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Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I

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Consider the 3-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory associated with a compact Lie group G and its quaternionic representation $\mathbf{M}$. Physicists study its Coulomb branch, which is a noncompact hyper-Kähler manifold, such as instanton moduli spaces on $\mathbb{R}^4$, SU(2)-monopole moduli spaces on $\mathbb{R}^3$, etc. In this paper and its sequel, we propose a mathematical definition of the coordinate ring of the Coulomb branch, using the vanishing cycle cohomology group of a certain moduli space for a gauged σ-model on the 2-sphere associated with $(G, \mathbf{M})$. In this first part, we check that the cohomology group has the correct graded dimensions expected from the monopole formula proposed by Cremonesi, Hanany and Zaffaroni [CHZ14]. A ring structure (on the cohomology of a modified moduli space) will be introduced in the sequel of this paper.

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1. Introduction

1(i). 3-dimensional $\mathcal{N} = 4$ SUSY gauge theories

Let $\mathbf{M}$ be a quaternionic representation (also called a *pseudoreal* representation) of a compact Lie group G. Let us consider the 4-dimensional gauge theory with $\mathcal{N} = 2$ supersymmetry associated with $(G, \mathbf{M})$. It has been studied by physicists for many years. It is closely related to pure mathematics, because the correlation function of its topological twist ought to give the Donaldson invariant [Don90] of 4-manifolds for $(G, \mathbf{M}) = (\mathrm{SU}(2), 0)$, as proposed by Witten [Wit88]. Thus physics and mathematics influence each other in this class of gauge theories. For example, Seiberg-Witten's ansatz [SW94] led to a discovery of a new invariant, namely the Seiberg-Witten invariant. It is associated with $(G, \mathbf{M}) = (\mathrm{U}(1), \mathbb{H})$. Nekrasov partition function [Nek03] gave a mathematically rigorous footing on Seiberg-Witten's ansatz, and hence has been studied by both physicists and mathematicians.

In this paper, we consider the 3-dimensional gauge theory with $\mathcal{N} = 4$ supersymmetry, obtained from the $4d$ theory, by considering on (3-manifold) $\times$ S^1_R and taking $R \to 0$. We denote the $3d$ gauge theory by $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ following [Tac], though it is used for the $4d$ gauge theory originally. One usually studies only asymptotically conformal or free theories in $4d$, while we do not have such restriction in $3d$. Also some aspects are easier, simplified and clarified in $3d$, and our hope is that to use understanding in $3d$ to study problems in $4d$.

We do not review what is $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$. It is a quantum field theory in dimension 3, and is not rigorously constructed mathematically.

Instead we will take the following strategy: physicists associate various mathematical objects to $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ and study their properties. They ask mathematicians to construct those objects, instead of $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ itself, in mathematically rigorous ways so that expected properties can be checked.

This strategy is posed explicitly in [MT12] for a particular object, which is very close to what we consider here. (See §3(iii) for detail on the precise relation.) Many physically oriented mathematical works in recent years are more or less take this strategy anyway.

1(ii). Topological twist

An example of mathematical objects is a topological invariant defined so that it ought to be a correlation function of a topologically twisted version of $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$. For the 4-dimensional $\mathcal{N}=2$ SUSY pure SU(2)-theory (i.e., $(G,\mathbf{M}) = (\mathrm{SU}(2),0)$), Witten claimed that the correlation function gives Donaldson invariants, as mentioned above. In [AJ90] Atiyah-Jeffrey understood the correlation function heuristically as the Euler class of an infinite rank vector bundle of an infinite dimensional space with a natural section s in the Mathai-Quillen formalism. The zero set of s is the moduli space of SU(2) anti-self-dual connections for $(G,\mathbf{M}) = (\mathrm{SU}(2),0)$, hence Witten's claim has a natural explanation as a standard result in differential topology applied formally to infinite dimension. It was also observed that the 3-dimensional story explains an Taubes' approach to the Casson invariant [Tau90], at least for homological 3-spheres. And 3 and 4-dimensional stories are nicely combined to Floer's instanton homology group [Flo88] and its relation to Donaldson invariants in the framework of a $(3+1)$-dimensional topological quantum field theory (TQFT) [Don02]. Another example of a similar spirit is the Seiberg-Witten invariant in dimensions 3/4 (see e.g., [Wit94] for $4d$ and [MT96] for $3d$). This is the case $(G,\mathbf{M}) = (\mathrm{U}(1),\mathbb{H})$.

Atiyah-Jeffrey's discussion is heuristic. In particular, it is not clear how to deal with singularities of the zero set of s in general. We also point out that a new difficulty, besides singularities of $\mathrm{Zero}(s)$, failure of compactness occurs in general. See §1(iv) and §6(ii) below. Therefore it is still an open problem to define topological invariants rigorously for more general $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$. It is beyond the scope of this paper. But we will use a naive or heuristic analysis of would-be topological invariants to help our understanding of $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$. We hope the study in this paper and its sequel [BFN16a] might be relevant to attack the problem of the definition of topological invariants.

1(iii). Coulomb branch

Instead of giving definitions of topological invariants, we consider the so-called Coulomb branch $\mathcal{M}_C$ of $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$.

Physically it is defined as a specific branch of the space of *vacua*, where the potential function takes its minimum. However it is only a classical description, and receives a quantum correction due to the integration of massive fields. At the end, there is no definition of $\mathcal{M}_C$, which mathematicians could understand in the literature up to now. However it is a physicists consensus that the Coulomb branch is a hyper-Kähler manifold with an SU(2)-action rotating hyper-Kähler structures I, J, K [SW97]. Physicists also found many hyper-Kähler manifolds as Coulomb branches of various gauge theories, such as toric hyper-Kähler manifolds, moduli spaces of monopoles on $\mathbb{R}^3$, instantons on $\mathbb{R}^4$ and ALE spaces, etc. (Reviewed below.)

1(iv). A relation to topological invariants

The Coulomb branch $\mathcal{M}_C$ might play some role in the study of would-be topological invariants for $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$. As topological invariants are currently constructed only $(G, \mathbf{M}) = (\mathrm{SU}(2), 0)$ and $(G, \mathbf{M}) = (\mathrm{U}(1), \mathbb{H})$, we could touch this aspect superficially. Nevertheless we believe that it is a good starting point.

First consider $\mathrm{Hyp}(0) /\!\!/\!\!/ \mathrm{SU}(2)$. Let us start with the 4-dimensional case. A pseudo physical review for mathematicians was given in [NY04, §1], hence let us directly go to the conclusion. The space of vacua is parametrized by a complex parameter u, and hence called the *u-plane* (Seiberg-Witten ansatz). We have a family of elliptic curves E_u parametrized by u, where E_u degenerates to rational curves at $u = \pm 2\Lambda^2$. The so-called prepotential of the gauge theory is recovered from the period integral of E_u. From the gauge theoretic view point, u is understood as a 'regularized' integration of a certain equivariant differential form over the framed moduli space of SU(2)-instantons on $\mathbb{R}^4$. More rigorously we define the integration by a regularization cooperating T^2-action on $\mathbb{R}^4$ (Nekrasov's Ω-background). The prepotential above determines the equivariant variable a as a function of u, and hence u as an inverse function of a.

Witten explained that the Donaldson invariant is given by a u-plane integral, and the contribution at the singularities $u = \pm 2\Lambda^2$ is given by the Seiberg-Witten invariant [Wit94]. See also [MW97] for a further development. This picture was mathematically justified, in a slightly modified way, for projective surfaces in [GNY08, GNY11].

Now we switch to the 3-dimensional case. As is mentioned above, we first consider $\mathbb{R}^3 \times S^1_R$ and take the limit $R \to 0$. Seiberg-Witten determines the space of vacua for large R as the *total space* of the family E_u for $u \in \mathbb{C}$ [SW97]. When we make $R \to 0$, points in E_u are removed, and get the moduli space of charge 2 centered monopoles on $\mathbb{R}^3$. This is a 4-dimensional hyper-Kähler manifold studied intensively by Atiyah-Hitchin [AH88]. This is the Coulomb branch $\mathcal{M}_C$ of the 3-dimensional gauge theory $\mathrm{Hyp}(0) \mathbin{/\mkern-6mu/} \mathrm{SU}(2)$. It means that the 3-dimensional gauge theory $\mathrm{Hyp}(0) \mathbin{/\mkern-6mu/} \mathrm{SU}(2)$ reduces at low energies to a sigma-model whose target is $\mathcal{M}_C$.

This picture gives us the following consequence for topological invariants. The partition function for the twisted $\mathrm{Hyp}(0) \mathbin{/\mkern-6mu/} \mathrm{SU}(2)$ is the Casson-Walker-Lescop invariant as above. On the other hand, the partition function for the sigma-model with target $\mathcal{M}_C$ is the topological invariant constructed by Rozansky-Witten [RW97]. Then as an analog of Seiberg-Witten = Donaldson in 4-dimension, it is expected that the Casson-Walker-Lescop invariant coincides with the Rozansky-Witten invariant.

The Rozansky-Witten invariant for $\mathcal{M}_C$ is a finite type invariant [Oht96, LMO98, LMMO99] of order 3, which is unique up to constant multiple. Therefore the coincidence of two invariants is not a big surprise. But it at least gives an expectation of a generalization to $G = \mathrm{SU}(r)$. The Coulomb branch of $\mathrm{Hyp}(0) \mathbin{/\mkern-6mu/} \mathrm{SU}(r)$ is the moduli space of charge r centered monopoles on $\mathbb{R}^3$ [CH97]. Neither the $\mathrm{SU}(r)$-Casson-Walker-Lescop invariant and the Rozansky-Witten invariant associated with $\mathcal{M}_C$ are not mathematically rigorously defined yet, but it is natural to expect that they coincide once they would be defined. For the former, singularities of moduli spaces must be treated appropriately. See [CLM90, BH98, BHK01, CLM02], for example, studying this problem. The convergence of an integral must be proved for Rozansky-Witten invariants, as the monopole moduli spaces are noncompact.

Let us continue examples from [SW97]. Let $G = \mathrm{U}(1)$, $\mathbf{M} = \mathbb{H}^N = \mathbb{C}^N \oplus (\mathbb{C}^*)^N$, the direct sum of N copies of the vector representation plus its dual. The Coulomb branch $\mathcal{M}_C$ is the multi-Taub-NUT space, which is $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$ as a complex variety when $N > 0$, and is $\mathbb{R}^3 \times S^1 = \mathbb{C} \times \mathbb{C}^*$ when $N = 0$.

The case $N = 0$ is trivial, so let us exclude it. There is a big distinction between $N = 1$ and $N > 1$ cases, where we have a singularity at the origin or not. For $N = 1$, the partition function gives the 3-dimensional Seiberg-Witten invariant. On the other hand, $\mathcal{M}_C$ is the Taub-NUT space, the associated Rozansky-Witten invariant is again finite type of order 3, hence should be equal to the Casson-Walker-Lescop invariant up to multiple again. Marino-Moore [MM99], Blau-Thompson [BT01] argued that the Casson-Walker-Lescop invariant is equal to the 3-dimensional Seiberg-Witten invariant, more precisely, the (regularized) sum over all Spin^c classes, as proved earlier by Meng-Taubes [MT96] for $b_1 > 0$. In the $b_1 = 0$ case, the definition of the Seiberg-Witten invariant is more subtle, and the claim was shown later by Marcolli-Wang [MW02].

For $N > 1$, the Coulomb branch $\mathcal{M}_C$ has a singularity at the origin, whose contribution to the Rozansky-Witten invariant needs to be clarified. In the Seiberg-Witten side, compactness of moduli spaces fails, as we will review in §6(ii). Therefore it is not yet clear how to define the invariant. The singularity comes from the Higgs branch $\mathcal{M}_H$, explained later. Therefore it is natural to expect that the Rozansky-Witten invariant for $\mathcal{M}_H$ enters the picture.

Let us note that $\mathcal{M}_C$ for $\mathrm{Hyp}(0) \mathbin{\#\!\#} \mathrm{SU}(2)$ can be defined as a limit of the total space of E_u, which is rigorously recovered from Nekrasov's partition function. This method could probably apply to if $(G, \mathbf{M})$ gives an asymptotically conformal or free theory in dimension 4, but not in general. For example, it is not clear how to do for $(\mathrm{U}(1), \mathbb{H})$.

1(v). (2 + 1)-dimensional TQFT

The Rozansky-Witten invariant associated with a hyper-Kähler manifold M is expected to fit in the framework of a $(2+1)$-dimensional TQFT. For a 2-manifold Σ, one associates a quantum Hilbert space $\mathcal{H}_\Sigma$, and an invariant of a 3-manifold X with boundary Σ takes value in $\mathcal{H}_\Sigma$. Then the gluing axiom is satisfied.

In [RW97, §5], it is proposed that

$$\mathcal{H}_{\Sigma_g} = (-1)^{1+g} \bigoplus_q H^q(M, (\textstyle\bigwedge^* V)^{\otimes g}),$$

where Σ_g is a 2-manifold of genus g, V is the natural $\mathrm{Sp}(\dim_{\mathbb{H}} M)$-bundle over M, and the sign $(-1)^{1+g}$ is introduced so that $\mathcal{H}_{\Sigma_g}$ has a correct $\mathbb{Z}/2$-graded vector space structure. Rozansky-Witten wrote that it is hazardous to apply this definition for noncompact M, like our Coulomb branch $\mathcal{M}_C$.

Nonetheless consider the case $g = 0$ assuming $\mathcal{M}_C$ is affine:

$$\mathcal{H}_{S^2} = \bigoplus_q (-1)^{1+q} H^q(\mathcal{M}_C, \mathcal{O}) = -\mathbb{C}[\mathcal{M}_C].$$

Let us give a nontrivial check for this hazardous assertion. By the gluing axiom, the invariant of $S^2 \times S^1$ is equal to the dimension of $\mathcal{H}_{S^2}$. On the other hand, from the knowledge of the Casson-Walker-Lescop invariant for $S^2 \times S^1$, it should be equal to $1/12$ if $\mathcal{M}_C$ is the Taub-NUT space or the

Atiyah-Hitchin manifold (see [RW97, (4.4), (5.15)] and [BT01, (2.3)]. Therefore we should have $\dim \mathbb{C}[\mathcal{M}_C] = -1/12$. For the Taub-NUT space, which is isomorphic to $\mathbb{C}^2$ as an affine variety, this is true after ζ-regularization:

$$(1.1) \qquad \left.\frac{1}{(1-t)^2}\right|_{t=1} = \left.\sum_{n=1}^{\infty} n t^{n-1}\right|_{t=1} = \zeta(-1) = -\frac{1}{12}.$$

The expression $1/(1-t)^2$ is coming from the natural grading on $\mathbb{C}[\mathcal{M}_C] = \mathbb{C}[x, y]$ as $\deg x = \deg y = 1$. See §4(ii) for more detail. A closely related observation on $-1/12$ can be found at [MM99, (6.16)], [BT01, §2.3]. It is interesting to look for a similar explanation for the Atiyah-Hitchin manifold, as well as a deeper understanding of this regularization process.

On the other hand, the Casson and Seiberg-Witten invariants are also expected to fit in the TQFT framework. The quantum Hilbert spaces $\mathcal{H}_\Sigma$ are the cohomology group of moduli spaces of flat SU(2)-bundles and solutions of the (anti-)vortex equation over Σ respectively [Ati88, Don99] (see also [Ngu14]). This cannot be literally true for the Casson case, as there is no nontrivial flat SU(2)-bundle on $\Sigma = S^2$. In the Seiberg-Witten case, $3d$ invariants depend on a choice of perturbation of the equation, and its dependence must be understood via the wall-crossing formula. We need to choose the corresponding perturbation of the (anti-)vortex equation to have a nonempty moduli space in $2d$. For a 'positive' (resp. 'negative') perturbation, moduli spaces for the vortex (resp. anti-vortex) equation are nonempty, and symmetric products of Σ. For $\Sigma = S^2$, the $(n-1)$th symmetric product $S^{n-1}\Sigma$ is $\mathbb{P}^{n-1}$, and its cohomology is n-dimensional. It is compatible with the above naive computation (1.1), as n appears in the middle.

If Casson-Walker-Lescop = Rozansky-Witten would be true as $(2+1)$-TQFT's, we conclude that

$$\mathbb{C}[\mathcal{M}_C \text{ of } \mathrm{Hyp}(0) \mathbin{/\!\!/\!\!/} \mathrm{SU}(2)] \overset{?}{=} H^*(\text{moduli spaces of flat bundles on } S^2),$$

and similarly for $\mathbb{C}[\mathcal{M}_C$ of $\mathrm{Hyp}(\mathbb{H}) \mathbin{/\!\!/\!\!/} \mathrm{U}(1)]$ and the cohomology group of moduli spaces of solutions of the (anti-)vortex equation on $\Sigma = S^2$. This could *not* be true for flat bundles as we have remarked above. For the (anti-)vortex equation we do not have an immediate contradiction, but it is too strong to be true, as the Coulomb branch seems to be independent of the choice of perturbation.

1(vi). Monopole formula

As we have explained just above, it seems difficult to use our current understanding of TQFT to determine $\mathcal{M}_C$ in a mathematically rigorous way.

A work, more tractable to mathematicians, has been done recently by Cremonesi, Hanany and Zaffaroni [CHZ14]. They write down a combinatorial expression, which gives the Hilbert series of the Coulomb branch $\mathcal{M}_C$. It is called *the monopole formula*.

The monopole formula is a formal Laurent power series

$$H_{G,\mathbf{M}}(t) \overset{\text{def.}}{=} \sum_{\lambda \in Y/W} t^{2\Delta(\lambda)} P_G(t;\lambda), \tag{1.2}$$

where Y is the coweight lattice of G, W is the Weyl group, and $\Delta(\lambda)$, $P_G(t;\lambda)$ are certain an integer and a rational function in t respectively. We postpone a detailed discussion of the monopole formula to §4(i). Let us give a brief comment here: (1.2) is a combinatorial expression, and mathematically makes sense contrary to the case of $\mathcal{M}_C$.

It is worthwhile to keep a physical origin of the monopole formula in mind. The monopole formula counts *monopole operators*, which are defined by fields having point singularities [BKW02]. Taking a radial coordinate system around a singular point, the singularity is modeled on a connection on S^2, whose topological charge is given by a coweight of G. This is the reason why the coweight λ appears in (1.2). We do not review physical origins of expressions $\Delta(\lambda)$, $P_G(t;\lambda)$. See [CHZ14, §2] and the references therein.

It is also clear that monopole operators belong to the quantum Hilbert space $\mathcal{H}_{S^2}$ for S^2. They form a ring (called a *chiral ring* in physics literature), by considering the topological quantum field theory associated with S^3 with three punctures.

Combining with our heuristic consideration in the TQFT framework, we will take the following strategy to find a definition of $\mathcal{M}_C$. We will start with the cohomology group of a moduli space, and look for its modification so that its Poincaré polynomial reproduces (1.2). Then we will study properties of the proposed Coulomb branch, whether they are compatible with physical expectations. We will propose such modification in this paper and its sequel [BFN16a].

Let us give a remark. In (1.2) we need to assume $2\Delta(\lambda) \geq 1$ for any $\lambda \neq 0$, a 'good' or 'ugly' theory in the sense of [GW09], hence no negative powers of t appear. Then (1.2) makes sense as a formal power series. This assumption fails for example the pure theory $(G, \mathbf{M}) = (\mathrm{SU}(2), 0)$, though $\mathcal{M}_C$ still exists and is the monopole moduli space as we discussed above. Our proposed definition, though motivated by (1.2), will make sense without this assumption.

1(vii). Higgs branch, $3d$ mirror symmetry and symplectic duality

There is a construction of a hyper-Kähler manifold from G and $\mathbf{M}$, which mathematicians can understand. It is the hyper-Kähler quotient construction [HKLR87]. (See §2(i) below for detail.) However the Coulomb branch $\mathcal{M}_C$ is *not* the hyper-Kähler quotient of $\mathbf{M}$ by G. In the above examples with $\mathbf{M} = 0$, the hyper-Kähler quotient $\mathbf{M}/\!\!/\!\!/G$ is just $\{0\}$ for any G. (Or $\emptyset$ if we consider only *free* orbits.) For $(G, \mathbf{M}) = (\mathrm{U}(1), \mathbb{H}^N)$, the hyper-Kähler quotient $\mathbf{M}/\!\!/\!\!/G$ is the closure of the minimal nilpotent orbit in $\mathfrak{sl}(N, \mathbb{C})$, or the cotangent bundle of $\mathbb{CP}^{N-1}$ if the real part of the level of the hyper-Kähler moment map is nonzero. (When $N = 1$, the former is $\{0\}$, and the latter is a single point.) In fact, the hyper-Kähler quotient of $\mathbf{M}$ by G arises as the *Higgs branch* $\mathcal{M}_H$, which is yet another mathematical object associated with the gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$.

For this class of 3-dimensional supersymmetric theories, it has been noticed that two theories often appear in pairs. It is called the *mirror symmetry* in 3-dimensional theories, as it is similar to more famous mirror symmetry between two Calabi-Yau's. The first set of examples was found by Intriligator and Seiberg [IS96]. In fact, the above is one of their examples, where the mirror theory is the gauge theory for $G = \mathrm{U}(1)^N/\,\mathrm{U}(1)$ with $\mathbf{M} = \mathbb{H}^N$ associated with the affine quiver of type $A_{N-1}^{(1)}$. (See §2(iv) below.)

When two theories A, B form a mirror pair, their Higgs and Coulomb branches are swapped:

$$\mathcal{M}_C^A = \mathcal{M}_H^B, \quad \mathcal{M}_H^A = \mathcal{M}_C^B.$$

This is indeed the case for our example. The hyper-Kähler quotient of $\mathbb{H}^N$ by $\mathrm{U}(1)^N/\,\mathrm{U}(1)$ is $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$, Kronheimer's construction of ALE spaces for type A [Kro89].

Remark 1.3. In order to the above equalities to be literally true, we need to replace the multi-Taub-NUT spaces by the corresponding ALE spaces. (The multi-Taub-NUT metric has a parameter g_{cl}, and it becomes the ALE space when $g_{\mathrm{cl}} \to \infty$.) This is because the mirror symmetry is a duality in infrared. See [IS96, §3.1] for a physical explanation why this is necessary. As we are only interested in the complex structure of $\mathcal{M}_C$, this process makes no change for us. We ignore this point hereafter.

Therefore the monopole formula (1.2) for the theory A computes the Hilbert series of the Higgs branch $\mathcal{M}_H^B$ of the mirror theory B. And there are lots of examples of mirror pairs, and we have a systematic explanation via branes, dualities, M-theory, etc. See e.g. [dBHOO97, PZ97, HW97, dBHO$^+$97] and also §§2(v), §3.

Unfortunately these techniques have no mathematically rigorous foundation, and hence there is no definition of the mirror, which mathematicians can understand. Moreover, the mirror of a gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ may not be of a form $\mathrm{Hyp}(\mathbf{M}') /\!\!/\!\!/ G'$ for some $\mathbf{M}'$ and G' in general, as we will explain in §3(i). Thus it seems that the mirror symmetry is more difficult to work with, and we will not use it to look for the mathematical definition of $\mathcal{M}_C^A$. In turn, we could hope that our proposed definition of $\mathcal{M}_C^A$ will shed some light on the nature of the $3d$ mirror symmetry.

Let us continue an example of a mirror pair. The pair in Figure 1, both of quiver types, is known to be mirror each other [dBHOO97]. Higgs branches are the k^{th} symmetric power of $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$ and the framed moduli space space of $\mathrm{SU}(N)$ k-instantons on $\mathbb{R}^4$, given by the ADHM description respectively. This includes the above example as $k = 1$ case. The former can be considered as the framed moduli space of $\mathrm{U}(1)$ k-instantons on $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$. Their Hilbert series has been computed. For $\mathcal{M}_H^A$, it can be written in terms of the Hilbert series of $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$ as it is a symmetric product. For $\mathcal{M}_H^B$, the Hilbert series is the K-theoretic Nekrasov partition function (or 5-dimensional partition function in physics literature), and can be computed via fixed point localization or the recursion by the blowup equation, e.g., see [NY05b].

$\mathcal{M}_H^A$: $\mathrm{U}(1)$ k-instantons on $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$ $\qquad$ $\mathcal{M}_H^B$: $\mathrm{SU}(N)$ k-instantons on $\mathbb{R}^4$

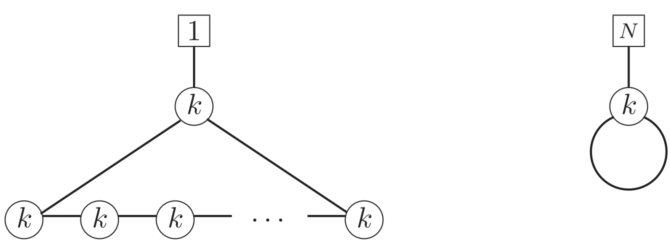

Figure 1: An example of a mirror pair

We should emphasize that it is not obvious to see why the monopole formula reproduces those results, as (1.2) looks very different from the known expression. For $\mathcal{M}_C^B$, this is possible after some combinatorial tricks. See §B. (It was checked in [CHZ14] for small k.) But it is not clear how to check for $\mathcal{M}_C^A$.

Let us mention that Braden et al. [BLPW14] expect that the mirror symmetry is related to the symplectic duality, which states an equivalence between categories attached to symplectic resolutions of two different conical hyper-Kähler manifolds $\mathcal{M}_H^A = \mathcal{M}_C^B$ and $\mathcal{M}_C^A = \mathcal{M}_H^B$. The definition of categories and the dual pair require also that both symplectic resolutions have torus action with *finite* fixed points. Since $\mathcal{M}_H^A$, $\mathcal{M}_C^A$ do not have symplectic resolutions nor torus action with finite fixed points in general, the symplectic duality deals with much more restrictive situations than ones considered here. If $\mathcal{M}_H^A$ satisfies these two conditions, it is natural to expect the same is true for $\mathcal{M}_C^A$, as twos are interchanged under the mirror symmetry, as will be explained in §5(i). Note that it is usually easy to check these conditions for $\mathcal{M}_H^A$, and we have lots of examples, say quiver varieties of type A or affine type A, toric hyper-Käher manifolds.

No general recipe to construct a symplectic duality pair was given in [BLPW14]. Since we will propose a definition of $\mathcal{M}_C^A$ in this paper, this defect will be fixed. Moreover we hope that we could give a better understanding on the symplectic duality, and gain a possibility to generalize it to more general cases when two conditions above are not satisfied. We have interesting sets of examples, where only one of two conditions is satisfied.

1(viii). Hikita conjecture

Recently Hikita [Hik15] proposes a remarkable conjecture. Suppose that G is a product of general linear groups, such as quiver gauge theories §2(iv) and abelian cases §2(vi). Using perturbation of the moment map equation, we can modify the hyper-Kähler quotient $\mathbf{M}/\!\!/\!\!/G$ to $\boldsymbol{\mu}^{-1}(\zeta)/G$. In many cases, it is a smooth manifold, and assume that this happens.

On the Coulomb branch side, we have an action of a torus T on $\mathcal{M}_C$, where T is the Pontryagin dual of $\pi_1(G)$. See (c) in §4(iii) below. Let $\mathcal{M}_C^T$ be the fixed point subscheme. Hikita conjectures that there exists a ring isomorphism

$$\mathbb{C}[\mathcal{M}_C^T] \overset{?}{\cong} H^*(\boldsymbol{\mu}^{-1}(\zeta)/G),$$

and checks for several nontrivial cases. This conjecture obviously has a similar flavor with our study, though a precise relation is not clear yet. The author is currently considering what happens if we consider the equivariant quantum cohomology group of $\boldsymbol{\mu}^{-1}(\zeta)/G$. It seems the quantized Coulomb branch defined in [BFN16a] plays a role.

1(ix). A proposal of a definition of $\mathcal{M}_C$

We take the cohomology group of moduli spaces of solutions of the generalized vortex equation for the gauged nonlinear σ-model on S^2 as a starting point for a definition of $\mathcal{M}_C$, as discussed in §1(v). The equation will be discussed in detail in §6. See (6.9). However we need to modify the definition, as we cannot obtain a reasonable answer for $\mathrm{Hyp}(0) /\!\!/\!\!/ \mathrm{SU}(2)$, as we have already remarked.

Our proposal here is the following modifications:

1) Drop the last equation in (6.9), which is related to the stability condition via the Hitchin-Kobayashi correspondence.
2) Consider the cohomology group with coefficients in the sheaf of a vanishing cycle.

Thus we propose

$$\mathbb{C}[\mathcal{M}_C] \stackrel{?}{=} H^{*+\dim\mathcal{F}-\dim\mathcal{G}_{\mathbb{C}}(P)}_{c,\mathcal{G}_{\mathbb{C}}(P)}\left(\left\{(A,\Phi)\,\middle|\, \begin{matrix}(\overline{\partial}+A)\Phi = 0\\ \boldsymbol{\mu}_{\mathbb{C}}(\Phi)=0\end{matrix}\right\}, \varphi_{\mathrm{CS}}(\mathbb{C}_{\mathcal{F}})\right)^*,$$

where $\overline{\partial}+A$ is a partial connection on a $G_{\mathbb{C}}$-bundle P on $S^2=\mathbb{P}^1$, Φ is a section of an associated vector bundle twisted by $\mathcal{O}_{\mathbb{P}^1}(-1)$, and $\boldsymbol{\mu}_{\mathbb{C}}$ is the complex moment map. And $\mathcal{F}$ is the space of all (A,Φ) imposing no equations, $\mathcal{G}_{\mathbb{C}}(P)$ the complex gauge group, and φ_{CS} is the vanishing cycle functor associated with the generalized Chern-Simons functional CS defined on $\mathcal{F}$. See §7 for more detail.

The moduli space above can be *loosely* regarded as the space parametrizing twisted holomorphic maps from $\mathbb{P}^1$ to the Higgs branch $\mathcal{M}_H$. It is literally true if we replace $\mathcal{M}_H = \boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/G_{\mathbb{C}}$ by the quotient stack $[\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}]$.

The vanishing cycle functor φ_{CS} with respect to the generalized Chern-Simons functional CS is strongly motivated by the one appearing in the theory of Donaldson-Thomas invariants for Calabi-Yau 3-categories.

The whole paper is devoted to explain why these modifications are natural. We see that (1) is inevitable even at this stage: (6.9) is just $F_A=0$ for $\mathrm{Hyp}(0) /\!\!/\!\!/ \mathrm{SU}(2)$. We cannot think of any reasonable modification, which gives us a non-trivial solution for S^2. For $\mathrm{Hyp}(\mathbb{H}) /\!\!/\!\!/ \mathrm{U}(1)$, moduli spaces depend on the choice of a stability condition, though the Coulomb branch should not. This problem apparently is related to the dependence of invariants of perturbation, mentioned at the end of §1(v). It has a similar flavor with the problem arising the definition of $\mathrm{SU}(r)$ Casson invariants, mentioned in §1(iv). Thus forgetting the equation and considering all connections seem the only reasonable candidate for the modification.

The definition of the multiplication, when $\mathbf{M}$ is of cotangent type as explained below, will be postponed to [BFN16a]. The goal of [BFN16a] will be to propose a definition of $\mathcal{M}_C$ as an affine scheme, i.e., the definition of its coordinate ring as a commutative ring. There remain lots to be done, in particular, we have no idea how to define a hyper-Kähler metric on $\mathcal{M}_C$ at this moment, though we could construct a natural noncommutative deformation (or quantization) of $\mathcal{M}_C$.

Also we are very far from checking our proposal reproduces various known examples already mentioned above, except $\mathbf{M} = 0$ and toric cases. Nevertheless we will reproduce the monopole formula (1.2), so we believe that our proposal passes the first check that it is a correct mathematical definition of Coulomb branches.

The paper is organized as follows. In §2 we review the hyper-Kähler quotient construction and examples of hyper-Kähler manifolds arising in this way. In §3 we give examples where the mirror of a gauge theory is not a gauge theory though it is still a reasonable theory. In §4 we review the monopole formula and its various properties. In particular, we start to list expected properties of the Coulomb branch suggested from the monopole formula. In §5 we review the monopole formula when a gauge theory has an additional flavor symmetry. We add a few properties to the list. Up to here, all materials are review of earlier works.

In §6 we consider a generalized Seiberg-Witten equation associated with a gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ and study the compactness property of the moduli space. We also study the dimension reduction of the equation to write down the generalized vortex equation, which will lead us to the proposed definition of the Coulomb branch. In §7 we observe that the complex part of the reduced equation on a Riemann surface arises the Euler-Lagrange equation of an analog of the holomorphic Chern-Simons functional, and hence it is natural to consider the analog of Donaldson-Thomas invariants, or more precisely the cohomology of the vanishing cycle. We then formally apply results on the vanishing cycle, known in the finite dimensional situation, to our case to reduce the equation further. In §8 we compute the dimension of the cohomology group and check that it reproduces the monopole formula.

§9 is a detour where we find that a few earlier works are nicely fit with the framework of §7 when the curve is the complex line $\mathbb{C}$.

In §A we give further examples of hyper-Kähler quotients related to instantons for classical groups. In §B we give a computation of the monopole formula in a particular example when the Coulomb branch is a symmetric product of a surface.

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2. Examples of hyper-Kähler quotients

In view of the $3d$ mirror symmetry, it is natural to expect that the Higgs branch $\mathcal{M}_H$ and the Coulomb branch $\mathcal{M}_C$ share similar properties. Therefore it is important to have examples of gauge theories whose Higgs branches (i.e., hyper-Kähler quotients of linear spaces) are well-understood. In this section we prepare notation and basics of hyper-Kähler quotients in the first three subsections, and then we review two important classes of gauge theories, quiver gauge theories and abelian theories. Further examples will be given in §A.

Here we consider hyper-Kähler quotients only in finite dimension. If we allow infinite dimensional ones, we have more examples, such as instanton moduli spaces for arbitrary gauge groups, solutions of Nahm's equation, etc. However it is not clear how to consider the corresponding Coulomb branches, in particular, the monopole formula introduced in §4 below.

2(i). Hyper-Kähler quotients of linear spaces

Let G be a compact Lie group with the Lie algebra $\mathfrak{g}$. Let $\mathbf{M}$ be its quaternionic representation. Let I, J, K denote multiplication by i, j, k, considered as linear operators on $\mathbf{M}$. A quaternionic representation of G is a representation such that the G-action commutes with I, J, K. We suppose $\mathbf{M}$ has a G-invariant inner product $(\ ,\ )$ which is hermitian with respect to all I,J,K. Therefore $\mathbf{M}$ is a hyper-Kähler manifold with a G-action preserving the hyper-Kähler structure. We have the hyper-Kähler moment map $\boldsymbol{\mu}\colon \mathbf{M} \to \mathfrak{g}^* \otimes \mathbb{R}^3$, vanishing at the origin:

$$\langle \xi, \boldsymbol{\mu}(\phi)\rangle = \frac{1}{2}((I\xi\phi, \phi), (J\xi\phi, \phi), (K\xi\phi, \phi)),$$

where $\phi \in \mathbf{M}$, $\xi \in \mathfrak{g}$, and $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. A hyper-Kähler moment map is, by definition, (a) G-equivariant, and (b) satisfying

$$\langle \xi, d\boldsymbol{\mu}_\phi(\dot{\phi})\rangle = (\omega_I(\xi^*, \dot{\phi}), \omega_J(\xi^*, \dot{\phi}), \omega_K(\xi^*, \dot{\phi})),$$

where $\dot{\phi}$ is a tangent vector, ξ^* is the vector field generated by ξ, and ω_I, ω_J, ω_K are Kähler forms associated with three complex structures I, J, K and the inner product. It is direct to see that two properties are satisfied in the above formula.

In the following, we only use the underlying complex symplectic structure. Let us give another formulation. Let $G_{\mathbb{C}}$ be the complexification of G. Let $\mathbf{M}$ be its complex representation, which has a complex symplectic form $\omega_{\mathbb{C}}$ preserved by $G_{\mathbb{C}}$. We have the complex moment map $\boldsymbol{\mu}_{\mathbb{C}}\colon \mathbf{M} \to \mathfrak{g}_{\mathbb{C}}$

$$\langle \xi, \boldsymbol{\mu}_{\mathbb{C}}(\phi)\rangle = \frac{1}{2}\omega_{\mathbb{C}}(\xi\phi, \phi),$$

where $\xi \in \mathfrak{g}_{\mathbb{C}}$. This is the complex part of the hyper-Kähler moment map under the identification $\mathbb{R}^3 \cong \mathbb{R} \oplus \mathbb{C}$.

We consider the hyper-Kähler quotient

$$\mathbf{M}/\!\!/\!\!/G \overset{\text{def.}}{=} \boldsymbol{\mu}^{-1}(0)/G \cong \boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/G_{\mathbb{C}},$$

where $/\!\!/G_{\mathbb{C}}$ is the affine algebro-geometric invariant theory quotient, and the second isomorphism follows from a result of Kempf-Ness (see e.g., [Nak99, Th. 3.12]). Let $\boldsymbol{\mu}^{-1}(0)^{\mathrm{reg}}$ be the (possibly empty) open subset of $\boldsymbol{\mu}^{-1}(0)$, consisting of points with trivial stabilizers. Then G acts freely on $\boldsymbol{\mu}^{-1}(0)^{\mathrm{reg}}$, and the quotient $\boldsymbol{\mu}^{-1}(0)^{\mathrm{reg}}/G$ is a smooth hyper-Kähler manifold. The Higgs branch of the 3d $N=4$ SUSY gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ is $\boldsymbol{\mu}^{-1}(0)^{\mathrm{reg}}/G$ or its closure in $\mathbf{M}/\!\!/\!\!/ G$ depending on the situation. We only consider $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ hereafter, but we actually mean $\boldsymbol{\mu}^{-1}(0)^{\mathrm{reg}}/G$ when we talk the hyper-Kähler structure on it.

The hyper-Kähler quotient $\mathbf{M}/\!\!/\!\!/ G$ has a natural $\mathrm{SU}(2) = \mathrm{Sp}(1)$-action induced from the $\mathbb{H}$-module structure of $\mathbf{M}$. It commutes with the G-action, and hence descends to the quotient. It rotates the hyper-Kähler structure. In the complex symplectic notation, its restriction to $\mathrm{U}(1)$-action is induced by the scalar multiplication on $\mathbf{M}$.

2(ii). Cotangent type

There is a class of quaternionic representations, which we call *cotangent type*. Let $\mathbf{N}$ be a complex representation of G. We put a G-invariant hermitian inner product on $\mathbf{N}$. Then $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ is a quaternionic representation of G. We define J by $J(x, y) = (-y^\dagger, x^\dagger)$ for $x \in \mathbf{N}$, $y \in \mathbf{N}^*$, where $x^\dagger \in \mathbf{N}^*$ is defined by $\langle x^\dagger, n\rangle = (n, x)$ for $n \in \mathbf{N}$, and $y^\dagger$ is defined so that $(x^\dagger)^\dagger = x$ for all $x \in \mathbf{N}$. Then J is skew-linear as the hermitian inner product is skew-linear in the second variable. We have $J^2 = -\mathrm{id}$ from the definition.

In the (complex) symplectic formulation, we start with a complex representation $\mathbf{N}$ of $G_{\mathbb{C}}$, and take $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ with a natural symplectic structure. Then $\mathbf{M}$ is naturally a representation of $G_{\mathbb{C}}$, and has a symplectic form preserved by $G_{\mathbb{C}}$.

2(iii). Complete intersection

Although $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/ G_{\mathbb{C}}$ makes sense as an affine scheme without any further condition, we do not expect they behave well in general. We propose to assume

- $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ is a complete intersection in $\mathbf{M}$.

More precisely it means as follows. We construct a Koszul complex from $\boldsymbol{\mu}_{\mathbb{C}}$:

$$
\begin{aligned}
0 \to \textstyle\bigwedge^{\dim \mathfrak{g}_{\mathbb{C}}} \mathfrak{g}_{\mathbb{C}} \otimes \mathcal{O}_{\mathbf{M}} \to \cdots \to \textstyle\bigwedge^2 \mathfrak{g}_{\mathbb{C}} \otimes \mathcal{O}_{\mathbf{M}} \\
\to \mathfrak{g}_{\mathbb{C}} \otimes \mathcal{O}_{\mathbf{M}} \to \mathcal{O}_{\mathbf{M}} \to \mathcal{O}_{\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)} \to 0.
\end{aligned}
$$

Our assumption says that this is exact. Under this assumption, the coordinate ring of $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ is given by

$$\mathbb{C}[\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)] = \sum_{i=0}^{\dim \mathfrak{g}_{\mathbb{C}}} (-1)^i \bigwedge^i \mathfrak{g}_{\mathbb{C}} \otimes \mathbb{C}[\mathbf{M}].$$

This is an equality of virtual $\mathbb{C}^* \times G_{\mathbb{C}}$-modules. Taking the $G_{\mathbb{C}}$-invariant part, we get the Hilbert series of $\mathbb{C}[\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/G_{\mathbb{C}}]$. This is the definition used in the K-theoretic Nekrasov partition function for instantons of classical groups. See [NS04].

We expect that this complete intersection assumption is related to the 'good' or 'ugly' condition, appearing in the monopole formula. But it is a superficial observation as we are discussing the Higgs branch now, while the monopole formula is about the Coulomb branch.

2(iv). Quiver gauge theory

Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. We can construct a quaternionic representation of $G = G_V = \prod_{i \in Q_0} \mathrm{U}(V_i)$ associated with Q_0-graded representations $V = \bigoplus V_i$, $W = \bigoplus W_i$. It is a cotangent type, and given in the complex symplectic description by

$$\begin{aligned}\mathbf{N} &= \bigoplus_{h \in Q_1} \operatorname{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)}) \oplus \bigoplus_{i \in Q_0} \operatorname{Hom}(W_i, V_i),\\ \mathbf{M} &= \bigoplus_{h \in Q_1} \operatorname{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)}) \oplus \operatorname{Hom}(V_{\mathrm{i}(h)}, V_{\mathrm{o}(h)})\\ &\qquad \oplus \bigoplus_{i \in Q_0} \operatorname{Hom}(W_i, V_i) \oplus \operatorname{Hom}(V_i, W_i).\end{aligned}$$

Here $\mathrm{o}(h)$ and $\mathrm{i}(h)$ are the outgoing and incoming vertices of the oriented edge $h \in Q_1$ respectively.

If Q is the Jordan quiver, the hyper-Kähler quotient $\boldsymbol{\mu}^{-1}(0)/G$ is the ADHM description of the framed moduli space of $\mathrm{SU}(W)$-instantons on $\mathbb{R}^4$, or more precisely its Uhlenbeck partial compactification (see e.g., [Nak99, Ch.3] and the reference therein). More generally, hyper-Kähler quotient $\mathbf{M}/\!\!/\!\!/G = \boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/G_{\mathbb{C}}$ is the quiver variety (with complex and stability parameters 0), introduced in [Nak94].

For a quiver gauge theory, $Z(\mathfrak{g}^*) = \{\zeta_{\mathbb{R}} \in \mathfrak{g}^* \mid \mathrm{Ad}_g^*(\zeta_{\mathbb{R}}) = \zeta_{\mathbb{R}} \text{ for any } g \in G\}$ is nontrivial, and is isomorphic to $\bigoplus_{i \in Q_0} \mathbb{R}\sqrt{-1}\,\mathrm{tr}_{V_i}$, where $\mathrm{tr}_{V_i} : \mathfrak{u}(V_i) \to$

$\sqrt{-1}\mathbb{R}$ is the trace. Then we can form a perturbed hyper-Kähler quotient $\boldsymbol{\mu}^{-1}(\zeta)/G$ for $\zeta \in \mathbb{R}^3 \otimes Z(\mathfrak{g}^*)$. If we decompose ζ as $(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ according to $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{C}$, we have an algebro-geometric description

$$\boldsymbol{\mu}^{-1}(\zeta)/G \cong \boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/\!\!/_{\zeta_{\mathbb{R}}} G_{\mathbb{C}},$$

where $/\!\!/_{\zeta_{\mathbb{R}}}$ is the GIT quotient with respect to the $\zeta_{\mathbb{R}}$-stability. (See [Nak94, §3].) If ζ is generic, $\boldsymbol{\mu}^{-1}(\zeta)/G$ is a smooth hyper-Kähler manifold whose metric is complete. Moreover, we have a projective morphism

$$\pi \colon \boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/\!\!/_{\zeta_{\mathbb{R}}} G_{\mathbb{C}} \to \boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/\!\!/ G_{\mathbb{C}} = \boldsymbol{\mu}^{-1}(0, \zeta_{\mathbb{C}})/G.$$

In an algebro-geometric approach to hyper-Kähler quotients, it is more natural to replace $\zeta_{\mathbb{R}}$ by a corresponding element for the group G, i.e., a character $\chi \colon G \to \mathrm{U}(1)$, where they are related by $\zeta_{\mathbb{R}} = d\chi$. The character χ defines a $G_{\mathbb{C}}$-equivariant structure on the trivial line bundle over $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$. We introduce the stability condition and form a GIT quotient with a natural projective morphism to $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/\!\!/ G$:

$$\pi \colon \boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/\!\!/_{\chi} G_{\mathbb{C}} \to \boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/\!\!/ G_{\mathbb{C}} \tag{2.1}$$

It is equipped with a relatively ample line bundle $\mathcal{L}_\chi$ in a natural way. See [Nak99, §3] for detail.

In Figure 1 we follow physicists convention. The underlying graph of the quiver is circled vertices and edges connecting them. (An orientation of the quiver is not relevant, and usually omitted.) Dimensions of V_i are put in circled vertices, while dimensions of W_i are put in the boxed vertices, connected to the corresponding circled vertices. It is more or less the same as the original convention in [KN90].

When $W = 0$, the scalar $\mathrm{U}(1)$ acts trivially, so we replace G by $\prod_{i \in Q_0} \mathrm{U}(V_i)/\,\mathrm{U}(1)$. This is the case for $\mathbf{M}$ used by Kronheimer [Kro89], mentioned in Introduction.

For a quiver gauge theory, good and ugly conditions were analyzed in [GW09, §2.4, §5.4]. It is conjectured, for example, that a quiver gauge theory of finite type is good or ugly if and only if

$$\dim W_i - \sum_j (2\delta_{ij} - a_{ij}) \dim V_j \geq -1 \tag{2.2}$$

for any $i \in Q_0$.[1] Here a_{ij} is the number of edges (regardless of orientation) between i and j if $i \neq j$, and its twice if $i = j$.

For quiver varieties $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/G_{\mathbb{C}}$ with $W = 0$, Crawley-Boevey [CB01, Th.1.1] gave a combinatorial condition for the complete intersection assumption above. It can be modified to cover the $W \neq 0$ case using the trick [CB01, the end of Introduction]. To state the result, let us prepare notation. Let $\mathbf{C} = (2\delta_{ij} - a_{ij})$ be the Cartan matrix. We denote the dimension vectors $(\dim V_i)_{i\in Q_0}$, $(\dim W_i)_{i\in Q_0}$ by $\mathbf{v}$, $\mathbf{w}$ respectively. A *root* is an element of $\mathbb{Z}^{Q_0}$ obtained from the coordinate vector at a loopfree vertex or $\pm$ an element of the fundamental region by applying a sequence of reflections at loopfree vertices. If there are no loops, this notion coincides with the usual notion of roots of the corresponding Kac-Moody Lie algebra by [Kac90, Th. 5.4].

Then $\mu_{\mathbb{C}}^{-1}(0)$ is a complete intersection if and only if the following is true:

- ${}^t\mathbf{v}(2\mathbf{w} - \mathbf{C}\mathbf{v}) \geq {}^t\mathbf{v}^0(2\mathbf{w} - \mathbf{C}\mathbf{v}^0) + \sum_k (2 - {}^t\beta^{(k)}\mathbf{C}\beta^{(k)})$ for any decomposition $\mathbf{v} = \mathbf{v}^0 + \sum_k \beta^{(k)}$ such that $\mathbf{w} - \mathbf{v}^0$ is a weight of an irreducible highest weight module $V(\mathbf{w})$ of the highest weight $\mathbf{w}$, and $\beta^{(k)}$ is a positive root.

(cf. [Nak09, Th.2.15(2)] for a closely related condition.) The dominance condition (2.2) is a necessary condition, from the decomposition $\mathbf{v} = \mathbf{v}^0 + \alpha_i$, but not a sufficient if there is $\beta^{(k)}$ with ${}^t\beta^{(k)}\mathbf{C}\beta^{(k)} < 2$, i.e., an imaginary root. Anyway these two conditions are closely related. This is the reason why we expect the complete intersection assumption and the 'good or ugly' condition are the same.

2(v). Type A quiver, nilpotent orbits and affine Grassmannian

As a special class of quiver gauge theories, type A (or linear) quiver gauge theories are important. The hyper-Kähler quotient, in other words, Higgs branch was identified with $\mathcal{O}_\mu \cap \mathcal{S}_\lambda$, where $\mathcal{O}_\mu$ is a nilpotent orbit and $\mathcal{S}_\lambda$ is Slodowy slice to another orbit $\mathcal{O}_\lambda$ of type A (see [Nak94, §8]). Here we assume $\boldsymbol{\mu}^{-1}(0)^{\mathrm{reg}} \neq \emptyset$, and two partitions λ, μ are defined from dimensions of V, W by an explicit formula.[2] Conversely any $\mathcal{O}_\mu \cap \mathcal{S}_\lambda$ for type A is described as a hyper-Kähler quotient.

---

[1] The conjecture is stated only for goodness, and the right hand side is replaced by 0 in [GW09].

[2] There is typo in [Nak94, §8]. μ must be replaced by its transpose.

Let us recall how the identification is constructed. The starting point is Kronheimer's realization [Kro90] of $\mathcal{O}_\mu \cap \mathcal{S}_\lambda$ as moduli spaces of SU(2)-equivariant instantons on $\mathbb{R}^4$. This construction works for any compact Lie groups. For type A, we apply the ADHM transform to these instantons. Suppose that an instanton corresponds to $(B_1, B_2, a, b) \in \mathbf{M} = \operatorname{Hom}(V, V)^{\oplus 2} \oplus \operatorname{Hom}(W, V) \oplus \operatorname{Hom}(V, W)$, the data for the Jordan quiver. If the original instanton is SU(2)-equivariant, V, W are representations of SU(2), and (B_1, B_2), a, b are SU(2)-linear. Here we mean the pair (B_1, B_2) is SU(2)-equivariant, when it is considered as a homomorphism in $\operatorname{Hom}(V, V \otimes \rho_2)$, where ρ_2 is the vector representation of SU(2). Let ρ_i be the i-dimensional irreducible representation of SU(2). We decompose V, W as $\bigoplus V_i \otimes \rho_i$, $\bigoplus W_i \otimes \rho_i$. Then a, b are maps between V_i and W_i. By the Clebsch-Gordan rule $\rho_i \otimes \rho_2 = \rho_{i-1} \oplus \rho_{i+1}$, (B_1, B_2) decomposes into maps between V_i and $V_{i-1} \oplus V_{i+1}$. The dimensions V_i, W_i are determined by λ, μ, as mentioned above. In a nutshell, the McKay quiver for SU(2) is the double of type A_∞ Dynkin graph (figure 2). Hence we get a quiver variety of type A_∞.

$$\overset{1}{\bullet} \leftrightarrows \overset{2}{\bullet} \leftrightarrows \overset{3}{\bullet} \leftrightarrows \overset{4}{\bullet} \leftrightarrows \cdots$$

Figure 2: McKay quiver for SU(2)

A quiver variety of type A can be obtained also from a framed moduli space of S^1-equivariant instantons on $\mathbb{R}^4$, where S^1 acts on $\mathbb{R}^4 = \mathbb{C}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. The reason is the same as above: (a) irreducible representations ρ_i of S^1 are parametrized by integers $i \in \mathbb{Z}$, i.e., weights, and (b) $\rho_i \otimes \mathbb{C}^2 = \rho_{i-1} \oplus \rho_{i+1}$, where $\mathbb{C}^2$ is the base manifold, identified with $\rho_1 \oplus \rho_{-1}$ as an S^1-module. Strictly speaking, McKay quiver for S^1 is slightly different from one for SU(2), and infinite in both direction (figure 3). But the quiver varieties remain the same as V is finite-dimensional.

$$\cdots \leftrightarrows \overset{-2}{\bullet} \leftrightarrows \overset{-1}{\bullet} \leftrightarrows \overset{0}{\bullet} \leftrightarrows \overset{1}{\bullet} \leftrightarrows \overset{2}{\bullet} \leftrightarrows \cdots$$

Figure 3: McKay quiver for S^1

By [BF10, §5] the framed moduli space of S^1-equivariant G-instantons on $\mathbb{R}^4$ is also identified with the intersection $\mathcal{W}^\mu_{G,\lambda}$ of a $G[[z]]$-orbit Gr^μ_G in the affine Grassmannian $\mathrm{Gr}_G = G((z))/G[[z]]$ with a transversal slice to another $G[[z]]$-orbit Gr_G^λ. Here we regard λ, μ as homomorphisms $S^1 \to G$, i.e., coweights of G.[3] This result is true for any G. Note that the identification of partitions with coweights in [BF10] is different from one in [Nak94, §10], which will be used below.

Thus we have identifications

$$\begin{array}{ccc} \mathcal{O}_\mu \cap \mathcal{S}_\lambda \longleftrightarrow & \begin{array}{c}\text{a framed moduli space of SU(2)-equivariant}\\ G\text{-instantons}\end{array} & \\ & \updownarrow & \\ & \text{a quiver variety of type } A & \\ & \updownarrow & \\ & \begin{array}{c}\text{a framed moduli space of } S^1\text{-equivariant}\\ G\text{-instantons}\end{array} & \longleftrightarrow \mathcal{W}^\mu_{G,\lambda}. \end{array}$$

The identifications $\mathcal{O}_\mu \cap \mathcal{S}_\lambda$ and $\mathcal{W}^\mu_{G,\lambda}$ with quiver varieties of type A were first found in [MV03]. Note however that horizontal arrows remain true for arbitrary G, while vertical ones are true only for type A. In the top row λ, μ are nilpotent orbits, while they are coweights in the bottom row. Therefore we cannot hope a vertical relation for general G. Therefore one should understand $\mathcal{O}_\mu \cap \mathcal{S}_\lambda \cong \mathcal{W}^\mu_{G,\lambda}$ as a composition of the natural horizontal identifications and accidental vertical ones. If we apply the ADHM transform to SU(2) and S^1-equivariant instantons for classical groups respectively, we will obtain different modifications of quiver varieties. It will be discussed in §§§A.1,A.2,A.3. For a finite subgroup $\Gamma \subset \mathrm{SU}(2)$, we can also consider Γ-equivariant instantons in the same way. See §A.4.

The mirror of this theory is given by $\mathcal{O}_{\lambda^t} \cap \mathcal{S}_{\mu^t}$, where λ^t, μ^t are transpose partitions. (It is not clear at this moment, what is the mirror if $\boldsymbol{\mu}^{-1}(0)^{\mathrm{reg}} = \emptyset$, i.e., μ is not necessarily dominant.) This mirror symmetry can be naturally extended to the case of quiver gauge theories of affine type A. It nicely fits with the level-rank duality of affine Lie algebras of type A via the author's work [Nak94]. This was observed by de Boer et al [dBHO⁺97, §3] based on brane configurations in string theories introduced by Hanany-Witten [HW97]. Further examples of the mirror symmetry will be given in §3.

[3] Here λ (resp. μ) corresponds to a homomorphism at ∞ (resp. 0) of $\mathbb{C}^2$. Since we follow the convention in [Nak94], this is opposite to [BF10]. In particular, $\mathcal{W}^\mu_{G,\lambda}$ is empty unless $\lambda \leq \mu$.

2(vi). Abelian theory

Let us take a collection of nonzero integral vectors $u_1, \ldots, u_d$ in $\mathbb{Z}^n$ such that they span $\mathbb{Z}^n$. We have an exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z}^{d-n} \xrightarrow{\alpha} \mathbb{Z}^d \xrightarrow{\beta} \mathbb{Z}^n \to 0, \tag{2.3}$$

where $\beta\colon \mathbb{Z}^d \to \mathbb{Z}^n$ is given by sending the coordinate vector e_i to u_i, and the kernel of $\mathbb{Z}^d \to \mathbb{Z}^n$ is identified with $\mathbb{Z}^{d-n}$ by taking a base. We have the corresponding exact sequence of tori:

$$1 \to G = \mathrm{U}(1)^{d-n} \xrightarrow{\alpha} T^d = \mathrm{U}(1)^d \xrightarrow{\beta} G_F = \mathrm{U}(1)^n \to 1, \tag{2.4}$$

where maps in (2.3) are induced homomorphisms between coweight lattices (or equivalently fundamental groups).

Let $\mathbf{M} = \mathbb{H}^d$ and let T^d act $\mathbb{H}$-linearly on $\mathbf{M}$ by multiplication. We consider $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$. It is called the *abelian theory*. It is of cotangent type with $\mathbf{N} = \mathbb{C}^d$.

The hyper-Kähler quotient $\mathbf{M} /\!\!/\!\!/ G$ is called a *toric hyper-Kähler manifold* and was introduced by Bielawski and Dancer [BD00]. Note that we have an action of G_F on $\mathbf{M} /\!\!/\!\!/ G$. This group is called a flavor symmetry group, and its importance will be explain in §5 below.

The space $Z(\mathfrak{g}^*)$ is nontrivial as in quiver gauge theories, and $\boldsymbol{\mu}^{-1}(\zeta_{\mathrm{Im}\,\mathbb{H}})/G$ is a hyper-Kähler orbifold for generic $\zeta_{\mathrm{Im}\,\mathbb{H}}$.

The abelian theory is a good example to understand the $3d$ mirror symmetry. We dualize the exact sequence (2.3) to get

$$1 \to G_F^\vee \xrightarrow{\beta^\vee} (T^d)^\vee \xrightarrow{\alpha^\vee} G^\vee \to 1,$$

where $\bullet^\vee$ denotes the dual torus, defined by $\pi_1(\bullet)^\wedge$. ($G^\vee$ is also $\mathrm{U}(1)^{d-n}$, but we would like to make our framework intrinsic.) Then we can consider another toric hyper-Kähler manifold $\mathbf{M} /\!\!/\!\!/ G_F^\vee$ with a $G^\vee$-action. It was proposed in [dBHO+97, §4] that $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ and $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G_F^\vee$ form a mirror dual theories. In particular, the Coulomb branch of $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ is $\mathbf{M} /\!\!/\!\!/ G_F^\vee$.

3. More on $3d$ mirror symmetry

It is important to have many examples of 3-dimensional mirror symmetric pairs, as they determine the Coulomb branches as the Higgs branches of mirror theories. We give more examples in this section.

3(i). Mirror could be a non-lagrangian theory

We first remark that the mirror of the gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ may not be of a form $\mathrm{Hyp}(\mathbf{M}') /\!\!/\!\!/ G'$ for some $\mathbf{M}'$ and G' in general. For example, if we replace the diagram in Figure 1 by the affine Dynkin diagram of type $E^{(1)}_{6,7,8}$ as in Figure 4, it is expected that $\mathcal{M}_C$ is the framed moduli space of $E_{6,7,8}$ k-instantons on $\mathbb{R}^4$. This example was found in [IS96] for $k=1$, and in [dBHO+97] for general k. It is widely accepted a common belief that

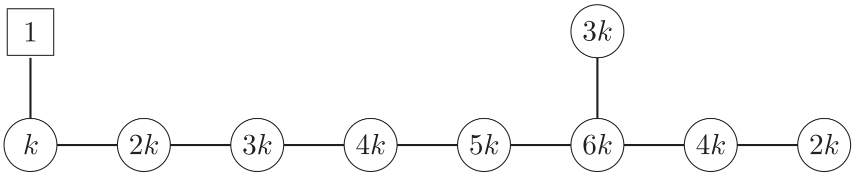

Figure 4: $\mathcal{M}_C$: E_8 k-instantons on $\mathbb{R}^4$.

there are no ADHM like description of instantons for exceptional groups. It means that the moduli spaces cannot be given by a hyper-Kähler quotient $\mathbf{M}' /\!\!/\!\!/ G'$ (with finite dimensional $\mathbf{M}'$, G'), hence the mirror theory B is not of a form $\mathrm{Hyp}(\mathbf{M}') /\!\!/\!\!/ G'$. In fact, the mirror theory B is known as a $3d$ Sicilian theory [BTX10], which does not have a conventional lagrangian description. See §3(iii) below. Nevertheless we can compute Hilbert series of instanton moduli spaces of exceptional types by the monopole formula.[4] This is even more exciting, as there is only a few way to compute them, say a conjectural blowup equation [NY05a, NY05b].

3(ii). Instantons on $\mathbb{R}^4/\Gamma$

Let us consider a quiver gauge theory of affine type. An affine quiver of type ADE arises as the McKay quiver of a finite subgroup Γ of $\mathrm{SU}(2)$. Hence the Higgs branch, the hyper-Kähler quotient of $\mathbf{M}$ by G, parametrizes Γ-equivariant $\mathrm{U}(\ell)$-instantons on $\mathbb{R}^4$ as in §2(v). In fact, this is a starting point of the work [KN90], which eventually leads to the study of quiver varieties [Nak94].

As we have mentioned already in §2(v), the mirror of a quiver gauge theory of affine type A is another quiver gauge theory again of affine type A. The precise recipe was given in [dBHO+97, §3.3].

[4] One need to modify the monopole formula to deal with non simply-laced groups. See [CFHM14].

From this example, together with Braverman-Finkelberg's proposed double affine Grassmannian [BF10], we will give an initial step towards the determination of the mirror of the quiver gauge theory of an arbitrary affine type as follows. It was mentioned in a vague form in [BLPW14, Rem. 10.13].

A framed moduli space of Γ-equivariant G-instantons on $\mathbb{R}^4$ has discrete data, a usual instanton number, as well as $\rho_0, \rho_\infty\colon \Gamma \to G$ homomorphisms from Γ to G given by Γ-actions on fibers at 0 and ∞. A quiver variety, that is the Higgs branch of a quiver gauge theory of affine type, corresponds to the case $G = \mathrm{U}(\ell)$. We regard ρ_0, ρ_∞ as ℓ-dimensional representations of Γ. They are given by dimension vectors $(\dim V_i)_{i\in Q_0}$, $(\dim W_i)_{i\in Q_0}$ by

$$\begin{aligned} \rho_\infty &= \bigoplus \rho_i^{\oplus \dim W_i}, \\ \rho_0 &= \bigoplus \rho_i^{\oplus u_i} \quad \text{with } u_i = \dim W_i - \sum_j (2\delta_{ij} - a_{ij}) \dim V_j, \end{aligned}$$

where $\{\rho_i\}$ is the set of isomorphism classes of irreducible representations of Γ, identified with Q_0 via McKay correspondence. If some u_i is negative, there is no genuine instanton. In other words, the Higgs branch $\mathcal{M}_H = \mathbf{M}/\!\!/\!\!/G$ contains no free orbits. We do not know what happens in the Coulomb branch without this assumption. Conversely u_i determines $\dim V$ modulo the kernel of the Cartan matrix $\mathbf{C}$, i.e., $\mathbb{Z}\delta$ for the (primitive) imaginary root δ. This ambiguity is fixed by specifying the instanton number as $\sum_j \delta_j \dim V_j$ where δ_j is the j^{th}-entry of δ.

Let $\mathfrak{g}$ be the complex simple Lie algebra of type ADE corresponding to Γ. Let $\mathfrak{g}_{\text{aff}}$ be the associated untwisted affine Lie algebra, containing the degree operator d. In [Nak94], an affine Lie algebra representation was constructed by quiver varieties. Dimension vectors give affine weights of $\mathfrak{g}_{\text{aff}}$ by

$$\lambda = \sum_i (\dim W_i)\Lambda_i, \qquad \mu = \sum_i (\dim W_i)\Lambda_i - (\dim V_i)\alpha_i,$$

where Λ_i (resp. α_i) is the i^{th} fundamental weight (resp. simple root). The weight λ is always dominant. We also have $\lambda \geq \mu$ by definition. If we assume $u_i \geq 0$ for all i as above, the second weight μ is also dominant.

On the other hand, Braverman and Finkelberg [BF10] associate a pair $(\lambda \geq \mu)$ of affine weights with ρ_0, ρ_∞ and instantons numbers as follows. They take $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$, where ℓ is the level of λ, also of μ as $\lambda \geq \mu$. They take G a simply-connected group, possibly of type $BCFG$. Then [BF10, Lemma 3.3] says a conjugacy class of a homomorphism $\mathbb{Z}/\ell\mathbb{Z} \to G$ corresponds to a dominant coweight $\overline{\lambda}$ of $G_{\mathbb{C}}$ with $\langle \overline{\lambda}, \theta\rangle \leq \ell$, where θ is the highest root of $\mathfrak{g}$. It can be regarded as a level ℓ weight of $\widehat{G}_{\mathbb{C}}^{\vee}$, the Langlands dual of the affine Kac-Moody group $\widehat{G}_{\mathbb{C}}$. Here $\widehat{G}_{\mathbb{C}}^{\vee}$ does not contain the degree operator. We assign dominant weights $\overline{\lambda}$, $\overline{\mu}$ of level ℓ to ρ_0, ρ_∞ respectively in this way. Then we extend them to λ, μ dominant weights of the full affine Kac-Moody group $G_{\mathrm{aff}}^{\vee}$ so that

$$\text{instanton number} = \ell\langle\lambda-\mu, d\rangle + \frac{(\overline{\lambda},\overline{\lambda})}{2} - \frac{(\overline{\mu},\overline{\mu})}{2}.$$

See [BF10, (4.3)]. This rule only determines $\langle\lambda-\mu,d\rangle$, but it is well-known that representation theoretic information depend only on the difference $\lambda-\mu$.

Thus a pair of affine weights $(\lambda \geq \mu)$ correspond to instanton moduli spaces in two ways, when G is of type ADE, first in [Nak94], second in [BF10], as we have just explained. Take the quiver gauge theory whose Higgs branch is the quiver variety associated with $(\lambda \geq \mu)$ in the first way. Then its Coulomb branch is expected to be the $\mathbb{Z}/\ell\mathbb{Z}$-equivariant instanton moduli space associated with $(\lambda \geq \mu)$ in the second way.

However this is not precise yet by the following reason. Since affine weights of the Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ may not give weights of $G_{\mathrm{aff}}^{\vee}$ if G is simply-connected, we need to replace G by its adjoint quotient. Then [BF10, Lemma 3.3] says a homomorphism $\mathbb{Z}/\ell\mathbb{Z} \to G$ corresponds to an element in the coset $\Lambda/W_{\mathrm{aff},\ell}$, where Λ is the coweight lattice of G, and $W_{\mathrm{aff},\ell}$ is the semi-direct product $W \ltimes \ell\Lambda$ of the Weyl group W and Λ. Here $\ell\Lambda$ acts on Λ naturally. If G is of *adjoint type*, Λ is the weight lattice of $G^{\vee}$, i.e., the weight lattice of $\mathfrak{g}$. But $W_{\mathrm{aff},\ell}$ is an extended affine Weyl group, i.e., the semi-direct product of the ordinary affine Weyl group and a group $\mathcal{T}$ consisting of affine Dynkin diagram automorphisms. Then a point in the coset $\Lambda/W_{\mathrm{aff},\ell}$ does not give an affine weight of $\widehat{G}_{\mathbb{C}}^{\vee}$. It only gives a $\mathcal{T}$-orbit.

When G is of type A_{r-1}, this inaccuracy can be fixed: we replace G by $\mathrm{U}(r)$, and $(\lambda \geq \mu)$ by a pair of generalized Young diagrams, in other words, dominant weights of $\widehat{\mathrm{GL}}(r,\mathbb{C})$. See [Nak09, App. A] for a detailed review. If we view both Higgs and Coulomb branches as quiver varieties of affine type A, the rule of the transform of dimension vectors is given by transpose of generalized Young diagrams, as reviewed in [Nak09, App. A]. It is the same as one in [dBHO$^+$97] up to a diagram automorphism.

If we take a gauge theory of *finite type* instead of affine type, λ, μ are dominant weights of the finite dimensional Lie algebra $\mathfrak{g}$ in [Nak94]. Then instead of [BF10], one can use just the ordinary geometric Satake correspondence, i.e., the affine Grassmannian for G of adjoint type. In terms of instantons, we use S^1-equivariant G-instantons on $\mathbb{R}^4$. Then λ, μ are regarded as dominant coweights of G, and correspond to S^1-actions on fibers at 0 and ∞. The inaccuracy disappears also in this case. This conjectural proposal was given in [BLPW14, Rem. 10.7], though their symplectic duality does not make sense in general outside type A, as we mentioned in Introduction.

3(iii). Sicilian theory

There is another class of 4-dimensional quantum field theories with $\mathcal{N}=2$ supersymmetry. They are called *theories of class S*. A theory $S_\Gamma(C, x_1, \rho_1, \dots, x_n, \rho_n)$ is specified with an ADE Dynkin diagram Γ, a punctured Riemann surface $(C, x_1, \dots, x_n)$ together with a homomorphism $\rho_i\colon \mathfrak{su}(2) \to \mathfrak{g}_\Gamma$ for each puncture x_i, where $\mathfrak{g}_\Gamma$ is the Lie algebra of a compact Lie group of type Γ. It is constructed as a dimensional reduction of a 6-dimensional theory associated with Γ, compactified on a Riemann surface C with defects at punctures specified by ρ_i. It is believed that $S_\Gamma(C, x_1, \rho_1, \dots, x_n, \rho_n)$ does not have a lagrangian description in general, hence is not a gauge theory studied in this paper. See [Tac] for a review aimed for mathematicians.

Physicists consider its Coulomb and Higgs branches. The Coulomb branch is expected to be the moduli space of solutions of Hitchin's self-duality equation on C with boundary condition at x_i given by ρ_i. On the other hand, it is asked in [MT12] what is the underlying complex symplectic manifold of its Higgs branch $\mathcal{M}_H(S_\Gamma(C, x_1, \rho_1, \dots, x_n, \rho_n))$. The underlying complex symplectic manifold is independent of the complex structure of $(C, x_1, \dots, x_n)$. It is supposed to satisfy various properties expected by physical considerations, most importantly it gives a $2d$ TQFT whose values are complex symplectic manifolds.

We can further compactify $S_\Gamma(C, x_1, \rho_1, \dots, x_n, \rho_n)$ on S^1_R and take limit $R \to 0$ to get a 3-dimensional quantum field theory with $\mathcal{N}=4$ supersymmetry. This is a $3d$ Sicilian theory mentioned above. It is expected that the Higgs branch is unchanged under the compactification by S^1_R.

When Γ is *not* exceptional, its mirror is supposed to be a certain gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ [BTX10]. Therefore the Higgs branch $\mathcal{M}_H(S_\Gamma(C, x_1, \rho_1, \dots, x_n, \rho_n))$ is the Coulomb branch of a gauge theory $\mathcal{M}_C(\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G)$, which we are studying in this paper.

Let us specify $\mathbf{M}$ and G. First suppose Γ is of type A_ℓ. If $C = S^2$, the mirror is the quiver gauge theory associated with the star shaped quiver with n legs. Entries of the dimension vector are ℓ at the central vertex, and given by ρ_i on the i^{th} leg specified by the rule as for the quiver construction of the nilpotent orbit $\rho_i(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$ (see §2(v)). If C has genus g, we add g loops at the central vertex. (Figure 5) Note also that quivers considered in §3(i) are of this type. For type D_ℓ, we modify this quiver as in §A.2.

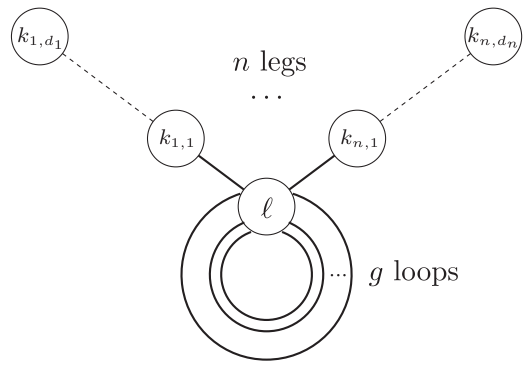

Figure 5: Mirror of a $3d$ Sicilian theory of type A_ℓ

The Coulomb branch $\mathcal{M}_C((S_\Gamma(C, x_1, \rho_1, \dots, x_n, \rho_n))$ of a $3d$ Sicilian theory is the Higgs branch of the gauge theory, which is the hyper-Kähler quotient $\mathbf{M}/\!\!/\!\!/G$. It is an additive version of the moduli space of homorphisms from the fundamental group $\pi_1(C \setminus \{x_1, \dots, x_n\})$ of the punctured Riemann surface to $\mathrm{GL}_{\ell+1}(\mathbb{C})$ or $\mathrm{SO}(2\ell, \mathbb{C})$ with prescribed conjugacy classes around punctures. (See [CB03].) There is an isomorphism between an open subset of $\mathbf{M}/\!\!/\!\!/G$ and the actual moduli space [Yam08], hence it is compatible with the expectation that $\mathcal{M}_C(S_\Gamma(C, x_1, \rho_1, \dots, x_n, \rho_n))$ is the Hitchin moduli space. When we make $R \to 0$, the Hitchin moduli space is replace by its additive version.

The monopole formula for this type of quivers is studied in [CHMZ14b].

4. Monopole formula

We discuss the monopole formula in detail in this section.

4(i). Definition

Let G be a compact Lie group. We assume G is connected hereafter for simplicity.[5] We choose and fix a maximal torus T and a set Δ^+ of positive

[5]The monopole formula for a disconnected group $\mathrm{O}(N)$ appears in [CHMZ14c].

roots. Let $Y = Y(T)$ be the coweight lattice of G. Let W denote the Weyl group.

Suppose that a quaternionic representation $\mathbf{M}$ (also called a *pseudoreal* representation) of G is given. We choose an $\mathbb{H}$-base $\{b\}$ of $\mathbf{M}$ compatible with the weight space decomposition.

We define two functions,[6] one depending on G and $\mathbf{M}$, another depending only on G, of a coweight $\lambda \in Y$ by

$$\begin{aligned} \Delta(\lambda) &\overset{\text{def.}}{=} -\sum_{\alpha\in\Delta^+} |\langle\alpha,\lambda\rangle| + \frac{1}{2}\sum_b |\langle \mathrm{wt}(b),\lambda\rangle|, \\ P_G(t;\lambda) &\overset{\text{def.}}{=} \prod \frac{1}{1-t^{2d_i}}, \end{aligned} \tag{4.1}$$

where $\langle\ ,\ \rangle$ is the pairing between weights and coweights, and the product in the second formula runs over exponents of the stabilizer $\mathrm{Stab}_G(\lambda)$ of λ. Since we take the absolute value in the second term, $\Delta(\lambda)$ is independent of the choice of b: it remains the same for jb. It is well-known that $P_G(t;\lambda)$ is equal to the Poincaré polynomial of the equivariant cohomology $H^*_{\mathrm{Stab}_G(\lambda)}(\mathrm{pt})$ of a point. Both $\Delta(\lambda)$ and $P_G(t;\lambda)$ are invariant under the Weyl group W action on Y.

We assume $2\Delta(\lambda) \geq 1$ for any $\lambda \neq 0$, a 'good' or 'ugly' theory in the sense of [GW09], hence no negative powers of t appear. (It is *good* if $2\Delta(\lambda) > 1$ and *ugly* if $2\Delta(\lambda) \geq 1$ and not good. But we do not see any differences of two conditions in this paper.) Since $\Delta(\lambda)$ is piecewise linear, there is only finitely many λ for a given $2\Delta(\lambda)$. Note also that $P_G(t;\lambda)$ can be expanded as a formal power series in t. Therefore

$$H_{G,\mathbf{M}}(t) \overset{\text{def.}}{=} \sum_{\lambda\in Y/W} t^{2\Delta(\lambda)} P_G(t;\lambda).$$

makes sense as a formal power series in t.

This elementary, but combinatorially complicated expression is the *monopole formula* for the *Hilbert series of the Coulomb branch* $\mathcal{M}_C \equiv \mathcal{M}_C(\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G)$.

Recall that physicists claim that the Coulomb branch $\mathcal{M}_C$ is a hyper-Kähler manifold with an SU(2)-action rotating hyper-Kähler structures I, J, K. One choose a complex structure I, and take $\mathrm{U}(1) \subset \mathrm{SU}(2)$, fixing I. Then the Hilbert series is the character of the coordinate ring $\mathbb{C}[\mathcal{M}_C]$ of $\mathcal{M}_C$ with respect to the U(1)-action, endowed with an affine scheme structure

[6] Following [CFHM14], we change t by t^2 from [CHZ14].

compatible with the complex structure I. Thus the main claim in [CHZ14] is

$$H_{G,\mathbf{M}}(t) = \operatorname{ch}_{\mathrm{U}(1)} \mathbb{C}[\mathcal{M}_C].$$

Remark 4.2. The good or ugly condition $2\Delta(\lambda) \geq 1$ for any $\lambda \neq 0$ means weights of $\mathbb{C}[\mathcal{M}_C]$ are nonnegative and the 0-weight space consists only on constant functions. If this is not satisfied, it is not clear whether (1.2) makes sense or not. However $\mathbb{C}[\mathcal{M}_C]$ itself might be well-defined. It is the case for $\mathrm{Hyp}(0) /\!\!/\!\!/ \mathrm{SU}(2)$ for example. The only trouble is that weight spaces might be infinite dimensional, and hence the character $\operatorname{ch}_{\mathrm{U}(1)} \mathbb{C}[\mathcal{M}_C]$ is not defined.

4(ii). Examples

Let us calculate $H_{G,\mathbf{M}}(t)$ for the simplest example. Let $G = \mathrm{U}(1)$ and $\mathbf{M} = \mathbb{H} = \mathbb{C} \oplus \mathbb{C}^*$, the vector representation plus its dual. We identify the coweight lattice $Y(\mathrm{U}(1))$ with $\mathbb{Z}$, and denote a coweight by m instead of λ.

There is no first term in $\Delta(m)$ as $\Delta^+ = \emptyset$. Thus $2\Delta(m) = |m|$. This is an ugly theory. The stabilizer of m is always $\mathrm{U}(1)$, thus $P_G(t;m) = 1/(1-t^2)$. Therefore

$$H_{\mathrm{U}(1),\mathbb{H}}(t) = \frac{1}{1-t^2} \sum_{m\in\mathbb{Z}} t^{|m|} = \frac{1}{(1-t)^2}.$$

In this case, Seiberg-Witten [SW97] claim that the Coulomb branch $\mathcal{M}_C$ is the Taub-NUT space. It is a 4-dimensional hyper-Kähler manifold with $\mathrm{Sp}(1)$-action, whose underlying complex manifold is $\mathbb{C}^2$. The subgroup commuting with the complex structure I is $\mathrm{U}(1)$ with the multiplication action on $\mathbb{C}^2$. Therefore the Hilbert series is $1/(1-t)^2$ as expected.

If we replace $\mathbf{M}$ by the direct sum of its N-copies, i.e., $\mathbf{M} = \mathbb{H}^N$, we get

$$H_{\mathrm{U}(1),\mathbb{H}^N}(t) = \frac{1+t^N}{(1-t^2)(1-t^N)} = \frac{1-t^{2N}}{(1-t^2)(1-t^N)^2}. \tag{4.3}$$

It is claimed that $\mathcal{M}_C$ is the multi-Taub-NUT space, which is $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$ as a complex variety. It is the surface $xy = z^N$ in $\mathbb{C}^3$, hence we recover the above formula if we set $\deg x = \deg y = N$, $\deg z = 2$.

In this case, the meaning of individual terms $t^{2\Delta(\lambda)} P_G(t;\lambda) = t^{N|m|}/(1-t^2)$ is also apparent from the exact sequence

$$0 \to \mathbb{C}[\mathcal{M}_C] \xrightarrow{z} \mathbb{C}[\mathcal{M}_C] \to \mathbb{C}[\{z=0\}] \to 0.$$

We have $(1-t^2)\,\mathrm{ch}_{\mathrm{U}(1)}\,\mathbb{C}[\mathcal{M}_C] = \mathrm{ch}_{\mathrm{U}(1)}\,\mathbb{C}[\{z=0\}]$. This explains $P_G(t;\lambda) = 1/(1-t^2)$. Since $\{z=0\} = \{xy=0\}$, we have $\mathbb{C}[\{z=0\}] = \mathbb{C}[x,y]/(xy)$. Then $t^{N|m|}$ corresponds to x^m for $m \geq 0$ and y^m for $m \leq 0$ (and 1 for $m=0$).

4(iii). Expected properties of $\mathcal{M}_C$

Let us give several expected properties of the Coulomb branch $\mathcal{M}_C$. First of all,

(a) $\mathcal{M}_C$ contains a hyper-Kähler manifold (or orbifold, more generally) with an SU(2)-action rotating I, J, K, as an open dense subset.

Once the Hilbert series is given, the dimension is given by the degree of the corresponding Hilbert polynomial. We expect

(b) $\dim_{\mathbb{H}} \mathcal{M}_C = \dim_{\mathbb{R}} T$.

It is a classical result that the fundamental group $\pi_1(G)$ of G is isomorphic to the quotient of the coweight lattice Y by the coroot lattice. We can refine the Hilbert series by remembering the class of λ in $\pi_1(G)$, i.e., the coordinate ring $\mathbb{C}[\mathcal{M}_C]$ has an additional $\pi_1(G)$-grading. Therefore we expect

(c) The Pontryagin dual $\pi_1(G)^\wedge = \mathrm{Hom}(\pi_1(G), \mathrm{U}(1))$ acts on $\mathcal{M}_C$, preserving the hyper-Kähler structure.

In [CHZ14, CHMZ14a], an additional variable z is introduced, and (1.2) is refined to

$$H_{G,\mathbf{M}}(t,z) = \sum_{\lambda \in Y/W} z^{J(\lambda)} t^{2\Delta(\lambda)} P_G(t;\lambda), \tag{4.4}$$

where J is the projection from Y to $\pi_1(G)$. In above examples, G is a product of unitary groups, hence, $\pi_1(G) = \mathbb{Z}^r$. Thus we expect the torus $\mathrm{U}(1)^r$ acts on $\mathcal{M}_C$. In the above formula, z is a (multi) variable for characters of the torus.

Let us check these expected properties for the abelian case. (a) is clear as $\mathcal{M}_C$ is supposed to be the hyper-Kähler quotient $\mathbf{M}/\!/\!/G_F^\vee$. We have

(b) $\dim_{\mathbb{H}} \mathbf{M}/\!/\!/G_F^\vee = d - n = \dim_{\mathbb{R}} G$.

(c) $\pi_1(G)^\wedge = (\mathbb{Z}^{d-n})^\wedge = G^\vee$ acts on $\mathbf{M}/\!/\!/G_F^\vee$.

Therefore these two proposed properties are satisfied.

In many examples, this group action can be enlarged to a nonabelien group action. Let us give a particular example.

Take a quiver gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ as in §2(iv). Since G is a product of unitary groups, $\pi_1(G)^\wedge$ is isomorphic to the product of $\mathrm{U}(1)$ for each vertex $i \in Q_0$. Therefore $\prod_{i\in Q_0} \mathrm{U}(1)$ acts on $\mathcal{M}_C$ from (c) above. It is expected that a larger group containing $\prod_{i\in Q_0} \mathrm{U}(1)$ acts on $\mathcal{M}_C$ as follows.

We consider the Weyl group $W(Q)$, naturally appeared in the context of quiver varieties [Nak94, §9]: Fix W, or more precisely $\dim W = (\dim W_i) \in \mathbb{Z}^{Q_0}$, but we allow V to change. We regard $(\dim W_i - \sum_j (2\delta_{ij} - a_{ij}) \dim V_j) \in \mathbb{Z}^{Q_0}$ as a weight of the Kac-Moody Lie algebra corresponding to Q. Then we change $\dim V$ given by the usual Weyl group action on weights. Concretely, for each vertex i without edge loops, we consider s_i, which change V by a new V' by the following rule: A) V'_j is the same as V_j if $j \neq i$. B) $\dim V'_i = \dim W_i + \sum_j a_{ij} \dim V_j - \dim V_i$. These s_i generates the Weyl group.

Now we fix V again, and introduce the subquiver $S = (S_0, S_1)$ of Q consisting of vertices i such that reflections s_i preserve $\dim V$ and edges between them. We suppose S is of finite type, i.e., the underlying graph is a disjoint union of ADE graphs. Let G'_S be the simply-connected[7] compact Lie group corresponding to S. Note that G'_S contains $\mathrm{U}(1)$'s corresponding to vertices in S, as a maximal torus. We then take other $\mathrm{U}(1)$'s corresponding to vertices *not* in S, and define G_S as the product of G'_S and those $\mathrm{U}(1)$'s. Now $\pi_1(G)^\wedge = \prod_{i\in Q_0} \mathrm{U}(1)$ is a maximal torus T_S of G_S.

We also consider the group Γ of the diagram automorphism of preserving both $\dim V$, $\dim W$. It acts on G_S by outer automorphisms. Then we expect

(d) $\Gamma \ltimes G_S$ acts on $\mathcal{M}_C$ extending the $\pi_1(G)^\wedge$-action, preserving the hyper-Kähler structure.

In the above example in Figure 1, the left one contains a type A_{N-1} subgraph in the bottom. Therefore $G_S = \mathrm{SU}(N) \times \mathrm{U}(1)$, where the extra $\mathrm{U}(1)$ comes from the upper circled vertex. We also have the overall $\mathrm{U}(1)$ in $\mathrm{SU}(2)$, and $\Gamma = \{\pm 1\}$ from diagram automorphisms. Thus their product should act on $\mathcal{M}_C^A$. This should be the same as the natural action on $\mathcal{M}_H^B$, the framed moduli space of $\mathrm{SU}(N)$-instantons on $\mathbb{R}^4$, where $\mathrm{SU}(N)$ acts by the change of framing, $\mathrm{U}(1) \times \mathrm{U}(1)$ acts on the base $\mathbb{R}^4 = \mathbb{C}^2$ with $(t, z) \cdot (x_1, x_2) = (tzx_1, tz^{-1}x_2)$, and the Γ-action is given by taking dual instantons. In the example in Figure 4, we have $E_8 \times \mathrm{U}(1) \times \mathrm{U}(1)$-action from this construction.

[7]The author does not know whether it naturally descends to a quotient or not.

Remark 4.5. Let us write $\pi_1(G) = Y/Y_{\mathrm{cr}}$, where Y (resp. Y_{cr}) is the coweight (resp. coroot) lattice of G. Therefore $\pi_1(G)^\wedge$ is the kernel of the homomorphism $Y^\wedge \to Y_{\mathrm{cr}}^\wedge$ between Pontryagin duals of Y, Y_{cr}. We have $Y^\wedge = (X \otimes \mathbb{R})/X$, where $X = \mathrm{Hom}(Y, \mathbb{Z})$ is the weight lattice of G. Therefore the coweight lattice $\mathrm{Hom}(\mathrm{U}(1), \pi_1(G)^\wedge)$ is the same as the kernel of the homomorphism $X \to \mathrm{Hom}(Y_{\mathrm{cr}}, \mathbb{Z})$ given by the pairing with coroots Y_{cr}. It is equal to the character group of G. In summary, we have a natural isomorphism

$$\mathrm{Hom}(\mathrm{U}(1), \pi_1(G)^\wedge) \cong \mathrm{Hom}(G, \mathrm{U}(1)) \cong \mathrm{Hom}(G_{\mathbb{C}}, \mathbb{C}^*). \tag{4.6}$$

Thus a coweight $\chi \in \mathrm{Hom}(\mathrm{U}(1), \pi_1(G)^\wedge)$ defines a character $G \to \mathrm{U}(1)$, and hence gives a stability condition and the corresponding GIT quotient of $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ by $G_{\mathbb{C}}$ as in (2.1).

Therefore an element of (4.6) plays two roles, one on $\mathcal{M}_H$, another on $\mathcal{M}_C$. This observation was essentially given already in [IS96, dBHOO97]. The former appears as a value of the hyper-Kähler moment map, or *Fayet-Iliopoulos parameter* in the physics terminology. On the other hand, when G_S acts on $\mathcal{M}_C$, an element of the Lie algebra of G_S is called *mass parameter*.

5. Flavor symmetry

5(i). Line bundles over Coulomb branches

Let us discuss a flavor symmetry following [CHMZ14a]. It means that we suppose that $\mathbf{M}$ is a quaternionic representation of a larger compact Lie group $\tilde{G}$, which contains the original group G as a normal subgroup. The quotient $G_F = \tilde{G}/G$ is called the *flavor symmetry group*. For a quiver gauge theory, we can take (at least) $G_F = \prod_{i \in Q_0} \mathrm{U}(W_i)/\mathrm{U}(1)$ (and $\tilde{G} = G \times G_F$), where $\mathrm{U}(1)$ is the overall scalar, which acts trivially on $\mathbf{M}/\!/\!/G$.

Note that we do not use G_F to take a quotient. The gauge theory $\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G$ has a G_F-symmetric QFT in the sense of [Tac]. (The reader needs to remember that $\mathbf{M}$ is a representation of $\tilde{G}$.) As a concrete mathematical consequence, for example, G_F acts as a symmetry group on the Higgs branch $\mathcal{M}_H(\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G)$, which is the hyper-Kähler quotient $\mathbf{M}/\!/\!/G$.

Let us turn to study the role of G_F playing on the Coulomb branch $\mathcal{M}_C(\mathrm{Hyp}(\mathbf{M}) /\!\!/\!\!/ G)$. As observed in [CHMZ14a], we can naturally put G_F in the monopole formula (1.2) as follows. Let us consider the short exact sequence of groups

$$1 \to G \xrightarrow{\alpha} \tilde{G} \xrightarrow{\beta} G_F \to 1.$$

(This is the same as (2.4) in the abelian case.) Let us fix a coweight λ_F of G_F and consider the inverse image $\beta^{-1}(\lambda_F)$. We consider $\Delta(\lambda)$ and $P_G(t;\lambda)$ for $\lambda \in \beta^{-1}(\lambda_F)$. Their definitions in (4.1) remain the same. For $\Delta(\lambda)$, the first term is the sum over Δ^+, positive roots of G, considered as positive roots of $\tilde{G}$. In the second term, we understand $\mathrm{wt}(b)$ as a weight of $\tilde{G}$, and paired with λ. We do not change $P_G(t;\lambda)$, we consider the stabilizer of λ in G, *not* in $\tilde{G}$. The sum is over $\beta^{-1}(\lambda_F)/W$, where W is the Weyl group of G. We get a function

$$H_{\tilde{G},\mathbf{M}}(t,\lambda_F) = \sum_{\lambda \in \beta^{-1}(\lambda_F)/W} t^{2\Delta(\lambda)} P_G(t;\lambda)$$

in t together with λ_F.

It was found in [CHMZ14a] (and more recent one [CHMZ14c]) that this generalization turned out to be very fruitful by two reasons:

First, let $\{\mathrm{Hyp}(\mathbf{M}_i /\!\!/\!\!/ G_i)\}_{i=1,2,\dots}$ be a collection of *simpler* gauge theories sharing the common flavor symmetry group G_F. (Thus $\mathbf{M}_i$ is a representation of $\tilde{G}_i$, and $G_F = \tilde{G}_i/G_i$.) We define a *complicated* gauge theory as $\mathrm{Hyp}(\bigoplus \mathbf{M}_i /\!\!/\!\!/ \prod' G_i)$, where $\prod' G_i$ is the fiber product of $\prod G_i$ and the diagonal G_F over $\prod G_F$. The Hilbert series of the complicated theory is written by those $H_{\tilde{G}_i,\mathbf{M}_i}(t;\lambda_F)$ ($i=1,2,3\dots$) of simpler theories as

$$\sum_{\lambda_F \in Y_F/W_F} t^{-2\sum_{\alpha\in\Delta_F^+}|\langle\alpha,\lambda_F\rangle|} P_{G_F}(t;\lambda_F) \prod_i H_{\tilde{G}_i,\mathbf{M}_i}(t;\lambda_F),$$

where Y_F, W_F, Δ_F^+ are the coweight lattice, Weyl group, the set of positive roots of G_F. This is clear from the form of the monopole formula.

Second, if $\mathrm{Hyp}(\mathbf{M} /\!\!/\!\!/ G)$ is a quiver gauge theory of type A (or its Sp/O version), the Hilbert series are written by Hall-Littlewood polynomials.

Combining two, one can write down the Hilbert series of Higgs branches of $3d$ Sicilian theories in terms of Hall-Littlewood polynomials, as an example of an application [CHMZ14b].

Since quiver varieties of type A are nilpotent orbits (and their intersection with Slodowy slices) as we mentioned in §2(iv), the appearance of Hall-Littlewood polynomials is very suggestive. They appear as dimensions of spaces of sections of line bundles over flag varieties. See [Bro93], where the Euler characteristic version was found earlier in [Hes80].

For an abelian gauge theory, the dual torus of G_F appears in the quotient construction of the Coulomb branch as $\mathcal{M}_C = \mathbf{M}/\!\!/\!\!/ G_F^\vee$ (§2(vi)). Therefore a coweight λ_F of G_F, which is a weight of $G_F^\vee$, defines a line bundle over the resolution $\boldsymbol{\mu}^{-1}(\zeta_{\mathrm{Im}\,\mathbb{H}})/G_F^\vee$ as

$$\boldsymbol{\mu}^{-1}(\zeta_{\mathrm{Im}\,\mathbb{H}}) \times_{G_F^\vee} \mathbb{C},$$

where $G_F^\vee$ acts on $\mathbb{C}$ by λ_F. It naturally has a connection, which is integrable for any of I, J, K [GN92]. In particular, it is a holomorphic line bundle with respect to I.

Based on these observations, it is natural to add the followings to the list of expected properties:

(e) We have a (partial) resolution of $\mathcal{M}_C$ whose Picard group is isomorphic to the coweight lattice Y_F of G_F. Moreover the character of the space of sections of a U(1)-equivariant holomorphic line bundle $\mathcal{L}_{\lambda_F}$ corresponding to a coweight λ_F is given by the monopole formula $H_{\tilde{G},\mathbf{M}}(t, \lambda_F)$. (Here we replace $\mathcal{M}_C$ if necessary so that $\mathcal{L}_{\lambda_F}$ is relatively ample.)

(f) The Weyl group W_F acts on the Picard group of the partial resolution above.

As is usual for hyper-Kähler manifolds, a resolution and a deformation are related by the hyper-Kähler rotation. Therefore we expect

(g) We have a deformation of $\mathcal{M}_C$ parameterized by the Cartan subalgebra $\mathfrak{h}_F$ of G_F. The Weyl group W_F acts on the homology group by the monodromy.

Let us check the compatibility of the conjecture with the $3d$ mirror symmetry. Recall that the group $\Gamma \ltimes G_S$ acting on $\mathcal{M}_C^A$ (see §4(iii)(d)). Since the definition of $\Gamma \ltimes G_S$ depends on the choice of the theory A, let us denote it by $\Gamma^A \ltimes G_S^A$. It is natural to expect that it is identified with G_F^B (the flavor symmetry group for the theory B) acting on $\mathcal{M}_H^B$. This is indeed the case for the example in Figure 1, where $\Gamma^A \ltimes G_S^A$ are $\{\pm 1\} \ltimes \mathrm{SU}(N) \times \mathrm{U}(1)$. The squared N gives us $\mathrm{SU}(N)$. The factor $\mathrm{U}(1)$ comes from an *internal* symmetry of the graph. The edge loop at the circled k gives the factor $\mathrm{End}(\mathbb{C}^k) \otimes \mathbb{C}^2$ in $\mathbf{M}$. Then $\mathrm{U}(1)$ is acting on $\mathbb{C}^2$ preserving the hyper-Kähler structure.[8] Finally $\{\pm 1\}$ is identified with the symmetry defined by transpose of linear maps in $\mathbf{M}$. (Since $\mathbf{M}$ is the cotangent type, transpose of an element in $\mathbf{N}$ is in $\mathbf{N}$.)

[8] A larger group $\mathrm{Sp}(1)$ acts on $\mathbb{C}^2$, but the author does not know how to see it in the monopole formula.

Since the mirror symmetry should be a duality, the role of G_F^B in $\mathcal{M}_C^B$ should be the same as the role of $\Gamma^A \ltimes G_S^A$ playing in $\mathcal{M}_H^A$. The diagram automorphism group Γ^A induces automorphisms on $\mathcal{M}_H^A$. This is clear.

Let us turn to G_S^A. Recall that we have defined G_S^A in two steps. We first define the maximal torus of G_S^A as $\pi_1(G)^\wedge$, where the group G is the one we take the quotient. Then we consider the Weyl group invariance in the second step. So consider $T_S^A \stackrel{\text{def.}}{=} \pi_1(G)^\wedge$ first. By (4.6) a coweight $\lambda_S \in \operatorname{Hom}(\mathrm{U}(1), T_S^A)$ defines a character of G, and hence gives a (partial) resolution and a line bundle on it for the Higgs branch $\mathcal{M}_H^A$ as in (2.1). This is exactly the property (e) for G_F^B and $\mathcal{M}_C^B$.

Moreover, if we take λ_S generic in $\operatorname{Hom}(\mathrm{U}(1), T_S)$, we expect that the Picard group of $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/_{\lambda_S} G_{\mathbb{C}}$ is isomorphic to $\operatorname{Hom}(\mathrm{U}(1), T_S^A)$ and we have an action of the Weyl group of G_S^A, as in (f).

5(ii). Abelian case

In [CHZ14, §6], the proposal (e) was checked for the abelian theory. Let us give a different proof.

We keep the notation in §2(vi). We choose and fix a coweight λ_F of G_F, considered also as a weight of $G_F^\vee$. We *define* the Coulomb branch $\mathcal{M}_C$ by $\mathbf{M}/\!\!/\!\!/G_F^\vee$ and its partial resolution $\tilde{\mathcal{M}}_C$ by $\boldsymbol{\mu}^{-1}(\lambda_F, 0)/G_F^\vee = \boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/_{\lambda_F}(G_F^\vee)_{\mathbb{C}}$. Here λ_F is considered as $(\operatorname{Lie} G_F^\vee)^*$ in the first description as a hyper-Kähler quotient, and the stability parameter in the second description as a GIT quotient. (The complex parameter is set 0.) We have a relative ample line bundle $\mathcal{L}_{\lambda_F} = \boldsymbol{\mu}^{-1}(\lambda_F, 0) \times_{G_F^\vee} \mathbb{C}$. Our goal is to check that the character of the space of sections of $\mathcal{L}_{\lambda_F}$ is given by $H_{\tilde{G},\mathbf{M}}(t, \lambda_F)$.

In order to have a clear picture, we consider the action of $\pi_1(G)^\wedge = G^\vee$ on $\mathcal{M}_C$. We take a lift of the action to the line bundle $\mathcal{L}_{\lambda_F}$. It means that we lift λ_F to a weight $\tilde{\lambda}_F$ of $(T^d)^\vee$ so that we have the induced $G^\vee = (T^d)^\vee/G_F^\vee$-action on $\boldsymbol{\mu}^{-1}(\lambda_F, 0) \times_{G_F^\vee} \mathbb{C}$. Then the space of sections is a representation of $\mathrm{U}(1) \times G^\vee$. The monopole formula is refined as

$$H_{\tilde{G},\mathbf{M}}(t, \lambda_F) = \sum_{\lambda \in \mathbb{Z}^{d-n}} z^\lambda t^{2\Delta(\lambda)} P_G(t; \lambda), \tag{5.1}$$

as in (4.4). Here we use the identification $\beta^{-1}(\lambda_F) = \tilde{\lambda}_F + \operatorname{Im}\alpha \cong \mathbb{Z}^{d-n}$, as we choose the lift $\tilde{\lambda}_F$. We understand z as a multi-variable, and z^λ means $z_1^{\lambda_1} \cdots z_{d-n}^{\lambda_{d-n}}$.

Since G is torus, the stabilizer of λ is always G itself. Therefore $P_G(t; \lambda) = 1/(1-t^2)^{\operatorname{rank} G}$. And the Weyl group is trivial, as we have already used above. We have

$$2\Delta(\lambda) = \sum_{i=1}^{d} \left|(\tilde{\lambda}_F + \alpha(\lambda))_i\right|,$$

where $(\tilde{\lambda}_F + \alpha(\lambda))_i$ is the i^{th}-component of $\tilde{\lambda}_F + \alpha(\lambda) \in \mathbb{Z}^d$.

Let us start the proof. A key is a *trivial* observation that the hyper-Kähler quotient of $\mathbf{M}$ by T^d is a single point at any level of the hyper-Kähler moment map. It is enough to check $d = 1$, then it is obvious that the solution $(x, y) \in \mathbb{C}^2$ of

$$\begin{cases} xy = \zeta_{\mathbb{C}}, \\ |x|^2 - |y|^2 = \zeta_{\mathbb{R}} \end{cases}$$

is unique up to $\mathrm{U}(1)$ for any $\zeta_{\mathbb{R}}$, $\zeta_{\mathbb{C}}$. Let us see this in the GIT picture. Let $\zeta_{\mathbb{C}} = 0$, as it becomes trivial otherwise. A function on $xy = 0$, which has weight $m \in \mathbb{Z}$ with respect to the $\mathbb{C}^*$-action $z \cdot (x, y) = (zx, z^{-1}y)$, is x^m if $m \geq 0$, y^{-m} if $m \leq 0$, up to constant multiple. This function has weight $|m|$ with respect to the dilatation action $t \cdot (x, y) = (tx, ty)$. Therefore with respect to the $\mathbb{C}^* \times \mathbb{C}^*$-action, the character of $\mathbb{C}[x, y]/(xy)$ is

$$\sum_{m \in \mathbb{Z}} z^m t^{|m|}. \tag{5.2}$$

This trivial observation can be applied to our situation by considering hyper-Kähler quotients of $\tilde{\mathcal{M}}_C$ by $G^\vee$, which are nothing but hyper-Kähler quotients of $\mathbf{M}$ by T^d. We have the complex moment map

$$\boldsymbol{\mu}_{\mathbb{C}}^{G^\vee} : \tilde{\mathcal{M}}_C \to (\operatorname{Lie} G^\vee)^* \otimes \mathbb{C} \cong \mathfrak{g}_{\mathbb{C}}.$$

We consider it as a set of functions defining the subvariety $(\mu_{\mathbb{C}}^{G^\vee})^{-1}(0)$. It is a complete intersection, and the space of sections on $(\mu_{\mathbb{C}}^{G^\vee})^{-1}(0)$ and that of $\tilde{\mathcal{M}}_C$ differ by the factor $1/(1-t^2)^{\operatorname{rank} G} = P_G(t; \lambda)$. Now we consider $(\mu_{\mathbb{C}}^{G^\vee})^{-1}(0)$ is a quotient of the subvariety $(\mu_{\mathbb{C}}^{T^d})^{-1}(0)$ in $\mathbf{M}$, where $\mu_{\mathbb{C}}^{T^d}$ is the complex moment map of the T^d-action on $\mathbf{M}$. The latter is given by $x_i y_i = 0$ $(i = 1, \dots, d)$ in the standard coordinate system $(x_i, y_i)_{i=1}^d$ of $\mathbf{M}$. A section of $\mathcal{L}_{\lambda_F}$ with weight λ with respect $G^\vee$ is a function on $x_i y_i = 0$ with weight $\tilde{\lambda}_F + \alpha(\lambda)$, hence is a monomial in either x_i or y_i according to the sign of $(\tilde{\lambda}_F + \alpha(\lambda))_i$. Now we deduce (5.1) by the same argument in (5.2). We replace z^m by z^λ, $t^{|m|}$ by $t^{2\Delta(\lambda)}$ respectively.

As a byproduct of this proof of the monopole formula, we obtain a linear base of the coordinate ring $\mathbb{C}[\mathcal{M}_C]$. It is given by monomials in components of the moment map $\boldsymbol{\mu}_{\mathbb{C}}^{G^\vee}$ and

$$\prod_i \left(x_i^{\alpha(\lambda)_i} \text{ or } y_i^{-\alpha(\lambda)_i} \right) \tag{5.3}$$

for each $\lambda \in \mathbb{Z}^{d-n}$. Here we take $x_i^{\alpha(\lambda)_i}$ if $\alpha(\lambda)_i \geq 0$ and $y_i^{-\alpha(\lambda)_i}$ otherwise.

Let us give an explicit presentation of the coordinate ring $\mathbb{C}[\mathcal{M}_C]$. It is an algebra over $\mathbb{C}[\mathfrak{g}_{\mathbb{C}}]$, the polynomial ring over $\mathfrak{g}_{\mathbb{C}}$ by the moment map $\boldsymbol{\mu}_{\mathbb{C}}^{G^\vee} : \mathcal{M}_C \to \mathfrak{g}_{\mathbb{C}}$. Let us denote the element (5.3) by z^λ. (The element gives the corresponding character z^λ, and there is no fear of confusion.) Then $\mathbb{C}[\mathcal{M}_C]$ is $\bigoplus_{\lambda\in\mathbb{Z}^{d-n}} \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]z^\lambda$ with multiplication

$$z^\lambda z^\mu = z^{\lambda+\mu}\pi\left(\prod_i \xi_i^{d_i(\lambda,\mu)}\right),$$

where $\pi\colon \mathbb{C}[\mathfrak{t}^d_{\mathbb{C}}] \to \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]$ is the projection induced by the inclusion $\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{t}^d_{\mathbb{C}}$, ξ_i is a standard linear coordinate function on $\mathfrak{t}^d_{\mathbb{C}}$, and

$$d_i(\lambda,\mu) = \begin{cases} 0 & \text{if } \alpha(\lambda)_i \times \alpha(\mu)_i \geq 0, \\ -\alpha(\mu)_i & \text{if } \alpha(\lambda)_i \geq 0,\ \alpha(\mu)_i \leq 0,\ \alpha(\lambda+\mu)_i \geq 0, \\ \alpha(\lambda)_i & \text{if } \alpha(\lambda)_i \geq 0,\ \alpha(\mu)_i \leq 0,\ \alpha(\lambda+\mu)_i \leq 0, \\ -\alpha(\lambda)_i & \text{if } \alpha(\lambda)_i \leq 0,\ \alpha(\mu)_i \geq 0,\ \alpha(\lambda+\mu)_i \geq 0, \\ \alpha(\mu)_i & \text{if } \alpha(\lambda)_i \leq 0,\ \alpha(\mu)_i \geq 0,\ \alpha(\lambda+\mu)_i \leq 0. \end{cases}$$

This is because the moment map $\boldsymbol{\mu}_{\mathbb{C}}^{G^\vee}$ is induced from $\boldsymbol{\mu}_{\mathbb{C}}^{T^d}$ on $\mathbf{M}$, which is given by $x_i y_i$. We write z^λ by $x_i^{\alpha(\lambda)_i}$ or $y_i^{-\alpha(\lambda)_i}$, and identify ξ_i with $x_i y_i$. Then we get the above formula of $d_i(\lambda,\mu)$ according to signs of $\alpha(\lambda)_i$, $\alpha(\mu)_i$, $\alpha(\lambda+\mu)_i$.

This formula is found by Hikita [Hik15, App. 2], Dimofte and Hilburn [DH14].

6. Topological twist

In this section we discuss PDE's relevant to the topologically twisted version of our gauge theories. For $(G,\mathbf{M}) = (\mathrm{U}(1),\mathbb{H})$, it is nothing but the Seiberg-Witten monopole equation [Wit94]. More general cases were discussed for example in [Tau99, Pid04, Hay13]. However their reductions to 2-dimension were not discussed before, as far as the author knows, except it is of course well known that the dimension reduction of the Seiberg-Witten equation is the (anti-)vortex equation. Therefore we will discuss equations from the scratch though our treatment is more or less a standard textbook material.

6(i). Generalized Seiberg-Witten equations

We consider only the case when the 4-manifold X is spin for simplicity. See Remark 6.2 for a framework close to the usual Seiberg-Witten equation for a spin^c 4-manifold under an additional assumption on $(G, \mathbf{M})$.

Recall that $\mathbf{M}$ is a quaternionic representation of G. Therefore $\mathrm{Sp}(1)$, the group of unit quaternions, acts on $\mathbf{M}$ commuting with the G-action. Then $\mathbf{M}$ can be made a representation of $\hat{G} \overset{\text{def.}}{=} \mathrm{Spin}(4) \times G = \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times G$ in two ways, by choosing $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1)$ either the first or second projection. Let us distinguish two representations, and denote them by $\mathbf{M}^+$ and $\mathbf{M}^-$. We write $\mathrm{Spin}(4) = \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ accordingly.

We regard $\mathbb{H}$ as an $\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$-module by left and right multiplication. The quaternion multiplication gives a $\hat{G}$-homomorphism $\mathbb{H} \otimes \mathbf{M}^- \to \mathbf{M}^+$. Combining this with $\mathbb{H} \otimes \mathbf{M}^+ \to \mathbf{M}^-$ given by $(x, m) \mapsto -\overline{x}m$, we have a Clifford module structure (over $\mathbb{R}$) on $\mathbf{M}^+ \oplus \mathbf{M}^-$.

We regard the hyper-Kähler moment map (see §2(i) for detail), as a map $\mu\colon \mathbf{M}^+ \to \mathfrak{g} \otimes \mathfrak{sp}(1)_+$ by identifying $\mathfrak{sp}(1)_+$ with the space of imaginary quaternions. It is equivariant under $\hat{G}$.

Let X be an oriented Riemannian 4-manifold. We assume X is spin, as mentioned above. Let $P \to X$ be a principal G-bundle. We have the associated $\hat{G}$-principal bundle, denoted by $\hat{P}$, by taking the fiber product with the double cover $P_{\mathrm{Spin}(4)}$ of the orthonormal frame bundle $P_{\mathrm{SO}(4)}$ of TX given by the spin structure. Let A be a connection on P. We extend it to a connection on $\hat{P}$, combining with the Levi-Civita connection on the $P_{\mathrm{Spin}(4)}$-part. We denote the extension also by A for brevity. We have induced covariant derivatives ∇_A on $\hat{P} \times_{\hat{G}} \mathbf{M}^+$ and $\hat{P} \times_{\hat{G}} \mathbf{M}^-$. Combining them with Clifford multiplication, we have Dirac operators $D_A^{\pm}\colon \Gamma(\hat{P} \times_{\hat{G}} \mathbf{M}^{\pm}) \to \Gamma(\hat{P} \times_{\hat{G}} \mathbf{M}^{\mp})$.

Let Φ be a section of $\hat{P} \times_{\hat{G}} \mathbf{M}^+$. Applying the hyper-Kähler moment map fiberwise, we regard $\mu(\Phi)$ as a section of $\hat{P} \times_{\hat{G}} (\mathfrak{g} \otimes \mathfrak{sp}(1)_+)$. Note that $\hat{P} \times_{\hat{G}} \mathfrak{sp}(1)_+ \cong P_{\mathrm{SO}(4)} \times_{\mathrm{SO}(4)} \mathfrak{sp}(1)_+$ is the bundle Λ^+ of self-dual 2-forms. Hence $\mu(\Phi)$ is a section of $\Lambda^+ \otimes (P \times_G \mathfrak{g})$.

Now we define a generalized Seiberg-Witten equation by

$$\begin{aligned} &D_A^+ \Phi = 0, \\ &F_A^+ = \boldsymbol{\mu}(\Phi). \end{aligned} \tag{6.1}$$

Here F_A^+ is the self-dual part of the curvature of A. More precisely it is of the connection on P, not of the extended one on $\hat{P}$.

Remark 6.2. Assume that G contains a central element acting by the multiplication of -1 on $\mathbf{M}$. This is true for $(G, \mathbf{M}) = (\mathrm{U}(1), \mathbb{H})$ corresponding to the usual Seiberg-Witten monopole equation. Then we can replace $\hat{G}$ by its quotient $\hat{G}' = \mathrm{Spin}(4) \times G/\pm(1,1)$. The example $G = \mathrm{U}(1)$ gives $\hat{G}' = \mathrm{Spin}^c(4)$. Then we consider a principal $\hat{G}'$-bundle $\hat{P}'$ together with an isomorphism $\hat{P}'/G \cong P_{\mathrm{SO}(4)}$ of $\mathrm{SO}(4) = \mathrm{Spin}(4)/\pm 1$-bundles. The generalized Seiberg-Witten equation still makes sense for a connection A on $\hat{P}'$ and a section Φ of $\hat{P}' \times_{\hat{G}'} \mathbf{M}^+$. Here A is supposed to induce the Levi-Civita connection on $P_{\mathrm{SO}(4)}$. In this framework, we do not need to assume X to be spin, an existence of a lift of $P_{\mathrm{SO}(4)}$ to a $\hat{G}'$-bundle $\hat{P}'$ is sufficient. This framework is used in the usual Seiberg-Witten equation.

When $\mathbf{M}$ is of the form $\mathbf{O} \otimes_{\mathbb{R}} \mathbb{H}$ for a real representation $\mathbf{O}$ of G, we have a further generalization. An example is $\mathbf{M} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}$. The point is $\mathbb{H}$ has two $\mathbb{H}$-module structures by left and right multiplications.

We regard $\mathbb{H}$ as a representation of $\mathrm{Spin}(4) = \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ as above. We consider $\mathbf{M}$ as a hyper-Kähler manifold by the left multiplication. And the moment map is regarded as $\boldsymbol{\mu}\colon \mathbf{M} \to \mathfrak{g} \otimes \mathfrak{sp}(1)_+$ as above. We have three possibilities to make $\mathbf{M} = \mathbf{O} \otimes_{\mathbb{R}} \mathbb{H}$ as an $\mathrm{Sp}(1)_+ \otimes \mathrm{Sp}(1)_-$-module. (They are possible topological twists in physics literature.) Let $(g_+, g_-) \in \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ and $x \in \mathbb{H}$. Then the action is either (1) $x \mapsto g_+ x$, (2) $x \mapsto g_+ x g_+^{-1}$, or (3) $x \mapsto g_+ x g_-^{-1}$. The case (1) is the same as the case studied above. In the cases (2), (3), it is better to replace $\hat{G}$ by $\mathrm{SO}(4) \times G$ as $(-1,-1) \in \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ acts trivially on $\mathbb{H}$.

The case (2) means that we regard $\mathbb{H}$ with $S^+ \otimes S^+ = \Lambda^0 \oplus \Lambda^+$. Then Φ is a section of $(\Lambda^0 \oplus \Lambda^+) \otimes (P \times_G \mathbf{O})$. Let us write it by $C \oplus B$. The Dirac operator D_A^+ is identified with $d_A \oplus d_A^*$ via the isomorphism $S^+ \otimes S^- = \Lambda^1$, where $S^\pm$ is the spinor bundle. Hence the generalized Seiberg-Witten equation is

$$\begin{aligned} &d_A C + d_A^* B = 0, \\ &F_A^+ = \boldsymbol{\mu}(C \oplus B). \end{aligned}$$

This is the equation in [VW94] if $\mathbf{M} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}$.

Now consider the case (3). Then $\mathbb{H}$ is identified with $S^+ \otimes S^- = \Lambda^1$. Hence Φ is a section of $\Lambda^1 \otimes (P \times_G \mathbf{O})$. The Dirac operator is identified with $d_A^* \oplus d_A^-$ via $S^- \otimes S^- = \Lambda^0 \oplus \Lambda^-$. Therefore the equation is

$$\begin{aligned} &d_A^- \Phi = 0 = d_A^* \Phi, \\ &F_A^+ = \boldsymbol{\mu}(\Phi). \end{aligned}$$

This is the equation in [KW07] if $\mathbf{M} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}$. For $\mathbf{M} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}$, there is a one-parameter family of equations parametrized by $t \in \mathbb{P}^1$. See [KW07, (3.29)].

6(ii). Roles of Higgs branch

Suppose X is compact. We return back to the equation (6.1).

From the Weitzenböck formula for D_A^+ for a solution (A, Φ) of generalized Seiberg-Witten equations, we have

$$
\begin{aligned}
0 &= \frac{1}{2}\Delta|\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4}|\Phi|^2 + (F_A^+\Phi, \Phi) \\
&= \frac{1}{2}\Delta|\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4}|\Phi|^2 + 2|\boldsymbol{\mu}(\Phi)|^2,
\end{aligned}
\tag{6.3}
$$

where s is the scalar curvature. For the ordinary Seiberg-Witten equation, we have $|\boldsymbol{\mu}(\Phi)|^2 = |\Phi|^4/4$. Then considering a point where $|\Phi|^2$ takes a maximum, one concludes $\sup|\Phi|^2 \le \sup_X \max(-s, 0)$. (See e.g., [Mor96, Cor. 5.2.2].)

The same argument works if there exists a constant C such that $|\phi|^4 \le C|\boldsymbol{\mu}(\phi)|^2$ for $\phi \in \mathbf{M}$. However this is not true in general. If there is nonzero ϕ such that $\boldsymbol{\mu}(\phi) = 0$, the inequality is not true. In fact, it is easy to check the converse. If $\boldsymbol{\mu}(\phi) = 0$ implies $\phi = 0$, the inequality holds. Thus we have proved

- If the Higgs branch $\mathcal{M}_H$ is $\{0\}$, Φ is bounded for a solution of (6.1).

The author learned this assertion during Witten's lecture at the Isaac Newton Institute for Mathematical Sciences in 1996, 18 years ago.

If $\mathcal{M}_H \neq \{0\}$, we should study behavior of a sequence of solutions (A_i, Φ_i) with $|\Phi_i| \to \infty$. We consider the normalized solution $\bar{\Phi}_i = \Phi_i/\|\Phi_i\|_{L^2}$. From (6.3) we derive a bound on $\sup|\bar{\Phi}_i|$, $\|\nabla_A \bar{\Phi}_i\|_{L^2}$, and also $\int_X |\boldsymbol{\mu}(\bar{\Phi}_i)|^2 \le C\|\Phi_i\|_{L^2}^{-2}$ for a constant C. Thus $\bar{\Phi}_i$ converges weakly in $W^{1,2}$ and strongly in L^p for any $p > 0$. Moreover $\boldsymbol{\mu}(\bar{\Phi}_i)$ converges to 0, i.e., the limit $\bar{\Phi}_\infty$ takes values in $\boldsymbol{\mu}^{-1}(0)$. Thus the Higgs branch $\mathcal{M}_H$ naturally shows up.

Further analyses are given in e.g., [Tau13] for a special case $(G, \mathbf{M}) = (\mathrm{SU}(2), \mathfrak{g} \otimes \mathbb{H})$. See also [HW14] for a special case $(G, \mathbf{M}) = (\mathrm{U}(1), \mathbb{H}^N)$ and dimension 3.

6(iii). Kähler surfaces

Let us consider the case X is a compact Kähler surface, following [Wit94, §4], [Mor96, Chapter 7].

The bundle Λ^+ of self-dual 2-forms decomposes as $\Lambda^0\omega \oplus \Lambda^{0,2}$, where ω is the Kähler form. It induces a decomposition $F_A^+ = (F_A^+)^{1,1} \oplus F_A^{0,2}$. The moment map $\boldsymbol{\mu}(\Phi)$ also decomposes as $\boldsymbol{\mu}_{\mathbb{R}}(\Phi) \oplus \boldsymbol{\mu}_{\mathbb{C}}(\Phi)$.

We replace $\hat{G}$ by its subgroup $\tilde{G} = \mathrm{U}(1) \times \mathrm{Sp}(1)_- \times G$. The principal bundle $P_{\mathrm{Spin}(4)}$ has the reduction $P_{\mathrm{U}(1)\times\mathrm{Sp}(1)_-}$ to $\mathrm{U}(1) \times \mathrm{Sp}(1)_-$-bundle. We have a square root $K_X^{-1/2}$ of the anti-canonical bundle K_X^{-1}, associated with the standard representation of $\mathrm{U}(1)$. On the other hand, the bundle associated with the vector representation of $\mathrm{Sp}(1)_-$ is $\Lambda^{0,1} \otimes K_X^{1/2}$. The bundle $\hat{P}$ likewise has a reduction $\tilde{P}$ to a $\tilde{G}$-bundle. It is also the fiber product of P and $P_{\mathrm{U}(1)\times\mathrm{Sp}(1)_-}$ over X. The connection A on P extends to a connection on $\tilde{P}$ as before, by combined with the Levi-Civita connection.

Since the multiplication of i on $\mathbf{M}$ commutes the action of $\mathrm{U}(1) \times G$, we can view $\mathbf{M}^+$ as a complex representation of $\tilde{G}$. Hence $\tilde{P} \times_{\tilde{G}} \mathbf{M}^+$ is a complex vector bundle. A direct calculation shows that $\boldsymbol{\mu}_{\mathbb{R}}(i\Phi) = \boldsymbol{\mu}_{\mathbb{R}}(\Phi)$, $\boldsymbol{\mu}_{\mathbb{C}}(i\Phi) = -\boldsymbol{\mu}_{\mathbb{C}}(\Phi)$.

We combine the complex structure i with the following which is a consequence of the Weitzenböck formula:

$$\begin{aligned}(6.4)\qquad &\int_X |D_A^+\Phi|^2 + \frac{1}{2}|F_A^+ - \boldsymbol{\mu}(\Phi)|^2 \\ &= \int_X |\nabla_A \Phi|^2 + \frac{1}{2}|F_A^+|^2 + \frac{1}{2}|\boldsymbol{\mu}(\Phi)|^2 + \frac{s}{4}|\Phi|^2.\end{aligned}$$

The right hand side is unchanged when we replace Φ by $i\Phi$. On the other hand, (A,Φ) is a solution of the generalized Seiberg-Witten equation if and only if the left hand side vanishes. Hence (A,Φ) is a solution if and only if both (A,Φ) and $(A,i\Phi)$ are solutions, that is

$$(6.5)\qquad \begin{aligned} &D_A^+\Phi = 0 = D_A^+(i\Phi), \\ &F_A^{0,2} = 0 = \boldsymbol{\mu}_{\mathbb{C}}(\Phi), \\ &(F_A^+)^{1,1} = \boldsymbol{\mu}_{\mathbb{R}}(\Phi). \end{aligned}$$

In particular, $P \times_G G_{\mathbb{C}}$ has a structure of a holomorphic principal bundle induced from A. Then $P \times_G \mathbf{M}$ and $\tilde{P} \times_{\tilde{G}} \mathbf{M}^+ = K_X^{-1/2} \otimes (P \times_G \mathbf{M})$ are holomorphic vector bundles.

We claim that Φ satisfies $D_A^+\Phi = 0 = D_A^+(i\Phi)$ if and only if $\overline{\Phi}$, viewed as a section of $(\tilde{P}\times_{\tilde{G}}\mathbf{M}^+)^*$, is holomorphic. It is enough to check the assertion at each point p, hence we take a holomorphic coordinate system (z_1, z_2) around p. We write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. We assume tangent vectors $\partial/\partial x_1$, $\partial/\partial y_1$, $\partial/\partial x_2$, $\partial/\partial y_2$ correspond to 1, i, j, k respectively under the isomorphism $T_pX \cong \mathbb{H}$. Then

$$\begin{aligned} D_A^+\Phi &= \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\cdot \nabla_{\frac{\partial}{\partial x_\alpha}}\Phi + \frac{\partial}{\partial y_\alpha}\cdot\nabla_{\frac{\partial}{\partial y_\alpha}}\Phi \\ &= -\nabla_{\frac{\partial}{\partial x_1}}\Phi + i\nabla_{\frac{\partial}{\partial y_1}}\Phi + j\nabla_{\frac{\partial}{\partial x_2}}\Phi + k\nabla_{\frac{\partial}{\partial y_2}}\Phi \\ &= -2\nabla_{\frac{\partial}{\partial z_1}}\Phi + 2\nabla_{\frac{\partial}{\partial \bar{z}_2}}\overline{\Phi}j. \end{aligned}$$

If we replace Φ by $i\Phi$, the first term is multiplied by i, and the second term by $-i$. Therefore the assertion follows.

Theorem 6.6. *A solution of the generalized Seiberg-Witten equation* (6.1) *on a compact Kähler surface X consists of a holomorphic $G_{\mathbb{C}}$-bundle $\mathcal{P}$, a holomorphic section $\overline{\Phi}$ of $K_X^{1/2}\otimes(\mathcal{P}\times_{G_{\mathbb{C}}}\mathbf{M})$ satisfying*

$$\begin{aligned} &\boldsymbol{\mu}_{\mathbb{C}}(\Phi) = 0, \\ &(F_A^+)^{1,1} = \boldsymbol{\mu}_{\mathbb{R}}(\Phi)\omega. \end{aligned}$$

The equation $(F_A^+)^{1,1} = \boldsymbol{\mu}_{\mathbb{R}}(\Phi)\omega$ corresponds to the stability condition by the Hitchin-Kobayashi correspondence. Therefore the moduli space of solutions has an algebro-geometric description.

Remark 6.7. For the original Seiberg-Witten equation for $(G,\mathbf{M}) = (\mathrm{U}(1),\mathbb{H})$, $\boldsymbol{\mu}_{\mathbb{C}}(\Phi) = 0$ can be written as $\alpha\overline{\beta} = 0$ where $\Phi = \alpha + j\beta$. Therefore we have either $\alpha = 0$ or $\beta = 0$. We do not have such a further reduction in general.

6(iv). Dimension reduction

Let us suppose the spin 4-manifold X is $\mathbb{R}\times Y$ for a spin 3-manifold Y. The 3-dimensional generalized Seiberg-Witten equation is an $\mathbb{R}$-invariant solution of (6.1). We regard $\mathfrak{so}(3) = \mathfrak{sp}(1)$ as the space of imaginary quaternions, and we have the Clifford multiplication $\mathfrak{so}(3)\otimes\mathbf{M}\to\mathbf{M}$. We take a principal G-bundle $P\to Y$ and consider the associated $\bar{G}$-bundle $\bar{P}$ with

$\bar{G} \overset{\text{def.}}{=} \operatorname{Spin}(3)\times G$, as in the 4-dimensional case. The 3-dimensional generalized Seiberg-Witten equation is a system of partial differential equations for a connection A on P and a section Φ of $\bar{P}\times_{\bar{G}}\mathbf{M}$:

$$\begin{aligned} &D_A\Phi = 0,\\ &* F_A = -\boldsymbol{\mu}(\Phi). \end{aligned} \tag{6.8}$$

Here $\boldsymbol{\mu}(\Phi)$ is a section of $\bar{P}\times_{\bar{G}}(\mathfrak{g}\otimes\mathfrak{so}(3))$, and considered as a $P\times_G\mathfrak{g}$-valued 1-form. The Hodge star $*$ send 2-forms to 1-forms in 3-dimension, the lower equation makes sense.

The same remark on the bound on Φ in 6(ii) applies also in the 3-dimensional case: We need to understand the behavior of solutions (A_i,Φ_i) with $|\Phi_i|\to\infty$ in order to define invariants rigorously.

For the Floer homology group, we consider the equation

$$\begin{aligned} \frac{d}{dt}\Phi &= -D_A\Phi,\\ \frac{d}{dt}A &= -*F_A - \boldsymbol{\mu}(\Phi) \end{aligned}$$

on $\mathbb{R}\times Y$. This is the gradient flow equation for a generalized Chern-Simons functional given by

$$\mathcal{E}(A,\Phi) = \int_Y \operatorname{CS}(A) + \frac{1}{2}(D_A\Phi,\Phi)\,d\operatorname{vol}.$$

Here CS is the (normalized) Chern-Simons functional satisfying

$$\frac{d}{dt}\int_Y \operatorname{CS}(A) = \int_Y (\frac{dA_t}{dt}\wedge F_{A_t}) = \int_Y (\frac{dA_t}{dt}, *F_{A_t})\,d\operatorname{vol}.$$

In the framework of a topological quantum field theory in $3d$, it is natural to expect that one should consider the generalized Seiberg-Witten equation (6.8) for a 3-manifold Y with cylindrical ends. As 'boundary conditions', one should look at the translation invariant solutions of (6.8) over $Y=\mathbb{R}\times C$ for a 2-dimensional Riemann manifold C [Don99, §2]. Since C is a Kähler manifold, we can apply the argument in §6(iii) to reduce the equation to

$$(6.9)\qquad \begin{aligned} &A \text{ defines a holomorphic } G_{\mathbb{C}}\text{-bundle } \mathcal{P} \text{ on } C,\\ &\overline{\Phi} \text{ is a holomorphic section of } K_C^{1/2}\otimes(\mathcal{P}\times_{G_{\mathbb{C}}}\mathbf{M}),\\ &\boldsymbol{\mu}_{\mathbb{C}}(\Phi)=0,\\ &*F_A=\boldsymbol{\mu}_{\mathbb{R}}(\Phi),\end{aligned}$$

where $*$ is the Hodge star operator in 2-dimension. The first condition is automatic as C is complex dimension 1, and hence $F_A^{0,2}$ is automatically 0.

This is the equation for the gauge nonlinear σ-model whose target is the Higgs branch $\mathbf{M}/\!\!/\!\!/G=\boldsymbol{\mu}^{-1}(0)/G$.

7. Motivic Donaldson-Thomas type invariants

From the $(2+1)$-dimensional TQFT framework explained in §1(v), the Coulomb branch $\mathcal{M}_C$ of $\mathrm{Hyp}(\mathbf{M}/\!\!/\!\!/G)$ is expected to be related to the cohomology of moduli spaces of solutions of (6.9). This is a starting point, and must be modified as we have explained in Introduction. We should get a reasonable answer for $(G,\mathbf{M})=(\mathrm{SU}(2),0)$ and we should recover the monopole formula (1.2) for good or ugly theories.

Our proposal is as in §1(ix). Besides reproducing (1.2), these modifications are natural in view of recent study of Donaldson-Thomas (DT) invariants, as we will explain below. Recall that DT invariants were introduced as complex analog of Casson invariants [DT98]. As we can use algebro-geometric techniques to handle singularities in moduli spaces, DT invariants are easier to handle than the original Casson invariant in a sense. Therefore it should be reasonable for us to model the theory of DT invariants.

7(i). Holomorphic Chern-Simons functional

Let C be a compact Riemann surface. We choose and fix a spin structure, i.e., the square root $K_C^{1/2}$ of the canonical bundle K_C. We also fix a (C^∞) principal $G_{\mathbb{C}}$-bundle P with a fixed reference partial connection $\overline{\partial}$. (See Remark 7.1(b) below, for a correct formulation.) A *field* consists of a pair

$\overline{\partial}+A$: a partial connection on P. So A is a C^∞-section of $\Lambda^{0,1}\otimes(P\times_{G_{\mathbb{C}}}\mathfrak{g}_{\mathbb{C}})$.

Φ : a C^∞-section of $K_C^{1/2}\otimes(P\times_{G_{\mathbb{C}}}\mathbf{M})$.

Let $\mathcal{F}$ be the space of all fields. There is a gauge symmetry, i.e., the group $\mathcal{G}_{\mathbb{C}}(P)$ of all (complex) gauge transformations of P natural acts on the space $\mathcal{F}$.

Our notation is slightly different from one in (6.9). We replace $\overline{\Phi}$ by Φ for brevity, and A is a partial connection instead of a connection.

Remarks 7.1. (a) When we have a flavor symmetry G_F, we slightly change the setting as follows. Recall we have an exact sequence $1 \to G \to \tilde{G} \to G_F \to 1$ of groups. Choose a principal $(G_F)_{\mathbb{C}}$-bundle P_F with a partial connection $\overline{\partial}_F$, and also a $\tilde{G}$-bundle $\tilde{P}$ with a partial connection $\overline{\partial}^{\sim}$. Then $\mathcal{F}$ consists of

> $\overline{\partial}^{\sim} + \tilde{A}$: a partial connection on $\tilde{P}$. Moreover, we assume that the induced connection on P_F is equal to $\overline{\partial}_F$.
>
> Φ : a C^∞-section of $K_C^{1/2} \otimes (\tilde{P} \times_{\tilde{G}_{\mathbb{C}}} \mathbf{M})$.

Since the notation is cumbersome, we will restrict ourselves to the case without flavor symmetry except we occasionally point out when a crucial difference arises.

(b) A C^∞ principal $G_{\mathbb{C}}$-bundle is classified by its 'degree', an element in $\pi_1(G)$. This is compatible with what is explained in §4(iii)(c). In particular, we need to take sum over all degrees to define the Coulomb branch. Since the sum occurs at the final stage, and is not relevant to most of calculations, we will not mention this point from now.

We define an analog of the holomorphic Chern-Simons functional by

$$\mathrm{CS}(A,\Phi) = \frac{1}{2}\int_C \omega_{\mathbb{C}}((\overline{\partial}+A)\Phi\wedge\Phi), \tag{7.2}$$

where $\omega_{\mathbb{C}}(\ \wedge\ )$ is the tensor product of the exterior product and the complex symplectic form $\omega_{\mathbb{C}}$ on $\mathbf{M}$. Since $(\overline{\partial}+A)\Phi$ is a C^∞-section of $\bigwedge^{0,1}\otimes K_C^{1/2}\otimes (P\times_{G_{\mathbb{C}}}\mathbf{M})$, $\omega_{\mathbb{C}}((\overline{\partial}+A)\Phi\wedge\Phi)$ is a C^∞-section of $\bigwedge^{0,1}\otimes K_C = \bigwedge^{1,1}$. Its integral is well-defined. This is invariant under the gauge symmetry $\mathcal{G}_{\mathbb{C}}(P)$.

When $\mathbf{M}$ is a cotangent type, we can slightly generalize the construction. Let us choose M_1, M_2 be two line bundles over C such that $M_1\otimes M_2 = K_C$. We modify Φ as

> Φ_1, Φ_2 : C^∞-sections of $M_1\otimes(P\times_{G_{\mathbb{C}}}\mathbf{N})$ and $M_2\otimes(P\times_{G_{\mathbb{C}}}\mathbf{N}^*)$ respectively.

Then

$$\mathrm{CS}(A, \Phi_1, \Phi_2) = \int_C \langle (\overline{\partial} + A)\Phi_1, \Phi_2\rangle. \tag{7.3}$$

It is a complex valued function on $\mathcal{F}$.

Remark 7.4. In this expression, analogy with the holomorphic Chern-Simons functional over a Calabi-Yau 3-fold X is clear. We consider a kind of dimension reduction from X to C, and connection forms in the reduced direction are changed to sections Φ_1, Φ_2.

Let us consider critical points of CS. We take a variation in the direction $(\dot{A}, \dot{\Phi})$:

$$\begin{aligned} & d\mathrm{CS}_{(A,\Phi)}(\dot{A}, \dot{\Phi}) \\ &= \frac{1}{2}\int_C \omega_{\mathbb{C}}((\overline{\partial} + A)\Phi \wedge \dot{\Phi}) + \omega_{\mathbb{C}}((\overline{\partial} + A)\dot{\Phi} \wedge \Phi) + \omega_{\mathbb{C}}(\dot{A}\Phi \wedge \Phi) \\ &= \int_C \omega_{\mathbb{C}}((\overline{\partial} + A)\Phi \wedge \dot{\Phi}) + \langle \dot{A} \wedge \boldsymbol{\mu}_{\mathbb{C}}(\Phi)\rangle. \end{aligned}$$

Therefore (A, Φ) is a critical point of CS if and only if the following two equations are satisfied:

$$\begin{aligned} (\overline{\partial} + A)\Phi &= 0, \\ \boldsymbol{\mu}_{\mathbb{C}}(\Phi) &= 0. \end{aligned} \tag{7.5}$$

The first equation means that Φ is a holomorphic section of $K_C^{1/2} \otimes (P \times_{G_{\mathbb{C}}} \mathbf{M})$ when we regard P as a holomorphic principal bundle by $\overline{\partial} + A$. Thus we have recovered the second and third equations in (6.9).

When we have a flavor symmetry G_F, the partial connection $\overline{\partial}_F$ on P_F is fixed, and we only arrow $\tilde{A}$ on $\tilde{P}$ to vary in the 'G-direction'. The Euler-Lagrange equation remains the same, if we understand $\overline{\partial} + A$ involves also $\overline{\partial}_F$.

Remark 7.6. When $\mathbf{M}$ is of an affine quiver type in §2(iv), the solution of (7.5) is analogous to quiver sheaves considered by Szendröi [Sze08]. When we further assume $\mathbf{M}$ is of Jordan quiver type, it is also analogous to ADHM sheaves considered by Diaconescu [Dia12b]. In these cases, hyper-Kähler quotients parametrize instantons on $\mathbb{C}^2/\Gamma$ (and its resolution and deformation) via the ADHM description [KN90], where Γ is a finite subgroup of

SU(2) corresponding to the quiver. We apply the ADHM description fiber-wise, critical points give framed sheaves on a compactified $\mathbb{C}^2/\Gamma$-bundle over C. We do not have the corresponding 3-fold for general $(G, \mathbf{M})$, but it is philosophically helpful to keep the 3-fold picture in mind.

Moreover, one could probably consider analog of Gromov-Witten invariants, namely invariants of quasimaps by Ciocan-Fontanine, Kim, and Maulik, again imposing a stability condition [CFKM14]. The twist by $K_C^{1/2}$ is not included, and they are analog of usual Gromov-Witten invariants for stable maps from $\mathbb{P}^1$ to $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\!\!/G_{\mathbb{C}}$.

Note that there are two bigger differences between our moduli and one in [Sze08, Dia12b, CFKM14]: we assume P to be a genuine G-bundle (or a vector bundle for $G = \mathrm{GL}(N)$), while it is just a coherent sheaf in [Sze08, Dia12b] and the base curve C is not fixed in [CFKM14]. And a stability condition in [Sze08, Dia12b, CFKM14] is not imposed here.

As is clear from Diaconescu and his collaborators [Dia12a, CDP12], a change of stability conditions is worth studying, but we cannot impose the stability condition in order to reproduce the monopole formula, as is clear from the special case $(G, \mathbf{M}) = (\mathrm{SU}(2), 0)$.

If the space $Z(\mathfrak{g}_{\mathbb{C}}^*) = \{\zeta_{\mathbb{C}} \in \mathfrak{g}_{\mathbb{C}}^* \mid \mathrm{Ad}_g^*(\zeta_{\mathbb{C}}) = \zeta_{\mathbb{C}} \text{ for any } g \in G\}$ is nontrivial, we could perturb CS by choosing a section $\tilde{\zeta}_{\mathbb{C}}$ of $K_C \otimes Z(\mathfrak{g}_{\mathbb{C}}^*)$:

$$\mathrm{CS}_{\tilde{c}}(A, \Phi) = \int_C \frac{1}{2}\omega_{\mathbb{C}}((\overline{\partial} + A)\Phi \wedge \Phi) - \langle A \wedge \tilde{\zeta}_{\mathbb{C}} \rangle.$$

The Euler-Lagrange equation is

$$\begin{aligned} &(\overline{\partial} + A)\Phi = 0, \\ &\boldsymbol{\mu}_{\mathbb{C}}(\Phi) = \tilde{\zeta}_{\mathbb{C}}. \end{aligned}$$

It is not clear whether this generalization is useful or not, but we will study the toy model case $C = \mathbb{C}$ in §9(ii).

7(ii). The main proposal

Our main proposal is that the coordinate ring of the Coulomb branch $\mathcal{M}_C$ of $\mathrm{Hyp}(M) /\!\!/\!\!/ G$ is the equivariant cohomology (with compact support) of the vanishing cycle associated with CS:

$$(7.7)\quad \mathbb{C}[\mathcal{M}_C] \overset{?}{=} H^{*+\dim \mathcal{F} - \dim \mathcal{G}_{\mathbb{C}}(P)}_{c, \mathcal{G}_{\mathbb{C}}(P)}(\text{space of solutions of (7.5)}, \varphi_{\mathrm{CS}}(\mathbb{C}_{\mathcal{F}}))^*,$$

up to certain degree shift explained later. Since $\mathcal{F}$ is infinite-dimensional, it is not clear whether the conventional definition of the vanishing cycle functor φ_f can be applied to our situation. We expect that it is possible to justify the definition, as the theory for usual complex Chern-Simons functional for connections on a compact Calabi-Yau 3-fold has been developed Joyce and his collaborators (see [BBD$^+$12, BBBJ13] and the references therein).

Since both $\dim \mathcal{F}$, $\dim \mathcal{G}_{\mathbb{C}}(P)$ are infinite (and its difference also as we will see below), we need to justify the meaning of the cohomological degree in (7.7). This will be done below by applying results for φ_f proved in a finite dimensional setting formally to our situation. Then cohomological degree in (7.7) will match with the grading on the left hand side coming from the U(1)-action, appeared in the monopole formula. We hope that this argument can be rigorously justified when the definition of the vanishing cycle functor is settled.

However, most importantly, it is not clear at this moment how to define multiplication in (7.7). In the subsequent paper [BFN16a], we replace $\mathbb{P}^1$ by the formal punctured disc $D^\times = \operatorname{Spec}\mathbb{C}((z))$, mimicking the affine Grassmannian. We also replace the cohomology group by the homology group. Then we can define multiplication by the convolution. If $\mathbf{M} = 0$, there is a natural pairing between the cohomology of the space for $\mathbb{P}^1$ (thick affine Grassmannian) and homology of the affine Grassmannian (cf. [Gin95, Ch.6]). It is not clear whether this remains true in general at this moment. The reason which we take the dual of the cohomology group in (7.7) comes only from the comparison with the $\mathbf{M} = 0$ case, and hence it is not strong in our current understanding.

Next we should have a Poisson structure in (7.7) from the complex symplectic form of $\mathcal{M}_C$. In [BFN16a], we consider $\mathbb{C}^*$-equivariant cohomology where $\mathbb{C}^*$ acts on $D^\times$ by rotation. It will give us a noncommutative deformation of the multiplication, and hence a Poisson structure in the non-equivariant limit. We call it the *quantized Coulomb branch*.

The equivariant cohomology group (7.7) is a module over $H^*_{\mathcal{G}_{\mathbb{C}}(P)}(\mathrm{pt})$. Therefore we should have a morphism from $\mathcal{M}_C$ to $\operatorname{Spec}(H^*_{\mathcal{G}_{\mathbb{C}}(P)}(\mathrm{pt}))$. It should factor through $\operatorname{Spec}(H^*_G(\mathrm{pt}))$ where $\mathcal{G}_{\mathbb{C}}(P) \to G_{\mathbb{C}}$ is given by an evaluation at a point in C, as $\operatorname{Ker}(\mathcal{G}_{\mathbb{C}}(P) \to G_{\mathbb{C}})$ acts freely on $\mathcal{F}$. Note that $H^*_G(\mathrm{pt}) = \mathbb{C}[\mathfrak{g}_{\mathbb{C}}]^G = \mathbb{C}[\mathfrak{h}_{\mathbb{C}}]^W$ is a polynomial ring, where $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Since we have $\dim \mathfrak{h}_{\mathbb{C}} = \dim \mathcal{M}_C/2$, it is natural to expect that $\mathcal{M}_C$ is an integrable system.

7(iii). Motivic universal DT invariants

Since (7.7) is too much ambitious to study at this moment, we consider less ambitious one, which is the motivic universal Donaldson-Thomas invariant defined by

$$\frac{[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[\mathcal{G}_{\mathbb{C}}(P)]_{\mathrm{vir}}}. \tag{7.8}$$

This is expected to be obtained by taking weight polynomials with respect to the mixed Hodge structure given by the Hodge module version of (7.7). The definition of motivic DT invariants have been worked out for degree zero by [BBS13], and also by [BBD$^+$12, BBBJ13] and the references therein for general cases.

Since we do not study the multiplication on (7.7) in this paper, our primary interest will be the Poincaré polynomial of (7.7) and its comparison with the monopole formula. For $C=\mathbb{P}^1$, we introduce a stratification on the space ($\mathcal{R}$ introduced below) in question such that each stratum has vanishing odd degree cohomology groups. Therefore the Poincaré polynomial is equal to the weight polynomial, hence (7.8) looses nothing for our purpose.

Since both $\mathcal{F}$, $\mathcal{G}_{\mathbb{C}}(P)$ are infinite dimensional, one still needs to justify the above definition, defined again via the vanishing cycle functor. Also note that $[\mathcal{G}_{\mathbb{C}}(P)]_{\mathrm{vir}}$ is the virtual motive of $\mathcal{G}_{\mathbb{C}}(P)$, i.e.,

$$(-\mathbb{L}^{\frac{1}{2}})^{-\dim \mathcal{G}_{\mathbb{C}}(P)}[\mathcal{G}_{\mathbb{C}}(P)], \tag{7.9}$$

which must be interpreted appropriately.

Maybe it is worthwhile to mention one more thing need to be justified at this stage. The moduli stack of holomorphic $G_{\mathbb{C}}$-bundles over C is of infinite type. For example, over $\mathbb{P}^1$, it can be reduced to $T_{\mathbb{C}}$-bundles, in other words, sums of line bundles, but we have no control for the degrees of those line bundles. Therefore we need to stratify crit(CS) by the Harder-Narashimhan filtration, and consider motivic invariants for strata, and then take their sum. The last sum is infinite, so we need to justify it. In practice, at least for $\mathbb{P}^1$, the 'good' or 'ugly' assumption means that we only get positive powers of $\mathbb{L}$ for each stratum, and we consider it as a formal power series in $\mathbb{L}$.

Remark 7.10. We call the above the *universal* DT invariant, as no stability condition is imposed following [MMNS12]. It is a hope that usual DT invariants and wall-crossing formula could be understood from the universal one as in [MMNS12]. In [MMNS12], the authors considered representations of a quiver, hence arbitrary objects in an abelian category. However, we do not include coherent sheaves (for $G = \mathrm{GL}$), it is not clear whether this is a reasonable hope or not.

7(iv). Cutting

We suppose that $\mathbf{M}$ is of cotangent type, so $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$. We consider the scalar multiplication on the factor $\mathbf{N}^*$. Then we have $(t \cdot v, t \cdot w) = t(v, w)$ for $t \in \mathrm{U}(1)$.

We have the induced $\mathbb{C}^*$-action on $\mathcal{F}$ given by

$$(A, \Phi) \mapsto (A, t \cdot \Phi).$$

Then CS is of weight 1:

$$\mathrm{CS}(A, t \cdot \Phi) = t\mathrm{CS}(A, \Phi).$$

Thus the result of Behrend-Bryan-Szendröi [BBS13] can be applied (at least formally) to get

$$[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}} = -(-\mathbb{L}^{\frac{1}{2}})^{-\dim \mathcal{F}} \left([\mathrm{CS}^{-1}(1)] - [\mathrm{CS}^{-1}(0)] \right).$$

We introduce the reduced space of fields by

$$\mathcal{F}_{\mathrm{red}} \overset{\mathrm{def.}}{=} \{(\overline{\partial} + A, \Phi_1)\} = \mathcal{F}^{\mathbb{C}^*}.$$

We have a projection $\mathcal{F} \to \mathcal{F}_{\mathrm{red}}$ induced from $\mathbf{M} \to \mathbf{N}$. Note that CS is linear in Φ_2 once $(\overline{\partial} + A, \Phi_1)$ is fixed. It is zero if $(\overline{\partial} + A)\Phi_1 = 0$, and nonzero otherwise. (See (7.3).) Therefore the argument by Morrison [Mor12] and Nagao [Nag11] can be (at least formally) applied to deduce

$$[\mathrm{CS}^{-1}(1)] - [\mathrm{CS}^{-1}(0)] = -(-\mathbb{L}^{\frac{1}{2}})^{2\dim(\mathrm{fiber})}[\mathcal{R}],$$

where $\mathcal{R}$ is the locus $\{(\overline{\partial} + A)\Phi_1 = 0\}$ in $\mathcal{F}_{\mathrm{red}}$. Hence

$$[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{2\dim(\mathrm{fiber}) - \dim \mathcal{F}}[\mathcal{R}]. \tag{7.11}$$

Note that $\mathcal{R}$ is the space of pairs of principal holomorphic $G_{\mathbb{C}}$-bundles $(P, \overline{\partial} + A)$ together with holomorphic sections Φ_1 of the associated bundle $M_1 \otimes (P \times_{G_{\mathbb{C}}} \mathbf{N})$.

If we take $M_1 = K_C$ (and hence $M_2 = \mathcal{O}_C$) and $\mathbf{N} = \mathbf{N}^* = \mathfrak{g}_{\mathbb{C}}$, $\mathcal{R}$ is the space of Higgs bundles, introduced first by Hitchin [Hit87].

The fiber in (7.11) is the space of Φ_2, i.e., all C^∞ sections of $M_2 \otimes (P \times_{G_\mathbb{C}} \mathbf{N}^*)$. This is obviously infinite dimensional, and hence dim(fiber) must be interpreted appropriately. Observe that $\dim \mathcal{F}$ is also infinite dimensional, however. Note that we also have $\dim \mathcal{G}(P)$ in (7.9). We combine them 'formally' to get

$$\begin{aligned} &2\dim(\text{fiber}) - \dim \mathcal{F} + \dim \mathcal{G}_\mathbb{C}(P) \\ &= \dim\{\Phi_2\} - \dim\{\Phi_1\} + \dim \mathcal{G}_\mathbb{C}(P) - \dim\{\overline{\partial} + A\}. \end{aligned} \tag{7.12}$$

Next note that

$$\begin{aligned} &\dim\{\Phi_2\} - \dim\{\Phi_1\} \\ &= \dim C^\infty(K_C \otimes M_1^* \otimes (P \times_{G_\mathbb{C}} \mathbf{N}^*)) - \dim C^\infty(M_1 \otimes (P \times_{G_\mathbb{C}} \mathbf{N})) \\ &= \dim C^\infty(\Lambda^{0,1} \otimes M_1 \otimes (P \times_{G_\mathbb{C}} \mathbf{N})) - \dim C^\infty(M_1 \otimes (P \times_{G_\mathbb{C}} \mathbf{N})), \end{aligned}$$

where the second equality is true because 'dim' is the same for the dual space. Now this is safely replaced by a well-defined number thanks to Riemann-Roch:

$$\begin{aligned} &-\operatorname{ind}(\overline{\partial} \text{ on } M_1 \otimes (P \times_{G_\mathbb{C}} \mathbf{N})) \\ &= -\deg(M_1 \otimes (P \times_{G_\mathbb{C}} \mathbf{N})) + \operatorname{rank}(M_1 \otimes (P \times_{G_\mathbb{C}} \mathbf{N}))(g-1), \end{aligned}$$

where g is the genus of C. Let us write this number by $-i(P_\mathbf{N})$. Similarly we have

$$\begin{aligned} &\dim \mathcal{G}_\mathbb{C}(P) - \dim\{\overline{\partial} + A\} = \operatorname{ind}(\overline{\partial} \text{ on } P \times_{G_\mathbb{C}} \mathfrak{g}_\mathbb{C}) \\ &= \dim G_\mathbb{C}(1-g). \end{aligned}$$

Let us denote this by $i(P)$. We have

$$\frac{[\operatorname{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[\mathcal{G}_\mathbb{C}(P)]_{\mathrm{vir}}} = (-\mathbb{L}^{\frac{1}{2}})^{i(P)-i(P_\mathbf{N})} \frac{[\mathcal{R}]}{[\mathcal{G}_\mathbb{C}(P)]}. \tag{7.13}$$

This argument can be 'lifted' to the case of the cohomology group of vanishing cycles thanks to a recent result by Davison [Dav13, Th. A.1]. Applying his result formally even though the fiber is infinite dimensional, we get

$$\begin{aligned} H^{*+\dim \mathcal{F} - \dim \mathcal{G}_\mathbb{C}}_{c,\mathcal{G}_\mathbb{C}(P)}(\mathcal{F}, \varphi_{\mathrm{CS}}(\mathbb{C}_\mathcal{F})) &\cong H^{*+\dim \mathcal{F} - \dim \mathcal{G}_\mathbb{C} - 2\dim(\text{fiber})}_{c,\mathcal{G}_\mathbb{C}(P)}(\mathcal{R}, \mathbb{C}) \\ &\cong H^{*-i(P)+i(P_\mathbf{N})}_{c,\mathcal{G}_\mathbb{C}(P)}(\mathcal{R}, \mathbb{C}). \end{aligned}$$

7(v). Second projection

There is a second projection $\mathcal{R} \to \mathcal{A}(P)$, where $\mathcal{A}(P)$ is the space of all (partial) connections. The fiber at $\overline{\partial} + A \in \mathcal{A}(P)$ is the space of holomorphic sections $\operatorname{Ker}(\overline{\partial} + A$ on $M_1 \otimes (P \times_{G_\mathbb{C}} \mathbf{N}))$. This depends on the isomorphism class of $\overline{\partial} + A$. Let us denote the isomorphism class by $[\mathcal{P}]$. If we write $\mathcal{P}$, it will be a representative of $[\mathcal{P}]$, considered as a holomorphic principal $G_\mathbb{C}$-bundle. Let

$$h^0(\mathcal{P}_\mathbf{N}) \overset{\text{def.}}{=} \dim H^0(M_1 \otimes (\mathcal{P} \times_{G_\mathbb{C}} \mathbf{N})).$$

Then we have

$$\begin{aligned}(7.14)\qquad & (-\mathbb{L}^{\frac{1}{2}})^{-i(P_\mathbf{N})+i(P)} \frac{[\mathcal{R}]}{[\mathcal{G}_\mathbb{C}(P)]} \\ &= \sum_{[\mathcal{P}] \in \mathcal{A}(P)/\mathcal{G}_\mathbb{C}(P)} (-\mathbb{L}^{\frac{1}{2}})^{i(P)-i(P_\mathbf{N})+2h^0(\mathcal{P}_\mathbf{N})} \frac{[\mathcal{G}_\mathbb{C}(P)\cdot \mathcal{P}]}{[\mathcal{G}_\mathbb{C}(P)]} \\ &= \sum_{[\mathcal{P}] \in \mathcal{A}(P)/\mathcal{G}_\mathbb{C}(P)} (-\mathbb{L}^{\frac{1}{2}})^{i(P)-i(P_\mathbf{N})+2h^0(\mathcal{P}_\mathbf{N})} \frac{1}{[\operatorname{Aut}(\mathcal{P})]}.\end{aligned}$$

This is a well-defined object.

Note that

$$\begin{aligned}&-i(P_\mathbf{N}) + 2h^0(\mathcal{P}_\mathbf{N}) = h^0(\mathcal{P}_\mathbf{N}) + h^1(\mathcal{P}_\mathbf{N}) \\ &= h^0(\mathcal{P}_\mathbf{N}) + h^0(\mathcal{P}_{\mathbf{N}^*}) = h^0(\mathcal{P}_\mathbf{M}),\end{aligned}$$

where $h^1(\mathcal{P}_\mathbf{N}) = \dim H^1(M_1 \otimes (\mathcal{P} \otimes_{G_\mathbb{C}} \mathbf{N}))$, and $\mathcal{P}_{\mathbf{N}^*}$, $\mathcal{P}_\mathbf{M}$ are vector bundles associated with representations $\mathbf{N}^*$, $\mathbf{M}$, and h^0 denotes the dimensions of their holomorphic sections. In particular, this combination is always nonnegative, and makes sense even when $\mathbf{M}$ is not necessarily cotangent type. Therefore it is reasonable to conjecture that this computation remains true even without assuming $\mathbf{M}$ is of cotangent type.

8. Computation for $C = \mathbb{P}^1$

8(i). Parametrization of $G_\mathbb{C}$-bundles

Let us assume $C = \mathbb{P}^1$. Then all holomorphic principal $G_\mathbb{C}$-bundles can be reduced to $T_\mathbb{C}$-bundles, and they are unique up to conjugation by the Weyl group. This is a result of Grothendieck [Gro57]. Note that line bundles are classified by its degree on $\mathbb{P}^1$. Therefore isomorphism classes of $G_\mathbb{C}$-bundles are parametrized by the coweight lattice Y of G modulo the Weyl group W, or *dominant* coweights in other words. A coweight is called a *magnetic charge* in the physics literature. Therefore a coweight is denoted by m in [CHZ14], but we denote a coweight by λ following a standard notation in mathematics.

Let us explain these more concretely. Suppose $G_{\mathbb{C}} = \mathrm{GL}(N)$. Then a $G_{\mathbb{C}}$-bundle is nothing but a rank N vector bundle. It decomposes into a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(\lambda_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda_N)$. As the order of sum does not matter, we may assume $\lambda_1 \geq \cdots \geq \lambda_N$. This is the dominance condition. We can make arbitrary $\lambda = (\lambda_1, \dots, \lambda_N)$ to a dominance one by the action of the symmetric group, which is the Weyl group of $\mathrm{GL}(N)$.

Keep in mind that rank and degree of the vector bundle are

$$\mathrm{rank} = N, \quad \deg = \sum \lambda_i.$$

The degree is called the *topological charge*, denoted by $J(\lambda)$ in [CHZ14, (2.7)]. See (4.4).

By our formula (7.14), the motivic invariant is the sum over all isomorphism classes, i.e., dominant coweights.

When we have a flavor symmetry G_F, we add a $(G_F)_{\mathbb{C}}$-bundle with a partial connection $\overline{\partial}_F$ as additional data. Its isomorphism class is also parametrized by a dominant coweight λ_F. It is a *background* flux, i.e., it enters in the formula, but it is fixed, and we do not sum over λ_F.

Let us note also that the calculation below is true in the level of Poincaré polynomials, not only as motives. Considering isomorphism classes of $G_{\mathbb{C}}$-bundles, we have a stratification of the space $\mathcal{R}$. Each stratum is a vector bundle over a $\mathcal{G}_{\mathbb{C}}(P)$-orbit, and hence has no odd cohomology groups. Therefore the short exact sequences associated with the stratification splits, hence the cohomology group is isomorphic (as a graded vector space) with the sum of cohomology groups of strata.

8(ii). Calculation of contributions from matters

Let $B(\mathbf{N})$ be a base of $\mathbf{N}$ compatible with weight space decomposition. For $b \in B(\mathbf{N})$, let $\mathrm{wt}(b)$ be its weight. Hence

$$\mathbf{N} = \bigoplus_b \mathbb{C}b.$$

We denote the pairing between weights and coweights by $\langle\ ,\ \rangle$. When a principal bundle $\mathcal{P}$ is reduced to a $T_{\mathbb{C}}$-bundle, the associated vector bundle decomposes as

$$\mathcal{P}_{\mathbf{N}} = \bigoplus_b \mathcal{P} \times_{T_{\mathbb{C}}} \mathbb{C}b,$$

a sum of line bundles. If λ is the coweight for $\mathcal{P}$, the degree of the line bundle is $\langle \lambda, \mathrm{wt}(b)\rangle + \deg M_1$. Then

$$h^0(\mathcal{P}_{\mathbf{N}}) + h^1(\mathcal{P}_{\mathbf{N}}) = \sum_b |\langle \lambda, \mathrm{wt}(b)\rangle + \deg M_1 + 1|.$$

This is because

$$\dim H^0(\mathcal{O}_{\mathbb{P}^1}(n)) + \dim H^1(\mathcal{O}_{\mathbb{P}^1}(n)) = |n+1|.$$

For our standard choice $M_1 = M_2 = \mathcal{O}_{\mathbb{P}^1}(-1)$, we have

$$\sum_b |\langle \lambda, \mathrm{wt}(b)\rangle|,$$

which coincides with the second term in $\Delta(\lambda)$ in (4.1) if we replace $-\mathbb{L}^{\frac{1}{2}}$ by t. (Recall $\Delta(\lambda)$ appears as $t^{2\Delta(\lambda)}$ in (1.2)).

The calculation remains the same for $\mathbf{M}$ not necessarily cotangent type.

8(iii). Calculation of automorphism groups ($G_{\mathbb{C}} = \mathrm{GL}(N, \mathbb{C})$ case)

Let us first suppose that $G_{\mathbb{C}}$ is $\mathrm{GL}(N,\mathbb{C})$. Then $\mathcal{P}$ is a usual vector bundle of rank N. We write $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$, a non-increasing sequence of integers. We also write $\lambda = (\cdots (-1)^{k_{-1}} 0^{k_0} 1^{k_1} \cdots)$, which is a modification of the usual notation for partitions, i.e., -1 appears k_{-1} times, 0 appears k_0 times, etc in the sequence λ.

In this notation, we have

$$\begin{aligned}\dim \mathrm{End}(\mathcal{P}) &= \sum_{\alpha,\beta} k_\alpha k_\beta \dim \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(\alpha), \mathcal{O}_{\mathbb{P}^1}(\beta)) \\ &= \sum_{\beta\geq\alpha} k_\alpha k_\beta (\beta - \alpha + 1).\end{aligned}$$

For $\mathrm{Aut}(\mathcal{P})$, we replace the block diagonal entries $\alpha = \beta$ by $\mathrm{GL}(k_\alpha)$. Therefore

$$[\mathrm{Aut}(\mathcal{P})] = [\mathrm{End}(\mathcal{P})] \prod_\alpha \frac{[\mathrm{GL}(k_\alpha)]}{[\mathrm{End}(\mathbb{C}^{k_\alpha})]}. \tag{8.1}$$

The ratio $[\mathrm{GL}(k)]/[\mathrm{End}(\mathbb{C}^k)]$ is easy to compute and well-known:

$$[\mathrm{GL}(k)]/[\mathrm{End}(\mathbb{C}^k)] = (\mathbb{L}^k - 1)(\mathbb{L}^k - \mathbb{L})\cdots(\mathbb{L}^k - \mathbb{L}^{k-1})\mathbb{L}^{-k^2}$$
$$= (-1)^k(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^k)\mathbb{L}^{-\frac{k(k+1)}{2}}.$$

Thus

$$\frac{(-\mathbb{L}^{\frac{1}{2}})^{i(P)}}{[\mathrm{Aut}(\mathcal{P})]} = (-1)^{\sum k_\alpha}(-\mathbb{L}^{\frac{1}{2}})^d \prod_\alpha \frac{1}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{k_\alpha})},$$

where

$$\begin{aligned} d &= i(P) - 2\dim\mathrm{End}(\mathcal{P}) + \sum k_\alpha(k_\alpha+1) \\ &= -2\sum_{\beta\ge\alpha} k_\alpha k_\beta(\beta-\alpha) - 2\sum_{\beta\ge\alpha} k_\alpha k_\beta + \sum_{\alpha,\beta} k_\alpha k_\beta + \sum k_\alpha(k_\alpha+1) \\ &= -2\sum_{\beta\ge\alpha} k_\alpha k_\beta(\beta-\alpha) + \sum k_\alpha. \end{aligned}$$

The first term is

$$-2\sum_{i<j}(\lambda_i - \lambda_j),$$

which is equal to the first term in $\Delta(\lambda)$ in (4.1), again if we replace $-\mathbb{L}^{\frac{1}{2}}$ by t.

The term

$$\prod_\alpha \frac{1}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{k_\alpha})}$$

is $P_{\mathrm{U}(N)}(t;\lambda)$ if we put $\mathbb{L} = t^2$.

Therefore up to the factor $(-1)^{\sum k_\alpha}(-\mathbb{L}^{\frac{1}{2}})^{\sum k_\alpha}$, it coincides with (1.2). Note that $\sum k_\alpha$ is $N = \operatorname{rank} G$, and hence is independent of λ.

Recall that the rank of the group is $\dim_{\mathbb{H}} \mathcal{M}_C$ (§4(iii)(3)). Therefore

$$H_{G,\mathbf{M}}(t) = (\mathbb{L}^{\frac{1}{2}})^{-\dim_{\mathbb{H}}\mathcal{M}_C} \frac{[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[\mathcal{G}_{\mathbb{C}}(P)]_{\mathrm{vir}}}.$$

The virtual motive is shifted already by $-\dim/2$, hence $H_{G,\mathbf{M}}(t)$ is shifted by $-\dim$ in total. This shift is unavoidable, as the monopole formula starts with 1 which corresponds to constant functions on $\mathcal{M}_C$. This convention is not taken in the cohomology side.

8(iv). Calculation of automorphism groups (general case)

Let us turn to a general case. The relevant calculation was done in [BD07, §7]. We reproduce it with shortcuts of a few arguments for the sake of the reader.

As in [BD07, Def. 2.4], we add the *torsor* relation to our motivic ring: $[\mathcal{P}] = [X][G]$ for a principal G bundle $\mathcal{P}$ over a variety X.

Let P, U, L be the parabolic subgroup, its unipotent radical and its Levi quotient associated with the dominant coweight λ. ($L = \mathrm{Stab}_G(\lambda)$ in the previous notation.) Since the underlying C^∞ principal $G_{\mathbb{C}}$-bundle P will not occur any more except through the formula $i(P)$ (which is $\dim G_{\mathbb{C}}$ in our case), there is no fear of confusion. Let $\mathfrak{u}$ be the Lie algebra of U, and $\mathcal{P}_{\mathfrak{u}}$ be the associated vector bundle. Then generalizing the computation for $G = \mathrm{GL}(N, \mathbb{C})$, we have

Lemma 8.2 ([Ram83, Prop. 5.2]). *We have* $\mathrm{Aut}(\mathcal{P}) = L \ltimes H^0(\mathbb{P}^1, \mathcal{P}_{\mathfrak{u}})$.

The proof is not difficult. It could be an exercise for the reader.

Since $\mathcal{P}$ is now a T-bundle, we have

$$\mathcal{P}_{\mathfrak{u}} = \bigoplus_{\substack{\alpha\in\Delta^+ \\ \langle\lambda,\alpha\rangle>0}} \mathcal{P}_{\mathfrak{g}_\alpha},$$

where $\mathcal{P}_{\mathfrak{g}_\alpha}$ is the line bundle associated with the root subspace $\mathfrak{g}_\alpha$. Its degree is $\langle\lambda, \alpha\rangle$. Hence

$$\dim H^0(\mathbb{P}^1, \mathcal{P}_{\mathfrak{u}}) = \sum_{\substack{\alpha\in\Delta^+ \\ \langle\lambda,\alpha\rangle>0}} (\langle\lambda, \alpha\rangle + 1)\,.$$

Therefore

$$\frac{[H^0(\mathbb{P}^1, \mathcal{P}_{\mathfrak{u}})]}{[U]} = \mathbb{L}^{2\langle\lambda,\rho\rangle},$$

where ρ is the half sum of positive roots. Note $2\langle\lambda, \rho\rangle = \sum_{\alpha\in\Delta^+}\langle\lambda, \alpha\rangle$. Since we take λ dominant, this is the first term (up to sign) of $\Delta(\lambda)$ in (4.1).

We substitute this into Lemma 8.2. Since $LU = P$,

$$[\mathrm{Aut}(\mathcal{P})] = \mathbb{L}^{2\langle\lambda,\rho\rangle}[P] = \mathbb{L}^{2\langle\lambda,\rho\rangle}\frac{[G]}{[G/P]},$$

where G/P is the partial flag variety. (We implicit used the torsor relation. See [BD07, p.637] for detail.)

The partial flag variety G/P has the Bruhat decomposition. Therefore we only need to compute ordinary cohomology. To connect with the ring of invariant polynomials, we use an isomorphism of equivariant cohomology groups:

$$H^*_G(G/P) \cong H^*_L(\mathrm{pt}).$$

The spectral sequence relating equivariant and ordinary cohomology groups collapses for G/P (as there is no odd degree cohomology). Hence $H^*_G(G/P) = H^*(G/P) \otimes H^*_G(\mathrm{pt})$. Thus we get

$$[G/P] = \frac{[H^*_L(\mathrm{pt})]}{[H^*_G(\mathrm{pt})]} = \frac{P_G(-\mathbb{L}^{\frac{1}{2}}; \lambda)}{P_G(-\mathbb{L}^{\frac{1}{2}}; 0)}.$$

From the special case $P = B$ of this computation, we have

$$[B] = [G] P_G(-\mathbb{L}^{\frac{1}{2}}; 0)(1 - \mathbb{L})^{\operatorname{rank} G},$$

as $[H^*_T(\mathrm{pt})] = (1 - \mathbb{L})^{-\operatorname{rank} G}$. On the other hand, B is the product of T and its unipotent radical. Hence $[B] = (\mathbb{L} - 1)^{\operatorname{rank} G} [\mathbb{L}]^{\frac{1}{2}(\dim G_{\mathbb{C}} - \operatorname{rank} G)}$. We get

$$[G] P_G(-\mathbb{L}^{\frac{1}{2}}; 0) = (-1)^{\operatorname{rank} G} [\mathbb{L}]^{\frac{1}{2}(\dim G_{\mathbb{C}} - \operatorname{rank} G)}.$$

Now we combine all these computation to get

$$\frac{(-\mathbb{L}^{\frac{1}{2}})^{i(P)}}{[\mathrm{Aut}(\mathcal{P})]} = (-1)^{\operatorname{rank} G} (-\mathbb{L}^{\frac{1}{2}})^{-4\langle \lambda, \rho \rangle + \operatorname{rank} G} P_G(-\mathbb{L}^{\frac{1}{2}}; \lambda).$$

This coincides with the contribution of terms not involving $\mathbf{M}$ in the monopole formula up to the factor $(-1)^{\operatorname{rank} G} (-\mathbb{L}^{\frac{1}{2}})^{\operatorname{rank} G}$.

9. Computation for $C = \mathbb{C}$

9(i). Motivic universal DT invariant for $\mathfrak{g}_{\mathbb{C}} \times \mathbf{M}$

Since we assume C is compact, we cannot apply the argument in §7 to $C = \mathbb{C}$. However we further consider the dimension reduction in the $\mathbb{C}$-direction to a point as in Remark 7.4. In practice, it means that we replace $\overline{\partial} + A$ by an element ξ in the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We thus arrive at

$$\mathrm{CS}(\xi, \Phi) = \frac{1}{2}\omega_{\mathbb{C}}(\xi\Phi, \Phi) = \langle \xi, \boldsymbol{\mu}_{\mathbb{C}}(\Phi)\rangle.$$

This is a function on a finite dimensional space $\mathcal{F} = \mathfrak{g}_{\mathbb{C}} \times \mathbf{M}$, and all computation in §7 become rigorous. We repeat our assertions in this setting:

1) (ξ, Φ) is a critical point of CS if and only if

$$\xi\Phi = 0, \quad \boldsymbol{\mu}_{\mathbb{C}}(\Phi) = 0.$$

2) The motivic universal DT invariant is defined by

$$\frac{[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[G_{\mathbb{C}}]_{\mathrm{vir}}}.$$

3) Assuming $\mathbf{M}$ is of cotangent type (i.e., $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$), we use a $\mathbb{C}^*$-action given by $(\xi, \Phi_1, \Phi_2) \mapsto (\xi, \Phi_1, t\Phi_2)$ to get

$$\frac{[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[G_{\mathbb{C}}]_{\mathrm{vir}}} = \frac{[\mathcal{R}]}{[G_{\mathbb{C}}]},$$

where $\mathcal{R}$ is the locus $\xi\Phi_1 = 0$ in $\mathcal{F}_{\mathrm{red}} = \{(\xi, \Phi_1) \in \mathfrak{g}_{\mathbb{C}} \times \mathbf{N}\}$.
Note that (7.12) vanishes in this setting.

4) We further have

$$\frac{[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[G_{\mathbb{C}}]_{\mathrm{vir}}} = \frac{[\mathcal{R}]}{[G_{\mathbb{C}}]} = \sum_{[\xi]\in\mathfrak{g}_{\mathbb{C}}/G_{\mathbb{C}}} \frac{\mathbb{L}^{\dim(\operatorname{Ker}\xi \text{ on } \mathbf{N})}}{[\mathrm{Stab}_{G_{\mathbb{C}}}(\xi)]}. \tag{9.1}$$

All these are rigorous now.

9(ii). Another $\mathbb{C}^*$-action

It is interesting to note that we have another $\mathbb{C}^*$-action $(\xi, \Phi) \mapsto (t\xi, \Phi)$, which makes sense without the cotangent type condition. It is of weight 1, even when we perturb $\boldsymbol{\mu}_{\mathbb{C}}$ by $\zeta_{\mathbb{C}} \in Z(\mathfrak{g}_{\mathbb{C}}^*)$. The cutting for this action was used by Mozgovoy [Moz11] for $(G, \mathbf{M})$ of quiver type.

Then the application of results on the cutting implies as in §7(iv)

$$\frac{[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[G_{\mathbb{C}}]_{\mathrm{vir}}} = (-\mathbb{L}^{\frac{1}{2}})^{-(\dim\mathbf{M} - 2\dim\mathfrak{g}_{\mathbb{C}})} \frac{[\boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})]}{[G_{\mathbb{C}}]}. \tag{9.2}$$

Note that $\dim\mathbf{M} - 2\dim\mathfrak{g}_{\mathbb{C}}$ is the expected dimension of $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/G_{\mathbb{C}}$.

Let us again suppose $\mathbf{M}$ is of cotangent type and consider the restriction of the projection $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}}) \to \mathbf{N}^*$ given by $(\Phi_1, \Phi_2) \mapsto \Phi_2$. When $\mathbf{M}$ is of quiver type, it was studied by Crawley-Boevey and his collaborators [CBH98, CB01, CBVdB04]. The argument can be modified to our setting as follows.

Let us fix $\Phi_2 \in \mathbf{N}^*$ and consider an exact sequence

$$0 \to \mathrm{Lie}(\mathrm{Stab}_{G_{\mathbb{C}}}(\Phi_2)) \to \mathfrak{g}_{\mathbb{C}} \to \mathbf{N}^*,$$

where the second arrow is the inclusion and the third arrow is the action of $G_{\mathbb{C}}$ on $\mathbf{N}^* : \xi \mapsto \xi\Phi_2$. We consider the transpose

$$\mathbf{N} \to \mathfrak{g}_{\mathbb{C}}^* \xrightarrow{t} \mathrm{Lie}(\mathrm{Stab}_{G_{\mathbb{C}}}(\Phi_2))^* \to 0.$$

From the definition of the moment map, the first arrow is given by $\Phi_1 \mapsto \boldsymbol{\mu}_{\mathbb{C}}(\Phi_1, \Phi_2)$. Let K_{Φ_2} denote the kernel of $\mathbf{N} \to \mathfrak{g}_{\mathbb{C}}^*$.

If we have a solution of $\boldsymbol{\mu}_{\mathbb{C}}(\cdot, \Phi_2) = \zeta_{\mathbb{C}}$ then $\zeta_{\mathbb{C}}$ is in the image of the first map. Hence we must have $t(\zeta_{\mathbb{C}}) = 0$ by the exact sequence. Moreover, if $t(\zeta_{\mathbb{C}}) = 0$, the space of solutions is an affine space modeled by K_{Φ_2}. Therefore we are led to introduce the following condition:

Definition 9.3. We say Φ_2 is $\zeta_{\mathbb{C}}$-*indecomposable* if $t(\zeta_{\mathbb{C}}) = 0$, in other words,

$$\langle \xi, \zeta_{\mathbb{C}} \rangle = 0 \quad \text{for any } \xi \in \mathrm{Lie}(\mathrm{Stab}_{G_{\mathbb{C}}}(\Phi_2)).$$

If $\mathbf{M}$ is of quiver type (with $W = 0$) and $\zeta_{\mathbb{C}}$ is generic, this is equivalent to that Φ_2 is indecomposable in the usual sense. See the proof of [CBVdB04, Prop. 2.2.1]. On the other hand, this imposes nothing if $\zeta_{\mathbb{C}} = 0$.

This condition is invariant under $G_{\mathbb{C}}$-action. Let $\mathbf{N}^*_{\zeta_{\mathbb{C}}\text{-ind}}$ be the constructible subset of $\mathbf{N}^*$ consisting of $\zeta_{\mathbb{C}}$-indecomposable Φ_2.

Let us go back to (9.2). We have

$$\begin{aligned} &(-\mathbb{L}^{\frac{1}{2}})^{-(\dim \mathbf{M} - 2\dim \mathfrak{g}_{\mathbb{C}})} \frac{[\boldsymbol{\mu}_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})]}{[G_{\mathbb{C}}]} \\ &= \sum_{[\Phi_2] \in \mathbf{N}^*_{\zeta_{\mathbb{C}}\text{-ind}}/G_{\mathbb{C}}} \frac{\mathbb{L}^{\dim K_{\Phi_2} - \dim \mathbf{N} + \dim \mathfrak{g}_{\mathbb{C}}}}{[\mathrm{Stab}_{G_{\mathbb{C}}}(\Phi_2)]} \\ &= \sum_{[\Phi_2] \in \mathbf{N}^*_{\zeta_{\mathbb{C}}\text{-ind}}/G_{\mathbb{C}}} \frac{[\mathrm{Lie}(\mathrm{Stab}_{G_{\mathbb{C}}}(\Phi_2))]}{[\mathrm{Stab}_{G_{\mathbb{C}}}(\Phi_2)]}, \end{aligned} \tag{9.4}$$

where we have used the exact sequence to compute the alternating sum of the dimension.

In the situation of [CBVdB04], we replace $G_{\mathbb{C}}$ by its quotient $G_{\mathbb{C}}/\mathbb{C}^*$ as explained at the end of §2(iv). If $\zeta_{\mathbb{C}}$ is generic and hence Φ_2 is indecomposable in the usual sense, $\mathrm{Stab}_{G_{\mathbb{C}}/\mathbb{C}^*}(\Phi_2)$ and its Lie algebra is isomorphic by the exponential. Then each term in the summand is 1. Hence the sum is the motif of the space of indecomposable modules. This is one of crucial steps in their proof of Kac's conjecture for indivisible dimension vectors. See [CBVdB04, Prop. 2.2.1].

Suppose $\zeta_{\mathbb{C}} = 0$, hence Φ_2 is an arbitrary element. When $(G, \mathbf{M})$ is of quiver type with $W = 0$, the right hand side of (9.4) was computed in [Moz11]. Let us introduce variables x_i for each $i \in Q_0$ and denote $\prod_i x_i^{v_i}$ by $x^{\mathbf{v}}$ for $\mathbf{v} = (v_i)_{i \in Q_0}$. Summing up to the motivic Donaldson-Thomas invariants for all dimension vectors, the generating function is given by

$$(9.5) \qquad \sum_{\mathbf{v}} \frac{[\mathrm{crit}(\mathrm{CS})]_{\mathrm{vir}}}{[G_{\mathbb{C}}]_{\mathrm{vir}}} x^{\mathbf{v}} = \exp\left(\sum_{d=1}^{\infty} \frac{\sum_{\mathbf{v}} a_{\mathbf{v}}(\mathbb{L}^d) x^{d\mathbf{v}}}{d(1-\mathbb{L}^{-d})} \right),$$

where $a_{\mathbf{v}}(q)$ is the Kac polynomial counting the number of absolute indecomposable representations of dimension vector $\mathbf{v}$ of the finite field $\mathbb{F}_q$. See [Moz11, Th. 1.1].

Note that Kac polynomials have combinatorial expressions very similar to the monopole formula due to Kac-Stanley (see [Kac83, p.90] and also [Hua00] for a detail of the proof).

Appendix A. Instantons for classical groups

We give further examples of hyper-Kähler quotients arising as instanton moduli spaces for SO / Sp groups on $\mathbb{R}^4$ with various equivariant structures.

A.1. The case of $\mathbb{R}^4$

It is well-known that the ADHM description of $\mathrm{SU}(N)$-instantons on $\mathbb{R}^4$ can be modified for SO / Sp groups.

For $\mathrm{SO}(N)$ k-instantons, we take the vector representation W of $\mathrm{SO}(N)$, the vector representation V of $\mathrm{Sp}(k)$ and set

$$(\mathrm{A.1}) \quad \mathbf{M} = \left\{ (B_1, B_2, a, b) \in \mathrm{Hom}(V,V)^{\oplus 2} \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W) \;\middle|\; \begin{array}{l} (B_\alpha v, v') = (v, B_\alpha v') \quad \text{for } v, v' \in V,\ \alpha = 1,2, \\ (aw, v) = (w, bv) \quad \text{for } v \in V,\ w \in W \end{array} \right\},$$

where (,) is either the symplectic form or orthogonal form on V or W respectively. Then the framed instanton moduli space is the hyper-Kähler quotient $\boldsymbol{\mu}^{-1}(0)/\operatorname{Sp}(k)$.

For $\operatorname{Sp}(N)$ k-instantons, we replace W by the vector representation of $\operatorname{Sp}(N)$, V by the vector representation of $\mathrm{O}(k)$. It should be warned that the group is not $\mathrm{SO}(k)$.

A.2. Nilpotent orbits and Slodowy slices for classical groups

Let us generalize discussions in §2(v) to classical groups. This straightforward generalization was mentioned [Nak94, Remark 8.5(4)] for the SU(2)-equivariant case, but not explicitly written down before.

Suppose (B_1, B_2, a, b) in (A.1) corresponds to an SU(2)-equivariant instanton. Then V, W are representations of SU(2), and (B_1, B_2), a, b are SU(2)-linear. Here we mean the pair (B_1, B_2) is SU(2)-equivariant, when it is considered as a homomorphism in $\operatorname{Hom}(V, V \otimes \rho_2)$, where ρ_2 is the vector representation of SU(2).

Let us decompose V, W as $\bigoplus V_i \otimes \rho_i$, $\bigoplus W_i \otimes \rho_i$ as in §2(v). Since ρ_i has a symplectic or orthogonal form according to the parity of i, we have either symmetric or orthogonal forms on V_i, W_i. For example, for $\mathrm{SO}(N)$-instantons, linear maps B_1, B_2, a, b can be put into a graph (see Figure A1) as before. Groups $\operatorname{Sp}(\frac{v_1}{2})$, $\mathrm{O}(w_1)$, etc in circles or squares indicate, we have

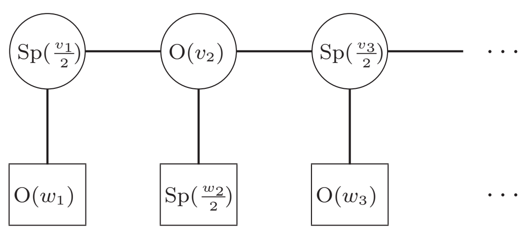

Figure A1: Intersection of a nilpotent orbit and a Slodowy slice

a symmetric or an orthogonal form on V_1, W_1, etc. (Here $v_i = \dim V_i$, $w_i = \dim W_i$.) The space $\mathbf{M}$ consists of linear maps, both directions for each edges, satisfying the transpose condition like in (A.1). And the group G, by which we take the hyper-Kähler quotient, is the product of groups in circles. It is not relevant here whether we should put either $\mathrm{O}(w_1)$,... or $\mathrm{SO}(w_1), \dots$, as we only need orthogonal forms. However, it is matter that for $\mathrm{O}(v_2), \dots$, as G is the product of groups in circles. As a special case $w_2 = w_3 = \cdots = 0$, we recover Kraft-Procesi's description of classical nilpotent orbits [KP82]. This special case also appeared later in [KS96].

A nilpotent orbit $\mathcal{O}_\lambda$ in SO corresponds to an even partition λ. It corresponds to that $\dim W_{\mathrm{even}}$ is even, and is compatible with that W_{even} has a symplectic form. Similarly $\mathcal{O}_\mu$ corresponds to an even partition as $\dim V_{\mathrm{odd}}$ is even.

A.3. Affine Grassmannian and S^1-equivariant instantons

The discussion in the previous subsection can be modified to the case of S^1-equivariant instantons on $\mathbb{R}^4$. We have a different quiver as the dual of ρ_i is ρ_{-i} for irreducible representations of S^1.

For an $\mathrm{SO}(r)$-instanton, the corresponding quiver is Figure A2. From

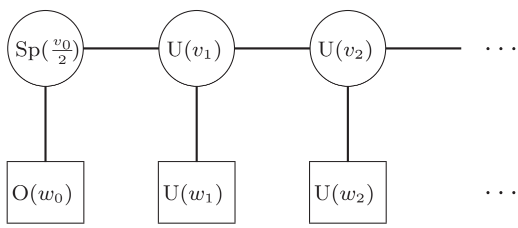

Figure A2: $\mathcal{W}^\mu_{G,\lambda}$ for $G = \mathrm{SO}(r)$

the isomorphisms $V \cong V^*$, $W \cong W^*$ given by the symplectic and orthogonal forms, we have $V_i \cong V^*_{-i}$, $W_i \cong W^*_{-i}$. Therefore we do not need to consider V_i, W_i for $i < 0$. The moment map takes value in $\mathbb{R}^3 \otimes (\mathfrak{sp}(v_0/2) \oplus \mathfrak{u}(v_1) \oplus \mathfrak{u}(v_2) \oplus \cdots)$. The $\mathfrak{u}(v_1) \oplus \mathfrak{u}(v_2) \oplus \cdots$-component is as for ordinary quiver gauge theories. The $\mathfrak{sp}(v_0/2)$-component is given by

$$B_{0,1}B_{1,0} - B^\vee_{1,0}B^\vee_{0,1} + i_0 i_0^\vee,$$

where $B_{0,1}\colon V_1 \to V_0$, $B_{1,0}\colon V_0 \to V_1$, $i_0\colon W_0 \to V_0$. And $\vee$ is defined by $(B_{1,0}v_0, v_{-1}) = (v_0, B^\vee_{1,0}v_{-1})$, by using the symplectic pairing $(\ ,\ )$ between V_1 and V_{-1}, V_0 and itself, etc. Thus $(G, \mathbf{M})$ for SU(2)-equivariant and S^1-equivariant instantons are different for classical groups.

A.4. The case of $\mathbb{R}^4/\Gamma$

Let Γ be a finite subgroup of SU(2). Consider a Γ-equivariant G-instanton on $\mathbb{R}^4$. The case $G = \mathrm{U}(\ell)$ was explained in §3(ii). So we explain how it can be modified to the case when G is SO / Sp.

As above, the isomorphisms $V \cong V^*$, $W \cong W^*$ induces $V_i \cong V_{i^*}^*$, $W_i \cong W_{i^*}^*$, where i^* is determined by the McKay correspondence as $\rho_i^* \cong \rho_{i^*}$. It fixes the trivial representation ρ_0, and hence induces a diagram involution on the finite Dynkin diagram. It is the same diagram involution given by the longest element w_0 of the Weyl group as $-w_0(\alpha_i) = \alpha_{i^*}$, where α_i is the simple root corresponding to the vertex i. In the labeling in [Kac90, Ch. 4], it is given by $i^* = \ell - i + 1$ for type A_ℓ, $1^* = 5$, $2^* = 4$, $3^* = 3$, $4^* = 2$, $5^* = 1$, $6^* = 6$ respectively. For type D_ℓ with odd ℓ, it is given by

$$i^* = \begin{cases} \ell - 1 & \text{if } i = \ell, \\ \ell & \text{if } i = \ell - 1, \\ i & \text{otherwise,} \end{cases}$$

It is the identity for other types.

When $i^* = i$, we determine whether $\rho_i \cong \rho_i^*$ is given by a symplectic or orthogonal form as follows: The trivial representation, assigned to $i = 0$, is orthogonal. For type A_ℓ with odd ℓ with $i = (\ell+1)/2$, it is orthogonal. For other types, ρ_{i_0} for the vertex i_0 adjacent to the vertex $i = 0$ in the affine Dynkin diagram is the representation given by the inclusion $\Gamma \subset \mathrm{SU}(2)$. Therefore $\rho_{i_0} \cong \rho_{i_0}^*$ is symplectic. If i is adjacent to i_0 and $i^* = i$, then $\rho_i \cong \rho_i^*$ is orthogonal. If j is adjacent to such an i with $j^* = j$, then $\rho_j \cong \rho_j^*$ is symplectic, and so on. From this rule, we determine either $V_i \cong V_i^*$, $W_i \cong W_i^*$ is symplectic or orthogonal.

We impose constraint on linear maps B, a, b between V_i, W_i's induced from (A.1) either as in §A.2 or §A.3.

Examples discussed in §A are not of cotangent type in general. We also remark that G is not connected if it contains $\mathrm{O}(N)$ as a factor. A criterion on the complete intersection property in §2(iii) is not known in general. See [Cho15] for special cases.

Appendix B. Hilbert series of symmetric products

We compute

$$H_k(t,z) = \sum_{\lambda=(\lambda_1 \geq \cdots \geq \lambda_k)} z^{\sum \lambda_i} t^{\sum_i N|\lambda_i|} P_{\mathrm{U}(k)}(t,\lambda).$$

We formally set $H_0(t,z) = 1$ and consider the generating function over all k. Our goal is to show that

$$\text{(B.1)} \qquad \sum_{k=0}^{\infty} H_k(t,z)\Lambda^k = \exp\left(\sum_{d=1}^{\infty} \frac{\Lambda^d}{d} H_1(t^d, z^d)\right).$$

The first term $H_1(t,z) = (1-t^{2N})/(1-t^2)(1-t^N z)(1-t^N z^{-1})$ is the Hilbert series of $\mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$, and the above is its plethystic exponential. Then it is the Hilbert series of symmetric powers.

Let us first note that there is a one-to-one correspondence between dominant coweights $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ and $(k_0, k_1, \cdots; m) \in \mathbb{Z}_{\geq 0}^{\infty} \times \mathbb{Z}$ such that

(a) $k_0 \neq 0$.

(b) $k_0 + k_1 + \cdots = k$. In particular, there are only finitely many nonzero k_α.

The correspondence is given by $m = \lambda_k$ and $k_\alpha = \#\{i \mid \lambda_i - \lambda_k = \alpha\}$. This is an extension of usual two types of presentations of partitions.

We write $(t;t)_k = (1-t)(1-t^2)\cdots(1-t^k)$ as usual. Under the bijection we have

$$\sum \lambda_i = \sum_{\alpha} k_\alpha(m+\alpha), \quad \sum |\lambda_i| = \sum_{\alpha} k_\alpha |m+\alpha|,$$

$$P_{\mathrm{U}(k)}(t,\lambda) = \prod_{\alpha=0}^{\infty} \frac{1}{(t^2;t^2)_{k_\alpha}}.$$

Hence

$$\begin{aligned} \sum_{k=0}^{\infty} H_k(t,z)\Lambda^k &= 1 + \sum_{m\in\mathbb{Z}} \sum_{\substack{k_0=1 \\ k_1,k_2,\cdots=0}}^{\infty} \prod_{\alpha=0}^{\infty} \frac{t^{N|m+\alpha|k_\alpha} z^{(m+\alpha)k_\alpha} \Lambda^{k_\alpha}}{(t^2;t^2)_{k_\alpha}} \\ &= 1 + \sum_{m\in\mathbb{Z}} \prod_{\alpha=0}^{\infty} \sum_{\substack{k_\alpha = 0 \text{ for } \alpha\neq 0 \\ k_\alpha = 1 \text{ for } \alpha = 0}}^{\infty} \frac{t^{N|m+\alpha|k_\alpha} z^{(m+\alpha)k_\alpha} \Lambda^{k_\alpha}}{(t^2;t^2)_{k_\alpha}}. \end{aligned}$$

By the q-binomial theorem (see e.g., [Mac95, Ch. I, §2, Ex. 4]) we have

$$\sum_{k_\alpha=0}^{\infty} \frac{t^{N|m+\alpha|k_\alpha} z^{(m+\alpha)k_\alpha} \Lambda^{k_\alpha}}{(t^2;t^2)_{k_\alpha}} = \frac{1}{(t^{N|m+\alpha|} z^{m+\alpha}\Lambda; t^2)_\infty},$$

where $(a;t)_\infty = (1-a)(1-at)\cdots$. For $\alpha = 0$, we subtract the first term $k_0 = 0$, which is 1. Hence this is equal to

$$1 + \sum_{m\in\mathbb{Z}} \left(\prod_{\alpha=0}^{\infty} \frac{1}{(t^{N|m+\alpha|} z^{m+\alpha}\Lambda; t^2)_\infty} - \prod_{\alpha=1}^{\infty} \frac{1}{(t^{N|m+\alpha|} z^{m+\alpha}\Lambda; t^2)_\infty} \right). \tag{B.2}$$

We separate the sum to two parts $m \geq 0$ and $m < 0$. First suppose $m \geq 0$. Note that $|m+\alpha| = m+\alpha$ in this case. We change $m+\alpha$ to α and get

$$\sum_{m=0}^{\infty} \left(\prod_{\alpha=m}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} - \prod_{\alpha=m+1}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} \right).$$

We cannot take the sum $\sum_{m=0}^{\infty}$ separately as each sum diverges. But it is possible once we subtract 1 from each term. Hence we get

$$\begin{aligned}
&\sum_{m=0}^{\infty} \left(\prod_{\alpha=m}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} - 1 \right) - \sum_{m=0}^{\infty} \left(\prod_{\alpha=m+1}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} - 1 \right) \\
&= \prod_{\alpha=0}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} - 1.
\end{aligned}$$

The last -1 cancels with 1 in (B.2).

Let us turn to $m < 0$. We have

$$\begin{aligned}
&\sum_{m<0} \left(\prod_{\alpha=0}^{\infty} \frac{1}{(t^{N|m+\alpha|} z^{m+\alpha}\Lambda; t^2)_\infty} - \prod_{\alpha=1}^{\infty} \frac{1}{(t^{N|m+\alpha|} z^{m+\alpha}\Lambda; t^2)_\infty} \right) \\
&= \prod_{\alpha=0}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} \sum_{m=1}^{\infty} \left(\prod_{\alpha=1}^{m} \frac{1}{(t^{N\alpha} z^{-\alpha}\Lambda; t^2)_\infty} - \prod_{\alpha=1}^{m-1} \frac{1}{(t^{N\alpha} z^{-\alpha}\Lambda; t^2)_\infty} \right).
\end{aligned}$$

We use the same trick as above. We subtract 1 from each term in the sum $\sum_{m=1}^{\infty}$ and separate the sum. We get

$$\prod_{\alpha=0}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} \left(\prod_{\alpha=1}^{\infty} \frac{1}{(t^{N\alpha} z^{-\alpha}\Lambda; t^2)_\infty} - 1 \right).$$

Adding 1 and terms with $m \geq 0$, $m < 0$, we get

$$\prod_{\alpha=0}^{\infty} \frac{1}{(t^{N\alpha} z^{\alpha}\Lambda; t^2)_\infty} \prod_{\alpha=1}^{\infty} \frac{1}{(t^{N\alpha} z^{-\alpha}\Lambda; t^2)_\infty}.$$

Now it is straightforward to check (B.1).

Note Added in Proof

After this paper was posted to ArXiv, several important progresses have been made. Bullimore et al. write a paper [BDG15], where it is argued from a physical intuition that $\mathbb{C}[\mathcal{M}_C]$ is embedded into a localization of another $\mathbb{C}[\mathcal{M}_C]$ of the abelian gauge theory where the gauge group G is replaced by its maximal torus T. This embedding is rigorously given in the definition of [BFN16a]. At the same time, [BDG15] discusses Coulomb branches of quiver gauge theories of type ADE when μ is *not* necessarily dominant. It clarifies a problem raised in §2(v).

The sequel [BFN16a], announced here, is written, and it is shown that the definition there reproduces many examples in quiver gauge theories [BFN16b]. Further examples have been studied in later papers by the author with collaborators.

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