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A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II — space-times of infinite lifetime

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The previous functional analytic construction of the fermionic projector on globally hyperbolic Lorentzian manifolds is extended to space-times of infinite lifetime. The construction is based on an analysis of families of solutions of the Dirac equation with a varying mass parameter. It makes use of the so-called mass oscillation property which implies that integrating over the mass parameter generates decay of the Dirac wave functions at infinity. We obtain a canonical decomposition of the solution space of the massive Dirac equation into two subspaces, independent of observers or the choice of coordinates. The constructions are illustrated in the examples of ultrastatic space-times and de Sitter space-time.

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1. Introduction

In the recent paper [9], the fermionic projector was constructed non-perturbatively in a space-time of finite lifetime. In the present paper, we extend the construction to space-times of infinite lifetime. In order to introduce the problem, we begin with the simplest possible example: the Minkowski vacuum. We thus consider the vacuum Dirac equation

$$(i\gamma^j \partial_j - m)\, \psi(x) = 0$$

in Minkowski space $(\mathscr{M}, \langle ., . \rangle)$. On solutions of the Dirac equation one has the scalar product

$$(\psi|\phi)_m := \int_{\mathbb{R}^3} (\overline{\psi} \gamma^0 \phi)(t, \vec{x})\, d^3x \tag{1.1}$$

(which by current conservation is independent of t; here $\overline{\psi} \equiv \psi^\dagger \gamma^0$ is the so-called adjoint spinor). Moreover, on wave functions with suitable decay at infinity (which do not need to be solutions of the Dirac equation), we can introduce a Lorentz invariant inner product by integrating over space-time,

$$<\!\psi|\phi\!> = \int_{\mathscr{M}} \overline{\psi(x)} \phi(x)\, d^4x\, . \tag{1.2}$$

In [9] we proceeded by representing the space-time inner product (1.2) with respect to the scalar product (1.1) as

$$<\!\psi|\phi\!> = (\psi | \mathcal{S} \phi)_m \tag{1.3}$$

with a signature operator $\mathcal{S}$. Then the positive and negative spectral subspaces of the operator $\mathcal{S}$ gave the desired splitting of the solution space into two subspaces. Unfortunately, in Minkowski space an identity of the form (1.3) makes no mathematical sense. Namely, the right side of (1.3) is defined only if ψ and ϕ are solutions of the Dirac equation. But on solutions, the left side of (1.3) is ill-defined because the time integral in (1.2) will in general diverge.

Our method to overcome this problem is to work with families of solutions with a varying mass parameter. This can be understood most easily if one takes the spatial Fourier transform,

$$\psi(t, \vec{x}) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3}\, \hat{\psi}(t, \vec{k})\, e^{i\vec{k}\vec{x}}\, .$$

Then a family of solutions has the representation

$$\hat{\psi}_m(t,\vec{k}) = c_+(\vec{k},m)\, e^{-it\omega(\vec{k},m)} + c_-(\vec{k},m)\, e^{it\omega(\vec{k},m)}$$

with suitable spinor-valued coefficients $c_\pm$, where we set $\omega(\vec{k},m) := \sqrt{|\vec{k}|^2+m^2}$. For a suitable class of solutions (for example families which are smooth and compactly supported in m and $\vec{x}$), the coefficients $c_\pm$ are smooth functions of m. If $m \neq 0$, the derivative $\partial_m \omega(\vec{k},m)$ is non-zero, implying that the phase factors $e^{\pm it\omega(m,\vec{k})}$ oscillate in m. The larger t is chosen, the faster these phase factors oscillate if m is varied. This implies that if we integrate over m by setting

$$(\mathfrak{p}\hat{\psi})(t,\vec{k}) = \int_I \hat{\psi}_m(t,\vec{k})\, dm\,, \tag{1.4}$$

we obtain destructive interference of a superposition of waves with different phases (here $I \subset \mathbb{R} \setminus \{0\}$ is an interval containing the support of $c_\pm(\vec{k},.)$). If t is increased, the integrand oscillates faster in m, so that the integral becomes smaller. We thus obtain decay in time. This intuitive picture that oscillations in the mass parameter give rise to decay for large times is made mathematically precise by the *mass oscillation property.* We shall prove that, using the mass oscillation property, one can give (1.3) a mathematical meaning by inserting suitable mass integrals,

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi_m | \mathcal{S}_m \phi_m)_m\, dm\,. \tag{1.5}$$

We thus obtain a family of bounded linear operators $\mathcal{S}_m$. For any fixed mass m, the positive and negative spectral subspaces of the operator $\mathcal{S}_m$ give rise to a canonical decomposition of the solution space into two subspaces.

It it the main purpose of this paper to make such ideas and methods applicable in the general setting of globally hyperbolic manifolds. After the preliminaries in Section 2, we begin by stating the most general assumptions on the Dirac operator in space-time under which mass oscillations can be studied, referred to as the *weak mass oscillation property* (Section 3). In this setting, the operators $\mathcal{S}_m$ cannot be defined for fixed m, but only the combination $\mathcal{S}_m\, dm$ is defined as an operator-valued measure. In Section 4 we introduce stronger assumptions (the *strong mass oscillation property*) which ensure that the operators $\mathcal{S}_m$ are bounded operators which are uniquely defined for any $m \in I$. We point out that we state the mass oscillation properties purely in terms of the solution spaces of the Dirac equation. This has the advantage that we do not need to make any assumptions on the asymptotic behavior of the metric at infinity. The strong mass oscillation property also makes it possible to define the fermionic projector as an integral operator with a distributional kernel.

In the last two sections we illustrate the abstract constructions by simple examples. Section 5 is devoted to the Dirac operator in ultrastatic space-times, possibly involving an arbitrary static magnetic field. We find that in this ultrastatic situation, the positive and negative spectral subspaces of the operator $\mathcal{S}_m$ coincide precisely with the solutions of positive and negative frequency. We thus obtain agreement with the "frequency splitting" commonly used in quantum field theory. Section 6 treats the Dirac operator in the de Sitter space-time. In this case, the positive and negative spectral subspaces of the operator $\mathcal{S}_m$ give a non-trivial interpolation between the spaces of positive and negative frequency as experienced by observers at asymptotic times $t \to \pm\infty$. In all these examples, the main task is to prove the mass oscillation properties. Establishing the weak mass oscillation property will always be an intermediate step for proving the strong mass oscillation property.

We finally remark that (1.4) and (1.5) can also be written with a Dirac distribution as

$$<\psi_m | \phi_{m'}> = \delta(m - m')\, (\psi_m | \mathcal{S}_m \phi_m)_m \,. \tag{1.6}$$

Such "δ-normalizations in the mass parameter" are commonly used in the perturbative treatment (see [4, 6] and [5, §2.1] or more recently [11]). The mass oscillation property makes it possible to give such normalizations a rigorous meaning in the non-perturbative treatment.

2. Preliminaries

As in [9], we let $(\mathcal{M}, g)$ be a smooth, globally hyperbolic Lorentzian spin manifold of dimension $k \geq 2$. For the signature of the metric we use the convention $(+, -, \ldots, -)$. We denote the corresponding spinor bundle by $S\mathcal{M}$. Its fibres $S_x\mathcal{M}$ are endowed with an inner product $\prec .|. \succ_x$ of signature (n, n) with $n = 2^{[k/2]-1}$ (where $[.]$ is the Gauß bracket; for details see [2, 14]), which we refer to as the spin scalar product. Clifford multiplication is described by a mapping γ which satisfies the anti-commutation relations,

$$\gamma \,:\, T_x\mathcal{M} \to \mathrm{L}(S_x\mathcal{M}) \qquad \text{with} \qquad \gamma(u)\,\gamma(v) + \gamma(v)\,\gamma(u) = 2\,g(u,v)\,\mathbb{1}_{S_x(\mathcal{M})} \,.$$

We again write Clifford multiplication in components with the Dirac matrices γ^j and use the short notation with the Feynman dagger, $\gamma(u) \equiv u^j \gamma_j \equiv \slashed{u}$. The metric connections on the tangent bundle and the spinor bundle are denoted by ∇. The sections of the spinor bundle are also referred to as wave functions. We denote the smooth sections of the spinor bundle by $C^\infty(\mathscr{M}, S\mathscr{M})$. Similarly, $C^\infty_0(\mathscr{M}, S\mathscr{M})$ denotes the smooth sections with compact support. On the wave functions, one has the Lorentz invariant inner product

$$\begin{aligned} <.|.> \,:\, C^\infty(\mathscr{M}, S\mathscr{M}) \times C^\infty_0(\mathscr{M}, S\mathscr{M}) \rightarrow \mathbb{C}\,, \\ <\psi|\phi> = \int_{\mathscr{M}} \prec \psi|\phi \succ_x \, d\mu_{\mathscr{M}}\,. \end{aligned} \tag{2.1}$$

The Dirac operator $\mathcal{D}$ is defined by

$$\mathcal{D} := i\gamma^j \nabla_j + \mathcal{B} \,:\, C^\infty(\mathscr{M}, S\mathscr{M}) \rightarrow C^\infty(\mathscr{M}, S\mathscr{M})\,,$$

where $\mathcal{B} \in \mathrm{L}(S_x)$ (the "external potential") can be any smooth and symmetric multiplication operator. For a given real parameter $m \in \mathbb{R}$ (the "mass"), the Dirac equation reads

$$(\mathcal{D} - m)\, \psi_m = 0\,. \tag{2.2}$$

For clarity, we always denote solutions of the Dirac equation by a subscript m. We mainly consider solutions in the class $C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M})$ of smooth sections with spatially compact support. On such solutions, one has the scalar product

$$(\psi_m|\phi_m)_m = 2\pi \int_{\mathscr{N}} \prec \psi_m | \slashed{\nu} \phi_m \succ_x \, d\mu_{\mathscr{N}}(x)\,, \tag{2.3}$$

where $\mathscr{N}$ denotes any Cauchy surface and ν its future-directed normal (due to current conservation, the scalar product is in fact independent of the choice of $\mathscr{N}$; for details see [9, Section 2]). Forming the completion gives the Hilbert space $(\mathcal{H}_m, (.|.)_m)$.

The *retarded* and *advanced Green's operators* $s^\wedge_m$ and $s^\vee_m$ are mappings (for details see for example [1])

$$s^\wedge_m, s^\vee_m \,:\, C^\infty_0(\mathscr{M}, S\mathscr{M}) \rightarrow C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M})\,.$$

Taking their difference gives the so-called causal fundamental solution k_m,

$$k_m := \frac{1}{2\pi i} \left(s^\vee_m - s^\wedge_m\right) \,:\, C^\infty_0(\mathscr{M}, S\mathscr{M}) \rightarrow C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m\,. \tag{2.4}$$

These operators can be represented as integral operators with distributional kernels; for example,

$$(k_m \phi)(x) = \int_{\mathscr{M}} k_m(x,y)\, \phi(y)\, d\mu_{\mathscr{M}}(y)\, .$$

The operator k_m is useful for two reasons. First, it can be used to construct a solution of the Cauchy problem:

Proposition 2.1. *Let $\mathscr{N}$ be any Cauchy surface. Then the solution of the Cauchy problem*

$$(\mathcal{D} - m)\, \psi_m = 0\, , \qquad \psi|_{\mathscr{N}} = \psi_{\mathscr{N}} \in C^\infty(\mathscr{N}, S\mathscr{M})$$

has the representation

$$\psi_m(x) = 2\pi \int_{\mathscr{N}} k_m(x,y)\, \not{\nu}\, \psi_{\mathscr{N}}(y)\, d\mu_{\mathscr{N}}(y)\, .$$

Second, the operator k_m can be regarded as the signature operator of the inner product (2.1) when expressed in terms of the scalar product (2.3):

Proposition 2.2. *For any $\psi_m \in \mathcal{H}_m$ and $\phi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$,*

$$(\psi_m \,|\, k_m \phi)_m = <\!\psi_m | \phi\!>\, .$$

Proposition 2.1 is stated and proved in [9, Section 2]. For the proof of Proposition 2.2 we refer to [3, Proposition 2.2] or [9, Section 3.1].

3. The weak mass oscillation property

3.1. Basic definitions

In a space-time of infinite life time, the space-time inner product $<\!\psi_m|\phi_m\!>$ of two solutions $\psi_m, \phi_m \in \mathcal{H}_m$ is in general ill-defined, because the time integral in (2.1) may diverge. In order to avoid this difficulty, we shall consider families of solutions with a variable mass parameter. The so-called mass oscillation property will make sense of the space-time integral in (2.1) after integrating over the mass parameter.

More precisely, we consider the mass parameter in a bounded open interval, $m \in I := (m_L, m_R)$. For a given Cauchy surface $\mathscr{N}$, we consider a function $\psi_{\mathscr{N}}(x,m) \in S_x\mathscr{M}$ with $x \in \mathscr{N}$ and $m \in I$. We assume that this wave function is smooth and has compact support in both variables, $\psi_{\mathscr{N}} \in C_0^\infty(\mathscr{N} \times I, S\mathscr{M})$. For every $m \in I$, we let $\psi(.,m)$ be the solution of the Cauchy problem for initial data $\psi_{\mathscr{N}}(.,m)$,

$$(\mathcal{D} - m)\,\psi(x,m) = 0\,, \qquad \psi(x,m) = \psi_{\mathscr{N}}(x,m) \;\; \forall\, x \in \mathscr{N}\,. \tag{3.1}$$

Since the solution of the Cauchy problem is smooth and depends smoothly on parameters, we know that $\psi \in C^\infty(\mathscr{M} \times I, S\mathscr{M})$. Moreover, due to finite propagation speed, $\psi(.,m)$ has spatially compact support. Finally, the solution is clearly compactly supported in the mass parameter m. We summarize these properties by writing

$$\psi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})\,, \tag{3.2}$$

where $C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ denotes the smooth wave functions with spatially compact support which are also compactly supported in I. We often denote the dependence on m by a subscript, $\psi_m(x) := \psi(x,m)$. Then for any fixed m, we can take the scalar product (2.3). On families of solutions $\psi, \phi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ of (3.1), we introduce a scalar product by integrating over the mass parameter,

$$(\psi|\phi) := \int_I (\psi_m|\phi_m)_m \,dm \tag{3.3}$$

(where dm is the Lebesgue measure). Forming the completion gives the Hilbert space $(\mathcal{H}, (.|.))$. It consists of measurable functions $\psi(x,m)$ such that for almost all $m \in I$, the function $\psi(.,m)$ is a weak solution of the Dirac equation which is square integrable over any Cauchy surface. Moreover, this spatial integral is integrable over $m \in I$, so that the scalar product (3.3) is well-defined. We denote the norm on $\mathcal{H}$ by $\|.\|$.

For the applications, it is useful to introduce a subspace of the solutions of the form (3.2):

Definition 3.1. We let $\mathcal{H}^\infty \subset C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$ be a subspace of the smooth solutions with the following properties:

(i) $\mathcal{H}^\infty$ is invariant under multiplication by smooth functions in the mass parameter,

$$\eta(m)\,\psi(x,m) \in \mathcal{H}^\infty \qquad \forall\, \psi \in \mathcal{H}^\infty,\; \eta \in C^\infty(I)\,.$$

(ii) The set $\mathcal{H}_m^\infty := \{\psi(.,m) \,|\, \psi \in \mathcal{H}^\infty\}$ is a dense subspace of $\mathcal{H}_m$, i.e.

$$\overline{\mathcal{H}_m^\infty}^{(.|.)_m} = \mathcal{H}_m \qquad \forall\, m \in I\,.$$

We refer to $\mathcal{H}^\infty$ as the **domain** for the mass oscillation property.

The simplest choice is to set $\mathcal{H}^\infty = C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$, but in some applications it is preferable to choose $\mathcal{H}^\infty$ as a proper subspace of $C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$.

Our motivation for considering a variable mass parameter is that integrating over the mass parameter should improve the decay properties of the wave function for large times (similar as explained in the introduction in the vacuum Minkowski space). This decay for large times should also make it possible to integrate the Dirac operator in the inner product (2.1) by parts without boundary terms,

$$<\mathcal{D}\psi|\phi> = <\psi|\mathcal{D}\phi>\,,$$

implying that the solutions for different mass parameters should be orthogonal with respect to this inner product. Instead of acting with the Dirac operator, it is technically easier to work with the operator of multiplication by m, which we denote by

$$T \;:\; \mathcal{H} \rightarrow \mathcal{H}\,, \qquad (T\psi)_m = m\,\psi_m\,.$$

In view of property (ii) in Definition 3.1, this operator leaves $\mathcal{H}^\infty$ invariant,

$$T|_{\mathcal{H}^\infty} \;:\; \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty\,.$$

Moreover, T is a symmetric operator, and it is bounded because the interval I is,

$$T^* = T \in \mathrm{L}(\mathcal{H})\,. \tag{3.4}$$

Finally, integrating over m gives the operation

$$\mathfrak{p} \;:\; \mathcal{H}^\infty \rightarrow C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})\,, \qquad \mathfrak{p}\psi = \int_I \psi_m\, dm\,.$$

The next definition should be regarded as specifying the minimal requirements needed for the construction of the fermionic projector (stronger assumptions which give rise to additional properties of the fermionic projector will be considered in Section 4 below).

Definition 3.2. The Dirac operator $\mathcal{D}$ on the globally hyperbolic manifold $(\mathscr{M}, g)$ has the **weak mass oscillation property** in the interval $I \subset \mathbb{R}$ with domain $\mathcal{H}^\infty$ (see Definition 3.1) if the following conditions hold:

(a) For every $\psi, \phi \in \mathcal{H}^\infty$, the function $\prec \mathfrak{p}\phi | \mathfrak{p}\psi \succ$ is integrable on $\mathscr{M}$. Moreover, for any $\psi \in \mathcal{H}^\infty$ there is a constant $c(\psi)$ such that

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| \leq c\,\|\phi\| \qquad \forall\, \phi \in \mathcal{H}^\infty\,. \tag{3.5}$$

(b) For all $\psi, \phi \in \mathcal{H}^\infty$,

$${<}\mathfrak{p}T\psi|\mathfrak{p}\phi{>} = {<}\mathfrak{p}\psi|\mathfrak{p}T\phi{>}\,. \tag{3.6}$$

Clearly, in a given space-time one must verify if the assumptions in this definition are satisfied. Before explaining in various examples how this can be done (see Sections 5 and 6), we now proceed by working out the consequence of the weak mass oscillation property abstractly.

3.2. A self-adjoint extension of $\mathcal{S}^2$

In view of the inequality (3.5), every $\psi \in \mathcal{H}^\infty$ gives rise to a bounded linear functional on $\mathcal{H}^\infty$. By continuity, this linear functional can be uniquely extended to $\mathcal{H}$. The Riesz representation theorem allows us to represent this linear functional by a vector $u \in \mathcal{H}$, i.e.

$$(u|\phi) = {<}\mathfrak{p}\psi|\mathfrak{p}\phi{>} \qquad \forall\, \phi \in \mathcal{H}\,.$$

Varying ψ, we obtain the linear mapping

$$\mathcal{S} \,:\, \mathcal{H}^\infty \to \mathcal{H}\,, \qquad (\mathcal{S}\psi|\phi) = {<}\mathfrak{p}\psi|\mathfrak{p}\phi{>} \quad \forall\, \phi \in \mathcal{H}\,.$$

This operator is symmetric because

$$(\mathcal{S}\psi|\phi) = {<}\mathfrak{p}\psi|\mathfrak{p}\phi{>} = (\psi|\mathcal{S}\phi) \qquad \forall\, \phi, \psi \in \mathcal{H}^\infty\,.$$

Moreover, (3.6) implies that the operators $\mathcal{S}$ and T commute,

$$\mathcal{S}\,T = T\,\mathcal{S} \,:\, \mathcal{H}^\infty \to \mathcal{H}\,. \tag{3.7}$$

For the construction of the fermionic projector we need a spectral calculus for the operator $\mathcal{S}$. Therefore, we would like to construct a self-adjoint extension of the operator $\mathcal{S}$. A general method for constructing self-adjoint extensions of symmetric operators is provided by the Friedrichs extension (see for example [15, §33.3]). Since this method only applies to semi-bounded operators, we are led to working with the operator $\mathcal{S}^2$. We thus introduce the scalar product

$$\langle \psi | \phi \rangle_{\mathcal{S}^2} = (\psi | \phi) + (\mathcal{S}\psi | \mathcal{S}\phi) \;:\; \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathbb{C}\,.$$

Clearly, the corresponding norm is bounded from below by the norm $\|.\|$. Thus, forming the completion gives a subspace of $\mathcal{H}$,

$$\mathcal{H}_{\mathcal{S}^2} := \overline{\mathcal{H}^\infty}^{\langle .|. \rangle_{\mathcal{S}^2}} \subset \mathcal{H}\,. \tag{3.8}$$

Proposition 3.3. *Introducing the operator $\mathcal{S}^2$ with domain of definition $\mathcal{D}(\mathcal{S}^2)$ by*

$$\begin{aligned} &\mathcal{D}\big(\mathcal{S}^2\big) = \Big\{ u \in \mathcal{H}_{\mathcal{S}^2} \quad \text{such that} \quad \big|\langle u | \phi \rangle_{\mathcal{S}^2}\big| \leq c(u)\, \|\phi\| \;\; \forall\, \phi \in \mathcal{H}_{\mathcal{S}^2} \Big\} \\ &\mathcal{S}^2 \;:\; \mathcal{D}(\mathcal{S}^2) \subset \mathcal{H} \to \mathcal{H}\,, \qquad (\mathcal{S}^2 \psi | \phi) = \langle \psi | \phi \rangle_{\mathcal{S}^2} - (\psi|\phi) \;\; \forall\, \phi \in \mathcal{H}_{\mathcal{S}^2}\,, \end{aligned}$$

this operator is self-adjoint. The operator T maps $\mathcal{D}(\mathcal{S}^2)$ to itself and commutes with $\mathcal{S}^2$,

$$\mathcal{S}^2\, T = T\, \mathcal{S}^2 \;:\; \mathcal{D}(\mathcal{S}^2) \to \mathcal{H}\,. \tag{3.9}$$

Proof. The self-adjointness of $\mathcal{S}^2$ follows exactly as in the standard construction of the Friedrichs extension (see for example [15, Theorem 33.3.4] for the operator $L := \mathcal{S}^2 + 1$).

Let us show that $T(\mathcal{D}(\mathcal{S}^2)) \subset \mathcal{D}(\mathcal{S}^2)$. Thus let $u \in \mathcal{D}(\mathcal{S}^2)$. Then $u \in \mathcal{H}_{\mathcal{S}^2}$, so that by definition (3.8) there is a series $u_n \in \mathcal{H}^\infty$ which converges to u in the topology given by $\langle .|. \rangle_{\mathcal{S}^2}$. Next, for any $\phi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$, we have the inequality

$$\begin{aligned} \langle T\phi | T\phi \rangle_{\mathcal{S}^2} &= (T\phi | T\phi) + (\mathcal{S}T\phi | \mathcal{S}T\phi) \\ &\overset{(3.7)}{=} (T\phi|\phi) + (T\mathcal{S}\phi | \mathcal{S}T\phi) \leq \|T\|^2_{\mathcal{H}}\, \langle \phi | \phi \rangle_{\mathcal{S}^2}\,, \end{aligned}$$

showing that the operator T is also bounded on $\mathcal{H}_{\mathcal{S}^2}$. As a consequence, the series Tu_n converges in $\mathcal{H}_{\mathcal{S}^2}$ to Tu. Moreover, it follows from (3.4) and (3.7) that

$$\begin{aligned}\langle T u_n | \phi \rangle_{\mathcal{S}^2} &= (T u_n | \phi) + (\mathcal{S} T u_n | \mathcal{S} \phi) \\ &= (u_n | T\phi) + (\mathcal{S} u_n | \mathcal{S} T \phi) = \langle u_n | T \phi \rangle_{\mathcal{S}^2} \,. \end{aligned} \tag{3.10}$$

Taking the limit $n \to \infty$, it follows that

$$\big| \langle T u | \phi \rangle_{\mathcal{S}^2} \big| \leq c(u) \, \|T\phi\| \leq c(u) \, \|T\| \, \|\phi\| \,.$$

We conclude that $Tu \in \mathcal{D}(\mathcal{S}^2)$.

To prove (3.9), we first evaluate the operator product on u_n. Then we know from (3.10) and (3.4) that $\mathcal{S}^2 T u_n = T \mathcal{S}^2 u_n$. Taking the limit $n \to \infty$ gives the result. □

The property (3.9) together with the fact that T is bounded guarantees that the resolvent of $\mathcal{S}^2$ commutes with T. More specifically,

$$\big[(\mathcal{S}^2 - i)^{-1}, T \big] = -(\mathcal{S}^2 - i)^{-1} \, \big[\mathcal{S}^2, T \big] \, (\mathcal{S} - i)^{-1} \,.$$

The operators $(\mathcal{S}^2 - i)^{-1}$ and T are both normal and bounded and commute with each other. The spectral theorem for bounded commuting normal operators (see for example [15, Sections 18 and 31.6], also cf. [16, Section VIII.5]) implies that there is a spectral measure E on $\sigma(\mathcal{S}^2) \times I$ such that

$$\big(\mathcal{S}^2 \big)^p \, T^q = \int_{\sigma(\mathcal{S}^2) \times I} \rho^p \, m^q \, dE_{\rho,m} \qquad \forall \, p, q \in \mathbb{N} \,. \tag{3.11}$$

3.3. The fermionic projector as an operator-valued measure

Acting with the operator k_m as defined in (2.4) for each m separately gives the operator

$$k \;:\; C^\infty_0(\mathscr{M} \times I, S\mathscr{M}) \to \mathcal{H} \,, \quad (k\psi)_m = k_m \psi_m \,.$$

This makes it possible to introduce the fermionic projector $P_\pm$ as an operator-valued measure on I. Namely, for any $f \in C^0(I)$ we define

$$\begin{aligned} \int_I f(m) \, dP_\pm(m) = \frac{1}{2} \int_{\sigma(\mathcal{S}^2) \times I} & f(m) \, \Big(\rho^{\frac{1}{2}} \pm \mathcal{S} \Big) \, \rho^{-\frac{1}{2}} \, dE_{\rho,m} \, k \\ & : \; C^\infty_0(\mathscr{M} \times I, S\mathscr{M}) \to \mathcal{H} \,. \end{aligned} \tag{3.12}$$

The next proposition explains the normalization of the fermionic projector. This normalization can be understood as the spatial normalization, expressed in a functional calculus form (for the spatial normalization see [9, Section 2.3] or the elementary discussion in [11, Section 2]).

Proposition 3.4. **(normalization)** *For any $s, s' \in \{\pm 1\}$ and all $f, g \in C^0(I)$ and $\psi, \phi \in C_0^\infty(\mathscr{M} \times I, S\mathscr{M})$,*

$$\begin{aligned}&\Big(\int_I f(m)\, dP_s(m)\, \psi \,\Big|\, \int_I g(m')\, dP_{s'}(m')\, \phi \Big) \\ &= \delta_{ss'} \int_I \overline{f(m)}\, g(m) <\psi_m \,|\, (dP_s(m)\, \phi)_m> .\end{aligned}$$

Proof. Using the continuous functional calculus, we obtain

$$\begin{aligned}&\Big(\int_I f(m)\, dP_\pm(m)\, \psi \,\Big|\, \int_I g(m')\, dP_\pm(m')\, \phi \Big) \\ &= \frac{1}{4} \int_{\sigma(\mathcal{S}^2) \times I} \overline{f(m)}\, g(m)\, \rho^{-1} \Big(k(\psi) \,\Big|\, \big(\rho^{\frac{1}{2}} \pm \mathcal{S}\big)^2\, dE_{\rho,m}\, k(\phi) \Big) \\ &\overset{(*)}{=} \frac{1}{2} \int_{\sigma(\mathcal{S}^2) \times I} \overline{f(m)}\, g(m)\, \rho^{-\frac{1}{2}} \Big(k(\psi) \,\Big|\, (\rho^{\frac{1}{2}} \pm \mathcal{S})\, dE_{\rho,m}\, k(\phi) \Big) \\ &= \Big(k(\psi) \,\Big|\, \int_I \overline{f(m)}\, g(m)\, dP_s(m)\, \phi \Big) = \int_I \overline{f(m)}\, g(m) <\psi_m \,|\, (dP_s(m)\, \phi)_m> ,\end{aligned}$$

where in $(*)$ we multiplied out $(\rho^{\frac{1}{2}} \pm \mathcal{S})^2$ and used that $\mathcal{S}^2 = \rho$. In the last step we applied (3.3) and Proposition 2.2. This gives the result in the case $s = s'$. The calculation for $s \neq s'$ is similar, but in $(*)$ we get zero. ☐

The following proposition, which is an immediate consequence of the continuous functional calculus, explains in which sense our construction is independent of the choice of the interval I.

Proposition 3.5. **(independence of the choice of I)** *Suppose that we have two mass intervals*

$$\check{I} = (\check{m}_L, \check{m}_R) \;\subset\; I = (m_L, m_R) .$$

We denote all the objects constructed in $\check{I}$ with an additional check and let $\check{\imath}$ and $\check{\pi}$ be the natural injection and projection operators,

$$\begin{aligned}&\check{\imath} \,:\, \check{\mathcal{H}} \rightarrow \mathcal{H} , \qquad \big(\check{\imath}(\psi)\big)(x,m) = \begin{cases} \psi(x,m) & \text{if } m \in \check{I} \\ 0 & \text{otherwise} . \end{cases} \\ &\check{\pi} \,:\, \mathcal{H} \rightarrow \check{\mathcal{H}} , \qquad \check{\pi}(\psi) = \psi|_{\mathscr{M} \times \check{I}} .\end{aligned}$$

Then

$$\int_{\check{I}} f(m)\, d\check{P}_\pm(m) = \check{\pi} \int_I f(m)\, dP_\pm(m)\, \check{\iota} \quad \forall\, f \in C^0(I)$$
$$\int_I f(m)\, dP_\pm(m) = \check{\iota} \int_{\check{I}} f(m)\, d\check{P}_\pm(m)\, \check{\pi} \quad \forall\, f \in C^0_0(\check{I})\,.$$

4. The strong mass oscillation property

4.1. Definition and general structural results

Definition 4.1. The Dirac operator $\mathcal{D}$ on the globally hyperbolic manifold $(\mathscr{M}, g)$ has the **strong mass oscillation property** in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$ (see Definition 3.1), if there is a constant $c > 0$ such that

$$|\langle\mathfrak{p}\psi|\mathfrak{p}\phi\rangle| \leq c \int_I \|\phi_m\|_m\, \|\psi_m\|_m\, dm \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty\,. \tag{4.1}$$

Theorem 4.2. *The following statements are equivalent:*

(i) *The strong mass oscillation property holds.*

(ii) *There is a constant $c > 0$ such that for all $\psi, \phi \in \mathcal{H}^\infty$, the following two relations hold:*

$$|\langle\mathfrak{p}\psi|\mathfrak{p}\phi\rangle| \leq c\, \|\psi\|\, \|\phi\| \tag{4.2}$$
$$\langle\mathfrak{p}T\psi|\mathfrak{p}\phi\rangle = \langle\mathfrak{p}\psi|\mathfrak{p}T\phi\rangle\,. \tag{4.3}$$

(iii) *There is a family of linear operators $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$ which are uniformly bounded,*

$$\sup_{m\in I} \|\mathcal{S}_m\| < \infty\,,$$

such that

$$\langle\mathfrak{p}\psi|\mathfrak{p}\phi\rangle = \int_I (\psi_m \,|\, \mathcal{S}_m\, \phi_m)_m\, dm \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty\,. \tag{4.4}$$

Proof. The implication (iii)⇒(i) follows immediately from the estimate

$$|\langle\mathfrak{p}\psi|\mathfrak{p}\phi\rangle| \leq \int_I \big|(\psi_m|\mathcal{S}_m\phi_m)_m\big|\, dm \leq \sup_{m\in I} \|\mathcal{S}_m\| \int_I \|\psi_m\|_m\, \|\phi\|_m\, dm\,.$$

In order to prove the implication (i)⇒(ii), we first apply the Schwarz inequality to (4.1) to obtain

$$
\begin{aligned}
|<\mathfrak{p}\psi|\mathfrak{p}\phi>| &\leq c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, dm \\
&\leq c \bigg(\int_I \|\phi_m\|_m^2 \, dm \bigg)^{\frac{1}{2}} \bigg(\int_I \|\psi_m\|_m^2 \, dm \bigg)^{\frac{1}{2}} = c \, \|\phi\| \, \|\psi\| \,,
\end{aligned}
$$

proving (4.2). Next, for given $N \in \mathbb{N}$ we subdivide the interval $I = (m_L, m_R)$ by choosing the intermediate points

$$
m_\ell = \frac{\ell}{N} \, (m_R - m_L) + m_L \,, \qquad \ell = 0, \ldots, N \,.
$$

Moreover, we choose non-negative test functions $\eta_1, \ldots, \eta_L \in C^\infty_0(\mathbb{R})$ which form a partition of unity and are supported in small subintervals, meaning that

$$
\sum_{\ell=1}^N \eta_\ell \big|_I = 1|_I \qquad \text{and} \qquad \operatorname{supp} \eta_\ell \subset (m_{\ell-2}, m_{\ell+1}) \,, \tag{4.5}
$$

where we set $m_{-1} = m_L - 1$ and $m_{N+1} = m_R + 1$. For any smooth function $\eta \in C^\infty_0(\mathbb{R})$ we define the operator $\eta(T) \in \mathrm{L}(\mathcal{H}) \,:\, \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$ by

$$
\big(\eta(T) \psi \big)_m = \eta(m) \, \psi_m \,.
$$

Then by linearity,

$$
\begin{aligned}
&<\mathfrak{p} T \psi | \mathfrak{p} \phi> - <\mathfrak{p} \psi | \mathfrak{p} T \phi> \\
&= \sum_{\ell, \ell'=1}^N \Big(<\mathfrak{p} \, T \, \eta_\ell(T) \, \psi \,|\, \mathfrak{p} \, \eta_{\ell'}(T) \, \phi> - <\mathfrak{p} \, \eta_\ell(T) \, \psi \,|\, \mathfrak{p} \, T \, \eta_{\ell'}(T) \, \phi> \Big) \\
&= \sum_{\ell, \ell'=1}^N \Big(<\mathfrak{p} \, \big(T - m_\ell \big) \, \eta_\ell(T) \, \psi \,|\, \mathfrak{p} \, \eta_{\ell'}(T) \, \phi> \\
&\qquad\qquad - <\mathfrak{p} \, \eta_\ell(T) \, \psi \,|\, \mathfrak{p} \, \big(T - m_\ell \big) \, \eta_{\ell'}(T) \, \phi> \Big) \,.
\end{aligned}
$$

Taking the absolute value and applying (4.1), we obtain

$$\begin{aligned}&\big|<\mathfrak{p}T\psi|\mathfrak{p}\phi> - <\mathfrak{p}\psi|\mathfrak{p}T\phi>\big| \\ &\leq c \sum_{\ell,\ell'=1}^N \int_I |m-m_\ell|\, \eta_\ell(m)\, \eta_{\ell'}(m)\, \|\phi_m\|_m\, \|\psi_m\|_m\, dm\,.\end{aligned}$$

In view of the second property in (4.5), we only get a contribution if $|\ell-\ell'| \leq 1$. Moreover, we know that $|m-m_\ell| \leq 2\,|I|/N$ on the support of η_ℓ. Thus

$$\begin{aligned}\big|<\mathfrak{p}T\psi|\mathfrak{p}\phi> - <\mathfrak{p}\psi|\mathfrak{p}T\phi>\big| &\leq \frac{6c\,|I|}{N} \sum_{\ell=1}^N \int_I \eta_\ell(m)\, \|\phi_m\|_m\, \|\psi_m\|_m\, dm \\ &= \frac{6c\,|I|}{N} \int_I \|\phi_m\|_m\, \|\psi_m\|_m\, dm\,.\end{aligned}$$

Since N is arbitrary, we obtain (4.3).

It remains to prove the implication (ii)⇒(iii). Combining (4.2) with the Fréchet-Riesz theorem, there is a bounded operator $\mathscr{S} \in \mathrm{L}(\mathscr{H})$ with

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = (\psi|\mathscr{S}\phi) \qquad \forall\, \psi,\phi \in \mathscr{H}^\infty\,.$$

The relation (4.3) implies that the operators $\mathscr{S}$ and T commute. Moreover, these two operators are obviously symmetric and thus self-adjoint. Hence the spectral theorem for commuting self-adjoint operators implies that there is a spectral measure F on $\sigma(\mathscr{S}) \times I$ such that

$$\mathscr{S}^p\, T^q = \int_{\sigma(\mathscr{S})\times I} \nu^p\, m^q\, dF_{\nu,m} \qquad \forall\, p,q \in \mathbb{N}\,. \tag{4.6}$$

For given $\psi,\phi \in \mathscr{H}^\infty$, we introduce the Borel measure $\mu_{\psi,\phi}$ on I by

$$\mu_{\psi,\phi}(\Omega) = \int_{\sigma(\mathscr{S})\times\Omega} \nu\, d(\psi|F_{\nu,m}\phi)\,. \tag{4.7}$$

Then $\mu_{\psi,\phi}(I) = (\psi|\mathscr{S}\phi)$ and

$$\mu_{\psi,\phi}(\Omega) = \int_{\sigma(\mathscr{S})\times I} \nu\, d\big(\chi_\Omega(T)\,\psi\,\big|\, F_{\nu,m}\,\chi_\Omega(T)\,\phi\big) = (\chi_\Omega(T)\,\psi\,|\,\mathscr{S}\,\chi_\Omega(T)\,\phi)\,.$$

Since the operator $\mathcal{S}$ is bounded, we conclude that

$$|\mu_{\psi,\phi}(\Omega)| \leq c\, \|\chi_\Omega(T)\,\psi\|\, \|\chi_\Omega(T)\,\phi\| \overset{(3.3)}{=} c \left(\int_\Omega \|\psi\|_m^2\, dm \int_\Omega \|\psi\|_{m'}^2\, dm' \right)^{\frac{1}{2}}$$

$$\leq c\,|\Omega| \left(\sup_{m\in\Omega} \|\psi_m\|_m \right) \left(\sup_{m'\in\Omega} \|\phi_{m'}\|_{m'} \right). \tag{4.8}$$

This shows that the measure μ is absolutely continuous with respect to the Lebesgue measure. The Radon-Nikodym theorem (see [17, Theorem 6.9] or [12, §VI.31]) implies that there is a unique function $f_{\psi,\phi} \in L^1(I, dm)$ such that

$$\mu_{\psi,\phi}(\Omega) = \int_\Omega f_{\psi,\phi}(m)\, dm\,. \tag{4.9}$$

Moreover, the estimate (4.8) gives the pointwise bound

$$|f_{\psi,\phi}(m)| \leq c\, \|\psi_m\|_m\, \|\phi_m\|_m\,.$$

Using this inequality, we can apply the Fréchet-Riesz theorem to obtain a unique operator $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$ such that

$$f_{\psi,\phi}(m) = (\psi_m | \mathcal{S}_m \phi_m)_m \qquad \text{and} \qquad \|\mathcal{S}_m\| \leq c\,. \tag{4.10}$$

Combining the above results, for any $\psi, \phi \in \mathcal{H}^\infty$ we obtain

$$\begin{aligned} <\mathfrak{p}\psi|\mathfrak{p}\phi> = (\psi|\mathcal{S}\phi) &= \int_{\sigma(\mathcal{S})\times I} \nu\; d(\psi\,|\,F_{\nu,m}\,\phi) \\ &= \int_I d\mu_{\psi,\phi} = \int_I f_{\psi,\phi}(m)\, dm = \int_I (\psi_m|\mathcal{S}_m\phi_m)_m\, dm\,. \end{aligned}$$

This concludes the proof. ◻

Comparing the statement of Theorem 4.2 (ii) with Definition 3.2, we immediately obtain the following result.

Corollary 4.3. *The strong mass oscillation property implies the weak mass oscillation property.*

We next show uniqueness as well as the independence of the choice of the interval I.

Proposition 4.4. (uniqueness of $\mathcal{S}_m$) *The family $(\mathcal{S}_m)_{m \in I}$ in the statement of Theorem 4.2 can be chosen such that for all $\psi, \phi \in \mathcal{H}^\infty$, the expectation value $f_{\psi,\phi}(m) := (\psi_m | \mathcal{S}_m \phi_m)_m$ is continuous in m,*

$$f_{\psi,\phi} \in C^0_0(I)\,. \tag{4.11}$$

The family $(\mathcal{S}_m)_{m \in I}$ with the properties (4.4) and (4.11) is unique. Moreover, choosing two intervals $\check{I}$ and I with $m \in \check{I} \subset I$ and $0 \notin \overline{I}$, and denoting all the objects constructed in $\check{I}$ with an additional check, we have

$$\check{\mathcal{S}}_m = \mathcal{S}_m\,. \tag{4.12}$$

Proof. Let us show that the function $f_{\psi,\phi}$ is continuous. To this end, we choose a function $\eta \in C^\infty_0(I)$. Then for any $\varepsilon > 0$ which is so small that $B_\varepsilon(\text{supp}\,\eta) \subset I$, we obtain

$$\begin{aligned}
&\int_I \Big(f_{\psi,\phi}(m+\varepsilon) - f_{\psi,\phi}(m) \Big)\, \eta(m)\, dm \\
&= \int_I f_{\psi,\phi}(m)\, \Big(\eta(m-\varepsilon) - \eta(m) \Big)\, dm \\
&\overset{(*)}{=} <\!\int_I \Big(\eta(m-\varepsilon) - \eta(m) \Big) \psi_m\, dm \,|\, \mathfrak{p}\phi\!> \\
&= <\!\int_I \eta(m)\, \Big(\psi_{m+\varepsilon} - \psi_m \Big)\, dm \,|\, \mathfrak{p}\phi\!>\,,
\end{aligned}$$

where in $(*)$ we used (4.6) and (4.7). Applying (4.2), we obtain

$$\left| \int_I \Big(f_{\psi,\phi}(m+\varepsilon) - f_{\psi,\phi}(m) \Big)\, \eta(m)\, dm \right| \leq c\, \|\psi_{+\varepsilon} - \psi\|\, \|\phi\|\, \sup_I |\eta|\,,$$

where the vector $\psi_{+\varepsilon} \in \mathcal{H}^\infty$ is defined by $(\psi_{+\varepsilon})_m := \psi_{m+\varepsilon}$. Since $\lim_{\varepsilon \searrow 0} \|\psi_{+\varepsilon} - \psi\| = 0$ and η is arbitrary, we conclude that $f_{\psi,\phi}$ is continuous (4.11). This continuity is important because it implies that the function $f_{\psi,\phi}$ is uniquely defined pointwise (whereas in (4.9) this function could be modified arbitrarily on sets of measure zero).

In order to prove (4.12), we first note that the spectral measures $dE_{\rho,m}$ and $dF_{\nu,m}$ (cf. (3.11) and (4.6)) are related to each other by

$$dE_{\rho,m} = dF_{\sqrt{\rho},m} + dF_{-\sqrt{\rho},m}\,.$$

A direct computation yields that the definitions (3.12) and (4.13) agree if the strong mass oscillation property holds (see also (4.7), (4.9) and (4.10)). The relation (4.12) then follows from Proposition 3.5. □

We remark that by considering higher difference quotients and taking the limit $\varepsilon \searrow 0$, one could even prove that $f_{\psi,\phi} \in C^\infty_0(I)$ is smooth, but this is not of relevance here.

4.2. Construction of the fermionic projector

Theorem 4.2 and Proposition 4.4 are very useful because for every $m \in I$ they provide a unique operator $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$, referred to as the **fermionic signature operator** corresponding to the mass m. This makes it possible to proceed with methods similar to [9]. From Definition 4.4, the operator $\mathcal{S}_m$ is obviously symmetric. Thus the spectral theorem gives rise to the spectral decomposition

$$\mathcal{S}_m = \int_{\sigma(\mathcal{S}_m)} \nu \, dE_\nu \,,$$

where E_ν is the spectral measure (see for example [16]). The spectral measure gives rise to the spectral calculus

$$f(\mathcal{S}_m) = \int_{\sigma(\mathcal{S}_m)} f(\nu) \, dE_\nu \,,$$

where f is a bounded Borel function.

Definition 4.5. Assume that the Dirac operator $\mathcal{D}$ on $(\mathscr{M}, g)$ satisfies the strong mass oscillation property (see Definition 4.1). We define the operators $P_\pm \,:\, C^\infty_0(\mathscr{M}, S\mathscr{M}) \to \mathcal{H}_m$ by

$$P_+ = \chi_{[0,\infty)}(\mathcal{S}_m)\, k_m \qquad \text{and} \qquad P_- = -\chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \tag{4.13}$$

(where χ denotes the characteristic function). The **fermionic projector** P is defined by $P = P_-$.

Proposition 4.6. *For all $\phi, \psi \in C^\infty_0(\mathscr{M}, S\mathscr{M})$, the operators $P_\pm$ are symmetric,*

$$<P_\pm \phi \,|\, \psi> = <\phi \,|\, P_\pm \psi> \,.$$

Moreover, the image of $P_\pm$ is the positive respectively negative spectral subspace of $\mathcal{S}_m$, i.e.

$$\begin{aligned} \overline{P_+(C^\infty_0(\mathscr{M}, S\mathscr{M}))} &= E_{(0,\infty)}(\mathcal{H}_m) \,, \\ \overline{P_-(C^\infty_0(\mathscr{M}, S\mathscr{M}))} &= E_{(-\infty,0)}(\mathcal{H}_m) \,. \end{aligned} \tag{4.14}$$

Proof. According to Proposition 2.2,

$$\begin{aligned} <P_-\phi \,|\, \psi> = (P_-\phi \,|\, k_m\psi)_m &= -\big(\chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m\phi \,\big|\, k_m\psi\big)_m \\ &= -\big(k_m\,\phi \,\big|\, \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m\psi\big)_m = <\phi \,|\, P_-\psi> . \end{aligned}$$

The proof for P_+ is similar. The relations (4.14) follow immediately from the fact that $k_m(C_0^\infty(\mathscr{M}, S\mathscr{M}))$ is dense in $\mathcal{H}_m$. □

4.3. Representation as a distribution and normalization

Similar as in [9, Theorem 3.12], the fermionic projector can be represented by a two-point distribution on $\mathscr{M}$. As usual, we denote the space of test functions (with the Fréchet topology) by $\mathcal{D}$ and define the space of distributions $\mathcal{D}'$ as its dual space.

Theorem 4.7. *Assume that the strong mass oscillation property holds. Then there is a unique distribution $\mathcal{P} \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$ such that for all $\phi, \psi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$,*

$$<\phi|P\psi> = \mathcal{P}(\phi \otimes \psi) .$$

Proof. According to Proposition 2.2 and Definition 4.5,

$$<\phi|P\psi> = (k_m\phi \,|\, P\psi) = -(k_m\phi \,|\, \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m\psi) .$$

Since the norm of the operator $\chi_{(-\infty,0)}(\mathcal{S}_m)$ is bounded by one, we conclude that

$$|<\phi|P\psi>| \le \|k_m\phi\|\, \|k_m\psi\| = (<\phi|k_m\phi>\, <\psi|k_m\psi>)^{\frac{1}{2}} ,$$

where in the last step we again applied Proposition 2.2. As $k_m \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$, the right side is continuous on $\mathcal{D}(\mathscr{M} \times \mathscr{M})$. We conclude that also $<\phi|P\psi>$ is continuous on $\mathcal{D}(\mathscr{M} \times \mathscr{M})$. The result now follows from the Schwartz kernel theorem (see [13, Theorem 5.2.1], keeping in mind that this theorem applies just as well to bundle-valued distributions on a manifold simply by working with the components in local coordinates and a local trivialization). □

Exactly as explained in [9, Section 3.5], it is convenient to use the standard notation with an integral kernel $P(x,y)$,

$$<\phi|P\psi> = \iint_{\mathscr{M}\times\mathscr{M}} \prec\phi(x)\,|\,P(x,y)\,\psi(y)\succ_x d\mu_{\mathscr{M}}(x)\,d\mu_{\mathscr{M}}(y)$$
$$(P\psi)(x) = \int_{\mathscr{M}} P(x,y)\,\psi(y)\,d\mu_{\mathscr{M}}(y)$$

(where $P(.,.)$ coincides with the distribution $\mathcal{P}$ above). In view of Proposition 4.6, we know that the last integral is not only a distribution, but a function which is square integrable over every Cauchy surface. Moreover, the symmetry of P shown in Proposition 4.6 implies that

$$P(x,y)^* = P(y,x)\,,$$

where the star denotes the adjoint with respect to the spin scalar product. Finally, exactly as shown in [9, Proposition 3.13], the spatial normalization property of Proposition 4.8 makes it possible to obtain a representation of the fermionic projector in terms of one-particle states. To this end, one chooses an orthonormal basis $(\psi_j)_{j\in\mathbb{N}}$ of the subspace $\chi_{(-\infty,0)}(\mathcal{S}_m) \subset \mathcal{H}_m$. Then

$$P(x,y) = -\sum_{j=1}^{\infty} |\psi_j(x)\succ\prec\psi_j(y)|$$

with convergence in $\mathcal{D}'(\mathscr{M}\times\mathscr{M})$.

We now specify the normalization of the fermionic projector. We introduce an operator Π by

$$\begin{aligned} &\Pi\,:\,\mathcal{H}_m \to \mathcal{H}_m\,, \\ &(\Pi\,\psi_m)(x) = -2\pi \int_{\mathscr{N}} P(x,y)\,\not\nu\,(\psi_m)|_{\mathscr{N}}(y)\,d\mu_{\mathscr{N}}(y)\,, \end{aligned} \tag{4.15}$$

where $\mathscr{N}$ is any Cauchy surface.

Proposition 4.8. (spatial normalization) *The operator* Π *is a projection operator on* $\mathcal{H}_m$*.*

Proof. According to Proposition 2.1, the spatial integral in (4.15) can be combined with the factor k_m in (4.13) to give the solution of the corresponding Cauchy problem. Thus

$$\Pi\,:\,\mathcal{H}_m \to \mathcal{H}_m\,, \qquad (\Pi\,\psi_m)(x) = \chi_{(-\infty,0)}(\mathcal{S}_m)\,\psi_m\,,$$

showing that Π is a projection operator. $\Box$

Instead of the spatial normalization, one could also consider the mass normalization (for details on the different normalization methods see [11]). To this end, one needs to consider families of fermionic projectors P_m indexed by the mass parameter. Then for all $\phi, \psi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$, we can use (4.4) and Proposition 2.2 to obtain

$$\begin{aligned} <\mathfrak{p}(P_m\phi) \,|\, \mathfrak{p}(P_{m'}\psi)> &= \int_I (P_m\phi \,|\, \mathcal{S}_m P_m \psi)_m \, dm \\ &= \int_I (k_m\phi \,|\, \mathcal{S}_m \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m\psi)_m \, dm \\ &= \int_I <\phi \,|\, \mathcal{S}_m \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m\psi> dm \\ &= -<\phi \,|\, \mathfrak{p}(\mathcal{S}_m P_m \psi)> , \end{aligned}$$

which can be written in a compact formal notation similar to (1.6) as

$$P_m \, P_{m'} = \delta(m-m')\,(-\mathcal{S}_m)\, P_m \,.$$

Due to the factor $(-\mathcal{S}_m)$ on the right, in general the fermionic projector does *not* satisfy the mass normalization condition. The mass normalization condition could be arranged by modifying the definition (4.13) to

$$\mathcal{S}_m^{-1}\, \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \,.$$

Here we prefer to work with the spatial normalization. For a detailed discussion of the different normalization methods we refer to [11, Section 2].

We finally remark that corresponding causal fermion systems can be constructed exactly as in [9, Section 4] by introducing regularization operators $(\mathfrak{R}_\varepsilon)_{\varepsilon>0}$, computing the local correlation operators $F^\varepsilon(x)$ and defining the universal measure by $d\rho = F^\varepsilon_* d\mu_{\mathscr{M}}$.

5. Example: ultrastatic space-times

In this section we prove that the strong mass oscillation property holds for the Dirac operator in complete ultrastatic space-times, even if an arbitrary static magnetic field is present. Thus we let $(\mathscr{M}, g)$ be a k-dimensional complete space-time which is ultrastatic in the sense that it is the product $\mathscr{M} = \mathbb{R} \times \mathscr{N}$ with a metric of the form

$$ds^2 = dt^2 - g_{\mathscr{N}} ,$$

where $g_{\mathscr{N}}$ is a Riemannian metric on $\mathscr{N}$. The completeness of $\mathscr{M}$ implies that also $\mathscr{N}$ is complete. Moreover, we assume that $\mathscr{N}$ is spin. Let $\mathcal{D}_{\mathscr{N}}$ denote the intrinsic Dirac operator on $\mathscr{N}$. In order to introduce the magnetic field, we let A be a smooth vector field on $\mathscr{N}$ (the "vector potential") and set

$$\mathcal{D}_A = \mathcal{D}_{\mathscr{N}} + \not{A} , \tag{5.1}$$

where the slash again denotes Clifford multiplication. Using standard elliptic theory (see [18, Proposition 8.2.7] and [10]), the operator $\mathcal{D}_A$ with domain $C_0^\infty(\mathscr{N}, S\mathscr{N})$ is essentially self-adjoint on the Hilbert space $L^2(\mathscr{N}, S\mathscr{N})$. Thus its closure, which we again denote by $\mathcal{D}_A$, is a self-adjoint operator with domain $\mathcal{D}(\mathcal{D}_A)$. The spectral theorem yields

$$\mathcal{D}_A = \int_{\sigma(\mathcal{D}_A)} \lambda \, dF_\lambda , \tag{5.2}$$

where dF_λ denotes the spectral-measure of $\mathcal{D}_A$.

The Dirac operator in the ultrastatic space-time $(\mathscr{M}, g)$ in the presence of the magnetic field A can be written in block matrix notation as

$$\mathcal{D} = \begin{pmatrix} i\partial_t & -\mathcal{D}_A \\ \mathcal{D}_A & -i\partial_t \end{pmatrix} . \tag{5.3}$$

Since the Dirac operator is time independent, we can separate the time dependence with a plane wave ansatz,

$$\psi(t,x) = e^{-i\omega t} \, \chi(x) .$$

The sign of ω gives a natural decomposition of the solution space into two subspaces. This is often referred to as "frequency splitting," and the subspaces are called the solutions of positive and negative energy, respectively.

This is the main result of this section.

Theorem 5.1. *On any interval* $I = (m_L, m_R)$ *with* $m_L, m_R > 0$*, the Dirac operator* (5.3) *has the strong mass oscillation property with domain*

$$\mathcal{H}^\infty := C^\infty_{\mathrm{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H} . \tag{5.4}$$

The operators $\mathcal{S}_m$ *in the representation* (4.4) *all have the spectrum* $\{\pm 1\}$*. The eigenspaces corresponding to the eigenvalues* ± 1 *coincide with the solutions of positive and negative frequency, respectively.*

We remark that the reason why the spectral decomposition of $\mathcal{S}_m$ gives the frequency splitting can already be understood in the perturbative treatment as explained in [4, Section 5]. As a corollary, the above theorem clearly yields the strong mass oscillation property for the Dirac operator in the Minkowski vacuum.

We now begin with preparations for the proof, which will be completed at the end of Section 5.2. The space-time inner product (2.1) and the scalar product (2.3) take the form

$$\begin{aligned} <\phi|\psi> &= \int_{-\infty}^{\infty} dt \int_{\mathscr{N}} \prec \psi | \phi \succ_{(t,x)} \, d\mu_{\mathscr{N}}(x) \\ &= \int_{-\infty}^{\infty} dt \left\langle \psi \right| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi \Big\rangle_{L^2(\mathscr{N}, S\mathscr{N})^2} \end{aligned} \tag{5.5}$$

$$\begin{aligned} (\phi|\psi)_m &= \int_{\mathscr{N}} \prec \psi | \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi \succ_{(t,x)} \, d\mu_{\mathscr{N}}(x) \\ &= 2\pi \, \langle \psi(t) | \phi(t) \rangle_{L^2(\mathscr{N}, S\mathscr{N})^2} \end{aligned} \tag{5.6}$$

(where in the last line t is arbitrary due to current conservation). In the following constructions, we will also work with the last scalar product without requiring that ϕ and ψ are solutions of the Dirac equation. In this case, the scalar product will depend on time, and we denote it by

$$(\phi|\psi)_t = 2\pi \, \langle \psi(t) | \phi(t) \rangle_{L^2(\mathscr{N}, S\mathscr{N})^2} .$$

We usually write the Dirac equation in the Hamiltonian form as

$$i\partial_t \psi = H\psi \qquad \text{with} \qquad H = \begin{pmatrix} 0 & \mathcal{D}_A \\ \mathcal{D}_A & 0 \end{pmatrix} + m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} .$$

Substituting the spectral decomposition (5.2), we get

$$H = \int_{\sigma(\mathcal{D}_A)} \begin{pmatrix} m & \lambda \\ \lambda & -m \end{pmatrix} dF_\lambda .$$

In order to bring the dynamics into a more explicit form, we diagonalize the 2×2-matrix,

$$\begin{pmatrix} m & \lambda \\ \lambda & -m \end{pmatrix} = \omega\, \Pi_+ - \omega\, \Pi_- \,,$$

where we set

$$\omega = \sqrt{\lambda^2 + m^2}\,. \tag{5.7}$$

The matrices and $\Pi_\pm$ are orthogonal projections, i.e.

$$\Pi_s\, \Pi_{s'} = \delta_{ss'}\, \Pi_s \qquad \forall\, s, s' \in \{\pm\}\,.$$

A short computation shows that

$$\Pi_\pm = \Pi_\pm(\lambda, m) = \frac{\mathbb{1}}{2} \pm \frac{1}{2\omega} \begin{pmatrix} m & \lambda \\ \lambda & -m \end{pmatrix}. \tag{5.8}$$

Applying the functional calculus, the solution of the Dirac equation of mass m with initial data $\psi_m|_{t=0} = \psi_m(0) \in C^\infty_0(\mathscr{N}, S\mathscr{M})$ can be written as

$$\psi_m(t) = e^{-itH(m)} \psi_m(0) = \int_{\sigma(\mathcal{D}_A)} U^t_m(\lambda)\, dF_\lambda\, \psi_m(0)\,, \tag{5.9}$$

where U^t_m is the unitary 2×2-matrix

$$U^t_m(\lambda) = e^{-it\omega(\lambda,m)}\, \Pi_+(\lambda, m) + e^{it\omega(\lambda,m)}\, \Pi_-(\lambda, m)\,. \tag{5.10}$$

5.1. The weak mass oscillation property using mass derivatives

In preparation for the strong mass oscillation property, we shall now prove the weak mass oscillation property. Let $\psi \in \mathscr{H}^\infty$ as defined in (5.4). Then

$$(\mathfrak{p}\psi)(t) = \int_I dm \int_{\sigma(\mathcal{D}_A)} U^t_m(\lambda)\, dF_\lambda\, \psi_m(0)\,. \tag{5.11}$$

For estimates of such expressions, it is helpful to observe that $U^t_m(\lambda)$ is a 2×2-matrix which commutes with the spectral measure dF_λ. In particular, the matrix entries of the inner integral in (5.11) can be written as

$$g(m) := \int_{\sigma(\mathcal{D}_A)} f(\lambda, m)\, dF_\lambda\, \psi(m) \;\in\; L^2(\mathscr{N}, S\mathscr{N}) \tag{5.12}$$

with $f \in C^\infty(I \times \mathbb{R})$ and $\psi \in C^\infty_0(\mathscr{N} \times I, S\mathscr{N})$ (where we use the notation $\psi(m) = \psi_m(.) \in C^\infty(\mathscr{N}, S\mathscr{N})$). In the next lemma it is shown that this function is differentiable and that we may interchange the differentiation with the integral. Since this is a somewhat subtle point, we give the proof in detail.

Lemma 5.2. *Let $\psi \in C^\infty_0(\mathscr{N} \times I, S\mathscr{N})$ be a smooth family of wave functions on $\mathscr{N}$. Moreover, let $f \in C^\infty(I \times \mathbb{R})$ be a smooth function such that f and all its mass derivatives are polynomially bounded, i.e. for all $p \in \mathbb{N}$ there is $\ell \in \mathbb{N}$ and a constant $c > 0$ such that*

$$|\partial^p_m f(\lambda, m)| \leq c\,\big(1 + \lambda^{2\ell}\big) \qquad \forall\, \lambda \in \mathbb{R},\; m \in I\,. \tag{5.13}$$

Then the function g defined by (5.12) *satisfies the bound*

$$\big\|g(m)\big\|_{L^2(\mathscr{N}, S\mathscr{N})} \leq c\,\big\|\big(1 + \mathcal{D}_A^{2\ell}\big)\psi\big\|_{L^2(\mathscr{N}, S\mathscr{N})}\,. \tag{5.14}$$

Moreover, the function g is smooth in m and

$$g^{(p)}(m) = \int_{\sigma(\mathcal{D}_A)} dF_\lambda\, \partial^p_m \Big(f(\lambda, m)\, \psi(m) \Big)\,. \tag{5.15}$$

Proof. For the proof of the bound (5.14), we may omit the mass dependence. Then the spectral calculus yields

$$\begin{aligned} \bigg\| \int_{\sigma(\mathcal{D}_A)} f(\lambda)\, dF_\lambda\, \psi \bigg\|^2_{L^2(\mathscr{N}, S\mathscr{N})} &= \int_{\sigma(\mathcal{D}_A)} |f(\lambda)|^2\, d\langle \psi | F_\lambda \psi \rangle_{L^2(\mathscr{N}, S\mathscr{N})} \\ &\overset{(5.13)}{\leq} c \int_{\sigma(\mathcal{D}_A)} \big(1 + \lambda^{2\ell}\big)^2\, d\langle \psi | F_\lambda \psi \rangle_{L^2(\mathscr{N}, S\mathscr{N})} = c\,\big\|\big(1 + \mathcal{D}_A^{2\ell}\big)\psi\big\|^2_{L^2(\mathscr{N}, S\mathscr{N})}\,. \end{aligned}$$

In order to prove that g is differentiable, we consider the difference quotient and subtract the expected derivative,

$$\phi_\varepsilon := \frac{g(m+\varepsilon) - g(m)}{\varepsilon} - \int_{\sigma(\mathcal{D}_A)} dF_\lambda\, \partial_m \big(f(\lambda, m)\, \psi(m) \big).$$

By rearranging the terms, we obtain

$$\phi_\varepsilon = \int_{\sigma(\mathcal{D}_A)} dF_\lambda \bigg[\frac{f(\lambda, m+\varepsilon)\, \psi(m+\varepsilon) - f(\lambda,m)\, \psi(m)}{\varepsilon} - \partial_m \Big(f(\lambda,m)\, \psi(m) \Big) \bigg]$$

$$= \int_{\sigma(\mathcal{D}_A)} dF_\lambda \bigg[\Big(\frac{f(\lambda, m+\varepsilon) - f(\lambda,m)}{\varepsilon} - \partial_m f(\lambda,m) \Big)\, \psi(m) \tag{5.16}$$

$$+ f(\lambda, m+\varepsilon) \Big\{ \frac{\psi(m+\varepsilon) - \psi(m)}{\varepsilon} - \partial_m \psi(m) \Big\} \tag{5.17}$$

$$+ \Big(f(\lambda, m+\varepsilon) - f(\lambda,m) \Big)\, \partial_m \psi(m) \bigg] . \tag{5.18}$$

The contribution (5.17) can be estimated immediately with the help of (5.14) (with $\psi(m)$ in (5.12) replaced by the expression in the curly brackets in (5.17)). We thus obtain

$$\|(5.17)\|_{L^2(\mathscr{N}, S\mathscr{N})} \leq c\, \bigg\| \big(1 + \mathcal{D}_A^{2\ell}\big) \Big(\frac{\psi(m+\varepsilon) - \psi(m)}{\varepsilon} - \partial_m \psi(m) \Big) \bigg\|_{L^2(\mathscr{N}, S\mathscr{N})} ,$$

and this converges to zero as $\varepsilon \searrow 0$ because ψ is smooth and has compact support. The term (5.18), on the other hand, is estimated by decomposing the λ-integral into the integrals over the regions $[-L, L]$ and $\mathbb{R} \setminus [-L, L]$ and estimating similar as in the proof of (5.14),

$$\bigg\| \int_{-L}^{L} dF_\lambda \Big(f(\lambda, m+\varepsilon) - f(\lambda,m) \Big)\, \partial_m \psi(m) \bigg\|_{L^2(\mathscr{N}, S\mathscr{N})}$$

$$\leq \|\partial_m \psi(m)\|_{L^2(\mathscr{N}, S\mathscr{N})} \sup_{(\lambda,m) \in [-L,L] \times I} \big| f(\lambda, m+\varepsilon) - f(\lambda,m) \big| . \tag{5.19}$$

Moreover, using again (5.13),

$$\bigg\| \int_{\mathbb{R} \setminus [-L,L]} dF_\lambda \Big(f(\lambda, m+\varepsilon) - f(\lambda,m) \Big)\, \partial_m \psi(m) \bigg\|^2_{L^2(\mathscr{N}, S\mathscr{N})}$$

$$\leq 4c^2 \int_{\mathbb{R} \setminus [-L,L]} \big(1 + \lambda^{2\ell}\big)^2\, d\big\langle \partial_m \psi(m), F_\lambda \partial_m \psi(m) \big\rangle_{L^2(\mathscr{N}, S\mathscr{N})}$$

$$\leq \frac{4c^2}{L^4} \int_{\mathbb{R} \setminus [-L,L]} \big(1 + \lambda^{2\ell+2}\big)^2\, d\big\langle \partial_m \psi(m), F_\lambda \partial_m \psi(m) \big\rangle_{L^2(\mathscr{N}, S\mathscr{N})}$$

$$= \frac{4c^2}{L^4}\, \big\| \big(1 + \mathcal{D}_A^{2\ell+2}\big) \psi \big\|^2_{L^2(\mathscr{N}, S\mathscr{N})} . \tag{5.20}$$

The term (5.20) can be made arbitrarily small by choosing L sufficiently large. The term (5.19), on the other hand, tends to zero as $\varepsilon \searrow 0$ for any fixed L due to the locally uniform convergence of $f(\lambda, m+\varepsilon)$ to $f(\lambda, m)$ (note that f is smooth in view of (5.10) and (5.8)). This shows that (5.18) tends to zero as $\varepsilon \searrow 0$. Finally, the contribution (5.16) can be estimated just as (5.18) by considering the regions $[-L, L]$ and $\mathbb{R} \setminus [-L, L]$ separately.

We conclude that in the limit $\varepsilon \searrow 0$, the vectors ϕ_ε converge to zero in $L^2(\mathscr{N}, S\mathscr{N})$. This shows (5.15) in the case $p = 1$. The relation for general p follows immediately by induction. ☐

Lemma 5.3. *The time evolution operator in the vacuum has the representation*

$$t^2\, U^t_m(\lambda) = \frac{\partial^2}{\partial m^2} A^t_m(\lambda) + \frac{\partial}{\partial m} B^t_m(\lambda) + C^t_m(\lambda) \tag{5.21}$$

with matrices A^t_m, B^t_m and C^t_m which are bounded uniformly in time by

$$\|A^t_m(\lambda)\| + \|B^t_m(\lambda)\| + \|C^t_m(\lambda)\| \leq c\,\big(1+\lambda^2\big) \qquad \forall\, m \in I$$

with a constant c which may depend on the choice of the interval I (here $\|.\|$ denotes any norm on the 2×2-matrices, and we again assume that $I = (m_L, m_R)$ with $m_L, m_R > 0$).

Proof. We can generate factors of t by differentiating the exponentials in (5.10) with respect to ω. With the help of (5.7), we can then rewrite the ω-derivatives as m-derivatives. We thus obtain

$$t^2\, e^{\pm i\omega t} = -\frac{\partial^2}{\partial \omega^2} e^{\pm i\omega t} = -\frac{\omega}{m}\frac{\partial}{\partial m}\left(\frac{\omega}{m}\frac{\partial}{\partial m} e^{\pm i\omega t}\right) .$$

A straightforward computation in which one uses the product rule inductively gives the result. ☐

Lemma 5.4. *For any $\psi \in \mathcal{H}^\infty$, there is a constant $C = C(\psi)$ such that*

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \frac{C}{1+t^2}\,.$$

Proof. Using that the operators U_m^t are unitary, we immediately obtain

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \int_I dm\, \|\psi_m\|_m\,.$$

In order to prove time decay, we apply the identity (5.21) to (5.11). Then Lemma 5.2 allows us to integrate by parts,

$$\begin{aligned} t^2(\mathfrak{p}\psi)|_t &= \int_{\mathscr{M}} dm \int_{\sigma(\mathcal{D}_A)} dF_\lambda \left(\frac{\partial^2}{\partial m^2} A_m^t + \frac{\partial}{\partial m} B_m^t + C_m^t\right) \psi_m(0) \\ &= \int_{\mathscr{M}} dm \int_{\sigma(\mathcal{D}_A)} dF_\lambda \Big(A_m^t(\lambda)\, \partial_m^2 \psi_m(0) \\ &\qquad\qquad - B_m^t(\lambda)\, \partial_m \psi_m(0) + C_m^t(\lambda)\, \psi_m(0)\Big). \end{aligned}$$

Now can use the estimate of Lemma 5.3 together with (5.14) to obtain

$$\begin{aligned} t^2\, \big\|(\mathfrak{p}\psi)|_t\big\|_t &\leq c \int_{\mathscr{M}} dm \sum_{a=0,2} \bigg\| \int_{\sigma(\mathcal{D}_A)} (1+\lambda^2)\, dF_\lambda\, \partial_m^a \psi_m(0) \bigg\|_t \\ &= c \int_{\mathscr{M}} dm \sum_{a=0,2} \Big\| (1+\mathcal{D}_A^2)\, \partial_m^a \psi_m(0) \Big\|_t\,, \end{aligned} \tag{5.22}$$

where in the last step we used the spectral calculus. □

Proposition 5.5. *On any interval $I = (m_L, m_R)$ with $m_L, m_R > 0$, the Dirac operator* (5.3) *has the weak mass oscillation property with domain* (5.4).

Proof. For every $\psi, \phi \in \mathcal{H}^\infty$, the Schwarz inequality gives

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| = \left| \int_{-\infty}^\infty \big((\mathfrak{p}\psi)|_t \,|\, \gamma^0\, (\mathfrak{p}\phi)|_t\big)_t\, dt \right| \leq \int_{-\infty}^\infty \big\|(\mathfrak{p}\psi)|_t\big\|_t\, \big\|(\mathfrak{p}\phi)|_t\big\|_t\, dt\,.$$

Applying Lemma 5.4 together with the estimate

$$\begin{aligned} \big\|(\mathfrak{p}\phi)|_t\big\|_t^2 &= \iint_{I\times I} (\phi_m|\phi_{m'})_t\, dm\, dm' \\ &\leq \frac{1}{2} \iint_{I\times I} \Big(\|\phi_m\|^2 + \|\phi_{m'}\|^2\Big)\, dm\, dm' = |I|\, \|\phi\|^2\,, \end{aligned}$$

we obtain the inequality (3.5) with

$$c = C\, \sqrt{|I|} \int_{-\infty}^\infty \frac{dt}{1+t^2} < \infty\,.$$

The identity (3.6) follows by integrating the Dirac operator in space-time by parts,

$$
\begin{aligned}
&\mathopen{<}\mathfrak{p}T\psi|\mathfrak{p}\phi\mathclose{>} = \mathopen{<}\mathfrak{p}\mathcal{D}\psi|\mathfrak{p}\phi\mathclose{>} = \mathopen{<}\mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi\mathclose{>} = \int_{\mathscr{M}} \prec \mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi \succ(x)\, d^4x \\
&\overset{(*)}{=} \int_{\mathscr{M}} \prec \mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi \succ(x)\, d^4x = \mathopen{<}\mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi\mathclose{>} = \mathopen{<}\mathfrak{p}\psi|\mathfrak{p}T\phi\mathclose{>}\,. \qquad (5.23)
\end{aligned}
$$

In $(*)$ we used that the Dirac operator is formally self-adjoint with respect to $\mathopen{<}.|.\mathclose{>}$. Moreover, we do not get boundary terms in view of the time decay in Lemma 5.4. □

5.2. The strong mass oscillation property using a Plancherel method

We now give the proof of Theorem 5.1. Before beginning, we point out that the method of working with mass derivatives in the previous section gave the inequality (3.5) with a constant c which depended on the derivatives of ψ (cf. (5.22)). For the strong mass oscillation property, however, this constant must depend only on the L^2-norm of ψ (see (4.2)). For this reason, working with mass derivatives and an integration-by-parts argument in the mass parameter is not appropriate for proving the strong mass oscillation property. Instead, we shall use the following Plancherel method.

First, in view of the decay established in Lemma 5.4, we know that for any $\psi, \phi \in \mathcal{H}^\infty$, the function $\prec \mathfrak{p}\psi|\mathfrak{p}\phi \succ$ is integrable. Moreover, the time integral can be carried out with the help of Plancherel's theorem,

$$
\begin{aligned}
\mathopen{<}\mathfrak{p}\psi|\mathfrak{p}\phi\mathclose{>} &= \int_{-\infty}^{\infty} dt \int_{\mathscr{N}} \prec \mathfrak{p}\psi|\mathfrak{p}\phi \succ_{(t,x)}\, d\mu(x) \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\langle \widehat{\mathfrak{p}\psi}(\omega) \middle| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \widehat{\mathfrak{p}\phi}(\omega) \right\rangle_{L^2(\mathscr{N}, S\mathscr{N})^2}, \qquad (5.24)
\end{aligned}
$$

where

$$
\widehat{\mathfrak{p}\psi}(\omega) = \int_{-\infty}^{\infty} (\mathfrak{p}\psi)(t)\, e^{i\omega t}\, dt\,.
$$

In order to compute this Fourier transform, we take the representation (5.9) and (5.10), integrate over the mass parameter, and rewrite the mass integral as an integral over ω,

$$\begin{aligned}(\mathfrak{p}\psi)(t) &= \sum_{s=\pm} \int_I dm \int_{\sigma(\mathcal{D}_A)} e^{-sit\omega(\lambda,m)}\, \Pi_s(\lambda,m)\, dF_\lambda\, \psi_m(0) \\ &= \sum_{s=\pm} \int_0^\infty \frac{dm}{d\omega}\, d\omega \int_{\sigma(\mathcal{D}_A)} \chi_{(m_L^2,m_R^2)}\big(\omega^2-\lambda^2\big) \\ &\qquad\qquad \times e^{-si\omega t}\, \Pi_s(\lambda,m)\, dF_\lambda\, \psi_m(0)\Big|_{m=\sqrt{\omega^2-\lambda^2}}\,,\end{aligned}$$

where the characteristic function $\chi_{(m_L^2,m_R^2)}\big(\omega^2-\lambda^2\big)$ vanishes unless $\sqrt{\omega^2-\lambda^2} \in I$. Using (5.7), we obtain

$$\begin{aligned}&= \int_{-\infty}^\infty d\omega\, e^{-i\omega t}\, \frac{|\omega|}{m} \int_{\sigma(\mathcal{D}_A)} \chi_{(m_L^2,m_R^2)}\big(\omega^2-\lambda^2\big) \\ &\qquad\qquad \times \Pi_s(\lambda,m)\, dF_\lambda\, \psi_m(0)\Big|_{m=\sqrt{\omega^2-\lambda^2},\ s=\text{sign}(\omega)}\,.\end{aligned}$$

This shows that

$$\begin{aligned}\widehat{\mathfrak{p}\psi}(\omega) &= 2\pi\, \frac{|\omega|}{m} \int_{\sigma(\mathcal{D}_A)} \chi_{(m_L^2,m_R^2)}\big(\omega^2-\lambda^2\big) \\ &\qquad\qquad \times \Pi_s(\lambda,m)\, dF_\lambda\, \psi_m(0)\Big|_{m=\sqrt{\omega^2-\lambda^2},\ s=\text{sign}(\omega)}\,.\end{aligned}$$

Using this formula in (5.24) and applying the spectral calculus for $\mathcal{D}_A$, we obtain

$$\begin{aligned}&<\mathfrak{p}\psi|\mathfrak{p}\phi> = 2\pi \int_{-\infty}^\infty d\omega\, \frac{\omega^2}{m^2} \int_{\sigma(\mathcal{D}_A)} \chi_{(m_L^2,m_R^2)}\big(\omega^2-\lambda^2\big) \\ &\times \bigg\langle \Pi_s(\lambda,m)\psi_m(0) \bigg| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dF_\lambda \Pi_s(\lambda,m)\phi_m(0) \bigg\rangle_{L^2(\mathcal{N},S\mathcal{N})^2} \bigg|_{m=\sqrt{\omega^2-\lambda^2}, s=\text{sign}(\omega)}\,.\end{aligned}$$

A short computation using (5.8) shows that

$$\Pi_s(\lambda,m) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Pi_s(\lambda,m) = \frac{m}{s\,|\omega|}\, \Pi_s(\lambda,m)\,.$$

Hence

$$
\begin{aligned}
<\mathfrak{p}\psi|\mathfrak{p}\phi> &= 2\pi \int_{-\infty}^{\infty} d\omega\, \frac{\omega}{m} \int_{\sigma(\mathcal{D}_A)} \chi_{(m_L^2, m_R^2)}\big(\omega^2 - \lambda^2\big) \\
&\qquad \times \big\langle \psi_m(0) \big| dF_\lambda\, \Pi_s(\lambda, m)\, \phi_m(0) \big\rangle_{L^2(\mathcal{N}, S\mathcal{N})^2} \Big|_{m=\sqrt{\omega^2-\lambda^2},\; s=\text{sign}(\omega)} \\
&= 2\pi \int_I dm \sum_{s=\pm} s \int_{\sigma(\mathcal{D}_A)} d\big\langle \psi_m(0) \,\big|\, F_\lambda\, \Pi_s(\lambda, m)\, \phi_m(0) \big\rangle_{L^2(\mathcal{N}, S\mathcal{N})^2} \,,
\end{aligned}
$$

where in the last step we transformed back to the integration variable m. Using (5.6), we obtain the identity

$$
<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I dm \int_{\sigma(\mathcal{D}_A)} d\big(\psi_m \,\big|\, F_\lambda\, \big(\Pi_+(\lambda, m) - \Pi_-(\lambda, m)\big)\, \phi_m\big)_m \,. \tag{5.25}
$$

The inequality (4.1) follows immediately by applying the Schwarz inequality and using that the matrices $\Pi_\pm$ have norm one.

Finally, comparing (5.25) with (4.4), one sees that the eigenvalues and corresponding eigenspaces of the operator $\mathcal{S}_m$ coincide precisely with those of the matrix $\Pi_+ - \Pi_-$. Hence $\mathcal{S}_m$ has the spectrum $\{\pm 1\}$, and in view of (5.10) the eigenspaces are precisely the subspaces of positive and negative frequency. This completes the proof of Theorem 5.1.

6. Example: de Sitter space-time

We consider the de Sitter space-time $\mathscr{M} = \mathbb{R} \times S^3$ with the line element

$$
ds^2 = dt^2 - R(t)^2\, ds^2_{S^3} \qquad \text{and} \qquad R(t) = \cosh t \,, \tag{6.1}
$$

where $ds^2_{S^3}$ is the line element on the three-dimensional unit sphere. This is a special case of the Friedmann-Robertson-Walker metric obtained for a specific choice of the scaling function. The Dirac operator was computed in [8] (see also [7, Appendix A]) to be

$$
\mathcal{D} = i\gamma^0 \left(\partial_t + \frac{3\dot{R}(t)}{2R(t)} \right) + \frac{1}{R(t)} \begin{pmatrix} 0 & \mathcal{D}_{S^3} \\ -\mathcal{D}_{S^3} & 0 \end{pmatrix} , \tag{6.2}
$$

where $\mathcal{D}_{S^3}$ is the Dirac operator on S^3. The inner products (2.1) and (2.3) take the form

$$<\psi|\phi> = \int_{-\infty}^{\infty} dt \int_{S^3} \prec\psi|\phi\succ(t,x)\, R(t)^3\, d\mu_{S^3}(x) \tag{6.3}$$

$$(\psi_m|\phi_m)_m = 2\pi \int_{S^3} \prec\psi|\gamma^0\phi\succ(t,x)\, R(t)^3\, d\mu_{S^3}(x)\,, \tag{6.4}$$

where $\prec\psi|\phi\succ = \psi^\dagger\gamma^0\phi$ and $\gamma^0 = \mathrm{diag}(1,1,-1,-1)$ (here $d\mu_{S^3}$ is the normalized volume measure on S^3).

In order to separate the Dirac equation (2.2), one uses that, being an elliptic operator on a bounded domain, the Dirac operator on S^3 has a purely discrete spectrum and finite-dimensional eigenspaces. More specifically, the eigenvalues are (see [14] or the detailed computations in [8, Appendix A]),

$$\sigma(\mathcal{D}_{S^3}) = \left\{ \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}, \ldots \right\}$$

with corresponding eigenspaces of dimensions

$$\dim\ker(\mathcal{D}_{S^3} - \lambda) = \lambda^2 - \frac{1}{4}\,.$$

Since the Dirac operator (6.2) obviously commutes with $\mathcal{D}_{S^3}$, the solution spaces can be decomposed into eigenspaces of $\mathcal{D}_{S^3}$. We use the notation

$$\mathcal{H}_m = \bigoplus_{\lambda\in\sigma(\mathcal{D}_{S^3})} \mathcal{H}_m^{(\lambda)}\,, \qquad \mathcal{H} = \bigoplus_{\lambda\in\sigma(\mathcal{D}_{S^3})} \mathcal{H}^{(\lambda)}\,.$$

We also refer to the eigenspaces of $\mathcal{D}_{S^3}$ as *spatial modes*. Next, we choose $\mathcal{H}^\infty$ as the proper subspace of $C^\infty_{\mathrm{sc},0}(\mathscr{M}\times S, S\mathscr{M})\cap\mathcal{H}$ of solutions composed of a *finite number of spatial modes*,

$$\mathcal{H}^\infty = \left\{ \psi \in C^\infty_{\mathrm{sc},0}(\mathscr{M}\times S, S\mathscr{M})\cap\mathcal{H} \;\middle|\; \psi \in \bigoplus\nolimits_{|\lambda|\leq\Lambda} \mathcal{H}^{(\lambda)} \text{ with } \Lambda\in\mathbb{R} \right\} \tag{6.5}$$

(this choice clearly has all the properties demanded in Definition 3.1; the reason for this choice will be explained after Lemma 6.2 below). This is our main result:

Theorem 6.1. *On any interval $I = (m_L, m_R)$ with $m_L, m_R > 0$, the Dirac operator in the de Sitter space-time has the strong mass oscillation property with domain* (6.5).

The remainder of this section is devoted to the proof of this theorem. Choosing a normalized eigenspinor $\phi^{(\lambda)}$ of $\mathcal{D}_{S^3}$ corresponding to the eigenvalue λ, we employ the ansatz

$$\psi_m = R(t)^{-\frac{3}{2}} \begin{pmatrix} u_1(m,t)\, \phi^{(\lambda)}(x) \\ u_2(m,t)\, \phi^{(\lambda)}(x) \end{pmatrix} \tag{6.6}$$

to obtain the coupled system of ordinary differential equations

$$i\,\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} m & -\lambda/R \\ -\lambda/R & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{6.7}$$

for the complex-valued functions u_1 and u_2. The inner products (6.3) and (6.4) become

$$<\psi|\tilde{\psi}> = \int_{-\infty}^{\infty} (\overline{u_1}\tilde{u}_1 - \overline{u_2}\tilde{u}_2)\, dt \tag{6.8}$$

$$(\psi_m|\phi_m)_m = 2\pi\, (\overline{u_1}\tilde{u}_1 + \overline{u_2}\tilde{u}_2) = 2\pi \langle u, \tilde{u}\rangle_{\mathbb{C}^2}\,. \tag{6.9}$$

Using that the matrix in (6.7) is Hermitian, one easily verifies that

$$\frac{d}{dt} \langle u, \tilde{u}\rangle_{\mathbb{C}^2} = 0\,, \tag{6.10}$$

showing that the scalar product (6.9) is indeed time independent. We refer to a wave function of the form (6.6) as a *single spatial mode*.

The asymptotics of solutions of (6.7) for large times can be described with a simple Grönwall-type estimate:

Lemma 6.2. *Asymptotically as $t \to \infty$, every solution of (6.7) is of the form*

$$u(t) = \begin{pmatrix} e^{-imt}\, f_1^{\pm} \\ e^{imt}\, f_2^{\pm} \end{pmatrix} + E^{\pm}(t) \tag{6.11}$$

with the error term bounded by

$$\|E^{\pm}(t)\| \leq \|f^{\pm}\| \Big(\exp\big(2\,|\lambda|\, e^{\mp t}\big) - 1 \Big) \tag{6.12}$$

(thus $E^{\pm}(t)$ decays exponentially as $t \to \pm\infty$).

Proof. Substituting into (6.7) the ansatz

$$u(t) = \begin{pmatrix} e^{-imt}\, f_1(t) \\ e^{imt}\, f_2(t) \end{pmatrix} , \tag{6.13}$$

we obtain for f the differential equation

$$\frac{df}{dt} = \frac{\lambda}{R} \begin{pmatrix} 0 & e^{2imt} \\ e^{-2imt} & 0 \end{pmatrix} f . \tag{6.14}$$

Taking the norm, we obtain the differential inequality

$$\left\| \frac{df}{dt} \right\| \leq \frac{|\lambda|}{R}\, \|f\| . \tag{6.15}$$

Let us first show that $f(t)$ has a limit as $t \to \pm\infty$. To this end, we first apply Kato's inequality to (6.15),

$$\frac{d}{dt}\, \|f\| \leq \frac{|\lambda|}{R}\, \|f\| . \tag{6.16}$$

We may assume that our solution is nontrivial, so that $\|f\| \neq 0$. Thus we may divide by $\|f\|$,

$$\frac{d}{dt} \log \|f\| \leq \frac{|\lambda|}{R} .$$

Since the scaling function grows exponentially for large t (cf. (6.1)), we conclude that $\|f\|$ is bounded and converges as $t \to \pm\infty$. Using this a-priori bound in (6.15), we infer that f has bounded variation, implying that $\lim_{t\to\pm\infty} f$ exists. We set

$$f^\pm = \lim_{t \to \pm\infty} f(t) .$$

In order to estimate $\|f - f^-\|$, we divide (6.16) by $\|f\|$ and integrate from t_0 to any $t > t_0$,

$$\|f(t)\| \leq \|f(t_0)\| \exp\left(\int_{t_0}^t \frac{|\lambda|}{R(\tau)}\, d\tau \right) .$$

Substituting this inequality into (6.15) gives

$$\left\| \frac{df}{dt} \right\| \leq \frac{|\lambda|}{R(t)}\, \|f(t_0)\| \exp\left(\int_{t_0}^t \frac{|\lambda|}{R(\tau)}\, d\tau \right) = \|f(t_0)\|\, \frac{d}{dt} \exp\left(\int_{t_0}^t \frac{|\lambda|}{R(\tau)}\, d\tau \right) .$$

Integrating on both sides from t_0 to some $t > t_0$ gives

$$\|f(t) - f(t_0)\| \leq \|f(t_0)\| \, \exp\left(\int_{t_0}^t \frac{|\lambda|}{R(\tau)}\, d\tau \right) .$$

Now we take the limit $t_0 \to -\infty$ to obtain

$$\big\|f(t) - f^-\big\| \leq \|f(t_0)\| \, \exp\left(\int_{-\infty}^t \frac{|\lambda|}{R(\tau)}\, d\tau \right) .$$

Employing the estimate

$$R(t) = \cosh t \geq \frac{e^{-t}}{2} ,$$

we conclude that

$$\big\|f(t) - f^-\big\| \leq \|f(t_0)\| \, \exp\big(2|\lambda|\, e^t\big) .$$

Using this estimate in (6.13) and comparing with (6.11) gives the desired estimate for E^-.

The estimate for E^+ is derived similarly. □

As is typical for a Grönwall estimate, the error bound (6.12) grows exponentially in λ. In particular, our estimate is not uniform in the spatial modes. It is not clear how to improve this estimate to for example polynomial growth in λ. This is the reason why with the choice (6.5) we always restrict attention to a finite number of spatial modes.

Lemma 6.3. *For every single mode $\psi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}^{(\lambda)}$, the corresponding coefficients $f^\pm$ in* (6.11) *are smooth in m,*

$$f^\pm \in C^\infty_0(I, \mathbb{C}^2) .$$

Proof. Evaluating ψ at time $t = 0$, we get a smooth family $u(m, t = 0)$. Consequently, the function f is smooth,

$$f|_{t=0} \in C^\infty_0(I, \mathbb{C}^2) .$$

Taking these initial conditions and solving the equation (6.14), we get a family of solutions which clearly depend smoothly on m. Differentiating (6.14) with respect to the mass and setting $f^{(p)} := \partial_m^p f$, we obtain

$$\frac{df^{(1)}}{dt} = \frac{\lambda}{R} \begin{pmatrix} 0 & e^{2imt} \\ e^{-2imt} & 0 \end{pmatrix} f^{(1)} + 2it\, \frac{\lambda}{R} \begin{pmatrix} 0 & e^{2imt} \\ -e^{-2imt} & 0 \end{pmatrix} f$$

and thus

$$\left\| \frac{df^{(1)}}{dt} \right\| \leq \frac{|\lambda|}{R}\, \|f^{(1)}\| + 2|t|\, \frac{|\lambda|}{R}\, \|f\| \,. \tag{6.17}$$

Again applying the Kato inequality, we obtain similar to (6.16) the differential inequality

$$\frac{d}{dt} \left(e^{-\int^t \frac{|\lambda|}{R}}\, \|f^{(1)}(t)\| \right) \leq e^{-\int^t \frac{|\lambda|}{R}}\, 2|t|\, \frac{|\lambda|}{R(t)}\, \|f(t)\| \,.$$

Integrating on both sides and using the exponential growth of $R(t)$ at infinity, we conclude that $\|f^{(1)}(t)\|$ converges as $t \to \pm\infty$. Using this fact in (6.17), we infer that also the vector $f^{(1)}(t)$ converges as $t \to \pm\infty$. The higher derivatives $f^{(p)}$ can be estimated inductively by differentiating (6.14) p times with respect to m, taking the norm, and integrating the resulting differential inequality. □

Lemma 6.4. *For every single mode $\psi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}^{(\lambda)}$, the corresponding function u in* (6.6) *satisfies the inequality*

$$\big\| (\mathfrak{p} u)(t) \big\| \leq \frac{c(\psi)}{1+t^2} \,. \tag{6.18}$$

Proof. Integrating the asymptotic expansion (6.11) over the mass parameter gives

$$\mathfrak{p} u = \int_I \begin{pmatrix} e^{-imt}\; f_1^\pm \\ e^{imt}\; f_2^\pm \end{pmatrix} dm + \int_I E^\pm\, dm \,. \tag{6.19}$$

The integral over the error term can be estimated by

$$\left\| \int_I E^\pm\, dm \right\| \leq |I|\, \sup_{m \in I} \|E^\pm\| \leq c(\psi)\, e^{\mp t} \,, \tag{6.20}$$

where in the last step we applied (6.12) and used that $\sup_m \|f^\pm\|$ is bounded by Lemma 6.3. Writing the first summand in (6.19) as

$$\int_I \begin{pmatrix} e^{-imt}\, f_1^\pm(t) \\ e^{imt}\, f_2^\pm(t) \end{pmatrix} dm = -\frac{1}{t^2} \int_I \left[\frac{d^2}{dm^2} \begin{pmatrix} e^{-imt} & 0 \\ 0 & e^{imt} \end{pmatrix} \right] \begin{pmatrix} f_1^\pm \\ f_2^\pm \end{pmatrix} dm \,,$$

we can integrate by parts to obtain the estimate

$$\left\| \int_I \begin{pmatrix} e^{-imt}\, f_1^\pm \\ e^{imt}\, f_2^\pm \end{pmatrix} dm \right\| \leq \frac{|I|}{t^2} \sup_{m \in I} \|\partial_m^2 f^\pm\| \leq \frac{c(\psi)}{t^2} \,.$$

Combining this estimate with (6.20), we obtain (6.18). □

Proposition 6.5. *The Dirac operator in the de Sitter space-time has the weak mass oscillation property with domain* (6.5)*.*

Proof. Suppose that $\psi, \phi \in \mathcal{H}^\infty$. Since ψ and ϕ only involve a finite number of spatial modes, we may restrict attention to one of them. Moreover, using orthonormality of the spatial eigenfunctions, we may assume that ϕ and ψ have the same spatial dependence. Then the Schwarz inequality yields

$$\|(\mathfrak{p}\phi)|_t\| \leq \int_I \|\phi(m)\|_m \, dm \leq \sqrt{|I|}\, \|\phi\| \,.$$

Combining this inequality with (6.3), (6.4) and Lemma 6.4, we obtain

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| \leq \sqrt{|I|}\, \|\phi\| \int_{-\infty}^{\infty} \|(\mathfrak{p}u)(t)\|\, dt \leq \|\phi\| \int_{-\infty}^{\infty} \frac{c}{1+t^2}\, dt \,,$$

proving (3.5).

The property (3.6) follows by integrating the Dirac operator by parts according to (5.23), where in $(*)$ we again use that the Dirac operator is formally self-adjoint with respect to $<.|.>$. Moreover, we do not get boundary terms in view of the time decay in (6.18). □

Our next task is to compute the inner product $<\mathfrak{p}\psi|\mathfrak{p}\tilde{\psi}>$ for two single modes $\psi, \tilde{\psi} \in \mathcal{H}^{(\lambda)}$ with the same spatial dependence. We write the result of Lemma 6.2 as

$$u(m,t) = \Theta(t) \begin{pmatrix} e^{-imt}\, f_1^+(m) \\ e^{imt}\, f_2^+(m) \end{pmatrix} + \Theta(-t) \begin{pmatrix} e^{-imt}\, f_1^-(m) \\ e^{imt}\, f_2^-(m) \end{pmatrix} + E_m(t)\,, \tag{6.21}$$

where Θ is the Heaviside function, and the error term decays exponentially as $t \to \pm\infty$,

$$\|E_m(t)\| \leq c\, e^{-|t|}\,. \tag{6.22}$$

For the function $\tilde{\psi}$ we use the same notation with an additional tilde.

Lemma 6.6. *For any single modes $\psi, \tilde{\psi} \in \mathcal{H}^{(\lambda)}$ with the same spatial dependence,*

$$<\mathfrak{p}\psi|\mathfrak{p}\tilde{\psi}> = \pi \sum_{s=\pm} \int_I \Big(\overline{f_1^s(m)}\, \tilde{f}_1^s(m) - \overline{f_2^s(m)}\, \tilde{f}_2^s(m) \Big)\, dm\,. \tag{6.23}$$

Proof. We first explain why the error terms $E_m(t)$ and $\tilde{E}_{\tilde{m}}(t)$ do not enter the formula. To this end, we again use the partition of unity $(\eta_\ell)_{\ell=1,\ldots,N}$ introduced in the proof of Theorem 4.2 (see (4.5)). Since we already know that the weak mass oscillation property holds, we conclude from (3.6) inductively that $<\mathfrak{p}T^p\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T^p\phi>$ for all p. Using the continuous functional calculus corresponding to the spectral theorem (3.11), we conclude that

$$<\mathfrak{p}\, \eta_\ell(T)\psi \,|\, \mathfrak{p}\, \eta_{\ell'}(T)\tilde{\psi}> = <\mathfrak{p}\, \big(\eta_\ell\, \eta_{\ell'}\big)(T)\, \psi \,|\, \mathfrak{p}\tilde{\psi}>\,,$$

which implies by the right side of (4.5) that

$$<\mathfrak{p}\, \eta_\ell(T)\psi \,|\, \mathfrak{p}\, \eta_{\ell'}(T)\tilde{\psi}> = 0 \qquad \text{unless } |\ell - \ell'| \leq 1\,.$$

Estimating the integrals of the error terms by

$$\left\| \int_I \eta_\ell(m)\, E_m(t) \right\| \leq \frac{|I|}{N}\, \sup_m \|E_m(t)\|$$

and using the bound (6.22), the contribution by the error terms tends to zero as $N \to \infty$.

It remains to consider the first two summands in (6.21). Because of the Heaviside functions, we only get the product of f_a^+ with $\tilde{f}_b^+$ and of f_a^-

with $\tilde{f}_b^-$. Moreover, the following argument shows why it suffices to consider the contributions where the lower indices coincide: For example, the contribution involving f_1^+ and $\tilde{f}_2^+$ is

$$A := \int_0^\infty dt \iint_{I\times I} \overline{f_1^+(m)}\, \tilde{f}_2^+(\tilde{m})\, e^{i(m+\tilde{m})t}\, dm\, d\tilde{m}\,.$$

Using the distributional equation

$$\int_0^\infty e^{i\omega t} = \pi\delta(\omega) + i\,\frac{\mathrm{PP}}{\omega} \tag{6.24}$$

(where PP denotes the principal value of the integral), we can use the fact that $m+\tilde{m}$ is bounded away from zero to obtain

$$A = \iint_{I\times I} \frac{\overline{f_1^+(m)}\tilde{f}_2^+(m)}{m+\tilde{m}}\, dm\, d\tilde{m}\,.$$

Again inserting the partition of unity $(\eta_\ell)_{\ell=1,\ldots,N}$ and taking the limit $N\to\infty$ gives zero. The other contributions for $a\neq b$ are treated similarly.

We conclude that it suffices to take into account the products of the form f_a^s with $\tilde{f}_a^s$ with $a=1,2$ and $s=\pm 1$. Thus

$$\begin{aligned}
&<\!\mathfrak{p}\psi|\mathfrak{p}\tilde{\psi}\!> \\
&= \int_0^\infty dt \iint_{I\times I} \Big(e^{i(m-m')t}\, \overline{f_1^+(m)}\, \tilde{f}_1^+(\tilde{m}) - e^{-i(m-m')t}\, \overline{f_2^+(m)}\, \tilde{f}_2^+(\tilde{m}) \Big) dm\, d\tilde{m} \\
&+ \int_{-\infty}^0 dt \iint_{I\times I} \Big(e^{i(m-m')t}\, \overline{f_1^-(m)}\, \tilde{f}_1^-(\tilde{m}) - e^{-i(m-m')t}\, \overline{f_2^-(m)}\, \tilde{f}_2^-(\tilde{m}) \Big) dm\, d\tilde{m}\,,
\end{aligned}$$

and applying (6.24) gives

$$<\!\mathfrak{p}\psi|\mathfrak{p}\tilde{\psi}\!> = \pi \int_I \Big(\overline{f_1^+}\tilde{f}_1^+ - \overline{f_2^+}\tilde{f}_2^+ + \overline{f_1^-}\,\tilde{f}_1^- - \overline{f_2^-}\,\tilde{f}_2^- \Big)\Big|_m\, dm \tag{6.25}$$

$$+\, i \iint_{I\times I} \frac{\mathrm{PP}}{m-\tilde{m}} \Big(\overline{f_1^+(m)}\, \tilde{f}_1^+(\tilde{m}) + \overline{f_2^+(m)}\, \tilde{f}_2^+(\tilde{m}) \tag{6.26}$$

$$- \overline{f_1^-(m)}\, \tilde{f}_1^-(\tilde{m}) - \overline{f_2^-(m)}\, \tilde{f}_2^-(\tilde{m}) \Big)\, dm\, d\tilde{m}\,. \tag{6.27}$$

Using current conservation (6.10) together with (6.11), we may evaluate the scalar product asymptotically as $t\to\pm\infty$ to obtain

$$\sum_{a=1,2} \overline{f_a^+(m)}\, \tilde{f}_a^+(\tilde{m}) = \sum_{a=1,2} \overline{f_a^-(m)}\, \tilde{f}_a^-(\tilde{m})\,.$$

This implies that the terms (6.26) and (6.27) cancel each other, giving the result. □

Proof of Theorem 6.1. Suppose that $\psi, \tilde{\psi} \in \mathcal{H}^\infty$. Then we decompose them into spatial modes, i.e.

$$\psi = \sum_{|\lambda|<|\Lambda|} \sum_{k=1}^{K(\lambda)} R(t)^{-\frac{3}{2}} \begin{pmatrix} (u_1)_k^{(\lambda)}(m,t)\, \phi_k^{(\lambda)}(x) \\ (u_2)_k^{(\lambda)}(m,t)\, \phi_k^{(\lambda)}(x) \end{pmatrix},$$

and similarly for $\tilde{\psi}$. Choosing the spatial wave functions $\phi_k^{(\lambda)}$ to be orthonormal, we can apply Lemma 6.6 to each mode to obtain

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\tilde{\psi}{>}| \leq \pi \sum_{|\lambda|<|\Lambda|} \sum_{k=1}^{K(\lambda)} \sum_{s=\pm} \int_I \big\|(f^s)_k^{(\lambda)}(m)\big\|\, \big\|(\tilde{f}^s)_k^{(\lambda)}(m)\big\|\, dm\,.$$

Using current conservation (6.10), we can compute the norms of $u_k^{(\lambda)}$ and $\tilde{u}_k^{(\lambda)}$ asymptotically as $t \to \pm\infty$ with the help of Lemma 6.2. This gives

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\tilde{\psi}{>}| \leq 2\pi \sum_{|\lambda|<|\Lambda|} \sum_{k=1}^{K(\lambda)} \int_I \big\|u_k^{(\lambda)}(m)\big\|\, \big\|\tilde{u}_k^{(\lambda)}(m)\big\|\, dm\,.$$

Applying the Schwarz inequality gives the result. □

We finally explain what the result of Lemma 6.6 means for the decomposition of the solution space. Comparing (6.23) with (4.4), one sees that now the spectral subspaces of the fermionic signature operator $\mathcal{S}_m$ no longer coincide with the solutions of positive and negative frequency. This is also clear because in the time-dependent setting of the de Sitter space-time, the "frequency" of a solution is only defined asymptotically as $t \to \pm\infty$, but not globally or at intermediate times. Instead, the sum over s in (6.21) corresponds to the fact that we must take a suitable "interpolation" of the frequency splittings as experienced by observers at asymptotic times $t \to \pm\infty$. Here the notion of "interpolation" can be understood similar as explained in [4, Section 5] and [9, Section 6].

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References

[1] C. Bär, N. Ginoux, and F. Pfäffle, *Wave Equations on Lorentzian Manifolds and Quantization*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2007.

[2] H. Baum, *Spinor structures and Dirac operators on pseudo-Riemannian manifolds*, Bull. Polish Acad. Sci. Math. **33** (1985), no. 3-4, 165–171.

[3] J. Dimock, *Dirac quantum fields on a manifold*, Trans. Amer. Math. Soc. **269** (1982), no. 1, 133–147.

[4] F. Finster, *Definition of the Dirac sea in the presence of external fields*, Adv. Theor. Math. Phys. **2** (1998), no. 5, 963–985. `arXiv:hep-th/9705006`.

[5] F. Finster, *The Principle of the Fermionic Projector*, AMS/IP Studies in Advanced Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2006. `arXiv:hep-th/0001048, hep-th/0202059, hep-th/0210121`.

[6] F. Finster and A. Grotz, *The causal perturbation expansion revisited: Rescaling the interacting Dirac sea*, J. Math. Phys. **51** (2010), 072301. `arXiv:0901.0334 [math-ph]`.

[7] F. Finster and C. Hainzl, *A spatially homogeneous and isotropic Einstein-Dirac Cosmology*, J. Math. Phys. **52** (2011), 042501. `arXiv:1101.1872 [math-ph]`.

[8] F. Finster and M. Reintjes, *The Dirac equation and the normalization of its solutions in a closed Friedmann-Robertson-Walker universe*, Class. Quantum Grav. **26** (2009), no. 10, 105021. `arXiv:0901.0602 [math-ph]`.

[9] F. Finster and M. Reintjes, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime*, Adv. Theor. Math. Phys. **19** (2015), no. 4, 761–803. `arXiv:1301.5420 [math-ph]`.

[10] F. Finster and C. Röken, *Self-adjointness of the Dirac Hamiltonian for a class of non-uniformly elliptic boundary value problems*, Annals of Mathematical Sciences and Applications **1** (2016), no. 2, 301–320. `arXiv:1512.00761 [math-ph]`.

[11] F. Finster and J. Tolksdorf, *Perturbative description of the fermionic projector: Normalization, causality and Furry's theorem*, J. Math. Phys. **55** (2014), no. 5, 052301. `arXiv:1401.4353 [math-ph]`.

[12] P. R. Halmos, *Measure Theory*, Springer, New York, 1974.

[13] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 256, Springer-Verlag, Berlin, 1990.

[14] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.

[15] P. D. Lax, *Functional Analysis*, Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2002.

[16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I, Functional analysis*, second ed., Academic Press Inc., New York, 1980.

[17] W. Rudin, *Real and Complex Analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[18] M. E. Taylor, *Partial Differential Equations. II*, second ed., Applied Mathematical Sciences, vol. 116, Springer, New York, 2011.

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