lambda X$ is indecomposable and for all modules Y with $F_\lambda Y \simeq F_\lambda X$ there is $g \in \mathbb{G}$ such that $Y \simeq X^g$.*

2.4.2. Important special cases. A special case, which is relevant for complete $\mathcal{N} = 2$ theories [8], is when \mathcal{A} is a *string* algebra. The indecomposable modules of string algebras have an explicit construction in terms of strings and bands [55] (see also [21]). For convenience of the reader we briefly review their construction in Appendix B. If C is a string (band) we write $M(C)$ (resp. $M(C, \mu, n)$) with $\mu \in \mathbb{C}$ and $n \in \mathbb{N}$ for the corresponding module.

Let \mathcal{A} be a string algebra with a freely acting automorphism group \mathbb{G} , and let $F: \mathcal{A} \rightarrow \mathcal{B} \equiv \mathcal{A}/\mathbb{G}$ be the associated Galois cover. The first observation is that \mathcal{B} is automatically a string algebra. Then, if $M(C) \in \text{mod}\mathcal{A}$ is a string module, its push down $F_\lambda M(C) \in \text{mod}\mathcal{B}$ is a string module with respect to the image string

$$(2.44) \quad F_\lambda M(C) = M(FC),$$

and hence is automatically indecomposable¹¹ (but the push down of a brick may be just indecomposable). On the contrary, the push down of a band is not necessarily a band, since it may be a non-trivial power of a shorter band. In facts, it is a $|\mathbb{G}_C|$ power, where $\mathbb{G}_C \equiv \mathbb{G}_{M(C, \mu, n)}$ is the isotropy subgroup of the covering band. Thus, a band module $M(C, \mu, n)$ pushes down to an indecomposable module $M(FC, \mu, n)$ if and only if its isotropy group is trivial, in agreement with **Fact 3**.

¹¹This may also be seen from **Fact 3**. Indeed, for all string module $M(C)$, one has $\mathbb{G}_{M(C)} = (1)$.

The pull back of a string of \mathcal{B} is the disconnected union of $|\mathbb{G}|$ strings of \mathcal{A} . By **Properties 2)** this means that the restriction of F_λ to strings is surjective (i.e. all string modules of \mathcal{B} are push down of string modules of \mathcal{A}). In other words, F_λ sets a $|\mathbb{G}|-to-1$ correspondence between string modules of \mathcal{A} and string modules of \mathcal{B} .

The pull back of a band is, in general, a disconnected union of bands which forms one orbit under \mathbb{G} . It produces an indecomposable module $M(C', \mu', n) \in \text{mod } \mathcal{A}$ precisely when this orbit contains a single element, that is, $\mathbb{G}_{M(C', \mu', n)} \equiv \mathbb{G}$. In this case

$$(2.45) \quad F^\lambda M(C, \mu, n) = M(F^{-1}C, \mu^{|\mathbb{G}|}, n).$$

Bricks. A band module is a brick if and only if it has the form $M(C, \mu, 1)$. Then, if $\mathbb{G}_C = (1)$, the indecomposable module $F_\lambda M(C, \mu, 1)$ is also a brick. If $M(C)$ is a string module of \mathcal{A} which is not a brick, the string module $F_\lambda M(C) \in \text{mod } \mathcal{B}$ cannot be a brick. On the contrary, if $M(C)$ is a brick, $F_\lambda M(C)$ may or may not be a brick. Let $C' = FC$. The module $M(C') \in \text{mod } \mathcal{B}$ may fail to be a brick only if C' contains a substring C'' which starts and ends at the same node, which then is a power of a band B . $M(C')$ is a brick if for all such bands B one of the following two conditions are fulfilled

$$(2.46) \quad \begin{aligned} & 1) \quad \sum_{i \in \text{sources of } B} (\mathbf{dim } M(C'))_i \neq \sum_{i \in \text{sinks of } B} (\mathbf{dim } M(C'))_i, \\ & 2) \quad \mathbf{dim } M(C')|_{\text{supp } M(B, \mu, 1)} \leq \mathbf{dim } M(B, \mu, 1), \end{aligned}$$

where sinks and sources are counted with their multiplicities in B .

Triangular string algebras. When \mathcal{A} (and hence \mathcal{B}) happens to be, in addition, a triangular algebra, the dimension vectors of its indecomposable modules are roots of the Tits form q_A which, in the string case, is positive semi-definite (string algebras are tame [37, 55]). Band modules correspond to imaginary roots of q_A , while string modules correspond to real roots unless they are in the projective closure of \mathbb{C} -families of band modules in which case they are imaginary. Hence

$$(2.47) \quad \begin{aligned} q_A(\mathbf{dim } X) = 0 & \Rightarrow q_B(\mathbf{dim } F_\lambda X) = 0 \\ q_A(\mathbf{dim } X) = 1 & \Rightarrow q_B(\mathbf{dim } F_\lambda X) = \begin{cases} 1 & F_\lambda X \text{ rigid: } \text{Ext}^1(X, X) = 0 \\ 0 & F_\lambda X \text{ has self-extensions.} \end{cases} \end{aligned}$$

2.4.3. Galois covers of dashed categories. By a *dashed category* we mean a triple $(\mathcal{J}, \mathcal{T}, \varpi)$ where \mathcal{J} and \mathcal{T} are bounded \mathbb{C} -categories such

that, as algebras, $\mathcal{J} \equiv \mathbb{C}Q/(\partial\mathcal{W})$ is the Jacobian algebra of a 2–acyclic quiver Q with superpotential \mathcal{W} , while \mathcal{T} is its factor defined by an admissible dashing of Q , i.e. $\mathcal{T} \equiv \mathcal{J}/(\text{dash})$. $\varpi: \mathcal{J} \rightarrow \mathcal{T}$ is then the canonical quotient functor.¹² Clearly one may also consider dashed categories which are only *locally* bounded.

Let \mathbb{G} be a group of admissible automorphisms of \mathcal{J} which preserves the ideal (dash), that is, which sends dashed (solid) arrows of Q into dashed (solid) arrows. Then \mathbb{G} is a group of admissible automorphisms of the factor \mathcal{T} , and we have a commutative diagram of functors

$$(2.48) \quad \begin{array}{ccc} \mathcal{J} & \xrightarrow{F} & \mathcal{J}/\mathbb{G} \\ \varpi \downarrow & & \downarrow \bar{\varpi} \\ \mathcal{T} & \xrightarrow{\bar{F}} & \mathcal{T}/\mathbb{G} \end{array}$$

The induced functors $\bar{F}^\lambda, \bar{F}_\lambda$ relate corresponding triangular sectors of $\text{mod } \mathcal{T}$ and $\text{mod } \mathcal{T}/\mathbb{G}$.

2.4.4. Covers of class $\mathcal{S}[A_1]$ dashed categories. We specialize to the case in which \mathcal{J} is a Jacobian category of a class $\mathcal{S}[A_1]$ $\mathcal{N} = 2$ QFT. Modulo a handful of exceptions,¹³ we may assume that the corresponding quiver Q has a unique block decomposition without type V blocks¹⁴; and that the *good* biquiver (Q, \tilde{Q}) associated to the factor \mathcal{T} has no type II^b bi–block, cfr. §.2.3.3. Then the Tits form $q_{\mathcal{T}}$ of \mathcal{T} is written (in a unique way) as a sum of squares associated to the several bi–blocks, Eqn. (2.29).

Let h be an automorphism of the bi–quiver. h maps a (bi)block decomposition into a (bi)block decomposition. Since the decomposition is unique, h maps each bi–block into a bi–block of the same Type, and hence acts by permutations of same–Type bi–blocks. Suppose a bi–block b is fixed by the

¹²‘Dashed categories’ are \mathbb{C} –categories. Its objects are the nodes of the quiver Q , and the morphisms are pairs of morphisms of the form $(f, \varpi(f))$.

¹³The argument may be extended to cover the exceptional cases. In facts, for the validity of the argument it suffices that the block decomposition is unique in some *class* of decompositions, e.g. the decompositions with minimal or maximal number of pairs of opposite arrows \Leftarrow , or the ones with minimal/maximal number of blocks of a given Type, *etc.*

¹⁴All our conclusions remain valid in presence of Type V blocks. However, the assumption that no such block is present allows to shorten our arguments. The straightforward extension to the case in which Type V blocks are present is left to the reader.

permutation, so that $h|_b$ is an automorphism of the bi-block b . Since no bi-block $\neq II^b$ has a non trivial automorphism acting freely on its nodes, h should fix some node of b . We conclude that all freely acting automorphisms g of the dashed category map bi-blocks into *distinct* same-Type bi-blocks. In particular, nodes i and gi belong to distinct bi-blocks for all $g \in \mathbb{G}$, $g \neq 1$.

Let $X \in \text{mod } \mathcal{T}$ be an indecomposable, so $q_{\mathcal{T}}(\mathbf{dim } X) = 0, 1$. One has

$$(2.49) \quad 2q_{\mathcal{T}}(\mathbf{dim } X) = \sum_{\substack{\text{orbits of} \\ 0, I, II}} \sum_{g \in \mathbb{G}} y_{gb}^2 + \frac{1}{2} \sum_{\substack{\text{orbits of} \\ III, IV}} \left\{ \sum_{g \in \mathbb{G}} w_{gb,1}^2 + \sum_{g \in \mathbb{G}} w_{gb,2}^2 \right\},$$

$$(2.50) \quad 2q_{\mathcal{T}/\mathbb{G}}(\mathbf{dim } F_{\lambda} X) = \sum_{\substack{\text{orbits of} \\ 0, I, II}} \left(\sum_{g \in \mathbb{G}} y_{gb} \right)^2 + \frac{1}{2} \sum_{\substack{\text{orbits of} \\ III, IV}} \left\{ \left(\sum_{g \in \mathbb{G}} w_{gb,1} \right)^2 + \left(\sum_{g \in \mathbb{G}} w_{gb,2} \right)^2 \right\}$$

where y_b , (resp. $w_{b,1}$, $w_{b,2}$) is the \mathbb{Z} -linear combinations of the $(\mathbf{dim } X)_i$ appearing in the polynomial P_b of the b -th block (see Eqns. (A.2)–(A.5)); the essential point is that $y_b, w_{b,1}, w_{b,w} \in \mathbb{Z}$ while $w_{b,1} = w_{b,2} \pmod 2$. If $\mathbf{dim } X$ is an imaginary root, $y_b = w_{b,s} = 0$ for all b , and so $\mathbf{dim } F_{\lambda} X$ is also an imaginary root. If $\mathbf{dim } X$ is a real root, we have three possibilities:

- i)* at one III/IV block b we have either $w_{b,1} = \pm 2, w_{b,2} = 0$ or $w_{b,1} = 0, w_{b,2} = \pm 2$, and at all other blocks $y_c = w_{c,1} = w_{c,2} = 0$;
- ii)* for a type III/IV block b and a 0/I/II block c we have $w_{b,s} = \pm 1$ and $y_c = \pm 1$, while $y_d = w_{d,1} = w_{d,2} = 0$ for $d \neq b, c$;
- iii)* for a pair of distinct I/II blocks¹⁵ b, c we have $y_b = \pm 1$ and $y_c = \pm 1$, all other variables being zero.

In cases *i)* and *ii)* $\mathbf{dim } F_{\lambda} X$ is automatically a real root of $q_{\mathcal{T}/\mathbb{G}}(\cdot)$. In case *iii)* $\mathbf{dim } F_{\lambda} X$ is also a real root, *unless* there exists a $g \in \mathbb{G}$ such that $c = gb$, while y_b and y_c have opposite signs. In this last case $\mathbf{dim } F_{\lambda} X$ is an imaginary root. The restriction of X to the two bi-blocks b and gb must

¹⁵The case of a pair of distinct Type 0 blocks is excluded since, at a Type 0 block b , $y_b \equiv (\mathbf{dim } F_{\lambda} X)_b \geq 0$. The same argument rules out pairs of Type I^p or Type II^p bi-blocks.

have one of two possible forms: if b is of Type I

$$(2.51) \quad X|_b \equiv \mathbb{C}^m \longrightarrow \mathbb{C}^{m+1} \quad X|_{gb} \equiv \mathbb{C}^{k+1} \longrightarrow \mathbb{C}^k$$

the arrow being mono in the left figure and epi in the right one, or, for Type II,

$$(2.52) \quad X|_b \equiv \begin{array}{ccc} & \mathbb{C}^\ell & \\ \uparrow & \cdots \searrow & \\ \mathbb{C}^{\ell+m+1} & \longleftarrow & \mathbb{C}^m \end{array} \quad X|_{gb} \equiv \begin{array}{ccc} & \mathbb{C}^k & \\ \uparrow & \cdots \searrow & \\ \mathbb{C}^{k+j} & \longleftarrow & \mathbb{C}^{j+1} \end{array}$$

the vertical arrows being epi, and the horizontal arrow mono for the left figure and epi on the kernel of the vertical arrow for the right one. In both cases the module

$$(2.53) \quad F_\lambda X|_{\substack{\text{orbit} \\ \text{of } b}} = \bigoplus_{h \in \mathbb{G}} X|_{hb}$$

admits a one-parameter deformation which for Type I makes the arrow to be iso, and for Type II makes the horizontal arrow iso on the kernel of the vertical arrow. Then in both cases, whenever $\mathbf{dim} X$ is a real root while $\mathbf{dim} F_\lambda X$ is an imaginary root, we conclude that $F_\lambda X$ belongs to the projective closure of a \mathbb{P}^1 family of indecomposable modules¹⁶. This result generalizes Eqn. (2.47) to $\mathcal{S}[A_1]$ dashed categories which are not (nor can be reduced to) string algebras.

2.4.5. $\Pi(\mathcal{J})$ versus $\pi_1(\Sigma)$. Let Σ be a surface with punctures and marks on the boundaries, Q the quiver of an ideal triangulation of Σ [47], and $\mathcal{J} = \mathbb{C}Q/(\partial\mathcal{W})$ the corresponding Jacobian algebra/category¹⁷. The authors of [56] pose the question of the relation between the fundamental group of the algebra, $\Pi(\mathcal{J})$, and the fundamental group of the corresponding surface $\pi_1(\Sigma)$. On the nose, there cannot be any simple relation: $\pi_1(\Sigma)$ is a topological invariant of the surface Σ , while $\Pi(\mathcal{J})$ depends on the particular ideal triangulation as the following example shows.

¹⁶ $F_\lambda X$ is indecomposable. Indeed, let $h \in \mathbb{G}_X$. h should fix both bi-blocks b and gb , but an automorphism fixing a bi-block necessarily fixes a node, while \mathbb{G}_X acts freely on the nodes. Hence $\mathbb{G}_X = (1)$, and the claim follows from **Fact 3**.

¹⁷We shall refer to \mathbb{C} -categories arising this way from an ideal triangulation of a surface with punctures and marks as a class $\mathcal{S}[A_1]$ category.

Example. The mutation class of the punctured disk with r marks on the boundary contains the Dynkin quiver of type D_r as well as the quiver consisting on a single oriented cycle of length r , C_r , with \mathcal{W} the cycle itself. The first Jacobian algebra, $\mathcal{J}_1 \equiv \mathbb{C}D_r$, has a tree quiver, and hence is simply-connected $\Pi(\mathcal{J}_1) = 0$, while the second one $\mathcal{J}_2 = \mathbb{C}C_r/(\partial\mathcal{W})$ is well known to have $\Pi(\mathcal{J}_2) = \mathbb{Z}$.

However, we are not really interested in $\Pi(\mathcal{J})$. This universal covering group describes, *via* the Galois correspondence (2.40), *all* possible coverings of \mathcal{J} . Most of the covering categories $\mathcal{A} \rightarrow \mathcal{J}$ described by the universal correspondence do not correspond to any $\mathcal{N} = 2$ QFT. A bounded \mathbb{C} -category \mathcal{A} should satisfy quite stringent conditions in order to correspond to a QFT: to the very least, its quiver Q should be 2-acyclic and its ideal I should arise from the differential of a non-degenerate superpotential \mathcal{W} (in facts, we know that there are additional, more restrictive, necessary conditions [5]). Morally speaking, the “universal covering group” we are really interested in, “ $\Pi(\mathcal{J})^{\text{QFT}}$ ”, is the group whose finite quotients are the Galois groups of the covers of \mathcal{J} by \mathbb{C} -categories which *do correspond* to $\mathcal{N} = 2$ QFTs. We shall not attempt here to construct rigorously a would be ‘universal quantum-field-theoretical Galois covering’. However, for the special case that \mathcal{J} is a class $\mathcal{S}[A_1]$ category, we describe all possible Galois covers $\mathcal{C} \rightarrow \mathcal{J}$ of the class $\mathcal{S}[A_1]$ category \mathcal{J} by connected bounded \mathbb{C} -categories \mathcal{C} which do correspond to *complete* $\mathcal{N} = 2$ QFTs.

Fact 4. *Let $F: \mathcal{J}' \rightarrow \mathcal{J}$ be a Galois cover of connected bounded \mathbb{C} -categories, where \mathcal{J}' , \mathcal{J} are Jacobian categories of class $\mathcal{S}[A_1]$ theories. In terms of Gaiotto UV surfaces Σ'_G , Σ_G and quadratic differentials ϕ'_2 , ϕ_2 [4], F is induced by a non-constant holomorphic map $\xi: \Sigma'_G \rightarrow \Sigma_G$ whose only branch points are at the irregular punctures of ϕ_2 . One has*

$$(2.54) \quad \phi'_2 = \xi^* \phi_2.$$

Equivalently, F is induced by an unbranched cover of bordered surfaces $c: \Sigma' \rightarrow \Sigma$ [8].

Fact 4 is proven in Appendix C. We recall that the unramified covers $c: \Sigma' \rightarrow \Sigma$ are classified by the normal subgroups of $\pi_1(\Sigma^\circ)$, where Σ° is Σ with the regular punctures ignored.

Let the quotient surface Σ have genus g , p punctures, and b boundaries carrying $\{\ell_i\}_{i=1}^b$ marks. Let $\xi: \Sigma'_G \rightarrow \Sigma_G$ be a covering of degree d which is ramified only over the irregular punctures corresponding to the boundaries of Σ [8]. The inverse image of the i -th irregular puncture is a set of s_i

ramification points with ramification numbers $m_{j,i} \geq 1$ ($j = 1, \dots, s_i$, $i = 1, \dots, b$) satisfying $\sum_j m_{j,i} = d$. The covering surface Σ' has then $b' = \sum_i s_i$ boundary components with $\ell'_{j,i} = m_{j,i} \ell_i$ marks on them. The topological invariants of the covering surface Σ' are

$$(2.55) \quad p' = dp, \quad b' = \sum_{i=1}^b s_i, \quad \ell'_{j,i} = m_{j,i} \ell_i,$$

$$(2.56) \quad 2g' - 2 = (2g - 2)d + (bd - b'), \quad \sum_{j=1}^{s_i} m_{ji} = d,$$

$$(2.57) \quad n' \equiv -3\chi(\Sigma') + \sum_{j,i} \ell'_{j,i} = nd.$$

Here n (n') is the number of ideal arcs in any ideal triangulation of Σ (Σ'). It follows that, if \mathbb{T} is an ideal triangulation of Σ , then $\mathbb{T}' \equiv \mathbf{c}^{-1}(\mathbb{T})$ is an ideal triangulation of Σ' .

By Hurwitz theory [58], the covering group \mathbf{G} of \mathbf{c} is a subgroup of the symmetric group \mathfrak{S}_d , acting *transitively* on $(1, 2, \dots, d)$, which is generated by the $b + 2g$ elements of \mathfrak{S}_d

$$(2.58) \quad \alpha_1, \dots, \alpha_b, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_g,$$

which form a representation of $\pi_1(\Sigma^\circ)$, i.e. satisfy the relation

$$(2.59) \quad \alpha_1 \cdots \alpha_b \beta_1 \gamma_1 \beta_1^{-1} \gamma_1^{-1} \cdots \beta_g \gamma_g \beta_g^{-1} \gamma_g^{-1} = 1,$$

and such that α_i belongs to the conjugacy class of \mathfrak{S}_d specified by the partition of d

$$(2.60) \quad m_{1,i} + m_{2,i} + \cdots + m_{s_i,i} = d.$$

So $\mathbf{G} = \pi_1(\Sigma^\circ)/N$ for some normal subgroup $N \triangleleft \pi_1(\Sigma^\circ)$. Conversely, any transitive subgroup $\mathbf{G} \subset \mathfrak{S}_d$ satisfying (2.58)–(2.60) corresponds to a cover \mathbf{c} .

By construction, \mathbf{G} is an automorphism group of the ideal triangulation \mathbb{T}' , hence of the covering Jacobian category \mathcal{J}' . However, it is not necessarily true that \mathbf{G} acts freely on the objects of \mathcal{J}' , i.e. on the arcs of \mathbb{T}' , as the following example shows:

Example 1. Let Σ be the pair of pants (a genus zero surface with three boundaries). By definition, a covering map $\xi: \Sigma'_G \rightarrow \Sigma_G \equiv \mathbb{P}^1$, which is branched only at the three irregular punctures, is given by a Belyi function ξ [59]. A well known example of degree d rational Belyi function is

the Chebyshev polynomial of the first kind, $T_d(z)$. Identifying the symmetric group \mathfrak{S}_d with the Weyl group W of $SU(d)$, we have $G = \langle \sigma_e, \sigma_o \rangle \subset W$, where σ_e (σ_o) is the product of the simple reflections at the even (odd) nodes of the bipartite Dynkin graph A_{d-1} . Clearly, $G \cong D_d$, the dihedral group of degree d , i.e. the symmetry group of a regular d -gon. D_d does not act freely on the vertices of the d -gon. Correspondingly, G does not act freely on the arcs of Γ' .

However, in this example we have a subgroup $\mathbb{G} \equiv \langle \text{Cox} \rangle \subset G$, i.e. the cyclic group \mathbb{Z}_d generated by the Coxeter element $\text{Cox} \equiv \sigma_e \sigma_o \in W$, which does act *freely* and *transitively* on the G -orbits. Then, as \mathbb{C} -categories, $\mathcal{J} = \mathcal{J}'/\mathbb{G}$, and it is the subgroup \mathbb{G} which is the *categorical* Galois group (while G is the Galois group in the usual sense). Hence

Fact 5. *The covering of bordered surfaces c corresponds to a Galois \mathbb{G} -cover of class $\mathcal{S}[A_1]$ categories iff there is a subgroup $\mathbb{G} \subseteq G \subseteq \mathfrak{S}_d$ which acts freely and transitively. In particular $d \equiv |\mathbb{G}|$.*

It is easy to find examples with $\mathbb{G} = G$ and Abelian (see section 3). We present a simple example of *non-Abelian* covering group G such that $\mathbb{G} \equiv G$.

Example 2. In the same context as in the previous example, consider the Belyi rational function of degree 6 given by the expression of the modular invariant $J(\tau)$ in terms of the Legendre $\Gamma(2)$ -modular function $\lambda(\tau)$

$$(2.61) \quad J(\lambda) = \frac{4}{27} \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2}.$$

One has $G \equiv PSL(2, \mathbb{Z})/\Gamma(2) \cong \mathfrak{S}_3$ [61]. Since G acts transitively as a subgroup of \mathfrak{S}_6 , and, since $|G| = 6$, G also acts freely, i.e. $\mathbb{G} = G$. In physical terms, in the minimal case (i.e. Σ without punctures and a single mark per boundary) Eqn. (2.61) yields the Galois \mathfrak{S}_3 -cover of $SU(2)^3$ SYM with one half-trifundamental hypermultiplet by $SU(2)^{13}$ SYM coupled to one half-trifundamental in each of the six representations of the form

$$(2.62) \quad \left(\overbrace{\mathbf{1}, \dots, \mathbf{1}}^{\text{even \# of 1's}}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \dots, \mathbf{1} \right)$$

and a fundamental quark coupled to each *odd-numbered* $SU(2)$ SYM system but the first and the 13-th ones which are each coupled to a D_3 AD system. If Σ contains more punctures and marks, Eqn. (2.61) gives a corresponding \mathfrak{S}_3 -cover of class $\mathcal{S}[A_1]$ theories.

2.4.6. A mere computational tool: quiver covers. In §.2.4.5 we discussed the Galois covers of Jacobian categories of $\mathcal{N} = 2$ QFTs by other Jacobian categories of $\mathcal{N} = 2$ QFTs which lead to covering maps between their BPS spectra. However, the covering techniques are useful to compute BPS spectra independently of the fact that the covering category is Jacobian or associated to a $\mathcal{N} = 2$ theory. From the computational point of view, what really matters is that the cover has a *simple* Representation Theory, not its physical interpretation.

An important special case is when we have a class $\mathcal{S}[A_1]$ with quiver Q and another class $\mathcal{S}[A_1]$ theory with quiver Q' , on which a group \mathbb{G} acts freely, such that $Q = Q'/\mathbb{G}$. The only reason why this is not a cover of $\mathcal{S}[A_1]$ categories is that, while the superpotential \mathcal{W}' on Q' is \mathbb{G} -invariant, its quotient is not equal to \mathcal{W} , and then

$$(2.63) \quad \mathbb{C}Q/(\partial\mathcal{W}) \equiv \left(\mathbb{C}Q'/(\partial F^*\mathcal{W})\right)/\mathbb{G} \neq \left(\mathbb{C}Q'/(\partial\mathcal{W}')\right)/\mathbb{G}.$$

From Appendix C, this happens when the corresponding geometric cover of Gaiotto surfaces, $\xi: \Sigma'_G \rightarrow \Sigma_G$, is *branched over some regular puncture of the quadratic differential ϕ_2* .

To compute the BPS spectrum of a class $\mathcal{S}[A_1]$ model with regular punctures it is convenient to use the trick that we call *gentling* and review in Appendix B.3. There the procedure is explained from two different points of view: the 4d gauge theory one, and the surgery of bordered surfaces Σ [8]. Here we rephrase it in terms of Gaiotto’s quadratic differentials ϕ_2 on the UV curve Σ_G . The trick is to deform each regular double pole of ϕ_2 into a order three pole with an infinitesimal coefficient

$$(2.64) \quad a_i \left(\frac{dz}{z - z_i}\right)^2 + \text{less singular} \longrightarrow \left(a_i + \frac{\epsilon}{z - z_i}\right) \left(\frac{dz}{z - z_i}\right)^2 + \text{less singular}$$

One then computes the easier BPS spectrum of the \mathbb{C} -category associated with a surface with no regular punctures. In the limit $\epsilon \rightarrow 0$ one almost gets back the original BPS spectrum, except for p vector multiplets which, geometrically, correspond to minimal WKB geodesics going around each of the p cubic punctures.

The effect of the gentling procedure is to replace $\pi_1(\Sigma^\circ)$ with $\pi_1(\Sigma)$ where now we think of the punctures of Σ as small holes. Although this would not give us a covering of BPS spectra in the strict sense, it would give us a computational tool which is practically equivalent to a *bona fide* cover since the discrepancy is just a known finite set of BPS states which should be

disregarded. To distinguish these situations from the genuine Galois covers of QFTs, we shall call them *quiver covers*.

2.4.7. Quotients of triangular factors and invariant Lie subalgebras. As discussed in §.2.3, to a triangular algebra (or to a dashed category) \mathcal{T} we may associate a Slodowy GIM Lie algebra \mathfrak{L}_A through its Cartan matrix A . If \mathbb{G} is an admissible group of automorphisms of \mathcal{T} , it induces an automorphism group of the Dynkin bi-graph of \mathfrak{L}_A and hence a group of outer automorphisms of the Lie algebra \mathfrak{L}_A . To the quotient functor $F: \mathcal{T} \rightarrow \mathcal{T}/\mathbb{G}$ there corresponds a Lie algebra embedding

$$(2.65) \quad \mathfrak{F}: \mathfrak{L}_A^{\mathbb{G}} \rightarrow \mathfrak{L}_A, \quad \mathfrak{L}_A^{\mathbb{G}} \equiv \mathfrak{L}_{A/\mathbb{G}},$$

where $\mathfrak{L}_A^{\mathbb{G}}$ is the \mathbb{G} -invariant Lie subalgebra. The Chevalley–Serre generators of $\mathfrak{L}_A^{\mathbb{G}}$

$$(2.66) \quad \{e_{\mathbb{G}i}, f_{\mathbb{G}i}, h_{\mathbb{G}i}\}$$

are labelled by the objects $\mathbb{G}i$ of \mathcal{T}/\mathbb{G} . Explicitly

$$(2.67) \quad e_{\mathbb{G}i} = \sum_{g \in \mathbb{G}} e_{gi}, \quad f_{\mathbb{G}i} = \sum_{g \in \mathbb{G}} f_{gi}, \quad h_{\mathbb{G}i} = \sum_{g \in \mathbb{G}} h_{gi}.$$

It is easy to see that the Dynkin bi-graph of $\mathfrak{L}_A^{\mathbb{G}}$ is the underlying bi-graph of the bi-quiver of \mathcal{T}/\mathbb{G} , i.e. $\mathfrak{L}_A^{\mathbb{G}} \equiv \mathfrak{L}_{A/\mathbb{G}}$.

We note that when \mathfrak{L}_A is an extended affine Lie algebra of simply-laced type, and the quiver \overline{Q} of the quotient category $\mathcal{T}/\mathbb{G} \equiv \mathbb{C}\overline{Q}/\overline{I}$ is 2-acyclic, $\mathfrak{L}_A^{\mathbb{G}}$ is also an extended affine Lie algebra of simply-laced type.

2.4.8. Pulled back central charge. Suppose we have a Galois cover $F: \mathcal{A} \rightarrow \mathcal{B}$ while on $\text{mod } \mathcal{B}$ we are given a central charge (stability function) i.e. a group homomorphism¹⁸ $Z: K^0(\text{mod } \mathcal{B}) \rightarrow \mathbb{C}$ which maps the positive cone of the Grothendieck group $K^0(\text{mod } \mathcal{B})$ into the upper half-plane. By *the pulled back central charge on mod A* we mean the group homomorphism $F^\lambda Z: K^0(\text{mod } \mathcal{A}) \rightarrow \mathbb{C}$ given by

$$(2.68) \quad F^\lambda Z([X]) = Z([F_\lambda X]).$$

¹⁸As always $K^0(\mathcal{A})$ stands for the Grothendieck group of the Abelian category \mathcal{A} . Given $X \in \mathcal{A}$ we write $[X] \in K^0(\mathcal{A})$ for its class.

3. First examples and applications

In this section we collect examples and some simple applications of the covering techniques, as a warm up for more serious computations in later sections.

The simplest possible non-free $\mathcal{N} = 2$ systems are the Argyres–Douglas models. The corresponding Jacobian algebras are representation–finite, and the $\mathcal{N} = 2$ models all whose Jacobian algebras are representation–finite are Argyres–Douglas models [8]. We have the following result

Fact [54]. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a Galois cover of bounded \mathbb{C} –linear categories with Galois group \mathbb{G} . Then \mathcal{B} is representation–finite if and only if (i) \mathbb{G} acts freely on the isoclasses of indecomposables and (ii) \mathcal{A} is representation–finite. In that case F_λ is surjective.*

In other words, if an Argyres–Douglas theory has a (finite) covering $\mathcal{N} = 2$ theory, that theory should also be an Argyres–Douglas model. From §.2.4.5, we know that the possible complete covers of Argyres–Douglas models (of type A, D) are controlled by the fundamental group of the corresponding surface, $\pi_1(\Sigma_{\text{AD}}^\circ)$. Since, Σ_{AD}° is a disk (with marks), these Argyres–Douglas theories do not have any non–trivial¹⁹ QFT covers. (Of course, for type D_p we have quiver covers in the sense of §.2.4.6; this is a very well known example of cover of \mathbb{C} –categories discussed at length, say, in ref.[53], or [57] Chapter V).

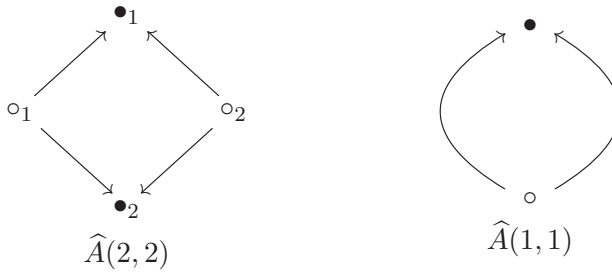
The next simpler class of theories are the $\mathcal{N} = 2$ gauge theories with gauge group $SU(2)$. These models do have interesting Galois covers.

3.1. \mathbb{Z}_k covers in $SU(2)$ SQCD

Let us start with our motivating example $N_f = 2$ SQCD with $G_{\text{gauge}} = SU(2)$ as a double cover of pure SYM. The quivers of the two theories are, respectively, the affine quivers of type $\widehat{A}(2, 2)$ and $\widehat{A}(1, 1)$ with alternating

¹⁹A *trivial* cover is one in which the covering theory consists of several non–interacting copies of the quotient theory. An example of trivial cover is the Galois cover of Argyres–Douglas theories $\mathbb{C}D_2 \rightarrow \mathbb{C}A_1$, i.e. two free hypers doubly covering one free hyper. See Appendix C for a list of such trivial exceptions.

orientation²⁰

(3.1) 

The quiver on the left has a freely acting \mathbb{Z}_2 automorphism which interchanges the nodes of the same color (black/white). From the figure, it is clear that its orbit category $\mathbb{C}\widehat{A}(2, 2)/\mathbb{Z}_2$ is the category of the quiver on the right, $\mathbb{C}\widehat{A}(1, 1)$. More generally, let p, q be two positive integers, and choose $k \mid \gcd(p, q)$. We have the \mathbb{Z}_k Galois cover

(3.2)
$$\mathbb{C}\widehat{A}(p, q) \longrightarrow \mathbb{C}\widehat{A}(p, q)/\mathbb{Z}_k \cong \mathbb{C}\widehat{A}(p/k, q/k)$$

(the orientation of the arrows is chosen in the obvious way to be consistent with the quotient). In this case, the covering $\mathcal{N} = 2$ theory is $SU(2)$ SYM coupled to two Argyres–Douglas systems of types D_p and D_q , while the quotient theory is $SU(2)$ SYM coupled to two AD systems²¹ of types $D_{p/k}$ and $D_{q/k}$ [8]. Choosing $(p/k, q/k) = (1, 1), (2, 1)$, and $(2, 2)$ the quotient theory is, respectively, $SU(2)$ SQCD with $N_f = 0, 1, 2$ fundamental quarks.

Hence $SU(2)$ SQCD coupled to at most two flavors (or, more generally, at most two Argyres–Douglas systems) admit \mathbb{Z}_k Galois covers for all $k \in \mathbb{N}$. These are their only covers by complete $\mathcal{N} = 2$ QFT (in fact, by any $\mathcal{N} = 2$ QFT) since, for all these models, $\Sigma \cong \Sigma^\circ = \mathbb{C}^\times$, and $\pi_1(\Sigma) \cong \pi_1(\Sigma^\circ) = \mathbb{Z}$.

For later reference, we describe the covering in terms of the corresponding 2d $(2, 2)$ theory [5, 8]. 4d $SU(2)$ SYM coupled to D_p and D_q AD systems corresponds to the $(2, 2)$ Landau–Ginzburg model

(3.3)
$$W(Z) = e^{pZ} + e^{-qZ}, \quad \text{with the identification } Z \sim Z + 2\pi i.$$

If $k \mid \gcd(p, q)$, the $(2, 2)$ model corresponding to the quotient theory $SU(2)$ SYM with $D_{p/k}$ and $D_{q/k}$ AD is the same Landau–Ginzburg model (3.3)

²⁰See [8] for our nomenclature about quivers. Since affine quivers are acyclic, the superpotential is zero.

²¹In our conventions [8], D_1 is the empty theory, and D_2 a doublet of free quarks.

but with the rescaled field identification

$$(3.4) \quad Z \sim Z + 2\pi i/k,$$

which gives a cover in the tt^* sense [6, 7].

In terms of Lie algebras \mathfrak{L}_A , the \mathbb{Z}_k cover (3.2) becomes the embedding of affine Kac–Moody algebras

$$(3.5) \quad A_{p/k+q/k-1}^{(1)} \equiv (A_{p+q-1}^{(1)})^{\mathbb{Z}_k} \longrightarrow A_{p+q-1}^{(1)}.$$

$N_f = 3$ SQCD (or more generally, $SU(2)$ SYM with two fundamental quarks and a D_p AD) corresponds to the disk D with two punctures and 2 marks (resp. p marks) on the boundary. We have $\pi_1(D^\circ) = (1)$, and hence its \mathbb{C} -category has no strict-sense cover by the \mathbb{C} -category of a $\mathcal{N} = 2$ QFT. Of course, since $\pi_1(D) = \langle x, y \rangle$, there are plenty of quiver covers in the sense of §.2.4.6. Likewise, $N_f = 4$ SQCD has only covers in the quiver sense; they are useful computational tools.

3.1.1. $SU(2)$ SQCD BPS spectra as Galois covers. We look at the maps between the BPS spectra of $SU(2)$ SYM coupled to different pairs of AD systems which are induced by the \mathbb{Z}_k covering functors F^λ, F_λ of §.2.4.1. The algebras associated to these models, $\mathbb{C}\widehat{A}(p, q)$, are hereditary (hence triangular with $q(\mathbf{x})$ a Kac–Moody Tits form), string, and class $\mathcal{S}[A_1]$, so all three special techniques of §§. 2.4.2, 2.4.4 apply.

An acyclic affine quiver corresponds to an asymptotically-free $SU(2)$ gauge theory [8]. The $SU(2)$ magnetic charge $m(X)$ of the BPS particle described by the module X is equal to the Ringel defect of the module [21], i.e. to the sum of the dimensions at the source nodes (the white ones) minus the sum of the dimensions at the sinks (the black nodes), cfr. Eqn. (2.6).

The BPS spectrum of a complete theory contains only hypermultiplets (corresponding to rigid bricks of the algebra) and vector multiplets (\mathbb{P}^1 families of bricks with self-extensions). In any chamber, an asymptotically-free $SU(2)$ gauge theory has at most *one* BPS vector multiplet, the W boson; in other words, the associated algebra \mathcal{J} has a unique \mathbb{P}^1 -family of non-rigid bricks W_μ ($\mu \in \mathbb{P}^1$). Being unique, this family is fixed by all automorphisms of \mathcal{J} , i.e. $g W_\mu \cong W_\mu$ for all $g \in \mathbb{Z}_k$. From §§. 2.4.2, 2.4.4 it follows that the W boson modules of the covering theory, \widetilde{W}_μ , are the pull back of the ones for the quotient theory, W_μ ,

$$(3.6) \quad \widetilde{W}_{\mu^k} = F^\lambda W_\mu.$$

Since F_λ is adjoint to F^λ , this implies that the push down functor F^λ preserves the magnetic charge of the module

$$(3.7) \quad m(F_\lambda X) = m(X).$$

Bricks with $m(X) \neq 0$ are rigid and cannot be in the closure of the W family (since the W boson has zero magnetic charge). By §§.2.4.2, 2.4.4 this implies that F_λ sets a k -to-1 correspondence between the dyons of the covering theory and the dyons of the quotient theory. This is our k -fold cover between dyon towers; the example $N_f = 2 \rightarrow N_f = 0$ SQCD corresponds to $k = 2$.

It remains to consider the rigid bricks of the covering theory with $m(X) = 0$ which correspond to the (finitely many) BPS states of the covering matter system [21]: in SQCD they are just the quark states. Let $X \in \text{mod } \mathbb{C}\widehat{A}(p, q)$ be such a brick. By §§. 2.4.2, 2.4.4, $F_\lambda X \in \text{mod } \mathbb{C}\widehat{A}(p/k, q/k)$ is indecomposable, but not necessarily a brick nor rigid. If $m(X) = 0$, $F_\lambda X$ is a rigid brick of the quotient category, if and only if $\mathbf{dim} F_\lambda X < \delta \equiv \mathbf{dim} W_\mu$ (for $k = 2$ this is equivalent to $\mathbf{dim} F_\lambda X \neq \delta$).

To the push down functor F_λ , restricted to the rigid bricks having vanishing magnetic charge, we may give a different interpretation. These bricks describe the BPS states of the covering matter $D_p \oplus D_q$ Argyres–Douglas system, and the Galois cover $F: \mathbb{C}\widehat{A}(p, q) \rightarrow \mathbb{C}\widehat{A}(p/k, q/k)$ induces a ‘covering’ of the respective matter systems

$$(3.8) \quad D_p \oplus D_q \rightarrow D_{p/k} \oplus D_{q/k}$$

which is not a strict-sense cover of $\mathcal{S}[A_1]$ categories (since, as shown at the beginning of this section, there are none) but merely a quiver cover in the sense of §.2.4.6. Then the would be push down f_λ maps bricks of the covering matter category into representations of the quotient matter quiver which may or may not satisfy the Jacobian relations $\partial\mathcal{W} = 0$. Only those which satisfy the relations are actual BPS states of the quotient matter sector; they are precisely the ones with $\mathbf{dim} f_\lambda X < \delta$, in agreement with the previous result.

Since (cfr. §§. 2.4.2, 2.4.4) F_λ is onto the rigid bricks of the quotient category, the full BPS spectral correspondence is

	covering model	quotient model	
(3.9)	vector multiplet	$F^\lambda W_\mu$	W_μ 1-to- k
	dyonic hypers	X	$F_\lambda X$ k -to-1
	$m = 0$ hypers	X	$F_\lambda X$, if $\mathbf{dim} F_\lambda X < \delta$ k -to-1

More precisely, if the covering model is equipped with the pull back $F^\lambda Z$ central charge, the stable particles of one column of Table (3.9) map in the corresponding stable particles of the other column. In the maximal chamber [19] all states in the table are stable, and the correspondence is fully realized.

For instance, in the $N_f = 2 \rightarrow N_f = 0$ cover, no rigid brick X with $m(X) = 0$ of the covering theory satisfies the condition $\mathbf{dim} F_\lambda X < \delta$, and the BPS spectrum of the quotient theory contains only the W boson and the dyonic towers. Of course, this is the right result.

3.2. \mathbb{Z}_k covers of pure SYM with simply-laced gauge groups

The above construction of \mathbb{Z}_k Galois covers of pure $SU(2)$ SYM may be extended directly to \mathbb{Z}_k covers of pure SYM with any simply-laced gauge group G . This is basically the idea of [15, 16]. Let us review their construction in the language of tt^* covers for the corresponding 2d (2,2) model as in Eqns. (3.3)–(3.4). The (2,2) model corresponding to SYM with gauge group G is [15, 70]

$$(3.10) \quad W(Z, X, Y) = e^Z + e^{-Z} + W_G(X, Y), \quad Z \sim Z + 2\pi i,$$

where the polynomial $W_G(X, Y)$ is the ADE minimal singularity corresponding to the simply-laced Lie group G . The tt^* \mathbb{Z}_k -Galois cover is obtained by replacing $Z \mapsto kZ$ in Eqn. (3.10) while keeping $Z \sim Z + 2\pi i$ [15, 16].

3.2.1. Quivers and their automorphisms. In the mutation class of quivers with superpotentials for the \mathbb{Z}_k cover of SYM with gauge group G there are two preferred ‘tensor product’ quivers [71]

$$(3.11) \quad \begin{aligned} &\text{triangle tensor product: } \widehat{A}(k, k) \boxtimes G, \\ &\text{square tensor product: } \widehat{A}(k, k) \square G, \end{aligned}$$

where G stands for the Dynkin quiver of the gauge group with an alternating orientation so that odd numbered nodes are sources and even numbered ones are sinks. The affine quiver $\widehat{A}(k, k)$ is also taken in the alternating orientation. These orientations are chosen for convenience; all orientations are mutation equivalent, and hence describe the same physics. Both product quivers have a ‘projection’ \mathfrak{p} on the second factor, i.e. on the Dynkin quiver G ; the inverse image of a simple root of G , $\mathfrak{p}^{-1}(\alpha_i)$, is a full affine subquiver $\widehat{A}(k, k)$. For both forms of the quiver we label the nodes as $\bullet_{a,i}, \circ_{a,i}$ with

$i \in G$ and $a \in \mathbb{Z}_k$ (the index a being defined mod k). Nodes $\bullet_{a,i}$ (resp. $\circ_{a,i}$) are the sinks (sources) of the i -th affine subquiver $\widehat{A}(k, k)$. The exchange matrix of the square quiver is

$$(3.12) \quad B_{\bullet_{a,i}, \bullet_{b,j}} = B_{\circ_{a,i}, \circ_{b,j}} = 0, \quad B_{\circ_{a,i}, \bullet_{b,j}} = (\delta_{a,b+1} - \delta_{ab})\delta_{ij} + C_{ij} \delta_{ab}$$

where C is the Cartan matrix of G . The superpotential \mathcal{W}_\square of the $\widehat{A}(k, k) \square G$ quiver is the sum of all its oriented squares. The exchange matrix of the triangle form is [21]

$$(3.13) \quad B_{\widehat{A}(k,k) \boxtimes G} = S_{\widehat{A}} \otimes S_G - S_{\widehat{A}}^t \otimes S_G^t$$

where $B_{\widehat{A}}(k, k)$ (resp. S_G) is the ‘Stokes’ matrix²², i.e. the matrix such that the quiver exchange matrix and the Tits matrix of the triangular algebra $\mathbb{C}\widehat{A}(k, k)$ (resp. $\mathbb{C}G$) are given by $B_{\widehat{A}} = S_{\widehat{A}} - S_{\widehat{A}}^t$ and $A_{\widehat{A}} = S_{\widehat{A}} + S_{\widehat{A}}^t$ (resp. $B_G = S_G - S_G^t$ and $A_G = S_G + S_G^t$). The superpotential \mathcal{W}_Δ of the $\widehat{A}(k, k) \boxtimes G$ quiver is a certain sum over its oriented triangles. $\mathcal{N} = 2$ super–Yang–Mills with simply laced gauge group G corresponds to the case $k = 1$ of the above construction [5, 20, 21].

Clearly both forms of the quiver, $\widehat{A}(k, k) \boxtimes G$ and $\widehat{A}(k, k) \square G$, have a \mathbb{Z}_k -automorphism group, inherited from the first factor in the tensor product, $\widehat{A}(k, k)$, which acts freely on the objects

$$(3.14) \quad \mathbb{Z}_k \ni s: (\bullet_{a,i}, \circ_{a,i}) \mapsto (\bullet_{a+s,i}, \circ_{a+s,i}).$$

Since this \mathbb{Z}_k action leaves invariant the superpotentials, \mathcal{W}_Δ and \mathcal{W}_\square , these quiver automorphisms extend to automorphisms of the Jacobian categories

$$(3.15) \quad \begin{aligned} \mathcal{J}_G^\Delta(k, k) &= \mathbb{C}\widehat{A}(k, k) \boxtimes G / (\partial\mathcal{W}_\Delta), \\ \mathcal{J}_G^\square(k) &= \mathbb{C}\widehat{A}(k, k) \square G / (\partial\mathcal{W}_\square). \end{aligned}$$

Hence, for all $\ell \mid k$ we may form Galois covers

$$(3.16) \quad F_\Delta: \mathcal{J}_G^\Delta(k, k) \longrightarrow \mathcal{J}_G^\Delta(k, k) / \mathbb{Z}_\ell \cong \mathcal{J}_G^\Delta(k/\ell, k/\ell)$$

$$(3.17) \quad F_\square: \mathcal{J}_G^\square(k) \longrightarrow \mathcal{J}_G^\square(k) / \mathbb{Z}_\ell \cong \mathcal{J}_G^\square(k/\ell)$$

whose quotient categories have the same ‘tensor product’ form with k replaced by k/ℓ , as it is obvious from the geometric description of the cover

²²In the math literature [32, 57] S is called the inverse of the Cartan matrix of the algebra; we avoid that terminology since it is very confusing in the present context.

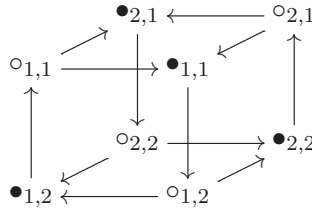


Figure 2: The quiver $\widehat{A}(2, 2) \square A_2$: a cube with oriented squares on the four lateral faces.

around Eqn. (3.10). Then both the cover and quotient categories correspond to well defined $\mathcal{N} = 2$ QFTs.

The triangle construction may be further generalized to tensor products of the form $\widehat{A}(p, q) \boxtimes G$ with $p \neq q$. Again, if $\ell \mid \text{gcd}(p, q)$, we have a Galois covering

$$(3.18) \quad F_\Delta: \mathcal{I}_G^\Delta(p, q) \longrightarrow \mathcal{I}_G^\Delta(p, q)/\mathbb{Z}_\ell \cong \mathcal{I}_G^\Delta(p/\ell, q/\ell).$$

The square category $\mathcal{I}_G^\square(k)$ may have a larger group of automorphisms acting freely on the objects. The simplest instance is $\widehat{A}(2, 2) \square A_2$ whose quiver is shown in Figure 2. The freely acting automorphisms of this category form the Klein Vierergruppe $V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

$$(3.19) \quad (\bullet_{1,i}, \circ_{1,i}) \leftrightarrow (\bullet_{2,i}, \circ_{2,i}) \quad \text{and} \quad (\bullet_{a,1}, \circ_{a,1}) \leftrightarrow (\bullet_{a,2}, \circ_{a,2}).$$

The quotient of $\mathcal{I}_{A_2}^\square(2)$ with respect to the full V_4 group has a non-2-acyclic quiver. So, while such quotients by enhanced automorphism groups may be quite useful to actually compute the BPS spectrum at special points in parameter space which preserve such enhanced symmetry, they do not describe Galois covers of BPS spectra of $\mathcal{N} = 2$ QFTs.

3.2.2. Covers of pure SYM from Minahan–Nemeshanski SCFT.

As explained in detail in [15, 16], the category $\mathcal{I}_G^\Delta(p, q)$ corresponds to $\mathcal{N} = 2$ SYM with group G which gauges the diagonal flavor (sub)group G of a pair of $\mathcal{N} = 2$ SCFTs of types $D_p(G)$ and $D_q(G)$. In the notations of [15, 16], $D_1(G)$ is the trivial SCFT. When the ‘matter’ sector consists of two copies of the *same* SCFT $D_p(G)$, it is more convenient to use the mutation equivalent square category $\mathcal{I}_G^\square(p)$.

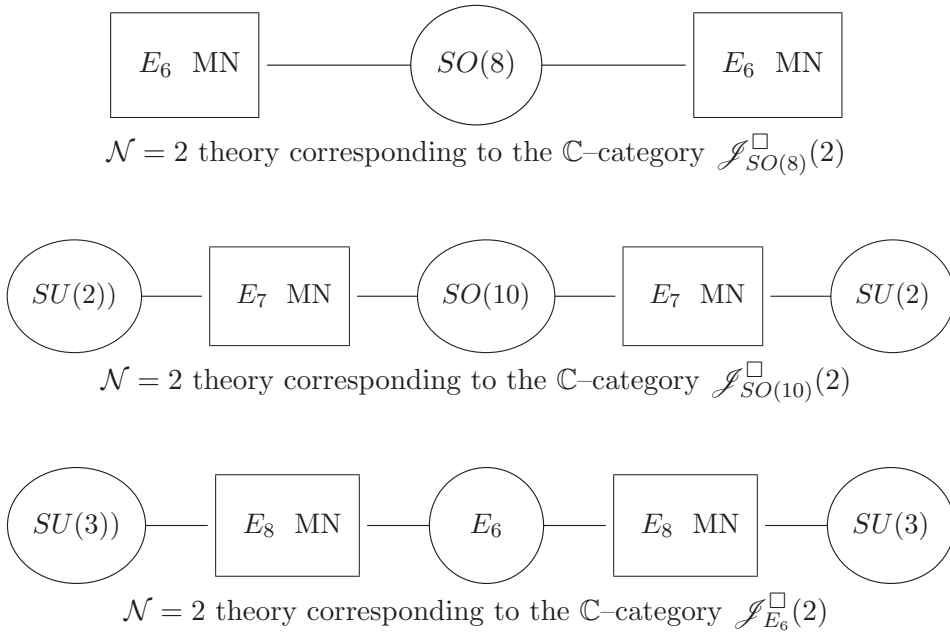


Figure 3: The $\mathcal{N} = 2$ theories whose BPS spectra double covers that of pure $\mathcal{N} = 2$ SYM with gauge groups $G = SO(8)$, $SO(10)$, and E_6 . The circles stands for $\mathcal{N} = 2$ SYM sectors, the rectangles for Minahan–Nemeschansky E_r SCFTs.

If $k \mid \gcd(p, q)$, the Galois cover

$$(3.20) \quad F: \mathcal{J}_G^\Delta(p, q) \rightarrow \mathcal{J}_G^\Delta(p/k, q/k)$$

then corresponds to replacing the ‘matter’ sector $D_p(G) \oplus D_p(G)$ with $D_{p/k}(G) \oplus D_{q/k}(G)$.

In particular, if $p = q = k$, we get a \mathbb{Z}_k -cover of the category corresponding to *pure* SYM with gauge group G , by the same gauge theory coupled to two copies of $D_k(G)$. If $G = SU(2)$ and $k = 2$, this reduces to the GMN double cover of $SU(2)$ SYM by the same theory coupled to two fundamental quarks. So the covers (3.20) are direct generalizations of our motivating example.

A more sophisticated example is obtained by replacing $SU(2)$ SYM by $SO(8)$ SYM and the two doublets of free quarks, $D_2(SU(2))$, by their $SO(8)$ counterpart $D_2(SO(8))$, which is [15, 16] the E_6 Minahan–Nemeschanski

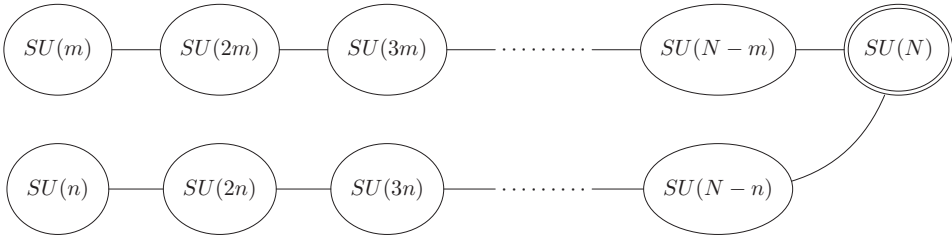


Figure 4: The SU linear quiver gauge theories which correspond to bounded \mathbb{C} -categories of the $\mathcal{J}_{SU(N)}^\Delta(p, q)$ class. m, n are divisors of N , and $p = N/m, q = N/n$. Circles denote $\mathcal{N} = 2$ SYM sectors and links bifundamental hypermultiplets.

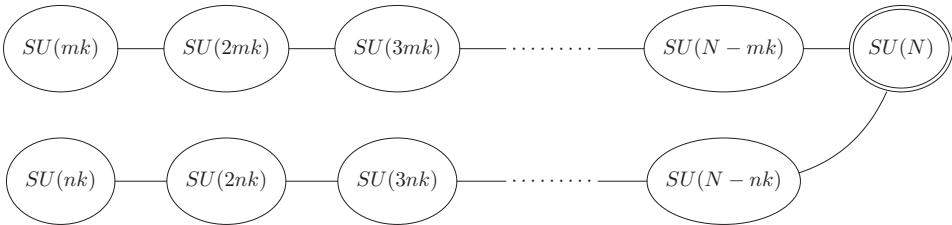


Figure 5: The \mathbb{Z}_k Galois quotient of the quiver gauge theory in Figure 4.

model [23]. Thus the $\mathcal{N} = 2$ theory obtained by gauging the $SO(8)_{\text{diag}} \subset E_6 \times E_6$ flavor symmetry of two copies of the E_6 Minahan–Nemeshanski SCFT corresponds to the category $\mathcal{J}_{SO(8)}^\square(2)$, which is a \mathbb{Z}_2 Galois cover of the pure $SO(8)$ SYM category $\mathcal{J}_{SO(8)}^\square(1)$.

Likewise, for $p = q = k = 2$ and gauge groups $G = SO(10)$ and $G = E_6$ we get ‘matter’ systems involving the Minahan–Nemeshanski theories with flavor symmetry, respectively, E_7 and E_8 [23, 24] together with some further SYM sectors, see Figure 3. Analogously, the double cover of the $SU(3)$ SYM \mathbb{C} -category, $\mathcal{J}_{SU(3)}^\square(2)$, corresponds to SYM gauging the diagonal flavor symmetry of two copies of the Argyres–Douglas system of type D_4 [15, 16].

3.2.3. Galois covers between SU and USp/SO linear quiver theories. The linear quiver $\mathcal{N} = 2$ gauge theories in Figure 4, where m, n , are divisors of N , have BPS quivers in the mutation class of $\widehat{A}(N/m, N/n) \boxtimes$

$SU(N)$. The beta–functions of all YM couplings vanish, but for the one corresponding to the $SU(N)$ group (the one in the double–line circle) which is just asymptotically free. Indeed the two ‘matter’ sectors, $D_{N/m}(SU(N))$ and $D_{N/n}(SU(N))$, correspond to the two disconnected quiver theories one obtains by deleting the asymptotically–free double–lined gauge node [16].

In the same fashion, if m, n are divisors of $N - 1$, the matter systems $D_{(N-1)/m}(SO(2N))$, $D_{(N-1)/n}(SO(2N))$ correspond to USp/SO linear quivers [72], see [16]. These quivers have different forms, depending on the parity of the two integers $(N - 1)/m$ and $(N - 1)/n$, see ref.[16] for details.

If there is an integer k such that $mk, nk \mid N$, the \mathbb{C} –category $\mathcal{J}_{SU(N)}^\Delta(N/m, N/n)$ associated to the $\mathcal{N} = 2$ theory in Figure 4 has a \mathbb{Z}_k Galois quotient which corresponds to a linear quiver model of the same kind, see Figure 5. The USp/SO linear quivers have similar \mathbb{Z}_k Galois covers, provided mk and nk both divide $(N - 1)$.

It is remarkable that pairs of linear quiver gauge theories — as the pair in Figures 4 and 5 — which look very different, have BPS spectra which (in \mathbb{Z}_k symmetric chambers) are simply related *via* the \mathbb{Z}_k covering functors F_λ and F^λ .

3.2.4. Galois covers of BPS spectra of SYM. The Galois cover of \mathbb{C} –categories

$$(3.21) \quad \mathcal{J}_G^\Delta(p, q) \rightarrow \mathcal{J}_G^\Delta(p/k, q/k)$$

induces covering functors F^λ, F_λ which relate the BPS spectra of two $\mathcal{N} = 2$ theories with the same gauge group G but different ‘matter’ content, at \mathbb{Z}_k symmetric points in the parameter space of the covering theory.

Finite chambers. Pure SYM with simply–laced gauge group G has a strongly–coupled finite chamber [20] whose spectrum consists of two hypermultiplet dyons per positive root of G . In facts, under the identification of the lattice of conserved charges for pure SYM with $\Gamma_{A_1^{(1)}} \otimes \Gamma_G$, where by Γ_G (resp. $\Gamma_{A_1^{(1)}}$) we mean the root lattice of the Lie algebra G (resp. $A_1^{(1)}$), the charge vectors of the stable dyons at strong YM coupling take the form

$$(3.22) \quad \alpha_a \otimes \beta \in \Gamma_{A_1^{(1)}} \otimes \Gamma_G \quad \begin{array}{l} \alpha_a \text{ is a simple root of } A_1^{(1)} \\ \beta \text{ a positive root of } G. \end{array}$$

More generally, the strong coupling BPS spectrum of G SYM coupled to $D_p(G)$ and $D_q(G)$ contains hypermultiplets whose charge vectors make $(p +$

q) copies of the positive roots of G

$$(3.23) \quad \alpha_a \otimes \beta \in \Gamma_{A_{p+q-1}^{(1)}} \otimes \Gamma_G \quad \begin{array}{l} \alpha_a \text{ is a simple root of } A_{p+q-1}^{(1)} \\ \beta \text{ a positive root of } G. \end{array}$$

Let $k \mid \text{gcd}(p, q)$ and consider the Galois group \mathbb{Z}_k acting on $\mathcal{J}^\Delta(p, q)$. The Galois group acts freely on the generators $\alpha_a \otimes \beta_i$ of the lattice $\Gamma_{A_{p+q-1}^{(1)}} \otimes \Gamma_G$ so that the push down functor $F_\lambda: \text{mod } \mathcal{J}_G^\Delta(p, q) \rightarrow \text{mod } \mathcal{J}_G^\Delta(p/k, q/k)$ send stable bricks into stable bricks, and gives a k -to-1 correspondence of the strong coupling spectra of the two theories. This result may be seen as a direct consequence of **Fact 3**.

The ‘dual’ weak coupling chamber. The models with a square product quiver $G' \square G$ have a ‘dual’ (in the sense of [5]) weak coupling chamber which has a very simple BPS spectrum. This remains true even if the Dynkin graph G' is affine as in the present class of theories. To get this ‘dual’ chamber for SYM coupled to $D_p(G)$, $D_q(G)$ AD one has to fine-tune the parameters to exceptional alignments in the Z -plane; hence the physical relevance of the resulting spectrum may be questioned. Mathematically, however, it makes sense, and the BPS spectrum consists of hypers and vector multiplets only with charge vectors

$$(3.24) \quad \alpha \otimes \beta_i \in \Gamma_{A_{p+q-1}^{(1)}} \otimes \Gamma_G \quad \begin{array}{l} \alpha \text{ is } \begin{cases} \text{a real Schur root of } A_{p+q-1}^{(1)} \text{ for hypers} \\ \text{imaginary Schur root of } A_{p+q-1}^{(1)} \text{ for vectors} \end{cases} \\ \beta_i \text{ is a simple root of } G. \end{array}$$

Then BPS spectrum is $r(G)$ copies of the spectrum of $SU(2)$ SYM with $D_p \oplus D_q$ matter at weak coupling, one copy per simple root of G . The spectral cover given by F_λ works exactly as in §.3.1.1 copy by copy. Again the idea that the cover of the BPS spectrum is realized chamber by chamber is confirmed.

More general chambers. The covering of BPS spectra works smoothly in all chambers. The details of the spectra of \mathbb{Z}_k covers of SYM will be presented elsewhere.

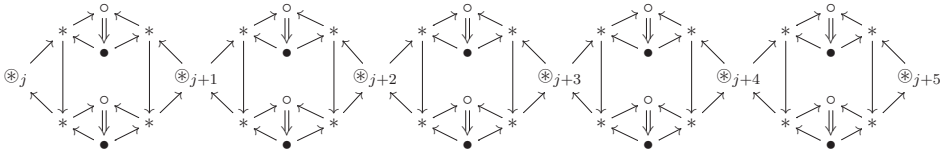


Figure 6: A piece of a triangulation quiver (with \mathcal{W}) for a surface Σ of genus $g = k + 1$ with k regular punctures; the pattern repeats for $j \in \mathbb{Z}$ and the nodes are periodically identified mod k , $\otimes_j \sim \otimes_{j+k}$. For all $m \mid k$, it is a *unbranched* \mathbb{Z}_m cover of the quiver (with \mathcal{W}) of a surface of genus $g' = k/m + 1$ and k/m punctures obtained by $\otimes_j \sim \otimes_{j+k/m}$.

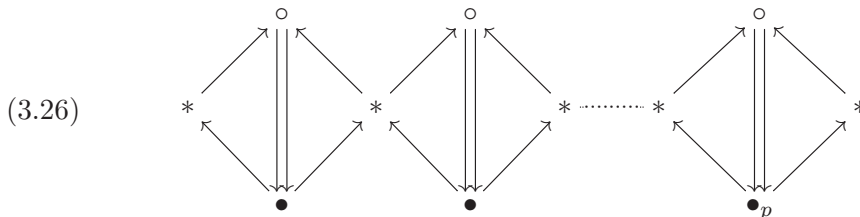
3.3. Abelian Galois covers of class $\mathcal{S}[A_1]$ models

Genus > 1 curves. Suppose we have a Gaiotto UV curve Σ_G with genus $g > 1$ and p regular punctures such that

$$(3.25) \quad k \mid (g - 1), \quad k \mid p,$$

which has a freely acting \mathbb{Z}_k automorphism group which permutes the punctures. The quotient $\Sigma'_G \equiv \Sigma_G/\mathbb{Z}_k$ is a Gaiotto surface with $(g' - 1) = (g - 1)/k$ and $p' = p/k$. The cover $\Sigma_G \rightarrow \Sigma'_G$ is *unbranched* (i.e. the corresponding $\mathcal{N} = 2$ gauge theories are coupled only to half-trifundamentals and bi-fundamentals, but no AD system).

To the geometric cover $\Sigma_G \rightarrow \Sigma'_G$ there correspond a Galois cover of triangulation categories. The \mathbb{Z}_k symmetric quiver for the minimal number of punctures ($p = k$) is represented in Figure 6. To add further punctures just replace in Figure 6 some \otimes nodes, or some pair of triangles forming a Kronecker subquiver, by the quiver of a multi-punctured cylinder



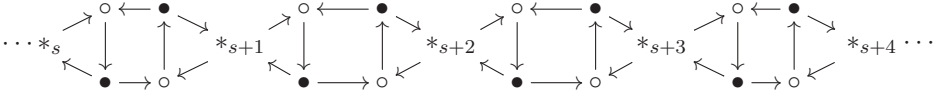


Figure 7: The \mathbb{Z}_k -symmetric triangulation quiver for the genus zero surface with two boundaries, each with k marks, and k regular punctures. The subscript s is periodically identified mod k . The superpotential \mathcal{W} is the sum of the oriented triangles and the oriented squares.

The generalization to irregular punctures may then be obtained by the gentling trick (Appendix B.3).

In conclusion, all $g > 1$ class $\mathcal{S}[A_1]$ categories with $p \geq g - 1$ regular punctures have natural \mathbb{Z}_k Galois covers, for $k \mid \text{gcd}(g - 1, p)$, which may be used to relate the BPS spectra of different theories (at special points in their parameter spaces).

Genus one curves. The Galois cover between genus one $\mathcal{S}[A_1]$ theories — corresponding to *isogenies* of elliptic curves — is discussed in section 6, where we describe their BPS spectra in detail. We refer to that section.

Genus zero curves. In the case of genus zero, according to §.2.4.5, to have a *strict sense* Galois cover we need at least two irregular punctures to serve as branching points of the cover. Then suppose (for simplicity) that we have precisely two irregular punctures, that is, Σ has two boundary components, respectively with r_1 and r_2 marks, and p punctures. The \mathbb{Z}_k -cover Σ' has also two boundary components, with kr_1 and kr_2 marks, and pk punctures.

As an illustration, the covering quiver with freely acting \mathbb{Z}_k -symmetry corresponding to $r_1 = r_2 = p = 1$ is represented in Figure 7. In this example, the covering theory is (in some regime) a generalized linear quiver with gauge group $SU(2)^{k+1}$, k bifundamental quarks, and two AD systems of type D_k ; the quotient model is just $SU(2)^2$ SYM coupled to a bifundamental. The covering BPS spectrum of this model (in a finite chamber) is described explicitly in §.5.2.2.

4. Non-Abelian \mathbb{G} : modular curves, *dessins d'enfants*

All examples of spectral Galois cover in the previous section had an *Abelian* Galois group \mathbb{G} . It is natural to look for examples where the Galois group

\mathbb{G} is genuinely non-Abelian. We already described a class of $\mathcal{S}[A_1]$ models which are $\mathbb{G} \equiv \mathfrak{S}_3$ covers of $\mathcal{N} = 2$ QFTs, namely the last example in §.2.4.5. In that case \mathbb{G} was realized as a degree 6 representation of the modular group $PSL(2, \mathbb{Z})$. In this section we consider in more detail the coverings between Jacobian categories whose Galois group \mathbb{G} arises from a finite-dimensional representation of $PSL(2, \mathbb{Z})$. We stress that the following construction may be applied to class $\mathcal{S}[G]$ theories for all $G = ADE$. However, to be very explicit, in this section we limit ourselves to class $\mathcal{S}[A_1]$ categories. The BPS spectra of the corresponding class $\mathcal{S}[A_1]$ theories have magical properties.

4.1. Principal modular curves

The last example of §.2.4.5 was based on the modular cover²³

$$(4.1) \quad \mathbb{P}^1 \cong X(2) \equiv \overline{\mathbb{H}}/\Gamma(2) \xrightarrow{\xi} \overline{\mathbb{H}}/PSL(2, \mathbb{Z}) \equiv X(1) \cong \mathbb{P}^1.$$

Here $X(N)$ is the (compactified) modular curve $\overline{\mathbb{H}}/\Gamma(N)$ where $\Gamma(N)$ is the level N principal congruence subgroup of the modular group $SL(2, \mathbb{Z})$ [60, 61]:

$$(4.2) \quad \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Going through the argument of §.2.4.5, we realize that we may replace $\Gamma(2)$ by any congruence subgroup Γ of $SL(2, \mathbb{Z})$ provided it is *normal* and *torsionless*. Both conditions hold for the principal congruence subgroups $\Gamma(N)$ for all $N \geq 2$, and we focus on this class of modular subgroups. The cover $X(N) \xrightarrow{\xi_N} X(1)$ has degree [60]

$$(4.3) \quad d(N) = \begin{cases} \frac{1}{2}N^3 \prod_{p|N} (1 - 1/p^2) & \text{if } N > 2, \\ 6 & \text{if } N = 2. \end{cases}$$

Note that $6 \mid d(N)$ and $N \mid d(N)$ for all $N \geq 2$. The genus $g(N)$ of the Riemann surface $X(N)$ is zero for $N = 1, 2$ while for $N > 2$

$$(4.4) \quad g(N) = 1 + \frac{(N - 6) d(N)}{12N},$$

²³Here $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup i\infty$, with \mathbb{H} the upper half plane.

from which we see that $g(N) = 0$ iff $N \leq 5$ and $g(N) = 1$ iff $N = 6$. The three Hurwitz partitions of $X(N) \xrightarrow{\xi_N} X(1)$ are

$$(4.5) \quad \begin{aligned} d(N) &= N + N + N + \cdots + N, & d(N) &= 2 + 2 + 2 + \cdots + 2, \\ d(N) &= 3 + 3 + 3 + \cdots + 3. \end{aligned}$$

Suppose we have a class $\mathcal{S}[A_1]$ theory defined by a quadratic differential ϕ_2 on $X(1) \equiv \mathbb{P}^1$ with three irregular poles at $\tau = i\infty$, i , and $e^{\pi i/3}$ of respective order $\ell_1 + 2$, $\ell_2 + 2$ and $\ell_3 + 2$, as well as k ordinary double poles at generic positions. The pulled back quadratic differential $\xi_N^* \phi_2$ defines a Gaiotto theory with UV surface $X(N)$ of genus $g(N)$, $d(N)$ k regular punctures, and

$$(4.6) \quad b(N) = \frac{5N + 6}{6N} d(N)$$

irregular punctures, i.e. boundary components for the associated bordered surface Σ . For $N > 2$, $d(N)/N$ boundary components have $N\ell_1$ marks, $d(N)/2$ of them have $2\ell_2$ marks, and $d(N)/3$ have $3\ell_3$ marks. The number of ideal arcs in the covering triangulation is then

$$(4.7) \quad n(N) = \left(3 + 3k + \sum_i \ell_i \right) d(N),$$

as it should. The categorical Galois group is

$$(4.8) \quad \mathbb{G} = PSL(2, \mathbb{Z}/N\mathbb{Z}).$$

Of course, $|\mathbb{G}| = d(N)$. In some corner of its parameter space, the resulting $\mathcal{N} = 2$ QFT is a gauge theory with gauge group

$$(4.9) \quad G_{\text{gauge}} = SU(2)^{m(N)}, \quad \text{where } m(N) = \frac{23}{12} d(N) + \frac{d(N)}{2N} + k d(N),$$

coupled (in a subtle way) to half-trifundamentals, bi-fundamentals, fundamentals, and Argyres–Douglas systems of types D_3 and D_N .

For instance, if we take the Galois group to be the icosahedral group $\mathfrak{A}_5 \equiv PSL(2, \mathbb{F}_5)$, corresponding to $N = 5$, and no regular punctures, $k = 0$, we get a $SU(2)^{126}$ gauge theory.

A special case is $N = 7$; the order of the symmetry group of the covering surface, $d(7)$, saturates the Hurwitz general bound on the order h of the

automorphism group of a Riemann surface of given genus g

$$(4.10) \quad h \leq 84(g - 1).$$

In this case the Galois group is the Hurwitz group [62]. The minimal theory with this symmetry has gauge group $SU(2)^{334}$.

4.2. The modular quivers $Q(N)$ and their flat cousins $Q(N)^{\text{fl}}$

For simplicity, in this subsection we assume $k = 0$ and $\ell_1 = \ell_2 = \ell_3 = 1$. The extension to the general case is, in principle, straightforward.

The fundamental domain of $\Gamma(N)$ has the form

$$(4.11) \quad \mathcal{F}(N) = \bigcup_{g \in \mathbb{G}} g(\mathcal{F})$$

where \mathcal{F} is the usual fundamental domain of the modular group $PSL(2, \mathbb{Z})$. The \mathbb{G} -invariant ideal triangulations of the covering surface Σ' are constructed by pulling back the ideal triangulations for the pair of pants with one mark on each boundary. Identifying the pair of pants with the fundamental domain \mathcal{F} with the cusp and elliptic points removed, a convenient (in fact, the most symmetric) ideal triangulation of \mathcal{F} , is shown in Figure 8; we denote it as $T_{\mathcal{F}}$. Then a nice \mathbb{G} -invariant triangulation $T(N)$ of Σ' is obtained by gluing together the images $g(T_{\mathcal{F}})$ for $g \in \mathbb{G}$, cfr. Eqn. (4.11).

From the figure it is obvious how to construct the triangulation quiver $Q(N)$ for the pulled back triangulation $T(N)$. Draw the *underlying* ideal triangulation $T(N)^{\text{fl}}$ of the fundamental domain $\mathcal{F}(N)$, i.e. the pull back of the triangulation $T_{\mathcal{F}}^{\text{fl}}$ which underlies $T_{\mathcal{F}}$: by this we mean the triangulation obtained by replacing marked boundaries with ordinary punctures. $T(N)^{\text{fl}}$ is just a convenient \mathbb{G} -invariant ideal triangulation of a genus $g(N)$ closed surface $\Sigma(N)^{\text{fl}}$ with $3d(N)$ punctures. Then replace the ideal arcs of $T(N)^{\text{fl}}$ with *pairs* of arcs (as in the left part of Fig. 8) and orient them in such a way that the boundaries of all triangles of $T(N)^{\text{fl}}$ form oriented cycles. Call *upper* (resp. *lower*) a triangle of $T(N)^{\text{fl}}$ if its boundary is counter-clockwise (resp. clockwise) oriented. The block decomposition of the quiver $Q(N)^{\text{fl}}$ of the underlying ideal triangulation $T(N)^{\text{fl}}$ consists of $d(N)$ Type II blocks associated to upper triangles and $d(N)$ Type II blocks for the lower triangles. In $Q(N)^{\text{fl}}$ an upper (lower) Type II block may be glued only to lower (upper) Type II blocks, and all their white nodes are glued in pairs in $Q(N)^{\text{fl}}$ (since $\Sigma(N)^{\text{fl}}$ is closed). We get the desired quiver $Q(N)$ from $Q(N)^{\text{fl}}$ by “separation of the glued white nodes in the vertical direction”,

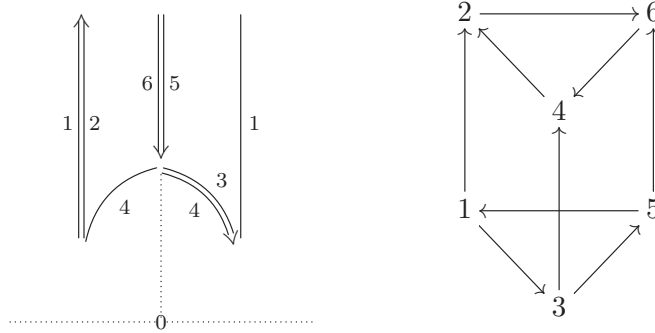


Figure 8: LEFT: the \mathbb{Z}_3 -symmetric ideal triangulation $T_{\mathcal{F}}$ of a pair of pants with one mark on each of the three boundaries as uniformized by the fundamental domain \mathcal{F} of the modular group (schematic). Ideal arcs with the same number are identified by the action of $PSL(2, \mathbb{Z})$. The arrows point toward the common vertex of two ideal sides of a triangle whose third side is a boundary arc. RIGHT: the triangulation quiver $Q(1)$ of the symmetric ideal triangulation: nodes of $Q(1)$ are numbered as the corresponding arcs in the triangulation.

that is, at each node of $Q(N)^{\text{fl}}$ we undo the gluing, insert a Type I block going from the white node of the lower Type II block to the white node of the upper Type II block, and glue back. For instance, if $Q(1)^{\text{fl}}$ is the quiver of the sphere with 3 punctures, obtained by gluing two oppositely oriented Type II blocks, the separation in the vertical direction produces $Q(1)$ which is the well-known prism quiver in Figure 8. We say that $Q(N)^{\text{fl}}$ is obtained by *flattening* $Q(N)$.

In principle, to compute the flattened out quiver $Q(N)^{\text{fl}}$, one has to pull back the triangulation $T_{\mathcal{F}}^{\text{fl}}$ to the fundamental domain of $\Gamma(N)$ by gluing the various copies through the decomposition (4.11). In practice, we may avoid going through that, and get $Q(N)^{\text{fl}}$ by elementary symmetry considerations.

To illustrate the procedure, we consider again our original example $N = 2$. Since the Hurwitz partitions for the cover $J = J(\lambda)$ of Eqn. (2.61) are $(2, 2, 2)$, $(2, 2, 2)$, and $(3, 3)$, the underlying triangulation $T(2)^{\text{fl}}$ is based on six punctures of valency 4 and two punctures of valency 6, while $T(2)^{\text{fl}}$ contains $6 \times 2 = 12$ (ordinary) triangles. The underlying graph $L(2)^{\text{fl}}$ of the flattened out quiver $Q(2)^{\text{fl}}$ is then given by the edges of a tiling of the sphere S^2 by 12 triangles, 6 squares, and 2 hexagons. $Q(2)^{\text{fl}}$ is then obtained by

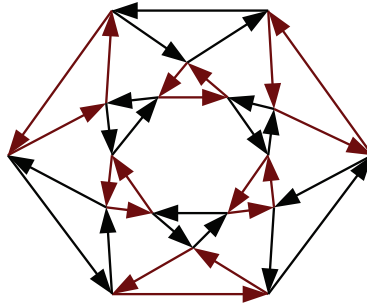


Figure 9: The flattened out quiver $Q(2)^{\text{fl}}$ for the modular \mathfrak{S}_3 -cover of the pair of pants with one mark per boundary. Upper (lower) triangles are drawn in red (black).

making all the 20 polygons of $L(2)^{\text{fl}}$ into oriented cycles. This determines uniquely $Q(2)^{\text{fl}}$ to be the diamond quiver in Figure 9. In the figure we have distinguished the arrows belonging to upper/lower Type II blocks by their color *red* resp. *black*. The automorphism group of $Q(2)^{\text{fl}}$ is the dihedral group D_{12} (which is an automorphism of the full Jacobian category of the sphere with 8 regular punctures); however only the subgroup $D_6 \cong \mathfrak{S}_3 \subset D_{12}$ preserves the coloring of the arrows, and hence is an automorphism of the vertically separated quiver $Q(2)$ (and category) of the Galois cover. \mathfrak{S}_3 acts freely on both the nodes and the Type II blocks. Hence we recover the correct Galois group \mathfrak{S}_3 of the QFT described at the end of §.2.4.5. The quiver $Q(2)$ is obtained by separating vertically the red and black blocks in Figure 9.

In the general case, the flattened quiver $Q(N)^{\text{fl}}$ is obtained by cyclically orienting the boundaries of a polygonal tiling of a surface $\Sigma(N)$ of genus $g(N)$, invariant under its automorphism group $PSL(2, \mathbb{Z}/N\mathbb{Z})$, consisting of the following polygonal tiles:

- $2d(N)$ triangles
- $\frac{d(N)}{2}$ 4-gons
- $\frac{d(N)}{3}$ hexagons
- $\frac{d(N)}{N}$ $2N$ -gons.

This tiling has $V = 3d(N)$ vertices, $E = 6d(N)$ edges, and $F = (17N + 6)d(N)/6N$ faces. The Euler characteristic of this tiling is

$$(4.12) \quad V - E + F = (6 - N)d(N)/6N \equiv 2 - 2g(N),$$

as it should. Since $\Sigma(N) \equiv X(N)$ is orientable, there exists an orientation of the edges of the tiling so that all polygonal faces have cyclically oriented boundaries. The tiling with this orientation is the flattened quiver $Q(N)^{\text{fl}}$. By vertical separation of upper/lower Type II blocks, one gets the desired \mathbb{G} -invariant quiver $Q(N)$.

4.3. Simple generalizations

Let N, M be coprime integers ≥ 2 . The modular cover $X(N) \xrightarrow{\xi_N} X(1)$ may be generalized to

$$(4.13) \quad X(NM) \xrightarrow{\xi_{N,M}} X(M).$$

$\xi_{N,M}$ is a cover of a genus $g(M)$ surface $X(M)$ by a genus $g(NM)$ one $X(NM)$ with degree

$$(4.14) \quad d(N, M) \equiv \frac{d(NM)}{d(M)} = \begin{cases} 2d(N) & \text{if } N, M \geq 3 \\ d(N) & \text{if } M = 2 \text{ or } N = 2, \end{cases}$$

branched over the $d(M)/M$ cusps with equal Hurwitz partitions

$$(4.15) \quad d(N, M) = N + N + \dots + N.$$

The Galois group of the cover is

$$(4.16) \quad \mathbb{G} = SL(2, \mathbb{Z}/N\mathbb{Z}).$$

Assuming the i -th boundary of the base bordered surface $\Sigma(M)$ have ℓ_i marks ($i = 1, \dots, d(M)/M$), and adding to it k regular punctures, we get a \mathbb{G} -cover QFT with gauge group

$$(4.17) \quad SU(2)^{m(N,M)} \quad \text{where } m(N, M) = \frac{NM + 2}{4MN} d(NM) + k \frac{d(NM)}{d(M)}.$$

Example. The minimal M for which the quotient surface $X(M)$ is not a sphere is $M = 6$. The smallest possible coprime integer is $N = 5$. This minimal non-spherical example corresponds to a Gaiotto theory defined by a torus with at least $d(6)/6 = 12$ irregular punctures (plus, possibly, regular

ones) covered by a Gaiotto theory on a surface of genus

$$(4.18) \quad g(30) = 577$$

having at least

$$(4.19) \quad \frac{d(30)}{30} = 288$$

irregular punctures. The *minimal* model in this class has gauge group

$$(4.20) \quad G_{\text{gauge}} = SU(2)^{2304},$$

and thousands of different matter sectors. It is remarkable that we may explicitly compute the BPS spectra of gauge theories of such enormous complexity!

4.4. Generalization: regular Grothendieck's *dessins d'enfants*

The above constructions may be generalized in various ways. As a payoff we get a more intrinsic geometric interpretation of the ‘separation in the vertical direction’ method to construct the relevant quivers. As in §.2.4.5 we consider covers of the sphere branched over three points. It is customary to set the three branch points in \mathbb{P}^1 at ∞ , 0, and 1 (before we set them at $i\infty$, i , and $e^{\pi i/3} \bmod PSL(2, \mathbb{Z})$). By definition a covering $\xi: \Sigma_G \rightarrow \mathbb{P}^1$ branched over $\infty, 1, 0$ is given by a normalized Belyi function ξ [59, 63]. It is a fundamental theorem of diophantine geometry that a smooth algebraic curve Σ_G is defined over the algebraic closure \mathbb{Q}^{al} of \mathbb{Q} if and only if there exists a Belyi function ξ on Σ_G [59, 63]. Moreover, in this case there is a subgroup Γ of some triangle group²⁴ $\Delta(p, q, r)$ such that $\Sigma_G \cong \mathbb{H}/\Gamma$ and [63, 66]

$$(4.21) \quad \xi: \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Delta(p, q, r) \cong \mathbb{P}^1$$

$$(4.22) \quad \Delta(p, q, r) \equiv \left\langle \sigma_0, \sigma_1, \sigma_\infty \mid \sigma_0^p = \sigma_1^q = \sigma_\infty^r = \sigma_0\sigma_1\sigma_\infty = 1 \right\rangle.$$

Belyi functions are encoded in Grothendieck's *dessins d'enfants* [63–65]. We define²⁵ a *dessin d'enfant* to be a bi-partite graph D on a compact

²⁴We always assume $\Delta(p, q, r)$ to be *minimal* i.e. p, q, r are the least common multiplexes of all ramifications orders of ξ above 0, 1, ∞ , respectively.

²⁵In the literature there are slightly different definitions. We use the one in ref.[65].

oriented 2–manifold Σ_G such that $\Sigma_G \setminus D$ is a disjoint union of simply connected sets. The vertices of D are colored alternatively in black and white. If $\xi: \Sigma_G \rightarrow \mathbb{P}^1$ is a Belyi function, the corresponding *dessin* is given by $\xi^{-1}([0, 1])$, which is a connected graph.

- a) the preimages of 0 and 1 are, respectively, the white and black vertices of D . The valency of a vertex is equal to the ramification order of ξ at these points;
- b) each component of $\Sigma_G \setminus D$ is a cell containing precisely one pole of ξ . The cell is bounded by $2m$ edges, where m is the order of the pole (ramification index over ∞).

The inverse correspondence *dessin* \rightarrow Belyi function is given by the Riemann existence theorem [67].

A *dessin* $D \subset \Sigma_G$ defines a *canonical triangulation* $T_{\text{can}}(D)$ of Σ_G whose edges are $\xi^{-1}(]0, 1[) \cup \xi^{-1}(]1, \infty[) \cup \xi^{-1}(] \infty, 0[)$. One constructs $T_{\text{can}}(D)$ by introducing a puncture in the center of each cell of $\Sigma_G \setminus D$ and connecting it to the vertices of D . $T_{\text{can}}(D)$ contains two kinds of triangles, *upper* and *lower*, according whether their vertices, ordered in the counter-clockwise direction, are labelled 0, 1, ∞ (resp. 0, ∞ , 1); ξ maps upper/lower triangles into the upper/lower half-plane [68].

Definition. $D \subset \Sigma_G$ a *dessin of enfant*. Its canonical quiver $Q(D)$ is obtained from the triangulation quiver of $T_{\text{can}}(D)$ by vertical separation of the upper/lower triangles (cfr. previous subsection). It describes the class $\mathcal{S}[A_1]$ theory obtained by pulling back $SU(2)^3$ with half a trifundamental by the Belyi function ξ associated to D .

Clearly, the modular quiver $Q(N)$ we obtained previously by direct considerations are just the canonical ones for the *dessins* associated to the modular covers.

The automorphism group \mathbf{G} of the *dessin d'enfant* D acts as an automorphism of the Jacobian category of the covering class $\mathcal{S}[A_1]$ theory. As we saw in **example 1** of §.2.4.5, in general \mathbf{G} does not act freely on the nodes of $Q(D)$, i.e. on the edges of D . By definition, \mathbf{G} acts transitively on the edges iff D is a regular *dessin* [65, 66]; for genus > 1 , this happens precisely when $\Gamma \equiv$ is normal in $\Delta(p, q, r)$ (then it is also torsionless). In this special case [66], Σ_G is a *quasi-platonic surface* i.e. a surface with many automorphisms [65, 66] and $\mathbb{G} = \Delta(p, q, r)/N$ is the group of automorphisms of the regular *dessin*, and also a group of automorphisms of the covering surface Σ_G , isomorphic to the monodromy group and the covering group of the normal

covering Belyi function

$$(4.23) \quad \xi: \Sigma_G \rightarrow \mathbb{H}/\Delta(p, q, r) \equiv \mathbb{P}^1.$$

By definition, in the regular case, the integers p , q , and r are divisors of the degree d of ξ .

Suppose the quotient theory is $SU(2)^3$ SYM coupled to half a trifundamental and three Argyres–Douglas systems of type D_{ℓ_1} , D_{ℓ_2} , and D_{ℓ_3} . Then its $\Delta(p, q, r)/N$ -cover given by a degree d Belyi function ξ , corresponding to a regular *dessin* (4.23), is a gauge theory with gauge group

$$(4.24) \quad SU(2)^m, \quad \text{where } m = \left(3 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) \frac{d}{2},$$

coupled to d half-trifundamentals together with d/p copies of $D_{p\ell_1}$ AD, d/q copies of $D_{q\ell_2}$ AD, and d/r copies of $D_{r\ell_3}$ AD.

Remark. The above discussion was focused on the special case of coverings branched over *three* irregular punctures (the case of two irregular punctures being Abelian and covered in the previous section). For $n > 3$ branching irregular punctures one needs to use a multi-color generalization of the *dessins d’enfants*, see ref.[69].

4.5. Finite chambers of *dessin d’enfant* cover $\mathcal{N} = 2$ QFTs

The BPS spectra of the $\mathcal{N} = 2$ QFTs defined by modular covers $X(N) \rightarrow X(1)$ of §.4.2 may be obtained, at \mathbb{G} -invariant points, by pulling back the BPS spectrum of $SU(2)^3$ SYM coupled to half a tri-fundamental in the corresponding chamber. The BPS spectrum of the last model is well known in many chambers, and may be easily determined for all choices of the central charge Z by the string/band construction (see Appendix B). We begin by describing the BPS spectrum in a class of finite chambers.

We note that the modular quivers $Q(N)$ decompose in the two complementary full subquivers $Q(N)_{\text{up}}$ and $Q(N)_{\text{down}}$ over, respectively, the nodes of the upper and lower Type II blocks. All arrows in $Q(N)$ which connect the two complementary subquivers are oriented from $Q(N)_{\text{down}}$ to $Q(N)_{\text{up}}$

$$(4.25) \quad \boxed{Q(N)_{\text{down}}} \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \boxed{Q(N)_{\text{up}}}$$

Both subquivers, $Q(N)_{\text{down}}$ and $Q(N)_{\text{up}}$ consist of the *disconnected union* of $d(N)$ copies of the cyclic form C_3 of the A_3 quiver (with its superpotential). Correspondingly the charge lattice Γ of the covering theory may be identified with

$$(4.26) \quad \Gamma \cong \bigoplus_{s=1}^{2d(N)} \Gamma_{C_3}, \quad \Gamma_{C_3} \cong \Gamma_{A_3} \text{ (the root lattice of } A_3\text{)}.$$

We choose the central charge so that²⁶

$$(4.27) \quad \arg Z(e_i) < \arg Z(e_j) \text{ for all } e_i \in Q(N)_{\text{down}} \text{ and } e_j \in Q(N)_{\text{up}}.$$

Then a module X is unstable unless $\text{supp } X \subset Q(N)_{\text{down}}$ or $\text{supp } X \subset Q(N)_{\text{up}}$. In fact, a necessary condition for stability is

$$(4.28) \quad \text{supp } X \subset \left\{ \text{a } C_3 \text{ connected component of } Q(N)_{\text{down}} \text{ or } Q(N)_{\text{up}} \right\}.$$

Thus, with respect to the identification (4.26), the stable modules X have charge vectors of the form

$$(4.29) \quad \begin{aligned} \dim X &= 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathbf{x} \oplus 0 \oplus \cdots \oplus 0 \in \bigoplus_{s=1}^{2d(N)} \Gamma_{C_3}, \\ \mathbf{x} \in \Gamma_{C_3} &\text{ is a charge vector of a stable module} \\ &\text{for } A_3 \text{ Argyres–Douglas.} \end{aligned}$$

Hence the BPS spectrum in these chambers consists of $2d(N)$ copies of the spectrum of the Argyres–Douglas of type A_3 . The various copies may correspond to spectra in different chambers of Argyres–Douglas, but at a \mathbb{G} -symmetric point all upper (lower) copies belong to the same AD chamber.

Of course, this is just the pull back of the usual finite chambers for the prism quiver $Q(1)$ in the right hand side of Figure 8. The consistency of this pulled back spectrum sets severe constraints on the structure of the quiver $Q(N)$, which essentially fix its structure to be the one we described in §.4.2.

Clearly, the same discussion holds, *mutatis mutandis*, for general *dessin d'enfant* covering of the $SU(2)^3$ theory with half a trifundamental, and in particular for the quasi-platonic coverings. Indeed, all canonical quivers

²⁶Note that this an open condition; our conclusions will apply to *open* BPS chambers. The subtleties of \mathbb{G} -points being on (the intersection point of several) walls of marginal stability, do not apply to the present situation.

$Q(D)$ decompose into two complementary subquivers, Q_{down} and Q_{up} which are disconnected unions of Type II blocks, and which are connected by arrows in the same direction as in (4.25). Hence

Fact. *All dessin d'enfant class $\mathcal{S}[A_1]$ $\mathcal{N} = 2$ model (see Eqn. (4.24) for its description as a gauge theory) has a chamber whose BPS spectrum consists of $2d$ copies of the A_3 Argyres–Douglas spectrum, where d is the degree of the associated Belyi function ξ .*

4.6. Dessins d'enfants models: general BPS chambers

The BPS spectrum of a regular *dessin d'enfant* $\mathcal{N} = 2$ model with $\ell_1 = \ell_2 = \ell_3 = 1$ may be determined in a rather explicit way in *any* BPS chamber. Indeed, the Jacobian algebra $\mathcal{J}(D)$ of the canonical quiver of a *dessin* is a string algebra, and its indecomposable modules may be constructed in terms of strings and bands. The special techniques introduced in §§.2.4.2, 2.4.4 apply to all these theories, leading to a rather explicit description of their BPS spectra in infinite as well finite chambers, both *per se* as well as covers of the well understood $SU(2)^3$ with a half trifundamental theory.

5. Galois covers and the quantum monodromy $\mathbb{M}(q)$

One purpose of BPS spectral covers is to prove/verify the WCF, which is essentially the same as computing the action of (the classical limit of) the quantum monodromy $\mathbb{M}(q)$ on the quantum torus algebra $\mathbb{T}_Q(q)$ [5].

We recall the basics of the formalism of ref.[5]. Given a 2–acyclic quiver Q with exchange matrix B and (reduced, non–degenerate) superpotential \mathcal{W} , the associated quantum torus algebra $\mathbb{T}_Q(q)$ is defined as follows [5, 73–75]: to each node i of Q (i.e. to each object of the Jacobian category $\mathcal{J} \equiv \mathbb{C}Q/(\partial\mathcal{W})$) we associate an invertible quantum operator Y_i , and set

$$(5.1) \quad \mathbb{T}_Q(q) = \mathbb{C}\langle Y_i, Y_j^{-1} \rangle / \mathbb{I}(q),$$

where $\mathbb{I}(q)$ is the ideal generated by the commutation relations (here $q \equiv e^{2\pi i\tau} \in \mathbb{C}^\times$)

$$(5.2) \quad Y_i Y_j = q^{B_{ij}} Y_j Y_i.$$

Replacing the quiver Q by its mutation at the k -th node, $\mu_k(Q)$, induces a correspondence²⁷ $\mathbb{T}_Q(q) \rightarrow \mathbb{T}_{\mu_k(Q)}(q)$ of quantum torus algebras given explicitly by the Fock–Goncharov mutation operation²⁸ ϕ_k [5, 20, 73–75]

$$(5.3) \quad \phi_k(Y_i) = \begin{cases} Y_k^{-1} & \text{if } i = k \\ Y_i & \text{if there are no arrows } i \rightarrow k \text{ in } \mu_k(Q) \\ q^{-m^2/2} Y_i Y_k^m & \text{if there are } m \geq 1 \text{ arrows } i \rightarrow k \text{ in } \mu_k(Q). \end{cases}$$

One extends the definition of the elements of the torus algebra to all charge vectors $\gamma \equiv \sum_i n_i e_i \in \Gamma$ by setting $Y_{e_i} = Y_i$ and defining

$$(5.4) \quad Y_{\gamma_1 + \gamma_2} = q^{-B(\gamma_1, \gamma_2)/2} Y_{\gamma_1} Y_{\gamma_2}, \quad \text{where } B(\gamma_1, \gamma_2) \equiv B_{ij} n_{1,i} n_{2,j}.$$

Then the quantum monodromy $\mathbb{M}(q)$ is the ordered product [5]

$$(5.5) \quad \mathbb{M}(q) = \widehat{\prod}_{(\gamma, s) \in \text{BPS}} \Psi(q^s Y_\gamma; q)^{(-1)^{2s}}$$

where the product is over all the (Clifford vacua of the) stable BPS multiplets with charge $\gamma \in \Gamma$ and spin s ; $\Psi(y; q)$ is the following version of the compact quantum dilogarithm [76]

$$(5.6) \quad \Psi(y; q) = \prod_{j=0}^{\infty} (1 - q^{j+1/2} y)^{-1},$$

with functional equation

$$(5.7) \quad \Psi(q^{\pm 1} y; q) = (1 - q^{\pm 1} y)^{\pm 1/2} \Psi(y; q),$$

and $\widehat{\prod}$ stands for the product ordered in increasing (cyclic) BPS phase $\arg Z(\gamma)$. The *refined* WCF [5, 25–27] is the statement that the conjugacy

²⁷Note that $\mathbb{T}_Q(q)$ is, up to unitary equivalence, a central extension of the algebra of quantum operators $e^{ix_a}, e^{2\pi i \tau p_a}$, where x_a, p_a are canonical operators in $L^2(\mathbb{R}^{\text{rank } B/2})$, satisfying $[x_a, p_b] = i\delta_{ab}$. Since rank B is a mutation invariant [47], all algebras $\mathbb{T}_{Q'}(q)$, with Q' in the mutation class of Q , are abstractly the ‘same’ algebra, and hence the correspondence $\mathbb{T}_Q(q) \rightarrow \mathbb{T}_{Q'}(q)$ makes sense.

²⁸We recall that the Y ’s before and after the modification belong to *distinct* quantum torus algebras; to avoid cluttering we shall not make the distinction manifest in the notation, we hope this will not cause confusion.

class of $\mathbb{M}(q)$ is independent of the chamber used to define it. The ordinary WCF is the classical limit, $q \rightarrow 1$, of the statement.

We stress a *caveat* which one should always keep in mind. The form (5.5) of the quantum monodromy as a product of quantum dilogarithms holds in the *generic* case, that is, under the assumption that all BPS states which have the same phase are mutually local

$$(5.8) \quad \textbf{(genericity condition): } \gamma_1, \gamma_2 \in \text{BPS} \quad \text{and} \\ \arg Z(\gamma_1) = \arg Z(\gamma_2) \Rightarrow \langle \gamma_1, \gamma_2 \rangle_D = 0.$$

Otherwise one should work with the (quantum) KS formula in its more general form²⁹ [1]. Although mathematically the stability conditions and the quantum monodromy make perfect sense at such non-generic points, and all general theorems (properly formulated) still hold, *physically* the BPS spectrum at such points is rather tricky since these points are exactly *on* instability walls separating different BPS chambers; the mathematically stable modules correspond to the BPS states which remain stable *on all sides* of the walls. This subtlety is important in our present context since we are studying $\mathcal{N} = 2$ models at points having \mathbb{G} symmetry which are typically very non-generic. One should be careful in stating the (possibly subtle) physical interpretation of the straightforward mathematical results.

5.1. A baby example: the covering $N_f = 2 \rightarrow N_f = 0$

Consider the quiver $Q^{(2)}$ in the left hand side of Figure (3.1). Its quantum torus algebra $\mathbb{T}_{Q^{(2)}}(q)$ has a very simple structure: the two operators

$$(5.9) \quad Y_{\bullet_1} Y_{\bullet_2}^{-1} \quad \text{and} \quad Y_{\circ_1} Y_{\circ_2}^{-1}$$

commute with everything and hence are just *c*-numbers in any irreducible representation. Hence, for some complex numbers t, s

$$(5.10) \quad Y_{\bullet_2} = t Y_{\bullet_1}, \quad Y_{\circ_2} = s Y_{\circ_1}.$$

Since the validity of the WCF is independent of the chosen representation of the torus algebra, we may take $s = t = -1$. Then the generators of $\mathbb{T}_{Q^{(2)}}$ are odd with respect to the Galois group $\mathbb{Z}_2 \equiv \{1, \iota\}$, which then acts on the

²⁹For *non-generic* phase configurations, the BPS counting indices $\Omega(\gamma)$ are not even expected to be integers, cfr. **Conjecture 6** in ref.[1].

quantum torus algebra as

$$(5.11) \quad \iota \cdot Y_\gamma \equiv Y_{\iota\gamma} = (-1)^{|\gamma|} Y_\gamma, \quad \text{where } |\gamma| \equiv \left| \sum_i n_i e_i \right| = \sum_i n_i.$$

If γ is a Schur root of $\widehat{A}(2, 2)$, that is, the dimension vector of a brick of the associated algebra, this is equivalent to

$$(5.12) \quad \iota \cdot Y_\gamma \equiv Y_{\iota\gamma} = (-1)^{m(\gamma)} Y_\gamma,$$

where $m(\gamma)$ is the magnetic component of the charge vector γ .

Let $N_f = 2$ SQCD be endowed with a pull back central charge³⁰ as in §.2.4.8, so that $\arg Z(\iota \cdot \gamma_j) = \arg Z(\gamma_j)$. Let γ_j be the charge of a stable BPS state which is not the W boson; then (cfr. Table (3.9)) $\iota \cdot \gamma_j$ is the charge of a *distinct* BPS particle which is also stable and has the same phase $\arg Z_{\gamma_j}$ and spin s_j . Their two quantum dilogs come together in the ordered product for $\mathbb{M}(q)$

$$(5.13) \quad \mathbb{M}(q) = \cdots \Psi(q^{s_j} Y_{\gamma_j}; q)^{(-1)^{2s_j}} \Psi(q^{s_j} Y_{\iota\gamma_j}; q)^{(-1)^{2s_j}} \cdots$$

If $m(\gamma_j) \neq 0$ (which implies $s_j = 0$, cfr. Eqn. (3.9)) the product of two dilogs simplifies to a single dilog of nome q^2

$$(5.14) \quad \mathbb{M}(q) = \cdots \Psi(Y_{2\gamma_j}; q^2) \cdots$$

The change of nome, $q \mapsto q^2$, reflects the fact that the subalgebra of $\mathbb{S}(q) \subset \mathbb{T}_{Q^{(2)}}(q)$ generated by the two elements

$$(5.15) \quad Y_\bullet = Y_{\bullet_1} Y_{\bullet_2}, \quad Y_\circ = Y_{\circ_1} Y_{\circ_2}$$

(and their inverse) is identified with the torus algebra of $N_f = 0$ SQCD, $\mathbb{T}_{Q^{(0)}}(q)$ up to the redefinition of the nome $q \mapsto q^2$

$$(5.16) \quad \mathbb{S}(q) = \mathbb{T}_{Q^{(0)}}(q^2).$$

Then the factor $\Psi(Y_{2\gamma_j}; q^2)$ in Eqn. (5.14) is identified with the element of $\mathbb{T}_{Q^{(0)}}(q^2)$ corresponding to the monodromy factor of the (single) hypermultiplet in the quotient theory having charge the push down of γ_j .

³⁰This is not a loss of generality: the quantum monodromy and hence the functional relation between quantum monodromies of different $\mathcal{N} = 2$ theories are *independent* of the chosen central charge.

It remains to consider the factors in $\mathbb{M}(q)$ from the BPS states of the cover $N_f = 2$ SQCD with zero magnetic charge. They have all the same³¹ phase $\arg Z(\gamma)$. So they contribute to the monodromy the factor³²

$$\begin{aligned}
 (5.17) \quad & \Psi(q^{-1/2}Y_{\bullet_1}Y_{\circ_1}; q) \Psi(q^{-1/2}Y_{\bullet_2}Y_{\circ_2}; q) \Psi(q^{-1/2}Y_{\bullet_1}Y_{\circ_2}; q) \\
 & \times \Psi(q^{-1/2}Y_{\bullet_2}Y_{\circ_1}; q) \Psi(q^{-3/2}Y_{\bullet_1}Y_{\circ_1}Y_{\bullet_2}Y_{\circ_2}; q)^{-1} \\
 & \times \Psi(q^{-1/2}Y_{\bullet_1}Y_{\circ_1}X_{\bullet_2}Y_{\circ_2}; q)^{-1} \\
 & = \Psi((q^{-1/2}Y_{\bullet_1}Y_{\circ_1})^2; q^2)^2 \Psi(q^{-1/2}(q^{-1/2}Y_{\bullet_1}Y_{\circ_1})^2; q)^{-1} \\
 & \times \Psi(q^{1/2}(q^{-1/2}Y_{\bullet_1}Y_{\circ_1})^2; q)^{-1}
 \end{aligned}$$

where we used Eqn. (5.10); this product belongs to the subalgebra $\mathbb{S}(q)$ and is identified with an element of the quantum torus algebra of the quotient theory i.e. pure SYM. Explicitly,

$$\begin{aligned}
 (5.18) \quad & \frac{\Psi(Y_{\bullet+\circ}; q^2)^2}{\Psi(q^{-1/2}Y_{\bullet+\circ}; q) \Psi(q^{1/2}Y_{\bullet+\circ}; q)} \\
 & = \prod_{n \geq 0} \frac{(1 - q^n Y_{\bullet+\circ}) (1 - q^{n+1} Y_{\bullet+\circ})}{(1 - q^{2n+1} Y_{\bullet+\circ})^2} \\
 & = \prod_{n \geq 0} (1 - q^{2n} Y_{\bullet+\circ}) \prod_{n \geq 0} (1 - q^{2n+2} Y_{\bullet+\circ}) \\
 & = \Psi((q^2)^{-1/2} Y_{\bullet+\circ}; q^2)^{-1} \Psi((q^2)^{1/2} Y_{\bullet+\circ}; q^2)^{-1},
 \end{aligned}$$

which is precisely the contribution from a single vector multiplet of charge $\delta = e_{\bullet} + e_{\circ}$, that is, the usual W boson of $SU(2)$ SYM. Thus, in all (pull back) BPS chambers, the quantum monodromy of the Galois cover $N_f = 2$ theory is equal — factor by factor — to the quantum monodromy of the quotient $N_f = 0$ model up the redefinition $q \mapsto q^2$. Therefore, the WCF for the Galois cover implies the WCF for the quotient, and we get the functional identity

$$(5.19) \quad \mathbb{M}^{(N_f=0)}(q^2) \doteq \mathbb{M}^{(N_f=2)}(q),$$

where \doteq means equality of actions in the \mathbb{Z}_2 -invariant subsector of a representation of the covering quantum torus algebra satisfying (5.11).

³¹Indeed, if X is a brick with $m(X) = 0$ it is either in the W family or $F_{\lambda}X$ is in the W family.

³²The order of the six factors is irrelevant since they commute.

5.2. Galois coverings, cluster mutations, and finite BPS chambers

The description of the quantum monodromy $\mathbb{M}(q)$ in [5] was based on quantum cluster algebras [73, 74] and the mutation algorithm [19, 20] (for the mathematical side see [75]).

To compare with the standard conventions³³ in cluster algebra theory, we should interchange the two square roots of q , $q^{1/2} \leftrightarrow -q^{1/2}$; this amounts to replacing the quantum dilog $\Psi(y; q)$ with the function $\mathbb{E}(y) \equiv \Psi(-y; q)$ [75]. The (initial) Y -seed of the quantum cluster algebra is identified with set of generators Y_i of the quantum torus algebra $\mathbb{T}_Q(q)$. A quantum cluster mutation is the composition of elementary mutations at single nodes of the quiver Q . The elementary quantum mutation at node $k \in Q$ of the Y -seed acts by

- i*) an elementary mutation of the 2-acyclic quiver with superpotential $Q \rightarrow \mu_k(Q)$ [30];
- ii*) an operation on the Y -seed $\mathcal{Q}_k: \mathbb{T}_Q \rightarrow \mathbb{T}_{\mu_k(Q)}$ given by the ‘right intertwiner’ [73–75]

$$(5.20) \quad \mathcal{Q}_k \cdot Y_i = \mathbb{E}(Y_k) \phi_k(Y_i) \mathbb{E}(Y_k)^{-1}$$

where ϕ_k is defined in Eqn. (5.3).

Explicitly (here B_{ij} is the exchange matrix of Q)

$$(5.21) \quad \mathcal{Q}_k \cdot Y_i = \begin{cases} Y_k^{-1} & \text{if } i = k \\ Y_i \prod_{s=1}^{B_{ki}} (1 + q^{-1/2+s} Y_k) & \text{if } B_{ki} \geq 0 \text{ in } \mu_k(Q) \\ q^{-B_{ik}^2/2} Y_i Y_k^{B_{ik}} \prod_{s=1}^{B_{ik}} (1 + q^{1/2-s} Y_k)^{-1} & \text{if } B_{ki} \leq 0 \text{ in } \mu_k(Q). \end{cases}$$

It follows from the arguments of [5, 19, 20] that if there exists a sequence k_1, k_2, \dots, k_s such that

$$(5.22) \quad \phi_{k_1} \phi_{k_2} \cdots \phi_{k_s}(Y_i) = Y_{\pi(i)}^{-1},$$

for some permutation π of the nodes of Q , then the quantum monodromy is $\mathbb{M}(q) = \mathbb{K}(q)^2$, where the half-monodromy operator $\mathbb{K}(q)$ is defined³⁴ by

³³We adopt Keller’s conventions [75]. The change of sign is required to make the *positivity property* of cluster algebras manifest (unfortunately it makes the physics less transparent).

³⁴More precisely: *defined up to the commutant of the quantum torus algebra*; see [81] for more details.

its adjoint action on $\mathbb{T}_Q(q)$

$$(5.23) \quad \text{Ad}(\mathbb{K}(q)) = \Pi^{-1} \mathcal{Q}_{k_1} \mathcal{Q}_{k_2} \cdots \mathcal{Q}_{k_s},$$

where the operator Π implements the automorphism $Y_i \rightarrow Y_{\pi(i)}$. We call a sequence of mutations with the property (5.22) a *green sequence for Q* .

Comparing with Eqn. (5.20), we see that $\mathbb{M}(q)$ is automatically a finite product of quantum dilogarithms $\mathbb{E}(Y_\gamma)$ [5]. Accordingly to §.5, each such dilogarithm is associated with a BPS hypermultiplet; then the BPS spectrum is *finite* consisting precisely of s hypermultiplets. Conversely, *all generic* finite chambers arise this way [19, 20]. The combinatorics of quantum cluster mutations, which act as products of quantum dilogarithms, describes quantum monodromies which satisfy the (**genericity condition**) of Eqn. (5.8). In particular, the quantum mutations in Eqn. (5.23) are *automatically* ordered according to the phase of the central charge of the corresponding BPS particle.

5.2.1. Mutations of 2–acyclic quivers with freely acting automorphisms. Now suppose the 2–acyclic finite quiver with superpotential (Q, \mathcal{W}) has a group \mathbb{G} of automorphisms, acting freely on the nodes, such that the \mathbb{G} –orbit category of its Jacobian category is again a bounded Jacobian category of a 2–acyclic quiver. Then, for all $i \in Q$ and $g \in \mathbb{G}$, $B_{i gi} = 0$; moreover, Q has no subquivers of the form $i \rightarrow j \rightarrow gi$. This means that the elementary quantum cluster mutations \mathcal{Q}_i and \mathcal{Q}_{gi} commute for all $g \in \mathbb{G}$, and we may unambiguously define

$$(5.24) \quad \mu_{\mathbb{G}i} = \prod_{g \in \mathbb{G}} \mu_{gi}, \quad \phi_{\mathbb{G}i} = \prod_{g \in \mathbb{G}} \phi_{gi}, \quad \mathcal{Q}_{\mathbb{G}i} = \prod_{g \in \mathbb{G}} \mathcal{Q}_{gi},$$

where all factors are distinct since \mathbb{G} acts freely. The mutations $\mu_{\mathbb{G}i}$ transform a quiver which is \mathbb{G} –symmetric into a quiver which is \mathbb{G} –symmetric; at the level of the quotient quiver, the mutation $\mu_{\mathbb{G}i}$ induces the elementary mutation at the node $\mathbb{G}i$ (which we also denote as $\mu_{\mathbb{G}i}$). The same remark applies to the associated change of basis $\phi_{\mathbb{G}i}$.

Suppose the covering $\mathcal{N} = 2$ theory has a *finite* BPS chamber which contains a \mathbb{G} –invariant point at which condition (5.8) holds. At the \mathbb{G} –invariant point, the BPS states form complete \mathbb{G} –orbits, while all BPS states in a \mathbb{G} –orbit have the same phase. Since the product (5.22) is automatically phase–ordered, all factors in the same \mathbb{G} –orbit should appear grouped together.

Then we have

$$(5.25) \quad \phi_{k_1} \phi_{k_2} \cdots \phi_{k_s} = \phi_{\mathbb{G}i_1} \phi_{\mathbb{G}i_2} \cdots \phi_{\mathbb{G}i_t}, \quad t = s/|\mathbb{G}|.$$

But the RHS is canonically interpreted as a green sequence for the quotient quiver, and hence as the BPS spectrum in a corresponding *finite* chamber of the quotient theory having $s/|\mathbb{G}|$ hypers.

Hence the generic finite chambers behave well under *strict-sense* Galois covers of $\mathcal{N} = 2$ theories. We recall an elementary example of Eqn. (5.25) which reproduce (for finite chambers) the results of §.3.1.1 and §.3.2.4 in terms of cluster algebra combinatorics. For $k \mid \gcd(p, q)$, consider the generalized SYM cover with simply-laced gauge group G , i.e.

$$(5.26) \quad \mathbb{C}\widehat{A}(p, q) \boxtimes G \longrightarrow \mathbb{C}\widehat{A}(p/k, q/k) \boxtimes G$$

whose green sequences (and hence finite BPS chambers) are given in Appendix B of ref.[16]. From their explicit form, it is pretty obvious³⁵ that green functions for covering and quotient models are related as in Eqn. (5.25).

5.2.2. Example: BPS spectra of branched covers of genus zero $\mathcal{S}[\mathbf{A}_1]$ models. As a new example, we consider the \mathbb{Z}_k Galois covers of $SU(2)^2$ SYM coupled to a bifundamental described by the quiver in Figure 7. We note that all mutations at white nodes \circ (resp. black nodes \bullet) commute. We then write ϕ_\circ , ϕ_\bullet and ϕ_* for the product of all mutations at nodes denoted by the same symbol in Figure 7; they are products of mutations of the form $\phi_{\mathbb{Z}_k i}$. We have

$$(5.27) \quad \phi_\circ \phi_\bullet \phi_* \phi_\circ(Y_i) = Y_{\pi(i)}^{-1},$$

where π is the permutation which fixes the nodes $*$ and interchanges the black/white pairs \bullet, \circ which are connected by an arrow belonging to an oriented triangle.

Eqn. (5.27) gives a finite chamber of the covering theory with $7k$ hypermultiplets which is a k -fold cover of the seven hyper finite chamber of $SU(2)^2$ with a bifundamental.

³⁵The finite chamber of $\mathbb{C}\widehat{A}(p, q) \boxtimes G$ consists of $(p + q)$ copies of the maximal chamber spectrum of Argyres–Douglas of type G [16], which is trivially a k -fold cover of the $(p + q)/k$ copies in the finite chamber of $\mathbb{C}\widehat{A}(p/k, q/k) \boxtimes G$.

5.3. \mathbb{Z}_k coverings of $SU(2)$ SQCD: friezes and SL_2 -tilings

Of course, we are more interested in infinite chambers. The example of §. 5.1 was particular simple since the quantum monodromies of the covering and quotient theories could be seen as acting on the same Hilbert space and hence compared directly. We wish to generalize the relation between the cover and quotient monodromies to the \mathbb{Z}_k covering of $SU(2)$ SYM coupled to arbitrary Argyres–Douglas systems of types $D_{p/k}$ and $D_{q/k}$ by SYM coupled to AD systems of types D_p , and D_q , which corresponds to the cover of \mathbb{C} -categories

$$(5.28) \quad \mathbb{C}\widehat{A}(p, q) \longrightarrow \mathbb{C}\widehat{A}(p/k, q/k), \quad \text{where } k \mid \gcd(p, q).$$

To this end, we take advantage of the description of the quantum monodromy $\mathbb{M}(q)$ in terms of quantum cluster algebras [5, 73–75, 77], and of the existence of linear recursions for cluster variables of affine quivers [78] which, in the present context, may be phrased in the suggestive language of *friezes* and SL_2 -tilings of the plane [79].

Although the results below will be valid for all covers (5.28), to simplify the presentation we first assume $p = q$. We consider the quiver $\widehat{A}(p, p)$ where sources and sinks alternate; we denote the sources as \circ_a and the sinks as \bullet_a and we take the index $a \in \mathbb{Z}$ with the mod p identifications $\circ_{a+p} \sim \circ_a$, $\bullet_{a+p} \sim \bullet_a$. The labels are chosen as follows

$$(5.29) \quad \cdots \longrightarrow \bullet_a \longleftarrow \circ_{a+1} \longrightarrow \bullet_{a+1} \longleftarrow \circ_{a+2} \longrightarrow \bullet_{a+2} \longleftarrow \circ_{a+3} \longrightarrow \bullet_{a+3} \longleftarrow \cdots$$

Mutations at nodes \bullet_a (resp. \circ_a) commute between themselves. Following Eqn. (5.24), we write

$$\begin{aligned} \mu_\bullet &= \prod_{a=1}^p \mu_{\bullet_a}, & \mu_\circ &= \prod_{a=1}^p \mu_{\circ_a}, & \phi_\bullet &= \prod_{a=1}^p \phi_{\bullet_a}, \\ \phi_\circ &= \prod_{a=1}^p \phi_{\circ_a}, & \mathcal{Q}_\bullet &= \prod_{a=1}^p \mathcal{Q}_{\bullet_a}, & \mathcal{Q}_\circ &= \prod_{a=1}^p \mathcal{Q}_{\circ_a}. \end{aligned}$$

One has $\mu_\bullet(Q) = \mu_\circ(Q) = Q^{\text{op}}$ and $\mu_\circ\mu_\bullet(Q) = Q$. Then

$$(5.30) \quad \phi_\bullet \begin{pmatrix} Y_{\circ_a} \\ Y_{\bullet_a} \end{pmatrix} = \begin{pmatrix} Y_{\circ_a} \\ Y_{\bullet_a}^{-1} \end{pmatrix}, \quad \phi_\circ\phi_\bullet \begin{pmatrix} Y_{\circ_a} \\ Y_{\bullet_a} \end{pmatrix} = \begin{pmatrix} Y_{\circ_a}^{-1} \\ Y_{\bullet_a} \end{pmatrix},$$

so that the half-monodromy is $\text{Ad}(\mathbb{K}(q)) = \mathcal{Q}_\circ \mathcal{Q}_\bullet$. Then

$$\begin{aligned}
 \mathcal{Q}_\bullet \begin{pmatrix} Y_{\circ_a} \\ Y_{\bullet_a} \end{pmatrix} &=_{\mathcal{Q}} \begin{pmatrix} Y_{\circ_a} \prod_{\substack{\circ_a \rightarrow \bullet_b \\ \text{in } Q}} (1 + q^{1/2} Y_{\bullet_b}) \\ Y_{\bullet_a}^{-1} \end{pmatrix}, \\
 \mathcal{Q}_\circ \begin{pmatrix} Y_{\circ_a} \\ Y_{\bullet_a} \end{pmatrix} &=_{\mu_\bullet(Q)} \begin{pmatrix} Y_{\circ_a}^{-1} \\ Y_{\bullet_a} \prod_{\substack{\circ_b \rightarrow \bullet_a \\ \text{in } Q}} (1 + q^{1/2} Y_{\circ_b}) \end{pmatrix}
 \end{aligned}
 \tag{5.31}$$

We promote each operator Y_i to a sequence of operators $\{\tilde{Y}_{i,n}\}_{n \in \mathbb{Z}}$ by setting $\tilde{Y}_{\bullet_a,0} \equiv Y_{\bullet_a}$, $\tilde{Y}_{\circ_a,0} \equiv Y_{\circ_a}$ and

$$\begin{aligned}
 \tilde{Y}_{\bullet_a,2n+1} &= \mathcal{Q}_\bullet \tilde{Y}_{\bullet_a,2n}, & \tilde{Y}_{\bullet_a,2n+2} &= \mathcal{Q}_\circ \tilde{Y}_{\bullet_a,2n+1}, \\
 \tilde{Y}_{\circ_a,2n+1} &= \mathcal{Q}_\bullet \tilde{Y}_{\circ_a,2n}, & \tilde{Y}_{\circ_a,2n+2} &= \mathcal{Q}_\circ \tilde{Y}_{\circ_a,2n+1}
 \end{aligned}
 \tag{5.32}$$

One has $\tilde{Y}_{\bullet_a,2n+1} = \tilde{Y}_{\bullet_a,2n}^{-1}$ and $\tilde{Y}_{\circ_a,2n} = \tilde{Y}_{\circ_a,2n-1}^{-1}$, so half the operators are redundant, and we simplify the notation by writing

$$Y_{\bullet_a,n} \equiv \tilde{Y}_{\bullet_a,2n}, \quad Y_{\circ_a,n} = \tilde{Y}_{\circ_a,2n+1}.
 \tag{5.33}$$

In terms of the sequence of operators $\{Y_{\bullet_a,n}, Y_{\circ_a,n}\}_{a,n \in \mathbb{Z}}$ the action of the quantum monodromy is given by shifts of the index n

$$\text{Ad}(\mathbb{M}(q)^m) Y_{i,n} = Y_{i,n+2m}, \quad i = \bullet_a, \circ_a.
 \tag{5.34}$$

From Eqn. (5.31), the sequence of operators $Y_{i,n}$ satisfies the quantum recursion relation

$$Y_{i,n} Y_{i,n+1} = \prod_{i \rightarrow j} (1 + q^{1/2} Y_{j,n+1}) \prod_{j \rightarrow i} (1 + q^{1/2} Y_{j,n}),
 \tag{5.35}$$

where all arrows are now referred to the original form of the quiver Q . Hence the action of the quantum monodromy $\mathbb{M}(q)$ on the quantum torus algebra $\mathbb{T}_Q(q)$ is described by the solutions to the recursion relations (5.35), see Eqn. (5.34).

Our goal is to show that the action of the monodromy for the quiver $\widehat{A}(p,p)$ is a k -fold cover of the action of the monodromy for the quiver $\widehat{A}(p/k,p/k)$. This is the same as saying that the solutions to the recursion relations (5.35) for $\widehat{A}(p,p)$ are ‘ k -fold covers’ of the solutions for $\widehat{A}(p/k,p/k)$. (We could be more general, and consider the k -fold cover $\widehat{A}(p,q)$ of $\widehat{A}(p/k,qk)$; we shall comment on such generalization momentarily). To show

this property of (5.35), we take advantage from the fact that its quantum solutions are determined, up to the correct prescription for the operator ordering, by the corresponding classical solutions, i.e. by their limit as $q \rightarrow 1$ [5]. In this limit, the half-monodromy action for $\widehat{A}(p, p)$, $Y_{i,n} \rightarrow Y_{i,n+2}$ is given by a rational map $\mathbb{C}^{2p} \xrightarrow{K(p)} \mathbb{C}^{2p}$ and likewise for $\widehat{A}(p/k, p/k)$ by a rational map $\mathbb{C}^{2p/k} \xrightarrow{K(p/k)} \mathbb{C}^{2p/k}$. Consider the diagonal map

$$(5.36) \quad \Delta: \mathbb{C}^{2p/k} \longrightarrow \overbrace{\mathbb{C}^{2p/k} \oplus \mathbb{C}^{2p/k} \oplus \dots \oplus \mathbb{C}^{2p/k}}^{k \text{ factors}} \cong \mathbb{C}^{2p}$$

which embeds $\mathbb{C}^{2p/k}$ into the \mathbb{Z}_k fixed subset of \mathbb{C}^{2p} . To prove the covering property of the quantum monodromy is equivalent to show that the following diagram of rational maps is commutative

$$(5.37) \quad \begin{array}{ccc} \mathbb{C}^{(p+q)/k} & \xrightarrow{K(p/k, q/k)} & \mathbb{C}^{(p+q)/k} \\ \Delta \downarrow & & \downarrow \Delta \\ \mathbb{C}^{p+q} & \xrightarrow{K(p, q)} & \mathbb{C}^{p+q} \end{array}$$

This is already evident from the form of Eqn. (5.35); indeed the recursion relation is invariant under translation of the first index

$$(5.38) \quad i \equiv \bullet_a, \circ_a \longrightarrow \bullet_{a+\ell}, \circ_{a+\ell} \equiv i[\ell]$$

for all $\ell \in \mathbb{N}$. Then, if the initial values $\{Y_{\bullet_a, 0}, Y_{\circ_a, 0}\}$ are chosen to be periodic in a of period ℓ , the solution $\{Y_{\bullet_a, n}, Y_{\circ_a, n}\}_{n \in \mathbb{Z}}$ is also periodic in a with the same period. By definition, $\{Y_{\bullet_a, 0}, Y_{\circ_a, 0}\}$ for the $\widehat{A}(p, q)$ case is periodic of period $p + q$, while for $\widehat{A}(p/k, p/k)$ of period $(p + q)/k$. Eqn. (5.37) then says that, if we make a *special* choice for the $(p + q)$ periodic initial datum of the $\widehat{A}(p, q)$ model which is actually $(p + q)/k$ periodic, we get a $(p + q)/k$ periodic solution which we may identify with a solution of the recursion for the $\widehat{A}(p/k, q/k)$ quiver. This is confirmed by the explicit solution of the recursion (5.35) that we now describe.

5.3.1. Solutions of the recursion: $SL(2, \mathbb{C})$ -tiling of the plane. It is convenient to parametrize the $Y_{i,n}$ in terms of new variables $X_{i,n}$ (in the cluster algebraic language, we are re-expressing the Y -seed in terms of the

X -seed [82]) as

$$(5.39) \quad Y_{i,n} = \prod_{i \rightarrow j} X_{j,n} \prod_{j \rightarrow i} X_{j,n-1}.$$

In terms of the $X_{i,n}$ the recursion becomes

$$(5.40) \quad X_{i,n+1}X_{i,n} = 1 + \prod_{i \rightarrow j} X_{j,n+1} \prod_{j \rightarrow i} X_{j,n}.$$

Written in this form, the recursion is well known: it is the *frieze* of the opposite quiver Q^{op} [78, 79]. Friezes exist for all acyclic quivers; they are defined by a source sequence [78]. In the example of the alternating $\widehat{A}(p, p)$ the source sequence is $(\bullet_1, \dots, \bullet_p, \circ_1, \dots, \circ_p)$. Written in terms of friezes, our formulae hold for all $\widehat{A}(p, q)$ covers.

For $\widehat{A}(p, q)$ the solutions to the frieze recursion (5.40) may be written in terms of SL_2 -tilings of the two dimensional square lattice \mathbb{Z}^2 [79]. Let R be a commutative ring with unit: an $SL(2, R)$ -tiling of the plane is an assignment of an element $r(i, j) \in R$ to each vertex $(i, j) \in \mathbb{Z}^2$ of the square lattice such that each plaquette in the lattice, seen as a 2×2 matrix, belongs to $SL(2, R)$, that is,

$$(5.41) \quad \begin{vmatrix} r(i, j) & r(i + 1, j) \\ r(i, j - 1) & r(i + 1, j - 1) \end{vmatrix} \\ \equiv r(i, j)r(i + 1, j - 1) - r(i + 1, j)r(i, j - 1) = 1.$$

The rule to express the solutions to the frieze (5.40) for a quiver $\widehat{A}(p, q)$ in terms of $SL(2, \mathbb{C})$ -tilings of the plane is as follows [79]. As in Figure (5.29), we consider the *infinite cover* $\widehat{A}(p, q)^\infty$ of the acyclic quiver $\widehat{A}(p, q)$ obtained by numbering its arrows by an integer mod $p + q$, and then forgetting the identifications $i \sim i + p + q$ of the nodes. $\widehat{A}(p, q)^\infty$ is then an infinite sequence, labelled by integers $k \in \mathbb{Z}$, of direct \rightarrow and inverse arrows \leftarrow which repeat in k with period $p + q$. We number nodes in $\widehat{A}(p, q)^\infty$ so that the k node lays between the $(k - 1)$ -th and k -th arrow. We embed $\widehat{A}(p, q)^\infty$ as a continuous path L along the links of the square plane lattice according to the rule that a direct arrow corresponds to a horizontal link and an inverse arrow to a vertical link, see Figure 10 for the example of $\widehat{A}(p, p)$ with the alternating orientation. The path L so obtained is called the *frontier* of the SL_2 -tiling. Sources/sinks of $\widehat{A}(p, q)^\infty$ are mapped to the corners of L .

The initial condition of the recursion relation is specified by assigning to the vertex k of the frontier L the complex number $X_{k,0}$, which repeats with

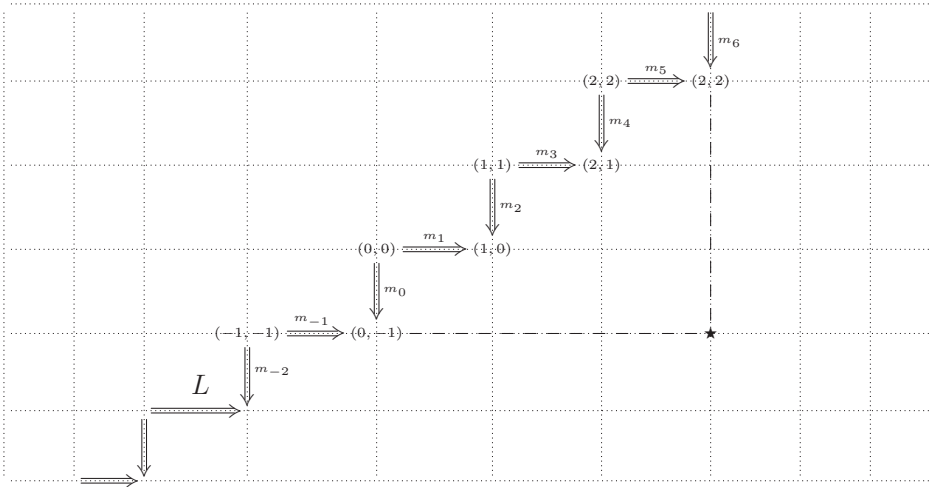


Figure 10: The frontier L in \mathbb{Z}^2 corresponding to the quiver $\widehat{A}(p, p)$ with the alternating orientation for any p . E.g. to the point \star are associated the sequences of initial values $X_{(-1,0)}, X_{(0,0)}, X_{(1,0)}, X_{(2,0)}, X_{(3,0)}, X_{(4,0)}, X_{(5,0)}$ and of matrices $m_0, m_1, m_2, m_3, m_4, m_5$.

periodicity $p + q$ in k , i.e.

$$(5.42) \quad X_{k+p+q,0} = X_{k,0}$$

To the k -th link of the frontier L we associate the 2×2 matrix m_k which also repeats with period $p + q$, i.e. $m_{k+p+q} = m_k$,

$$(5.43) \quad m_k = \begin{cases} \begin{pmatrix} X_{k-1,0} & 1 \\ 0 & X_{k,0} \end{pmatrix} & \text{the } k\text{-th arrow is direct } (\equiv \text{horizontal link in } L) \\ \begin{pmatrix} X_{k,0} & 0 \\ 1 & X_{k-1,0} \end{pmatrix} & \text{the } k\text{-th arrow is inverse } (\equiv \text{vertical link in } L). \end{cases}$$

The $SL(2, \mathbb{C})$ -tiling with boundary condition $X_{i,0}$ is then constructed by assigning to each vertex (i, j) below the frontier³⁶ the complex number $c(i, j)$ determined as follows. One connects the given point (i, j) (denoted by \star in Figure 10) to the frontier L by drawing the vertical and horizontal segments; the intersection points determine a finite segment $L_{(i,j)}$ in the frontier L , hence a finite sequence of points with attached values $X_{k_0}, X_{k_0+1}, \dots, X_{k_0+\ell+1}$, as well as of links with associated matrices $m_{k_0+1}, m_{k_0+2}, \dots, m_{k_0+\ell+1}$, see Figure 10 for an example. Consider the subsequence obtained by forgetting the first and the last matrices, and let $M(i, j)$ the product of the remaining ones

$$(5.44) \quad M(i, j) = m_{k_0+1} m_{k_0+2} \cdots m_{k_0+\ell}.$$

Then

$$(5.45) \quad c(i, j) = \frac{1}{\prod_{s=1}^{\ell} X_{k_0+s,0}} (1 \ X_{k_0,0}) M(i, j) \begin{pmatrix} 1 \\ X_{k_0+\ell+1} \end{pmatrix},$$

is the unique $SL(2, \mathbb{C})$ -tiling on the plane which reduces to the assigned values $X_{k,0}$ on the frontier L [79]. This is obvious for the alternating case, and shown to be true in general in ref.[79].

Let $(i_k, j_k) \in \mathbb{Z}^2$ be the point on the frontier L which corresponds to the node $k \in \widehat{A}(p, q)^\infty$. Then the explicit solution of the frieze (5.40), and hence to the (classical) Y -seed recursion (5.35) is [79]

$$(5.46) \quad X_{k,n} = c(i_k + n, j_k - n), \quad n \in \mathbb{Z}.$$

The solution is periodic in k , $X_{k+p+q,n} = X_{k,n}$ since the boundary condition was periodic. Again, the statement is obvious for the alternating $\widehat{A}(p, p)$ and true in general by ref.[79].

From the SL_2 -tiling viewpoint, the Galois covering property of the action of the monodromy is just the self-similarity of the SL_2 -tiling under ‘period rescaling’ $(p, q) \rightarrow (p\ell, q\ell)$ which is an immediate consequence of the identity of infinite quivers

$$(5.47) \quad \widehat{A}(p, q)^\infty \equiv \widehat{A}(p\ell, q\ell)^\infty.$$

Note that from the explicit solution (5.45) it follows that the variables $X_{k,n}$ (and hence $Y_{k,n}$), as a function of n satisfy a linear recursion relation of

³⁶The pattern above the frontier is symmetric, so we may limit ourselves to the pattern below it [79].

finite length [79] (see also [78]). Comparing with Eqn. (5.34), we see that the monodromy acts by linear recursions, a fact whose physical meaning is discussed in [77].

5.3.2. Extension to general class $\mathcal{S}[A_1]$ theories. The previous explicit construction of the (classical limit of the) action of the monodromy $\mathbb{M}(q)$ in terms of friezes may be extended to all class $\mathcal{S}[A_1]$ theories; the extension is particularly easy when all punctures on the Gaiotto surface Σ_G are irregular.

Above we have thought of the Euclidean algebra $\mathbb{C}\widehat{A}(p, q)$ as a hereditary algebra; but it is also a string algebra and hence all its indecomposable modules may be described in the string/band language (see Appendix B). The points of the lattice \mathbb{Z}^2 below the frontier (cfr. Figure 10) are naturally identified (modulo periodicity) with the *string modules* of $\mathbb{C}\widehat{A}(p, q)$: to each point (i, j) we associate the string given by the sequence of direct/inverse arrows in the corresponding segment of frontier $L_{(i,j)}$ (see Figure 10). Thus we see that the map

$$(5.48) \quad (\text{string module}) \longmapsto (\text{frieze variable}) \equiv X\text{-seed}$$

is given by replacing each direct (resp. inverse) arrow in the string by a 2×2 matrix m_k having the form in the upper line (resp. bottom line) of Eqn. (5.43), and multiplying by the appropriate factors for the two end points of the string. That this prescription works in general (in the string case) is proven in ref.[?]. From the sequence of X -seeds we get the sequence of Y -seeds in the standard way [82]; then the classical action of $\mathbb{M}(q)$ is given by a shift of the Y -seed sequence.

In particular, this shows the covering property for the action of the monodromy of the *dessins d'enfants* examples of section 4, whose algebras are string algebras.

5.4. Covers of $\mathcal{N} = 2$ gauge theories with gauge group G

The above analysis may be extended to the \mathbb{Z}_k coverings of G gauge theories

$$(5.49) \quad \mathbb{C}\widehat{A}(p, q) \boxtimes G \rightarrow \mathbb{C}\widehat{A}(p/k, q/k) \boxtimes G$$

discussed in §.3.2. The monodromy of pure SYM also acts by linear recursions (called Q -systems [83, 84]), and the situation looks rather similar to the one discussed above as a consequence of asymptotic freedom of the associated $\mathcal{N} = 2$ QFTs [77]. For simplicity, we consider the case $p = q$ and

use the square form of the quiver $\widehat{A}(p, p) \square G$. Again, as in §.5.3 we replace the alternating affine quiver $\widehat{A}(p, p)$ by its infinite cover $\widehat{A}(p, p)^\infty$, so that we work with the infinite cover product quiver $\widehat{A}(p, p)^\infty \square G$. By standard manipulations [5, 71], similar to the ones in §.5.3, we reduce the action of the quantum monodromy in the classical limit to the recursion relation of the (A_∞, G) Y -system

$$(5.50) \quad Y_{k,a,n-1} Y_{k,a,n+1} = \frac{(1 + Y_{k-1,a,n})(1 + Y_{k+1,a,n})}{\prod_{b \in G} (1 + Y_{k,b,n}^{-1})^{A_{ab}}}, \quad k \in \mathbb{Z}, \quad a \in G, \quad n \in \mathbb{Z},$$

where the matrix A_{ab} is related to the Cartan matrix of the (simply-laced) group G by

$$(5.51) \quad C_{ab} = 2\delta_{ab} - A_{ab}.$$

In this notation the monodromy acts by a shift in n

$$(5.52) \quad Y_{k,a,n} \xrightarrow{\mathbb{M}} Y_{k,a,n+2h(G)}$$

where $h(G)$ is the Coxeter number of G .

One reduces the Y -system for the infinite cover $\widehat{A}(p, p)^\infty \square G$ to the Q -system for the finite quiver $\widehat{A}(p, p) \square G$ just by taking the initial values of the recursion³⁷, $Y_{k,a,0}$ and $Y_{k,a,1}$, to be periodic in k of period $2p$. Again, the identity (5.47) (with $p = q$) guarantees the covering property for the action of the monodromies in the \mathbb{Z}_k covering of G gauge theories (5.49) (i.e. the analogue of the diagram (5.37) is commutative).

6. BPS spectra of genus one class $\mathcal{S}[A_1]$ theories

In this section we illustrate the Galois cover of BPS spectra in the context of the class $\mathcal{S}[A_1]$ theories [3, 4] defined by a meromorphic quadratic differential ϕ_2 over a genus one curve E . The techniques of section 2 will allow to describe the BPS spectra both explicitly and elegantly.

We start by considering the theory $T_{1,p}$ defined by a quadratic differential with $p \geq 1$ second order poles (regular punctures). In some corner of its parameter space, $T_{1,p}$ is the quiver gauge theory in Figure 11 with gauge groups $SU(2)$ on the nodes and bifundamental quarks on the links.

³⁷In the physical problem only half the data $\{Y_{k,a,0}, Y_{k,a,1}\}$ are independent.

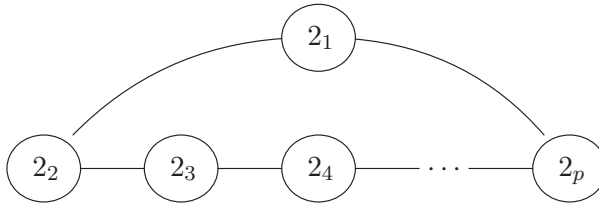


Figure 11: The (gauge) quiver representations of the $T_{1,p}$ theory. Each circle represents a $SU(2)$ $\mathcal{N} = 2$ SYM sector, while each link stands for a bifundamental quark hypermultiplet.

Let $d \mid p$. We claim that (at special points in parameter space) the $\mathcal{N} = 2$ theory $T_{1,p}$ is a Galois \mathbb{Z}_d -cover of the $T_{1,p/d}$ theory. In particular, taking $d = p$, $T_{1,p}$ is a \mathbb{Z}_p -cover of $T_{1,1}$, which is $\mathcal{N} = 2^*$ SYM with gauge group $SU(2)$. Thus we may construct the $T_{1,p}$ BPS spectrum (at \mathbb{Z}_p symmetric points) by pulling back the known spectrum of $\mathcal{N} = 2^*$ which has a simple Lie theoretic description.

The \mathbb{Z}_p -cover may be described in terms of Gaiotto geometries: the elliptic curves $E_p, E_{p/d}$ associated, respectively, to $T_{1,p}$ and $T_{1,p/d}$ are isogenous,

$$(6.1) \quad E_p \xrightarrow{\phi} E_{p/d}, \quad \deg \phi = d.$$

ϕ , being an isogeny, is a (holomorphic) covering map and an Abelian group homomorphism; its kernel is a \mathbb{Z}_d subgroup of E_p , which is the Galois group of the cover. The curve $E_{p/d}$ has p/d regular punctures at $z_1, \dots, z_{p/d}$ (which we assume to be in general position), and their pre-images in E_p form a set of p regular punctures invariant under the covering group \mathbb{Z}_d . The $T_{1,p}$ theory at a \mathbb{Z}_d symmetric point is specified by a quadratic differential of the form

$$(6.2) \quad \phi_2(E_p) = \phi^* \phi_2(E_{p/d})$$

for some quadratic differential $\phi_2(E_{p/d})$ on $E_{p/d}$ having second order poles at the z_i . For instance, in the special case $d = p$, the quadratic differential which specifies the $T_{1,p}$ theory takes the form $\phi^*(\lambda \wp(z, \tau) dz^2 + \mu dz^2)$ for some constants λ, μ .

In the $d = p$ case we may see the Gaiotto geometry as a pair (E_p, C) where E_p is an elliptic curve and $C \subset E_p$ is a \mathbb{Z}_p subgroup. As it is well known [60], the isomorphism classes of such pairs, called *enhanced elliptic curves* for $\Gamma_0(p)$, are parametrized by the upper half plane \mathcal{H} modulo the

congruence modular subgroup $\Gamma_0(p)$

$$(6.3) \quad \tau \sim \frac{a\tau + b}{c\tau + d}, \quad \tau \in \mathcal{H},$$

where

$$(6.4) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c = 0 \pmod{p} \right\}.$$

A given point τ in the modular curve $Y_0(p) \equiv \mathcal{H}/\Gamma_0(p)$ corresponds to the (isomorphism class) of the \mathbb{Z}_p invariant Gaiotto geometry given by a complex torus of periods $(1, \tau)$ with regular punctures at the p points $z_k = k/p$ ($k = 0, 1, \dots, p - 1$). The cyclic covering map ϕ in Eqn. (6.1) then is simply the map between the tori of periods $(1, \tau)$ and $(1, p\tau)$ given by multiplication by the integer p , $z \mapsto pz$.

6.1. Categorical covers and duality frames

Let us consider the particular quiver Q_p in the mutation class of the $T_{1,p}$ model

$$(6.5) \quad \begin{array}{ccccccc} & & \circ_{a-1} & & \circ_a & & \circ_{a+k} \\ & \nearrow & \parallel & \nwarrow & \nearrow & \nwarrow & \nearrow \\ \cdots & *_{a-2} & & *_{a-1} & & *_{a+k-1} & & *_{a+k} \cdots \\ & \nwarrow & \parallel & \nearrow & \nwarrow & \parallel & \nearrow \\ & & \bullet_1 & & \bullet_a & & \bullet_{a+k} \end{array}$$

$A_a \quad B_a$

$\xi_a \quad \eta_a \quad \xi_a^* \quad \eta_a^*$

where the index a in the node labels takes value in \mathbb{Z} with periodic identification mod p

$$(6.6) \quad *_{a+p} \sim *_{a}, \quad \circ_{a+p} \sim \circ_a, \quad \bullet_{a+p} \sim \bullet_a.$$

In the usual Lagrangian regime, where $T_{1,p}$ is the weakly coupled gauge theory in Figure 11, we identify each Kronecker subquiver $\circ_a \rightrightarrows \bullet_a$ of Q_p with a vertex of the gauge quiver, and the node $*_a$ with the flavor symmetry associated to the a -th bifundamental link. This is *not* the situation we wish to study: in this paper we are mainly interested in other ‘strongly coupled’ regimes of the $T_{1,p}$ model which have a nicer Galois cover interpretation.³⁸

³⁸Of course, the ordinary weak coupling may also be seen as a Galois cover of $T_{1,1}$.

The quiver Q_p has a \mathbb{Z}_p automorphism group, acting freely on the nodes, given by

$$(6.7) \quad \mathbb{Z}_p \ni k: (\circ_a, \bullet_a, *a) \mapsto (\circ_{a+k}, \bullet_{a+k}, *_{a+k}).$$

Without loss of generality³⁹ we may choose the free superpotential parameters $\lambda_a \in \mathbb{C}^\times$ associated to the punctures (see [19, 48]) to be all equal, $\lambda_a = \lambda \neq 0$; then the superpotential

$$(6.8) \quad \mathcal{W}_p = \sum_{a \in \mathbb{Z}_p} \left(\xi_a^* A_a \xi_a + \eta_a^* B_a \eta_a + \lambda \eta_a^* A_a \eta_a \xi_{a+1}^* B_{a+1} \xi_{a+1} \right)$$

is also \mathbb{Z}_p -invariant, and the group \mathbb{Z}_p acts by automorphisms of the (bounded) Jacobian category $\mathcal{J}_p \equiv \mathbb{C}Q_p/(\partial\mathcal{W}_p)$. This group acts freely on the objects of \mathcal{J}_p . If $d \mid p$, the category \mathcal{J}_p is a Galois cover of the category $\mathcal{J}_{p/d}$ with Galois group $\mathbb{Z}_d \simeq \mathbb{Z}_p/\mathbb{Z}_{p/d}$. In particular, \mathcal{J}_p is a \mathbb{Z}_p -cover of the category \mathcal{J}_1 corresponding to $SU(2) \mathcal{N} = 2^*$ whose (unique) quiver is the Markov quiver Q_1



endowed with the superpotential \mathcal{W}_1 . The three nodes $\circ, \bullet, *$ of Q_1 correspond, respectively, to the three orbits of \mathbb{Z}_p on the nodes of Q_p i.e. $\{\circ_a\}, \{\bullet_a\}$, and $\{*_a\}$.

A stability function on Q_1 is specified by three points in the upper half-plane, $Z_\circ, Z_\bullet,$ and Z_* . The pulled back central charge on Q_p is then

$$(6.10) \quad Z_{\circ_a} = Z_\circ, \quad Z_{\bullet_a} = Z_\bullet, \quad Z_{*_a} = Z_* \quad \text{for all } a = 1, 2, \dots, p.$$

Of course, this central charge is very *non-generic*; the corresponding BPS spectrum is much simpler than the one in generic BPS chambers. The choice (6.10) is nevertheless technically natural [22], since it corresponds to an enhancement of symmetry by \mathbb{Z}_p .

For the $\mathcal{N} = 2^*$ theory all BPS chambers are essentially equivalent, up to the action of the \mathbb{Z}_3 automorphism of the category \mathcal{J}_1 and the $SL(2, \mathbb{Z})$

³⁹The BPS spectrum is independent of the λ_a as long as they are not zero.

S -duality group. However these equivalent regimes pull back to physically *inequivalent* chambers for the $T_{1,p}$ theory, so the covering trick allows us to explore several distinct physical situations by a single computation.

A $SL(2, \mathbb{Z})$ -frame for $\mathcal{N} = 2^*$ is specified by the vector multiplet which plays the role of weakly coupled W boson, or, equivalently, by declaring which \mathbb{P}^1 family of bricks of $\text{mod } \mathcal{J}_1$ is associated to the W boson. The most common choice (the ‘minimal’ W) is the family of regular bricks with support on one of the three Kronecker subquivers of the Markoff quiver Q_1 . However this is not the most canonical choice [21]; we shall work with the ‘canonical’ choice of W i.e. the family of modules $W(\mu : \lambda)$

$$(6.11) \quad W(\mu : \lambda): \quad \begin{array}{ccc} \mathbb{C} & & \\ \downarrow & \swarrow 0 & \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \mathbb{C} \\ & \searrow (\mu \ 0) & \\ \mathbb{C}^2 & & \end{array}, \quad (\mu : \lambda) \in \mathbb{P}^1.$$

In terms of the central charge, the stability of $W(\mu : \lambda)$ requires

$$(6.12) \quad \arg Z_* < \arg Z_\bullet < \arg Z_\circ.$$

In this S -duality frame the magnetic charge of a module X is

$$(6.13) \quad m(X) = \dim X_\circ - \dim X_*.$$

In the ‘minimal’ W frame, the \mathbb{Z}_p -cover produces the weakly coupled Lagrangian regime of the $T_{1,p}$ theory: we have p light⁴⁰, mutually local, W bosons associated to the p Kronecker subquivers of Q_p , and the only other states which are local relatively to all W bosons are p bifundamental hypermultiplets. All other BPS particles are magnetically charged.

On the contrary, the \mathbb{Z}_p -cover of the ‘canonical’ W frame leads to a *strongly coupled* version of $T_{1,p}$ with a *single* light (i.e. weakly coupled) W boson which is magnetically charged with respect to all the original Lagrangian $SU(2)$ groups. The stable W boson is the pull back of the one for

⁴⁰By a *light* state in the covering theory we mean a particle whose mass is bounded as the coupling constant g_{YM}^2 of the quotient $\mathcal{N} = 2^*$ theory goes to zero (with respect to the chosen S -duality frame).

$\mathcal{N} = 2^*$ ⁴¹. The charge vector of the unique light W is

$$(6.14) \quad \sum_{a \in \mathbb{Z}_p} \left(e_{\circ_a} + 2 e_{\bullet_a} + e_{*_a} \right),$$

and the magnetic charge with respect to the associated weakly coupled $SU(2)$ gauge group

$$(6.15) \quad m(X) = \sum_{a \in \mathbb{Z}_p} \left(\dim X_{\circ_a} - \dim X_{*_a} \right).$$

The BPS spectrum contains only finitely many hypermultiplets which are local with respect to this W boson; in fact the stable BPS states with zero cover magnetic charge are in one-to-one correspondence with the BPS states of two copies of the Argyres–Douglas system of type D_{2p} (the correspondence does not preserve the charge vectors and the Dirac pairing, so the theory does *not* look, in this frame, $SU(2)$ SYM coupled to two D_{2p} AD systems).

Geometrically, the pair (E_p, C) which corresponds to this ‘strongly coupled’ regime is given, modulo the congruence modular group $\Gamma_0(p)$, by a $\tau \in Y_0(p)$ in a neighborhood of the cusp at infinity, $\tau = i\infty$. The usual weak coupling is the cusp at the origin $\tau = 0$.

For $\mathcal{N} = 2^*$ with the central charge (6.12), all stable modules have vanishing arrows $* \rightarrow \circ$, i.e. are modules of the triangular factor $\mathcal{J}_1^{\text{tr}}$ of \mathcal{J}_1 specified by the bi-quiver



Thus we have a triangular chamber, and the BPS spectrum is controlled by its Tits form. In particular, the spectrum may be described in Lie theoretic terms: the charge vectors of BPS particles are roots of the toroidal Lie algebra $A_1^{(1,1)}$.

The pull back of this triangular chamber for $\mathcal{N} = 2^*$ produces a triangular chamber for $T_{1,p}$. Note that acting with a \mathbb{Z}_3 automorphism of the

⁴¹This property fails for the ‘minimal’ W frame. The point is that the isotropy subgroup of the cover light W ’s in the ‘minimal’ frame, $\mathbb{G}_{W_{\text{min}}}$, is trivial, whereas for the canonical case $\mathbb{G}_{W_{\text{can}}} \cong \mathbb{G}$.

category \mathcal{J}_1 and then pulling back to \mathcal{J}_p , we get non-isomorphic triangular factor of \mathcal{J}_p (however their Tits forms are \mathbb{Z} -equivalent).

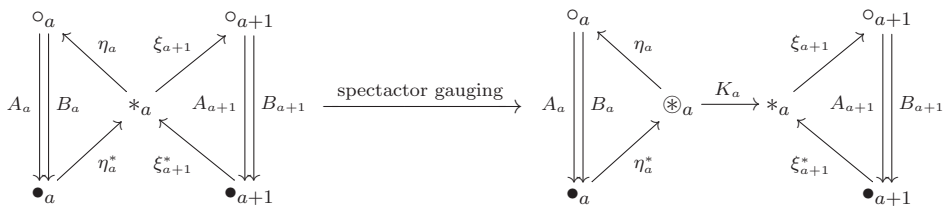
We claim that by pulling back the description of the $\mathcal{N} = 2^*$ spectrum in terms of the root system $A_1^{(1,1)}$, we get a description of the BPS spectrum of $T_{1,p}$ (in our chamber) in terms of root systems of *extended affine Lie algebras*. To show the claim, we begin by checking the triangularity of the \mathbb{Z}_p -symmetric chamber of $T_{1,p}$.

6.2. Triangularity of the \mathbb{Z}_p -symmetric BPS spectra

The first observation is that we may replace the Jacobian algebra \mathcal{J}_p of the superpotential in Eqn. (6.8) with the Jacobian algebra⁴² \mathcal{J}_p^g obtained by forgetting the order six terms in \mathcal{W}_p . This is an instance of the gentling trick of Appendix B.3: each node $*_a$ corresponds to a flavor $SU(2)$ which is gaugeable. We may weakly gauge all the $SU(2)^p$ flavor symmetry by coupling p spectator $SU(2)$ SYM sectors and compute the BPS spectrum of the fully gauged theory. In the limit of vanishing spectator gauge coupling, $g_{\text{spec}} \rightarrow 0$, the BPS spectrum consists of heavy states carrying spectator magnetic charge, which decouple at $g_{\text{spec}} = 0$, and states of zero spectator magnetic charge which survive the decoupling limit. The surviving states consist of the BPS spectra of the several decoupled sectors, i.e. the original model with $SU(2)^p$ flavor symmetry plus the p spectator W bosons.

At the level of the quiver with superpotential the spectator gauging of the a -th flavor $SU(2)$ modifies the quiver (6.5) at the node $*_a$ by splitting it into two nodes connected by an arrow $\otimes_a \xrightarrow{K_a} *_a$

(6.17)



while the new superpotential is given by Eqn. (6.8) with the order six terms omitted. The BPS states of the fully gauged theory which have bounded masses as $g_{\text{spec}} \rightarrow 0$ correspond to the stable modules of the Jacobian algebra of the modified quiver such that all new arrows K_a are isomorphisms. Being

⁴²The algebra \mathcal{J}_p^g is not finite dimensional. However the relevant modules are in fact modules of a finite-dimensional *string* algebra [21], as the argument in the text shows.

an isomorphism, K_a identifies nodes \oplus_a and $*_a$, so that the relevant modules may be seen as stable \mathcal{J}_p^g -modules, i.e. modules of the original quiver with the purely cubic superpotential. This light sector of the fully gauged theory should contain the BPS spectrum of the original model plus p spectator W bosons.

In \mathcal{J}_p^g the relations take a very simple form

$$(6.18) \quad A_a \xi_a = \xi_a \xi_a^* = \xi_a^* A_a = B_a \eta_a = \eta_a \eta_a^* = \eta_a^* B_a = 0 \quad \forall a.$$

Let X be a stable module of \mathcal{J}_p^g which is not a spectator W . We write $X|_{\text{Kr}_a}$ for the Kr -module obtained by restriction to the a -th Kronecker subquiver. By Eqn. (2.3) we have

$$(6.19) \quad X|_{\text{Kr}_a} = p_a \oplus t_a \oplus q_a.$$

We claim that

$$(6.20) \quad \eta_a^*|_{q_a} = \xi_a^*|_{q_a} = 0, \quad \text{and dually} \quad \text{Im } \eta_a, \text{Im } \xi_a \subset t_a \oplus q_a.$$

This is a consequence of the relations (6.18) and the fact that, restricted to q_a (resp. p_a) the arrows A_a, B_a are surjective (resp. injective). From Eqn. (6.20) it follows that each q_a (resp. p_a) is a submodule (resp. quotient) of X . Since X is stable

$$(6.21) \quad \arg Z(q_a) < \arg Z(X) < \arg Z(p_a),$$

which is impossible in view of (6.12) unless, for each a , either q_a or p_a vanishes. The pulled back central charge (6.12) corresponds to a special \mathbb{Z}_p symmetric point such that

$$(6.22) \quad \arg Z(p_a) < \arg Z(q_b) \quad \text{for all } a, b.$$

Then, if $p_a \neq 0$ (resp. $q_a \neq 0$) for one a then $q_b = 0$ (resp. $p_b \neq 0$) for all $b \in \mathbb{Z}_p$. In the next subsection we shall relax the condition (6.22) which is too strong from a physical perspective.

From §.2.1.2 we have $t_a = \oplus_s R_{n_s}(\lambda_s)$. Comparing the relations (6.18) with the structure of $R_{n_s}(\lambda_s)$, we conclude that if t_a contains a direct summand $R_{n_s}(\lambda_s)$ with $\lambda_s \neq 0, \infty$ then X belongs to the \mathbb{P}^1 family of a magnetically charged vector multiplet with support on the a -th Kronecker subquiver⁴³ Kr_a . For all other stable modules λ_s is equal to 0 or ∞ . Since X

⁴³In the weak coupling Lagrangian regime, a state with these quantum numbers would be interpreted as the W boson of the a -th gauge group.

is a brick, there are no direct summands $R_{n_s}(0), R_{n_s}(\infty)$ with $n_s > 1$. Indeed, let π and $(\cdot)|$ be the projection and restriction to the direct summand $R_{n \geq 2}(\infty)$. Taking into account the relations (6.18), in presence of such a summand we have the situation

(6.23)

$$\begin{array}{ccc}
 & \mathbb{C}^{n \geq 2} & \\
 \pi \xi_a \nearrow & \parallel & \nwarrow 0 \\
 *_{a-1} & A_a \neq 0 & *_{a-1} \\
 \xi_a^* \searrow & \text{Id} & \nearrow 0 \\
 & \mathbb{C}^{n \geq 2} &
 \end{array}
 \quad \text{with } A_a \xi_a = \xi_a^* A_a = 0, \text{ and } A_a \text{ nilpotent,}$$

which shows that multiplication by A_a at the nodes of the a -th Kronecker subquiver and by 0 elsewhere is a non-zero nilpotent endomorphism. Then t_a is the direct sum of copies of $R_1(0)$ and $R_1(\infty)$. In view of (6.18), for each $R_1(\infty)$ summand we have a situation as in (6.23) with $n = 1$ and $A_a = 0$ (for $R_1(0)$ the situation is symmetric with $\xi_a \leftrightarrow \eta_a, \xi_a^* \leftrightarrow \eta_a^*$). We distinguish three possibilities

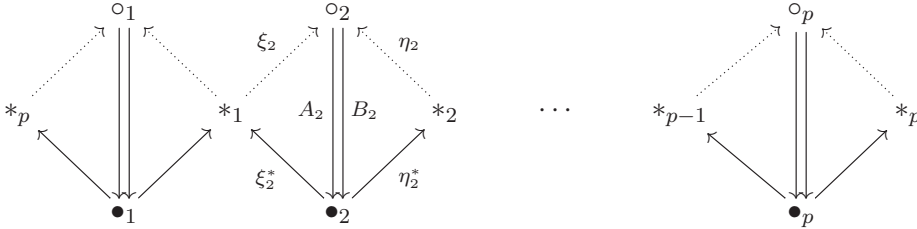
(6.24) a) $R_1^q(\infty) : \xi_a^*| = 0,$ b) $R_1^p(\infty) : \pi \xi_a = 0,$ c) $R_1^r(\infty) : \pi \xi_a, \xi_a^*| \neq 0.$

In Appendix D we show that no stable representation (which is not a spectator W) contains summands of the form c). $R_1^q(\infty), R_1^q(0)$ are subrepresentations of X while $R_1^p(\infty), R_1^p(0)$ are quotients. Arguing as in Eqn. (6.21), we get a contradiction if X contains (even at different Kroneckers) summands of both kinds, R_1^q and R_1^p ; we also get a contraction if we have non-trivial summands of the form R_1^q and p_a or R_1^p and q_a . Hence, for all $a \in \mathbb{Z}_p, X|_{\mathbb{K}r_a}$ is either: 1) a direct sum of a preinjective module q_a and regular modules of the form $R_1^q(\infty), R_1^q(0)$, or 2) a direct sum of a preprojective module p_a and regular modules of the form $R_1^p(\infty), R_1^p(0)$.

Consider first the possibility 1). Suppose that $X_{*a} \neq 0$ for some a . X has a quotient Y with support on $*_a$; since $\arg Z_* \leq \arg Z(M)$ for all indecomposable modules M , with equality iff M is the simple with support on $*_a$, we deduce that X , being stable, is such a simple module. On the other hand, if $X_{*a} = 0$ for all a , X , being indecomposable, has support on a single Kronecker subquiver. In both cases the arrows ξ_a, η_a vanish for all a .

In the possibility 2) the arrows ξ_a, η_a vanish by definition. Then, with our present choices, we get a triangular chamber with respect to the triangular

factor algebra defined by the bi-quiver
(6.25)



The covering BPS spectrum is then controlled by the Tits form of the above triangular algebra which happens to be the Tits form of an extended affine Lie algebra of nullity $p + 1$. We describe these Lie theoretical aspects in the Subsection 6.4. Before going to that, we have to discuss the physical significance of the above computation.

6.3. Physical interpretation: perturbing away from the \mathbb{Z}_p -point

Although the above analysis of the \mathbb{Z}_p -symmetric point is mathematically robust, its physical significance may be questioned on the grounds that, at such a point, there are stable modules with the same BPS phase, $\arg Z(X)$, which are *not* mutually local. For the present class of models, this is an automatic consequence of \mathbb{Z}_p -symmetry, which implies special alignments of the BPS phases in the Z -plane.

To get the physically sound interpretation of the mathematical result, we have to slightly perturb this alignment in the Z -plane to a situation in which any two stable representations, X_1, X_2 with $\arg Z(X_1) = \arg Z(X_2)$ are mutually local, i.e. $\langle \mathbf{dim} X_1, \mathbf{dim} X_2 \rangle_D = 0$. Of course, there are many possible directions in which we may perturb, which correspond to different BPS spectra, because the \mathbb{Z}_p points lay at the intersection of several walls of marginal stability.

The perturbations which lead to BPS spectra which are simple to describe are the ones which keep the chamber triangular (in the broader sense). We start from these perturbations, which are rather special. First of all, we require the phases to satisfy the order

$$(6.26) \quad \arg Z_{*a} < \arg Z_{\bullet b} < \arg Z_{oc}, \quad \text{for all } a, b, c.$$

In addition to this phase-order condition, to ensure that no R_1^r summand appears, we need a few special alignments in Z -plane (cfr. Appendix D)

$$(6.27) \quad \arg(Z_{\circ_{a-1}} + Z_{\bullet_{a-1}} + Z_{*a}) = \arg(Z_{\circ_a} + Z_{\bullet_a} + Z_{*a}), \quad a = 1, 2, \dots, p.$$

Calling ϵ the overall size of the perturbation, we remain free to choose the complex numbers $Z_{*a} = Z_* + O(\epsilon)$, $Z_{\bullet_a} = Z_\bullet + O(\epsilon)$, and real numbers $t_a = 1 + O(\epsilon)$ ($a = 2, 3, \dots, p$), such that

$$(6.28) \quad \begin{aligned} Z_{\circ_a} &= t_a Z_{\circ_{a-1}} + (t_a Z_{\bullet_{a-1}} - Z_{\bullet_a}) + (t_a - 1)Z_{*a} \\ &\text{for } a \geq 2, \quad \text{while } Z_{\circ_1} = Z_\circ. \end{aligned}$$

In this perturbed chamber, the relation (6.22) does not hold in general; it may be violated for those pairs a, b such that

$$(6.29) \quad \arg(Z_{\circ_a} + Z_{\bullet_a}) > \arg(Z_{\circ_b} + Z_{\bullet_b}),$$

and in this case only if the dimensions of both summands p_a and q_b are *parametrically large* of order $O(1/\epsilon)$. At any rate, it remains true (by construction) that we have no R_1^r summands, and hence, for all a , $X|_{\mathbb{K}r_a}$ is either: *i*) a direct sum of p_a and copies of R_1^p , or *ii*) a direct sum of q_a and copies of R_1^q . We attach a sign $\epsilon_a(X) = \pm 1$ to the a -th Kronecker subquiver to distinguish the two possibilities: for case *i*) we set $\epsilon_a(X) = +1$, while for case *ii*) we set $\epsilon_a(X) = -1$. Then if $\epsilon(X)_a = +1$ (resp. $\epsilon_a(X) = -1$) $X|_{\mathbb{K}r_a}$ is a quotient (resp. submodule) of X , and the arrows ξ_a, η_a (resp. ξ_a^*, η_a^*) vanish. So all (non-spectator) stable brick X is triangular with respect to a X -dependent triangular factor $\mathcal{T}[\epsilon_a(X)]$ of \mathcal{J} whose Tits form $q(\mathbf{x}; \{\epsilon_a(X)\})$ is \mathbb{Z} -equivalent to the Tits form $q(\mathbf{x})$ of the bi-quiver (6.25)

$$(6.30) \quad q(x_{\circ_a}, x_{\bullet_a}, x_{*a}; \{\epsilon_a\}) = q(\epsilon_a x_{\circ_a}, \epsilon_a x_{\bullet_a}, x_{*a}).$$

However, stable non-spectator modules X having $\epsilon_a(X) = -1$ for some a , i.e. which are not modules of the original triangular factor (6.25), have charges, hence masses, parametrically large of order $O(1/\epsilon)$. Note that these very heavy states do not form full \mathbb{Z}_p -multiplets as do states with masses which remain bounded as $\epsilon \rightarrow 0$. On the other hand, after the perturbation, the stable states with the same BPS phase are mutually local; indeed, choosing the Z_i to be \mathbb{Q} -linearly independent, the only alignments are those in Eqn. (6.30) which correspond to mutually local BPS states.

In conclusion, in this class of perturbed chambers, BPS spectrum is still controlled by the Tits form $q(\mathbf{x})$, and hence described by the root system of an extended affine Lie algebra, and it consists of two distinct sectors:

- 1) states with masses $O(1)$ which form \mathbb{Z}_p -multiplets (with $O(\epsilon)$ mass splittings inside the \mathbb{Z}_p -multiplets). They have $\epsilon_a(X) \equiv +1$ for all a ;
- 2) parametrically heavy states, $m = O(1/\epsilon)$ which do not form complete \mathbb{Z}_p -multiplets. In this case some (but not all) signs $\{\epsilon_a(X)\}_{a=1}^p$ must be equal -1 .

The BPS spectrum at the \mathbb{Z}_p -point computed in §.6.2 corresponds to the sector 1) of particles whose mass remains bounded as we switch off the perturbation. For $\epsilon \sim 0$, the physics of the perturbed chamber is the same as the one obtained by taking our covering computation at the \mathbb{Z}_p -symmetric point at its face level (up to $O(\epsilon)$ corrections); that computation is exact asymptotically as $\epsilon \rightarrow 0$. This clarifies the precise physical meaning of the spectrum computed at the \mathbb{Z}_p -symmetric point.

If we consider a more general perturbation of order ϵ , where the equalities (6.27) hold only up to $O(\epsilon)$ corrections, it is not longer true that we get a triangular chamber, and hence the full spectrum cannot be simply described in terms of root systems of Lie algebras. Nevertheless, it is easy to see that it remains true that the BPS spectrum decomposes in two sectors: a triangular sector with masses of order $O(1)$ as $\epsilon \rightarrow 0$, which form representations of \mathbb{Z}_p and coincides with our ‘mathematical’ spectrum at the \mathbb{Z}_p -point, and a second sector, whose detailed structure is rather involved, but which contains only particles with parametrically large masses $O(1/\epsilon)$ which do not form \mathbb{Z}_p representations. Besides being the result of the analysis of module stability, this picture is physically very natural, and it is expected to hold quite in general.

The situation is analogue⁴⁴ to weak coupling in $\mathcal{N} = 2$ gauge theories; also there as $\epsilon \equiv g_{YM}^2 \rightarrow 0$ the spectrum decomposes in two parts: states with bounded masses which form complete representations of the gauge group (which is interpreted as a flavor symmetry in the limit) and a complicated dyonic sector whose masses are of order $O(1/\epsilon)$.

The physical conclusion, is that we may trust the BPS spectrum computed at a \mathbb{Z}_p -point by pulling back the spectrum of the quotient theory. In any nearby BPS chamber the physical spectrum is given by the pull back one plus essentially decoupled super-heavy states. The decoupling of the heavy junk is exact asymptotically as $\epsilon \rightarrow 0$.

⁴⁴In fact this is more than just an analogy.

6.4. Relation with extended affine and Kac–Moody Lie algebras

Let us look at the Tits form q of the triangular algebra associated to the bi–quiver (6.25). From §.2.3.3, q is the integral quadratic form on $\Gamma \simeq \mathbb{Z}^{3p}$

$$(6.31) \quad q(x_{\circ_a}, x_{\bullet_a}, x_{*_a}) = \frac{1}{2} \sum_{a=1}^p \left((x_{\bullet_a} - x_{\circ_a} - x_{*_a})^2 + (x_{\bullet_a} - x_{\circ_a} - x_{*_a-1})^2 \right).$$

Being a sum of squares, the Tits forms is manifestly positive semi–definite. By §.2.3.2, a positive semi–definite Tits form is uniquely identified, up to \mathbb{Z} –equivalence, by its type \mathfrak{g}_r and nullity κ . The form in Eqn. (6.31) has type A_{2p-1} and nullity $\kappa = p + 1$. To see this, note that there is an element of $\text{rad } q$ associated to each Kronecker, i.e. $x_{\circ_a} = x_{\bullet_a} = 1$ and zero elsewhere, so, working mod $\text{rank } q$, we may set $x_{\circ_a} = 0$ for all a . q reduces to the Tits form of the Kac–Moody algebra whose Dynkin graph is obtained by deleting all \circ_a nodes in the quivers: (6.25) reduces to the affine Dynkin graph $A_{2p-1}^{(1)}$. Being affine this Lie algebra has Tits form of nullity 1 (and type A_{2p-1}). Hence the nullity of the Tits form q is one more the number of Kronecker subquivers; $\text{rad } q$ is generated by the dimension vectors $\delta_a = e_{\circ_a} + e_{\bullet_a}$ of the p W bosons of the usual weak coupling plus the dimension vector $\sum_a (e_{*_a} + 2e_{\bullet_a} + e_{\circ_a})$ of the W boson which is light in our pulled–back triangular regime.

This result is most conveniently stated in terms of Lie algebras. The Tits form in Eqns. (6.31) is identified with the Tits forms of the extended affine Lie algebra $A_{2p-1}^{[p+1]}$. $\mathbf{x} \in \mathbb{Z}^{\kappa+r}$ is a *root* iff $q(\mathbf{x}) \leq 1$ and its support is connected. A root \mathbf{x} is *real* iff $q(\mathbf{x}) = 1$, and *imaginary* iff $q(\mathbf{x}) = 0$. The imaginary roots form the sublattice $\text{rad } q$. From the recursive construction, Eqn. (2.25), it is clear that the real roots have the form $\alpha + \delta$ where α is a root of the Lie algebra \mathfrak{g}_r and δ is an element of the lattice of imaginary roots $\text{rad } q$. Equivalently, we may separate the generator $\delta \in \text{rad } q$ associated to the light W , $\delta = \sum_a (e_{*_a} + 2e_{\bullet_a} + e_{\circ_a})$, and write the roots in the form $\hat{\alpha}_n + \rho$ where $\hat{\alpha}_n = \alpha + n\delta$ is a root of the affine Lie algebra $\mathfrak{g}_r^{(1)}$ and $\rho \in \text{rad } q/\mathbb{Z}\delta$.

6.4.1. The GIM Lie algebra of the $g = 1$ $\mathcal{S}[A_1]$ models. We write explicitly the Chevalley generators of the *extended affine Lie algebra* $A_{2p-1}^{[p+1]}$ associated to the triangular factor (6.25). We write α_i for the simple roots of A_{2p-1} , $\alpha_\theta \equiv \sum_i \alpha_i$ for its maximal root, and $\{E_i, F_i, H_i\}_{i=1}^{2p-1}$ for its Chevalley generators. The Chevalley generators of \mathfrak{L}_p are

$$\begin{aligned}
 (6.32) \quad & e_{\circ_a} = E_{2a-1}, \quad f_{\circ_a} = F_{2a-1}, \quad h_{\circ_a} = H_{2a-1} && a = 1, 2, \dots, p \\
 & e_{*_a} = E_{2a}, \quad f_{*_a} = F_{2a}, \quad h_{*_a} = H_{2a}, && a = 1, 2, \dots, p-1 \\
 & e_{*_p} = F_\theta t_0, \quad f_{*_p} = E_\theta t_0^{-1}, \quad h_{*_p} = -H_\theta + \text{central} \\
 & e_{\bullet_a} = F_{2a-1} t_a, \quad f_{\bullet_a} = E_{2a-1} t_a^{-1}, \quad h_{\bullet_a} = -H_{2a-1} + \text{central} && a = 1, 2, \dots, p,
 \end{aligned}$$

where t_0, t_1, \dots, t_p are coordinates on the torus $(\mathbb{C}^*)^{p+1}$. One checks that all the Serre–Slodowy relations (2.18)–(2.21) following from the Dynkin bi-graph (6.25) are satisfied.

6.4.2. Dyonic towers, Kac–Moody algebras, and Schur roots. From the discussion around Eqn. (2.16) and the above analysis it follows that the stable \mathcal{J}_p -modules X are controlled by the Tits form q . Hence, at weak coupling, the charge vectors $\mathbf{dim} X$ of the BPS particles satisfies

$$(6.33) \quad q(\dim X_{\circ_a}, \dim_{\bullet_a}, \dim X_{*_a}) = \begin{cases} 1 & \text{for hypermultiplets} \\ 0 & \text{vector multiplets,} \end{cases}$$

and there are no higher spin BPS states since q is semi-definite. In view of §.6.4, the quantum numbers of the hypermultiplets (resp. vector multiplets) are real roots (resp. imaginary roots) of the extended affine Lie algebra $A_{2d-1}^{[d+1]}$. More precisely, the charge vectors should be *Schur roots* for the appropriate orientation of the bi-quiver.

Let $\delta_a = e_{\bullet_a} + e_{\circ_a}$. $\delta_a \in \text{rad } q$, so $\Gamma_{\text{el}} \equiv \bigoplus_a \mathbb{Z}\delta_a$ is a sublattice of $\text{rad } q$; in the Lagrangian regime, Γ_{el} would be identified with the lattice of the gauge electric charge. The Tits form q on Γ induces a quadratic form \bar{q} on the quotient $\Gamma/\Gamma_{\text{el}}$ which is the Tits form of the Kac–Moody algebra $A_{2p-1}^{(1)}$ whose Dynkin graph is obtained deleting the \circ_a nodes.

Let $\mathbf{x} \in \Gamma$ the charge vector of a BPS particle. Forgetting its Lagrangian electric charge, amounts to considering the residue class $\bar{\mathbf{x}} \in \Gamma/\Gamma_{\text{el}}$. The reduced charge vector $\bar{\mathbf{x}}$ of a stable BPS particle (in our pulled back chamber) is necessarily a *Schur*⁴⁵ root of $A_{2p-1}^{(1)}$, real roots being hypermultiplets and imaginary ones vector multiplets. Considering the reduced charge vector $\bar{\mathbf{x}}$, instead of \mathbf{x} , amounts to forgetting the information about the (Lagrangian) electric charge of the state, while keeping the information about its (Lagrangian) magnetic and flavor charges, (m_a, f_i) . This reduced information is

⁴⁵The notion of Schur root depends on the orientation of the Dynkin graph of $A_{2p-1}^{(1)}$. Here the orientation is the one of the Dynkin quiver obtained from the original quiver by collapsing the Kronecker subquivers to a single node \bullet_a and deleting the dashed arrows.

encoded in the Schur roots of $A_{2p-1}^{(1)}$. The condition of the roots being Schur gives restrictions that we now discuss.

The reduced charge vector is a Schur root of $A_{2p-1}^{(1)}$. Let $\alpha_i, i = 0, 1, \dots, 2p - 1$ be the simple roots of the Kac–Moody algebra $A_{2p-1}^{(1)}$. The element of the affine root lattice

$$(6.34) \quad \begin{aligned} \gamma(m_a, f_a) &:= \sum_{a=0}^{p-1} \left((\dim X_{\bullet_a} - \dim X_{\circ_a}) \alpha_{2a} + \dim X_{*_a} \alpha_{2a+1} \right) \\ &\equiv \sum_{a=0}^{p-1} \left(-m_a \alpha_{2a} + f_a \alpha_{2a+1} \right), \end{aligned}$$

is a positive root of the Kac–Moody algebra $A_{2p-1}^{(1)}$. In (6.34) we use the nomenclature for the different charges which is standard at the ordinary weakly coupled Lagrangian point, i.e. m_a is the a 'th Lagrangian magnetic charge and f_a is the Lagrangian a -th quark flavor charge. More precisely, $\gamma(m_a, f_a)$ is a *Schur root* for the orientation of the $\widehat{A}(p, p)$ quiver in which the $2a$ -th node is a *source*. The Ringel defect [21] of this affine quiver is

$$(6.35) \quad \mu(m_a, f_a) = - \sum_a (f_a + m_a),$$

and the list of the Schur roots is [31]:

- i*) the minimal imaginary root $\delta = \sum_i \alpha_i$ which corresponds to a dyonic vector–multiplet;
- ii*) the real positive roots with $\mu(m_a, f_a) \neq 0$;
- iii*) the real positive roots with $\mu(m_a, f_a) = 0$ and $|m_a| \leq 1, f_a \leq 1$ for all a .

The above are *necessary* conditions for a reduced charge vector $\gamma(m_a, f_a)$ to correspond to quantum numbers of stable BPS dyons. For each $\gamma(m_a, f_a)$ satisfying these conditions, we need to determine the sets $\{e_a\}$ of Lagrangian electric charges such that (e_a, m_a, f_a) are the quantum numbers of a stable BPS state. A convenient way of working is to reduce to the stability problem for a representation of the effective $\widehat{A}(p, p)$ with the orientation specified by the $\epsilon_a(X)$'s and an effective central charge $Z^{\text{eff}}(\{e_a\})$ which depends on the electric charges $\{e_a\}$. A representation of $\widehat{A}(p, p)_{\{e_a\}}$ with dimension vector $(|m_a|, f_a)$ which is stable for $Z^{\text{eff}}(\{e_a\})$ corresponds to a $(e_a, |m_a|, f_a)$ BPS state.

6.5. Irregular punctures

The analysis may be easily extended to class $\mathcal{S}[A_1]$ theories with p irregular punctures corresponding to poles of degree $s + 1$ of the quadratic differential ϕ_2 where $s > 1$ [8]. In one ‘weakly coupled’ corner of its parameter space, this model is $SU(2)^{2p}$ $\mathcal{N} = 2$ SYM coupled to p half tri-fundamentals and p copies of the Argyres–Douglas systems of type D_{s+1} [8]. Here we discuss a different regime which has a nice Galois cover interpretation.

The \mathbb{Z}_p -symmetric form of the quiver, $Q_{p,s}$, is obtained from Q_p by replacing each node $*_a$ with a A_s subquiver $*_{a,1} \xrightarrow{K_{a,1}} *_{a,2} \xrightarrow{K_{a,2}} \dots \xrightarrow{K_{a,s-2}} *_{a,s-1} \xrightarrow{K_{a,s-1}} *_{a,s}$, i.e.

$$(6.36) \quad \begin{array}{c} \dots \\ \swarrow \\ *_{a,1} \xrightarrow{K_{a,1}} \dots \xrightarrow{K_{a,s-1}} *_{a,s} \begin{array}{c} \nearrow \xi_{a+1} \circ_{a+1} \eta_{a+1} \\ \parallel A_{a+1} B_{a+1} \\ \searrow \xi_{a+1}^* \bullet_{a+1} \eta_{a+1}^* \end{array} *_{a+1,1} \longrightarrow \dots \longrightarrow *_{a+1,s} \begin{array}{c} \nearrow \circ_{a+2} \\ \parallel \\ \searrow \bullet_{a+2} \end{array} \dots \\ \swarrow \\ \dots \end{array}$$

where again the index $a \in \mathbb{Z}$ is periodically identified $a \sim a + p$. $Q_{p,s}$ is endowed with the superpotential

$$(6.37) \quad \mathcal{W}_{p,s} = \sum_{p \in \mathbb{Z}_p} \left(A_a \xi_a \xi_a^* + B_a \eta_a \eta_a^* \right).$$

Clearly we have a free action on the nodes of the \mathbb{Z}_p automorphism group of the $Q_{1,s}$ quiver

$$(6.38) \quad \mathbb{Z}_p \ni k : (\circ_a, \bullet_a, *_{a,j}) \longmapsto (\circ_{a+k}, \bullet_{a+k}, *_{a+k,j}) \quad a \in \mathbb{Z}_p, \quad j = 1, 2, \dots, s,$$

which induces a \mathbb{Z}_p automorphism of the Jacobian category $\mathcal{J}_{p,s} = \mathbb{C}Q_{p,s}/(\partial\mathcal{W}_{p,s})$. The orbit category $\mathcal{J}_{p,s}/\mathbb{Z}_p$ is the Jacobian category $\mathbb{C}Q_{1,s}/(\partial\mathcal{W}_{1,s})$, where $Q_{1,s}$ is the quiver

$$(6.39) \quad \begin{array}{c} \circ_3 \dots \dots \circ_{s-1} \\ \nearrow \quad \quad \quad \searrow \\ *_2 \quad \quad \quad *_{s-1} \\ \parallel \quad \quad \quad \parallel \\ \circ \quad \quad \quad \circ \\ \searrow \quad \quad \quad \nearrow \\ *_1 \quad \quad \quad *_s \\ \bullet \end{array}$$

By comparing with the previous $s = 1$ case, we see that in the regime

$$(6.40) \quad \arg Z_{*i} < \arg Z_{\bullet} < \arg Z_{\circ},$$

the arrows with target \circ vanish. Hence we have a triangular chamber with Tits form the one of the toroidal Lie algebra $A_s^{(1,1)}$.

The \mathbb{Z}_p cover of this chamber is also triangular. Its BPS spectrum is controlled by the Tits form of the extended affine Lie algebra $A_{p(s+1)-1}^{[p+1]}$ in agreement with **Fact 2** of §.2.3.3.

Acknowledgements

The authors thank the Simons Center for Geometry and Physics for hospitality while working on this paper, and thank the organizers of the Simons Summer Workshop and of the Quiver Variety program. SC thanks the Department of Physics of Harvard University for hospitality. The work of MDZ is supported by the NSF grant PHY-1067976.

Appendix A. Nullity κ and type \mathfrak{g}_r of class $\mathcal{S}[A_1]$ triangular factors

Here we prove **Fact 2**. Let Q be a triangulation quiver of a surface Σ having genus g , p punctures, b boundaries (with $p + b > 0$), and $\ell_i \geq 1$ marks on the i -th boundary. The number of nodes of Q is

$$(A.1) \quad n \equiv 6(g - 1) + 3(p + b) + \sum_i \ell_i = -3\chi + \sum_i \ell_i,$$

where χ is the Euler characteristic of Σ . We write $n_{\mathfrak{b}}$ for the number of blocks of Type \mathfrak{b} in the decomposition of the quiver Q . We consider a *good* biquiver obtained by dashing some arrows of Q as described in §.2.3.3.

From Figure 1, a good bi-quiver block decomposition defines a group homomorphism

$$(A.2) \quad \beta: \mathbb{Z}^n \rightarrow \mathbb{L} \equiv \mathbb{Z}^{n_0+n_I+n_{II}} \oplus L_2^{(n_{III}+n_{IV})} \oplus \mathbb{Z}^{3n_V}$$

$$(A.3) \quad \beta: x_i \mapsto (y_a, w_{m,1}, w_{m,2}, z_{s,1}, z_{s,2}, z_{s,3}),$$

where L_2 is the rank 2 lattice $L_2 = \{(w_1, w_2) \in \mathbb{Z}^2 \mid w_1 = w_2 \pmod{2}\}$, such that the Tits form $q(x_i)$ of the bi-quiver is the pull-back

$$(A.4) \quad q_A = \beta^* \mathfrak{q}$$

of the \mathbb{Q} -quadratic form on the lattice \mathbb{L}

$$(A.5) \quad \mathfrak{q} = \frac{1}{2} \sum_a y_a^2 + \frac{1}{4} \sum_m (w_{m,1}^2 + w_{m,2}^2) + \frac{1}{2} \sum_{s,a} z_{s,a}^2.$$

Lemma. *One has*

$$(A.6) \quad \text{rank } \mathbb{L} = \chi + n$$

Proof. The number of nodes in Q , n , is half the total number of white nodes in the several blocks plus the number of black ones

$$(A.7) \quad n \equiv -3\chi + \sum_i \ell_i = \frac{1}{2}(n_0 + 2n_I + 3n_{II} + n_{III} + 2n_{IV} + n_V) \\ + (2n_{III} + 2n_{IV} + 4n_V),$$

while

$$(A.8) \quad \sum_i \ell_i = 2n_0 + n_I + n_{III},$$

since (cfr. Table 1) a Type 0 block corresponds to a triangle with two sides on the boundary, a Type I block to a triangle with one side on the boundary, and Type III to a punctured di-gon with one side on the boundary, while the other block Types correspond to internal ‘puzzle pieces’ in the ideal triangulation [47]. Then

$$(A.9) \quad n + \chi = n - \frac{1}{3} \left(n - \sum_i \ell_i \right) \\ = (n_0 + n_I + n_{II}) + 2(n_{III} + n_{IV}) + 3n_V \equiv \text{rank } \mathbb{L}.$$

Since the \mathbb{Q} -quadratic form \mathfrak{q} is *positive definite*, the index of nullity of the extended affine Lie algebra having Tits form $q(x_i)$ is

$$(A.10) \quad \kappa = n - \text{rank}_{\mathbb{Q}} \beta \equiv -\chi + \dim_{\mathbb{Q}} \text{coker } \beta,$$

and $\kappa = -\chi$ iff β is surjective (over \mathbb{Q}). Let β be the $(n + \chi) \times n$ matrix giving the \mathbb{Q} -linear map extending (A.2). One has $\dim_{\mathbb{Q}} \text{coker } \beta = \dim_{\mathbb{Q}} \ker \beta^t$. From Figure 1 it is clear that, for a connected bi-quiver, $\ker \beta^t$ cannot be non zero whenever in the block decomposition of Q there are black nodes, that is, blocks of Types III , IV , V . \square

The case $n_{III} = n_{IV} = n_V = 0$. In this case a non-zero zero-eigenvector of β^t , $(y_1, \dots, y_{n+\chi})$ should satisfy $|y_a| = |y_b|$ for all a, b . Hence coker β is at most one-dimensional and κ at most $-\chi + 1$. Suppose then that $\kappa = -\chi + 1$. By flipping the sign of some y_a , if necessary, we reduce to the situation that

$$(A.11) \quad \mathfrak{q} = \frac{1}{2} \sum_a y_a^2$$

$$(A.12) \quad \text{im } \beta = \{(y_a) \in \mathbb{Q}^{n+\chi} \mid y_1 + y_2 + \dots + y_{n+\chi} = 0\},$$

and from the classification of the non-negative forms we know that for $-\chi \geq 0$ the form $\mathfrak{q}|_{\text{im } \beta}$ should be isometric to the Tits form of an affine Lie algebra of type either A or D , while for $\chi = 1$ to the Tits form of a finite-type Lie algebra of type A or D . In both cases this condition is satisfied by the type A algebra with isometries

$$(A.13) \quad -\chi \geq 0 \quad y_i = x_i - x_{i+1}, \quad x_{i+n+\chi} \equiv x_i$$

$$(A.14) \quad -\chi = -1 \quad x_i = y_i - y_{i+1}, \quad i = 1, \dots, n,$$

but not by D type. We conclude that $\kappa = -\chi + 1$ implies A type. Conversely, $\kappa = -\chi$ implies D -type.

Appendix B. Quick review of strings and bands

B.1. Indecomposable modules of string algebras

A finite-dimensional algebra \mathcal{A} is called a *string algebra* [55] iff it has the form $\mathbb{C}Q/I$ where:

- S1:** each node of the quiver Q is the starting point of at most two arrows;
- S2:** each node of the quiver Q is the ending point of at most two arrows;
- S3:** for each arrow α in Q there is at most one arrow β with $\alpha\beta \notin I$;
- S4:** for each arrow α in Q there is at most one arrow β with $\beta\alpha \notin I$;
- S5:** the ideal I is generated by zero-relations.

Every string algebra is, in particular, special biserial and tame. Physically, tame means that the BPS spectrum consists only of hypermultiplets and vector-multiplets, higher spin states been forbidden. A string algebra has two kinds of indecomposable representations [55]:

- 1) *string* representations, which without free parameters;

2) *band* representations which come in one-parameter families.

By a *string* C we mean a *walk* (generalized path) in the quiver Q where we are allowed to pass from one node to another one either following an arrow connecting them or going backwards such an arrow. We write a string as a sequence of nodes connected by direct or inverse arrows, i.e. in the form

$$(B.1) \quad C: \quad i_0 \xleftarrow{\phi_{\alpha_1}} i_1 \xrightarrow{\phi_{\alpha_2}} i_2 \xrightarrow{\phi_{\alpha_3}} \cdots \cdots i_{n-1} \xleftarrow{\phi_{\alpha_\ell}} i_\ell,$$

where the nodes in the sequence are labelled as $\{i_0, i_1, i_2, \dots, i_\ell\}$ and the direct/inverse arrows as $\{\phi_{\alpha_1}, \phi_{\alpha_2}, \dots, \phi_{\alpha_\ell}\}$. A string is not allowed to contain:

C1: adjacent direct/inverse arrow pairs, as $\cdots i \xrightarrow{\phi_a} j \xleftarrow{\phi_a} i \cdots$ or $\cdots i \xleftarrow{\phi_a} j \xrightarrow{\phi_a} i \cdots$;

C2: adjacent pairs of (direct or inverse) arrows whose composition vanishes by the relations $\partial\mathcal{W} = 0$.

A string C is identified with its inverse C^{-1} (i.e. the string obtained by reading C starting from the right instead than from the left). Given a string C of length $\ell + 1$, the corresponding string module $M(C)$ is obtained in the following way. For each vertex v of Q let

$$(B.2) \quad I_v = \{a \mid \text{the } a\text{-th node in the string } C \text{ is } v\} \subset \{0, 1, 2, \dots, \ell\}.$$

As $M(C)_v$ we take the vector space of dimension $\#I_v$ with base vectors z_a , $a \in I_v$. We define

$$(B.3) \quad \begin{cases} \phi_{\alpha_a}(z_a) = z_{a+1} & \phi_{\alpha_a} \text{ is direct} \\ \phi_{\alpha_a}(z_{a+1}) = z_a & \phi_{\alpha_a} \text{ is inverse} \end{cases}$$

Finally, for all arrows γ and vectors z_a such that $\gamma(z_a)$ is not of the form in (B.3), we set $\gamma(z_a) = 0$.

A *band* is a cyclic sequence C of nodes of Q and direct/inverse arrows, again forbidding sequences as in **C1** and **C2** above. Moreover, C is required to be such that all powers C^n are well-defined (cyclic) strings, but C itself is not the power of a string of smaller length. C is identified up to cyclic rearrangement and overall inversion. Given such a band C of length ℓ , we define a family $M(C, \lambda, n)$ of indecomposable band modules labelled by an

integer $n \geq 1$ and a point $\lambda \in \mathbb{C}^*$. Explicitly, we set

$$M(C, \lambda, n)_v = \bigoplus_{a \in I_v} Z_a, \quad (a = 1, 2, \dots, \ell),$$

where the Z_a 's are copies of \mathbb{C}^n , and

$$(B.4) \quad \begin{cases} \phi_{\alpha_a} \Big|_{Z_a} : Z_a \xrightarrow{1} Z_{a+1} & \phi_{\alpha_a} \text{ direct } a \neq \ell \\ \phi_{\alpha_a} \Big|_{Z_{a+1}} : Z_{a+1} \xrightarrow{1} Z_a & \phi_{\alpha_a} \text{ inverse } a \neq \ell \\ \phi_{\alpha_\ell} \Big|_{Z_\ell} : Z_\ell \xrightarrow{J(\lambda)} Z_1 & \phi_{\alpha_\ell} \text{ direct} \\ \phi_{\alpha_\ell} \Big|_{Z_1} : Z_1 \xrightarrow{J(\lambda)^{-1}} Z_\ell & \phi_{\alpha_\ell} \text{ inverse} \end{cases}$$

where $J(\lambda)$ is the $n \times n$ Jordan block of eigenvalue $\lambda \neq 0$. All other linear maps which are not defined in (B.4) are set to zero.

If \mathcal{A} is a string algebra, the above string and band modules give a complete list of indecomposable (and pairwise non-isomorphic) \mathcal{A} -modules [55]. Note also that a brick has necessarily $n = 1$ since $J(\lambda) \in \text{End } M(C, \lambda, n)$ is non trivial for $n \geq 2$.

B.2. The Assem–Bruštle–Charbonneau–Plamondon theorem

A gentle algebra is a string algebra where the ideal I may be generated by zero-relations of length two (i.e. involving just two arrows). All Jacobian algebras of class $\mathcal{S}[A_1]$ theories with only irregular punctures are gentle:

Theorem [56]. *Let (Q, \mathcal{W}) be a quiver with superpotential corresponding to a Gaiotto A_1 -theory with only irregular punctures and at least one such puncture. Then the Jacobian algebra $\mathcal{A} \equiv \mathbb{C}Q/(\partial\mathcal{W})$ is gentle.*

Thus for such an ‘irregular’ Gaiotto theory the BPS states are stable modules of a gentle algebra which may be explicitly constructed in terms of strings/bands. A string module corresponds to an open WKB trajectory [3] on the Gaiotto surface of the corresponding class $\mathcal{S}[A_1]$ theory, see [49, 56] for the explicit construction of the string module out of the WKB geodesic and *viceversa*. Likewise, a family $M(C, \lambda, 1)$ of representations a closed WKB geodesic associated to a primitive⁴⁶ homotopy class of closed WKB geodesics

⁴⁶By primitive homotopy class we mean a class which is not a non-trivial integral multiple of an integral class. Note also that the identification $C \leftrightarrow C^{-1}$ corresponds

on Σ . Hence the theorem of ref.[56] just says that the WKB analysis of the BPS spectra is exact in the present context.

B.3. Gentling class $\mathcal{S}[A_1]$ algebras

The Assem–Bruštle–Charbonneau–Plamondon theorem may sound a little odd to a physicist, since WKB is exact for all class $\mathcal{S}[A_1]$ theory whether or not the Gaiotto surface Σ_G has regular punctures. Therefore, *for all class $\mathcal{S}[A_1]$ category, it should be possible to construct all relevant modules in terms of strings/bands.*

We consider a general $\mathcal{S}[A_1]$ theory based on a ultraviolet curve Σ of genus g with p regular punctures (double poles of the quadratic differential ϕ_2) and a number b of irregular punctures (poles of order $c_i \geq 3$) with $p + b > 0$. For $p \neq 0$ the algebra is not gentle. However, to study the stable representations of the corresponding algebras $\mathbb{C}Q/(\partial\mathcal{W})$ we can effectively replace the algebra by a gentle one. We proceed as follows. Each puncture carries a flavor $SU(2)$ group, so that our theory has a flavor symmetry $SU(2)^p$. We couple p ‘spectator’ copies of $SU(2)$ SYM, each copy weakly gauging the flavor $SU(2)$ of one puncture. In the limit of very small spectator YM coupling, $g_{\text{spect}} \rightarrow 0$, we get a BPS spectrum consisting of the BPS spectrum of the original theory, together with p spectator W bosons, as well as particles having $O(1/g_{\text{spect}}^2)$ masses which are magnetically charged under the spectator gauge groups.

On the other hand, the weakly gauged model corresponds to the $\mathcal{S}[A_1]$ theory based on the same surface Σ with the p ordinary punctures replaced by *cubic* irregular ones. The weakly coupled model, having no more regular punctures, corresponds to a gentle algebra. We may compute the BPS spectrum of the gauged model using the simple Representation Theory of this gentle algebra and then recover the BPS spectrum of the original theory by taking $g_{\text{spec}} \rightarrow 0$, that is, by projecting out states of non-zero spectator magnetic charge as well as the spectator W bosons. We shall refer to this procedure as gentling.

The original vs. the gentled algebra. We wish to describe more in detail the effect of gentling on the Jacobian algebra $\mathbb{C}Q/(\partial\mathcal{W})$. To fully specify the Jacobian algebra, we fix one quiver Q in the mutation class of the original theory, corresponding to a particular ideal triangulation of its Gaiotto surface Σ . From the point of view of Σ , the gentling procedure may

to the fact that the WKB geodesics have no orientation due to the \mathbb{Z}_2 -monodromy of the differential $\sqrt{\phi_2}$.

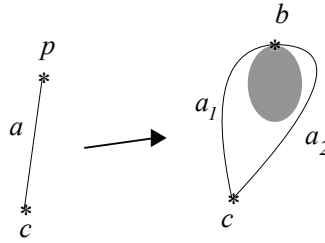


Figure B1: Gauging the $SU(2)$ symmetry associated to a puncture p replaces one arc a of the triangulation with a pair of arcs a_1, a_2 with the same starting point (c) forming a triangle with the boundary (grey).

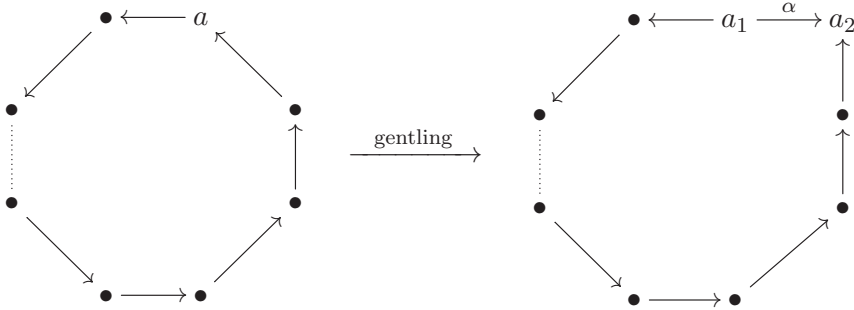
be seen as a *local* operation at each of the p punctures⁴⁷: we cut a small disk near the puncture in such a way that the puncture becomes a marked point on the disk boundary (Figure B1). One can always choose a given ideal triangulation of the surface so obtained that matches the previous one up to the doubling one of the ideal arcs originating from the puncture, a , in two copies, a_1, a_2 , which form the two sides of a triangle whose third side is the disk boundary (see Figure B1). At the level of the quiver, this replaces the node corresponding to the ideal arc a with two nodes associated to the two copies of that arc, a_1, a_2 , forming a Type I block $a_1 \xrightarrow{\alpha} a_2$ [47].

The procedure being local at each puncture, the global geometry of Σ is irrelevant, and to analyze the effect of gentling on the Representation Theory of the corresponding BPS quiver, we may replace the surface Σ with a small disk around the given puncture p_0 . Let r be the number of segments of ideal arcs incident in p_0 ; locally in the small disk around p_0 the geometry is the same as in the ‘radial’ ideal triangulation of the punctured disk with r marked points on its boundary whose quiver is the *cyclic* orientation of the affine Dynkin graph $A_{r-1}^{(1)}$, which belongs to the mutation class of the D_r AD model. Thus all gentling operation is modeled on the gauging of the $SU(2)$ flavor symmetry of a D -type AD model in a region of its parameter space covered by the cyclic form of its quiver. Gentling just replaces⁴⁸ the

⁴⁷For simplicity, here we assume the puncture is not the internal vertex of a self-folded triangle [47]. This is not a restriction: all surfaces have ideal triangulations without self-folded triangles.

⁴⁸The discussion in the text is valid under the assumption that the arc a does not start and end in the same puncture p_0 . When it does, the segment $a_1 \xrightarrow{\alpha} a_2$ appears twice in the covering affine quiver which is then of the form $\widehat{A}(r, 2)$.

cyclic orientation of $A_{r-1}^{(1)}$ by the acyclic affine quiver $\widehat{A}(r, 1)$
 (B.5)



We stress that this does not necessarily mean that the original quiver had a cyclic $A_{r-1}^{(1)}$ subquiver which is replaced with a $\widehat{A}(r, 1)$ subquiver by gentling; indeed two segments incident in p may correspond to the two ends of the *same* arc, and some nodes and arrows in Eqn. (B.5) should be identified in pair. The above description is valid for a suitable *cover* of the quiver. Indeed, the representation corresponding to the spectator W boson for the gentling of the puncture p_0 is precisely the band module of the gentled algebra given by the RHS of Figure (B.5). In what follows, by a slight abuse of notation W_{spec} denotes either the spectator W boson, or the corresponding representation of the gentled algebra, or the corresponding band, according to the context.

Properties of the representations. If X is a representation of the gentled quiver, the spectator magnetic charge of the corresponding particle is [21]

$$(B.6) \quad m(X)_{\text{spec}} \equiv \frac{1}{2} \langle \dim X, \dim W_{\text{spec}} \rangle_D = \dim X_{a_1} - \dim X_{a_2}.$$

Let X be an indecomposable representation of the gentled algebra and $X|_p$ be its restriction to the $\widehat{A}(r_p, 1)$ cover subquiver associated to the gentled p puncture (cfr. (B.5)). One has $X|_p = \oplus_s Y_s$ where Y_s are indecomposables of the Euclidean algebra $\mathbb{C}\widehat{A}(r_p, 1)$. The following statement is a standard property of the *light category* [21]

Lemma. *Suppose X has zero spectator magnetic charge, $m(X)_{\text{spec}} = \dim X_{a_1} - \dim X_{a_2} = 0$ and that it is stable for parametrically small spectator coupling g_{spec}^2 . Then $m(Y_s)_{\text{spec}} \equiv \dim(Y_s)_{a_1} - \dim(Y_s)_{a_2} = 0$, for all s .*

From the properties of the light category [21] we also know that, for a representation X of zero spectator magnetic charge which is stable as $g_{\text{spec}}^2 \rightarrow 0$, the extra arrow α is either an isomorphism or $\dim X$ is n times the charge vector of W_{spec} (equal to the image under the cover map of the minimal imaginary root of $\widehat{A}(r, 1)$). In the second case, only $n = 1$ gives a brick, which belongs to the family of the spectator W . For all other such bricks X of the gentled algebra, α is an isomorphism, and X is canonically identified with a representation of the original quiver.

Appendix C. Proof of Fact 4

In this appendix we prove **Fact 4**.

C.1. The associated topological cover

Assume we have a Galois cover $\mathcal{C} \rightarrow \mathcal{J}$ where \mathcal{C} and \mathcal{J} are the \mathbb{C} -categories of two class $\mathcal{S}[A_1]$ QFTs. Then $\mathcal{C} = \mathbb{C}Q' / (\partial\mathcal{W}')$ for a connected triangulation quiver Q' and corresponding superpotential \mathcal{W}' . Let us first look at the conditions for Q' to cover the quiver Q of \mathcal{J} ; later we shall check under which conditions the covering map $Q' \rightarrow Q$ is a covering map of the full Jacobian categories.

For simplicity, we assume that Q' has a unique block decomposition.⁴⁹ The blocks of given Type b then form orbits under \mathbb{G} . We form a collection \mathfrak{C} of blocks by taking one chosen representative for each orbit. Let $\{\circ_{b,i}\}$ be the set of white nodes of the representative block $b \in \mathfrak{C}$. Node $\circ_{b,i}$ is glued in Q' to node $\circ_{gc,j}$ for some $c \in \mathfrak{C}$ and $g \in \mathbb{G}$. Suppose $b = c$; since $b \neq gc$, one has $g \neq 1$. Consider the subquiver of $Q'_{b,g} \subset Q'$ obtained by gluing the two same-Type blocks b and gb at the node $\circ_{b,i} \equiv \circ_{gb,j}$. Restricted to $Q'_{b,g}$, g is an automorphism acting freely on nodes. By inspection of the 10 possible gluings of two same-Type blocks, such an automorphism exists only in four cases:

- i)* both nodes of two Type I blocks are glued together with arrows in opposite directions to give a pair of opposite arrows which correspond to two disconnected nodes. This is the quiver of the Argyres–Douglas theory of type D_2 ;

⁴⁹Again, this is not a substantial limitation since the remark in footnote 13 applies.

- ii) all nodes of two Type II blocks are glued together with *opposite* orientations to form the (non 2-acyclic [19, 20]) quiver of the T_2 theory;
- iii) all nodes of two Type II blocks are glued together with the same orientation to form the Markov quiver of the $SU(2)$ $\mathcal{N} = 2^*$ theory;
- iv) all white nodes of two Type IV blocks are glued with opposite orientations to form the $(\mathbb{Z}_2)^3$ -symmetric form of the quiver of $SU(2)$ SQCD with $N_f = 4$.

In these 4 cases, all white nodes have being already paired up so, since Q' is connected, $Q' \equiv Q'_{b,g}$. We conclude that, for all quivers Q' (admitting a unique decomposition) which are not in the list *i)–iv)*, we have $b \neq c$, that is, two blocks in the decomposition of Q' which share a white node belong to *distinct* \mathbb{G} -orbits.

Let Q' with a freely acting automorphism group \mathbb{G} , which has a unique decomposition and is not in the above list of four. The quiver $Q = Q'/\mathbb{G}$ is then constructed as follows. We have the collection \mathfrak{C} of chosen representative of \mathbb{G} orbits of blocks, which we identify with the collection of blocks for Q'/\mathbb{G} . Two blocks $b, c \in \mathfrak{C}$ are glued together by identifying the white nodes $\circ_{b,i} \sim \circ_{c,j}$ iff there is a $g \in \mathbb{G}$ (necessarily unique) such that $\circ_{b,i}$ and $\circ_{gc,j}$ are identified in Q' . By the previous argument, the white nodes which get identified belong to distinct blocks, and hence form an admissible block decomposition of the Q'/\mathbb{G} quiver, which is then an ideal triangulation quiver of some surface Σ_{quot} *endowed with a preferred block decomposition* \mathfrak{C} .

Given a block decomposition \mathfrak{C} of a triangulation quiver Q , we construct a surface with punctures and marks on the boundaries $\Sigma_{\mathfrak{C}}$. This is done simply by taking for each block in \mathfrak{C} the corresponding n -gon from Table 1 and gluing them together by identifying in pairs their sides according to the rule given by their correspondence with glued pairs of white nodes of the blocks in \mathfrak{C} .

Hence starting with a surface Σ' and one of its triangulation quivers Q' (which is not one of the exceptions above) having a freely acting automorphism group \mathbb{G} , by the procedure above we uniquely construct a surface (with punctures and marks) $\Sigma_{\mathfrak{C}}$. Moreover, the procedure gives us a covering map

$$(C.1) \quad \gamma: \Sigma' \rightarrow \Sigma_{\mathfrak{C}},$$

given in the following way. For each block Type $\mathfrak{b} \in \mathfrak{B}$ let $S_{\mathfrak{b}}$ be the n -gon in Table 1. We write $S_{\mathfrak{b}}(b)$ for the n -gon associated to the block b in the decomposition of Q' . From each chosen block representative $b \in \mathfrak{C}$ and $g \in \mathbb{G}$

we have a homeomorphism

$$(C.2) \quad \gamma_{b,g}: S_{\mathfrak{b}}(gb) \rightarrow S_{\mathfrak{b}}(b),$$

which sends the segments of the boundary (and the arcs of the internal triangulation) into corresponding segments (and arcs). The family of maps $\{\gamma_{b,g} \mid b \in \mathfrak{C}, g \in \mathbb{G}\}$ glue into a continuous map

$$(C.3) \quad \gamma: \Sigma' \rightarrow \Sigma_{\mathfrak{C}},$$

which is a covering map which may be branched only at the base points of the ideal triangulation (i.e. punctures and marks) which are not the inner puncture of a self-folded triangle. This is the map to be constructed.

A covering map of (punctured and marked) surfaces $\gamma: \Sigma' \rightarrow \Sigma$, branched only at punctures, defines a quotient map between the corresponding quivers. For instance, the \mathbb{Z}_{pq} -symmetric quiver of the punctured disk with pq marks on the boundary, which is an oriented pq cycle, is a \mathbb{Z}_p -cover (branched over the puncture) of the punctured disk with q marks. However, this covering of quivers do not extend to a cover of Jacobian categories, since the superpotential on the quotient quiver is not the quotient superpotential. This is the general case; the term in the superpotential [48] associated to a puncture (not inside a self-folded triangle) has degree equal to the number s of ideal arcs incident in the puncture; under a cover branched on this puncture with local branch number m , this number reduced from s to s/m , and agreement is possible only if $m = 1$ i.e. there is no branching.

The conclusion is: *(with the mentioned exceptions) the non-trivial Galois covers, $F: \mathcal{C} \rightarrow \mathcal{J}$ between class $\mathcal{S}[A_1]$ categories \mathcal{C}, \mathcal{J} are in one-to-one to unbranched covers between the corresponding bordered surfaces $f: \Sigma(\mathcal{C}) \rightarrow \Sigma(\mathcal{J})$ where the punctures (resp. marks) on $\Sigma(\mathcal{C})$ are taken to be the pre-images of the punctures (marks) on $\Sigma(\mathcal{J})$.*

Gaiotto viewpoint. We replace the bordered surfaces $\Sigma(\mathcal{C}), \Sigma(\mathcal{J})$ by the corresponding Gaiotto curves obtained by replacing the i -th boundary component with $\ell_i \geq 1$ marks with an irregular puncture at which the quadratic differential has a higher pole of the form

$$(C.4) \quad \phi_2 \Big|_{z \sim z_i} \sim \text{const.} (z - z_i)^{-\ell_i} \left(\frac{d(z - z_i)}{z - z_i} \right)^2.$$

Locally around a branch point of order m , the cover looks like $z - z_i = w^m + \dots$, so

$$(C.5) \quad w^* \phi_2 = \text{const.}' w^{-m\ell_i} \left(\frac{dw}{w} \right)^2,$$

which correctly corresponds to a boundary component with a number of marks which is the original one, ℓ_i times the local degree of the covering map, m .

C.2. Exceptional cases

There are two sources of exceptions:

- A) the list of four special Σ' i – iv). The quotient quivers of iii) and iv) are not 2–acyclic (this does *not* mean they are not useful covers). The quiver of ii) is not 2–acyclic from the start. It remains i): the D_2 Argyres–Douglas model (a doublet of free hypermultiplet) double covers the A_1 Argyres–Douglas model (i.e. a free hyper). Of course, the statement that the BPS spectrum of two copies of the free hyper makes two copies of the BPS spectrum of one free hyper, it by no means a surprise.
- B) The quiver Q' has more than one block decomposition. Suppose we have a Galois cover of triangulation quivers $Q' \rightarrow Q'/\mathbb{G}$. Fix a particular block decomposition of the quotient quiver; the pull back of the this block decomposition is a particular block decomposition of Q' with the property that \mathbb{G} acts by permutation of same–Type blocks. This is enough to make the previous argument work. It is possible that Q'/\mathbb{G} is the triangulation quiver of several topologically distinct surfaces; fixing a block decomposition implicitly chooses a topological surface in this set, and we describe the surface cover as a cover of this particular surface; thus, in this case, we may describe in many ways as covering of surfaces the same covering of class $\mathcal{S}[A_1]$ categories.

Appendix D. $g = 1$: no R_1^r summands at \mathbb{Z}_p –symmetric points

Here we show that no R_1^r appears in a module of the Jacobian algebra \mathcal{J}_p^g which is stable in our pulled back chamber and is not a spectator W .

D.1. Strings and bands

The model with the flavor symmetry fully gauged has a gentle Jacobian algebra, whose indecomposable modules may be constructed using strings and bands (see [21] for a review). Hence the relevant modules of \mathcal{J}_p^g may be described in the same language. We write \overline{Q} for the (two-side) ideal in $\mathcal{J}_p^g \equiv \mathbb{C}Q_p/(\partial\mathcal{W}_{\text{cubic}})$ which is the image of the arrow ideal of $\mathbb{C}Q_p$. We say that a \mathcal{J}_p^g -module M has order $n \in \mathbb{N}$ iff

$$(D.1) \quad \overline{Q}^n \cdot M \neq 0, \quad \overline{Q}^{n+1} \cdot M = 0.$$

Lemma. *Let $X \in \text{mod } \mathcal{J}_p^g$ be a brick which is not a spectator W . Then its order is at most 5. Then X is canonically identified with a brick of the string algebra $\mathcal{J}_p^g/\overline{Q}^6$.*

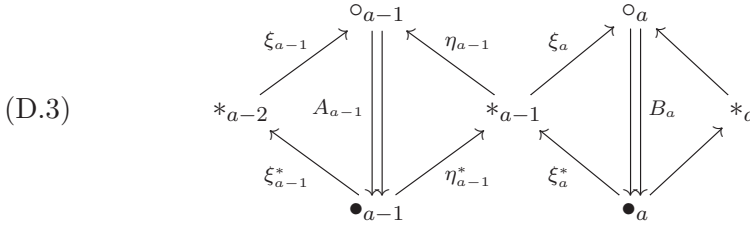
Proof. For $a = 1, 2, \dots, p$ consider the families $\Lambda^{(a)} = \{\lambda_{\circ_b}^{(a)}, \lambda_{\bullet_b}^{(a)}, \lambda_{*_b}^{(a)}\}_{b=1}^p$ of linear maps defined by

$$(D.2) \quad \begin{aligned} \lambda_{\circ_a}^{(a)} &= \eta_a \xi_{a+1}^* B_{a+1} \xi_{a+1} \eta_a^* A_a, & \lambda_{\circ_{a+1}}^{(a)} &= \xi_{a+1} \eta_a^* A_a \eta_a \xi_{a+1}^* B_{a+1}, & \lambda_{\circ_b}^{(a)} &= 0 \text{ for } b \neq a, a+1 \\ \lambda_{\bullet_a}^{(a)} &= A_a \eta_a \xi_{a+1}^* B_{a+1} \xi_{a+1} \eta_a^*, & \lambda_{\bullet_{a+1}}^{(a)} &= B_{a+1} \xi_{a+1} \eta_a^* A_a \eta_a \xi_{a+1}^*, & \lambda_{\bullet_b}^{(a)} &= 0 \text{ for } b \neq a, a+1 \\ \lambda_{*_a}^{(a)} &= \eta_a^* A_a \eta_a \xi_{a+1}^* B_{a+1} \xi_{a+1} + \xi_{a+1}^* B_{a+1} \xi_{a+1} \eta_a^* A_a \eta_a, & \lambda_{*_b}^{(a)} &= 0 \text{ for } b \neq a; \end{aligned}$$

one checks that $\Lambda^{(a)} \in \text{End}_{\text{mod } \mathcal{J}_p^g} X$. So, if X is a $\text{mod } \mathcal{J}_p^g$ -brick, $(\Lambda^{(a)})_i = \lambda^{(a)} \in \mathbb{C}$ for all a and all nodes i in the support of X . We lift X to a $\text{mod } \mathcal{J}_p^{\text{fg}}$ -module by reinserting the isomorphisms K_b at all nodes $*_b \in \text{supp } X$. Since $\text{mod } \mathcal{J}_p^{\text{fg}}$ is gentle the brick X may be explicitly constructed as the string module of a string C or as a band module of the form $M(C, 1, \lambda)$. To produce a non zero $(\Lambda^{(a)})_i$ the string/band module should contain a substring given, up to cyclic reordering, by the sequence of arrows (direct or inverse) C_a making the band of the a -th spectator boson. Essentially by definition, the resulting $(\Lambda^{(a)})_i = \lambda^{(a)} \in \mathbb{C}^\times$ if and only if it is the band module $M(C_a, 1, \lambda^{(a)})$, which corresponds to the a -th spectator W boson. All other bricks of $\text{mod } \mathcal{J}_p^{\text{fg}}$, such that K_a are isomorphisms, must have $(\Lambda^{(a)})_i = 0$ for all a , which implies that the corresponding strings/bands do not contain any of the C_a as a substring. Identifying such bricks of $\text{mod } \mathcal{J}_p^{\text{fg}}$ with bricks of $\text{mod } \mathcal{J}_p^g$, we conclude that all products of six arrows appearing in Eqn. (D.2) (including the ones in $\lambda_{*_a}^{(a)}$) vanish. By diagram chasing one checks that these six arrows products generate the ideal \overline{Q}^6 . The last statement follows from the fact that \mathcal{J}_p^g failed to be a string algebra [55] only because it was not finitely dimensional. □

D.2. No R_1^r

We draw only the relevant piece of the quiver:



We write $R_1^{(a)}$ for a R_1 summand of the restriction of X to the a -th Kronecker subquiver Kr_a . If C is the string (band) corresponding to X , each $R_1^{(a)}$ is represented by a maximal⁵⁰ segment of C in Kr_a consisting of a single arrow (direct or inverse). We focus on the case in which this arrow is the *second* arrow of the a -th Kronecker, B_a , the other case being related by an automorphism of the quiver. In the R_1^r case $R_1^{(a)}$ appears in C inside the substring $\dots \xleftarrow{\xi_a^*} \xleftarrow{B_a} \xleftarrow{\xi_a} \dots$. Note that if $\xleftarrow{\xi_a}$ is the last arrow in the string, or if it is followed by $\xrightarrow{\eta_{a-1}}$, we have a destabilizing quotient with support at the node $*_{a-1}$. Hence C should contain the substring

(D.4)
$$\dots \xleftarrow{\xi_a^*} \xleftarrow{B_a} \xleftarrow{\xi_a} \xleftarrow{\eta_{a-1}^*} \dots$$

Consider the part of the string which precedes $\xleftarrow{\xi_a^*}$. We have three possibilities: $\xleftarrow{\xi_a^*}$ is the first arrow of the string; or it is preceded by $\dots \xrightarrow{\eta_{a-1}^*}$; or it is preceded by $\dots \xleftarrow{\eta_{a-1}}$. In the last case, $\xleftarrow{\eta_{a-1}}$ cannot be the first arrow, nor cannot be preceded by $\xrightarrow{\xi_{a-1}}$, since in both cases we have a destabilizing submodule with support at \circ_{a-1} . Thus in the third case we must have

(D.5)
$$\dots \xleftarrow{A_{a-1}} \xleftarrow{\eta_{a-1}} \xleftarrow{\xi_a^*} \xleftarrow{B_a} \xleftarrow{\xi_a} \xleftarrow{\eta_{a-1}^*} \dots$$

But any string/band module X which contains this substring has $\overline{Q}^6 \cdot X \neq 0$, and hence cannot be a non-spectator brick. We remain with the first two

⁵⁰Here and below *maximality* is meant with respect to inclusion in the a -th Kronecker Kr_{a-1} , i.e. any strictly larger segment of C contains arrows different from $\xleftarrow{A_a}$ and $\xrightarrow{B_a}$.

cases, where we have a *subrepresentation* Y of X

$$(D.6) \quad \dots \overbrace{\left\langle \frac{\xi_a^*}{\leftarrow} \left\langle \frac{B_a}{\leftarrow} \right\rangle \right\rangle}^{Y \text{ subrepr.}} \left\langle \frac{\xi_a}{\leftarrow} \left\langle \frac{\eta_{a-1}^*}{\leftarrow} \right\rangle \right\rangle \dots$$

with central charge $Z_o + Z_\bullet + Z_*$.

The arrow $\leftarrow^{\eta_{a-1}^*}$ is followed by a maximal string segment in Kr_{a-1} which corresponds to either a $R_1^{(a-1),r}$, or a $R_1^{(a-1),p}$, or a $P_n^{(a)}$ direct summand of $X|_{a-1}$. In the first case $\overline{Q}^6 \cdot X \neq 0$ and hence X is not a non-spectator brick. In the second and third cases we have, respectively, the substrings

$$(D.7) \quad \dots \overbrace{\left\langle \frac{\xi_a^*}{\leftarrow} \left\langle \frac{B_a}{\leftarrow} \right\rangle \right\rangle}^{Y \text{ subrepr.}} \left\langle \frac{\xi_a}{\leftarrow} \overbrace{\left\langle \frac{\eta_{a-1}^*}{\leftarrow} \left\langle \frac{A_{a-1}}{\leftarrow} \right\rangle \right\rangle}^{V \text{ quotient}} \right\rangle$$

$$(D.8) \quad \dots \overbrace{\left\langle \frac{\xi_a^*}{\leftarrow} \left\langle \frac{B_a}{\leftarrow} \right\rangle \right\rangle}^{Y \text{ subrepr.}} \left\langle \frac{\xi_a}{\leftarrow} \overbrace{\left\langle \frac{\eta_{a-1}^*}{\leftarrow} \left(\left\langle \frac{A_{a-1}}{\leftarrow} \left\langle \frac{B_{a-1}}{\leftarrow} \right\rangle \right\rangle \right)^n}^{U_n \text{ quotient}} \right\rangle \dots \quad n = 0, 1, 2, \dots$$

which contain, respectively, a quotient of X of the form V or, respectively, U_n whose central charges are

$$(D.9) \quad Z(V) = Z_o + Z_\bullet + Z_*, \quad Z(U_n) = nZ_o + (n + 1)Z_\bullet + Z_*$$

Since $Z(Y) = Z(V)$ a module of the first form is never stable, while the stability of a module of the second form requires (in particular)

$$(D.10) \quad \begin{aligned} \arg Z(U_n) &\equiv \arg(nZ_o + (n + 1)Z_\bullet + Z_*) \\ &> \arg(Z_o + Z_\bullet + Z_*) \equiv Z(Y) \end{aligned}$$

i.e.

$$(D.11) \quad \arg(n(Z_o + Z_\bullet) + Z_*) > \arg(Z_o + Z_\bullet), \quad n \geq 0$$

which is impossible since $\arg Z_\bullet < \arg Z_o$.

References

- [1] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, [arXiv:0811.2435](#).
- [2] D. Gaiotto, G. W. Moore, and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Commun. Math. Phys.* **299** (2010), 163–224, [arXiv:0807.4723](#).
- [3] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, [arXiv:0907.3987](#).
- [4] D. Gaiotto, *$N = 2$ dualities*, *JHEP* **1208** (2012), 034, [arXiv:0904.2715](#).
- [5] S. Cecotti, A. Neitzke, and C. Vafa, *R-Twisting and 4d/2d correspondences*, [arXiv:1006.3435](#).
- [6] S. Cecotti and C. Vafa, *Topological-anti-topological fusion*, *Nucl. Phys.* **B367** (1991), 359–461.
- [7] S. Cecotti and C. Vafa, *On classification of $\mathcal{N} = 2$ supersymmetric theories*, *Commun. Math. Phys.* **158** (1993), 569–644, [arXiv:hep-th/9211097](#).
- [8] S. Cecotti and C. Vafa, *Classification of complete $\mathcal{N} = 2$ supersymmetric theories in 4 dimensions*, *Surveys in differential geometry* **18** (2013), [arXiv:1103.5832](#).
- [9] S. Cecotti, P. Fendley, K. A. Intriligator, and C. Vafa, *A New supersymmetric index*, *Nucl. Phys.* **B386** (1992), 405, [arXiv:hep-th/9204102](#).
- [10] K. Hori, A. Iqbal, and C. Vafa, *D-branes and mirror symmetry*, [arXiv:hep-th/0005247](#).
- [11] D. Gaiotto, G. W. Moore, and A. Neitzke, *Framed BPS states*, *Adv. Theor. Math. Phys.* **17** (2013), 241, [arXiv:1006.0146](#) [[hep-th](#)].
- [12] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing in coupled 2d-4d systems*, *JHEP* **1212** (2012), 082, [arXiv:1103.2598](#) [[hep-th](#)].
- [13] D. Gaiotto, G. W. Moore, and A. Neitzke, *Spectral networks*, *Annales Henri Poincaré* **14** (2013), 1643, [arXiv:1204.4824](#) [[hep-th](#)].
- [14] D. Gaiotto, G. W. Moore, and A. Neitzke, *Spectral networks and snakes*, *Annales Henri Poincaré* **15** (2014), 61, [arXiv:1209.0866](#) [[hep-th](#)].

- [15] S. Cecotti, and M. Del Zotto, *Infinitely many $N = 2$ SCFT with ADE flavor symmetry*, JHEP **1301** (2013), 191, [arXiv:1210.2886](#).
- [16] S. Cecotti, M. Del Zotto, and S. Giacomelli, *More on the $N = 2$ superconformal systems of type $D_p(G)$* , JHEP **1304** (2013), 153, [arXiv:1303.3149](#).
- [17] P. Gabriel, *The universal cover of a representation-finite algebra*, in: Representation of algebras, Lectures Notes in Mathematics **903**, Springer (1981), pp. 68–105.
- [18] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Inven. Math. **65** (1982), 331–378.
- [19] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, and C. Vafa, *BPS quivers and spectra of complete $N = 2$ quantum field theories*, Commun. Math. Phys. **323** (2013), 1185–1227, [arXiv:1109.4941](#).
- [20] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, and C. Vafa, *$N = 2$ quantum field theories and their BPS quivers*, Advances in Theor. Math. Phys. **18** (2014), 27–127, [arXiv:1112.3984](#).
- [21] S. Cecotti, *Categorical tinkertoys for $N = 2$ gauge theories*, Int. J. Mod. Phys. **A28** (2013), 1330006, [arXiv:1203.6743](#)
- [22] G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, NATO Sci. Ser. B **59** (1980), 135.
- [23] J. Minahan and D. Nemeschansky, *An $N = 2$ superconformal fixed point with E_6 global symmetry*, Nucl. Phys. **B482** (1996), 142–152, [arXiv:hep-th/9608047](#).
- [24] J. Minahan and D. Nemeschansky, *Superconformal fixed points with E_n global symmetry*, Nucl. Phys. **B489** (1997), 24–46, [arXiv:hep-th/96010076](#).
- [25] S. Cecotti and C. Vafa, *BPS wall crossing and topological strings*, (2009), [arXiv:0910.2615](#) [[hep-th](#)].
- [26] T. Dimofte and S. Gukov, *Refined, motivic, and quantum*, Lett. Math. Phys. **91** (2010), 1, [arXiv:0904.1420](#) [[hep-th](#)].
- [27] T. Dimofte, S. Gukov, and Y. Soibelman, *Quantum wall crossing in $N = 2$ gauge theories*, Lett. Math. Phys. **95** (2011), 1, [arXiv:0912.1346](#) [[hep-th](#)].

- [28] F. Denef, *Quantum quivers and Hall/Holes Halos*, arXiv:hep-th/0206072.
- [29] M. Del Zotto and A. Sen, *About the absence of exotics and the Coulomb branch formula*, arXiv:1409.5442 [hep-th].
- [30] H. Derksen, J. Wyman, and A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, *Selecta Mathematica* **14** (2008), 59–119.
- [31] W. Crawley–Boevey, *Lectures on representations of quivers*, available on line at <http://www1.maths.leeds.ac.uk/pmtwc/quivlecs.pdf>.
- [32] C. M. Ringel, *Tame algebras and integral quadratic forms*, *Lecture Notes in Mathematics* **1099**, Springer (1984), Berlin.
- [33] D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras*, volume 2: *Tubes and concealed algebras of Euclidean type*, *London Mathematical Society Student Text* **71**, Cambridge University Press (2007).
- [34] V. Kac, *Infinite root systems, representations of graphs, and invariant theory*, *Invent. Math.* **56** (1980), 57–92.
- [35] V. Kac, *Infinite root systems, representations of graphs, and invariant theory. II*, *J. Algebra* **78** (1982), 141–162.
- [36] V. Kac, *Root systems, Representations of Quivers and Invariant Theory*, (Montecatini, 1982). *Lecture Notes in Mathematics* **996**, Springer (1983), pages 74–108.
- [37] J. de la Peña, *Integral quadratic forms and the representation type of an algebra*, 2006 ICTP lectures, available on line at <http://webusers.imj-prg.fr/~bernhard.keller/ictp2006/lecturenotes/delapena-all.pdf>.
- [38] S. Cecotti and M. Del Zotto, *On Arnold’s 14 ‘exceptional’ $\mathcal{N} = 2$ superconformal gauge theories*, *JHEP* **1110** (2011), 099, arXiv:1107.5747.
- [39] M. Del Zotto, *More Arnold’s $\mathcal{N} = 2$ superconformal gauge theories*, *JHEP* **1111** (2011), 115, arXiv:1110.3826.
- [40] S. Cecotti, and M. Del Zotto, *The BPS spectrum of the 4d $\mathcal{N} = 2$ SCFT’s $H_1, H_2, D_4, E_6, E_7, E_8$* , *JHEP* **1306**(2013), 75, arXiv:1304.0614.

- [41] V. Kac, *Infinite dimensional Lie Algebras*, 3rd edition, Cambridge University Press (1990).
- [42] P. Slodowy, *Beyond Kac–Moody algebras and inside*, Can. Math. Soc. Conf. Proc. **5** (1986), 361–371, CMP 18:10.
- [43] P. Slodowy, *Kac–Moody algebras, assoziiert Gruppen und Verallgemeinerungen*, Habilitationsschrift, Universität Bonn, 1984.
- [44] B. Allison, S. Azam, S. Berman, Y. Gao, and A. Pianizola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. **126** (1997).
- [45] E. Neher, *Lectures on extended affine Lie algebras*, arXiv:1003.2352 [math.RA]
- [46] R. Moody, S. Rao, and T. Yokonuma, *Toroidal Lie algebras and vertex representations*, Geom. Dedicata **35** (1990), no. 1-3, 283–307.
- [47] S. Fomin, M. Shapiro, and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, Acta Math. **201** (2008), 83–146.
- [48] D. Labardini–Fragoso, *Quivers with potentials associated to triangulated surfaces*, arXiv:0803.1328 [math.RT].
- [49] D. Labardini–Fragoso, *Quivers with potentials associated to triangulated surfaces. Part II: Arc representations*, arXiv:0909.4100 [math.RT].
- [50] C. Geiss, D. Labardini-Fragoso, and J. Schroer, *The representation type of Jacobian algebras*, arXiv:1308.0478.
- [51] R. Martinez-Villa and J. A. de la Peña, *The universal cover of a quiver with relations*, J. Pure and Appl. Algebra **30** (1983), 277–292.
- [52] P. Dowbor and A. Skowroński, *Galois coverings of representation infinite algebras*, Comment. Math. Helv. **62** (1987), 311–337.
- [53] A. Skowroński, *Periodicity in representation theory of algebras*, ICTP Lecture Notes 2006 available on-line at <http://webusers.imj-prg.fr/~bernhard.keller/ictp2006/lecturenotes/skowronski.pdf>.
- [54] J. A. de la Peña, *Group actions on algebras and module categories*, Rev. Un. Mat. Argen. **48** (2007), 1–20.
- [55] M. C. R. Butler and C. M. Ringel, *Auslander–Reiten sequences with few middle terms and applications to string algebras*, Comm. in Algebra **15** (1987), 145–179.

- [56] I. Assem, T. Brüstle, G. Charbonneau–Jodoin, and P.-G. Plamondon, *Gentle algebras arising from surface triangulations*, [arXiv:0903.3347](https://arxiv.org/abs/0903.3347).
- [57] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, Volume 1: *Techniques of Representation Theory*, London Mathematical Society Student Text **65**, Cambridge University Press (2006).
- [58] A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, *Math. Ann.* **39** (1891), 1–60.
- [59] G.V. Belyi, *On Galois extensions of a maximal cyclotomic field*, *Math. USSR Izvestija* **14** (1980), 247–265.
- [60] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics **228**, Springer (2005).
- [61] H. M. Farkas and I. Kra, *Theta constants, Riemann surfaces and the modular group*, Graduate Studies in Mathematics **37**, AMS (2001).
- [62] R. Stekolshchik, *Notes on Coxeter transformations and the McKay correspondence*, Springer Monographs in Mathematics, (2008).
- [63] P. Beazley Cohen, C. Itzykson, and J. Wolfart, *Fuchsian triangle groups and Grothendieck dessins. Variations on a theme of Belyi*, *Comm. Math. Phys.* **163** 605–627, (1994).
- [64] A. Grothendieck, *Esquisse d'un programme*, pp. 5–48 in L. Schneps, P. Lochak (eds.) *Geometric Galois Actions. 1*, London Math. Lecture Note Series **242**, Cambridge University Press (1997).
- [65] J. Wolfart, *ABC for polynomials, dessins d'enfants, and uniformization — a survey*, in: *Elementare und analytische Zahlentheories*, *Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main* **20**, Stuttgart (2006).
- [66] J. Wolfart, *Triangle groups and Jacobians of CM type*, available on line at <http://www.math.uni-frankfurt.de/~wolfart/Artikel/jac.pdf>.
- [67] A. K. Zvonkin, *Belyi functions: examples, properties, and applications*, available on line at <https://www.labri.fr/perso/zvonkin/Research/belyi.pdf>.
- [68] N. Magot and A. K. Zvonkin, *Belyi functions for Archimedean solids*, *Discrete Math.* **217** (2000), 249–271.

- [69] E. Pervova and C. Petronio, *On the existence of branched coverings between surfaces with prescribed branch data. I*, Alg. and Geom. Topology **6** (2006), 1957–1985.
- [70] Y. Tachikawa and S. Terashima, *Seiberg–Witten geometries revisited*, arXiv:1108.2315 [hep-th].
- [71] B. Keller, *The periodicity conjecture for pairs of Dynkin diagrams*, arXiv:1001.1531 [math.RT].
- [72] Y. Tachikawa, *Six–dimensional DN theory and four–dimensional SO–USp quivers*, JHEP **0907** (2009), 067, arXiv:0905.4074.
- [73] V. V. Fock and A. B. Goncharov, *The quantum dilogarithm and representations of quantum cluster varieties*, Inven. Math. **175** (2009), 223–286.
- [74] V. V. Fock and A. B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. scient. de l’ENS **42** (2009), 865–930.
- [75] B. Keller, *On cluster theory and quantum dilogarithm identities*, arXiv:1102.4148.
- [76] L. D. Faddeev and R. M. Kashaev, *Quantum dilogarithm*, Modern Phys. Lett. **A9** (1994), 427–434.
- [77] S. Cecotti and M. Del Zotto, *Y systems, Q systems, and 4D $\mathcal{N} = 2$ supersymmetric QFT*, J. Phys. A **47** (2014), no. 47, 474001, arXiv:1403.7613 [hep-th].
- [78] B. Keller and S. Scherotzke, *Linear recurrence relations for cluster variables of affine quivers*, arXiv:1004.0613.
- [79] I. Assem, C. Reutenauer, and D. Smith, *Frisés*, arXiv:0906.2026.
- [80] I. Assem, G. Dupont, R. Schiffler, and D. Smith, *Friezes, strings and cluster variables*, arXiv:1009.3341.
- [81] S. Cecotti, C. Cordova, and C. Vafa, *Braids, walls, and mirrors*, arXiv:1110.2115 [hep-th].
- [82] S. Fomin and A. Zelevinsky, *Cluster algebras IV: Coefficients*, arXiv:math/0602259 [math.RA].
- [83] R. Kedem, *Q–systems as cluster algebras*, J. Math. Phys. A: Math. Theor. **41** (2008), 1940011, arXiv:0712.2695 [math.RT].

- [84] P. Di Francesco and R. Kedem, *Q-systems, heaps paths and cluster positivity*, *Comm. Math. Phys.* **293** (2009), 727820, [arXiv:0811.37027](#) [[math.CO](#)].

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
VIA BONOMEA 265, 34100 TRIESTE, ITALY
E-mail address: cecotti@sissa.it

JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY
CAMBRIDGE, MA 02138, USA
E-mail address: eledelz@gmail.com