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Symplectic log Calabi–Yau surface — deformation class

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Dedicated to Professor Shing-Tung Yau on the occasion of his 65th birthday

We study the symplectic analogue of log Calabi–Yau surfaces and show that the symplectic deformation classes of these surfaces are completely determined by the homological information.

1. Introduction

In [2] and [6], Auroux and Gross–Hacking–Keel proposed a way to interpret mirror symmetry for Looijenga pair (X, D), where X is a smooth projective surface over $\mathbb{C}$ and D is an effective reduced anti-canonical divisor on X with maximal boundary. Under mirror symmetry, certain symplectic invariants of $X - D$ are conjectured to be related to holomorphic invariants of its mirror. In this regard, Pascaleff showed in [24] that the symplectic cohomology of $X - D$ is, as a vector space, isomorphic to the global sections of the structure sheaf of its mirror. A step towards a deeper understanding of mirror symmetry for Looijenga pairs would be to classify them. The moduli spaces of such pairs were studied by Looijenga in [16] and Gross–Hacking–Keel in [7]. Friedman gave an excellent survery in [4]. Since one direction of mirror symmetry concerns about the symplectic invariants of $X - D$ instead of the holomorphic invariants, we would like to establish, in this paper, a classification for 'symplectic log Calabi–Yau surfaces' (including 'symplectic Looijenga pairs' as a special case). From symplectic point of view, we have the following definition of log Calabi–Yau surfaces.

For a connected closed symplectic 4 dimensional manifold (X, ω), which we assume throughout the whole paper, a **symplectic divisor** D is a connected configuration of finitely many closed embedded symplectic surfaces (called irreducible components) $D = C_1 \cup \cdots \cup C_k$. D is further required to have the following two properties: No three different C_i intersect at a point and any intersection between two irreducible components is transversal and positive. The orientation of each C_i is chosen to be positive with respect to ω.

Definition 1.1. A **symplectic log Calabi–Yau surface** (X, D, ω) is a closed symplectic real dimension four manifold (X, ω) together with a symplectic divisor D representing the homology class of the Poincare dual of $c_1(X, \omega)$.

A symplectic Looijenga pair (X, D, ω) is a symplectic log Calabi–Yau surface such that each irreducible component of D is a sphere.

Let (X, D, ω) be a symplectic log Calabi–Yau surface. By Theorem A of [15] or [22] and the adjunction formula, it is easy to show (Lemma 3.1) that X is uniruled with base genus 0 or 1, and D is a torus or a cycle of spheres. And if (X, D, ω) is a symplectic Looijenga pair then X is rational.

Similar to studying the moduli space under complex deformation in the complex category, we would like to classify symplectic log Calabi–Yau surfaces up to symplectic deformation equivalence.

Definition 1.2. A **symplectic homotopy** (resp. **symplectic isotopy**) of (X, D, ω) is a smooth one-parameter family of symplectic divisors (X, D_t, ω_t) with $(X, D_0, \omega_0) = (X, D, \omega)$ (resp. such that in addition $\omega_t = \omega$ for all $t \in [0, 1]$). (X', D', ω') is said to be **symplectic deformation equivalent** to (X, D, ω) if it is symplectomorphic to (X, D_1, ω_1) for some symplectic homotopy (X, D_t, ω_t) of (X, D, ω). The symplectic deformation equivalence is called **strict** if the symplectic homotopy is a symplectic isotopy.

Definition 1.3. Two symplectic log Calabi–Yau surfaces (X^i, D^i, ω^i) for $i = 1, 2$ are said to be **homological equivalent** if there is a diffeomorphsim $\Phi : X^1 \to X^2$ such that $\Phi_*[C^1_j] = [C^2_j]$ for all $j = 1, \ldots, k$. The homological equivalence is called **strict** if $\Phi^*[\omega^2] = [\omega^1]$. We call Φ a (strict) homological equivalence.

Here is the main result of this paper.

Theorem 1.4. *Let (X^i, D^i, ω^i) be symplectic log Calabi–Yau surfaces for $i = 1, 2$. Then (X^1, D^1, ω^1) is (resp. strictly) symplectic deformation equivalent to (X^2, D^2, ω^2) if and only if they are (resp. strictly) homological equivalent.*

Moreover, the symplectomorphism in the (resp. strict) symplectic deformation equivalence has same homological effect as the (resp. strict) homological equivalence.

We remark that when D is a smooth divisor, the relative Kodaira dimension $\kappa(X, D, \omega)$ was introduced in [14] and it was noted there that this notion could be extended to nodal divisors. With this extension understood, symplectic Calabi–Yau surfaces have relative Kodaira dimension $\kappa = 0$ (cf. Theorem 3.28 in [14]). Moreover, Theorem 1.4 is also valid when $\kappa(X, D, \omega) = -\infty$. This will be treated in the sequel. Coupled with the techniques developed in [11], [12], some applications to Stein fillings will also be treated in the sequel.

The paper is organized as follows. In Section 2 we introduce marked divisors and establish the invariance of their deformation class under blow-up/down in Proposition 2.10. This reduces Theorem 1.4 to the minimal cases. In Section 3, we classify the deformation classes of minimal models and finish the proof of Theorem 1.4.

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2. Symplectic deformation equivalence of marked divisors

We study the symplectic deformation equivalence property in a general setting, which was initiated by Ohta and Ono in [23]. Here we provide details using the notion of marked divisor, which encodes the blow-down information. We will show that the deformation class of marked symplectic divisors is stable under various operations.

2.1. Homotopy and blow-up/down of symplectic divisors

2.1.1. Homotopy. Parallel to the two types of homotopy of a symplectic divisor (X, D, ω) mentioned in the introduction,

- Symplectic isotopy (X, D_t, ω), and
- Symplectic homotopy (X, D_t, ω_t).

We also consider the more restrictive homotopies keeping D fixed:

- $D-$symplectic isotopy (X, D, ω_t) with constant $[\omega_t]$, and
- $D-$symplectic homotopy (X, D, ω_t)

In particular, $D-$symplectic homotopies/isotopies are symplectic homotopies. To compare these notions we introduce the following terminology.

Definition 2.1. Two symplectic homotopies (X^1, D^1_t, ω^1_t) and (X^2, D^2_t, ω^2_t) are *symplectomorphic* if there exist a one parameter family of symplectomorphism $\Phi_t : (X^1, \omega^1_t) \to (X^2, \omega^2_t)$ such that $\Phi_t(D^1_t) = D^2_t$ for all $t \in [0, 1]$.

Lemma 2.2. *A symplectic homotopy (resp. isotopy) of a symplectic divisor is symplectomorphic to a $D-$symplectic homotopy (resp. isotopy) and vice versa.*

Proof. A $D-$symplectic homotopy is a symplectic homotopy by definition, and by Moser lemma a $D-$symplectic isotopy is symplectomorphic to a symplectic isotopy.

On the other hand, a symplectic homotopy (X, D_t, ω_t) gives rise to a smooth isotopy $\Phi : D \times [0, 1] \to X$. Since the intersections of D are transversal and no three of the components intersect at a common point, we can apply the smooth isotopy extension theorem to extend Φ to a smooth ambient isotopy $\Phi = \{\Phi_t\} : X \times [0, 1] \to X$. Then we get a $D-$symplectic homotopy $(X, D, \Phi_t^* \omega_t)$ which is symplectomorphic to (X, D_t, ω_t) via the one parameter family of symplectomorphisms $\{\Phi_t\}$. Similarly, a symplectic isotopy is symplectomorphic to a $D-$symplectic isotopy. □

Lemma 2.2 implies that symplectic isotopies (resp. homotopies) are the same as $D-$symplectic isotopies (resp. homotopies), up to symplectomorphism. This simple observation will be repeatedly used.

2.1.2. Toric and non-toric blow-up/down. Throughout the paper, we use the following terminology for symplectic blow-up/down of $D \subset (X, \omega)$.

A **toric blow-up** (resp. **non-toric blow-up**) of D is the total (resp. proper) transform of a symplectic blow-up centered at an intersection point (resp. at a smooth point) of D.

Here, for blow-up at a smooth point p on the divisor D, it means that we first do a C^0 small perturbation of D to D' fixing p and then we do a symplectic blow-up of a ball centered at p such that D' coincide, in the local coordinates given by the ball, with a complex subspace. Similarly, for blow-up at an intersection point, a C^0 small perturbation is performed so that D' is ω-orthogonal at p and D' coincide, in the local coordinates given by the ball, with two complex subspaces.

To describe the corresponding blow-down operations, recall that an embedded symplectic sphere with self-intersection -1 is called an exceptional sphere. The homology class of an exceptional sphere is called an exceptional class.

A **toric blow-down** refers to blowing down an exceptional sphere contained in D that intersects exactly two other irreducible components and exactly once for each of them. Moreover, we require that the intersections are positive and transversal. Such an exceptional sphere is called a toric exceptional sphere.

A **non-toric blow-down** refers to blowing down an exceptional sphere not contained in D that intersects exactly one irreducible component of D and exactly once with the intersection being positive and transversal. Such an exceptional sphere is called a non-toric exceptional sphere.

More precisely, for blow-down of a toric or non-toric exceptional sphere E, we first perturb our symplectic divisor D to another symplectic divisor D' (or perturbing E) such that the intersections of D' and E are ω-orthogonal (In the case that E is an irreducible component of D, we require E has ω-orthogonal intersections with all other irreducible components). Then, we will do the symplectic blow-down of E and D' will descend to a symplectic divisor.

Definition 2.3. An exceptional class e is called **non-toric** if e has trivial intersection pairing with all but one of the homology classes of the irreducible components of D and the only non-trivial pairing is 1.

An exceptional class e is called **toric** if e is homologous to an irreducible component of D such that e pairs non-trivially with the classes of exactly two other irreducible components of D and these two pairings are 1.

Clearly, the homology class of a toric (non-toric) exceptional sphere is a toric (non-toric) exceptional class. Conversely, we have the following observations.

For a toric exceptional class e, the component of D with class e is obviously a toric exceptional sphere in the class e. For a non-toric exceptional class e, we also have an exceptional sphere in the class e, at least when D is $\omega-$orthogonal.

Lemma 2.4. *(cf. Theorem 1.2.7 of [20]) Let D be an ω-orthogonal symplectic divisor. There is a non-empty subspace $\mathcal{J}(D)$ of the space of ω-tamed almost complex structure making D pseudo-holomorphic such that for any non-toric exceptional class e, there is a residue subset $\mathcal{J}(D,e) \subset \mathcal{J}(D)$ so that e has an embedded J-holomorphic representative for all $J \in \mathcal{J}(D,e)$.*

Proof. It is immediate to prove that e is D-good in the sense of Definition 1.2.4 in [20] if e is non-toric. Theorem 1.2.7 of [20] then implies the result. □

2.2. Deformation of marked divisors

When we blow down an exceptional sphere, we encode the process by marking the descended symplectic divisor.

Definition 2.5. A marked symplectic divisor consists of a five-tuple

$$\Theta = (X, D, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$$

such that

• $D \subset (X, \omega)$ is a symplectic divisor,

• p_j, called centers of marking, are points on D (intersection points of D allowed),

• $I_j : (B(\delta_j), \omega_{std}) \to (X, \omega)$, called coordinates of marking, are symplectic embeddings sending the origin to p_j such that $I_j^{-1}(D) = \{x_1 = y_1 = 0\} \cap B(\delta_j)$ (resp. $I_j^{-1}(D) = (\{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\}) \cap B(\delta_j)$) if p_j is a smooth (resp. an intersection) point of D. Moreover, we require that the image of I_j are disjoint. Here, $B(\delta_j)$ is the standard symplectic ball of radius δ_j.

If p_j is an intersection point of D, then we define the symplectic embedding $I_j^{re} = I_j \circ re$, where $re(x_1, y_1, x_2, y_2) = (-x_2, -y_2, x_1, y_1)$ interchanges the two subspaces $\{x_1 = y_1 = 0\}$ and $\{x_2 = y_2 = 0\}$. If p_j is a smooth point of D, then we define $I_j^{re} = I_j$. For simplicity, we denote a marked symplectic divisor as (X, D, p_j, ω, I_j) or Θ and also call it a marked divisor if no confusion would arise.

Definition 2.6. Let $\Theta = (X, D, p_j, \omega, I_j)$ be a marked divisor. A $D-$**symplectic homotopy** (resp. $D-$**symplectic isotopy**) of Θ is a 4-tuple (X, D, p_j, ω_t) such that ω_t is a smooth family of symplectic forms (resp. cohomologous symplectic forms) on X with $\omega_0 = \omega$ and D being ω_t-symplectic for all t.

If $\Theta^2 = (X^2, D^2, p_j^2, \omega^2, I_j^2)$ is another marked symplectic divisor and there is a symplectomorphism sending the 4-tuple $(X^2, D^2, p_j^2, \omega^2)$ to a 4-tuple (X, D, p_j, ω_1) which is symplectic homotopic (resp. isotopic) to Θ, then we say that Θ and Θ^2 are $D-$**symplectic deformation equivalent** (resp. **strict** $D-$**symplectic deformation equivalent**).

A symplectic divisor can be viewed as a marked divisor without markings.

Lemma 2.7. *Two symplectic divisors are (strict) deformation equivalent if and only if they are (strict) D-deformation equivalent as marked symplectic divisors.*

Proof. It follows directly from Lemma 2.2. □

For marked divisors, both $D-$symplectic deformation equivalence and its strict version do not involve the symplectic embeddings I_j. We have the following seemingly stronger definition of deformation.

Definition 2.8. Let $\Theta = (X, D, p_j, \omega, I_j)$ be a marked divisor. A **strong $D-$symplectic homotopy** (resp. **strong $D-$symplectic isotopy**) of Θ is a 5-tuple $(X, D, p_j, \omega_t, I_{j,t})$ such that

• the 4-tuple (X, D, p_j, ω_t) is a $D-$symplectic homotopy (resp. isotopy) of Θ,

• D is ω_t-orthogonal, and

• $I_{j,t} : B(\epsilon_j) \to (X, \omega_t)$ are symplectic embedding sending the origin to p_j, $I_{j,0} = I_j|_{B(\epsilon_j)}$ and $(I_{j,t})^{-1}(D) = \{x_1 = y_1 = 0\} \cap B(\epsilon_j)$ (resp. $(I_{j,t})^{-1}(D) = (\{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\}) \cap B(\epsilon_j))$ if p_j is a smooth point (resp. p_j is an intersection point), for some $\epsilon_j < \delta_j$.

If $\Theta^2 = (X^2, D^2, p_j^2, \omega^2, I_j^2)$ is another marked sympelctic divisor and there is a symplectomorphism sending $(X^2, D^2, p_j^2, \omega^2, (I_j^2)^{\#})$ to $(X, D, p_j, \omega_1, I_{j,1})$, where $(I_j^2)^{\#}$ is the unique choice between I_j^2 and $(I_j^2)^{re}$ such that the existence of symplectomorphism is possible, then we say that Θ and Θ^2 are **strong $D-$symplectic deformation equivalent** (resp. **strong strict $D-$symplectic deformation equivalent**).

Lemma 2.9. *If $\Theta = (X, D, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ and $\Theta^2 = (X^2, D^2, \{p_j^2\}_{j=1}^l, \omega^2, \{I_j^2\}_{j=1}^l)$ are (strict) $D-$symplectic deformation equivalent, then they are strong (strict) $D-$symplectic deformation equivalent.*

Proof. We will only do the case when $l = 1$. The general case is similar. We denote p_1 as p, I_1 as I and I_1^2 as I^2.

By assumption, there is a $D-$symplectic homotopy (X, D, p, ω_t) of Θ such that there is a symplectomorphism sending (X, D, p, ω_1) to $(X^2, D^2, p_1^2, \omega^2)$. Therefore, without loss of generality, we can assume $(X, D, p, \omega_1) = (X^2, D^2, p_1^2, \omega^2)$.

The proof is easier when p is a smooth point of D so we only prove the case when p is an intersection point of D. Moreover, by possibly replacing I^2 with $(I^2)^{re}$, we can assume the irreducible component of D corresponding to $\{x_1 = y_1 = 0\}$ in chart I is the same as that of I^2.

The idea of the proof goes as follows. First, we find a smooth family of symplectic embeddings of small ball $\Phi_t : (B(\delta), \omega_{std}) \to (X, \omega_t)$ sending the origin to p such that $\Phi_0 = I|_{B(\delta)}$ and $\Phi_1 = I^2|_{B(\delta)}$. Then, we find another family of symplectic forms ω_t' such that the 4-tuple (X, D, p, ω_t') is still a $D-$symplectic homotopy of Θ with $\omega_1' = \omega_1$ and D is ω_t'-orthogonal for all t. A corresponding symplectic embeddings I'_t for (X, D, p, ω'_t) will be constructed based on Φ_t such that the 5-tuple $(X, D, p, \omega'_t, I'_t)$ is a strong $D-$sympelctic homotopy between Θ and Θ^2 and this will finish the proof.

We begin our construction of Φ_t. By the one-parameter family version of Moser lemma, there exist a sufficiently small $\epsilon > 0$ and a smooth family of symplectic embeddings $\Phi = \{\Phi_t\} : (B(\epsilon), \omega_{std}) \to (X, \omega_t)$ sending the origin to p for all $t \in [0, 1]$. Moreover, Φ_0 can be chosen to coincide with $I|_{B(\epsilon)}$. This is not yet the Φ_t we want.

Notice that Φ_1 is a symplectic embedding of $(B(\epsilon), \omega_{std})$ to (X, ω_1) sending the origin to p and so is $I^2|_{B(\epsilon)}$. By possibly choosing a smaller ϵ, there is a symplectic isotopy of embeddings from Φ_1 to $I^2|_{B(\epsilon)}$ sending the origin to p for all time, by the trick in Exercise 7.22 of [18] (This is the trick to prove the space of symplectic embeddings of small balls is connected). By smoothing the concatenation of Φ_t with this symplectic isotopy, we can assume that $\Phi_1 = I^2|_{B(\epsilon)}$.

We need to further modify Φ_t by another concatenation. We consider the family of local divisors Let $F_t = \Phi_t^{-1}(D)$ in the standard coordinates in $(B(\epsilon), \omega_{std})$. Let M_t be the ordered 2-tuple of the symplectic tangent spaces to the two branches of F_t at the origin. Since $\Phi_0 = I|_{B(\epsilon)}$ and $\Phi_1 = I^2|_{B(\epsilon)}$, M_t is a loop. Let $-M_t$ be the inverse loop of M_t in the space of ordered 2-tuples of positively transversal intersecting two dimensional symplectic vector subspaces. We can find an isotopy of symplectic embeddings Ψ_t from Φ_1 to Φ_1 in (X, ω_1) such that the corresponding ordered 2-tuple of the symplectic tangent spaces of $\Psi_t^{-1}(D)$ at the origin is $-M_t$. By concatenating Φ_t with Ψ_t, we can assume at the beginning that the Φ_t we chose is such that M_t is null-homotopic. This is the Φ_t we want which gives a nice family of Darboux balls in (X, ω_t).

To construct ω'_t, we will isotope the one parameter family of local divisors F_t (fixing both ends) to another one parameter family of symplectic divisors $F_{1,t}$ such that it coincides with $F_0 = F_1$ near the origin for all t. First, we perform a one-parameter family of C^1 small perturbations to make F_t coincide with a symplectic vector subspace in a sufficiently small ball $(B(\epsilon_2), \omega_{std})$, where $\epsilon_2 < \epsilon$. In other words, F_t coincides with M_t in $B(\epsilon_2)$. Since M_t is null-homotopic, there is a homotopy $W_{r,t}$ between M_t $(r = 0)$ and the constant path $M_0 = M_1$ $(r = 1)$ such that $W_{r,0} = W_{r,1} = M_0$ for all r. Hence, we can perform a one-parameter family of Lemma 5.10 of [21] (See its proof) to obtain a 3-parameter family of submanifolds $U_{r,s,t}$ in $B(\epsilon_2)$ such that $U_{r,s,t} = W_{s,t}$ outside a fixed small compact set containing the origin, $U_{r,s,t} = W_{r,t}$ close to the origin and $U_{r,r,t} = W_{r,t}$. As in the proof of

Lemma 5.10 of [21], from $U_{r,s,t}$ one can construct an $s-$parameter of symplectic isotopy $F_{s,t} \subset B(\epsilon_2)$ such that

• $F_{0,t} = F_t$,

• $F_{s,t}$ is a pair of symplectic submanifolds positively intersecting at the origin for all $s, t \in [0, 1]$,

• $F_{1,t} = F_0 = F_1 = M_0 = M_1$ inside $B(\epsilon_4)$ for all t,

• $F_{s,0} = F_{s,1} = F_0 = F_1$, and

• the isotopy is supported inside $B(\epsilon_3)$,

where $0 < \epsilon_4 < \epsilon_3 < \epsilon_2$.

Due to the last bullet, we obtain a $2-$parameter family of marked divisors $D_{s,t}$ with $D_{0,t} = D_t, D_{s,0} = D_{s,1} = D$, and satisfying the bullets 2 and 3 above near the marked point (recall we assume there is only one marking for simplicity).

The effect of the symplectic isotopy from D_t $(s = 0)$ to $D_{1,t}$ $(s = 1)$ can be converted through symplectomorphism, as in Lemma 2.2, to replace (X, D, p, ω_t) $(s = 0)$ by an another $D-$symplectic homotopy (X, D, p, ω'_t) $(s = 1)$. More precisely, for the 1-parameter family of isotopy $D_{s,t}$ parameterized by t, we can find a 1-parameter family of ambient isotopy $\Delta = \{\Delta_s\}_{t\in[0,1]} = \{\Delta_{s,t}\}$, $\Delta_{s,t} : X \to X$ extending the 1-parameter family of isotopy $D_{s,t}$ (in particular, for fixed t_0, Δ_{s,t_0} is an ambient isotopy extension of D_{s,t_0}) such that $\Delta_{0,t} = \Delta_{s,0} = \Delta_{s,1} = Id_X$. Then we define $\omega'_t = \Delta^*_{1,t}\omega_t$.

By construction, we have

• $\omega'_i = \omega_i$ for $i = 0, 1$,

• D is positively ω'_t-orthogonal for all t

• there is a family of symplectic embedding $\Phi'_t : B(\epsilon_4) \to (X, \omega_t)$ such that $\Phi'^{-1}_t(D) = F_0$ for all t, and

• $\Phi'_0 = I|_{B(\epsilon_4)}$ and $\Phi'_1 = I^2|_{B(\epsilon_4)}$

In particular, if we let $I'_t = \Phi'_t$, then $(X, D, p, \omega'_t, I'_t)$ is a strong $D-$symplectic homotopy between Θ and Θ^2. The strict version follows similarly. □

The ultimate goal for this section is the following proposition, which will be proved after discussing various operations for marked divisors in the next subsection.

Proposition 2.10. *Let $\Theta = (X, D, p_j, \omega, I_j)$ and $\Theta^2 = (X^2, D^2, p^2_j, \omega^2, I^2_j)$ be two marked divisors both with l marked points.*

(i) Up to moving inside the $D-$symplectic deformation class, we can blow down a toric or non-toric exceptional class in Θ (and Θ^2) to obtain a marked divisor $\hat{\Theta}$ (resp. $\hat{\Theta}^2$) with an extra marked point (For toric exceptional class,

original marked points on the exceptional sphere will be removed after blow-down).

(ii) Moreover, if the blow down divisors $\hat{\Theta}$ *and* $\hat{\Theta}^2$ *are* $D-$*symplectic deformation equivalent such that the extra marked points correspond to each other via the equivalence, then* Θ *and* Θ^2 *are* $D-$*symplectic deformation equivalent.*

2.3. Operations on marked divisors

This subsection studies various operations on marked divisors as well as their stability properties with respect to $D-$symplectic deformation.

- **Perturbations**

The following fact will be frequently used.

Lemma 2.11. *Perturbations of a marked divisor preserve the strict* $D-$*symplectic deformation class.*

Proof. A perturbation of a marked divisor is determined by a symplectic isotopy of the corresponding underlying unmarked divisor and isotopies of the centers (points) on the symplectic isotopy. By Lemma 2.2, the perturbed divisor is symplectomorphic to the original divisor, up to a $D-$symplectic isotopy. The result follows. □

- **Marking addition**

A marking addition of a marked divisor $(X, D, \{p_j\}_{j=1}^{l}, \omega, \{I_j\}_{j=1}^{l})$ is another marked divisor $(X, D, \{p_j\}_{j=1}^{l+1}, \omega, \{I_j\}_{j=1}^{l+1})$ with the additional marking (p_{l+1}, I_{l+1}).

Lemma 2.12. *Let* $(X, D, \{p_j\}_{j=1}^{l}, \omega, \{I_j\}_{j=1}^{l})$ *be a marked divisor. If the two marked divisors* $(X, D, \{p_j\}_{j=1}^{l} \cup \{q_1\}, \omega, \{I_j\}_{j=1}^{l} \cup \{I_{q_1}\})$ *together with* $(X, D, \{p_j\}_{j=1}^{l} \cup \{q_2\}, \omega, \{I_j\}_{j=0}^{l} \cup \{I_{q_2}\})$ *are obtained by adding markings* (q_1, I_{q_1}) *and* (q_2, I_{q_2}) *respectively, then they are strict* $D-$*symplectic deformation equivalent if*

- *the centers* q_1 *and* q_2 *coincide (intersection points of* D *allowed), or*
- q_1 *and* q_2 *are distinct smooth points of the same irreducible component.*

Proof. If q_1 and q_2 are the same point of D, then the claim is trivial since Definition 2.6 only involves the centers of marking, but not the coordinates of markings.

If q_1 and q_2 are smooth points of the same irreducible component, say C_1, then we need to show that the 4-tuple $(X, D, \{p_j\}_{j=1}^l \cup \{q_2\}, \omega)$ is symplectomorphic to a $D-$symplectic isotopy of $(X, D, \{p_j\}_{j=1}^l \cup \{q_1\}, \omega)$. For this purpose, we find a symplectic isotopy of D fixing C_1 setwise, fixing intersection points and $\{p_j\}$ pointwise and moving q_1 to q_2. Using the smooth isotopy extension theorem as in Lemma 2.2, this isotopy of symplectic divisor gives rise to a smooth isotopy Φ_t of X. The desired $D-$symplectic isotopy is obtained by taking the $D-$symplectic isotopy to be $(X, D, \{p_j\}_{j=1}^l \cup \{q_1\}, \Phi_t^*\omega)$ and the symplectomorphism to be $\Phi_1 : (X, D, \{p_j\}_{j=1}^l \cup \{q_1\}, \Phi_1^*\omega) \to (X, D, \{p_j\}_{j=1}^l \cup \{q_2\}, \omega)$. □

We note that marking addition at an intersection point of a marked divisor is not always possible because the intersection might not be ω-orthogonal. However, by Lemma 2.11, marking addition at a non-marked intersection point is always possible at the cost of choosing another representative in the strict $D-$symplectic deformation class because a C^0 small perturbation of a symplectic divisor is sufficient to make the intersection points ω-orthogonal ([5]).

- **Marking moving**

Sometimes, it is useful to be able to move an intersection point.

Lemma 2.13. *Let $(X, D = C_1 \cup C_2 \cup \cdots \cup C_k, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ be a marked divisor. Let $[C_2]^2 = -1$ and $p_1 = C_1 \cap C_2$. For any smooth point $\overline{p_1}$ on C_2, there is a marked divisor $(X, \overline{D} = \overline{C_1} \cup C_2 \cup \cdots \cup C_k, \{\overline{p_1}\} \cup \{p_j\}_{j=2}^l$, $\omega', \{\overline{I_j}\}_{j=1}^l)$ such that $\overline{p_1} = \overline{C_1} \cap C_2$, where $\omega' = \omega$ and $C_1 = \overline{C_1}$ away from a small open neighborhood of C_2. Moreover, these two marked divisors are in the same $D-$symplectic deformation equivalence class.*

Proof. By Lemma 2.11 we may assume that the intersection points of D are ω-orthogonal. In particular, if C_j intersects C_2, then C_j coincides with a fiber of the symplectic normal bundle of C_2 when identifying the symplectic normal bundle with a tubular neighborhood of C_2.

Choose an ω-compatible almost complex structure J integrable near C_2 which coincides with $(I_j)_*(J_{std})$ for all j and making the symplectic normal bundle a holomorphic vector bundle. We blows down C_2 and identify the ball obtained by blowing down C_2 as a chart $(B(\epsilon), \omega_{std}, J_{std})$. In this chart, C_j descends to the union of complex vector subspaces V_j each of which corresponds to an intersection point of $C_2 \cap C_j$. On the other hand, $\overline{p_1}$ being a point on C_2 represents a complex vector subspace $V_{\overline{p_1}}$ in this chart. We take a smooth family of complex vector subspaces W_t from V_1 to $V_{\overline{p_1}}$ avoiding V_j for all $j \neq 1$. Applying the trick in Lemma 5.10 of [21] with $N = N' = \emptyset$, $i = 1$, S being the center of $B(\epsilon)$, S_1 being the descended C_1, $W_t = W_1^t$, we obtain an isotopy of symplectic manifolds C^t supported in $B(\epsilon)$ from the descended C_1 (i.e. $C^{t=0}$) to some $C^{t=1} = \tilde{C}_1$ such that C^t coincides with W_t near the origin of $B(\epsilon)$ for all t. By blowing up $B(\epsilon_2) \subset B(\epsilon)$ for some sufficiently small ϵ_2, we can lift this symplectic isotopy to a $D-$symplectic deformation from $(X, D = C_1 \cup C_2 \cup \cdots \cup C_k, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ to $(X, \overline{D} = \overline{C_1} \cup C_2 \cup \cdots \cup C_k, \{\overline{p_1}\} \cup \{p_j\}_{j=2}^l, \omega', \{\overline{I_j}\}_{j=1}^l)$ such that $\overline{p_1} = \overline{C_1} \cap C_2$, where $\overline{C_1}$ is the proper transform of $\tilde{C}_1$. □

- **Canonical blow-up**

Given a marked divisor with l markings, there are l canonical blow-ups we can do, namely, blow-ups using the symplectic embeddings I_j and hence the blow-up size is $B(\delta_j)$. A canonical blow-up of a marked divisor is still a marked divisor with one less the number of p_j's.

Lemma 2.14. *If* $\Theta = (X, D, \{p_j\}_{j=1}^l, \omega, \{I_j\}_{j=1}^l)$ *and* $\Theta^2 = (X^2, D^2, \{p_j^2\}_{j=1}^l, \omega^2, \{I_j^2\}_{j=1}^l)$ *are $D-$symplectic deformation equivalent, then so are the marked divisors obtained by canonical blow-ups using* I_1 *and* I_1^2.

Proof. By Lemma 2.9, Θ and Θ^2 are strong $D-$symplectic deformation equivalent. By blowing up using $I_{1,t}$, we obtain a $D-$symplectic deformation equivalence between the blown-up marked divisors. □

2.4. Proof of Proposition 2.10

Proof of Proposition 2.10. For a non-toric class e, we can find by Lemma 2.4, a pseudo-holomorphic representative E such that D is at the same time pseudo-holomorphic, after possibly applying Lemma 2.11 to deform Θ within the strict $D-$symplectic deformation class. By positivity of intersection, E intersects exactly one irreducible component of D and the intersections is positively transversally once and hence a non-toric exceptional curve. By perturbing E, we can assume E has ω-orthogonal intersection with D. We can get a marked divisor after blowing down E with a marked point corresponds to the contracted E.

For a toric class e, we again apply Lemma 2.11 to deform Θ within its strict $D-$symplectic deformation class such that every intersection is ω-orthogonal. The irreducible component E of D in the class e is a toric exceptional sphere. Hence, E intersects two other irreducible components of D once. We apply Lemma 2.13 to find another representative of Θ in the $D-$symplectic deformation class such that after we blow down the exceptional curve, the intersection point corresponding to the exceptional curve is an ω-orthogonal intersection point so this descended divisor is still a marked divisor (recall, a marking for a marked divisor at an intersection point requires the intersection point is an ω-orthogonal intersection).

Finally, suppose the blow down divisors are $D-$symplectic deformation equivalent. We want to do canonical blow-ups and marking additions to recover our original divisor D and D^2. Notice that, marking additions are needed because when one blow down a divisor which originally has markings on it, the marking will not persist after the blow-down. Therefore, when we blow up the symplectic ball back, we need marking additions to get back the original marked divisor. We remark that we may not get back exactly the pair of D and D^2 by just canonical blow-ups and marking additions but we can get some pair in the same $D-$symplectic deformation equivalence classes by Lemma 2.11.

Since $D-$symplectic deformation equivalence is stable under canonical blow-ups (Lemma 2.14) and marking additions (Lemma 2.12), we conclude that Θ is $D-$symplectic deformation equivalent to Θ^2. □

3. Minimal models

We first collect some facts, which should be well known to experts.

Lemma 3.1. *Let (X, D, ω) be a symplectic log Calabi–Yau surface. Then X is rational or an elliptic ruled surface, and D is either a torus or a cycle of spheres. If (X, D, ω) is a symplectic Looijenga pair, then (X, ω) is rational.*

Proof. Since D is symplectic and $[D] = PD(c_1(X, \omega))$, we have $c_1(X, \omega) \cdot [\omega] = [D] \cdot [\omega] > 0$. By Theorem A of [15] or [22], X is rational or ruled.

Write $D = C_1 \cup C_2 \cdots \cup C_k$, where each C_i is a smoothly embedded closed symplectic genus g_i surface. By adjunction, we have $[C_i] \cdot [D] = [C_i]^2 + 2 - 2g_i$. Therefore, we have

$$[C_i] \cdot \left(\sum_{j \neq i} [C_j] \right) = 2 - 2g_i \geq 0.$$

In particular, we have $g_i \leq 1$ for all i. Since we assumed D is connected (we always assume a symplectic divisor is connected), D is either a torus or a cycle of spheres. Here a cycle of spheres means that the dual graph is a circle and each vertex has genus 0.

If X is not rational, then X admits an S^2-fibration structure over a Riemann surface of positive genus. After possibly smoothing, we get a torus T representing the class $c_1(X)$. Moreover, $c_1(X)(f) = 2$ where f is the fiber class. The projection from T to the base is of non-zero degree. Therefore, the base genus of X is at most 1.

If (X, D, ω) is a symplectic Looijenga pair, then at least one of the sphere component pairs positively with the fiber class (by $c_1(X)(f) = 2$ again). Hence, the base genus is 0 and X is rational. □

For a cycle with k spheres we will also call it a $k-$gon, and a torus a $1-$gon. If we allow some C_i to be positively immersed, then by adjunction we see that the only possibility is a single sphere with one positive double point, which we call a degenerated 1-gon.

The following observations are straightforward.

Lemma 3.2. *The operations of toric blow-up, non-toric blow-up, toric blow-down and non-toric blow-down all preserve being symplectic log Calabi–Yau.*

In the next subsection it is convenient to apply a slightly more general version of toric blow-down: Suppose a component C of a bi-gon D is an exceptional sphere. The generalized toric blow down of D along C is blowing down C, which results in a degenerated 1-gon. Notice that the homology class of a degenerated 1-gon is still Poincare dual to the first Chern class.

3.1. Minimal reductions

Definition 3.3. A symplectic log Calabi–Yau surface (X, D, ω) is called a **minimal model** if either (X, ω) is minimal, or (X, D, ω) is a symplectic Looijenga pair with $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Lemma 3.4. *Every symplectic log Calabi–Yau surface can be transformed to a minimal model via a sequence of non-toric blow-downs followed by a sequence of toric blow-downs.*

Proof. **Non-toric blow-down**. Suppose e is an exceptional class intersecting each component of D non-negatively. Then e is a non-toric exceptional class by adjunction.

By Lemma 2.4, there is an ω-compatible almost complex structure J such that D is J-holomorphic (possibly after perturbation of D) and e has an embedded J-holomorphic sphere representative E. Thus we can perform non-toric blow-down along E.

By iterative non-toric blow-downs, we end up with a symplectic log Calabi–Yau surface (X_0, D_0, ω_0) such that any exceptional class pairs negatively with some component of D.

Toric blow-down. If X_0 is not minimal and not diffeomorphic to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, then for any ω_0-compatible J_0 making D_0 J_0-holomorphic, the exceptional class with minimal ω_0-area has an embedded J_0-holomorphic representative, by Lemma 1.2 of [25]. Therefore, this embedded representative must coincide with an irreducible component C of D_0.

Therefore if D_0 is a torus then X_0 must be minimal. So from now on we assume that D_0 is a cycle of spheres, ie. (X_0, D_0, ω_0) is a Looijenga pair.

Suppose that C intersects two other components of D_0 and hence a toric exceptional sphere. In this case we perform toric blow down along C to get another symplectic Looijenga pair (X_0', D_0', ω_0'). We claim that there is no exceptional class in X_0' that pairs all irreducible components of D_0' non-negatively. If there were one, by Lemma 2.4, after possibly perturbing D_0' to be $\omega_0'-$orthogonal, then there would be an embedded pseudo-holomorphic representative E_0' intersecting exactly one irreducible component of D_0' transversally at a smooth point. This E_0' can be lifted to the symplectic log Calabi–Yau surface (X_0, D_0, ω_0) because the contraction of C becomes an intersection point of D_0', which is away from E_0'. Contradiction. Therefore, we can continue to perform toric blow-down until the ambient manifold is minimal, diffeomorphic to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ or the minimal area exceptional sphere intersect only one irreducible component of the divisor.

We now consider the case that the minimal area expectional sphere C only intersects with one component of the divisor D_0, then D_0 must be a bigon. We claim that $X_0 = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ in this case, and hence (X_0, D_0, ω_0) is minimal, according to Definition 3.3. To see why $X_0 = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, we apply a generalized toric blow-down along C to obtain (X_0', D_0', ω_0') where D_0' is a degenerated 1-gon. We next show that (X_0', ω_0') is minimal. After possibly perturbing the nodal point of D_0' to be $\omega_0'-$orthogonal so D_0' can be made a pseudo-holomorphic nodal sphere, the analysis above also shows that there is no exceptional class in X_0' that intersects $[D_0']$ non-negatively. Since D_0' represents the Poincáre dual of $c_1(X_0', \omega_0')$, there are also no exceptional class intersecting $[D_0']$ negatively. Thus, it means that $X_0' = \mathbb{C}P^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$. If X_0' is $\mathbb{S}^2 \times \mathbb{S}^2$, then D_0' is obtained by blowing down a component of a bi-gon D_0 in $X_0 = \mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$. In this case there are three exceptional class in (X_0, ω_0) with pairwise intersecting number 1. It is simple to check by adjunction that any exceptional class not represented by any of the two components of D_0 is non-toric. But this situation would not appear due to our procedure which performs non-toric blow down first. Hence the only possibility is that $X_0' = \mathbb{C}P^2$, from which it follows that $X_0 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

In summary, we can do iterative toric blow-downs from (X_0, D_0, ω_0) to obtain a symplectic Looijenga pair (X_b, D_b, ω_b) such that either (X_b, ω_b) is minimal or X_b is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. □

From Lemma 3.1, Lemma 3.2, Lemma 3.4 and adjunction formula, we can enumerate the minimal symplectic log Calabi–Yau surfaces up to the homology of the irreducible components.

• Case (A): The base genus of X is 1. D is a torus.

• Case (B): $X = \mathbb{C}P^2$. $c_1 = 3H$. Then the symplectic log Calabi–Yau are

$(B1)$ D is a torus,

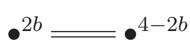
$(B2)$ D consists of a $H-$sphere and a $2H-$sphere, or

$(B3)$ D consists of three $H-$spheres.

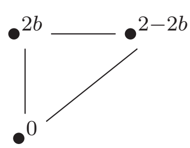
• Case (C): $X = \mathbb{S}^2 \times \mathbb{S}^2$, $c_1 = 2f + 2s$, where f and s are the homology classes of the two factors. By adjunction, the homology $af + bs$ of any embedded symplectic sphere satisfies $a = 1$ or $b = 1$. Symplectic log Calabi–Yau surfaces are

$(C1)$ D is a torus.

$(C2)$ If D has two irreducible components C_1 and C_2, then the only possible case (modulo obvious symmetry) is $[C_1] = f + bs$ and $[C_2] = f + (2-b)s$. Its graph is given by

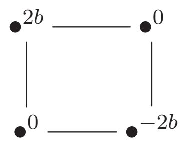


$(C3)$ If D has three irreducible components C_1, C_2 and C_3, then the only possible case (modulo obvious symmetry) is $[C_1] = f + bs$, $[C_2] = f + (1-b)s$ and $[C_3] = s$. Its graph is given by



$(C4)$ If D has four irreducible components, then the only possible case (modulo obvious symmetry) is $[C_1] = f - bs$, $[C_2] = f + bs$, $[C_3] = s$ and

$[C_3] = s$. Its graph is given by

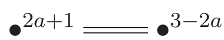


It is not hard to draw contradiction if D has 5 or more irreducible components.

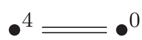
• Case (D): $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. $c_1 = f + 2s$, where f and s are fiber class and section class, respectively, such that $f^2 = 0$, $f \cdot s = 1$ and $s^2 = 1$. By adjunction, the homology $af + bs$ of an embedded symplectic sphere satisfies $b = 1$ or $b = 2 - 2a$.

$(D1)$ D cannot be a torus because it would not be minimal.

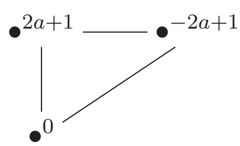
$(D2)$ If D has two irreducible components C_1 and C_2, then the only two possible cases (modulo obvious symmetry) are $([C_1], [C_2]) = (af + s, (1 - a)f + s)$ and $([C_1], [C_2]) = (f, 2s)$. The graphs are given by



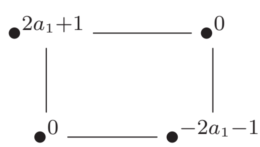
and



$(D3)$ If D has three irreducible components, then the only possible case (modulo obvious symmetry) is $[C_1] = af + s$, $[C_2] = -af + s$ and $[C_3] = f$.



$(D4)$ If D has four irreducible components, then the only possible case (modulo obvious symmetry) is $[C_1] = af + s$, $[C_2] = -(a + 1)f + s$, $[C_3] = f$ and $[C_4] = f$.



It is not hard to draw contradiction if D has 5 or more irreducible components.

3.2. Deformation classes of minimal models

In this section, we study the symplectic deformation classes of minimal symplectic log Calabi–Yau surfaces.

Proposition 3.5. *Let* $(X, D = C_1 \cup \cdots \cup C_k, \omega)$ *be a minimal symplectic log Calabi–Yau surface. If* $\overline{D} = \overline{C_1} \cup \cdots \cup \overline{C_k} \subset (X, \omega)$ *is another symplectic divisor representing the first Chern class such that* $[C_i] = [\overline{C_i}]$ *for all* i*. Then* (X, D, ω) *is symplectic deformation equivalent to* $(X, \overline{D}, \omega)$*.*

The proof of Proposition 3.5 is separated into two cases, Proposition 3.6 and Proposition 3.9.

3.2.1. Isotopy in rational surfaces.

Proposition 3.6. *Suppose* (X, D, ω) *and* $(X, \overline{D}, \omega)$ *satisfy the assumptions of Proposition 3.5 such that, in addition,* X *is rational, then* D *is symplectic isotopic to* $\overline{D}$*.*

The proof of Proposition 3.6 when D is a torus is given by [28] and Theorem B and Theorem C of [27]. We only need to deal with symplectic Looijenga pairs.

Recall that cohomologous symplectic forms on a rational or ruled 4-manifold are symplectomorphic (cf. [29], [10] and the survey [26]). Therefore it suffices to consider the following 'standard symplectic models' for $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

• $\mathbb{S}^2 \times \mathbb{S}^2$ model:

When X is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, we define the product symplectic form $\omega_\lambda = (1+\lambda)\sigma \times \sigma$ with σ a symplectic form on the second factor with area 1 and $\lambda \geq 0$. Let E_0 be the class of the first factor, F be the class of the second factor and $E_{2k} = E_0 - kF$ for $0 \leq k \leq l$, where l is the integer with $l - 1 < \lambda \leq l$. For $0 \leq k \leq l$, let U_k be the set of ω_λ-compatible almost complex structure such that E_{2k} is represented by an embedded pseudo-holomorphic sphere.

• $\mathbb{C}P^2$ model:

When X is diffeomorphic to $\mathbb{C}P^2$, we use a multiple of the Fubini–Study form, $c\omega_{FS}$.

• $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ model:

When X is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, we use ω_λ to denote a form obtained by blowing up $(\mathbb{C}P^2, (2+\lambda)\omega_{FS})$ with size $1+\lambda$. So the line class H has area $2+\lambda$ and the exceptional class E_1 has area $1+\lambda$, where $\lambda >$

-1. Let $F = H - E_1$ be the fiber class and let also $E_{2k+1} = E_1 - kF$ for $0 \le k \le l$, where l is again the integer with $l - 1 < \lambda \le l$. Similarly, let U_k be the space of ω_λ-compatible almost complex structure such that E_{2k+1} is represented by an embedded pseudo-holomorphic sphere.

Proposition 3.7. *(Proposition 2.3 and Corollary 2.8 of [1], see also Proposition 6.4 of [13]) Let (X, ω_λ) be one of the above two cases. For each $0 \le k \le l$, U_k is non-empty and path connected. As a result, any two embedded symplectic spheres C_0 and C_1 representing the same class E_j for some $0 \le j \le 2l + 1$ are symplectic isotopic to each other.*

Lemma 3.8. *Let (X, ω_λ) be as in Proposition 3.7. Assume $C_0, C_1 \subset X$ are two embedded symplectic spheres representing the same class E_j for some $0 \le j \le 2l + 1$. Then there is a Hamiltonian diffeomorphism of (X, ω_λ) sending C_0 to C_1.*

Proof. By Proposition 3.7, we can find a symplectic isotopy $C_t \subset X$ from C_0 to C_1. We can extend this symplectic isotopy from a neighborhood of C_0 to a neighborhood of C_1 by a Moser type argument (See e.g. Chapter 3 of [18]). Our aim is to extend this symplectic isotopy to an ambient symplectic isotopy in order to obtain the result.

We first extend this symplectic isotopy to an ambient diffeomorphic isotopy $\Phi : X \times [0, 1] \to X$. By considering the pull-back form $\Phi^*\omega_\lambda$, we can identify $C_0 = \Phi_t^{-1}(C_t)$ for all t in the family of symplectic manifold $(X \times \{t\}, \Phi^*\omega_\lambda|_{X\times\{t\}})$, as in Lemma 2.2. We denote $\Phi^*\omega_\lambda|_{X\times\{t\}}$ as ω_λ^t. By definition, ω_λ^t is fixed near C_0 for all t. Identify a tubular neighborhood of C_0 with a symplectic normal bundle. Then, choose a smooth family of ω_λ^t-compatible almost complex structure J_t on X such that J_t is fixed near C_0 and the fibers of the normal bundle of C_0 are J_t-holomorphic. Pick a point p_0 on C_0. Let the J_t holomorphic sphere representing the fiber class F and passing through p_0 be C_t^F. Since the fiber class with a single point constraint has Gromov–Witten invariant one or minus one, C_t^F forms a symplectic isotopy by Gromov compactness. By Lemma 3.2.1 of [20] (let C_0 be C^{S_1} and $[C_t^F]$ be B_1), we can assume that the intersection between C_0 and C_t^F is ω_λ^t-orthogonal, after possibly perturbing J_t.

Now, $\Phi(C_0, t) \cup \Phi(C_t^F, t) = C_t \cup \Phi(C_t^F, t)$ is an ω_λ orthogonal symplectic isotopy in (X, ω_λ) (Strictly speaking, C_t^F is the image of another diffeomorphic isotopy Ψ such that $C_t^F = \Psi(C_0^F, t)$ and $C_0 = \Psi(C_0, t)$, then the isotopy we want is $\Phi(\Psi(\cdot, t), t)$). We can extend this symplectic isotopy to a neighborhood of it by another Moser type argument since $\Phi(C_0, t)$ intersects $\Phi(C_t^F, t)$

ω_λ-orthogonally. We have the exact sequence

$$H^1(C_0 \cup C_0^F, \mathbb{R}) = 0 \to H^2(X, C_0 \cup C_0^F, \mathbb{R}) \to H^2(X, \mathbb{R}) \to H^2(C_0 \cup C_0^F, \mathbb{R})$$

where the last arrow is an isomorphism and hence $H^2(X, C_0 \cup C_0^F, \mathbb{R}) = 0$. By Banyaga extension theorem (See e.g. [18]), there is an ambient symplectic isotopy extending the symplectic isotopy $C_t \cup \Phi(C_t^F, t)$. Finally, this ambient symplectic isotopy is a Hamiltonian isotopy because $H^1(X) = 0$. □

Proof of Proposition 3.6. As seen in the previous section, D and $\overline{D}$ have at most four irreducible components. We are going to prove Proposition 3.6 by dividing it into the cases of two, three or four irreducible components. The proof for bigons is written with details, while the proof for triangles or rectangles being similar to that of bigons will be sketched.

• Bigons

First, let $(X, \omega) = (\mathbb{S}^2 \times \mathbb{S}^2, c\omega_\lambda)$ for some constant c, $D = C_1 \cup C_2$, $\overline{D} = \overline{C_1} \cup \overline{C_2}$ and $[C_i] = [\overline{C_i}]$ for $i = 1, 2$. Without loss of generality, we may assume $[C_1]^2 \leq [C_2]^2$. From the enumeration, we have $[C_1] = F + (2 - b_1)E_0$ and $[C_2] = F + b_1 E_0$ for some $b_1 \geq 1$, or $[C_1] = (2 - a_1)F + E_0$ and $[C_2] = a_1 F + E_0$ for some $a_1 \geq 1$. We consider the latter case and the first case can be treated similarly.

We first consider $a_1 \geq 2$. By Lemma 3.8, after composing a Hamiltonian diffeomorphism, we can assume C_1 and $\overline{C_1}$ completely coincide. Fix an ω-tamed almost complex structure J_0 making $C_1 = \overline{C_1}$ pseudo-holomorphic and integrable near C_1. Consider the set of ω-tamed almost complex structure $\mathcal{J}$ agree with J_0 near C_1. Fix $J \in \mathcal{J}$, we want to inspect all possible degenerations of J-holomorphic nodal curve representing $[C_2]$. By positivity of intersection, positivity of area and adjunction, the homology class $aF + bE_0$ of any J-holomorphic curve has non-negative coefficient for the E_0 factor (i.e. $b \geq 0$). Therefore, the irreducible components (possibly not simple) of any J-holomorphic curve representing $[C_2]$ give rise to a decomposition $[C_2] = (s_1 F + E_0) + s_2 F + \cdots + s_m F$, where $s_j > 0$ for $2 \leq j \leq m$ (by positivity of intersection with $[C_1]$). If $s_1 \leq 0$, then $s_1 F + E_0 = [C_1]$ by positivity of intersection with $[C_1]$. The sum of non-negative Fredholm index of the underlying curve of each individual component is given by $Ind_{nodal} = (4s_1 + 2) + 2(m - 1)$ when $s_1 \geq 0$, and $Ind_{nodal} = 2(m - 1)$ when $s_1 < 0$ because the class $s_1 F + E_0$ is primitive and the underlying curve for $s_j F$ has homology F (the index formula for a pseudo-holomorphic curve with class A is $2c_1(A) - 2$). On the other hand, the index of the class $[C_2]$ is given by $Ind_{C_2} = 2(2a_1 + 2) - 2 = 4(\sum_{i=1}^m s_i) + 2 = (4s_1 + 2) + 4(\sum_{i=2}^m s_i)$.

If $s_1 \geq 0$ and $m \geq 2$, we have

$$Ind_{nodal} + 2 \leq (4s_1 + 2) + 4\left(\sum_{i=2}^{m} s_i\right) = Ind_{C_2}$$

If $s_1 < 0$, we have $s_1 = 2 - a_1$ and hence

$$\begin{aligned} Ind_{nodal} + 2 = 2(m-1) + 2 &\leq 2\left(\sum_{i=2}^{m} s_i\right) + 2 \\ &= 2(a_1 - (2 - a_1)) + 2 = 4a_1 - 2 < Ind_{C_2} \end{aligned}$$

Therefore, any degeneration happens in codimension two or higher.

Then we can apply the standard pseudo-holomorphic curve argument to obtain a symplectic isotopy from C_2 to $\overline{C_2}$ transversal to C_1 for all time along the isotopy and finish the proof. Since we could not find a reference that fits exactly to our situation (Proposition 1.2.9(ii) of [20] is a very closely related one), we provide some details here. We will basically follow [19] together with Lemma 3.2.2 and Proposition 3.2.3 of [20].

We perturb C_2 and $\overline{C_2}$ so that they have $2a_1 + 1$ distinct intersection points and call these intersection points $\{p_j\}_{j=1}^{2a_1+1}$. We form the universal moduli space for genus 0 curve representing the class $[C_2]$ with $2a_1 + 1$ point constraints $\{p_j\}_{j=1}^{2a_1+1}$ with respect to the space of almost complex structures $\mathcal{J}$. We want to pick $J, \overline{J} \in \mathcal{J}$ that are regular for all underlying (marked) simple curves appearing in a degeneration of $[C_2]$ except $C_1 = \overline{C_1}$ such that C_2 is J-holomorphic and $\overline{C_2}$ is $\overline{J}$-holomorphic.

To find J and $\overline{J}$, we note the following two facts. For any $J \in \mathcal{J}$ (resp. $\overline{J} \in \mathcal{J}$) making C_2 J-holomorphic (resp. making $\overline{C_2}$ $\overline{J}$-holomorphic), the Fredholm operator taking the point constraints $\{p_j\}_{j=1}^{2a_1+1}$ into account is regular by automatic transversality (See Theorem 3.1 and Proposition 3.2 of [13], and also [8], [9]). On the other hand, for a generic choice of J (resp. $\overline{J}$) making C_1 and C_2 J-holomorphic (resp. $C_1 = \overline{C_1}$ and $\overline{C_2}$ $\overline{J}$-holomorphic), each simple curve other than C_1 and C_2 (resp. other than C_1 and $\overline{C_2}$) in any degeneration has a somewhere injective point away from C_1 and C_2 (resp. away from $\overline{C_1}$ and $\overline{C_2}$) and hence is regular (See Chapter 3.4 of [19]). As a result, we can find $J, \overline{J} \in \mathcal{J}$ as desired.

For such $J, \overline{J}$, there is a regular smooth path $J_t \in \mathcal{J}$ (regular in the sense of Definition 6.2.10 of [19]) such that the parametrized moduli space of J_t-holomorphic curves representing $[C_2]$ and passing through $\{p_j\}_{j=1}^{2a_1+1}$ forms a non-empty one dimensional smooth manifold. Since degeneration happens in codimension 2 or higher, if we choose J_t to be also regular with respect to the lower dimensional strata, the one dimensional moduli space is also compact.

Thus, there is a family of embedded J_t-holomorphic spheres C^t all of which passing through $\{p_j\}_{j=1}^{2a_1+1}$. By positivity of intersection, C^t is the only J_t-holomorphic family passing through $\{p_j\}_{j=1}^{2a_1+1}$, hence we have a symplectic isotopy from C_2 to $\overline{C_2}$. Finally, by applying Lemma 3.2.1 of [20] to $\{C^t\}$ to get another symplectic isotopy $\{C^{t'}\}$ transversal to C_1, we get that the intersection pattern of $\{C^{t'}\} \cup C_1$ is unchanged along the symplectic isotopy. This finishes the proof when $a_1 \geq 2$.

The case that $a_1 = 1$ can be treated similarly, which is easier and only requires an analogue of Proposition 3.7 and Lemma 3.8 for symplectic sphere with non-negative self-intersection (See e.g Proposition 3.2 of [13]).

Now, we consider $(X, \omega) = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, c\omega_\lambda)$ for some constant c, $D = C_1 \cup C_2$, $\overline{D} = \overline{C_1} \cup \overline{C_2}$ and $[C_i] = [\overline{C_i}]$ for $i = 1, 2$. By the enumeration, there are two possible cases.

The first one is when $[C_1] = [\overline{C_1}] = (1 - a_1)f + s = (2 - a_1)F + E_1$ and $[C_2] = [\overline{C_2}] = a_1 f + s = (a_1 + 1)F + E_1$. By symmetry, it suffices to consider $a_1 \geq 1$. If $a_1 \geq 2$, we apply Lemma 3.8 and assume C_1 completely coincides with $\overline{C_1}$. Again, we inspect all possible J-holomorphic degenerations of C_2 for J making C_1 J-holomorphic. A direct index count as before shows that any degeneration of C_2 has at least codimension two. Therefore, the same method applies. The case that $a_1 = 1$ is dealt similarly.

The other case is $[C_1] = [\overline{C_1}] = f = F$ and $[C_2] = [\overline{C_2}] = 2s = 2F + 2E_1$. This cannot cause additional trouble as they have non-negative self-intersection numbers. One can deal with this similar to the previous cases.

The case that $X = \mathbb{C}P^2$ is analogous and easier.

• Triangles and Rectangles

Now, we consider $X = \mathbb{S}^2 \times \mathbb{S}^2$ or $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and assume $D, \overline{D}$ has three or four irreducible components. We observe that, there is at most one component with negative self-intersection number and one with positive self-intersection numbers in all cases. Moreover, the homology class of the component with negative self-intersection number is of the form $E_i + jF$ for some j and $i = 0, -1$. If there is a negative self-intersection component, we can apply Lemma 3.8 and assume the negative self-intersection components for D and $\overline{D}$ completely coincide. Then we study all the possible J-holomorphic degeneration of the **positive** curve for J making the negative component J-holomorphic. One can show that the degeneration happens in at least codimension two by index count. Therefore, we can find a relative pseudo-holomorphic isotopy Φ_t from the positive self-intersection component of D

to the positive self-intersection component of $\overline{D}$. At the same time, since the remaining components of D and $\overline{D}$ are sphere fibers, which cannot have any pseudo-holomorphic degeneration, the pseudo-holomorphic isotopy Φ_t can be extended to a pseudo-holomorphic isotopy from D to $\overline{D}$. Hence, the result follows when there is a negative self-intersection component. The remaining cases are all similar and simpler, including the case when $X = \mathbb{C}P^2$. □

3.2.2. Elliptic ruled surfaces. In this subsection, we want to finish the proof of Proposition 3.5 for the torus type.

Proposition 3.9. *Suppose (X, D, ω) and $(X, \overline{D}, \omega)$ are minimal symplectic log Calabi–Yau surfaces such that X is elliptic ruled. Then they are symplectic deformation equivalent to each other.*

We first describe the complement of D following [30]. Any ω-compatible almost complex structure J provides us a J-holomorphic ruling, meaning that there is a sphere bundle map $\pi : X \to \mathbb{T}^2$ such that fibers are J-holomorphic. Usher proves in [30] (Lemma 3.5) that, if D is J-holomorphic, $\pi|_D$ is a two to one covering and in particular D is transversal to the J-holomorphic sphere foliation. If a tubular neighborhood of D is taken out, we have a J-holomorphic annulus foliation, which defines an annulus bundle $X - P(D)$ over the torus $\mathbb{T}^2$. We want to identify this annulus bundle.

Equip the orientation of $\mathbb{T}^2$ such that $\pi|_D$ is orientation preserving, where the orientation of D is determined by J. Choose a positively oriented basis $\{t, u\} \in H_1(D, \mathbb{Z})$ and $\{v, w\} \in H_1(\mathbb{T}^2, \mathbb{Z})$ such that $\pi_* t = v$ and $\pi_* u = 2w$. Let $\mathbb{A} = \{z \in \mathbb{C} | \frac{1}{2} \leq |z| \leq 2\}$. The monodromy of this annulus bundle around the loop corresponding to v is orientation preserving and does not flip the boundary. Therefore, the monodromy is isotopic to the identity. Similarly, the monodromy of this annulus bundle around the loop corresponding to w is orientation preserving but flip the boundary components due to $\pi_* u = 2w$. Therefore, the monodromy is isotopic to the map sending z to z^{-1}. This annulus bundle is isomorphic as an annulus bundle to (See the paragraph before Lemma 3.6 of [30])

$$\mathbb{S}^1 \times \frac{\mathbb{R} \times \mathbb{A}}{(x+1, z) \sim (x, z^{-1})}$$

if X is the smoothly trivial sphere bundle, and isomorphic to

$$\frac{\mathbb{R} \times \mathbb{S}^1 \times \mathbb{A}}{(x+1, e^{i\theta}, z) \sim (x, e^{i\theta}, e^{i\theta} z^{-1})}$$

if X is the smoothly non-trivial sphere bundle.

Let $\overline{D}$ be another connected symplectic torus representing $c_1(X)$. For $\overline{D}$, we can also define $\overline{J}$, $\overline{\pi}$, $\overline{\mathbb{T}^2}$, $\overline{t}, \overline{u}, \overline{v}, \overline{w}$ as above. Let $\tau : \mathbb{T}^2 \to \overline{\mathbb{T}^2}$ be a diffeomorphism sending v and w to $\overline{v}$ and $\overline{w}$, respectively. By construction, the pull-back annulus bundle $\tau^*(\overline{X} - P(\overline{D})) \to \mathbb{T}^2$ has the same monodromy (up to isotopy) as $X - P(D) \to \mathbb{T}^2$ over the one-skeleton. The existence of an annulus bundle isomorphism from $X - P(D)$ to $\tau^*(X - \overline{D})$ covering the identity of $\mathbb{T}^2$ reduces to whether $X - P(D)$ and $\tau^*(\overline{X} - \overline{D})$ are isomorphic annulus bundle (covering some diffeomorphism of the base), which is true because there is only one class of isomorphic annulus bundle for a choice of monodromies over one skeleton (and it is explicitly described above in our case). Therefore, we have a bundle isomorphism $F : X - P(D) \to X - P(\overline{D})$ covering τ. On the other hand, since the image of $\tau_* \circ \pi_*|_{H_1(D,\mathbb{Z})}$ equals the image of $\overline{\pi}_*|_{H_1(\overline{D},\mathbb{Z})}$, there are two lifts of τ to $\tilde{\tau}_i : D \to \overline{D}$ such that $\overline{\pi} \circ \tilde{\tau}_i = \tau \circ \pi$, for $i = 1, 2$. Then, there is a unique way, up to isotopy, to get a sphere bundle isomorphism $\tilde{F} : X \to X$ extending F and $\tilde{\tau}_1$ (or, F and $\tilde{\tau}_2$) by following the pseudo-holomorphic foliation. In particular, we have $\tilde{F}(D) = \overline{D}$.

Using $\tilde{F}$, we can identify $\overline{D} \subset (X, \omega)$ with $D \subset (X, \tilde{F}^*\omega)$. Proposition 3.9 will follow if we can find a symplectic deformation equivalence from (X, D, ω) to $(X, D, \tilde{F}^*\omega)$, which can be obtained by the following lemma.

Lemma 3.10. *Let $\pi : (X, \omega_i, J_i) \to B$ be a symplectic surface bundle over surface such that J_i is ω_i-compatible and fibers are J_i holomorphic for both $i = 0, 1$. Moreover, we assume the orientation of fibers induced by J_0 and J_1 are the same and the orientation of the total space induced by ω_0^2 and ω_1^2 are the same. Assume $D \subset (X, \omega_i)$ is a J_i holomorphic surface for $i = 0, 1$. and $\pi|_D$ is submersive. Then there is a smooth family of (possibly non-homologous) symplectic forms ω_t on X making D symplectic for all $t \in [0, 1]$ joining ω_0 and ω_1.*

Proof. Fix a point $p \in X$ and consider a non-zero tangent vector $v \in T_pX$ which does not lie in the vertical tangent sub-bundle T_pX^{vert}. Since fibers are J_i holomorphic, we have $Span\{v, J_iv\} \cap T_pX^{vert} = \{0\}$. Choose a volume form (symplectic form) ω_B on B. Since π is a submersion, $\pi_*Span\{v, J_iv\} = T_{\pi(p)}B$. Therefore, we have $\omega_B(\pi_*(v), \pi_*(J_iv)) \neq 0$. By possibly changing the sign of ω_B, we can assume $\omega_B(\pi_*(v), \pi_*(J_iv)) > 0$. Moreover, this inequality is true for any $v \in T_pX$ not lying in T_pX^{vert}. By continuity, $\omega_B(\pi_*(v), \pi_*(J_iv)) > 0$ for any $p \in X$ and any $v \in T_pX - T_pX^{vert}$ for both $i = 0, 1$.

Now, we apply the Thurston trick. For any $K \geq 0$, we let $\omega_i^K = \omega_i + K\pi^*\omega_B$, which is clearly closed. It is also non-degenerate because it is non-degenerate for the vertical tangent sub-bundle and for any $p \in X$, and any $v \in T_pX - T_pX^{vert}$, we have $\omega_i^K(v, J_iv) = \omega_i(v, J_iv) + K\omega_B(\pi_*(v), \pi_*(J_iv)) > 0$. The first term being greater than zero is by compatibility and the second term being non-negative is due to $K \geq 0$ and the first paragraph. Notice that D is symplectic with respect to ω_i^K for both $i = 0, 1$ because $\pi|_D$ is submersive and D is J_i-holomorphic.

Now, we consider $\omega_t^K = (1-t)\omega_0^K + t\omega_1^K$, which is clearly closed and non-degenerate for TX^{vert}. For $v \in T_pX - T_pX^{vert}$, we have $\omega_t^K(v, J_0v) = (1-t)\omega_0(v, J_0v) + t\omega_1(v, J_0v) + K\omega_B(\pi_*v, \pi_*J_0v)$. We know that the first and the third terms on the right hand side are non-negative but we have no control on the second term. However, there is a sufficiently large K such that $\omega_t^K(v, J_0v) > 0$ for all $p \in X$ and $v \in T_pX - T_pX^{vert}$ and for all t because the sphere subbundle of TX is compact. By smoothening out the piecewise smooth family from ω_0 to ω_0^K, ω_t^K and from ω_1^K to ω_1, we finish the proof. □

We remark that Lemma 3.10 can be viewed as a relative version of Proposition 4.4 in [17] in dimension four.

3.3. Proof of Theorem 1.4

We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let (X^i, D^i, ω^i) be symplectic log Calabi–Yau surfaces for $i = 1, 2$, which are homological equivalent via a diffeomorphism Φ.

Let $\{e_1, \ldots, e_\beta\}$ be a maximal set of pairwisely orthogonal non-toric exceptional classes in X. We can choose an almost complex structure J^1 (possibly after deforming D^1) such that D^1 is J^1-holomorphic and all e_j has embedded J^1-holomorphic representative, by Lemma 2.4. Since (X^1, D^1, ω^1) and (X^2, D^2, ω^2) are homological equivalent via Φ, $\{\Phi_*(e_j)\}$ is a maximal set of pairwisely orthogonal non-toric exceptional classes. We can find an ω^2-tamed almost complex structure (possibly after deforming D^2) J^2 such that D^2 is J^2-holomorphic and the $\Phi_*(e_j)$ has embedded J^2-holomorphic representative. After blowing down the J^i-holomorphic representatives of e_j, and $\Phi_*(e_j)$ for all $1 \leq j \leq \beta$, we obtain two symplectic log CY surfaces $(\overline{X^1}, \overline{D^1}, \overline{\omega^1})$ and $(\overline{X^2}, \overline{D^2}, \overline{\omega^2})$.

Clearly, $(\overline{X^1}, \overline{D^1}, \overline{\omega^1})$ and $(\overline{X^2}, \overline{D^2}, \overline{\omega^2})$ are homological equivalent for some natural choice of diffeomorphism $\overline{\Phi}$. Now, a component in $\overline{D^1}$ is exceptional if and only if the corresponding component in $\overline{D^2}$ is exceptional.

By Lemma 3.4, we pass to minimal models $(\overline{X_b^i}, \overline{D_b^i}, \overline{\omega_b^i})$ by toric blow-downs. By identifying $\overline{X_b^1}$ and $\overline{X_b^2}$ with a natural choice of diffeomorphism $\overline{\Phi_b}$, the homology classes of the components of $\overline{D_b^1}$ and $\overline{D_b^2}$ are the same.

By Proposition 1.2.15 of [20] or Theorem 2.9 of [3], up to a D-symplectic homotopy (ie. a deformation of $\overline{\omega_b^2}$ keeping $\overline{D_b^2}$ symplectic), we can assume $[\overline{\omega_b^1}] = \overline{\Phi_b}^*[\overline{\omega_b^2}]$. Therefore, $\overline{X_b^1}$ and $\overline{X_b^2}$ are actually symplectomorphic ([29], [10]) and we thus can choose $\overline{\Phi_b}$ to be a symplectomorphism from $(\overline{X_b^1}, \overline{\Phi_b}^{-1}(\overline{D_b^2}), \overline{\omega_b^1})$ to $(\overline{X_b^2}, \overline{D_b^2}, \overline{\omega_b^2})$. Therefore, we conclude that $(\overline{X_b^1}, \overline{D_b^1}, \overline{\omega_b^1})$ and $(\overline{X_b^2}, \overline{D_b^2}, \overline{\omega_b^2})$ are symplectic deformation equivalent, by applying Proposition 3.5 to $(\overline{X_b^1}, \overline{D_b^1}, \overline{\omega_b^1})$ and $(\overline{X_b^1}, \overline{\Phi_b}^{-1}(\overline{D_b^2}), \overline{\omega_b^1})$. Further, by Lemma 2.7, they are $D-$symplectic deformation equivalent.

Now we record the sequence of non-toric and toric blow-downs by markings $\overline{D_b^1}$ and $\overline{D_b^2}$. As marked divisors, they are $D-$symplectic deformation equivalent by Lemma 2.12. Finally, by Proposition 2.10 (and viewing unmarked divisors as marked divisors without markings), (X^1, D^1, ω^1) is $D-$symplectic deformation equivalent to (X^2, D^2, ω^2), and hence symplectic deformation equivalent to (X^2, D^2, ω^2) by Lemma 2.7. Tracing the steps, we see that the symplectomorphism in the symplectic deformation equivalence between (X^1, D^1, ω^1) and (X^2, D^2, ω^2) has the same homological effect as Φ.

Now, assume (X^1, D^1, ω^1) is *strictly* homological equivalent to (X^2, D^2, ω^2) via a diffeomorphism Φ. It means that Φ is a homological equivalence and $\Phi^*[\omega^2] = [\omega^1]$. We first note that, up to symplectic isotopy of D^1 and D^2, which preserves the strict D-symplectic deformation class (Lemma 2.11), we can assume D^i are ω^i-orthogonal. We have shown that there is a $D-$symplectic homotopy (X^1, D^1, ω_t^1) of (X^1, D^1, ω^1) and a symplectomophism $\Psi : (X^1, D^1, \omega_1^1) \to (X^2, D^2, \omega^2)$ with the same homological effect as Φ. Therefore, we have $[\omega^1] = \Phi^*[\omega^2] = \Psi^*[\omega^2] = [\omega_1^1]$. By Theorem 1.2.12 of [20], ω_t^1 can be chosen such that $[\omega_t^1]$ is constant for all t. By Corollary 1.2.13 of [20], there is a symplectic isotopy (X^1, D_t^1, ω^1) such that $D_0^1 = D^1$ and (X^1, D_1^1, ω^1) is symplectomorphic to (X^1, D^1, ω_1^1) and hence to (X^2, D^2, ω^2). Therefore, the result follows. □

In the case $X^1 = X^2 = X$, Theorem 1.4 implies the symplectic deformation class of (X, D, ω) is uniquely determined by the homology classes $\{[C_j]\}_{j=1}^k$ modulo the action of diffeomorphism on $H_2(X, \mathbb{Z})$. The fact the the homology classes of D completely determine the symplectic deformation equivalent class can be regarded as in the same spirit of Torelli type theorems in a weak sense.

If $(X^1, \omega^1) = (X^2, \omega^2) = (X, \omega)$ and $[C_j^1] = [C_j^2]$ for all j, we can take the strict homological equivalence to be identity and hence the symplectomorphism from (X, D^1, ω) to the time-one end of the symplectic isotopy of (X, D^2, ω) in Theorem 1.4 has trivial homological action. Therefore, the number of symplectic isotopy classes of homological equivalent log Calabi–Yau surfaces in (X, ω) is bounded above by the number of connected components of Torelli part of the symplectomorphism group of (X, ω).

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