$(0,2)$-deformations and the G-Hilbert scheme

BENJAMIN GAINES

We study first order deformations of the tangent sheaf of resolutions of Calabi-Yau threefolds that are of the form $\mathbb{C}^3/\mathbb{Z}_r$, focusing on the cases where the orbifold has an isolated singularity. We prove a lower bound on the number of deformations for any crepant resolution of this orbifold. We show that this lower bound is achieved when the resolution used is the G-Hilbert scheme, and note that this lower bound can be found using methods from string theory. These methods lead us to a new way to construct the G-Hilbert scheme using the singlet count.

1. Introduction

When G is an abelian group, the orbifold $\mathbb{C}^3/G$ is a normal toric variety. It can have several crepant resolutions, and the process for finding these resolutions is explained in detail in [6]. We will review these methods in Section 2. For each of these resolutions, we are interested in the moduli space of the tangent bundle $\mathcal{T}$. We will calculate the dimension of $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T})$, to determine the number of first order deformations of the tangent sheaf.

There are many reasons to be interested in this moduli space, but our primary motivation comes from string theory. One of the predictions that geometric string theory makes is that there is connection between the orbifold conformal theory, independent of the projective resolution chosen, and the moduli space. In $N = (2,2)$ theories, this statement is well understood and follows from the McKay correspondence, as in [3, 4]. In the case of $(0,2)$-theories, this relationship is less well understood. One of our goals is to look for a relationship between the conformal field theory and the number of deformations of the tangent bundle of X, as predicted in [8]. Following Aspinwall, we call this the $(0,2)-$McKay correspondence [1].

In this paper, we prove that the conformal field theory orbifold count will serve as a lower bound for the dimension of the moduli space. We also find that the conformal field theory gives a distinguished quiver, and that

this quiver can be used to construct a particular resolution for X. We will then show that this particular resolution is in fact the G-Hilbert scheme, and that for this resolution the $(0,2)-$McKay correspondence holds. This gives the following main result:

Theorem 1.1. *Among all crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_r$ with r prime, the G-Hilbert scheme has minimal dimension for $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T})$. This dimension exactly matches the prediction given by the conformal field theory of the orbifold.*

In the course of proving this theorem, we find a few other interesting results about the G-Hilbert scheme, in the case where G is an abelian subgroup of $SL(3, \mathbb{C})$. In particular, we see that there is a very easy way to find the number of first order deformations of the G-Hilbert scheme, using the knockout method of [7]. We also show that the fan for the G-Hilbert scheme can be constructed using the singlets from the conformal field theory.

2. Crepant resolution of an orbifold

2.1. Quotient presentation of toric variety

We begin with a review of the quotient presentation of a toric variety. Let Σ be a simplicial fan, with 3 dimensional lattice N. Let $\{\rho_i\}$ denote the set of one-dimensional cones (or rays) of Σ, with their minimal generators given by u_{ρ_i}, and $|\Sigma(1)|$ being the number of rays. Let M be the dual lattice to N, and $\{P_i\}$ be the set of primitive collections of Σ. A primitive collection is a set of rays so that the entire set does not lie in a cone of Σ, but every proper subset does. We can now describe the toric variety associated with Σ as a quotient $X = (\mathrm{Spec}(S) - V(B))/G$, using the method of Cox [5]. In this notation, S is the homogeneous coordinate ring of the variety, B is the irrelevant ideal, and G is the quotient group.

- $S = \mathbb{C}[x_{\rho_i}]$, where each ray ρ_i corresponds to a variable x_{ρ_i}.

- $G = \mathrm{Hom}_{\mathbb{Z}}(Cl(X), \mathbb{C}^*)$, with a group action given by

$$\left\{ (t_\rho) \in (\mathbb{C}^*)^{|\Sigma(1)|} \;\middle|\; \prod_\rho t_\rho^{<e_i, u_\rho>} = 1 \text{ for } 1 \leq i \leq 3 \right\}$$

  where $\{e_i\}$ are a basis for M.

- $B = \bigcap(I_{P_i})$, where I_{P_i} is the ideal generated by the elements of the primitive collection P_i.

In particular, we note that both S and G depend only on the one-dimensional cones of Σ.

2.2. Exceptional divisors

Let $X = \mathbb{C}^3/G$, with $G = \mathbb{Z}_r$ and having a group action given by $(\zeta_r^{a_1}, \zeta_r^{a_2}, \zeta_r^{a_3})$ where ζ_r is a primitive r-th root of unity. We are interested in resolving $\mathbb{C}^3/G$, and getting a smooth variety. To begin, we'll look at the case where $a_1 = 1$, $a_2 = a$, and $a_3 = b$. As long as some a_i and r are relatively prime, we can renumber to get this case. We note that if X has an isolated singularity, this is always true. Since $\mathbb{C}^3/G$ is a simplicial toric variety, there is a fan in $\mathbb{R}^3$ associated to it. Such a fan must consist of a single cone, with 3 rays. To find minimal generators for these rays, we must make sure the group action described in the previous section gives $(\zeta_r^{a_1}, \zeta_r^{a_2}, \zeta_r^{a_3})$. We do this by choosing $u_1 = (r, -a, -b)$, $u_2 = (0,1,0)$, and $u_3 = (0,0,1)$, where the lattice N is just $\mathbb{Z}^3$.

We must now find all of the lattice points in the interior of the convex hull of these 3 minimal generators. We note that these points all lie in a hyperplane, and therefore can be used to find a crepant resolution of X. Each of these points will be the minimal generator of a ray in $\mathbb{R}^3$, which corresponds to an exceptional divisor of the resolved variety.

Proposition 2.1. *The interior points can be labeled by*

$$\{(i, -s, -t) \mid i, s, t \in \mathbb{Z}, 0 < i < r, rs < ia < r(s+1),$$
$$rt < ib < r(t+1), i - s - t = 1\}$$

Proof. We will now prove the proposition. We can find interior points by starting at one vertex, $(r, -a, -b)$ and adding $d_1(-r, 1+a, b)$ and $d_2(-r, a, 1+b)$, satisfying $d_1 + d_2 \leq 1$, $d_1, d_2 \geq 0$, ignoring the cases corresponding to vertices. We therefore see that every interior point is of the form $(r(1 - d_1 - d_2), (d_1 + d_2 - 1)a + d_1, (d_1 + d_2 - 1)b + d_2)$. To find just the lattice points, we need all of these values to live on the lattice N. In this case, the lattice was $\mathbb{Z}^3$. Let $r(1 - d_1 - d_2) = i$. After a few algebraic manipulations, this leads to the conditions that $d_1 r \equiv ia \mod r$, $d_2 r \equiv ib \mod r$, and $d_1 + d_2 = 1 - \frac{i}{r}$. We therefore have $d_1 = \frac{ia}{r} - s$, $d_2 = \frac{ib}{r} - t$, for some s, t integers.

Thus, the set of interior lattice points is labeled by

$$\{(i, -s, -t) \mid i, s, t \in \mathbb{Z}, 0 < i < r, rs < ia < r(s+1),$$
$$rt < ib < r(t+1), i - s - t = 1\}$$

We will call these interior points u_4 through u_N (and will call the three vertices u_1, u_2, u_3). □

As an example, let's look at the orbifold $X = \mathbb{C}^3/\mathbb{Z}_{11}$, where the group action is given by $(\zeta_{11}, \zeta_{11}^2, \zeta_{11}^8)$. In this case, the minimal generators for the rays of the cone are $u_1 = (11, -2, -8), u_2 = (0, 1, 0)$, and $u_3 = (0, 0, 1)$. Using the process described above, we must find the set

$$\{(i, -s, -t) \mid i, s, t \in \mathbb{Z}, 11s < 2i < 11s + 11,$$
$$11t < 8i < 11t + 11, i - s - t = 1\}.$$

Checking all possible values for i, we find that the interior points are

$$u_4 = (1, 0, 0), \quad u_5 = (2, 0, -1), \quad u_6 = (3, 0, -2),$$
$$u_7 = (6, -1, -4), \quad u_8 = (7, -1, -5).$$

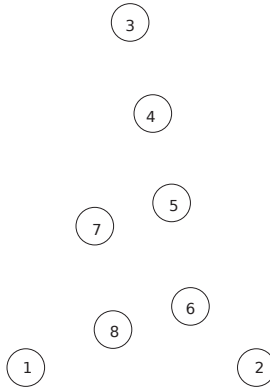These can be seen in the figure below.

Figure 2.1. $\mathbb{C}^3/\mathbb{Z}_{11}$ with interior lattice points.

Note that if the singularity is not isolated, it is possible for the group action to be given by $(\zeta_r^{a_1}, \zeta_r^{a_2}, \zeta_r^{a_3})$, where each of the a_i is not relatively prime to r. In this case a similar theorem holds (by finding interior points on a sublattice of $\mathbb{Z}^3$), but for reasons we will expound on later, we will limit our focus to orbifolds with isolated singularities.

2.3. Resolving the singularity

Now that we have found all of the exceptional divisors, we can find a resolution of the singularity. To do this, we find some triangulation of the points

$u_1, \ldots, u_N$, and then take the fan over this triangulation. This gives a new fan with the same lattice N, which corresponds to the crepant resolution associated with this particular triangulation. Since all of these fans have the same one dimensional rays (with minimal generators $\{u_1, \ldots, u_N\}$), they have the same homogeneous coordinate ring S and quotient group G. We have $S = \mathbb{C}[x_1, \ldots, x_N]$, where each x_i corresponds to a ray. To find G, we first construct an $n \times 3$ matrix A, whose rows are the coordinates of these points.

$$A = \begin{pmatrix} r & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ (u_4)_1 & (u_4)_2 & (u_4)_3 \\ \vdots & \vdots & \vdots \\ (u_N)_1 & (u_N)_2 & (u_N)_3 \end{pmatrix}$$

The cokernel of A will give the class group $Cl(X)$ of any resolution with these one-dimensional cones, and $G = Hom_{\mathbb{Z}}(Cl(X), \mathbb{C}^*)$. This tells us that $G \cong (\mathbb{C}^*)^{N-3}$, and what the group action of G is. Let Φ be a matrix with coefficients in $\mathbb{Z}$ and rows given by a basis of the cokernel of A. In the physics literature, this is known as the charge matrix. This matrix will be $(N - 3) \times N$, and so using some linear algebra we can reduce it to another matrix Φ', where

$$\Phi' = \begin{pmatrix} \nu_1^4 & \nu_2^4 & \nu_3^4 & -1 & 0 & \cdots & 0 \\ \nu_1^5 & \nu_2^5 & \nu_3^5 & 0 & -1 & \cdots & 0 \\ & & & & \vdots & & \\ \nu_1^N & \nu_2^N & \nu_3^N & 0 & 0 & \cdots & -1 \end{pmatrix}$$

for some rational numbers $\nu_1^i, \nu_2^i, \nu_3^i$ (note: the superscript is an index here, not a power).

The matrix Φ gives a grading for S, where the multidegree for x_j is a vector given by the jth column of Φ. We will call this vector the charge of x_j, and denote it by q_j.

Let's go back to our example, of $\mathbb{C}^3/\mathbb{Z}_{11}$. Using the result from the previous section, we know there are 8 rays in the fan of any crepant resolution,

so $S = \mathbb{C}[x_1, \ldots, x_8]$. We also find that

$$A = \begin{pmatrix} 11 & -2 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & 0 & -2 \\ 6 & -1 & -4 \\ 7 & -1 & -5 \end{pmatrix} \quad \Phi' = \begin{pmatrix} \frac{1}{11} & \frac{2}{11} & \frac{8}{11} & -1 & 0 & 0 & 0 & 0 \\ \frac{2}{11} & \frac{4}{11} & \frac{5}{11} & 0 & -1 & 0 & 0 & 0 \\ \frac{3}{11} & \frac{6}{11} & \frac{2}{11} & 0 & 0 & -1 & 0 & 0 \\ \frac{6}{11} & \frac{1}{11} & \frac{4}{11} & 0 & 0 & 0 & -1 & 0 \\ \frac{7}{11} & \frac{3}{11} & \frac{1}{11} & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

To determine the irrelevant ideal, however, we need to know the particular triangulation we are using.

Different triangulations lead to different maximal cones, and therefore a primitive collection for one fan may not be a primitive collection for the fan of an alternate resolution. The differences between these irrelevant ideals means that not all crepant resolutions have the same number of first order deformations of the tangent sheaf. We will focus on the existence of deformations that do not depend on the particular resolution chosen, and the deformations of the G-Hilbert scheme, a particular resolution.

3. Studying the tangent sheaf

We now talk briefly about how to find first order deformations of the tangent sheaf of a toric variety X. We recall that each first order deformation corresponds to an element of $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T})$, where $\mathcal{T}$ is the tangent bundle of X [10]. Let q_i be the charge of x_i, as described above. We will use as the index i or j for u_1, u_2, u_3, the vertices, and index α or β for $u_4, \ldots, u_N$, the interior points. Following [1] we review the argument that $\mathcal{D} := \bigoplus_{i=1}^3 \mathcal{O}(q_i) \oplus \bigoplus_{\alpha=4}^N \mathcal{O}_{D_\alpha}(q_\alpha)$ is a deformation of the tangent bundle.

There is an exact sequence

$$0 \to \mathcal{O}^{\oplus N-3} \to \bigoplus_{i=1}^N \mathcal{O}(q_i) \to \mathcal{T} \to 0$$

and we can define $\mathcal{W} = \mathcal{O}(q_1) \oplus \mathcal{O}(q_2) \oplus \mathcal{O}(q_3)$. Since $\mathcal{W} \hookrightarrow \bigoplus_{i=1}^N \mathcal{O}(q_i)$, we have an induced map $f : \mathcal{W} \to \mathcal{T}$, where the cokernel of f is the cokernel of $\mathcal{O}^{\oplus N-3} \to \bigoplus_{\alpha=4}^N \mathcal{O}(q_\alpha)$. We note that the map is given by the diagonal

matrix $(x_4, \ldots, x_N)$. Let $D_i = V(x_i)$. Then we know that $0 \to \mathcal{O}(-q_\alpha) \xrightarrow{x_\alpha} \mathcal{O} \to \mathcal{O}_{D_\alpha} \to 0$ is exact, and combining these exact sequences gives

$$0 \to \bigoplus_{i=1}^{3} \mathcal{O}(q_i) \to \mathcal{T} \to \bigoplus_{\alpha=4}^{N} \mathcal{O}_{D_\alpha}(q_\alpha) \to 0$$

an exact sequence. Therefore, $\mathcal{D} := \bigoplus_{i=1}^{3} \mathcal{O}(q_i) \oplus \bigoplus_{\alpha=4}^{N} \mathcal{O}_{D_\alpha}(q_\alpha)$ is a deformation of the tangent bundle. So by studying $\mathrm{Ext}^1(\mathcal{D}, \mathcal{D})$, we can find all of the deformations we are looking for. In particular, these are only the framed deformations of $\mathcal{T}$ (see [1]).

We again look at our example. Here, we have

$$\mathcal{D} = \bigoplus_{i=1}^{3} \mathcal{O}(q_i) \oplus \bigoplus_{\alpha=4}^{8} \mathcal{O}_{D_\alpha}(q_\alpha)$$

where the q_i are the columns from Φ, defined in the previous section. We note that each $\mathcal{O}_{D_\alpha}$ depends on the resolution chosen, so $\mathcal{D}$ is not independent of the triangulation.

3.1. Deformations of the tangent sheaf

We can find first order deformations of $\mathcal{D}$ (and thus first order framed deformations of $\mathcal{T}$) by looking at the direct sum of Ext^1s on each possible pairing of components. There are four types of pairings that occur. Following Aspinwall, we show they can all be computed using cohomology (we will do one of these computations in detail below):

1: $\mathrm{Ext}^1(\mathcal{O}(q_i), \mathcal{O}(q_j)) = 0$

2: $\mathrm{Ext}^1(\mathcal{O}(q_i), \mathcal{O}_{D_\alpha}(q_\alpha)) = 0$

3: $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i)) = H^0(\mathcal{O}_{D_\alpha}(q_i))$

4: $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta)) = H^0(C_{\alpha\beta}, N_\alpha) = H^0(\mathcal{O}_{C_{\alpha\beta}}(q_\beta))$,
where N_α is the normal bundle of $C_{\alpha\beta}$ in D_α.

We will show (3) above, that $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i)) = H^0(\mathcal{O}_{D_\alpha}(q_i))$. First, recall that

$$\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i)) \cong \mathcal{H}om^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$$

Next, we can shift one of the exact sequences described in the previous section to find

$$0 \to \mathcal{O} \xrightarrow{x_\alpha} \mathcal{O}(q_\alpha) \to \mathcal{O}_{D_\alpha}(q_\alpha) \to 0,$$

This means that in the derived category, $\mathcal{O}_{D_\alpha}(q_\alpha)$ is isomorphic to $\underline{\mathcal{O}} \xrightarrow{x_\alpha} \mathcal{O}(q_\alpha)$ (where the underline represents position 0)[9]. Thus, $\mathcal{H}om(\overline{\mathcal{O}_{D_\alpha}}(q_\alpha), \mathcal{O}(q_i))$ is given by

$$\underline{\mathcal{O}(q_i - q_\alpha)} \xrightarrow{x_\alpha} \mathcal{O}(q_i)$$

We recognize this as $\mathcal{O}_{D_\alpha}(q_i)[-1]$, just a twist of the sequence described above shifted over by 1. Therefore, we have

$$\mathcal{H}om^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i)) \cong \mathcal{O}_{D_\alpha}(q_i) \cong H^0(\mathcal{O}_{D_\alpha}(q_i))$$

The other computations can be done in a similar fashion, and can be seen in [1]. We see that only two of these are non-zero, so those are the only ones we need to study. We also see that we only need to look at global sections of twisted line bundles to determine what these deformations actually are.

4. Number of deformations of the tangent sheaf

First look at the deformations that come from $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$. We begin by finding a basis for $H^0(\mathcal{O}_{D_\alpha}(q_i))$. Such a basis is given by the set of monomials in D_α, whose degree in D_α is equal to the degree of x_i in D_α. Since $D_\alpha = X \cap V(x_\alpha)$, this is the set of all Laurent monomials in $\mathbb{C}(x_1, \ldots, x_N)$ that do not contain x_α, and which have non-negative power for all x_ρ that share a cone with x_α. We know that $\deg(x_i)$ is given by the i-th column of Φ (which we will denote Φ_i).

For $x_1^{c_1} \cdots x_N^{c_N}$ to have the correct multidegree, we must have $\sum c_j(\Phi_j) = \Phi_i$. But since the rows of Φ and Φ' span the same complex subspace, its sufficient to find coefficients c_j that make this true for Φ' (again, requiring $c_\alpha = 0$).

We now look at this row by row. We first must have $c_1\nu_1^\alpha + c_2\nu_2^\alpha + c_3\nu_3^\alpha = \nu_i^\alpha$, where c_j is the power of x_j in the Laurent monomial. This equation follows directly from the row corresponding to u_α, since the only columns that are nonzero in this row are x_1, x_2, x_3, and x_α, with $c_\alpha = 0$. We note that the c_j do not have to all be non-negative, only those where u_j and u_α share a cone.

Once we have (c_1, c_2, c_3), there is a unique c_β for every u_β so that $c_1\nu_1^\beta + c_2\nu_2^\beta + c_3\nu_3^\beta - c_\beta = \nu_i^\beta$. Thus we have c_β as the power of x_β, in order to have

the correct multidegree. Some of the c_β can also be negative, though which negative powers are allowed depends on the specific resolution we chose. This leads to the following proposition.

Proposition 4.1. *Every deformation from* $\text{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$ *corresponds to some triple of integers* (c_1, c_2, c_3) *satisfying* $c_1\nu_1^\alpha + c_2\nu_2^\alpha + c_3\nu_3^\alpha = \nu_i^\alpha$ *with the above non-negativity conditions.*

Next, we look at $\text{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$. To classify these deformations, we need a basis for $H^0(\mathcal{O}_{C_{\alpha\beta}}(q_\beta))$. This will be the set of Laurent monomials in $\mathbb{C}(x_1, \ldots, x_N)$ that do not contain x_α or x_β, and which have non-negative power for the two x_ρ that share a cone with x_α and x_β. We proceed as above, and find that the equations that must be satisfied are $c_1\nu_1^\alpha + c_2\nu_2^\alpha + c_3\nu_3^\alpha = 0$ and $c_1\nu_1^\beta + c_2\nu_2^\beta + c_3\nu_3^\beta = -1$. These c_j will be the power of x_j in the Laurent monomial. The only c_j that must be non-negative are those that share a cone with both u_α and u_β.

Given (c_1, c_2, c_3), there is a unique c_γ for each other interior points u_γ such that $c_1\nu_1^\gamma + c_2\nu_2^\gamma + c_3\nu_3^\gamma - c_\gamma = 0$. As long as c_γ is non-negative for any u_γ sharing a cone with both u_α and u_β, $x_j^{c_j}$ gives a basis element of $H^0(\mathcal{O}_{C_{\alpha\beta}}(q_\beta))$, and thus a deformation. We can summarize this in another proposition.

Proposition 4.2. *Every deformation from* $\text{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$ *corresponds to some triple of integers* (c_1, c_2, c_3) *satisfying* $c_1\nu_1^\alpha + c_2\nu_2^\alpha + c_3\nu_3^\alpha = 0$ *and* $c_1\nu_1^\beta + c_2\nu_2^\beta + c_3\nu_3^\beta = -1$ *with the above non-negativity conditions.*

The two previous propositions then give an easy way to describe any deformation of $\mathcal{D}$, and therefore the tangent sheaf.

Theorem 4.3. *Every deformation of the tangent sheaf of a crepant resolution of X corresponds to a unique triple of integers* (c_1, c_2, c_3).

In the case of $\mathbb{C}^3/\mathbb{Z}_{11}$, we now look at this triple for one deformation of each type. The triple $(2, 3, 0)$ gives a Laurent monomial $x_1^2 x_2^3 x_5 x_6^2 x_7 x_8^2$, and restricting to D_4 (in any resolution) this gives a basis element of $\text{Ext}^1(\mathcal{O}_{D_4}(q_4), \mathcal{O}(q_3))$. Similarly, the triple $(3, 1, -2)$ gives the Laurent monomial $x_1^3 x_2 x_3^{-2} x_6 x_7^2 x_8^3$, and restricting to C_{45} this gives a basis element of $\text{Ext}^1(\mathcal{O}_{D_4}(q_4), \mathcal{O}_{D_5}(q_5))$ as long as $x_3 = 1$ on C_{45}.

4.1. Non-isolated singularities

If there is a non-isolated singularity, that means that there is some lattice point u_α on the boundary of the convex hull of (u_1, u_2, u_3). This then implies that $\nu_i^\alpha = 0$ for some i. But when we take this together with the propositions proved above, it means that there are infinitely many elements of $\text{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_j))$ for $j \neq i$. In particular, the monomials $x_i^n x_j$ will have the proper multidegree for all integers n, in any crepant resolution. Since the main purpose of this paper is counting the number of deformations, we limit our focus to resolutions of orbifolds with isolated singularities.

5. Computing singlets

We will now bring in some methods from physics, which will help us to count the triples that satisfy the constraints from the previous section. Let $G = \mathbb{Z}_r$ acting on $\mathbb{C}^3$, with G generated by $g = (\zeta_r^{a_1}, \zeta_r^{a_2}, \zeta_r^{a_3})$, as before. We associate a "twisted sector" with each element of G, where the j/r twisted sector refers to g^j. We begin by studying the j/r-twisted sector. We note that this means we twist by

$$(\exp(2\pi i\nu_1), \exp(2\pi i\nu_2), \exp(2\pi i\nu_3))$$

where we define ν_i to be a rational number such that $r\nu_i \equiv ja_i \mod r$ and $0 \leq \nu_i < 1$. We will show in the following section that these ν_i are in fact the same as the ν_i^α found in Φ' previously, for an appropriate choice of α. We define

$$\widetilde{\nu}_i := \begin{cases} \nu_i - \frac{1}{2} : \nu_i \leq \frac{1}{2} \\ \nu_i - \frac{3}{2} : \nu_i > \frac{1}{2} \end{cases}$$

This follows from the physics, and is explained in more detail in [2]. We define the energy of the twisted vacuum to be

$$E = \frac{1}{2} \sum_{i=1}^3 (\nu_i(1 - \nu_i) + \widetilde{\nu}_i(1 + \widetilde{\nu}_i)) - \frac{5}{8}$$

with charge given by $q = -\frac{3}{2} - \sum \widetilde{\nu}_i$. For physical reasons we will not go into, we only look at the cases where $\sum \nu_i = 1$. We can find all the singlets for a given twisted sector by using a generating function. The number of

singlets is the coefficient of $q^0 z^0$ in

$$q^E z^{-\frac{3}{2} - \sum_i \tilde{\nu}_i} \prod_i \frac{(1 + q^{1+\tilde{\nu}_i} z^{-1})(1 + q^{-\tilde{\nu}_i} z)}{(1 - q_i^\nu)(1 - q^{1-\nu_i})}$$

By analyzing this partition function, we are able to arrive at a much simpler condition for counting the number of singlets for each twisted sector.

Proposition 5.1. *Every singlet in a twisted sector with (ν_1, ν_2, ν_3) corresponds to one of the following two cases:*

Case *1: Determined by a unique non-negative triple of integers (c_1, c_2, c_3) such that $c_1\nu_1 + c_2\nu_2 + c_3\nu_3 = \nu_i$ for some ν_i.*

Case *2: Determined by a unique triple of integers (c_1, c_2, c_3) where exactly one $c_i < -1$, and $c_1\nu_1 + c_2\nu_2 + c_3\nu_3 = c_i + 1$.*

6. Twisted sectors and interior points

6.1. Action by generator

We now take another look at $\mathbb{C}^3/\mathbb{Z}_r$, with group action given by $(\zeta_r^{a_1}, \zeta_r^{a_2}, \zeta_r^{a_3})$. We focus on the case of isolated singularities, where we always have $a_1 = 1$, $a_2 = a$, and $a_3 = b$. We recall the fan in $\mathbb{R}^3$ associated to this variety has one dimensional cones with minimal generators $(r, -a, -b), (0, 1, 0)$, and $(0, 0, 1)$. We know that there are $r - 1$ possible twisted sectors, corresponding to twisting by $\frac{1}{r}, \frac{2}{r}, \ldots, \frac{r-1}{r}$. Looking at the $\frac{i}{r}$−twisted sector, we have $\nu_1 = \frac{i}{r}$, $r\nu_2 \equiv ia \mod r$, and $r\nu_3 \equiv ib \mod r$, with $0 \leq \nu_j < 1$. As in the previous section, we only are interested in the case where $\sum \nu_i = 1$. For each of these twisted sectors, look at the point $(\nu_1 r, \nu_2 - a\nu_1, \nu_3 - b\nu_1)$. All points of this type are in the convex hull of the three minimal generators, and these are in fact all lattice points in the convex hull. This follows from a result of Ito and Reid, relating crepant exceptional divisors of the resolution of an orbifold $\mathbb{C}^3/G$ to the conjugacy classes of G [12]. Using the language of our paper, this result is stated below.

Proposition 6.1. *There is a one to one correspondence between the twisted sectors and the interior lattice points of the cone associated to $\mathbb{C}^3/\mathbb{Z}_r$.*

We note that the interior point with first coordinate i corresponds to the $\frac{i}{r}$−twisted sector. If the singularity is not isolated, the above proposition

still holds. In that case, we find the interior lattice points (that is, minimal generators for the irreducible exceptional divisors) have coordinates given by $(\nu_1 r, a_1\nu_2 - a_2\nu_1, a_1\nu_3 - a_3\nu_1)$.

6.2. Relation to charge matrix

We have now seen that each interior points u_α corresponds to a particular twisted sector, and that this twisted sector has a triple of rational numbers (ν_1, ν_2, ν_3) associated with it. We call these $(\nu_1^\alpha, \nu_2^\alpha, \nu_3^\alpha)$, for u_α and have

$$u_\alpha = (\nu_1^\alpha r, -\nu_1^\alpha a + \nu_2^\alpha, -\nu_1^\alpha b + \nu_3^\alpha)$$

Here are those triples for our running example:

u_α	ν_1^α	ν_2^α	ν_3^α
u_4	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{8}{11}$
u_5	$\frac{2}{11}$	$\frac{4}{11}$	$\frac{5}{11}$
u_6	$\frac{3}{11}$	$\frac{6}{11}$	$\frac{2}{11}$
u_7	$\frac{6}{11}$	$\frac{1}{11}$	$\frac{4}{11}$
u_8	$\frac{7}{11}$	$\frac{3}{11}$	$\frac{1}{11}$

Computing the cokernel of A, using these coordinates for the u_α, we can show these ν_i^α are the same as the variables with the same name used in the matrix Φ' previously.

7. Counting deformations of the tangent sheaf

We saw earlier that every deformation of the tangent sheaf corresponds to a unique triple of integers (c_1, c_2, c_3), and that each such triple satisfies equations involving the ν_i^α, and certain negativity conditions. In particular, which x_i or x_α were allowed to have negative power was dependent on the particular resolution chosen. But this implies that any any actual monomial of $\mathbb{C}(x_1, \ldots, x_N)$ (with all powers ≥ 0) will correspond to a deformation for any resolution. We will now find all such monomials, by showing that any triple (c_1, c_2, c_3) of non-negative integers, satisfying the appropriate equations, will have non-negative powers for all x_α.

Theorem 7.1. $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$ *has a minimum number of elements for any crepant resolution of* $\mathbb{C}^3/G$, *and the G-Hilbert scheme achieves that minimum.*

We prove the first part immediately, and the second part later in the paper, when we study the G-Hilbert scheme. For ease of notation, we will assume $i = 3$ and $\alpha = 4$. Let $c_1, c_2, c_3 \geq 0$, and $c_1 \nu_1^4 + c_2 \nu_2^4 + c_3 \nu_3^4 = \nu_3^4$. The key step in this proof uses the following lemma.

Lemma 7.2. *If $c_1, c_2, c_3 \geq 0$, then every $c_\beta \geq 0$.*

Combining this lemma with Proposition 4.1, any triple (c_1, c_2, c_3) of non-negative integers that satisfies $c_1 \nu_1^4 + c_2 \nu_2^4 + c_3 \nu_3^4 = \nu_3^4$ will give an element of $H^0(\mathcal{O}_{D_4}(q_3))$ for any resolution. Counting the number of triples of this type thus gives the lower bound for the dimension of $\mathrm{Ext}^1(\mathcal{O}_{D_4}(q_4), \mathcal{O}(q_3))$. We now prove the lemma.

Proof. Let the group action be given by (a_1, a_2, a_3), and say $c_1 \nu_1^\alpha + c_2 \nu_2^\alpha + c_3 \nu_3^\alpha = \nu_i^\alpha$. Without loss of generality, we can assume $i = 3$. We look at all triples that have each of $c_1, c_2, c_3 \geq 0$. There is always one such triple, $(0, 0, 1)$. This corresponds to the Laurent monomial x_3, and will obviously always occur (all the c_β are 0). All the other triples must have $c_3 = 0$, so our equation becomes $c_1 \nu_1^\alpha + c_2 \nu_2^\alpha = \nu_3^\alpha$. We know that $\nu_1^\alpha + \nu_2^\alpha + \nu_3^\alpha = 1$. Therefore, we can rewrite the above as $(c_1 + 1)\nu_1^\alpha + (c_2 + 1)\nu_2^\alpha = 1$. Let $c_1' = c_1 + 1$, $c_2' = c_2 + 1$.

Now, we use the fact that $r\nu_1^\alpha = ia_1 \mod r$ and $r\nu_2^\alpha = ia_2 \mod r$, so we can rewrite the equation as $c_1'(ia_1 \mod r) + c_2'(ia_2 \mod r) = r$. Thus, $c_1'(ia_1 - n_i r) + c_2'(ia_2 - m_i r) = r$, and $i(c_1' a_1 + c_2' a_2) = r(1 + n_i + m_i)$. We note all of the constants used are integers. Since i does not divide r, it must divide $(1 + n_i + m_i)$ and we thus have $c_1' a_1 + c_2' a_2 = rk$ where k is an integer, with $k \geq 1$.

Next we show that for any β, $c_1 \nu_1^\beta + c_2 \nu_2^\beta \geq \nu_3^\beta$. This is equivalent to showing $c_1' \nu_1^\beta + c_2' \nu_2^\beta \geq 1$, by the same argument as above. We again have $r\nu_1^\beta = ja_1 \mod r$, $r\nu_2^\beta = ja_2 \mod r$, for some j. So if we multiply both sides by r, we get another equivalent condition, $c_1'(ja_1 - n_j r) + c_2'(ja_2 - m_j r) \geq r$. The left hand side can then be written as $j(c_1' a_1 + c_2' a_2) - r(c_1 n_j + c_2 m_j)$. But this is just $r(jk - c_1 n_j + c_2 m_j)$. Therefore, we see that the left hand side is in fact divisible by r. We know that $c_1'(ja_1 \mod r) + c_2'(ja_2 \mod r) > 0$, and therefore it must be at least r. So we have shown $c_1'(ja_1 - n_j r) + c_2'(ja_2 - m_j r) \geq r$, and equivalently, $c_1 \nu_1^\beta + c_2 \nu_2^\beta \geq \nu_3^\beta$. This implies that all the c_β are non-negative, since $c_1 \nu_1^\beta + c_2 \nu_2^\beta - c_\beta = \nu_3^\beta$. $\square$

We also note that the conditions necessary for the theorem were a triple of integers (c_1, c_2, c_3) with $c_1, c_2, c_3 \geq 0$ and $c_1 \nu_1^\alpha + c_2 \nu_2^\alpha + c_3 \nu_3^\alpha = \nu_i^\alpha$. Such

a triple also uniquely determined all singlets of case 1. This leads to the following:

Theorem 7.3. *Every singlet of case 1 gives a distinct first order deformation of the tangent sheaf of any crepant resolution of $\mathbb{C}^3/G$.*

7.1. Other singlets

Unfortunately, Theorem 7.1 does not extend to singlets of case 2. Recall, such a singlet is identified by a triple of integers (c_1, c_2, c_3) where exactly one $c_i < -1$, and $c_1\nu_1^\alpha + c_2\nu_2^\alpha + c_3\nu_3^\alpha = c_i + 1$. We find that whether such a singlet corresponds to a deformation of $\mathcal{T}$ depends on the particular triangulation chosen. However, there is still a relationship to deformations.

Proposition 7.4. *If (c_1, c_2, c_3) identify a singlet, then they also uniquely identify a deformation of the tangent sheaf of some particular resolution of $\mathbb{C}^3/G$.*

Proof. We have already shown this proposition for singlets of case 1. We now show this for singlets of case 2. Say we have a triple of integers as described above. For ease of reading, we assume that $i = 3$, so $c_1\nu_1^\alpha + c_2\nu_2^\alpha + c_3\nu_3^\alpha = c_3 + 1$, where c_3 is negative. Since we know that $\nu_3^\alpha = 1 - \nu_1^\alpha - \nu_2^\alpha$, we can rewrite this as $(c_1 - c_3)\nu_1^\alpha + (c_2 - c_3)\nu_2^\alpha = 1$. We also know the coordinates of u_α are $(\nu_1^\alpha r, a_1\nu_2^\alpha - a_2\nu_1^\alpha, a_1\nu_3^\alpha - a_3\nu_1^\alpha)$. Now look at

$$u_\beta = (-c_3 - 1)u_\alpha \qquad \text{and} \qquad u_\gamma = (-c_3)u_\alpha.$$

After calculating, we find $c_1\nu_1^\gamma + c_2\nu_2^\gamma + c_3\nu_3^\gamma = 0$, and $c_1\nu_1^\beta + c_2\nu_2^\beta + c_3\nu_3^\beta = -1$.

We note that these are exactly the equations we needed for (c_1, c_2, c_3) to give a deformation from the group $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$. $\square$

While these particular deformations do not show up for every crepant resolution of $\mathbb{C}^3/G$, we note that there are always at least this many deformations. This can be seen from the geometry in the following section.

This proposition leads to the question of whether there is a resolution with a one to one correspondence between first order deformations and singlets. We will show the answer is yes, and that the G-Hilbert scheme is such a resolution. In order to do this, we will first construct a quiver associated with the original orbifold $\mathbb{C}^3/G$.

8. Quiver construction of a resolution

We will now use the two types of singlets to construct a quiver associated to the orbifold $\mathbb{C}^3/G$. This quiver will then lead to a distinguished triangulation of $\{u_1, \ldots, u_N\}$, and thus a particular resolution of the orbifold.

As a reminder, for each sector (and thus each u_α), we had two possible cases of singlets:

Case 1: Determined by a unique non-negative triple of integers (c_1, c_2, c_3) such that $c_1 \nu_1^\alpha + c_2 \nu_2^\alpha + c_3 \nu_3^\alpha = \nu_i^\alpha$ for some ν_i^α.

Case 2: Determined by a unique triple of integers (c_1, c_2, c_3) where exactly one $c_i < -1$, and $c_1 \nu_1^\alpha + c_2 \nu_2^\alpha + c_3 \nu_3^\alpha = c_i + 1$.

For each singlet of case 1, include an arrow from u_α to u_i. We note that there is always at least one arrow to each vertex, since $(1,0,0), (0,1,0)$, and $(0,0,1)$ all meet the criteria. Below we have the quiver at this stage, for the $\mathbb{C}^3/\mathbb{Z}_{11}$ example we have been working with.
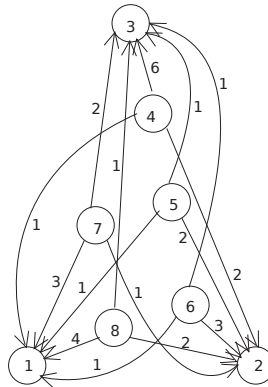

Figure 8.1. First step of quiver for $\mathbb{C}^3/\mathbb{Z}_{11}$.

Now, for each singlet of case 2 we know exactly one c_i is negative. Extend the line from u_i through u_α until it passes through $-c_i$ nodes total, in addition to u_i. For each singlet with this value for c_i, include an arrow from this node to the node one closer to u_i. We note that if there is a singlet with $c_i = -n$, then there will be singlets with $c_i = -m$ for all $m < n$, so there are arrows between all nodes along this line, pointing towards u_i. Let's look at our example again, concentrating first on u_4. In this case, we had $\nu_1^4 = \frac{1}{11}$, $\nu_2^4 = \frac{2}{11}$, and $\nu_3^4 = \frac{8}{11}$. The only triples (c_1, c_2, c_3) that satisfy the condition for case 2 are $(5, 0, -2), (3, 1, -2), (1, 2, -2), (1, 0, -3)$, and $(0, 1, -3)$. This means we must extend the line from u_3 to u_4 through two more nodes (u_5

and u_6), with arrows of multiplicity 3 from u_5 to u_4, and multiplicity 2 from u_6 to u_5. These arrows, along with those corresponding to the other u_α, are included in the final quiver.
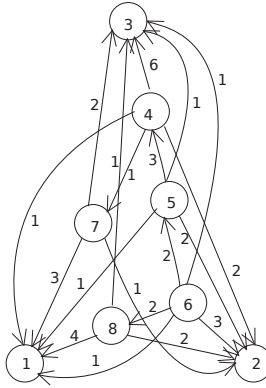

Figure 8.2. $\mathbb{C}^3/\mathbb{Z}_{11}$ quiver.

8.1. Choosing a triangulation

We now use this quiver to construct a triangulation of the point set $\{u_1, \ldots, u_N\}$. First, we suppress all arrows to vertices of multiplicity one. Since they always exist we don't lose any information by not including them, but now the quiver is planar. Next, replace every arrow with an edge and add in the 3 boundary edges, connecting the vertices to each other. Finally, take a regular tessellation of any remaining triangles that are not smooth. In this way, we get a smooth triangulation of the triangle with vertices $\{u_1, u_2, u_3\}$.

Here is the triangulation for $\mathbb{C}^3/\mathbb{Z}_{11}$. We include the multiplicities the arrows previously had, since they will prove useful later on.
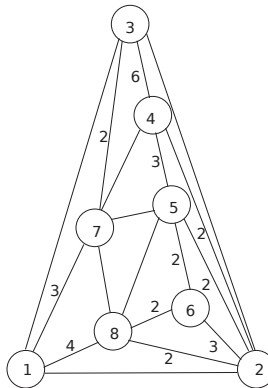

Figure 8.3. $\mathbb{C}^3/\mathbb{Z}_{11}$ triangulation.

If we now take a fan over this triangulation, we get a crepant resolution of the orbifold $\mathbb{C}^3/G$. We will show that this resolution is actually the G-Hilbert scheme. To do this, we will expand on the knockout method for constructing the fan of the G-Hilbert scheme described in [7], with a slight modification.

9. G-Hilbert scheme

9.1. Jung-Hirzebruch continued fraction

The G-Hilbert scheme is a particular resolution of $\mathbb{C}^3/G$, which parameterizes the 0-dimensional subschemes of $\mathbb{C}^n$ that are invariant under the group action, have length $|G|$, and have global sections that give a regular representation of G [11]. Since this scheme is a toric variety, we know that there is a fan associated with it. [7] gives us a combinatorial method for constructing this associated fan, which we now review.

The first step in constructing the fan of the G-Hilbert scheme is finding the Jung-Hirzebruch continued fraction associated to each vertex. We include here a quick review of how the continued fraction is computed. For each vertex (u_1, u_2, and u_3), we look at the two dimensional cone whose origin is the vertex and whose walls are the rays pointing to the other two vertices. This cone will likely be singular, but as a toric surface there's a well known method for resolving it using the continued fraction. Choose a basis so that one ray has minimal generator $(1,0)$ and the other has generator $(-c, r)$. We can then write r/c as a continued fraction, $r/c = [[b_1, \ldots, b_m]]$. This means that

$$\frac{r}{c} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots}}$$

We can find the b_i through the equations $r = c(b_1) - d_1, c = d_1(b_2) - d_2, d_1 = d_2(b_3) - d_3, \ldots, d_{m-2} = d_{m-1}b_m - d_m$. There are two more values associated with this continued fraction, which we will define recursively. We define P_i by setting $P_0 = 1, P_1 = b_1$, and $P_i = b_i P_{i-1} - P_{i-2}$. Similarly, we define Q_i by setting $Q_0 = 0, Q_1 = 1$, and $Q_i = b_i Q_{i-1} - Q_{i-2}$. In particular, note that this implies $P_i = b_i(P_{i-1} + P_{i+1})$ and $Q_i = b_i(Q_{i-1} + Q_{i+1})$

For each vertex, r is the order of the underlying group G, and the particular c can be found through the group action. We then have m rays from the vertex to interior points, and label each ray with the number b_i, which we call its strength. The coordinates of the interior points, in the basis we chose earlier, will be (P_i, Q_i). We do this for the other two vertices as well, so each vertex has a number of labeled rays to particular interior points.

Let's look at our example, $\mathbb{C}^3/\mathbb{Z}_{11}$. At vertex u_3, we have $11/2 = [[6,2]]$ since $11 = 2(6) - 1$, and $2 = 1(2) - 0$. So the edges have strength $6, 2$ respectively. At u_2 the continued fraction is $11/7 = [[2,3,2,2]]$, and at u_1 it is $11/4 = [[3,4]]$.
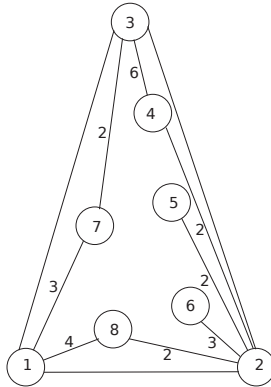

Figure 9.1. First step in constructing G-Hilbert scheme.

9.2. Smooth triangulation

Once we have the lines from vertices, extend each line further inside of the triangle. As it goes through interior points, reduce the strength by 1 for every other line that goes through the same interior point. If a line would ever have 1 or 0 strength, do not include it. Finally, take a regular tessellation of any remaining regular triangles. The fan over this triangulation gives the G-Hilbert scheme. (Thus far, this is exactly the knockout method from [7]).

We now modify this method slightly by subtracting an additional 1 the first time an edge goes between 2 interior points. This results in all the numbers on interior edges being 1 less than they are determined to be in [7], and means it is now possible to have interior edges of strength 1.

With this modification, here is the triangulation (with strengths) for the G-Hilbert scheme of $\mathbb{C}^3/\mathbb{Z}_{11}$.

We see that this is the same triangulation as was found in the previous section, constructed using the singlets. Not only is the triangulation the same, but the numbers along each edge are also the same. The remainder of the paper will focus on proving that this is always the case, and understanding what this tells us about the $(0,2)-$McKay correspondence.
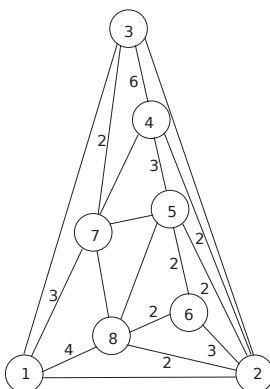
Figure 9.2. Triangulation for G-Hilbert scheme of $\mathbb{C}^3/\mathbb{Z}_{11}$.

10. Deformations of the tangent sheaf of the G-Hilbert scheme

Every framed deformation of the tangent sheaf $\mathcal{T}$ corresponds to an element of $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$ or an element of $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$. The above method of constructing the Hilbert scheme then gives the dimension for each of these groups, as follows. The strength of the edge between u_α and u_i gives the dimension of $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$. The strength of the edge between u_α and u_β gives the dimension of $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$, where u_β is closer to the vertex that gave this edge (see previous section). We will now summarize and prove this relationship.

Theorem 10.1. *The numbers on edges as described above will be exactly the dimension of the Ext^1 group corresponding to the given vertices, and therefore count the number of framed deformations.*

10.1. Proof of Theorem

We first show that the numbers on edges that go from u_α to u_i match up with the dimension of $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$.

Call the u_i vertex $(0,0)$, where corresponding coordinates for u_α are (x,y), and the two neighboring vertices are (a,b) and (c,d). By definition of continued fraction, we know that $a + c = Nx$ and $b + d = Ny$, where N is the strength of the edge. Now let's move the origin to (x,y), as in [1]. In this case, we know the dimension is given by $x_2y_3 - x_3y_2 + 2$. Now the coordinates are $(-x, -y)$ for top one, $(a-x, b-y)$ and $(c-x, d-y)$. We compute $x_2y_3 - x_3y_2$, and find $(a-x)(d-y) - (b-y)(c-x)$. We know $x =$

$\frac{a+c}{N}$ and $y = \frac{b+d}{N}$, so this becomes

$$\frac{1}{N^2}(((N-1)a - c)((N-1)d - b) - ((N-1)c - a)((N-1)b - d))$$

which is just $\frac{1}{N}(N-2)(ad - bc)$. But we note that $(Nx, Ny) \times (c, d) = ((a,b)+(c,d)) \times (c,d) = ad - bc$, and $(Nx, Ny) \times (c, d) = N(x, y) \times (c, d) = N$, so $x_2 y_3 - x_3 y_2 = N - 2$. Therefore, the dimension of $\text{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$ is $N - 2 + 2 = N$. Thus, we have shown that the strength N, is exactly the dimension of the Ext group for edges to vertices.

Now look at edges that do not go to vertices. Begin by setting one of the nodes on the edge, u_α, as the origin. We then have coordinates for the other three nodes in this system, $u_i = (a, b)$, $u_j = (c, d)$ and $u_\beta = (x, y)$. Since we are using a smooth triangulation, we know that $(a, b) + (c, d) = N(x, y)$ for some N. If this number is 0 or negative, then instead set u_β as the origin, which will give an integer $N \geq 1$. Note that the positive N satisfying this equation came from using the Hirzebruch-Jung resolution and contracting, and therefore this positive N is the (unmodified) strength as originally found in [7]. This also means that in the modified version, the strength of the edge is $N - 1$. As in [1], the dimension of $\text{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$ is given by $x_2 y_3 - x_3 y + 1$. After computing as in the previous case, we find this is equal to $N - 1$. Therefore, the numbers on each interior edge as described in the previous method do give the dimension of $\text{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$. $\square$

By studying the geometry of this triangulation, we see that any other triangulation will have at least the same total number of interior deformations. This is because the only edges we can change are those with strength 0, and therefore are part of a regular tessellation. Any flop will introduce new deformations on bordering strength 0 edges, and as there are always at least two of those, the total number of interior deformations cannot decrease.

11. Relationship between singlets and the G-Hilbert scheme

We will now prove that the triangulation given by the singlets, from Section 8, is in fact the triangulation which gives the G-Hilbert scheme. To show the triangulation is the same, it will be enough to show that the numbers on each edge from the modified knockout method (Section 9), and the quiver method (Section 8) agree. This is because in both cases, the triangulation is uniquely determined by these numbers. To do so, we first find a relationship between the ν_i^α and the P_i that arise from the continued fraction. For ease of notation, we focus on the continued fraction based at vertex u_3, and

the edges to u_3. In particular, this means whenever we write P_i, we will be referring to that continued fraction. All of the results generalize to the other vertices, but writing the computations in complete generality is a bit unwieldy.

11.1. Edges to a vertex

Proposition 11.1. *For u_α that appears as the j-th term in the Hirzebruch-Jung resolution based at u_3, we have $\nu_1^\alpha = \frac{P_j}{r}$, $\nu_2^\alpha = \frac{d_j}{r}$, $\nu_3^\alpha = \frac{r - P_j - d_j}{r}$, and $r = P_j d_{j-1} - P_{j-1} d_j$*

Proof. We will now prove the above relations. $\nu_1^\alpha = \frac{P_i}{r}$, since $P_i = (u_\alpha)_1 = \nu_1^\alpha r$. Similarly, we have $\nu_2 = \frac{aP_i \mod r}{r}$. We can then show by induction that $P_i a \equiv d_i \mod r$, for all $i > 1$. Since every $d_i < r$, and $0 \le \nu_2^\alpha < 1$, we must actually have $\nu_2^\alpha = \frac{d_i}{r}$. The condition on ν_3^α follows from $\sum_i \nu_i^\alpha = 1$. Finally, we know that $P_1 d_0 - P_0 d_1 = b_1 a - d_1 = r$, by the first relation of the Hirzebruch-Jung resolution. We note we also have $d_{n-1} = b_{n+1} d_n - d_{n+1}$. Therefore $P_n d_{n-1} - P_{n-1} d_n = b_{n+1} P_n d_n - P_n d_{n+1} - P_{n-1} d_n = (b_{n+1} P_n - P_{n-1}) d_n - P_n d_{n+1}$. But $b_{n+1} P_n - P_{n-1} = P_{n+1}$ by the recursion relation, so this is $P_{n+1} d_n - P_n d_{n+1}$. Since we have shown all of these combinations are the same, they are thus all equal to r. We see that we can then rewrite ν_3 as $\frac{P_i(d_{i-1} - 1) - (P_{i-1} + 1) d_i}{r}$. □

Now, recall the way we determine the number of arrows from u_α to u_3 in the singlet construction is by finding all non-negative triples with $c_1 \nu_1 + c_2 \nu_2 + c_3 \nu_3 = \nu_3$. We always have $c_1 = c_2 = 0, c_3 = 1$. We want to count how many triples there are so that $c_1 \nu_1 + c_2 \nu_2 = \nu_3$. To do so, we will use the relationships we just found. This gives $c_1 \frac{P_i}{r} + c_2 \frac{d_i}{r} = \frac{P_i(d_{i-1} - 1) - (P_{i-1} + 1) d_i}{r}$ and thus

$$c_1 P_i + c_2 d_i = P_i(d_{i-1} - 1) - (P_{i-1} + 1) d_i$$

Now say that $c_2 = P_i n - (P_{i-1} + 1)$, where n is some integer. We then have

$$c_1 P_i + (P_i n - (P_{i-1} + 1)) d_i = P_i(d_{i-1} - 1) - (P_{i-1} + 1) d_i$$

A bit more algebra gives $c_1 P_i + P_i n d_i = P_i(d_{i-1} - 1)$, and we can divide out by P_i to get

$$c_1 + n d_i = d_{i-1} - 1$$

We now see that when $n = 1$, this gives $c_1 = d_{i-1} - 1 - d_i \ge 0$, since d_i is decreasing sequence, and $c_2 = P_i - P_{i-1} - 1 \ge 0$ since P_i is increasing sequence. Therefore, $n = 1$ gives a triple that works.

Next look at $n = b_i - 1$. This gives $c_1 + (b_i - 1)d_i = d_{i-1} - 1$. We recall that $b_i d_i = d_{i-1} + d_{i+1}$. So this becomes $c_1 + d_{i-1} - d_i + d_{i+1} = d_{i-1} - 1$, and gives $c_1 = d_i - d_{i+1} - 1$. $c_1 \geq 0$, since d_i is decreasing, and $c_2 = P_i(b_i - 1) - (P_{i-1} + 1) \geq 0$, since $b_i > 1$. So $n = b_i - 1$ also gives a triple that works.

We see that n can be any integer from 1 to $b_i - 1$, giving $b_i - 1$ triples that work. Adding in the earlier case where $c_3 = 1$, we get a total of b_i singlets of this type. But b_i is exactly the strength that we determined by the Hirzebruch-Jung resolution. Thus, for every edge from an interior point to the vertex x_3, the two methods agree.

Incidentally, this also proves the second part of Theorem 7.1.

Theorem 7.1. $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$ *has a minimum number of elements, and the G-Hilbert scheme achieves that minimum.*

This is because we have just shown that for the G-Hilbert scheme, we have

$$|\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))|$$
$$= |\{(c_1, c_2, c_3) \in \mathbb{Z}^3 \mid c_1, c_2, c_3 \geq 0, c_1\nu_1^\alpha + c_2\nu_2^\alpha + c_3\nu_3^\alpha = \nu_i^\alpha\}|.$$

11.2. Interior edges

Next, look at the number on an interior edge. By the modified rule above, we find this number by looking at the strength on the edge one closer to a vertex, subtracting one for each other edge that goes to the shared node, and subtracting an additional 1 when we go from an edge adjacent to a vertex to an interior one. If the number ever drops to 0, we do not include that edge.

Recall that the way we determine what singlets occur between two interior nodes was by finding all possible $c_1, c_2 \geq 0$ such that

$$c_1\nu_1^\alpha + c_2\nu_2^\alpha = (1 - c_3)\nu_3^\alpha + c_3$$

where u_α lies on the edge extended into a line, bordering u_3. First, let $n = -c_3$ and assume that the number of possibilities such that $c_1\nu_1^\alpha + c_2\nu_2^\alpha = (n+1)\nu_3^\alpha - n$ is m. We want the number of possible d_i so that

$$d_1\nu_1^\alpha + d_2\nu_2^\alpha = (n+2)\nu_3^\alpha - (n+1) = (n+1)\nu_3^\alpha - n - (1 - \nu_3^\alpha)$$

By the method described in Section 8, each pair (d_1, d_2) gives an arrow on the "next" interior edge. But since $1 = \nu_1^\alpha + \nu_2^\alpha + \nu_3^\alpha$, we see this is equal to $(n+1)\nu_3^\alpha - n - \nu_1^\alpha - \nu_2^\alpha$. So we have $(d_1 + 1)\nu_1^\alpha + (d_2 + 1)\nu_2^\alpha = (c_1)\nu_1^\alpha +$

$c_2 \nu_2^\alpha$, and see that the pairs (d_1, d_2) that work are exactly $(c_1 - 1, c_2 - 1)$. Note, we still have the requirement that $d_1, d_2 \geq 0$. So the pairs that are valid are exactly those with both $c_1, c_2 \neq 0$. If none of the triples had $c_1, c_2 = 0$, the number of singlets is still m. If one could be 0, the number is $m - 1$, and if both could be 0, the number is $m - 2$. Now, if there is a combination with $c_1 = 0$, that means the node lies on the extension of an edge that comes from u_1. Similarly, if there is a singlet with $c_2 = 0$, that means the node lies on the extension of an edge from u_2. We therefore see that the strength of each edge changes exactly the same way as it does in the knockout method. Finally, we note that when we are going from an "exterior" edge (to a vertex) to a purely interior one, the strength decreases by an additional one. This is because the singlet corresponding to $c_1 = c_2 = 0, c_3 = 1$ occurs in addition to those of the form $c_1 \nu_1 + c_2 \nu_2 = \nu_3$, and has no analog among the singlets of type 2. So the number on the interior edges decreases exactly according to the modified knockout rule described in Section 9.

We have therefore proven the following theorem:

Theorem 11.2. *The singlets associated to the orbifold $\mathbb{C}^3/G$ can be used to construct the fan for the G-Hilbert scheme, along with the* Ext-*quiver, in a new way.*

11.3. Singlets and first order deformations

We have proven that the distinguished triangulation given by the singlets is in fact the G-Hilbert scheme. In addition, by the theorems of Section 7, we have also shown that every first order deformation of the G-Hilbert scheme corresponds to a particular singlet. This has a number of useful implications. First of all, it implies that the quiver constructed in Section 6 is actually the Ext$-$quiver for the G-Hilbert scheme. That is, if we associate each summand V_i of $\mathcal{D} = \bigoplus_{i=1}^3 \mathcal{O}(q_i) \oplus \bigoplus_{\alpha=4}^8 \mathcal{O}_{D_\alpha}(q_\alpha)$ with the node u_i, there are $(\dim \mathrm{Ext}^1(V_i, V_j))$ arrows from node i to node j.

Second, this shows the $(0, 2)$-McKay correspondence always works for the G-Hilbert scheme. The total number of first order deformations of the tangent sheaf is the same as the number found from the conformal field theory. In addition, we have the further result that the number of singlets of case 1 gives the number of deformations that come from $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}(q_i))$, while the number of singlets of case 2 gives the number of deformations coming from $\mathrm{Ext}^1(\mathcal{O}_{D_\alpha}(q_\alpha), \mathcal{O}_{D_\beta}(q_\beta))$.

Also, we now have proven all parts of Theorem 1.1:

Theorem 1.1. *Among all crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_r$ with r prime, the G-Hilbert scheme has minimal dimension for $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T})$. This dimension exactly matches the prediction given by the conformal field theory of the orbifold.*

12. Future directions

We would like to understand more about the resolutions where the count does not agree. This can happen in two ways. First, there are resolutions where each singlet does not correspond to a deformation. This can be seen from the proof of proposition 7.5, where we saw that the non-negativity conditions were not necessarily satisfied. Second, there are resolutions which have deformations that do not satisfy the conditions for the singlets. For example, we can look at the resolution of $\mathbb{C}^3/\mathbb{Z}_{11}$ corresponding to the following triangulation.
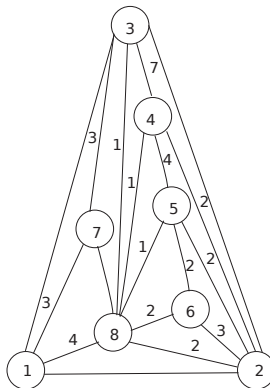

Figure 12.1. Triangulation for alternate resolution of $\mathbb{C}^3/\mathbb{Z}_{11}$.

We see that this triangulation does not have any deformations coming from $\mathrm{Ext}^1(\mathcal{O}_{D_4}(q_4), \mathcal{O}_{D_7}(q_7))$, so there is a singlet that does not correspond to any deformation. It also has a deformation from $\mathrm{Ext}^1(\mathcal{O}_{D_4}(q_4), \mathcal{O}_{D_8}(q_8))$, which does not come from any singlet, so it embodies both of the issues discussed above.

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Mathematics Department, Iona College
715 North Avenue, New Rochelle, NY 10801, USA
E-mail address: gaines.benjamin@gmail.com