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Applications of affine structures to Calabi-Yau moduli spaces

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In this paper, we review our recent results and the methods of proofs in [15] in which it is proved that the Hodge metric completion of the moduli space of polarized and marked Calabi–Yau manifolds, i.e. the Torelli space, is a complex affine manifold. As applications it is proved that the period map from the Torelli space and the extended period map from its completion space, both are injective into the period domain, and that the period map from the moduli space of polarized Calabi–Yau manifolds with level m structure is also injective.

1. Introduction

In this paper, a compact projective manifold M of complex dimension n with $n \geq 3$ is called Calabi–Yau, if it has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$. We fix a lattice Λ with an pairing Q_0, where Λ is isomorphic to $H^n(M_0, \mathbb{Z})/\mathrm{Tor}$ for some fixed Calabi–Yau manifold M_0, and Q_0 is the intersection pairing.

A polarized and marked Calabi–Yau manifold is a triple (M, L, γ) consisting of a Calabi–Yau manifold M, an ample line bundle L over M, and a marking γ defined as an isometry of the lattices

$$\gamma : (\Lambda, Q_0) \to (H^n(M, \mathbb{Z})/\mathrm{Tor}, Q).$$

Let $\mathcal{Z}_m$ be a smooth irreducible component of the moduli space of polarized Calabi–Yau manifolds with level m structure with $m \geq 3$. For example, see Section 2 of [27] for the construction of $\mathcal{Z}_m$. We define the Teichmüller space of Calabi–Yau manifolds to be the universal cover of $\mathcal{Z}_m$, which can be easily proved to be independent of the choice of m. We will denote by $\mathcal{T}$ the Teichmüller space of Calabi–Yau manifolds.

Let $\mathcal{T}'$ be a smooth irreducible component of the moduli space of equivalence classes of marked and polarized Calabi–Yau manifolds. We call $\mathcal{T}'$ the Torelli space of Calabi–Yau manifolds in this paper. The Torelli space $\mathcal{T}'$ is also called the framed moduli as discussed in [1]. We will see that the Torelli space $\mathcal{T}'$ is the most natural space to consider the period map and to study the Torelli problem.

We will assume that both $\mathcal{Z}_m$ and $\mathcal{T}'$ contain the polarized Calabi–Yau manifold that we start with. We will see that the Torelli space $\mathcal{T}'$ is a natural covering space of $\mathcal{Z}_m$, therefore $\mathcal{T}$ is also the universal cover of $\mathcal{T}'$. See Section 2 for details. The relations of these spaces can be put into the following commutative diagram,

$$\begin{tikzcd} \mathcal{T} \arrow[dd, "\pi_m"] \arrow[dr, "\pi"] & \\ & \mathcal{T}', \arrow[dl, "\pi'_m"] \\ \mathcal{Z}_m & \end{tikzcd} \tag{1}$$

with π_m, π'_m and π the corresponding covering maps.

Let D be the period domain of polarized Hodge structures of the n-th primitive cohomology of M. Let us denote the period map on the smooth moduli space $\mathcal{Z}_m$ by

$$\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma,$$

where Γ denotes the global monodromy group which acts properly and discontinuously on D. Recall that Γ is the image of the monodromy representation

$$\rho : \pi_1(\mathcal{Z}_m) \to \Gamma \subseteq \mathrm{Aut}(H_{\mathbb{Z}}, Q).$$

Then we can lift the period map $\Phi_{\mathcal{Z}_m}$ to the period map $\Phi : \mathcal{T} \to D$ from the universal cover $\mathcal{T}$ of $\mathcal{Z}_m$, such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\Phi"] \arrow[d, "\pi_m"] & D \arrow[d, "\pi_D"] \\
\mathcal{Z}_m \arrow[r, "\Phi_{\mathcal{Z}_m}"] & D/\Gamma.
\end{tikzcd}$$

Similarly we define the period map

$$\Phi' : \mathcal{T}' \to D$$

on the Torelli space $\mathcal{T}'$ by the definition of marking, such that the above diagram and diagram (1) fit into the following commutative diagram

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, "\Phi"] \arrow[dr, "\pi"] \arrow[dd, "\pi_m"'] & & D \arrow[dd, "\pi_D"] \\
& \mathcal{T}' \arrow[ur, "\Phi'"] \arrow[dl, "\pi'_m"] & \\
\mathcal{Z}_m \arrow[rr, "\Phi_{\mathcal{Z}_m}"] & & D/\Gamma.
\end{tikzcd} \tag{2}$$

There is a natural metric, called Hodge metric h, on D, which is a complete homogeneous metric induced from the Killing form as studied in detail in [9]. By local Torelli theorem of Calabi–Yau manifolds, both $\Phi_{\mathcal{Z}_m}$ and Φ are nondegenerate. Clearly Φ' is also nondegenerate, and the pull-back of the Hodge metric to $\mathcal{T}'$ by Φ' also defines a Kähler metric on the Torelli space $\mathcal{T}'$. These Kähler metrics are still called the Hodge metrics in this paper.

One of our crucial constructions is the global holomorphic affine structures on the Teichmüller space, the Torelli space, and the Hodge metric completion space of the Torelli space. Here we first outline the construction of affine structure on the Teichmüller space to give the reader some basic ideas of our method. See Section 3 of [15] for details.

We first fix a base point $p \in \mathcal{T}$ with its Hodge structure $\{H_p^{k,n-k}\}_{k=0}^n$ as the reference Hodge structure. With this fixed base point $\Phi(p) = o \in D$, we identify the unipotent subgroup $N_+ = \exp(\mathfrak{n}_+)$ with its orbit in $\check{D}$. By using a method of Harish-Chandra, we prove that the image of the period map has the property that,

$$\Phi(\mathcal{T}) \subseteq N_+ \cap D,$$

is a bounded subset in N_+. In this case we will simply say that the period map Φ is bounded. Here the Euclidean structure on $\mathfrak{n}_+$ and N_+ are induced from the Hodge metric on D by the identification of $\mathfrak{n}_+$ to the holomorphic tangent space $\mathrm{T}_o^{1,0}D$ of D at the base point o. We also consider N_+ as a complex Euclidean space such that the exponential map

$$\exp : \mathfrak{n}_+ \to N_+$$

is an isometry.

We then introduce the abelian subalgebra $\mathfrak{a} \subset \mathfrak{n}_+$, which is defined by the image of the differential of the period map at the base point $p \in \mathcal{T}$,

$$\mathfrak{a} = d\Phi_p(\mathrm{T}_p^{1,0}\mathcal{T}) \subset \mathrm{T}_o^{1,0}D \simeq \mathfrak{n}_+,$$

and

$$A = \exp(\mathfrak{a}) \subset N_+$$

the corresponding abelian Lie group. We consider $\mathfrak{a}$ and A as an Euclidean subspace of $\mathfrak{n}_+$ of N_+ respectively. Denote the projection map by

$$P : N_+ \cap D \to A \cap D$$

and define $\Psi = P \circ \Phi$.

With local Torelli theorem for Calabi–Yau manifolds, we show that the holomorphic map

$$\Psi : \mathcal{T} \to A \cap D \subset A \simeq \mathbb{C}^N$$

is nondegenerate on $\mathcal{T}$. Thus $\Psi : \mathcal{T} \to A \cap D$ induces a holomorphic affine structure on $\mathcal{T}$ by pulling back the affine structure on $\mathbb{C}^N$.

To proceed further, we must consider the Hodge metric completions of moduli spaces. More precisely, let $\mathcal{Z}_m^H$ be the Hodge metric completion of the smooth moduli space $\mathcal{Z}_m$ and let $\mathcal{T}_m^H$ be the universal cover of $\mathcal{Z}_m^H$ with the universal covering map

$$\pi_m^H : \mathcal{T}_m^H \to \mathcal{Z}_m^H.$$

We will prove that $\mathcal{Z}_m^H$ can be identified to the Griffiths completion space of finite monodromy as introduced in Theorem 9.6 in [8]. In [15], the space $\mathcal{T}_m^H$ is called the Hodge metric completion space. In fact, it is proved in [15] that $\mathcal{T}_m^H$ is the completion space of the Torelli space with respect to the Hodge metric on it. Sometimes we simply call $\mathcal{T}_m^H$ the completion space for convenience.

We need to study the extended period maps, which fit into the following commutative diagram,

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_m"] \arrow[d, "\pi_m"] & \mathcal{T}_m^H \arrow[r, "\Phi_m^H"] \arrow[d, "\pi_m^H"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}_m^H \arrow[r, "\Phi_{\mathcal{Z}_m}^H"] & D/\Gamma, \end{tikzcd} \tag{3}$$

where

$$\Phi_{\mathcal{Z}_m}^H : \mathcal{Z}_m^H \to D/\Gamma$$

is the extension map of the period map $\Phi_{\mathcal{Z}_m}$, and

$$i : \mathcal{Z}_m \to \mathcal{Z}_m^H$$

is the inclusion map with i_m a lifting of $i \circ \pi_m$, and Φ_m^H is a lifting of the map $\Phi_{\mathcal{Z}_m}^H \circ \pi_m^H$. It is elementary to show that there is a suitable choice of i_m and Φ_m^H such that

$$\Phi = \Phi_m^H \circ i_m.$$

As a corollary of the boundedness of the period map Φ, we know that Φ_m^H is actually a bounded holomorphic map from $\mathcal{T}_m^H$ to $N_+ \cap D$. The first result proved in [15] is the following theorem.

Theorem 1.1. *For any $m \geq 3$, the complete complex manifold $\mathcal{T}_m^H \simeq A \cap D$ is a complex affine manifold, which is a bounded domain in $A \simeq \mathbb{C}^N$. Moreover, the holomorphic map*

$$\Phi_m^H : \mathcal{T}_m^H \to N_+ \cap D$$

is an injection. As a consequence, the complex manifolds $\mathcal{T}_m^H$ and $\mathcal{T}_{m'}^H$ are biholomorphic to each other for any $m, m' \geq 3$.

This theorem allows us to get rid of the subscript m, and define the complete complex manifold $\mathcal{T}^H$ with respect to the Hodge metric by $\mathcal{T}^H = \mathcal{T}^H_m$, the holomorphic map $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$ by $i_{\mathcal{T}} = i_m$, and the extended period map $\Phi^H : \mathcal{T}^H \to D$ by $\Phi^H = \Phi^H_m$ for any $m \geq 3$. Then diagram (3) becomes

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_{\mathcal{T}}"] \arrow[d, "\pi_m"] & \mathcal{T}^H \arrow[r, "\Phi^H"] \arrow[d, "\pi_m^H"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}_m^H \arrow[r, "\Phi^H_{\mathcal{Z}_m}"] & D/\Gamma. \end{tikzcd} \tag{4}$$

By these definitions, Theorem 1.1 implies that $\mathcal{T}^H$ is a complex affine manifold and that

$$\Phi^H : \mathcal{T}^H \to N_+ \cap D$$

is a holomorphic injection.

For Theorem 1.1, we remark that one technical difficulty of our arguments is to prove that $\mathcal{T}^H_m$ is independent of m. For this purpose we introduced the smooth complete manifold $\mathcal{T}^H_m$ equipped with the Hodge metric. Moreover we prove the existence of an affine structure on $\mathcal{T}^H_m$, which is given by pulling back the affine structure on $\mathbb{C}^N$ through the holomorphic map

$$\Psi^H_m : \mathcal{T}^H_m \to A \cap D \subset A \simeq \mathbb{C}^N,$$

where $\Psi^H_m = P \circ \Phi^H_m$ is shown to be nondegenerate by using affine structures on $\mathcal{T}^H_m$ and A, and the local Torelli theorem for Calabi–Yau manifolds.

By using the completeness of $\mathcal{T}^H_m$ with the Hodge metric and Corollary 2 of Griffiths and Wolf in [10], we show that Ψ^H_m is a covering map. Finally, we prove that Ψ^H_m is injective using the simply-connectedness of $A \cap D$. In fact we show that $A \cap D$ is diffeomorphic to a Euclidean space, which follows from the proof of Lemma 3.1 by using Harish-Chandra's method.

In diagram (4), it is easy to show that the map $i_{\mathcal{T}} = i_m$ is a covering map onto its image. Then we prove that the Torelli space $\mathcal{T}'$ is biholomorphic to the image $\mathcal{T}_0 = i_{\mathcal{T}}(\mathcal{T})$ of $i_{\mathcal{T}}$ in $\mathcal{T}^H$. Here the markings and level structures of the Calabi–Yau manifolds come into play substantially. From this we define an injective map

$$\pi^0 : \mathcal{T}' \to \mathcal{T}^H$$

such that diagram (4) together with diagram (2) gives the following commutative diagram

$$
\begin{tikzcd}
\mathcal{T} \arrow[rr, "i_{\mathcal{T}}"] \arrow[dr, "\pi"] \arrow[dd, "\pi_m"'] & & \mathcal{T}^H \arrow[r, "\Phi^H"] \arrow[dd, "\pi_m^H"] & D \arrow[dd, "\pi_D"] \\
& \mathcal{T}' \arrow[ur, "\pi^0"] \arrow[urr, "\Phi'"] \arrow[dl, "\pi'_m"] & & \\
\mathcal{Z}_m \arrow[rr, "i"] & & \mathcal{Z}_m^H \arrow[r, "\Phi_{\mathcal{Z}_m}^H"] & D/\Gamma.
\end{tikzcd}
\tag{5}
$$

With the injectivity of Φ^H and π^0, we have the global Torelli theorem on the Torelli space of Calabi–Yau manifolds.

Theorem 1.2. *The global Torelli theorem holds on the Torelli space of Calabi–Yau manifolds, i.e. the period map*

$$\Phi' : \mathcal{T}' \to D$$

is injective. Moreover, the complete complex affine manifold $\mathcal{T}^H$ is the completion space of $\mathcal{T}'$ with respect to the Hodge metric, and it is a bounded domain in $A \simeq \mathbb{C}^N$.

Actually we proved that $\mathcal{T}^H \simeq A \cap D$ is a bounded pseudoconvex domain in A. As a direct corollary, we have the global Torelli theorem on the moduli space $\mathcal{Z}_m$ of polarized Calabi–Yau manifolds with level m structure for any $m \geq 3$.

Corollary 1.3. *The global Torelli theorem holds on the moduli space $\mathcal{Z}_m$ of polarized Calabi–Yau manifolds with level m structure, i.e. the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$ is injective.*

This paper can be considered as a review of the results and the main ideas of proofs in [15]. In Section 2, we review the definition of the period domain of polarized Hodge structures, and briefly describe the constructions of the moduli space $\mathcal{Z}_m$, the Teichmüller space $\mathcal{T}$ and the Torelli space $\mathcal{T}'$ of Calabi–Yau manifolds, as well as the definitions of the period maps and the Hodge metrics on the moduli space and the Teichmüller space respectively. We also introduce the Hodge metric completion space $\mathcal{T}_m^H$ and study the extended map Φ_m^H on $\mathcal{T}_m^H$. In Section 3, we discuss our ideas to prove that the image of the period map

$$\Phi : \mathcal{T} \to D$$

lies in $N_+ \cap D$ as a bounded subset in the Euclidean space N_+ with Euclidean metric induced from the Hodge metric.

In Section 4, we describe the main ideas of the proof that there exists a global holomorphic affine structure on $\mathcal{T}$, as well as an affine structure on $\mathcal{T}_m^H$. Then we deduce that the extended period map

$$\Phi_m^H : \mathcal{T}_m^H \to D$$

is injective. In Section 5, we define the completion space $\mathcal{T}^H$ and the extended period map Φ^H. We discuss the main steps to show that $\mathcal{T}^H$ is the Hodge metric completion space of the Torelli space $\mathcal{T}'$, which is also a complex affine manifold, and that Φ^H is a holomorphic injection, which extends the period map

$$\Phi' : \mathcal{T}' \to D.$$

From this the global Torelli theorems for polarized Calabi–Yau manifolds on the Torelli space and the moduli spaces with level m structures for any $m \geq 3$ follow directly.

2. Moduli, Teichmüller, Torelli spaces and the period map

In Section 2.1, we recall the definition and some basic properties of the period domain. In Section 2.2, we discuss the constructions of the Teichmülller space and the Torelli space of Calabi–Yau manifolds based on the works of Popp [22], Viehweg [30] and Szendröi [27] on the moduli spaces of Calabi–Yau manifolds. In Section 2.3, we define the period maps from the Teichmüller space and the Torelli space to the period domain.

In Section 2.4, we define the Hodge metric completion spaces of Calabi–Yau manifolds and study the extended period map over the completion space. We remark that most of the results in this section are standard and can be found from the literature we refer in the subjects.

2.1. Period domain of polarized Hodge structures

We first review the construction of the period domain of polarized Hodge structures. We refer the reader to §3 in [23] for more details.

A pair (M, L) consisting of a Calabi–Yau manifold M of complex dimension n with $n \geq 3$ and an ample line bundle L over M is called a polarized Calabi–Yau manifold. By abusing notation, the Chern class of L will also be denoted by L and thus $L \in H^2(M, \mathbb{Z})$. We fix a lattice Λ with a pairing Q_0, where Λ is isomorphic to $H^n(M_0, \mathbb{Z})/\mathrm{Tor}$ for some Calabi–Yau manifold M_0 and Q_0 is defined by the cup-product. For a polarized Calabi–Yau manifold (M, L), we define a marking γ as an isometry of the lattices

$$\gamma : (\Lambda, Q_0) \to (H^n(M, \mathbb{Z})/\mathrm{Tor}, Q). \tag{6}$$

Definition 2.1. Let the pair (M, L) be a polarized Calabi–Yau manifold, then we call the triple (M, L, γ) a polarized and marked Calabi–Yau manifold.

For a polarized and marked Calabi–Yau manifold M with background smooth manifold X, the marking identifies $H^n(M, \mathbb{Z})/\mathrm{Tor}$ isometrically to the fixed lattice Λ. This gives us a canonical identification of the middle dimensional de Rham cohomology of M to that of the background manifold X, that is,

$$H^n(M, \mathbb{F}) \cong H^n(X, \mathbb{F}),$$

where the coefficient ring $\mathbb{F}$ can be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Since the polarization L is an integer class, it defines a map

$$L : H^n(X, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q}), \quad A \mapsto L \wedge A.$$

We denote by $H^n_{pr}(X) = \ker(L)$ the primitive cohomology groups, where the coefficient ring can also be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. We let

$$H^{k,n-k}_{pr}(M) = H^{k,n-k}(M) \cap H^n_{pr}(M, \mathbb{C})$$

and denote its dimension by $h^{k,n-k}$. We have the Hodge decomposition

$$H^n_{pr}(M, \mathbb{C}) = H^{n,0}_{pr}(M) \oplus \cdots \oplus H^{0,n}_{pr}(M), \tag{7}$$

such that $H^{n-k,k}_{pr}(M) = \overline{H^{k,n-k}_{pr}(M)}$. It is easy to see that for a polarized Calabi–Yau manifold, since $H^2(M, \mathcal{O}_M) = 0$, we have

$$H^{n,0}_{pr}(M) = H^{n,0}(M), \quad H^{n-1,1}_{pr}(M) = H^{n-1,1}(M).$$

The Poincaré bilinear form Q on $H^n_{pr}(X,\mathbb{Q})$ is defined by

$$Q(u,v)=(-1)^{\frac{n(n-1)}{2}}\int_X u\wedge v$$

for any d-closed n-forms u,v on X. Furthermore, Q is nondegenerate and can be extended to $H^n_{pr}(X,\mathbb{C})$ bilinearly. Moreover, it also satisfies the Hodge-Riemann bilinear relations

$$Q\left(H^{k,n-k}_{pr}(M),H^{l,n-l}_{pr}(M)\right)=0 \quad \text{unless } k+l=n, \tag{8}$$

$$\left(\sqrt{-1}\right)^{2k-n}Q\left(v,\overline{v}\right)>0 \quad \text{for } v\in H^{k,n-k}_{pr}(M)\setminus\{0\}. \tag{9}$$

Let $f^k=\sum_{i=k}^n h^{i,n-i}$, denote $f^0=m$, and

$$F^k=F^k(M)=H^{n,0}_{pr}(M)\oplus\cdots\oplus H^{k,n-k}_{pr}(M),$$

from which we have the decreasing filtration

$$H^n_{pr}(M,\mathbb{C})=F^0\supset\cdots\supset F^n.$$

We know that

$$\dim_{\mathbb{C}}F^k=f^k, \tag{10}$$

$$H^n_{pr}(X,\mathbb{C})=F^k\oplus\overline{F^{n-k+1}}, \quad \text{and} \quad H^{k,n-k}_{pr}(M)=F^k\cap\overline{F^{n-k}}. \tag{11}$$

In terms of the Hodge filtration, the Hodge-Riemann relations (8) and (9) are

$$Q\left(F^k,F^{n-k+1}\right)=0, \quad \text{and} \tag{12}$$

$$Q\left(Cv,\overline{v}\right)>0 \quad \text{if} \quad v\neq 0, \tag{13}$$

where C is the Weil operator given by $Cv=\left(\sqrt{-1}\right)^{2k-n}v$ for $v\in H^{k,n-k}_{pr}(M)$. The period domain D for polarized Hodge structures with data (10) is the space of all such Hodge filtrations

$$D=\left\{F^n\subset\cdots\subset F^0=H^n_{pr}(X,\mathbb{C})\mid (10),(12) \text{ and } (13) \text{ hold}\right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D}=\left\{F^n\subset\cdots\subset F^0=H^n_{pr}(X,\mathbb{C})\mid (10) \text{ and } (12) \text{ hold}\right\}.$$

The period domain $D \subseteq \check{D}$ is an open subset. From the definition of period domain we naturally get the Hodge bundles on $\check{D}$ by associating to each point in $\check{D}$ the vector spaces $\{F^k\}_{k=0}^n$ in the Hodge filtration of that point. Without confusion we will also denote by F^k the bundle with F^k as the fiber for each $0 \leq k \leq n$.

Remark 2.2. Here we would like to remark the notation change for the primitive cohomology groups. As mentioned above, for a polarized Calabi–Yau manifold we have

$$H_{pr}^{n,0}(M) = H^{n,0}(M), \quad H_{pr}^{n-1,1}(M) = H^{n-1,1}(M).$$

For simplicity, we will also use $H^n(M, \mathbb{C})$ and $H^{k,n-k}(M)$ to denote the primitive cohomology groups $H_{pr}^n(M, \mathbb{C})$ and $H_{pr}^{k,n-k}(M)$ respectively. Moreover, cohomology will mean primitive cohomology in the rest of the paper.

2.2. Moduli, Teichmüller and Torelli spaces

We first recall the concept of universal family of compact complex manifolds in deformation theory. We refer to page 8-10 in [25] and page 94 in [22] for equivalent definitions and more details.

A family of compact complex manifolds $\pi : \mathcal{W} \to \mathcal{B}$ is called *versal* at a point $p \in \mathcal{B}$ if it satisfies the following conditions:

1) If given a complex analytic family $\iota : \mathcal{V} \to \mathcal{S}$ of compact complex manifolds with a point $s \in \mathcal{S}$ and a biholomorphic map

$$f_0 : V = \iota^{-1}(s) \to U = \pi^{-1}(p),$$

then there exists a holomorphic map g from a neighbourhood $\mathcal{N} \subseteq \mathcal{S}$ of the point s to $\mathcal{B}$ and a holomorphic map $f : \iota^{-1}(\mathcal{N}) \to \mathcal{W}$ with $\iota^{-1}(\mathcal{N}) \subseteq \mathcal{V}$ such that they satisfy that $g(s) = p$ and $f|_{\iota^{-1}(s)} = f_0$ with the following commutative diagram

$$\begin{tikzcd} \iota^{-1}(\mathcal{N}) \arrow[r, "f"] \arrow[d, "\iota"] & \mathcal{W} \arrow[d, "\pi"] \\ \mathcal{N} \arrow[r, "g"] & \mathcal{B}. \end{tikzcd}$$

2) For all g satisfying the above condition, the tangent map $(dg)_s$ is uniquely determined.

The family $\pi : \mathcal{W} \to \mathcal{B}$ is called *universal* at a point $p \in \mathcal{B}$ if (1) is satisfied and (2) is replaced by

(2') The map g is uniquely determined.

If a family $\pi : \mathcal{W} \to \mathcal{B}$ is versal (universal resp.) at every point $p \in \mathcal{B}$, then it is called a *versal family* (*universal family* resp.) on $\mathcal{B}$.

Let (M, L) be a polarized Calabi–Yau manifold. Recall that a marking of (M, L) is defined as an isometry

$$\gamma : (\Lambda, Q_0) \to (H^n(M, \mathbb{Z})/\mathrm{Tor}, Q).$$

For any integer $m \geq 3$, we follow the definition of Szendröi [27] to define an m-equivalent relation of two markings on (M, L) by

$$\gamma \sim_m \gamma' \text{ if and only if } \gamma' \circ \gamma^{-1} - \mathrm{Id} \in m \cdot \mathrm{End}(H^n(M, \mathbb{Z})/\mathrm{Tor}),$$

and denote $[\gamma]_m$ to be the set of all the equivalent classes of such γ. Then we call $[\gamma]_m$ a level m structure on the polarized Calabi–Yau manifold.

Two triples $(M, L, [\gamma]_m)$ and $(M', L', [\gamma']_m)$ are equivalent if there exists a biholomorphic map $f : M \to M'$ such that

$$f^*L' = L,$$
$$f^*\gamma' \sim_m \gamma,$$

where $f^*\gamma'$ is given by

$$\gamma' : (\Lambda, Q_0) \to (H^n(M', \mathbb{Z})/\mathrm{Tor}, Q)$$

composed with

$$f^* : (H^n(M', \mathbb{Z})/\mathrm{Tor}, Q) \to (H^n(M, \mathbb{Z})/\mathrm{Tor}, Q).$$

We denote by $[M, L, [\gamma]_m]$ the equivalent class of the polarized Calabi–Yau manifolds with level m structure $(M, L, [\gamma]_m)$.

For deformation of polarized Calabi–Yau manifold with level m structure, we reformulate Theorem 2.2 in [27] as the following theorem, in which we only put the statements we need in this paper. One can also look at [22] and [30] for more details about the construction of moduli spaces of Calabi–Yau manifolds.

Theorem 2.3. *Let $[M, L, [\gamma]_m]$ be a polarized Calabi–Yau manifold with level m structure $[\gamma]_m$ for $m \geq 3$. Then there exists a connected quasi-projective complex manifold $\mathcal{Z}_m$ with a universal family of Calabi–Yau manifolds,*

$$f_m : \mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m, \tag{14}$$

which contains $[M, L, [\gamma]_m]$ as a fiber and is polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$.

As discussed in [27], $\mathcal{Z}_m$ is a smooth irreducible component of the moduli space of polarized Calabi–Yau manifolds with level m structure.

For $m \geq 3$, it is not hard to show that the universal cover of $\mathcal{Z}_m$ is independent of m by using the universal property of the moduli spaces $\mathcal{Z}_m$. So we will denote by $\mathcal{T}$ the universal covering space of $\mathcal{Z}_m$ for any $m \geq 3$. We call $\mathcal{T}$ the *Teichmüller space of Calabi–Yau manifolds.* We also denote by $\varphi : \mathcal{U} \to \mathcal{T}$ the pull-back family of the family (14) via the covering $\pi_m : \mathcal{T} \to \mathcal{Z}_m$.

In summary, we have the following proposition.

Proposition 2.4. *The Teichmüller space $\mathcal{T}$ of Calabi–Yau manifolds is a connected and simply connected complex manifold, and the family*

$$\varphi : \mathcal{U} \to \mathcal{T} \tag{15}$$

which contains M as a fiber, is a universal family.

We remark that the family $\varphi : \mathcal{U} \to \mathcal{T}$ being universal at each point is essentially due to the local Torelli theorem for Calabi–Yau manifolds. In fact, the Kodaira-Spencer map of the family $\mathcal{U} \to \mathcal{T}$

$$\kappa : T_p^{1,0}\mathcal{T} \to H^{0,1}(M_p, T^{1,0}M_p),$$

is an isomorphism for each $p \in \mathcal{T}$. Then by theorems in page 9 of [25], we conclude that the family $\mathcal{U} \to \mathcal{T}$ is versal at each $p \in \mathcal{T}$. Since

$$H^0(M, \Theta_M) = H^0(M, \Omega_M^{n-1}) = H^{n-1,0}(M) = 0,$$

we conclude from Theorem 1.6 of [25] that the family $\mathcal{U} \to \mathcal{T}$ is universal at each $p \in \mathcal{T}$. Moreover, the well-known Bogomolov-Tian-Todorov result ([28] and [29]) implies that $\dim_{\mathbb{C}}(\mathcal{T}) = N = h^{n-1,1}$. We refer the reader to

Chapter 4 in [14] for more details about deformation of complex structures and the Kodaira-Spencer map.

Recall that a polarized and marked Calabi–Yau manifold is a triple (M, L, γ), where M is a Calabi–Yau manifold, L is a polarization on M, and γ is a marking

$$\gamma : (\Lambda, Q_0) \to (H^n(M, \mathbb{Z})/\mathrm{Tor}, Q).$$

Two triples (M, L, γ) and (M', L', γ') are equivalent if there exists a biholomorphic map $f : M \to M'$ with

$$f^* L' = L, \quad f^* \gamma' = \gamma,$$

where $f^*\gamma'$ is given by

$$\gamma' : (\Lambda, Q_0) \to (H^n(M', \mathbb{Z})/\mathrm{Tor}, Q)$$

composed with

$$f^* : (H^n(M', \mathbb{Z})/\mathrm{Tor}, Q) \to (H^n(M, \mathbb{Z})/\mathrm{Tor}, Q).$$

We denote by $[M, L, \gamma]$ the equivalent class of the polarized and marked Calabi–Yau manifold (M, L, γ). In this paper, we define the Torelli space as follows.

Definition 2.5. The Torelli space $\mathcal{T}'$ of Calabi–Yau manifolds is the irreducible smooth component of the moduli space of the equivalent classes of polarized and marked Calabi–Yau manifolds, which contains $[M, L, \gamma]$.

By mapping $[M, L, \gamma]$ to $[M, L, [\gamma]_m]$, we have a natural covering map

$$\pi'_m : \mathcal{T}' \to \mathcal{Z}_m.$$

From this we see easily that $\mathcal{T}'$ is a smooth and connected complex manifold. We also get a pull-back universal family $\varphi' : \mathcal{U}' \to \mathcal{T}'$ on the Torelli space $\mathcal{T}'$ via the covering map π'_m.

Recall that we have defined the Teichmüller space $\mathcal{T}$ to be the universal covering space of $\mathcal{Z}_m$ with covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. Therefore we can lift π_m via the covering map $\pi'_m : \mathcal{T}' \to \mathcal{Z}_m$ to get a covering map $\pi : \mathcal{T} \to \mathcal{T}'$, such that the following diagram,

$$
\begin{tikzcd}
\mathcal{T} \arrow[dr, "\pi"] \arrow[dd, "\pi_m"] & \\
& \mathcal{T}' \arrow[dl, "\pi'_m"] \\
\mathcal{Z}_m &
\end{tikzcd}
\tag{16}
$$

commutes.

2.3. The period maps

For the family $f_m : \mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$, we denote each fiber by $[M_s, L_s, [\gamma_s]_m] = f_m^{-1}(s)$ and $F_s^k = F^k(M_s)$ for any $s \in \mathcal{Z}_m$. With some fixed point $s_0 \in \mathcal{Z}_m$, the period map is defined as a morphism $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$ by

$$s \mapsto \tau^{[\gamma_s]}(F_s^n \subseteq \cdots \subseteq F_s^0) \in D,$$

where $\tau^{[\gamma_s]}$ is an isomorphism between the complex vector spaces

$$\tau^{[\gamma_s]} : H^n(M_s, \mathbb{C}) \to H^n(M_{s_0}, \mathbb{C}),$$

which depends only on the homotopy class $[\gamma_s]$ of the curve γ_s between s and s_0. Then the period map is well-defined with respect to the monodromy representation

$$\rho : \pi_1(\mathcal{Z}_m) \to \Gamma \subseteq \mathrm{Aut}(H_{\mathbb{Z}}, Q).$$

It is well-known that the period map has the following properties:

1) locally liftable;
2) holomorphic, i.e. $\partial F_z^i / \partial \overline{z} \subset F_z^i$, $0 \le i \le n$;
3) Griffiths transversality: $\partial F_z^i / \partial z \subset F_z^{i-1}$, $1 \le i \le n$.

We define the horizontal tangent bundle $\mathrm{T}_h^{1,0} D$ as a subbundle of $\mathrm{T}^{1,0}D$, in terms of the Hodge bundles $F^k \to D$, $0 \le k \le n$ by

$$\mathrm{T}_h^{1,0} D \simeq \mathrm{T}^{1,0} D \cap \bigoplus_{k=1}^{n} \mathrm{Hom}(F^k/F^{k+1}, F^{k-1}/F^k). \tag{17}$$

From Griffiths transversality 3), we see that the tangent map

$$(d\Phi_{\mathcal{Z}_m})_q : \mathrm{T}^{1,0}_q \mathcal{Z}_m \to \mathrm{T}^{1,0}_s D$$

takes values in $\mathrm{T}^{1,0}_{h,s} D$ for any $q \in \mathcal{Z}_m$ and $s = \Phi_{\mathcal{Z}_m}(q)$.

From 1) and the fact that $\mathcal{T}$ is the universal cover of $\mathcal{Z}_m$, we can lift the period map to $\Phi : \mathcal{T} \to D$ such that the diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\Phi"] \arrow[d, "\pi_m"] & D \arrow[d, "\pi"] \\ \mathcal{Z}_m \arrow[r, "\Phi_{\mathcal{Z}_m}"] & D/\Gamma \end{tikzcd}$$

is commutative.

From the definition of marking in (6), we also have a well-defined period map $\Phi' : \mathcal{T}' \to D$ from the Torelli space $\mathcal{T}'$ by defining

$$p \mapsto \gamma_p^{-1}(F_p^n \subseteq \cdots \subseteq F_p^0) \in D, \tag{18}$$

where the triple $[M_p, L_p, \gamma_p]$ is the fiber over $p \in \mathcal{T}'$ of the analytic family $\mathcal{U}' \to \mathcal{T}'$, and the marking γ_p is an isometry from a fixed lattice Λ to $H^n(M_p, \mathbb{Z})/\mathrm{Tor}$, which extends $\mathbb{C}$-linearly to an isometry from $H = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ to $H^n(M_p, \mathbb{C})$. Here

$$\gamma_p^{-1}(F_p^n \subseteq \cdots \subseteq F_p^0) = \gamma_p^{-1}(F_p^n) \subseteq \cdots \subseteq \gamma_p^{-1}(F_p^0) = H$$

denotes a Hodge filtration of H.

Then we have the following commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\Phi"] \arrow[dr, "\pi"] \arrow[dd, "\pi_m"'] & & D \arrow[dd, "\pi_D"] \\ & \mathcal{T}' \arrow[ur, "\Phi'"] \arrow[dl, "\pi'_m"] & \\ \mathcal{Z}_m \arrow[rr, "\Phi_{\mathcal{Z}_m}"] & & D/\Gamma, \end{tikzcd} \tag{19}$$

where the maps π_m, π'_m and π are all natural covering maps between the corresponding spaces as in (16).

Before closing this section, we state a simple lemma concerning the monodromy group Γ. We refer [15] for its proof.

Lemma 2.6. *Let γ be the image of some element of $\pi_1(\mathcal{Z}_m)$ in Γ under the monodromy representation. Suppose that γ is finite, then γ is trivial. Therefore for $m \geq 3$, we can assume that Γ is torsion-free and D/Γ is smooth.*

2.4. Hodge metric completion and extended period maps

By the work of Viehweg in [30], we know that $\mathcal{Z}_m$ is quasi-projective and consequently we can find a smooth projective compactification $\overline{\mathcal{Z}}_m$ such that $\mathcal{Z}_m$ is Zariski open in $\overline{\mathcal{Z}}_m$ and the complement $\overline{\mathcal{Z}}_m \backslash \mathcal{Z}_m$ is a divisor of normal crossings. Therefore, $\mathcal{Z}_m$ is dense and open in $\overline{\mathcal{Z}}_m$ with the complex codimension of the complement $\overline{\mathcal{Z}}_m \backslash \mathcal{Z}_m$ at least one.

As in [9], the Hodge metric is a complete homogeneous metric induced by the Killing form. By local Torelli theorem for Calabi–Yau manifolds, we know that the period maps $\Phi_{\mathcal{Z}_m}, \Phi$ both have nondegenerate differentials everywhere. Thus it follows from [16] that the pull-backs of h by $\Phi_{\mathcal{Z}_m}$ and Φ to $\mathcal{Z}_m$ and $\mathcal{T}$ respectively are both well-defined Kähler metrics.

By abuse of notation, we still call these pull-back metrics the Hodge metrics. Let us denote $\mathcal{Z}_m^H$ to be the completion of $\mathcal{Z}_m$ with respect to the Hodge metric. By definition $\mathcal{Z}_m^H$ is the smallest complete space with respect to the Hodge metric that contains $\mathcal{Z}_m$. Then $\mathcal{Z}_m^H \subseteq \overline{\mathcal{Z}}_m$ and the complex codimension of the complement $\mathcal{Z}_m^H \backslash \mathcal{Z}_m$ is at least one.

In order to understand $\mathcal{Z}_m^H$ and the extended period map $\Phi_{\mathcal{Z}_m}^H$ well, we introduce another two extensions of $\mathcal{Z}_m$, which will be proved to be biholomorphic to $\mathcal{Z}_m^H$.

Let $\mathcal{Z}'_m \supseteq \mathcal{Z}_m$ be the maximal subset of $\overline{\mathcal{Z}}_m$ to which the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$ extends continuously and let $\Phi_{\mathcal{Z}'_m} : \mathcal{Z}'_m \to D/\Gamma$ be the extended map. Then one has the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_m \arrow[r, "i"] \arrow[rr, bend left, "\Phi_{\mathcal{Z}_m}"] & \mathcal{Z}'_m \arrow[r, "\Phi_{\mathcal{Z}'_m}"] & D/\Gamma. \end{tikzcd}$$

with $i : \mathcal{Z}_m \to \mathcal{Z}'_m$ the inclusion map.

Since $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a divisor with simple normal crossings, for any point in $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ we can find a neighborhood U of that point, which is isomorphic to a polycylinder Δ^n, such that

$$U \cap \mathcal{Z}_m \simeq (\Delta^*)^k \times \Delta^{N-k}.$$

Let T_i, $1 \leq i \leq k$ be the image of the i-th fundamental group of $(\Delta^*)^k$ under the monodromy representation, then the T_i's are called the Picard-Lefschetz transformations. Let us define the subspace $\mathcal{Z}''_m \subset \overline{\mathcal{Z}}_m$, introduced by Griffiths in [8], which contains $\mathcal{Z}_m$ and the points in $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ around which the Picard-Lefschetz transformations are of finite order, hence trivial by Lemma 2.6.

In Section 1.4 of [15], we proved the equivalence of the extended moduli spaces constructed as above. Moreover, we proved the following lemma.

Lemma 2.7. *We have $\mathcal{Z}'_m = \mathcal{Z}''_m = \mathcal{Z}^H_m$ which is an open complex submanifold of $\overline{\mathcal{Z}}_m$ with codim$_{\mathbb{C}}(\overline{\mathcal{Z}}_m \setminus \mathcal{Z}^H_m) \geq 1$. The subset $\mathcal{Z}^H_m \setminus \mathcal{Z}_m$ consists of the points around which the Picard-Lefschetz transformations are trivial. Moreover the extended period map*

$$\Phi^H_{\mathcal{Z}_m} : \mathcal{Z}^H_m \to D/\Gamma$$

is proper and holomorphic.

Let $\mathcal{T}^H_m$ be the universal cover of $\mathcal{Z}^H_m$ with the universal covering map $\pi^H_m : \mathcal{T}^H_m \to \mathcal{Z}^H_m$. Thus $\mathcal{T}^H_m$ is a connected and simply connected complete complex manifold with respect to the Hodge metric. We will call $\mathcal{T}^H_m$ the Hodge metric completion space with level m structure. Recall that the Teichmüller space $\mathcal{T}$ is the universal cover of the moduli space $\mathcal{Z}_m$ with the universal covering map denoted by $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. Thus we have the following commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_m"] \arrow[d, "\pi_m"] & \mathcal{T}^H_m \arrow[r, "\Phi^H_m"] \arrow[d, "\pi^H_m"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}^H_m \arrow[r, "\Phi^H_{\mathcal{Z}_m}"] & D/\Gamma, \end{tikzcd} \tag{20}$$

where i is the inclusion map, i_m is a lifting map of $i \circ \pi_m$, π_D is the covering map and Φ^H_m is a lifting map of $\Phi^H_{\mathcal{Z}_m} \circ \pi^H_m$. In particular, Φ^H_m is a continuous map from $\mathcal{T}^H_m$ to D.

We notice that the lifting maps $i_{\mathcal{T}}$ and Φ^H_m are not unique, it is easy to show that there exist suitable choices of i_m and Φ^H_m such that $\Phi = \Phi^H_m \circ i_m$. We will fix the choices of i_m and Φ^H_m such that $\Phi = \Phi^H_m \circ i_m$ in the rest of the paper. Without confusion of notations, we denote $\mathcal{T}_m := i_m(\mathcal{T})$ and the restriction map $\Phi_m = \Phi^H_m|_{\mathcal{T}_m}$. Then we also have $\Phi = \Phi_m \circ i_m$. Moreover, we have the following proposition. See Section 1.4 of [15] for the detailed proof.

Proposition 2.8. *The image $\mathcal{T}_m$ equals to the preimage $(\pi_m^H)^{-1}(\mathcal{Z}_m)$, and*

$$i_m : \mathcal{T} \to \mathcal{T}_m$$

is a covering map.

Proposition 2.8 implies that $\mathcal{T}_m$ is an open complex submanifold of $\mathcal{T}_m^H$ and $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}_m^H \setminus \mathcal{T}_m) \geq 1$. Moreover, we have that $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is an analytic subvariety of $\mathcal{T}_m^H$.

In fact, since $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a union of simple normal crossing divisors, from Lemma 2.6, we see that the subset $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ consists of normal crossing divisors in $\mathcal{Z}_m^H$ around which the monodromy group is trivial. Therefore $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is an analytic subvariety of $\mathcal{Z}_m^H$. On the other hand, from Proposition 2.8, we know that $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is the inverse image of $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ under the covering map

$$\pi_m^H : \mathcal{T}_m^H \to \mathcal{Z}_m^H,$$

this implies that $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is an analytic subvariety of $\mathcal{T}_m^H$.

We will call $\Phi_m^H : \mathcal{T}_m^H \to D$ the extended period map. Then by using the closedness of the horizontal tangent bundle $\mathrm{T}_h^{1,0}D$ in $\mathrm{T}^{1,0}D$, we have

Lemma 2.9. *The extended period map*

$$\Phi_m^H : \mathcal{T}_m^H \to D$$

satisfies the Griffiths transversality.

3. Boundedness of the period maps

In Section 3.1, we review some basic properties of the period domain from Lie group and Lie algebra point of view. We fix a base point $p \in \mathcal{T}$ and introduce the unipotent space $N_+ \subseteq \check{D}$, which is biholomorphic to complex Euclidean space $\mathbb{C}^d$. In Section 3.2, we define

$$\check{\mathcal{T}} = \Phi^{-1}(N_+ \cap D),$$

and consider, in terms of the notations of Lie algebras in Section 3.1,

$$\mathfrak{p}_+ = \mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b}) = \mathfrak{p} \cap \mathfrak{n}_+ \subseteq \mathfrak{n}_+$$

and $\exp(\mathfrak{p}_+) \subseteq N_+$ as complex Euclidean subspaces.

Let

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

be the induced projection map and $\Phi_+ = P_+ \circ \Phi|_{\check{\mathcal{T}}}$. We first explain the proof that the image of

$$\Phi_+ : \check{\mathcal{T}} \to \exp(\mathfrak{p}_+) \cap D$$

is bounded in $\exp(\mathfrak{p}_+)$ with respect to the Euclidean metric on $\exp(\mathfrak{p}_+) \subseteq N_+$. In fact we actually proved that $\exp(\mathfrak{p}_+) \cap D$ is bounded in $\exp(\mathfrak{p}_+)$. Then we review the proof of the boundedness of the image of

$$\Phi : \check{\mathcal{T}} \to N_+ \cap D$$

in N_+ otained by proving the finiteness of the map $P_+|_{\Phi(\check{\mathcal{T}})}$. From linear algebra we have that $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\operatorname{codim}_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) \geq 1$. Then we can apply the Riemann extension theorem to get the boundedness of the image of

$$\Phi : \mathcal{T} \to N_+ \cap D$$

in N_+.

As mentioned in the introduction, when the images of these maps are bounded sets in the corresponding complex Euclidean spaces, we will simply say that these maps are bounded.

3.1. Period domain from Lie algebras and Lie groups

Let us briefly recall some properties of the period domain from Lie group and Lie algebra point of view. All of the results in this section is well-known to the experts in the subject. The purpose to give details is to fix notations. One may either skip this section or refer to [9] and [23] for most of the details.

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over $\mathbb{Q}$. Let us simply denote $H_{\mathbb{C}} = H^n_{pr}(M, \mathbb{C})$ and $H_{\mathbb{R}} = H^n_{pr}(M, \mathbb{R})$. The group over $\mathbb{C}$ is

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

which acts on $\check{D}$ transitively. The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

which acts transitively on D as well.

Consider the period map $\Phi : \mathcal{T} \to D$. Fix a point $p \in \mathcal{T}$ with the image

$$o := \Phi(p) = \{F_p^n \subset \cdots \subset F_p^0\} \in D.$$

The points $p \in \mathcal{T}$ and $o \in D$ will be referred as the base points or the reference points. A linear transformation $g \in G_{\mathbb{C}}$ preserves the base point if and only if $gF_p^k = F_p^k$ for each k. Thus it gives the identification

$$\check{D} \simeq G_{\mathbb{C}}/B \quad \text{with} \quad B = \{g \in G_{\mathbb{C}} |\ gF_p^k = F_p^k, \text{ for any } k\}.$$

Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D} \quad \text{with} \quad V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion

$$G_{\mathbb{R}}/V = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B \subseteq G_{\mathbb{C}}/B.$$

The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ can be described as

$$\mathfrak{g} = \{X \in \mathrm{End}(H_{\mathbb{C}}) |\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\}.$$

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{X \in \mathfrak{g} |\ XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_{\mathbb{R}} \subseteq G_{\mathbb{C}}$, $\mathfrak{g}_0$ becomes the Lie algebra of $G_{\mathbb{R}}$. One observes that the reference Hodge structure $\{H_p^{k,n-k}\}_{k=0}^n$ of $H^n(M, \mathbb{C})$ induces a Hodge structure of weight zero on $\mathfrak{g}$, namely,

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g} | XH_p^{r,n-r} \subseteq H_p^{r+k,n-r-k}\}.$$

Since the Lie algebra $\mathfrak{b}$ of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_p^n \subset \cdots \subset F_p^0\}$, one thus has

$$\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k}.$$

The Lie algebra $\mathfrak{v}_0$ of V is

$$\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}.$$

With the above isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra

$$\mathfrak{n}_+ := \oplus_{k\geq 1}\mathfrak{g}^{-k,k}.$$

Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We take the unipotent group

$$N_+ = \exp(\mathfrak{n}_+).$$

We shall review and collect some facts about the structure of simple Lie algebra $\mathfrak{g}$ in our case. Again one may refer to [9] and [23] for more details. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the Weil operator, which is defined by

$$\theta(X) = (-1)^p X \quad \text{ for } X \in \mathfrak{g}^{p,-p}.$$

Then θ is an involutive automorphism of $\mathfrak{g}$, and is defined over $\mathbb{R}$. The $(+1)$ and (-1) eigenspaces of θ will be denoted by $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Moreover, set

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

The fact that θ is an involutive automorphism implies

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Let us consider $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then $\mathfrak{g}_c$ is a real form for $\mathfrak{g}$. Recall that the killing form $B(\cdot, \cdot)$ on $\mathfrak{g}$ is defined by

$$B(X, Y) = \mathrm{Trace}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \quad \text{ for } X, Y \in \mathfrak{g}.$$

A semisimple Lie algebra is compact if and only if its Killing form is negative definite. Thus it is not hard to check that $\mathfrak{g}_c$ is actually a compact real form of $\mathfrak{g}$, while $\mathfrak{g}_0$ is a noncompact real form. Recall that $G_\mathbb{R} \subseteq G_\mathbb{C}$ is the subgroup which correpsonds to the subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$. Let us denote the connected subgroup $G_c \subseteq G_\mathbb{C}$ which corresponds to the subalgebra $\mathfrak{g}_c \subseteq \mathfrak{g}$.

Let us denote the complex conjugation of $\mathfrak{g}$ with respect to the compact real form $\mathfrak{g}_c$ by τ_c, and the complex conjugation of $\mathfrak{g}$ with respect to the noncompact real form $\mathfrak{g}_0$ by τ_0.

The intersection $K = G_c \cap G_\mathbb{R}$ is then a compact subgroup of $\mathbb{G}_\mathbb{R}$, whose Lie algebra is $\mathfrak{k}_0 = \mathfrak{g}_\mathbb{R} \cap \mathfrak{g}_c$. With the above notations, Schmid showed in [23] that K is a maximal compact subgroup of $G_\mathbb{R}$, and it meets every connected component of $G_\mathbb{R}$. Moreover, $V = G_\mathbb{R} \cap B \subseteq K$.

3.2. Boundedness of the period maps

Now let us fix the base point $p \in \mathcal{T}$ with $\Phi(p) = o \in D$. Then after fixing the base point o, N_+ can be viewed as a subset in $\check{D}$ by identifying it with its orbit in $\check{D}$ with the base point $\Phi(p) = o$. We define

$$\check{\mathcal{T}} = \Phi^{-1}(N_+ \cap D).$$

At the base point $\Phi(p) = o \in N_+ \cap D$, we have identifications of the tangent spaces

$$\mathrm{T}_o^{1,0} N_+ = \mathrm{T}_o^{1,0} D \simeq \mathfrak{n}_+ \simeq N_+.$$

Then the Hodge metric on $\mathrm{T}_o^{1,0} D$ induces an Euclidean metric on N_+. In the proof of the following lemma, we require all the root vectors to be unit vectors with respect to this Euclidean metric.

Let

$$\mathfrak{p}_+ = \mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b}) = \mathfrak{p} \cap \mathfrak{n}_+ \subseteq \mathfrak{n}_+,$$

or equivalently,

$$\mathfrak{p}_+ = \oplus_{k \text{ odd}, \, k \geq 1} \mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+ \simeq \oplus_{k \geq 1} \mathfrak{g}^{-k,k},$$

denote a subspace of $\mathrm{T}_o^{1,0} D \simeq \mathfrak{n}_+$, and $\mathfrak{p}_+$ can be viewed as an Euclidean subspace of $\mathfrak{n}_+$. Similarly $\exp(\mathfrak{p}_+)$ can be viewed as an Euclidean subspace of N_+ with the induced metric from N_+. Define the projection map

$$P_+ : \; N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

by

$$P_+ = \exp \circ \, p_+ \circ \exp^{-1} \tag{21}$$

where $\exp^{-1} : \; N_+ \to \mathfrak{n}_+$ is the inverse of the isometry

$$\exp : \; \mathfrak{n}_+ \to N_+,$$

and

$$p_+ : \; \mathfrak{n}_+ \to \mathfrak{p}_+$$

is the projection map from the complex Euclidean space $\mathfrak{n}_+$ to its Euclidean subspace $\mathfrak{p}_+$.

The restricted period map

$$\Phi : \check{\mathcal{T}} \to N_+ \cap D,$$

composed with the projection map P_+, gives a holomorphic map

$$\Phi_+ : \check{\mathcal{T}} \to \exp(\mathfrak{p}_+) \cap D, \tag{22}$$

where $\Phi_+ = P_+ \circ \Phi|_{\check{\mathcal{T}}}$.

Because the period map is a horizontal map, and the geometry in the horizontal direction of the period domain D is similar to Hermitian symmetric space as discussed in detail in [9], the proof of the following lemma is basically an analogue of the proof of the Harish-Chandra embedding theorem for Hermitian symmetric spaces, see for example [20].

Lemma 3.1. *The image of the holomorphic map*

$$\Phi_+ : \check{\mathcal{T}} \to \exp(\mathfrak{p}_+) \cap D$$

is bounded in $\exp(\mathfrak{p}_+)$ *with respect to the Euclidean metric on* $\exp(\mathfrak{p}_+) \subseteq N_+$.

In fact we proved that $\exp(\mathfrak{p}_+) \cap D$ is a bounded domain in complex Euclidean space $\exp(\mathfrak{p}_+)$. See Lemma 2.9 in [15] for the detailed proof.

The following lemma is proved in [15], by using Griffiths transversality and the fact that the projection map

$$\pi : D \to G_{\mathbb{R}}/K$$

is a Riemannian submersion, together with some basic results in the book of Grauert–Remmert [5]. .

Lemma 3.2. *The restricted map*

$$P_+|_{\Phi_m^H(\mathcal{T}_m^H)} : N_+ \cap \Phi_m^H(\mathcal{T}_m^H) \to \exp(\mathfrak{p}_+) \cap D$$

is a finite holomorphic map onto its image, i.e. it is a proper holomorphic map with finite fibers.

Combining the previous results we get the boundedness of the period map restricted to $\check{\mathcal{T}}$.

Theorem 3.3. *The image of the restriction of the period map*

$$\Phi : \check{\mathcal{T}} \to N_+ \cap D$$

is bounded in N_+ with respect to the Euclidean metric on N_+.

Let $p \in \mathcal{T}$ be the base point with $\Phi(p) = \{F_p^n \subseteq F_p^{n-1} \subseteq \cdots \subseteq F_p^0\}$. Let $q \in \mathcal{T}$ be any point with $\Phi(q) = \{F_q^n \subseteq F_q^{n-1} \subseteq \cdots \subseteq F_q^0\}$. By using linear algebra, we prove that $\Phi(q) \in N_+$ if and only if F_q^k is isomorphic to F_p^k for all $0 \leq k \leq n$. From this we get the following lemma.

Lemma 3.4. *The subset $\check{\mathcal{T}}$ is an open dense submanifold in $\mathcal{T}$, and $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.*

The Riemann extension theorem immediately gives us the following needed boundedness.

Corollary 3.5. *The image of*

$$\Phi : \mathcal{T} \to D$$

lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.

Recall that in Proposition 2.8 in Section 2.4, we have proved that

$$\Phi_m : \mathcal{T}_m \to D$$

is holomorphic with

$$\Phi_m(\mathcal{T}_m) = \Phi_m^H(i_m(\mathcal{T})) = \Phi(\mathcal{T})$$

and $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}_m^H \setminus \mathcal{T}_m) \geq 1$. Moreover, from the argument following Proposition 2.8, we get that $\mathcal{T}_m^H \setminus \mathcal{T}_m$ is an analytic subvariety of $\mathcal{T}_m^H$.

On the other hand, Corollary 3.5 implies that the image of

$$\Phi_m : \mathcal{T}_m \to N_+ \cap D$$

is bounded in N_+ which implies the following easy corollary.

Corollary 3.6. *The image of*

$$\Phi_m^H : \mathcal{T}_m^H \to D$$

lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.

4. Affine structures and injectivity of extended period map

In Section 4.1, we introduce the abelian subalgebra $\mathfrak{a}$, the abelian Lie group $A = \exp(\mathfrak{a})$, and the projection map

$$P : N_+ \cap D \to A \cap D.$$

Then from local Torelli for Calabi–Yau manifolds, we get that the holomorphic map

$$\Psi : \mathcal{T} \to A \cap D \subset A \simeq \mathbb{C}^N$$

is nondegenerate, therefore defines a global affine structure on $\mathcal{T}$.

In Section 4.2, we consider the extended period map

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D,$$

where $\Psi_m^H = P \circ \Phi_m^H$. Then by using the affine structure, local Torelli for Calabi–Yau manifolds and extension of Hodge bundles we get that Ψ_m^H is nondegenerate and hence defines a global affine structure on $\mathcal{T}_m^H$. A lemma of Griffiths-Wolf in [10] tells us that the completeness of $\mathcal{T}_m^H$ with Hodge metric implies that

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D$$

is a covering map. In Section 4.3, we explain the idea to prove that Ψ_m^H is an injection by using the holomorphic affine structure on $\mathcal{T}_m^H$. As a corollary, we get that the holomorphic map Φ_m^H is an injection.

4.1. Affine structure on the Teichmüller space

Let us consider

$$\mathfrak{a} = d\Phi_p(\mathrm{T}_p^{1,0}\mathcal{T}) \subseteq \mathrm{T}_o^{1,0} D \simeq \mathfrak{n}_+$$

where p is the base point in $\mathcal{T}$ with $\Phi(p) = o$. Then by Griffiths transversality, $\mathfrak{a} \subseteq \mathfrak{g}^{-1,1}$ is an abelian subalgebra of $\mathfrak{n}_+$ determined by the tangent map of the period map

$$d\Phi : \mathrm{T}^{1,0}\mathcal{T} \to \mathrm{T}^{1,0} D.$$

Consider the corresponding Lie group

$$A \triangleq \exp(\mathfrak{a}) \subseteq N_+.$$

Then A can be considered as a complex Euclidean subspace of N_+ with the induced Euclidean metric from N_+.

Define the projection map

$$P : N_+ \cap D \to A \cap D$$

by

$$P = \exp \circ\, p \circ \exp^{-1}$$

where $\exp^{-1} : N_+ \to \mathfrak{n}_+$ is the inverse of the isometry

$$\exp : \mathfrak{n}_+ \to N_+,$$

and

$$p : \mathfrak{n}_+ \to \mathfrak{a}$$

is the projection map from the complex Euclidean space $\mathfrak{n}_+$ to its Euclidean subspace $\mathfrak{a}$.

The period map $\Phi : \mathcal{T} \to N_+ \cap D$ composed with the projection map P gives a holomorphic map

$$\Psi : \mathcal{T} \to A \cap D \tag{23}$$

where $\Psi = P \circ \Phi$. Similarly we define

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D$$

by $\Psi_m^H = P \circ \Phi_m^H$. We will prove that the map in (23) defines a global affine structure on the Teichmüller space $\mathcal{T}$. First we review the definition of complex affine structure on a complex manifold.

Definition 4.1. Let M be a complex manifold of complex dimension n. If there is a coordinate cover $\{(U_i,\, \varphi_i);\, i \in I\}$ of M such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$ whenever $U_i \cap U_k$ is not empty, then $\{(U_i,\, \varphi_i);\, i \in I\}$ is called a complex affine coordinate cover on M and it defines a holomorphic affine structure on M.

First we have the following theorem. We refer the reader to [15], Theorem 3.2, for its proof.

Theorem 4.2. *The holomorphic map*

$$\Psi : \mathcal{T} \to A \cap D \subset A \simeq \mathbb{C}^N$$

is nondegenerate, therefore defines a global holomorphic affine structure on $\mathcal{T}$.

We remark that this affine structure on $\mathcal{T}$ depends on the choice of the base point p. Affine structures on $\mathcal{T}$ defined in this ways by fixing different base point may not be compatible with each other.

4.2. Affine structure on the Hodge metric completion space

First recall the diagram (20):

(24)
$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_m"] \arrow[d, "\pi_m"] & \mathcal{T}_m^H \arrow[r, "\Phi_m^H"] \arrow[d, "\pi_m^H"] & D \arrow[d, "\pi_D"] \\ \mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}_m^H \arrow[r, "\Phi_{\mathcal{Z}_m}^H"] & D/\Gamma. \end{tikzcd}$$

By Corollary 3.6, we define the holomorphic map

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D$$

by composing the extended period map $\Phi_m^H : \mathcal{T}_m^H \to N_+ \cap D$ with the projection map $P : N_+ \cap D \to A \cap D$. We also define the holomorphic map

$$\Psi_m : \mathcal{T}_m \to A \cap D$$

by restricting $\Psi_m = \Psi_m^H|_{\mathcal{T}_m}$, which is given by $\Psi_m = P \circ \Phi_m$.

Next, recall that $\mathcal{T}_m \subset \mathcal{T}_m^H$ is an open complex submanifold of $\mathcal{T}_m^H$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}_m^H \setminus \mathcal{T}_m) \geq 1$, i_m is a covering map onto $\mathcal{T}_m$, and

$$\mathcal{T}_m = i_m(\mathcal{T}) = (\pi_m^H)^{-1}(\mathcal{Z}_m).$$

We can choose a small neighborhood U of any point in $\mathcal{T}_m$ such that

$$\pi_m^H : U \to V = \pi_m^H(U) \subset \mathcal{Z}_m,$$

is a biholomorphic map. We can shrink U and V simultaneously such that $\pi_m^{-1}(V) = \cup_\alpha W_\alpha$ and $\pi_m : W_\alpha \to V$ is also biholomorphic. Choose any W_α

and denote it by $W = W_\alpha$. Then $i_m : W \to U$ is a biholomorphic map. Since

$$\Psi = \Psi_m^H \circ i_m = \Psi_m \circ i_m,$$

we have $\Psi|_W = \Psi_m|_U \circ i_m|_W$. Theorem 4.2 implies that $\Psi|_W$ is biholomorphic onto its image, if we shrink W, V and U again. Therefore

$$\Psi_m|_U : U \to A \cap D$$

is biholomorphic onto its image. By pulling back the affine coordinate chart in $A \simeq \mathbb{C}^N$, we get an induced affine structure on $\mathcal{T}_m$ such that Ψ_m is an affine map.

In conclusion, we have the following lemma.

Lemma 4.3. *The holomorphic map*

$$\Psi_m : \mathcal{T}_m \to A \cap D$$

is a local embedding. In particular, Ψ_m defines a global holomorphic affine structure on $\mathcal{T}_m$.

Next we will show that the affine structure induced by

$$\Psi_m : \mathcal{T}_m \to A \cap D$$

can be extended to a global affine structure on $\mathcal{T}_m^H$, which is precisely induced by the extended period map

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D.$$

Definition 4.4. Let M be a complex manifold and $N \subset M$ a cloesd subset. Let $E_0 \to M \setminus N$ be a holomorphic vector bundle. Then E_0 is called holomorphically trivial along N, if for any point $x \in N$, there exists an open neighborhood U of x in M such that $E_0|_{U \setminus N}$ is holomorphically trivial.

We need to following elementary lemma, which is Proposition 4.4 from [18], to proceed.

Lemma 4.5. *The holomorphic vector bundle $E_0 \to M \setminus N$ can be extended to a unique holomorphic vector bundle $E \to M$ such that $E|_{M \setminus N} = E_0$, if and only if E_0 is holomorphically trivial along N.*

On the period domain D we have the Hodge bundle which is the horizontal subbundle of the tangent bundle of D,

$$\bigoplus_{k=1}^{n} \mathrm{Hom}(F^k/F^{k+1}, F^{k-1}/F^k), \tag{25}$$

in which the differentials of the period maps Φ, Φ_m^H and Φ_m take values. From local Torelli for Calabi–Yau manifolds, we know that the image of the differential of the period map at each point is the Hodge subbundle $\mathrm{Hom}(F^n, F^{n-1}/F^n)$. On the other hand, as the variation of Hodge structure from geometry, the Hodge bundles naturally exist on $\mathcal{T}$ and $\mathcal{T}_m$.

Let us denote the restriction to $A \cap D$ of the Hodge subbundle $\mathcal{H} = \mathrm{Hom}(F^n, F^{n-1}/F^n)$ on D by

$$\mathcal{H}_A = \mathrm{Hom}(F^n, F^{n-1}/F^n)|_{A\cap D}.$$

By the definition of A, the holomorphic tangent bundle of $A \cap D$ is naturally isomorphic to $\mathcal{H}_A$ as described explicitly in the proof of Theorem 4.2 in [15]. Theorem 4.2 and Lemma 4.3 give the natural isomorphisms of the holomorphic vector bundles over $\mathcal{T}$ and $\mathcal{T}_m$ respectively

$$d\Psi : \mathrm{T}^{1,0}\mathcal{T} \simeq \Psi^*\mathcal{H}_A,$$
$$d\Psi_m : \mathrm{T}^{1,0}\mathcal{T}_m \simeq \Psi_m^*\mathcal{H}_A.$$

We remark that the period map $\Phi_{\mathcal{Z}_m}^H : \mathcal{Z}_m^H \to D/\Gamma$ can be lifted to the universal cover to get $\Phi_m^H : \mathcal{T}_m^H \to D$ due to the fact that around any point in $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$, the Picard-Lefschetz transformation, or equivalently, the monodromy is trivial, which is given in Lemma 2.7. Hence the Hodge bundle $(\Psi_m^H)^*\mathcal{H}_A$ is the natural extension of $\Psi_m^*\mathcal{H}_A$ over $\mathcal{T}_m^H$. More precisely we have the following lemma.

Lemma 4.6. *The isomorphism* $\mathrm{T}^{1,0}\mathcal{T}_m \simeq \Psi_m^*\mathcal{H}_A$ *of holomorphic vector bundles over* $\mathcal{T}_m$ *has a unique extension to an isomorphism of holomorphic vector bundles over* $\mathcal{T}_m^H$ *with*

$$\mathrm{T}^{1,0}\mathcal{T}_m^H \simeq (\Psi_m^H)^*\mathcal{H}_A.$$

With the same notation $\mathcal{H} = \mathrm{Hom}(F^n, F^{n-1}/F^n)$ to denote the corresponding Hodge bundles on $\mathcal{T}_m$ and $\mathcal{T}_m^H$, the above lemma simply tells us that the isomorphism of bundles $\mathrm{T}^{1,0}\mathcal{T}_m \simeq \mathcal{H}$ on $\mathcal{T}_m$ extends to isomorphism on $\mathcal{T}_m^H$,

$$\mathrm{T}^{1,0}\mathcal{T}_m^H \simeq \mathcal{H}.$$

Then we can prove the following theorem by using the extension of the period map and the fact that $\Psi_m : \mathcal{T}_m \to A \cap D$ is an affine map.

Theorem 4.7. *The holomorphic map*

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D$$

is nondegenerate. Hence Ψ_m^H defines a global affine structure on $\mathcal{T}_m^H$.

For the proof of the above theorem, see Theorem 3.7 in [15]. In the remark following the proof there, one can also see the geometric origin of Theorem 4.7 due to the special feature of the period map of Calabi–Yau manifolds.

Recall the following lemma due to Griffiths and Wolf, which is proved as Corollary 2 in [10].

Lemma 4.8. *Let $f : X \to Y$ be a local diffeomorphism of connected Riemannian manifolds. Assume that X is complete for the induced metric. Then $f(X) = Y$, f is a covering map and Y is complete.*

This lemma, together with Theorem 4.7 proved above, gives the following corollary.

Corollary 4.9. *The holomorphic map*

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D$$

is a universal covering map, and the image $\Psi_m^H(\mathcal{T}_m^H) = A \cap D$ is complete with respect to the Hodge metric.

It is important to note that the flat connections which correspond to the global holomorphic affine structures on $\mathcal{T}$, on $\mathcal{T}_m$ or on $\mathcal{T}_m^H$ are in general not compatible with the corresponding Hodge metrics on them.

4.3. Injectivity of the period map on the Hodge metric completion space

The main purpose of this section is Theorem 4.10 stated below. The idea of proof is to show directly that $A \cap D$ is simply connected, which together with Corollary 4.9 implies the theorem.

Theorem 4.10. *For any $m \geq 3$, the holomorphic map*

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D$$

is an injection and hence a biholomorphic map.

Proof. As explicitly described in the proof of Lemma 3.2 as given in [15], the natural projection

$$\pi : D \to G_{\mathbb{R}}/K,$$

when restricted to the underlying real manifold of $\exp(\mathfrak{p}_+) \cap D$, is given by the diffeomorphism

$$\pi_+ : \exp(\mathfrak{p}_+) \cap D \longrightarrow \exp(\mathfrak{p}_0) \xrightarrow{\simeq} G_{\mathbb{R}}/K, \tag{26}$$

which is defined by mapping $\exp(Y)\bar{o}$ to $\exp(X)\bar{o}$ for $Y \in \mathfrak{p}_+$ and $X \in \mathfrak{p}_0$ with the relation that

$$X = T_0(Y + \tau_0(Y))$$

for some real number T_0. Here $\bar{o}$ is the base point in $\exp(\mathfrak{p}_+) \cap D$, and $\exp(Y)\bar{o}$ and $\exp(X)\bar{o}$ denote the left translations.

By Griffiths transversality, one has

$$\mathfrak{a} \subset \mathfrak{g}^{-1,1} \subset \mathfrak{p}_+ \text{ and } \mathfrak{a}_0 = \mathfrak{a} + \tau_0(\mathfrak{a}) \subset \mathfrak{p}_0.$$

Then $A \cap D$ is a submanifold of $\exp(\mathfrak{p}_+) \cap D$, and the diffeomorphism π_+ maps $A \cap D \subseteq \exp(\mathfrak{p}_+) \cap D$ diffeomorphically to its image $\exp(\mathfrak{a}_0)$ inside $G_{\mathbb{R}}/K$, from which one has the diffeomorphism

$$A \cap D \simeq \exp(\mathfrak{a}_0)$$

induced by π_+. Since $\exp(\mathfrak{a}_0)$ is simply connected, one concludes that $A \cap D$ is also simply connected.

Now since $\mathcal{T}_m^H$ is simply connected and

$$\Psi_m^H : \mathcal{T}_m^H \to A \cap D$$

is a covering map, we conclude that Ψ_m^H must be a biholomorphic map. □

Since $\Psi_m^H = P \circ \Phi_m^H$, we also have the following corollary.

Corollary 4.11. *The extended period map*

$$\Phi_m^H : \mathcal{T}_m^H \to N_+ \cap D$$

is an injection.

5. Global Torelli

In this section, we first explain the proof that $\mathcal{T}_m^H$ does not depend on the choice of the level m, so that we can denote the Hodge metric completion space $\mathcal{T}^H$ by $\mathcal{T}^H = \mathcal{T}_m^H$, and the extended period map Φ^H by $\Phi^H = \Phi_m^H$ for any $m \geq 3$. Therefore $\mathcal{T}^H$ is a complex affine manifold and that Φ^H is a holomorphic injection.

As a direct corollary, we derive the global Torelli theorem for the injectivity of the period maps from the Torelli space $\mathcal{T}'$ and from its completion space $\mathcal{T}^H$ to the period domain. Another corollary is the global Torelli theorem on the moduli space of polarized Calabi–Yau manifolds with level m structure for any $m \geq 3$.

For any two integers $m, m' \geq 3$, let $\mathcal{Z}_m$ and $\mathcal{Z}_{m'}$ be the smooth quasi-projective manifolds as in Theorem 2.3 and let $\mathcal{Z}_m^H$ and $\mathcal{Z}_{m'}^H$ be their completions with respect to the Hodge metric. Let $\mathcal{T}_m^H$ and $\mathcal{T}_{m'}^H$ be the universal cover spaces of $\mathcal{Z}_m^H$ and $\mathcal{Z}_{m'}^H$ respectively. From Theorem 4.10, we know that both $\mathcal{T}_m^H$ and $\mathcal{T}_{m'}^H$ are biholomorphic to $A \cap D$. Hence we have the following proposition.

Proposition 5.1. *For any $m \geq 3$, the complex manifold T_m^H is complete equipped with the Hodge metric, and is biholomorphic to $A \cap D$. So for any integers m, $m' \geq 3$, the complex manifolds $\mathcal{T}_m^H$ and $\mathcal{T}_{m'}^H$ are biholomorphic to each other.*

Proposition 5.1 shows that $\mathcal{T}_m^H$ is independent of the choice of the level m structure, and it allows us to introduce the following notations.

We define the complex manifold $\mathcal{T}^H = \mathcal{T}_m^H$, the holomorphic map

$$i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}^H$$

by $i_{\mathcal{T}} = i_m$, and the extended period map

$$\Phi^H : \mathcal{T}^H \to D$$

by $\Phi^H = \Phi^H_m$ for any $m \geq 3$. In particular, with these new notations, we have the commutative diagram

$$
\begin{tikzcd}
\mathcal{T} \arrow[r, "i_{\mathcal{T}}"] \arrow[d, "\pi_m"] & \mathcal{T}^H \arrow[r, "\Phi^H"] \arrow[d, "\pi_m^H"] & D \arrow[d, "\pi_D"] \\
\mathcal{Z}_m \arrow[r, "i"] & \mathcal{Z}_m^H \arrow[r, "\Phi^H_{\mathcal{Z}_m}"] & D/\Gamma.
\end{tikzcd}
\tag{27}
$$

The main result of this section is the following,

Theorem 5.2. *The complex manifold $\mathcal{T}^H$ is a complex affine manifold which can be embedded into $A \simeq \mathbb{C}^N$ and it is the completion space of the Torelli space $\mathcal{T}'$ with respect to the Hodge metric. Moreover, the extended period map*

$$\Phi^H : \mathcal{T}^H \to N_+ \cap D$$

is a holomorphic injection.

Proof. By the definition of $\mathcal{T}^H$ and Theorem 4.10, it is easy to see that $\mathcal{T}^H$ is a complex affine manifold, which can be embedded into $A \simeq \mathbb{C}^N$. It is also not hard to see that the injectivity of Φ^H follows directly from Corollary 4.11 by the definition of Φ^H. Now we only need to show that $\mathcal{T}^H$ is the Hodge metric completion space of $\mathcal{T}'$, which follows from the following lemma. □

The proof of the following lemma depends crucially on the level structures. See Lemma 4.3 in [15] for its proof.

Lemma 5.3. *Let $\mathcal{T}_0 \subset \mathcal{T}^H$ be defined by $\mathcal{T}_0 := i_{\mathcal{T}}(\mathcal{T})$. Then $\mathcal{T}_0$ is biholomorphic to the Torelli space $\mathcal{T}'$.*

From this we get directly the following global Torelli theorem on the Torelli space.

Corollary 5.4 (Global Torelli theorem). *The period map*

$$\Phi' : \mathcal{T}' \to D$$

is injective.

As another direct corollary, we have the global Torelli theorem on the moduli space $\mathcal{Z}_m$ of polarized Calabi–Yau manifolds with level m structure for any $m \geq 3$.

Corollary 5.5. *The period map*

$$\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$$

is injective.

Corollary 5.5 is derived from Corollary 5.4 by using the equivariance of the period map $\Phi^H : \mathcal{T}^H \to D$ with respect to the induced action of the fundamental group $\pi_1(\mathcal{Z}_m^H)$ on $\mathcal{T}^H$ and the action of the monodromy group Γ on D. See Corollary 4.5 of [15] for the detail.

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