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Quasilocal angular momentum and center of mass in general relativity

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For a spacelike 2-surface in spacetime, we propose a new definition of quasi-local angular momentum and quasi-local center of mass, as an element in the dual space of the Lie algebra of the Lorentz group. Together with previous defined quasi-local energy-momentum, this completes the definition of conserved quantities in general relativity at the quasi-local level. We justify this definition by showing the consistency with the theory of special relativity and expectations on an axially symmetric spacetime. The limits at spatial infinity provide new definitions for total conserved quantities of an isolated system, which do not depend on any asymptotically flat coordinate system or asymptotic Killing field. The new proposal is free of ambiguities found in existing definitions and presents the first definition that precisely describes the dynamics of the Einstein equation.

1. Introduction

In [14], Penrose considered the question of how to define quasi-local energy-momentum and quasi-local angular momentum to be important in classical general relativity. In other words, given a space-like two dimensional surface in a space-time, we would like to associate a four-vector that represents the energy-momentum, and a quantity that measures the angular momentum of the region enclosed by the surface. For the definition of quasi-local mass, Penrose, Hawking, and others have made important contributions, see [17] and the references therein. The work of Wang-Yau [18] on quasi-local energy-momentum is the most satisfactory one. However, the definition of quasi-local angular momentum is more complicated.

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In fact, for an asymptotically flat space-time, there was the definition of total angular momentum due to Arnowitt-Deser-Misner, and studied by Ashtekar-Hansen, Regge-Teitelboim, [1, 2, 16], etc. Closely related to angular momentum is the concept of center of mass. Assuming the space-time is asymptotically flat, there were various definitions proposed by Huisken-Yau, Regge-Teitelboim, Beig-ÓMurchadha, Christodoulou, and Schoen [3, 9, 12, 13, 16]. All these definitions depend heavily on the asymptotically flat coordinate or asymptotic Killing field. In particular, there are issues of finiteness, well-definedness, and physical validity among these definitions. The definitions by various references are not completely consistent.

Following the approach given by Wang-Yau on quasi-local energy-momentum, we propose a new definition of quasi-local angular momentum and center of mass, which does not rely on special coordinates or Killing fields. Based on this we manage to resolve the above-mentioned classical problems in defining total angular momentum for asymptotically flat space-times. At both the quasi-local and total level, the definitions satisfy many highly desirable properties, and capture the dynamics of Einstein equations.

2. Review of quasi-local energy-momentum

In order to be consistent with our previous work, we first review the definition of quasi-local energy-momentum in [18]. Consider a spacetime region M that is foliated by a family of spacelike hypersurface Ω_t for t in the time interval $[t', t'']$. The boundary of M consists of $\Omega_{t'}$, $\Omega_{t''}$, and ${}^3B$. Let u^μ denote the future pointing timelike unit normal to Ω_t. Assume u^μ is tangent to ${}^3B$. Denote the boundary of Ω_t by Σ_t which is the intersection of Ω_t and ${}^3B$. Let v^μ denote the outward pointing spacelike unit normal of Σ_t such that $u_\mu v^\mu = 0$. Denote by k the trace of the 2-dimensional extrinsic curvature of Σ_t in Ω_t in the direction of v^μ. Denote the Riemannian metric, the extrinsic curvature, and the trace of the extrinsic curvature on Ω_t by $g_{\mu\nu}$, $K_{\mu\nu} = \nabla_\mu u_\nu$, and $K = g^{\mu\nu} K_{\mu\nu}$, respectively. Let t^μ be a timelike vector field satisfying $t^\mu \nabla_\mu t = 1$. t^μ can be decomposed into the lapse function and shift vector $t^\nu = N u^\nu + N^\nu$. Let S denote the action for M, then the Hamiltonian at t'' is given by $\mathfrak{H} = -\frac{\partial S}{\partial t''}$. The calculation in Brown-York [4] (see also Hawking-Horowitz [11]) leads to

$$\mathfrak{H} = -\frac{1}{8\pi} \int_{\Sigma_{t''}} [Nk - N^\mu v^\nu (K_{\mu\nu} - K g_{\mu\nu})] \tag{1}$$

on a solution M of the Einstein equation. To define quasilocal energy, we need to find a reference action S_0 that corresponds to fixing the metric on ${}^3B$ and compute the corresponding reference Hamiltonian $\mathfrak{H}_0$. The energy is then $E = -\frac{\partial}{\partial t''}(S - S_0) = \mathfrak{H} - \mathfrak{H}_0$. We defined quasi-local energy using general isometric embeddings into $\mathbb{R}^{3,1}$ as reference configurations. In particular, t^ν is obtained by transplanting a Killing vector field in $\mathbb{R}^{3,1}$ to the physical spacetime through the embedding.

Suppose Σ is a spacelike surface in a time-orientable spacetime M, u^ν is a future-pointing timelike unit normal, and v^ν is a spacelike unit normal with $u^\nu v_\nu = 0$ along Σ. We assume v^ν is the outward normal of a spacelike hypersurface Ω that is defined locally near Σ. We recall the *mean curvature vector field* $h^\nu = -kv^\nu + pu^\nu$ in [18] and its companion $j^\nu = ku^\nu - pv^\nu$. Both are normal vector fields along Σ which are defined independent of the choice of u^ν and v^ν.

Consider a reference isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ of Σ. Fix a constant timelike unit vector t_0^ν in $\mathbb{R}^{3,1}$, and choose a preferred pair of normals u_0^ν and v_0^ν along $i(\Sigma)$ in the following way: Take a spacelike hypersurface Ω_0 with $\partial\Omega_0 = i(\Sigma)$ and such that the outward pointing spacelike unit normal v_0^ν of $\partial\Omega_0$ satisfies $(t_0)_\nu v_0^\nu = 0$. Let u_0^ν be the future pointing timelike unit normal of Ω_0 along $i(\Sigma)$. We can similarly form h_0^ν and j_0^ν in terms of the corresponding geometric quantities on Ω_0 and $i(\Sigma)$. $({u_0}^\nu, {v_0}^\nu)$ along $i(\Sigma)$ in $\mathbb{R}^{3,1}$ is the reference normal gauge we shall fix, and it depends on the choice of the pair (i, t_0^ν).

When the mean curvature vector h^ν of Σ in M is spacelike, a reference isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ and $t_0^\nu \in \mathbb{R}^{3,1}$ determine a canonical future-pointing timelike normal vector field $\bar{u}^\nu$ in M along Σ. Indeed, there is a unique $\bar{u}^\nu$ that satisfies

$$h_\nu \bar{u}^\nu = (h_0)_\nu {u_0}^\nu \tag{2}$$

where h_0^ν is the mean curvature vector of $i(\Sigma)$ in $\mathbb{R}^{3,1}$. Physically, (2) means the expansions of $\Sigma \subset M$ and $i(\Sigma) \subset \mathbb{R}^{3,1}$ along the respective directions $\bar{u}^\nu$ and ${u_0}^\nu$ are the same. Take $\bar{v}^\nu$ to be the spacelike normal vector that is orthogonal to $\bar{u}^\nu$ and satisfies $\bar{v}^\nu h_\nu < 0$, and take a spacelike hypersurface $\bar{\Omega}$ in M such that $\bar{v}^\nu$ is the outward normal. We can similarly form $\bar{K}_{\mu\nu}$ and $\bar{k}$ from the corresponding data on $\bar{\Omega}$. The trace of the 2-dimensional extrinsic curvature $\bar{k}$ of Σ with respect to $\bar{v}^\nu$ is then given by $\bar{k} = -\bar{v}^\nu h_\nu > 0$.

4-vectors in $\mathbb{R}^{3,1}$ and M, along $i(\Sigma)$ and Σ respectively, can be identified through

$${u_0}^\nu \to \bar{u}^\nu, {v_0}^\nu \to \bar{v}^\nu, \tag{3}$$

and the identification of tangent vectors on $i(\Sigma)$ and Σ. t_0^ν in $\mathbb{R}^{3,1}$ is identified with $N_0\bar{u}^\nu + N_0^\nu$ in M. In terms of the lapse N_0 and shift N_0^ν, the quasi-local energy with respect to (i, t_0^ν) (equation (6) in [18]) is

$$\frac{1}{8\pi}\int_\Sigma (k_0 - \bar{k})N_0 - (v_0^\mu (K_0)_{\mu\nu} - \bar{v}^\mu \bar{K}_{\mu\nu})N_0^\nu. \tag{4}$$

(4) can be expressed in terms of the norm of the mean curvature vector and an associated connection one-form. The norm of the mean curvature vector is denoted by $|H| = \sqrt{k^2 - p^2} > 0$. h^ν and j^ν also define a connection one-form α_H of the normal bundle of Σ by

$$\alpha_H = \frac{1}{k^2 - p^2}\pi_\alpha^\beta (h^\nu \nabla_\beta j_\nu) \tag{5}$$

where $\pi_\alpha^\beta = \delta_\alpha^\beta + u^\beta u_\alpha - v^\beta v_\alpha$ is the projection from the tangent bundle of N onto the tangent bundle of Σ. We choose local coordinates $\{u^a\}_{a=1,2}$ on Σ and express this one-form as $(\alpha_H)_a$ and the induced Riemannian metric on Σ as a symmetric $(0, 2)$ tensor σ_{ab}. The definition of quasi-local conserved quantities depends only on the data $(\sigma_{ab}, |H|, (\alpha_H)_a)$.

Suppose the mean curvature vector H_0 of $i(\Sigma)$ is also spacelike (this unnecessary assumption makes the exposition easier), and denote the corresponding data by $|H_0|$ and $(\alpha_{H_0})_a$. Denote the components of the isometric embedding i by (X^0, X^1, X^2, X^3), each as a smooth function on Σ. Let $\eta_{\mu\nu}$ be the Minkowski metric and $\tau = -t_0^\mu \eta_{\mu\nu} X^\nu$ be the time function on Σ with respect to a constant unit future timelike vector t_0^ν. The quasi-local energy (4) can be written as

$$\begin{aligned}\frac{1}{8\pi}\int_\Sigma &\Big\{(\cosh\theta_0|H_0| - \cosh\theta|H|)\sqrt{1 + |\nabla\tau|^2}\\ &- [\nabla_a\theta_0 - \nabla_a\theta + (\alpha_{H_0})_a - (\alpha_H)_a]\nabla^a\tau\Big\}\end{aligned} \tag{6}$$

where $\theta_0 = \sinh^{-1}\frac{-\Delta\tau}{|H_0|\sqrt{1+|\nabla\tau|^2}}$ and $\theta = \sinh^{-1}\frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$. $\Delta\tau = \nabla^a\nabla_a\tau$ and $|\nabla\tau|^2 = \sigma^{ab}\nabla_a\tau\nabla^a\tau$, where ∇_a is the covariant derivative of σ_{ab}. Expression (6) is the same as (4) with $N_0 = \sqrt{1 + |\nabla\tau|^2}$ and $N_0^\nu = -\nabla^a\tau$.

3. Definition and properties of quasi-local conserved quantities

In order to define quasi-local conserved quantities, we introduce a function ρ and a one-form j_a on Σ defined as follows:

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}. \tag{7}$$

and

$$j_a = \rho\nabla_a\tau - \nabla_a\left[\sinh^{-1}\left(\frac{\rho\Delta\tau}{|H_0||H|}\right)\right] - (\alpha_{H_0})_a + (\alpha_H)_a. \tag{8}$$

Notice that $\theta_0 - \theta = \sinh^{-1}(\frac{\rho\Delta\tau}{|H_0||H|})$ by the addition formula of the sinh function. In terms of these, the quasi-local energy is $\frac{1}{8\pi}\int_\Sigma(\rho + j_a\nabla^a\tau)$.

We consider (i, t_0^μ) as a quasi-local observer and minimize quasi-local energy among all such observers. A critical point of the quasi-local energy satisfies the optimal isometric embedding equation :

Definition 1 [6, 7, 19] - An embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ satisfies the optimal isometric embedding equation for $(\sigma_{ab}, |H|, (\alpha_H)_a)$ if the components of i, X^0, X^1, X^2, X^3, as functions on Σ, satisfy $\eta_{\mu\nu}\nabla_a X^\mu \nabla_b X^\nu = \sigma_{ab}$ and there exists a future unit timelike constant vector t_0^ν such that $\tau = -t_0^\mu\eta_{\mu\nu}X^\nu$ satisfies

$$\nabla^a j_a = 0. \tag{9}$$

Such an optimal isometric embedding may not be unique, but it is shown in [7] that this is locally unique if $\rho > 0$.

The quasi-local mass and quasi-local energy-momentum 4-vector with respect to (i, t_0^ν) are $m = \frac{1}{8\pi}\int_\Sigma \rho$ and

$$p^\nu = \frac{1}{8\pi}\int_\Sigma \rho\, t_0^\nu,$$

respectively. If Σ bounds a spacelike domain in the flat spacetime, m is zero and p^ν vanishes.

Let (x^0, x^1, x^2, x^3) denote the standard coordinate system on $\mathbb{R}^{3,1}$.

Definition 2 - Let $K_{\alpha\gamma}$ be an element of the Lie algebra of the Lorentz group with $K_{\alpha\gamma} = -K_{\gamma\alpha}$. Let $K = K_{\alpha\gamma}\eta^{\gamma\beta}x^\alpha\frac{\partial}{\partial x^\beta}$ be the corresponding Killing *field in* $\mathbb{R}^{3,1}$*. The conserved quantity corresponding to* (i, t_0^ν, K) *is* $K_{\alpha\gamma}\Phi^{\alpha\gamma}$ *where*

$$\Phi^{\alpha\gamma} = -\frac{1}{8\pi}\int_\Sigma (\rho X^{[\alpha} t_0^{\gamma]} + j^a X^{[\alpha}\nabla_a X^{\gamma]}), \tag{10}$$

where $j^a = \sigma^{ab} j_b$.

For a spacelike 2-surface in $\mathbb{R}^{3,1}$, $\rho = 0$ and $j_a = 0$, thus all quasi-local conserved quantities vanish with respect to its own isometric embedding. By the definition, p^ν and $\Phi^{\alpha\gamma}$ transform equivariantly when the pair (i, t_0^μ) is acted by a Lorentz transformation. When the optimal isometric embedding i is shifted by a translation in $\mathbb{R}^{3,1}$ such that $X^\mu \mapsto X^\mu + b^\mu$ for some constant vector b^μ, $\Phi^{\alpha\gamma}$ is changed by

$$\Phi^{\alpha\gamma} \mapsto \Phi^{\alpha\gamma} - \frac{1}{2} b^\alpha p^\gamma + \frac{1}{2} b^\gamma p^\alpha.$$

For a surface of symmetry in an axially symmetric spacetime, the quasi-local angular momentum is the same as the Komar angular momentum, and the quasi-local center of mass lies on the axis of symmetry.

4. Conserved quantities at spatial infinity

On an asymptotically flat initial data set (M, g, k), we take the limit of quasi-local conserved quantities on coordinate spheres to define total conserved quantities. For each family of solutions $(i_r, t_0^\nu(r))$ of optimal isometric embeddings for Σ_r such that the isometric embedding i_r converges to the standard embedding of a round sphere of radius r in $\mathbb{R}^3$ when $r \to \infty$, we define:

Definition 3 -Suppose $\lim_{r\to\infty} t_0^\nu(r) = t_0^\nu = L_0{}^\nu$ *for a Lorentz transformation* $L_\mu{}^\nu$*. Denote* $L_{\mu\gamma} = L_\mu{}^\nu \eta_{\nu\gamma}$*. The total center of mass of* (M, g, k) *is defined to be*

$$C^i = \frac{1}{m}\lim_{r\to\infty}[\Phi^{i\gamma}(r) L_{0\gamma} + \Phi^{0\gamma}(r) L_{i\gamma}], \quad i = 1, 2, 3 \tag{11}$$

and the total angular momentum is defined to be

$$J_i = \lim_{r\to\infty} \epsilon_{ijk}[\Phi^{j\gamma}(r) L_{k\gamma} - \Phi^{k\gamma}(r) L_{j\gamma}], \quad i, j, k = 1, 2, 3. \tag{12}$$

C^i corresponds to the conserved quantity of a boost Killing field, while J_i corresponds to that of a rotation Killing field with respect to the direction of the energy-momentum 4-vector.

5. Finiteness of total conserved quantities

In this section, we prove finiteness of the newly defined total conserved quantities for vacuum asymptotically flat initial data sets of order one.

Definition 4 - (M, g, k) is asymptotically flat of order one if there is a compact subset C of M such that $M \backslash C$ is diffeomorphic to $\mathbb{R}^3 \backslash B$, and in terms of the coordinate system $\{x^i\}_{i=1,2,3}$ on $M \backslash C$, $g_{ij} = \delta_{ij} + \frac{g_{ij}^{(-1)}}{r} + \frac{g_{ij}^{(-2)}}{r^2} + o(r^{-2})$ and $k_{ij} = \frac{k_{ij}^{(-2)}}{r^2} + \frac{k_{ij}^{(-3)}}{r^3} + o(r^{-3})$, where $r = \sqrt{\sum_{i=1}^3 (x^i)^2}$.

Transforming into spherical coordinates $(r, \theta, \phi) = (r, u^1, u^2)$, on each level set of r, Σ_r, we can use $\{u^a\}_{a=1,2}$ as coordinate system to express the geometric data we need in order to define quasi-local conserved quantities:

$$
\begin{aligned}
\sigma_{ab} &= r^2 \tilde{\sigma}_{ab} + r\sigma_{ab}^{(1)} + \sigma_{ab}^{(0)} + o(1) \\
|H| &= \frac{2}{r} + \frac{h^{(-2)}}{r^2} + \frac{h^{(-3)}}{r^3} + o(r^{-3}) \qquad (13)\\
\alpha_H &= \frac{\alpha_H^{(-1)}}{r} + \frac{\alpha_H^{(-2)}}{r^2} + o(r^{-2}),
\end{aligned}
$$

where $\tilde{\sigma}_{ab}$ is the standard round metric on S^2, and $\sigma_{ab}^{(1)}$, $\sigma_{ab}^{(0)}$, $h^{(-2)}$, $h^{(-3)}$, $\alpha_H^{(-1)}$, and $\alpha_H^{(-2)}$ are all considered as geometric data on S^2. It was proved in [6] that for such an initial data set, there exists a unique family of optimal isometric embeddings $(i_r, t_0^\nu(r))$ of Σ_r such that the components of i_r are given

$$
(0, r\tilde{X}^1, r\tilde{X}^2, r\tilde{X}^3) + o(r), \qquad (14)
$$

where $\tilde{X}^i, i = 1, 2, 3$ are $\sin\theta \sin\phi$, $\sin\theta\cos\phi$, and $\cos\theta$, respectively, in the standard coordinates (θ, ϕ) on S^2. Similar expansions for $|H_0|$ and α_{H_0} can be obtained.

We recall from [6, 20] that the total energy-momentum vector p^ν satisfies $\lim_{r\to\infty} t_0^\nu(r) = \frac{1}{m} p^\nu$ with $m = \sqrt{-p_\nu p^\nu}$, and the components are given by

$$
p^0 = \frac{1}{8\pi} \int_{S^2} (h_0^{(-2)} - h^{(-2)}) \quad \text{and} \quad p^i = \frac{1}{8\pi} \int_{S^2} \tilde{X}^i \widetilde{div}(\alpha_H^{(-1)}). \qquad (15)
$$

It is shown to be the same as the ADM energy-momentum vector on the initial data set (M, g, k) in [20].

Unlike translating Killing fields which define energy and linear momentum, the expression of boost and rotation Killing fields involves the coordinate functions. Therefore existing definitions of total angular momentum and total center of mass are in general infinite and not well-defined unless additional conditions [2, 10, 16] are imposed at spatial infinity.

Recall that a vacuum initial data set (M, g, k) satisfies

$$R(g) + (tr_g k)^2 - |k|_g^2 = 0 \quad \text{and} \quad \nabla_g^i (k_{ij} - (tr_g k) g_{ij}) = 0 \tag{16}$$

where $R(g)$ is the scalar curvature of g_{ij}.

Theorem 1 - The new total angular momentum and new center of mass are always finite on any vacuum asymptotically flat initial data set of order one.

Proof. By the expansions of geometric data (13) and (14), the optimal isometric embedding equation (9) decomposes into linear equations on S^2 which can be solved by spherical harmonics. The total center of mass and total angular momentum are finite if

$$\int_{S^2} \tilde{X}^i (h_0^{(-2)} - h^{(-2)}) = 0 \quad \text{and} \quad \int_{S^2} \tilde{X}^i \left(\tilde{\epsilon}^{ab} \tilde{\nabla}_b (\alpha_H^{(-1)})_a \right) = 0, \tag{17}$$

where $\tilde{\nabla}$ is the covariant derivative of $\tilde{\sigma}_{ab}$ and $\tilde{\epsilon}_{ab}$ is the area form on S^2. The proof of these two equalities makes use of the optimal isometric embedding equation and the vacuum constraint equation and the details can be found in [8]

The new total angular momentum vanishes on any hypersurface in $\mathbb{R}^{3,1}$: a property that is rather unique among known definitions. Indeed, it is shown in [5] that there exists asymptotically flat hypersurface in $\mathbb{R}^{3,1}$ with finite non-zero ADM angular momentum. In [8], we show that the new total angular momentum is the same on any strictly spacelike hypersurface in the Kerr spacetime.

6. Conserved quantities and the vacuum Einstein equation

Let (M, g, k) be a vacuum initial data set. The vacuum Einstein equation is formulated as an evolution equation of the pair $(g(t), k(t))$ that satisfies $g(0) = g$, $k(0) = k$ and

$$\begin{aligned} \partial_t g_{ij} &= -2N k_{ij} + \nabla_i \gamma_j + \nabla_j \gamma_i \\ \partial_t k_{ij} &= -\nabla_i \nabla_j N + N \left(R_{ij} + (tr_g k) k_{ij} - 2 k_{il} k^l{}_j \right) \end{aligned} \tag{18}$$

where N is the lapse function and γ is the shift vector.

Theorem 2- Suppose (M, g, k) is a vacuum asymptotically flat initial data set of order one. Let $(M, g(t), k(t))$ be the solution to the initial value problem

$g(0) = g$ and $k(0) = k$ for the vacuum Einstein equation with lapse function $N = 1 + O(r^{-1})$ and shift vector $\gamma = \gamma^{(-1)}r^{-1} + O(r^{-2})$. We have

$$\partial_t C^i(t) = \frac{p^i}{p^0} \quad \textit{and} \quad \partial_t J_i(t) = 0 \tag{19}$$

for $i = 1, 2, 3$ where p^ν is the ADM energy momentum 4-vector of (M, g, k).

Proof. Again, we solve the linearized optimal isometric embedding equation (9) by the expansions of geometric data (13). It suffices to show

$$\begin{aligned} &\partial_t \left[\frac{1}{8\pi} \int_{S^2} \tilde{X}^i \rho^{(-3)}(t) \right] = \frac{p^i}{t_0^0} \quad \text{and} \\ &\partial_t \left[\int_{S^2} (\tilde{X}^i \tilde{\nabla}_a \tilde{X}^j - \tilde{X}^i \tilde{\nabla}_a \tilde{X}^j) \tilde{\sigma}^{ab} j_b^{(-2)}(t) \right] = 0. \end{aligned} \tag{20}$$

The first equality is proved using the vacuum constraint equation and the Einstein evolution equation and the second one is proved using the conservation law of quasi-local angular momentum. The details of these proofs can be found in [8].

Remark. Assuming the total center of mass and angular momentum is finite initially, Theorem 2 holds under the weaker assumption $g = \delta + O(r^{-1})$ and $k = O(r^{-2})$, see [8].

7. Properties of the new total conserved quantities and conclusions

Our work not only gives the precise definitions of angular momentum and center of mass at the quasi-local level, the limits of which also define new total conserved quantities for any isolated gravitating system. The definitions of total angular momentum and total center of mass are new and free from the ambiguities and difficulties of known definitions. To the best of our knowledge, the new total center of mass gives the unique definition that satisfies the highly desirable equation (19). This is the first time when the fundamental formula $p = mv$ is shown to be consistent with the Einstein equation. Though we only consider the case of spatial infinity in the current paper. The definition can be readily extended to the case of null infinity by taking the corresponding limit of the quasi-local definition.

The quasi-local definition involves solving the nonlinear optimal isometric embedding equation. However, for an isolated system, the equation decomposes into linear equations of the form

$$\Delta(\Delta + 2)f = g$$

for functions f and g on S^2 and Δ the Laplace operator on S^2. The solutions of these linear equations are completely understood and the calculation only involves elementary spherical harmonics. Current numerical method can easily handle these calculations and definitions at null infinity can be in principle measured by gravitational wave detectors. We thus expect the new definitions of total angular momentum and total center of mass will have applications in the physical communities of numerical relativity and astrophysics.

To summarize, the new total angular momentum and total center of mass on an asymptotically flat initial data set (M, g, k) satisfy the following properties:

1. The definition depends only on the geometric data (g, k) and the foliation of surfaces at infinity, and in particular does not depend on an asymptotically flat coordinate system or the existence of an asymptotically Killing field.

2. All total conserved quantities vanish on any spacelike hypersurface in the Minkowski spacetime, regardless of the asymptotic behavior.

3. The new total angular momentum and total center of mass are always finite on any vacuum asymptotically flat initial data set of order one.

4. The total angular momentum satisfies the conservation law. In particular, the total angular momentum on any strictly spacelike hypersurface of the Kerr spacetime is the same.

5. Under the vacuum Einstein evolution of initial data sets, the total angular momentum is conserved and the total center of mass obeys the dynamical formula $\partial_t C^i(t) = \frac{p^i}{p^0}$ where p^ν is the ADM energy-momentum four vector.

References

[1] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, 227–265, Wiley, New York.

[2] A. Ashtekar and R. O. Hansen, *A unified treatment of spatial and null infinity in general relativity: universal structure, asymptotic symmetries*

and conserved quantities at spatial infinity, J. Math. Phys. **19** (1978), no. 7, 1542–1566.

[3] R. Beig and N. ÓMurchadha, *The Poincaré group as the symmetry group of canonical general relativity*, Ann. Physics **174** (1987), no. 2, 463–498.

[4] J. D. Brown and J. W. York, *Quasi-local energy and conserved charges derived from the gravitational action*, Phys. Rev. D (3) **47** (1993), no. 4, 1407–1419.

[5] P.-N. Chen, L.-H. Huang, M.-T. Wang, and S.-T. Yau, *On the validity of the definition of angular momentum in general relativity*, Ann. Henri Poincaré **17** (2016), no. 2, 253–270. `arXiv:1401.0597`.

[6] P.-N. Chen, M.-T. Wang, and S.-T. Yau, *Evaluating quasilocal energy and solving optimal embedding equation at null infinity*, Comm. Math. Phys. **308** (2011), no. 3, 845–863.

[7] P.-N. Chen, M.-T. Wang, and S.-T. Yau, *Minimizing properties of critical points of quasi-local energy*, Comm. Math. Phys. **329** (2014), no. 3, 919–935, `arXiv:1302.5321`.

[8] P.-N. Chen, M.-T. Wang, and S.-T. Yau, *Conserved quantities in general relativity: from the quasi-local level to spatial infinity*, Comm. Math. Phys. **338** (2015), no. 1, 31–80 `arXiv:1312.0985`.

[9] D. Christodoulou, in *Zürich Lectures in Advanced Mathematics*, European Mathematical Society (EMS), Zürich, 2008.

[10] P. T. Chruściel, *On angular momentum at spatial infinity*, Class. Quantum Grav. **4** (1987), no. 6, L205–L210.

[11] S. W. Hawking and G. T. Horowitz, *The gravitational Hamiltonian, action, entropy and surface terms*, Class. Quantum Grav. **13**, no. 6, (1996).

[12] L.-H. Huang, *On the center of mass of isolated physical systems with general asymptotics*, Class. Quantum Grav. **26** (2009), no. 1, 015012, 25 pp.

[13] G. Huisken and S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Invent. Math. **124**, 281–311 (1996).

[14] R. Penrose, in *Seminar on Differential Geometry*, pp. 631–668, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982

[15] R. Penrose, *Quasilocal mass and angular momentum in general relativity*, Proc. Roy. Soc. London Ser. A **381** (1982), no. 1780, 53–63.

[16] T. Regge and C. Teitelboim, *Role of surface integrals in the Hamiltonian formulation of general relativity*, Ann. Phys. **88** (1974), 286–318.

[17] L. B. Szabados, *Quasi-local energy-momentum and angular momentum in general relativity*, Living Rev. Relativity, **12** (2009) No 4.

[18] M.-T. Wang and S.-T. Yau, *Quasilocal mass in general relativity*, Phys. Rev. Lett. **102** 021101 (2009), no. 2.

[19] M.-T. Wang and S.-T. Yau, *Isometric embeddings into the Minkowski space and new quasi-local mass*, Comm. Math. Phys. **288** (2009), no. 3, 919–942.

[20] M.-T. Wang and S.-T. Yau, *Limit of quasilocal mass at spatial infinity*, Comm. Math. Phys. **296** (2010), no.1, 271–283.

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