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The SYZ mirror symmetry and the BKMP remodeling conjecture

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The Remodeling Conjecture proposed by Bouchard-Klemm-Mariño-Pasquetti (BKMP) relates the A-model open and closed topological string amplitudes (open and closed Gromov-Witten invariants) of a symplectic toric Calabi-Yau 3-fold to Eynard-Orantin invariants of its mirror curve. The Remodeling Conjecture can be viewed as a version of all genus open-closed mirror symmetry. The SYZ conjecture explains mirror symmetry as T-duality. After a brief review on SYZ mirror symmetry and mirrors of symplectic toric Calabi-Yau 3-orbifolds, we give a non-technical exposition of our results on the Remodeling Conjecture for symplectic toric Calabi-Yau 3-orbifolds. In the end, we apply SYZ mirror symmetry to obtain the descendent version of the all genus mirror symmetry for toric Calabi-Yau 3-orbifolds.

1. Introduction

1.1. The SYZ conjecture

Mirror symmetry relates the A-model on a Calabi-Yau n-fold $\mathcal{X}$, defined by the symplectic structure on $\mathcal{X}$, to the B-model on a mirror Calabi-Yau n-fold $\check{X}$, defined by the complex structure on $\check{X}$. Strominger-Yau-Zaslow proposed that Mirror Symmetry is T-duality [110] in the following sense. There exist $\pi : \mathcal{X} \to B$ and $\check{\pi} : \check{\mathcal{X}} \to B$, where $\dim_{\mathbb{R}} B = n$, such that over a dense open subset $U \subset B$, $Y := \pi^{-1}(U) \to U$ and $\check{Y} := \check{\pi}^{-1}(U) \to U$ are dual special Lagrangian n-torus fibrations. In the semi-flat case, $U = B$, $Y \cong T^*B/\Lambda$ as a symplectic manifold and $\check{Y} \cong TB/\Lambda^\vee$ as a complex manifold, where Λ and $\Lambda^\vee$ are dual lattices, and for each $b \in B$, $\pi^{-1}(b) = T_b^*B/\Lambda$ and $\check{\pi}^{-1}(b) = T_bB/\Lambda^\vee$. In the general case, there is quantum correction to the complex structure on $\check{Y}$ coming from holomorphic disks in $\mathcal{X}$ bounded by Lagrangian n-tori $\pi^{-1}(b)$, $b \in U$.

Mirror symmetry has been extended to certain non-Calabi-Yau manifolds. When $\mathcal{X}$ is Fano (or more generally semi-Fano), the mirror is a Landau-Ginzburg model, which can also be constructed by SYZ transformation.

1.2. SYZ mirror symmetry for toric Calabi-Yau 3-folds

Let $(\mathcal{X}, \omega)$ be a symplectic toric Calabi-Yau 3-manifold/orbifold, where ω is the symplectic form. There are two families of mirrors, both of which can be constructed by the SYZ transformation.

Landau-Ginzburg mirror. The mirror B-model to the toric Calabi-Yau 3-orbifold $(\mathcal{X}, \omega)$ is a 3-dimensional Landau-Ginzburg model on $(\mathbb{C}^*)^3$ given by the superpotential

$$W = H(X, Y, q)Z.$$

The Calabi-Yau condition ensures that W is in this form. The complex parameter q is related to the Kähler parameter of $\mathcal{X}$ by the mirror map. The Landau-Ginzburg mirror $((\mathbb{C}^*)^3, W)$ can be constructed by applying SYZ transformation to $\mu : \mathcal{X} \to \Delta \subset \mathbb{R}^3$, where μ is the moment map of the Hamiltonian $U(1)^3$-action on $(\mathcal{X}, \omega)$, and $\Delta = \mu(\mathcal{X})$ is the moment polyhedron.

Hori-Vafa mirror. By Hori-Vafa [66], the mirror of $(\mathcal{X}, \Omega)$ is a non-compact Calabi-Yau 3-fold $(\check{\mathcal{X}}, \Omega)$, where

$$\check{\mathcal{X}} = \{(u, v, X, Y) : u, v \in \mathbb{C}, X, Y \in \mathbb{C}^*, uv = H(X, Y)\}$$

is a hypersurface in $\mathbb{C}^2 \times (\mathbb{C}^*)^2$, and

$$\Omega = \mathrm{Res}_{\check{\mathcal{X}}} \left(\frac{1}{uv - H(X, Y)} du \wedge dv \wedge \frac{dX}{X} \wedge \frac{dY}{Y} \right)$$

is a holomorphic 3-form on $\check{\mathcal{X}}$. The Hori-Vafa mirror $(\check{\mathcal{X}}, \Omega)$ can be constructed by applying SYZ transformation to the Gross fibration [26, 27].

The two "equivalent" mirrors come with no surprise since as a toric variety $\mathcal{X}$ should have a Landau-Ginzburg mirror, while as a Calabi-Yau

3-fold Hori-Vafa showed that they both could be reduced to a mirror curve

$$C_q = \{(X, Y) \in (\mathbb{C}^*)^2 : H(X, Y, q) = 0\} \subset (\mathbb{C}^*)^2.$$

In particular

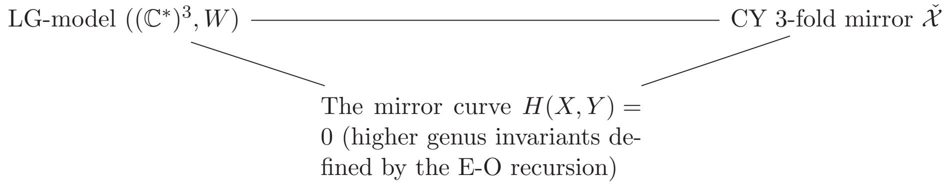
$$\int_\Gamma e^{-W} \frac{dXdYdZ}{XYZ} = \int_{\widetilde{\Gamma}} \Omega = \int_\gamma y dx, \tag{1.1}$$

where $x = -\log X$, $y = -\log Y$, and Lagrangian cycles $\Gamma \subset (\mathbb{C}^*)^3$, $\widetilde{\Gamma} \subset \check{\mathcal{X}}$, $\gamma \subset C_q$ are related by a series of dimensional reductions.

1.3. The BKMP remodeling conjecture

It is usually difficult to obtain higher genus invariants of the B-model. A standard way is to apply the BCOV holomorphic anomaly equations [15]. These equations do not have unique solutions (holomorphic ambiguity). One needs to fix the boundary conditions via the input from mirror symmetry, like Gromov-Witten invariants in low degrees in the large radius limit point, and to utilize the so-called “gap conditions” at the conifold point. To mathematically prove the all genus mirror symmetry under this B-model approach is currently beyond reach since the A-side theory of BCOV (and the A-side theory at the conifold point) is still a mystery to mathematicians.

The Eynard-Orantin topological recursion is an algorithm which produces higher genus invariants for a spectral curve [46]. Applying the Eynard-Orantin topological recursion to the mirror curve C_q of a symplectic toric Calabi-Yau 3-orbifold $(\mathcal{X}, \omega)$, we obtain a version of the B-model, related but not a priori the same as the BCOV theory on the Hori-Vafa mirror $(\check{\mathcal{X}}, \Omega)$ of $(\mathcal{X}, \omega)$. The Bouchard-Klemm-Mariño-Pasquetti (BKMP) remodeling conjecture [20, 21] says that all genus B-model topological strings on $(\check{\mathcal{X}}, \Omega)$ are essentially Eynard-Orantin invariants [46] of C_q. Using mirror symmetry, BKMP relates the Eynard-Orantin invariant $\omega_{g,n}$ of the mirror curve to a generating function $F_{g,n}^{\mathcal{X},\mathcal{L}}$ of open GW invariants (A-model topological open string amplitudes) counting holomorphic maps from bordered Riemann surfaces with g handles and n holes to $\mathcal{X}$ with boundaries in an Aganagic-Vafa Lagrangian brane $\mathcal{L}$. The correspondence between $\omega_{g,n}$ and $F_{g,n}^{\mathcal{X},\mathcal{L}}$ can be extend to the case when $n = 0$. In this case, $\omega_{g,0}$ is defined to be the free energy and the A-model potential becomes the closed Gromov-Witten potential $F_g^{\mathcal{X}}$. Therefore, the remodeling conjecture gives us an all genus open-closed mirror symmetry for toric Calabi-Yau 3-orbifolds. The three equivalent B-models mirror to $\mathcal{X}$ are illustrated as below.



The BKMP remodeling conjecture was proved for $\mathbb{C}^3$ at all genus g independently by L. Chen [28] and J. Zhou [116] in the $n > 0$ case (open string sector), and by Bouchard-Catuneanu-Marchal-Sułkowski [19] in the $n = 0$ case (closed string sector). In 2012, Eynard-Orantin provided a proof of the BKMP remodeling conjecture for all symplectic smooth toric Calabi-Yau 3-folds [47].

The SYZ T-duality transformation [9, 78, 110] associates a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ to a Lagrangian cycle $\mathrm{SYZ}(\mathcal{F}) \subset (\mathbb{C}^*)^3$. A coherent sheaf on $\mathcal{X}$ is admissible if there exists $\gamma(\mathcal{F}) \subset C_q$ such that

$$\int_{\mathrm{SYZ}(\mathcal{F})} e^{\frac{W+x+fy}{z}} \frac{dXdYdZ}{XYZ} = \int_{\gamma(\mathcal{F})} e^{\frac{x+fy}{z}} ydx$$

and $x + fy$ is bounded below on $\gamma(\mathcal{F})$. Here f is an integer and z is a negative real number, so that the integral on the right hand side converges. The remodeling conjecture has a descendant version: given n admissible coherent sheaves $\mathcal{F}_1, \ldots, \mathcal{F}_n$ on $\mathcal{X}$, the Laplace transform of $\omega_{g,n}$ along $\gamma(\mathcal{F}_1), \ldots, \gamma(\mathcal{F}_n)$ is a generating function of genus g descendant Gromov-Witten invariants with n insertions $\kappa(\mathcal{F}_1), \ldots, \kappa(\mathcal{F}_n)$, where $\kappa(\mathcal{F}_i)$ is the so-called Gamma class of $\mathcal{F}_i$.

In the rest of this paper, we will give a non-technical exposition of our results on the remodeling conjecture for toric Calabi-Yau 3-orbifolds [51, 52], which is a version of all genus open-closed mirror symmetry. We will also discuss the all genus mirror symmetry of free energies and descendant potentials.

2. Toric Calabi-Yau 3-orbifolds and their mirror curves

2.1. Toric Calabi-Yau 3-orbifolds

A Calabi-Yau 3-fold X is toric if it contains the algebraic torus $\mathbb{T} = (\mathbb{C}^*)^3$ as a Zariski dense open subset, and the action of $\mathbb{T}$ on itself extends to

X. All Calabi-Yau 3-folds are non-compact. There is a rank 2 subtorus $\mathbb{T}' \subset \mathbb{T}$ which acts trivially on the canonical line bundle of X. We call $\mathbb{T}'$ the Calabi-Yau torus. Then $\mathbb{T} \cong \mathbb{T}' \times \mathbb{C}^*$. Let $\mathbb{T}'_{\mathbb{R}} \cong U(1)^2$ be the maximal compact subgroup of $\mathbb{T}'$.

Let $M' = \mathrm{Hom}(\mathbb{T}', \mathbb{C}^*) \cong \mathbb{Z}^2$ and $N' = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}')$ be the character lattice and the cocharacter lattice of $\mathbb{T}'$, respectively. Then M' and N' are dual lattices. Let X_Σ be a toric Calabi-Yau 3-fold defined by a simplicial fan $\Sigma \subset N'_{\mathbb{R}} \times \mathbb{R}$, where $N'_{\mathbb{R}} := N' \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$ can be identified with the Lie algebra of $\mathbb{T}'_{\mathbb{R}}$. Then X_Σ has at most quotient singularities. We assume that X_Σ is semi-projective, i.e., X_Σ contains at least one $\mathbb{T}$ fixed point, and X_Σ is projective over its affinization $X_0 := \mathrm{Spec} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma})$. Then the support of the fan Σ is a strongly convex rational polyhedral cone $\sigma_0 \subset N'_{\mathbb{R}} \times \mathbb{R} \cong \mathbb{R}^3$, and X_0 is the affine toric variety defined by the 3-dimensional cone σ_0. There exists a convex polytope $P \subset N'_{\mathbb{R}} \cong \mathbb{R}^2$ with vertices in the lattice $N' \cong \mathbb{Z}^2$, such that σ_0 is the cone over $P \times \{1\} \subset N'_{\mathbb{R}} \times \mathbb{R}$, i.e. $\sigma_0 = \{(tx, ty, t) : (x, y) \in P, t \in [0, \infty)\}$. The fan Σ determines a triangulation of P: the 1-dimensional, 2-dimensional, and 3-dimensional cones in Σ are in one-to-one correspondence with the vertices, edges, and faces of the triangulation of P, respectively. This triangulation of P is known as the toric diagram or the dual graph of the simplicial toric Calabi-Yau 3-fold X_Σ.

Let $\Sigma(d)$ be the set of d-dimensional cones in Σ, and let $p = |\Sigma(1)| - 3$. Then X_Σ is a GIT quotient

$$X_\Sigma = \mathbb{C}^{3+p} /\!\!/ G_\Sigma = (\mathbb{C}^{3+p} - Z_\Sigma)/G_\Sigma$$

where G_Σ is a p-dimensional subgroup of $(\mathbb{C}^*)^{3+p}$ and Z_Σ is a Zariski closed subset of $\mathbb{C}^{3+p}$ determined by the fan Σ. If X_Σ is a smooth toric Calabi-Yau 3-fold then $G_\Sigma \cong (\mathbb{C}^*)^p$ and G_Σ acts freely on $\mathbb{C}^{3+p} - Z_\Sigma$. In general we have $(G_\Sigma)_0 \cong (\mathbb{C}^*)^p$, where $(G_\Sigma)_0$ is the connected component of the identity, and the stabilizers of the G_Σ-action on $\mathbb{C}^{3+p} - Z_\Sigma$ are at most finite and generically trivial. The stacky quotient

$$\mathcal{X} = [(\mathbb{C}^{3+p} - Z_\Sigma)/G_\Sigma]$$

is a toric Calabi-Yau 3-orbifold; it is a toric Deligne-Mumford stack in the sense of Borisov-Chen-Smith [18].

2.2. Toric crepant resolution and extended Kähler classes

Given a semi-projective simplicial toric Calabi-Yau 3-fold X_Σ which is not smooth, there exists a subdivision Σ' of Σ, such that

$$X_{\Sigma'} = (\mathbb{C}^{3+p+s} - Z_{\Sigma'})/G_{\Sigma'} \longrightarrow X_\Sigma = \big((\mathbb{C}^{3+p} - Z_\Sigma) \times (\mathbb{C}^*)^s\big)/G_{\Sigma'}.$$

is a crepant toric resolution, where $X_{\Sigma'}$ is a smooth toric Calabi-Yau 3-fold, $s = |\Sigma'(1)| - |\Sigma(1)|$, and $G_{\Sigma'} \cong (\mathbb{C}^*)^{p+s}$. $X_{\Sigma'}$ and X_Σ are GIT quotients of the same $G_{\Sigma'}$-action on $\mathbb{C}^{3+p+s}$ with respect to different stability conditions.

Let $K_{\Sigma'} \cong U(1)^{p+s}$ be the maximal compact subgroup of $G_{\Sigma'} \cong (\mathbb{C}^*)^{p+s}$. The $G_{\Sigma'}$-action on $\mathbb{C}^{3+p+s}$ restricts to a Hamiltonian $K_{\Sigma'}$-action on the Kähler manifold $(\mathbb{C}^{3+p+s}, \omega_0 = \sqrt{-1}\sum_{i=1}^{3+p+s} dz_i \wedge d\bar{z}_i)$, with moment map $\widetilde{\mu} : \mathbb{C}^{3+p+s} \to \mathbb{R}^{p+s}$. There exist two (open) cones C and C' in $\mathbb{R}^{p+s}$ such that

$$\begin{aligned} &\tilde{\mu}^{-1}(\vec{r})/K_{\Sigma'} \\ &= \begin{cases} (\mathbb{C}^{3+p+s} - Z_{\Sigma'})/G_{\Sigma'} = X_{\Sigma'}, & \vec{r} \in C', \\ \big((\mathbb{C}^{3+p} - Z_\Sigma) \times (\mathbb{C}^*)^s\big)/G_{\Sigma'} = (\mathbb{C}^{3+p} - Z_\Sigma)/G_\Sigma = X_\Sigma, & \vec{r} \in C \end{cases} \end{aligned}$$

$C' \subset \mathbb{R}^{p+s} = H^2(X_{\Sigma'}; \mathbb{R})$ is the Kähler cone of $X_{\Sigma'}$ and $C \subset \mathbb{R}^{p+s}$ is the extended Kähler cone of X_Σ.

The parameter $\vec{r} \in C$ determines a Kähler form $\omega(\vec{r})$ on the toric Calabi-Yau 3-orbifold $\mathcal{X} = [(\mathbb{C}^{3+p} - Z_\Sigma)/G_\Sigma]$. The $p+s$ parameters $\vec{r} = (r_1, \ldots, r_{p+s})$ are extended Kähler parameters of $\mathcal{X}$, where $r_1, \ldots, r_p$ are Kähler parameters of $\mathcal{X}$. The A-model closed string flat coordinates are complexified extended Kähler parameters

$$\tau_a = -r_a + \sqrt{-1}\theta_a, \quad a = 1, \ldots, p+s.$$

2.3. Toric graphs

The action of the Calabi-Yau torus $\mathbb{T}'$ on $\mathcal{X}$ restricts to a Hamiltonian $\mathbb{T}'_{\mathbb{R}}$-action on the Kähler orbifold $(\mathcal{X}, \omega(\vec{r}))$, with moment map $\mu' : \mathcal{X} \to M'_{\mathbb{R}} = \mathbb{R}^2$. The 1-skeleton $\mathcal{X}^1$ of the toric Calabi-Yau 3-fold $\mathcal{X}$ is the union of 0-dimensional and 1-dimensional orbits of the $\mathbb{T}$-action on $\mathcal{X}$. $\Gamma := \mu'(\mathcal{X}^1) \subset \mathbb{R}^2$ is a planar trivalent graph, which is known as the toric graph of the symplectic toric Calabi-Yau 3-orbifold $(\mathcal{X}, \omega(\vec{r}))$. The toric diagram depends only on the complex structure on $\mathcal{X}$, where as the toric graph depends also on the symplectic structure of $\mathcal{X}$.

2.4. Aganagic-Vafa Lagrangian branes

An Aganagic-Vafa Lagrangian brane in a toric Calabi-Yau 3-orbifold $\mathcal{X}$ is a Lagrangian sub-orbifold of the form

$$\mathcal{L} = [\widetilde{L}/K_{\Sigma'}] \subset \mathcal{X} = [\widetilde{\mu}^{-1}(\vec{r})/K_{\Sigma'}]$$

where

$$\widetilde{L} = \Big\{ (z_1, \ldots, z_{3+p+s}) \in \widetilde{\mu}^{-1}(\vec{r}) : \sum_{i=1}^{3+p+s} \hat{l}_i^1 |z_i|^2 = c_1, \ \sum_{i=1}^{3+p+s} \hat{l}_i^2 |z_i|^2 = c_2, \ \arg(z_1 \cdots z_{3+p+s}) = c_3 \Big\},$$

c_1, c_2, c_3 are constants, and

$$\sum_{i=1}^{3+p+s} \hat{l}_i^\alpha = 0, \quad \alpha = 1, 2.$$

The compact 2-torus $\mathbb{T}'_{\mathbb{R}} \cong U(1)^2$ acts on $\mathcal{L}$, and $\mu'(\mathcal{L})$ is a point on the toric graph $\Gamma = \mu'(\mathcal{X}^1)$ which is not a vertex. $\mathcal{L}$ intersects a unique 1-dimensional $\mathbb{T}$ orbit $\mathfrak{l} \subset \mathcal{X}$. We have $\mathfrak{l} \cong \mathbb{C}^* \times \mathcal{B}\mathbb{Z}_m$ for some positive integer m. When $m = 1$, $\mathcal{L} \cong S^1 \times \mathbb{C}$ is smooth; when $m > 1$, $\mathcal{L}$ is smooth away from $\mathcal{L} \cap \mathfrak{l} \cong S^1 \times \mathcal{B}\mathbb{Z}_m$.

2.5. Chen-Ruan orbifold cohomology

Let $U = \mathbb{C}^{3+p} - Z_\Sigma$, so that $\mathcal{X} = [U/G_\Sigma]$. Given $v \in G_\Sigma$, let $U^v = \{z \in U : v \cdot z = z\}$. The inertia stack of $\mathcal{X}$ is

$$\mathcal{I}\mathcal{X} = \bigcup_{v \in \mathrm{Box}(\Sigma)} \mathcal{X}_v$$

where $\mathrm{Box}(\Sigma) = \{v \in G_\Sigma : U^v \neq \emptyset\}$ and $\mathcal{X}_v = [U^v/G_\Sigma]$.

We consider cohomology with $\mathbb{C}$-coefficient. As a graded $\mathbb{C}$-vector space, the Chen-Ruan orbifold cohomology [31] of $\mathcal{X}$ is

$$H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{C}) = \bigoplus_{v \in \mathrm{Box}(\Sigma)} H^*(\mathcal{X}_v; \mathbb{C})[2\mathrm{age}(v)], \quad \mathrm{age}(v) \in \{0, 1, 2\}.$$

Let $\mathfrak{g} := |\mathrm{Int}(P) \cap N'|$ be the number of lattice points in $\mathrm{Int}(P)$, the interior of the polytope P, and let $\mathfrak{n} := |\partial P \cap N'|$ be the number of lattice points on ∂P, the boundary of the polytope P. Then

$$\begin{aligned}
p &= |\Sigma(1)| - 3 = \dim_{\mathbb{C}} H^2(X_\Sigma; \mathbb{C}),\\
p + s &= |\Sigma'(1)| - 3 = |P \cap N'| - 3 = \dim_{\mathbb{C}} H^2(X_{\Sigma'}; \mathbb{C}) = \dim_{\mathbb{C}} H^2_{\mathrm{CR}}(\mathcal{X}; \mathbb{C})\\
&= \mathfrak{g} + \mathfrak{n} - 3,\\
\mathfrak{g} &= |\mathrm{Int}(P) \cap N'| = \dim_{\mathbb{C}} H^4(X_{\Sigma'}) = \dim_{\mathbb{C}} H^4_{\mathrm{CR}}(\mathcal{X}; \mathbb{C}),\\
\chi &= |\Sigma'(3)| = 2\mathrm{Area}(P) = \dim_{\mathbb{C}} H^*(X_{\Sigma'}; \mathbb{C}) = \dim_{\mathbb{C}} H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{C})\\
&= 1 + p + s + \mathfrak{g} = 2\mathfrak{g} - 2 + \mathfrak{n}.
\end{aligned}$$

2.6. The mirror curve

Following the notation in Section 2.5, the convex polytope $P \subset N'_{\mathbb{R}} \cong \mathbb{R}^2$ defines a polarized toric surface (S, L), where S is a toric variety and L is an ample line bundle. We have

$$\chi(S, L) = h^0(S, L) = |P \cap N'| = 3 + p + s.$$

The mirror curve $H(X, Y)$ is given by

$$H(X, Y) = \sum_{(m,n) \in P \cap N'} a_{m,n} X^m Y^n, \ a_{m,n} \in \mathbb{C}^*.$$

So $H(X, Y) \in H^0((\mathbb{C}^*)^2, \mathcal{O}_{(\mathbb{C}^*)^2})$ is the restriction of a section $s \in H^0(S, L)$. The compactified mirror curve is $s^{-1}(0) \subset S$.

The element $(t_1, t_2, t_3) \in (\mathbb{C}^*)^3$ acts on the section $H(X, Y)$ by

$$H(X, Y) \mapsto t_3 H(t_1 X, t_2 Y).$$

Modulo this action, the mirror curve is parametrized by $p + s$ elements $q = (q_1, \ldots, q_{p+s}) \in (\mathbb{C}^*)^{p+s}$. For generic q, the mirror curve C_q is a Riemann surface of genus $\mathfrak{g}$ with $\mathfrak{n}$ punctures, and the compactified mirror curve $\overline{C}_q$ is a smooth hypersurface in the toric surface S. The Euler characteristic of C_q is

$$\chi(C_q) = 2 - 2\mathfrak{g} - \mathfrak{n} = -\dim_{\mathbb{C}} H^*(\mathcal{X}; \mathbb{C}) = -\chi(X).$$

2.7. Framings

The framing $f \in \mathbb{Z}$ specifies a 1-dimensional subgroup

$$\mathbb{T}_f = \ker(\mathsf{f}) \subset \mathbb{T}',$$

where the character $\mathsf{f} = \mathsf{w}_1' - f\mathsf{w}_2' \in M' = \mathrm{Hom}(N', \mathbb{Z})$. It induces a surjective group homomorphism

$$(\mathbb{T}')^\vee \cong (\mathbb{C}^*)^2 \to (\mathbb{T}_f)^\vee \cong \mathbb{C}^*, \quad (X, Y) \mapsto XY^f.$$

Other than several finite number of choices of f, the function

$$\hat{X} := XY^f : C_q \to \mathbb{C}^*$$

is holomorphic Morse, i.e. it has simple ramification points. We have

$$|\mathrm{Crit}(\hat{X})| = -\chi(C_q) = 2\mathfrak{g} - 2 + \mathfrak{n} = \dim_{\mathbb{C}} H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{C}).$$

Around each ramification point $p_0 \in \mathrm{Crit}(\hat{X})$, one writes

$$\hat{x} = \hat{x}(p_0) + \zeta_0^2,$$

where ζ_0 is the local coordinate around p_0. We denote $\check{u}^{p_0} = \hat{x}(p_0)$. It depends on the complex parameter q, and is a *canonical coordinate* of the B-model. For any p in the neighborhood of p_0 we define $\bar{p}$ by

$$\zeta_0(p) = -\zeta_0(\bar{p}).$$

We also define a multi-valued holomorphic 1-form on C_q

$$\Phi = \log Y \frac{d\hat{X}}{\hat{X}}.$$

3. Gromov-Witten invariants of Toric Calabi-Yau 3-orbifolds

3.1. Open Gromov-Witten invariants and A-model open potentials

Let $\mathcal{L}$ be an Aganagic-Vafa Lagrangian brane in a toric Calabi-Yau 3-orbifold $\mathcal{X}$. Then $\mathcal{L}$ is homotopic to $S^1 \times \mathcal{B}\mathbb{Z}_m$, so

$$H_1(\mathcal{L}; \mathbb{Z}) = \pi_1(\mathcal{L}) = \mathbb{Z} \times \mathbb{Z}_m.$$

Open GW invariants of $(\mathcal{X}, \mathcal{L})$ count holomorphic maps

$$u : \left(\Sigma, x_1, \ldots, x_\ell, \partial\Sigma = \coprod_{j=1}^{n} R_j\right) \to (\mathcal{X}, \mathcal{L})$$

where Σ is a bordered Riemann surface with stacky points $x_i = B\mathbb{Z}_{r_i}$ and $R_j \cong S^1$ are connected components of $\partial\Sigma$. These invariants depend on the following data:

1) the topological type (g, n) of the coarse moduli of the domain, where g is the genus of Σ and n is the number of connected components of $\partial\Sigma$,
2) the degree $\beta' = u_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$,
3) the winding numbers $\mu_1, \ldots, \mu_n \in \mathbb{Z}$ and the monodromies $k_1, \ldots, k_n \in \mathbb{Z}_m$, where $(\mu_j, k_j) = u_*[R_j] \in H_1(\mathcal{L}; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_m$,
4) the framing $f \in \mathbb{Z}$ of $\mathcal{L}$.

We call the pair $(\mathcal{L}, f)$ a framed Aganagic-Vafa Lagrangian brane. We write $\vec{\mu} = ((\mu_1, k_1), \ldots, (\mu_n, k_n))$. Let $\mathcal{M}_{g,\ell}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ be the moduli space parametrizing maps described above, and let $\overline{\mathcal{M}}_{g,\ell}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ be the partial compactification: we allow the domain Σ to have nodal singularities, and an orbifold/stacky point on Σ is either a marked point x_j or a node; we require the map u to be stable in the sense that its automorphism group is finite. Evaluation at the i-th marked point x_i gives a map $\mathrm{ev}_i : \overline{\mathcal{M}}_{(g,n),\ell}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}) \to \mathcal{I}\mathcal{X}$.

Given $\gamma_1, \ldots, \gamma_\ell \in H^*_{\mathrm{CR},\mathbb{T}'}(\mathcal{X}; \mathbb{C})$, we define

$$\langle \gamma_1, \ldots, \gamma_\ell \rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta,\vec{\mu}} := \int_{[\overline{\mathcal{M}}_{(g,n),\ell}(\mathcal{X},\mathcal{L}|\beta',\vec{\mu})^{\mathbb{T}'_{\mathbb{R}}}]^{\mathrm{vir}}} \left. \frac{\prod_{i=1}^{\ell} \mathrm{ev}_i^* \gamma_i}{e_{\mathbb{T}'_{\mathbb{R}}}(N^{\mathrm{vir}})} \right|_{(\mathbb{T}_f)_{\mathbb{R}}} \in \mathbb{C} v^{\sum_{i=1}^{\ell} \frac{\deg \gamma_i}{2} - 1}$$

where v is the generator of $H^2(\mathcal{B}(\mathbb{T}_f)_{\mathbb{R}}; \mathbb{Z}) = H^2(\mathcal{B}U(1); \mathbb{Z}) \cong \mathbb{Z}$.

For $\boldsymbol{\tau} = \sum_{a=1}^{p+s} \tau_a e_a \in H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$, we define generating functions $F^{\mathcal{X},(\mathcal{L},f)}_{g,n}$ of open Gromov-Witten invariants as follows.

$$\begin{aligned} &F^{\mathcal{X},(\mathcal{L},f)}_{g,n}(Z_1,\dots,Z_n,\boldsymbol{\tau}) \\ &= \sum_{\beta',\ell\geq 0} \sum_{(\mu_j,k_j)\in\mathbb{Z}\times\mathbb{Z}_m} \frac{\langle \boldsymbol{\tau}^\ell \rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta,(\mu_1,k_1)\cdots,(\mu_n,k_n)}}{\ell!} \\ &\quad \cdot \otimes_{j=1}^n \Big(Z_j^{\mu_j} (-(-1)^{\frac{-k_j}{m}}) \mathbf{1}'_{\frac{-k_j}{m}} \Big) \in H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m;\mathbb{C})^{\otimes n} \end{aligned}$$

where $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m;\mathbb{C}) = \oplus_{k=0}^{m-1} \mathbb{C}\mathbf{1}'_{\frac{k}{m}}$.

3.2. Primary closed Gromov-Witten invariants and A-model free energies

We define genus g, degree β primary closed Gromov-Witten invariants:

$$\langle \boldsymbol{\tau}^\ell \rangle^{\mathcal{X}}_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,\ell}(\mathcal{X},\beta)^{\mathbb{T}'_{\mathbb{R}}}]^{\mathrm{vir}}} \frac{\prod_{i=1}^{\ell} \mathrm{ev}_i^* \boldsymbol{\tau}}{e_{\mathbb{T}'_{\mathbb{R}}}(N^{\mathrm{vir}})} \Bigg|_{(\mathbb{T}_f)_{\mathbb{R}}} \in \mathbb{C}.$$

This closed Gromov-Witten invariant can be viewed as the case when $n = 0$ i.e. there is no boundary on the domain curve. The A-model genus g free energy $F^{\mathcal{X}}_g$ is a generating function of primary genus g closed Gromov-Witten invariants.

$$F^{\mathcal{X}}_g(\boldsymbol{\tau}) = \sum_{\beta,\ell\geq 0} \frac{\langle \boldsymbol{\tau}^\ell \rangle^{\mathcal{X}}_{g,\beta}}{\ell!}.$$

The BKMP remodeling conjecture builds the mirror symmetry for the open Gromov-Witten potentials $F^{\mathcal{X},(\mathcal{L},f)}_{g,n}(Z_1,\dots,Z_n,\boldsymbol{\tau})$ as well as free energies $F^{\mathcal{X}}_g(\boldsymbol{\tau})$.

3.3. Descendant closed Gromov-Witten invariants

Given $\gamma_1,\dots,\gamma_n$, we define a generating function of genus g, n-point descendant closed Gromov-Witten invariants:

$$\left\langle\!\!\left\langle \frac{\gamma_1}{z_1-\psi_1},\dots,\frac{\gamma_n}{z_n-\psi_n} \right\rangle\!\!\right\rangle^{\mathcal{X}}_{g,n} = \sum_{\beta,\ell\geq 0} \frac{1}{\ell!} \left\langle \frac{\gamma_1}{z_1-\psi_1},\dots,\frac{\gamma_n}{z_n-\psi_n}, \boldsymbol{\tau}^\ell \right\rangle^{\mathcal{X}}_{g,\beta},$$

where $\psi_i = c_1(\mathbb{L}_i)$ and $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n+\ell}(\mathcal{X},\beta)$ is line bundle whose fiber at moduli point $[u:(C,x_1,\ldots,x_{n+\ell})\to\mathcal{X}]$ is the cotangent line $T^*_{x_i}C$ at the i-th marked point to (the coarse moduli space of) the domain curve.

We will state an extension of the remodeling conjecture to higher genus descendent potentials $\left\langle\!\!\left\langle \frac{\gamma_1}{z_1-\psi_1},\ldots,\frac{\gamma_n}{z_n-\psi_n}\right\rangle\!\!\right\rangle^{\mathcal{X}}_{g,n}$.

4. Eynard-Orantin invariants of the mirror curve

4.1. Fundamental normalized differential of the second kind

In this subsection, we recall the definition of the fundamental normalized differential of the second kind $B(p_1,p_2)$ (see e.g. [53]) for a general compact Riemann surface $\overline{C}$.

Let $\overline{C}$ be a compact Riemann surface of genus $\mathfrak{g}$. When $\mathfrak{g}>0$, let $A_1,B_1,\ldots,A_{\mathfrak{g}},B_{\mathfrak{g}}$ be a symplectic basis of $(H_1(\overline{C};\mathbb{C}),\cdot)$:

$$A_i\cdot A_j = B_i\cdot B_j = 0,\quad A_i\cdot B_j = -B_j\cdot A_i = \delta_{ij}$$

where $\cdot$ is the intersection pairing. For our purpose, we need to consider $H_1(\overline{C};\mathbb{C})$ instead of the integral first homology group $H_1(\bar{C};\mathbb{Z})$. We assume that the Lagrangian subspace $\bigoplus_{i=1}^{\mathfrak{g}}\mathbb{C}A_i$ of $H_1(\overline{C};\mathbb{C})$ is transversal to the Lagrangian subspace

$$H^{1,0}(\overline{C})^{\perp} := \{\gamma\in H_1(\overline{C};\mathbb{C}) : \langle\theta,\gamma\rangle = 0 \quad \forall\theta\in H^{1,0}(\overline{C})\}$$

where $\langle\ ,\ \rangle : H^1(\overline{C};\mathbb{C})\times H_1(\overline{C};\mathbb{C})$ is the natural pairing; this assumption holds when $A_1,\ldots,A_{\mathfrak{g}}\in H_1(\overline{C};\mathbb{R})$.

The fundamental normalized differential of the second kind $B(p_1,p_2)$ on $\bar{C}$ is characterized by the following properties:

1) $B(p_1,p_2)$ is a bilinear symmetric meromorphic differential on $\overline{C}_q\times\overline{C}_q$.

2) $B(p_1,p_2)$ is holomorphic everywhere except for a double pole along the diagonal. If p_1,p_2 have local coordinates z_1,z_2 in an open neighborhood U of $p\in\overline{C}_q$ then

$$B(p_1,p_2) = \left(\frac{1}{(z_1-z_2)^2} + a(z_1,z_2)\right)dz_1dz_2$$

where $a(z_1,z_2)$ is holomorphic on $U\times U$ and symmetric in z_1,z_2.

3) $\displaystyle\int_{p_1\in A_i} B(p_1,p_2) = 0$, $i=1,\ldots,\mathfrak{g}$.

In fact, we do not need a particular choice of A-cycles. We just need to specify a Lagrangian subspace of $(H_1(\overline{C};\mathbb{C}),\cdot)$ transversal to the Lagrangian subspace $H^{1,0}(\overline{C};\mathbb{C})^\perp \subset H_1(\overline{C};\mathbb{C})$ such that the period of $B(p_1,p_2)$ along any element in this subspace is zero.

The fundamental differential $B(p_1,p_2)$ also satisfies the following properties:

(4) If f is a meromorphic function on $\overline{C}$ then

$$df(p_1) = \mathrm{Res}_{p_2\to p_1} B(p_1,p_2)f(p_2).$$

(5) $\displaystyle\int_{p_1\in B_i} B(p_1,p_2) = 2\pi\sqrt{-1}\omega_i(p_2)$, where ω_i is the unique holomorphic 1-form on $\overline{C}$ such that $\int_{A_j}\omega_i = \delta_{ij}$.

4.2. Choice of A-cycles on the compactified mirror curve

The mirror theorem for semi-projective toric orbifolds [33] relates the 1-primary 1-descendant function (the J-function)

$$\left\langle\!\!\left\langle 1, \frac{\phi_a}{z-\psi}\right\rangle\!\!\right\rangle_{0,2}^{\mathcal{X}} \phi^a$$

to certain hypergeometric I-function $I^{\mathcal{X}}(q,z)$ under the mirror map

$$\tau_a = \frac{1}{2\pi\sqrt{-1}}\int_{A_a}\Phi = \begin{cases} \log q_a + h_a(q), & a=1,\ldots,p \\ q_a(1+h_a(q)), & a=p+1,\ldots,p+s. \end{cases}$$

which as the prescribed leading term behavior (all $h_a(q)$ are power series in q).

It is a well-known fact that these mirror maps are given by such period integrals where $A_a \in H_1(C_q;\mathbb{C})$. The inclusion $C_q \hookrightarrow \overline{C}_q$ induces a surjective group homomorphism $H_1(C_q;\mathbb{C}) \cong \mathbb{C}^{2\mathfrak{g}+\mathfrak{n}-1} \to H_1(\overline{C}_q) \cong \mathbb{C}^{2\mathfrak{g}}$ where the kernel is generated by the $\mathfrak{n}$ loops around the $\mathfrak{n}$ points in $\overline{C}_q \setminus C_q$; each of these $\mathfrak{n}$ loops is contractible in $\overline{C}_q$, and the sum of these $\mathfrak{n}$ loops is homologous to zero in C_q. The images of $A_a \in H_1(C_q;\mathbb{C})$ in $H_1(\overline{C}_q;\mathbb{C})$ span a Lagrangian subspace $L_A \subset H_1(\overline{C}_q;\mathbb{C})$ transversal to the Lagrangian subspace $H^{1,0}(\overline{C}_q)^\perp \subset H_1(\overline{C}_q;\mathbb{C})$. We use the Lagrangian subspace L_A to define our fundamental normalized differential of the second kind $B(p_1,p_2)$ for the purpose of constructing higher genus B-model invariants.

4.3. The Eynard-Orantin topological recursion

We use the fundamental differential B prescribed above to run the Eynard-Orantin topological recursion. It starts with two initial data (unstable cases)

$$\omega_{0,1} = 0, \quad \omega_{0,2} = B.$$

The stable cases ($2g - 2 + n > 0$) are defined recursively by the Eynard-Orantin topological recursion:

$$\begin{aligned}
&\omega_{g,n}(p_1, \dots, p_n) \\
:= &\sum_{p_0 \in \mathrm{Crit}(\hat{X})} \mathrm{Res}_{p \to p_0} \frac{\int_{\xi=p}^{\bar{p}} B(p_n, \xi)}{2(\Phi(p) - \Phi(\bar{p}))} \Bigg(\omega_{g-1,n+1}(p, \bar{p}, p_1, \dots, p_{n-1}) \\
&+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1,\dots,n-1\}}} \omega_{g_1,|I|+1}(p, p_I) \omega_{g_2,|J|+1}(\bar{p}, p_J) \Bigg).
\end{aligned}$$

The resulting $\omega_{g,n}$ for $2g - 2 + n > 0$ is a symmetric meromorphic form on $(\overline{C}_q)^n$. They are holomorphic on $(\overline{C}_q \setminus \mathrm{Crit}(\hat{X}))^n$ and satisfy the following properties:

1) For any $j \in \{1, \dots, n\}$ and any $p_0 \in \mathrm{Crit}(\hat{X})$,

$$\mathrm{Res}_{p_j \to p_0} \omega_{g,n}(p_1, \dots, p_n) = 0.$$

2) For any $j \in \{1, \dots, n\}$ and any $i \in \{1, \dots, \mathfrak{g}\}$,

$$\int_{p_j \in A_i} \omega_{g,n}(p_1, \dots, p_n) = 0.$$

4.4. B-model open potentials

For $\ell \in \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, let

$$\psi_\ell := \frac{1}{m} \sum_{k=0}^{m-1} e^{\frac{2\pi\sqrt{-1}k\ell}{m}} \mathbf{1}'_{\frac{k}{m}}.$$

Then $\{\psi_\ell : \ell = 0, 1 \dots, m-1\}$ is a canonical basis of $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m; \mathbb{C})$.

Recall that $\mathcal{L}$ intersects a unique 1-dimensional orbit $\mathfrak{l}$ of the $\mathbb{T}$-action on $\mathcal{X}$. We assume that the closure $\bar{\mathfrak{l}}$ of $\mathfrak{l}$ in $\mathcal{X}$ is non-compact, so that $\mathcal{L}$ is an "outer" brane. Then the 2-dimensional cone associated to $\bar{\mathfrak{l}}$ corresponds an edge e on the boundary of the polytope P, and $|e \cap N'| = m + 1$. Let $D \subset S$ be the torus invariant divisor associated to the edge e. For generic q, the compactified mirror curve $\overline{C}_q$ intersects D transversally at m points $\bar{p}_0, \ldots, \bar{p}_{m-1}$. For $\ell \in \{0, 1, \ldots, m-1\}$, there exist open neighborhoods U_ℓ of $\bar{p}_\ell$ in the compactified mirror curve $\overline{C}_q$ and U of 0 in $\mathbb{P}^1 = \mathbb{C}^* \cup \{0, \infty\}$ such that $\hat{X}|_{U_\ell} : U_\ell \to U$ is biholomorphic. Let $\rho_\ell := (\hat{X}|_{U_\ell})^{-1} : U \to U_\ell$. We define B-model topological open string partition functions as follows.

1) disk invariants

$$\check{F}_{0,1}(q; X) := \sum_{\ell \in \mathbb{Z}_m} \int_0^X \left(\big(\log Y(\rho_\ell(X')) - \log Y(\bar{p}_\ell) \big) \frac{dX'}{X'} \right) \psi_\ell$$

which take values in $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m; \mathbb{C})$.

2) annulus invariants

$$\check{F}_{0,2}(q; X_1, X_2) := \sum_{\ell_1, \ell_2 \in \mathbb{Z}_m} \int_0^{X_1} \int_0^{X_2} \left((\rho_{\ell_1} \times \rho_{\ell_2})^* \omega_{0,2} - \frac{dX_1' dX_2'}{(X_1' - X_2')^2} \right) \psi_{\ell_1} \otimes \psi_{\ell_2}$$

which take values in $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m; \mathbb{C})^{\otimes 2}$.

3) $2g - 2 + n > 0$

$$\check{F}_{g,n}(q; X_1, \ldots, X_n) := \sum_{\ell_1, \cdots, \ell_n \in \mathbb{Z}_m} \int_0^{X_1} \cdots \int_0^{X_n} (\rho_{\ell_1} \times \cdots \times \rho_{\ell_n})^* \omega_{g,n} \psi_{\ell_1} \otimes \cdots \otimes \psi_{\ell_n}$$

which take values in $H^*_{\mathrm{CR}}(\mathcal{B}\mathbb{Z}_m; \mathbb{C})^{\otimes n}$.

Each of the m^n components of $\check{F}_{g,n}(q; X_1, \ldots, X_n)$ is a power series in $q_1, \ldots, q_{p+s}, X_1, \ldots, X_n$ which converges in an open neighborhood of the origin.

4.5. B-model free energies

For $g \geq 2$, the B-model free energy is defined as

$$\check{F}_g(q) = \frac{1}{2g-2} \sum_{p_0 \in \mathrm{Crit} \hat{X}} \mathrm{Res}_{p \to p_0} \omega_{g,1}(p) \widetilde{\Phi}(p),$$

where

$$d\widetilde{\Phi} = \Phi.$$

Notice that the function $\widetilde{\Phi}$ locally defined around each critical point of $\hat{X}$ has some ambiguities, since Φ is multi-valued, and $\widetilde{\Phi}$ is determined by Φ up to a constant. However, the residue is well-defined since is does not depend on these ambuities.

For $g = 1$, the free energy is defined up to a constant

$$\check{F}_1(q) = -\frac{1}{2}\log\tau_B - \frac{1}{24}\sum_{p_0\in\mathrm{Crit}\hat{X}} \log h_1^{p_0}.$$

Here the Bergmann τ-function τ_B is defined up to a constant by

$$\frac{\partial \log\tau_B}{\partial \check{u}^{p_0}} = \mathrm{Res}_{p\to p_0}\frac{B(p,\bar{p})}{d\hat{x}(p)}.$$

When $g = 0$, the prepotential F_0 is characterized by

$$\frac{\partial \check{F}_0}{\partial \tau_a} = \int_{p\in B_a} \Phi(p).$$

Notice that since Φ is a multi-valued differential form, and it satisfies the following

$$\frac{\partial \Phi(p)}{\partial \tau_a} = \int_{p'\in B_a} \omega_{0,2}(p,p').$$

The prepotential $\check{F}_0$ defined this way is only determined up to a quadratic polynomial in τ_a.

5. All genus open-closed mirror symmetry

In this section, $(\mathcal{L}, f)$ is an outer Aganagic-Vafa Lagrangian brane in $\mathcal{X}$, so that the closure of $\mathfrak{l} = \mathbb{C}^* \times \mathcal{B}\mathbb{Z}_m$ contains a unique $\mathbb{T}$ fixed point. Let G be the stabilizer of this fixed point. Then G is a finite abelian group which contains $\mathbb{Z}_m$ as a subgroup. When $\mathcal{X}$ is smooth, we have $m = 1$ and G is trivial.

5.1. All genus open-closed mirror symmetry: the remodeling conjecture

Conjecture 1 (Bouchard-Klemm-Mariño-Pasqetti [20, 21]).

$$\check{F}_{g,n}(q; X_1, \ldots, X_n) = (-1)^{g-1+n}|G|^n F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau}; Z_1, \ldots, Z_n)$$

where (q, X_j) *and* $(\boldsymbol{\tau}, Z_j)$ *are related by the open-closed mirror map:*

$$\tau_a = \frac{1}{2\pi\sqrt{-1}} \int_{A_a} \Phi = \begin{cases} \log q_a + h_a(q), & a = 1 \ldots, p \\ q_a(1 + h_a(q)), & a = p+1, \ldots, p+s \end{cases}$$

$$\log Z_j = \log X_j + h_0(q)$$

where $h_0(q), h_1(q), \ldots, h_{p+s}(q)$ *are explicit power series in* q *convergent in a neighborhood of the origin in* $\mathbb{C}^{p+s}$*. Notice that when* $n = 0$*, this is a statement about closed Gromov-Witten mirror symmetry (and the right-hand side does not depend on* $(\mathcal{L}, f)$*). When* $(g, n) = (1, 0)$ *and* $(0, 0)$*, the statement is valid up to a constant and a quadratic polynomial in* τ_a*, respectively.*

Indeed, the above statement is more general than the original conjecture in [20, 21], where they conjecture about non-gerby branes (the $m = 1$ case).

Conjecture 1 was proved when $\mathcal{X} = \mathbb{C}^3$ independently by L. Chen [28] and J. Zhou [116]. In 2012, Eynard-Orantin provided a proof of the BKMP remodeling conjecture for all symplectic smooth toric Calabi-Yau 3-folds [47]. In the orbifold case, the authors prove Conjecture 1 first for affine toric Calabi-Yau 3-orbifolds [51] and later for all semi-projective toric Calabi-Yau 3-orbifolds [52].

We now give a brief outline of the proof of Conjecture 1 in [52]. Givental proved a quantization formula for total descendant potential of equivariant GW theory of GKM manifolds [57–59]. (See also the book by Lee-Pandharipande [77].) The third author generalized this formula to GKM orbifolds [120]. The quantization formula is equivalent to a graph sum formula of the total descendant potential, which implies a graph sum formula

$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)} = \sum_{\vec{\Gamma} \in G_{g,n}} \frac{w_A(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|},$$

where $G_{g,n}$ is a certain set of decorated stable graphs. The unique solution $\{\omega_{g,n}\}$ to the Eynard-Orantin topological recursion can be expressed as a sum over graphs [43–45, 70]. We expand the graph sum formula in [43, Theorem 3.7] (which is equivalent to [44, Theorem 5.1]) at punctures $\{\bar{p}_\ell : \ell \in \mathbb{Z}_m\}$, and obtain a graph sum formula

$$\check{F}_{g,n} = \sum_{\vec{\Gamma} \in G_{g,n}} \frac{w_B(\vec{\Gamma})}{|\mathrm{Aut}(\vec{\Gamma})|}.$$

Finally, we use the genus-zero mirror theorem for smooth toric DM stacks [33] to prove

$$w_B(\vec{\Gamma}) = (-1)^{g-1+n} |G|^n w_A(\vec{\Gamma})$$

for all decorated graphs $\vec{\Gamma}$.

5.2. Descendant version of the all genus mirror symmetry

Iritani [68] studies the oscillatory integral and shows the following

$$\int_{\mathrm{SYZ}(\mathcal{F})} e^{\frac{\widetilde{W}}{z}} \Omega = \left\langle\!\!\left\langle \frac{\kappa(\mathcal{F})}{z-\psi} \right\rangle\!\!\right\rangle_{0,1},$$

where $\mathcal{F}$ is a $\mathbb{T}_f$-equivariant coherent sheaf on $\mathcal{X}$. Here the SYZ is the SYZ T-dual functor, which takes a $\mathbb{T}_f$-equivariant coherent sheaf on $\mathcal{X}$ and produces a Lagrangian brane in $(\mathbb{C}^*)^3$.[1] The *equivariantly perturbed* superpotential W is given by

$$\widetilde{W} = W - \log X - f \log Y.$$

Let $(\mathbb{T}_f)_{\mathbb{R}} \cong U(1)$ be the maximal torus of $\mathbb{T}_f \cong \mathbb{C}^*$, and let $\mu_{(\mathbb{T}_f)_{\mathbb{R}}} : \mathcal{X} \to \mathbb{R}$ be the moment map of the Hamiltonian $(\mathbb{T}_f)_{\mathbb{R}}$-action on $(\mathcal{X}, \omega)$. We say a $\mathbb{T}_f$-equivariant coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ is admissible if (i) $\mu_{(\mathbb{T}_f)_{\mathbb{R}}}(\mathrm{supp}(\mathcal{F})) \subset \mathbb{R}$ is bounded below, and (ii) the Lagrangian brane $\mathrm{SYZ}(\mathcal{F})$ reduces to a cycle $\gamma(\mathcal{F})$ on the mirror curve C_q, while the oscillatory integral could be done on the curve

$$\int_{\mathrm{SYZ}(L)} e^{\frac{\widetilde{W}}{z}} \frac{dXdYdZ}{XYZ} = \int_{\gamma(\mathcal{F})} e^{\frac{\hat{x}}{z}} y dx.$$

Condition (i) implies that $\hat{x}$ is bounded below on $\gamma(\mathcal{F})$, so the integral on the RHS converges when $z \in (-\infty, 0)$.

[1] Iritani [68] does not explicitly states this identity under the SYZ transform, but instead he matches the cases $\mathcal{F} = \mathcal{O}_{\mathcal{X}}$ and a skyscraper sheaf. He then applies the monodromy to $\mathcal{O}_{\mathcal{X}}$ to obtain other line bundles on $\mathcal{X}$. These sheaves generate the K-theory group.

Using this result and combining with the remodeling conjecture, we have

Theorem 2 (Descendant version of the all genus mirror symmetry for $\mathcal{X}$).

$$\int_{\gamma(L_1)\times\cdots\times\gamma(L_n)} e^{\frac{\hat{x}_1}{z_1}+\cdots+\frac{\hat{x}_n}{z_n}} \omega_{g,n} = \left\langle\!\!\left\langle \frac{\kappa(L_1)}{z_1-\psi_1}\cdots\frac{\kappa(L_n)}{z_n-\psi_n} \right\rangle\!\!\right\rangle_{g,n}.$$

To obtain this theorem, one observes that when integrating $\omega_{g,n}$ we are simply integrating the leaf terms of $w_A(\vec{\Gamma})$, since only leaf terms are forms while all other graph component contributions are scalars. The genus 0 oscillatory integral theorem from [68] turns these leafs into genus 0 descendants, and the graph becomes precisely the graph for higher genus descendant potentials.

References

[1] D. Abramovich and B. Fantechi, *Orbifold techniques in degeneration formulas*, arXiv:1103.5132.

[2] D. Abramovich, T. Graber, and A. Vistoli, *Algebraic orbifold quantum products*, Orbifolds in mathematics and physics (Madison, WI, 2001), 1–24, Contemp. Math. **310**, Amer. Math. Soc., Providence, RI, 2002.

[3] D. Abramovich, T. Graber, and A. Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398.

[4] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa, *The topological vertex*, Comm. Math. Phys. **254** (2005), no. 2, 425–478.

[5] M. Aganagic, A. Klemm, and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, Z. Naturforsch. **A 57** (2002), 1–28.

[6] M. Aganagic, M. Mariño, and C. Vafa, *All loop topological string amplitudes from Chern-Simons theory*, Comm. Math. Phys. **247** (2004), no. 2, 467–512.

[7] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*, arXiv:hep-th/0012041.

[8] M. Alim and J. D. Länge, *Polynomial structure of the (open) topological string partition function*, J. High Energy Phys. **2007**, no. 10, 045, 13pp.

[9] D. Arinkin and A. Polishchuk, *Fukaya category and Fourier transform*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, AMS/IP Stud. Adv. Math., Vol. 23, 2001, pp. 261–274.

[10] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28.

[11] V. V. Batyrev and D. A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math. J. **75** (1994), no. 2, 293–338.

[12] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), no. 3, 601–617.

[13] K. Behrend and B. Fantechi, *Intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88.

[14] K. Behrend and Y. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), no. 1, 1–60.

[15] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic anomalies in topological field theories*, Nuclear Phys. **B 405** (1993), no. 2-3, 279–304.

[16] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165** (1994), no. 2, 311–427.

[17] M. Bershadsky and A. Radul, *Conformal field theories with additional $\mathbb{Z}_n$ symmetry*, Int. J. Mod. Phys. **A 02**, (1987), 165.

[18] L. Borisov, L. Chen, and G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. **18** (2005), no. 1, 193–215.

[19] V. Bouchard, A. Catuneanu, O. Marchal, and P. Sułkowski, *The remodeling conjecture and the Faber-Pandharipande formula*, Lett. Math. Phys. **103** (2013), 59–77.

[20] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Remodeling the B-model*, Comm. Math. Phys. **287** (2009), no. 1, 117–178.

[21] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Topological open strings on orbifolds*, Comm. Math. Phys. **296** (2010), no. 3, 589–623.

[22] A. Brini, *Open topological strings and integrable hierarchies: Remodeling the A-model*, Comm. Math. Phys. **312** (2012), no. 3, 735–780.

[23] J. Bryan, C. Cadman, and B. Young, *The orbifold topological vertex*, Adv. Math. **229** (2012), no. 1, 531–595.

[24] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. **B 359** (1991), no. 1, 21–74.

[25] S. Cecotti and C. Vafa, *Massive orbifolds*, Mod. Phys. Lett. **A 07** (1992), 1715.

[26] K. Chan, C.-H. Cho, S.-C. Lau, and H.-H. Tseng, *Gross fibrations, SYZ mirror symmetry, and open Gromov-Witten invariants for toric Calabi-Yau orbifolds*, preprint, `arXiv:1306.0437`. J. Differential Geom. **103** (2016), no. 2, 207–288.

[27] K. Chan, S.-C. Lau, and N. C. Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds*, J. Differential Geom. **90** (2012), no. 2, 177–250. `MR2899874`

[28] L. Chen, *Bouchard-Klemm-Marino-Pasquetti conjecture for* $\mathbb{C}^3$, `arXiv:0910.3739`.

[29] B. Chen, A.-M. Li, S. Sun, and G. Zhao, *Relative orbifold Gromov-Witten theory and degeneration formula*, `arXiv:1110.6803`.

[30] W. Chen and Y. Ruan, *Orbifold Gromov-Witten theory*, *Orbifolds in mathematics and physics*, 25–85, Contemp. Math. **310**, Amer. Math. Soc., Providence, RI, 2002.

[31] W. Chen and Y. Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004), no. 1, 131.

[32] T.-M. Chiang, A. Klemm, S.-T. Yau, and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, Adv. Theor. Math. Phys. **3** (1999), 495–565.

[33] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng, *A mirror theorem for toric stacks*, Compos. Math. **151** (2015), no. 10, 1878–1912.

[34] K. J. Costello, *Renormalization and effective field theory*, Mathematical Surveys and Monographs, Vol. 170, American Mathematical Society, Providence, RI, 2011.

[35] K. J. Costello and S. Li, *Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model*, arXiv:1201.4501.

[36] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**, American Mathematical Society, Providence, RI, 1999.

[37] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. No. **36** (1969), 75–109.

[38] D. E. Diaconescu and B. Florea, *Localization and gluing of topological amplitudes*, Comm. Math. Phys. **257** (2005), 119–149.

[39] L. Dixon, D. Friedan, E. Martinec, and S. Shenker, *The conformal field theory of orbifolds*, Nucl. Phys. **B 282** (1987), 13–73.

[40] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds*, Nuclear Phys. **B 261** (1985), no. 4, 678–686.

[41] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds II*, Nuclear Phys. **B 274** (1986), no. 2, 285–314.

[42] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), 31–47, Oxford Univ. Press, Oxford, 1998.

[43] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the special curves topological recursion procedure*, Comm. Math. Phys. **328** (2014), no. 2, 669–700.

[44] B. Eynard, *Intersection numbers and spectral curves*, arXiv:1104.0176.

[45] B. Eynard, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*, Commun. Number Theory Phys. **8** (2014), no. 3, 541–588.

[46] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452.

[47] E. Eynard and N. Orantin, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, Comm. Math. Phys. **337** (2015), no. 2, 483–567.

[48] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math **139** (2000), no. 1, 173–199.

[49] B. Fang and C.-C. M. Liu, *Open Gromov-Witten invariants of toric Calabi-Yau 3-folds*, Comm. Math. Phys. **323** (2013), no. 1, 285–328.

[50] B. Fang, C.-C. M. Liu, and H.-H. Tseng, *Open-closed Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks*, `arXiv:1212.6073`.

[51] B. Fang, C.-C. M. Liu, and Z. Zong, *All genus open-closed mirror symmetry for affine toric Calabi-Yau 3-orbifolds*, `arXiv:1310.4818`.

[52] B. Fang, C.-C. M. Liu, and Z. Zong, *On the remodeling conjecture for toric Calabi-Yau 3-orbifolds*, `arXiv:1604.07123`.

[53] J. D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin-New York, 1973.

[54] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), no. 5, 933–1048.

[55] A. B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **1996**, no. 13, 613–663.

[56] A. B. Givental, *A mirror theorem for toric complete intersections*, *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, 141–175, Progr. Math. **160**, Birkhäuser Boston, Boston, MA, 1998.

[57] A. B. Givental, *Semisimple Frobenius structures at higher genus*, IMRN (2001), no. 23, 1265–1286.

[58] A. B. Givental, *Gromov-Witten invariants and quantization of quadratic hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568.

[59] A. B. Givental, *Symplectic geometry of Frobenius structures*, *Frobenius manifold*, 91–112, Aspects Math., **E36**, Friedr. Vieweg, Wiesbanden 2004.

[60] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518.

[61] T. Graber and E. Zaslow, *Open-string Gromov-Witten invariants: calculations and a mirror 'theorem'*, Orbifolds in mathematics and physics (Madison, WI, 2001), 107–121, Contemp. Math. **310**, Amer. Math. Soc., Providence, RI, 2002.

[62] M. Gross, L. Katzarkov, and H. Ruddat, *Towards mirror symmetry for varieties of general type*, `arXiv:1202.4042`.

[63] S. Hamidil and C. Vafa, *Interactions on orbifolds*, Nuclear Physi. **B 279** (1987), 465–513.

[64] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[65] K. Hori, A. Iqbal, and C. Vafa, *D-branes and mirror symmetry*, `arXiv:hep-th/0005247`.

[66] K. Hori and C. Vafa, *Mirror symmetry*, `arXiv:hep-th/0002222`.

[67] M.-X. Huang, A. Klemm, and S. Quackenbush, *Topological string theory on compact Calabi-Yau: modularity and boundary conditions*, Homological mirror symmetry, 45–102, Lecture Notes in Phys. **757**, Springer, Berlin, 2009.

[68] H. Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), no. 3, 1016–1079.

[69] T. J. Jarvis and T. Kimura, *Orbifold quantum cohomology of the classifying space of a finite group*, Orbifolds in mathematics and physics (Madison, WI, 2001), 123–134, Contemp. Math. **310**, Amer. Math. Soc., Providence, RI, 2002.

[70] I. Kostov and N. Orantin, *CFT and topological recursion*, J. High Energy Phys. **2010**, no. 11, 056, 30pp.

[71] J. Kaczorowski and A. Perelli, *A uniform version of Stirling's formula*, Funct. Approx. Comment. Math. **45** (2011), part 1, 89–96.

[72] S. Katz and C.-C. M. Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, Adv. Theor. Math. Phys. **5** (2001), no. 1, 1–49.

[73] A. Klemm and E. Zaslow, *Local mirror symmetry at higher genus*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), 183–207, AMS/IP Stud. Adv. Math. **23**, Amer. Math. Soc., Providence, RI, 2001.

[74] V. G. Knizhnik, *Analytic fields on Riemann surfaces. II*, Comm. Math. Phys. **112**, no. 4 (1987), 567–590.

[75] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[76] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), 335–368, Progr. Math. **129**, Birkhäuser Boston, Boston, MA, 1995.

[77] Y. P. Lee and R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory and Virasoro constraints.*

[78] N.-C. Leung, S.-T. Yau, and E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Adv. Theor. Math. Phys. **4** (2000) 1319–1341.

[79] A. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), no. 1, 151–218.

[80] J. Li, *Stable Morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. **57** (2001), 509–578.

[81] J. Li, *A degeneration formula of Gromov-Witten invariants*, J. Diff. Geom. **60** (2002), 199-293.

[82] J. Li, *Recent progress in GW-invariants of Calabi-Yau threefolds*, Current developments in mathematics, 2007, 77–99, Int. Press, Somerville, MA, 2009.

[83] J. Li, C.-C. M. Liu, K. Liu, and J. Zhou, *A mathematical theory of the topological vertex*, Geom. Topol. **13** (2009), no. 1, 527–621.

[84] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998), no. 1, 119–174.

[85] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47–83, First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998.

[86] B. H. Lian, K. Liu, and S.-T. Yau, *Mirror principle I*, Asian J. Math. **1** (1997), no. 4, 729–763.

[87] B. H. Lian, K. Liu, and S.-T. Yau, *Mirror principle III*, Asian J. Math. **3** (1999), no. 4, 771–800.

[88] C.-C. M. Liu, *Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for an S^1-equivariant pair*, `arXiv:math/0211388`.

[89] C.-C. M. Liu, *Localization in Gromov-Witten theory and orbifold Gromov-Witten theory*, Handbook of Moduli, Volume II, 353–425, Adv. Lect. Math., (ALM) **25**, International Press and Higher Education Press, 2013.

[90] C.-C. M. Liu, K. Liu, and J. Zhou, *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), no. 2, 289–340.

[91] C.-C. M. Liu, K. Liu, and J. Zhou, *A formula of two-partition Hodge integrals*, J. Amer. Math. Soc. **20** (2007), no. 1, 149–184.

[92] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys. **2008** , no. 3, 060, 34pp.

[93] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory I*, Compos. Math. **142** (2006), no. 5, 1263–1285.

[94] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory II*, Compos. Math. **142** (2006), no. 5, 1286–1304.

[95] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Invent. Math. **186** (2011), no. 2, 435–479.

[96] D. Maulik and R. Pandharipande, *A topological view of Gromov-Witten theory*, Topology **45** (2006), no. 5, 887–918.

[97] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, 271–328, Progr. Math. **36**, Birkhäuser Boston, Boston, MA, 1983.

[98] A. Okounkov and R. Pandharipande, *Hodge integrals and invariants of the unknot*, Geom. Topol. **8** (2004), 675–699.

[99] N. Orantin, *Symplectic invariants, Virasoro constraints and Givental decomposition*, `arXiv:0808.0635`.

[100] A. Popa, *The genus one Gromov-Witten invariants of Calabi-Yau complete intersections*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1149–1181.

[101] S.-S. Roan, *On Calabi-Yau orbifolds in weighted projective spaces*, Internat. J. Math. **1** (1990), no. 2, 211–232.

[102] D. Ross, *Localization and gluing of orbifold amplitudes: the Gromov-Witten orbifold vertex*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1587–1620.

[103] D. Ross, *On the Gromov-Witten/Donaldson-Thomas correspondence and Ruan's conjecture for Calabi-Yau 3-orbifolds*, Comm. Math. Phys. **340** (2015), no. 2, 851–864.

[104] D. Ross and Z. Zong, *The gerby Gopakumar-Mariño-Vafa formula*, Geom. Topol. **17** (2013), no. 5, 2935–2976.

[105] D. Ross and Z. Zong, *Cyclic Hodge integrals and loop Schur functions*, Adv. Math. 285 (2015), 14481486. `arXiv:1401.2217`.

[106] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves*, Proceedings of 6th Gökova Geometry-Topology Conference, Turkish J. Math. **23** (1999), no. 1, 161–231.

[107] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995), no. 2, 259–367.

[108] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. **130** (1997), no. 3, 455–516.

[109] B. Siebert, *Gromov-Witten invariants of general symplectic manifolds*, `arXiv:dg-ga/9608005`.

[110] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Physics B **479** (1), 243–259.

[111] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations*, J. Differential Geom. **54** (2000), no. 2, 367–438.

[112] H.-H. Tseng, *Orbifold quantum Riemann-Roch, Lefschetz and Serre*, Geom. Topol. **14** (2010), no. 1, 1–81.

[113] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.

[114] S. Yamaguchi, S.-T. Yau, *Topological string partition functions as polynomials*, J. High Energy Phys. **2004**, no. 7, 047, 20pp.

[115] E. Zaslow, *Topological orbifold models and quantum cohomology rings*, Comm. Math. Phys. **156** (1993), no. 2, 301–331.

[116] J. Zhou, *Local mirror symmetry for one-legged topological vertex*, arXiv:0910.4320; *Local Mirror symmetry for the topological vertex*, arXiv:0911.2343.

[117] J. Zhou, *Open string invariants and mirror curve of the resolved conifold*, arXiv:1001.0447.

[118] A. Zinger, *The reduced genus 1 Gromov-Witten invariants of Calabi-Yau hypersurfaces*, J. Amer. Math. Soc. **22** (2009), no. 3, 691–737.

[119] Z. Zong, *Generalized Mariño-Vafa Formula and Local Gromov-Witten Theory of Orbi-curves*, J. Differential Geom. **100** (2015), no. 1, 161–190.

[120] Z. Zong, *Equivariant Gromov-Witten theory of GKM orbifolds*, arXiv:1604.07270.

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