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p-adic Berglund-Hübsch duality

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Berglund-Hübsch duality is an example of mirror symmetry between orbifold Landau-Ginzburg models. In this paper we study a D-module-theoretic variant of Borisov's proof of Berglund-Hübsch duality. In the p-adic case, the D-module approach makes it possible to endow the orbifold chiral rings with the action of a non-trivial Frobenius endomorphism. Our main result is that the Frobenius endomorphism commutes with Berglund-Hübsch duality up to an explicit diagonal operator.

1. Introduction

Berglund-Hübsch duality was originally introduced [3] as a generalization of the Greene-Plesser construction [4] of mirror pairs. Let $W(x) \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ be an invertible polynomial defining a Calabi-Yau hypersurface X and let $G \subset (\mathbb{C}^*)^n$ be a group fixing W. Then the Berglund-Hübsch dual of the orbifold of X by G is the hypersurface X^T, defined by the "transpose" invertible polynomial $W^T(x) \in \mathbb{C}[x]$, orbifolded by an explicitly constructed group $G^T \subset (\mathbb{C}^*)^n$ fixing W^T. As shown in [6] and [8], the Berglund-Hübsch construction can be further generalized to Landau-Ginzburg models with invertible potentials (not necessarily of Calabi-Yau type) as follows. For any invertible polynomial $W(x)$ the bigraded chiral ring of the orbifold Landau-Ginzburg model $(W(x), G)$ is isomorphic to the (twisted) chiral ring of the orbifold Landau-Ginzburg model $(W^T(x), G^T)$.

In the context of the vertex algebra approach to mirror symmetry [1], Borisov [2] has shown that, as an isomorphism of bigraded vector spaces (that is, disregarding the multiplicative structure), Berglund-Hübsch duality can be lifted to the level of chains. Let $\mathbb{C}[x, y]_0$ be the quotient of $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by the ideal $\langle x_1y_1, \ldots, x_ny_n\rangle$ and let $\bigwedge(\mathbb{C}^n)$ be the standard exterior representation of the Clifford algebra with generators e_i,

$e_i^\vee$ and relations $e_i e_j^\vee + e_j^\vee e_i = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Borisov's construction hinges on the differential

$$\delta_\infty = \sum_{i=1}^{n} x_i \partial_{x_i} W(x) \otimes e_i + \sum_{i=1}^{n} y_i \otimes e_i^\vee \tag{1.1}$$

acting on $\mathbb{C}[x, y]_0 \otimes \bigwedge(\mathbb{C}^n)$. As shown in [2], $(\mathbb{C}[x, y]_0 \otimes \bigwedge(\mathbb{C}^n), \delta_\infty)$ contains a copy of the standard Koszul resolution of the Milnor ring $\mathbb{C}[x]/dW$ in such a way that the inclusion is a quasi-isomorphism. The starting point for this paper is to deform δ_∞ to

$$\delta_\pi = \sum_{i=1}^{n} (x_i \partial_{x_i} + \pi x_i \partial_{x_i} W(x)) \otimes e_i + \sum_{i=1}^{n} (y_i \partial_{y_i} + \pi y_i) \otimes e_i^\vee ,$$

where $\pi \in \mathbb{C}^*$ is an arbitrary constant. As it turns out, the complex $(\mathbb{C}[x, y]_0 \otimes \bigwedge(\mathbb{C}^n), \delta_\pi)$ contains a copy of the de Rham complex of the D-module $\mathbb{C}[x]e^{\pi W(x)}$. The quasi-isomorphism (see e.g. [9]) between the latter and the Milnor ring allows us to provide an alternate chain-level realization of Berglund-Hübsch duality. More precisely, our method yields a chain-level proof of the "total unprojected" (in the terminology of [8]) Berglund-Hübsch duality, from which the usual "projected" duality of [2] can be obtained by restricting to the invariant sectors as in [8].

The key difference between our construction and [2] emerges if one replaces $\mathbb{C}[x]$ with the ring $\mathbb{C}_p^\dagger\langle x\rangle$ of p-adic overconvergent power series. While the de Rham cohomology of the D-module $\mathbb{C}_p^\dagger\langle x\rangle e^{\pi W(x)}$ (where now π is a fixed $(p-1)$-th root of $-p$) is still isomorphic to the p-adic Milnor ring, the de Rham chain model has extra structure: a non-trivial Frobenius endomorphism which descends to cohomology. In this paper we show that the Frobenius endomorphisms extends naturally to a chain map Fr acting on the full chain complex $\mathbb{C}_p^\dagger\langle x, y\rangle_0 \otimes \bigwedge(\mathbb{C}_p^n)$. It is then natural to ask how the Frobenius endomorphism interacts with the Berglund-Hübsch duality quasi-isomorphism Δ. Our main result is that, at the level of cohomology, Δ and Fr commute up to an explicit diagonal operator whose entries are non-negative integer powers of p.

The interplay between the cohomological Frobenius and Berglund-Hübsch duality was first noticed in [10] and used to explore some arithmetic consequences of Berglund-Hübsch duality in the spirit of [12]. The present work originated as an attempt to understand the results of [10] at the level of chains. We hope to further investigate the arithmetic implications of our construction in future work.

This paper is organized as follows. In Section 2 we review some basic facts about invertible polynomials defined over a field $\mathbb{F}$. In Section 3 and Section 4 we introduce our "de Rham" version of Borisov's complex attached to a suitable matrix A, which we denote by $\mathcal{B}_A(\mathbb{F})$. In Section 5 we point out that $\mathcal{B}_A(\mathbb{F})$ is the total complex of a $\mathbb{Z} \times \mathbb{Z}$-bigraded bicomplex. In Section 6 we show that $\mathcal{B}_A(\mathbb{F})$ is quasi isomorphic to the de Rham cohomology of a certain D-module. To do this we follow the analogous argument given by Borisov in [2]. However, the bigrading of [2] is no longer preserved by our differentials and this is why we need the bigrading introduced in Section 3 instead. In Section 7 we prove that $\mathcal{B}_A(\mathbb{F})$ is quasi-isomorphic to a subcomplex $\mathcal{C}_A(\mathbb{F})$ which is in turn canonically isomorphic to $\mathcal{C}_{A^T}(\mathbb{F})$. Together with the results of Section 5, this proves unprojected Berglund-Hübsch duality. In Sections 8 and 9 we specialize to the p-adic case and observe that the constructions of the previous sections can be extended by replacing polynomials with overconvergent p-adic power series. While not changing cohomology, this allows for the extra room needed in order to define a natural chain-level Frobenius endomorphism Fr à la Dwork (see e.g. [9], [11]) whose compatibility with Berglund-Hübsch duality is then addressed. Finally, in Section 10 we illustrate our constructions by working out two simple examples.

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2. Invertible polynomials

Let $\mathbb{F}$ be a field and consider the map

$$W\colon \mathrm{GL}_n(\mathbb{Z}_{\geq 0}) \to \mathbb{F}[x] = \mathbb{F}[x_1, \ldots, x_n]$$

defined by

$$A \mapsto W_A(x) = \sum_{i=1}^{n} x^{e_i A},$$

where $\{e_i\}_{1 \leq i \leq n}$ is the standard basis of $\mathbb{Z}^n$, and for $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n_{\geq 0}$ we write $x^v = x_1^{v_1} \cdots x_n^{v_n}$. For simplicity, we assume that $\operatorname{char} \mathbb{F} = 0$

or $\operatorname{char}\mathbb{F} > \det A$. A matrix $A \in \mathrm{GL}_n(\mathbb{Z}_{\geq 0})$ is *Berglund-Hübsch* over $\mathbb{F}$ if $W_A(x)$ is an *invertible polynomial*, i.e. if $W_A(x)$ is quasi-homogeneous and $(\partial_1 W_A(x), \ldots, \partial_n W_A(x))$ is a regular sequence in $\mathbb{F}[x]$. For each $n \in \mathbb{Z}_{\geq 0}$ we let

$$\mathrm{BH}(\mathbb{F}) = \bigcup_n \mathrm{BH}_n(\mathbb{F}) \,,$$

where

$$\mathrm{BH}_n(\mathbb{F}) = \{A \in \mathrm{GL}_n(\mathbb{Z}_{\geq 0}) \mid A \text{ is Berglund-Hübsch over } \mathbb{F}\} \,.$$

Remark 2.1. Berglund-Hübsch matrices satisfy the following properties.

1) If $A \in \mathrm{BH}_n(\mathbb{F})$ and $B \in \mathrm{BH}_m(\mathbb{F})$, then $A \oplus B \in \mathrm{BH}_{n+m}(\mathbb{F})$.
2) If
$$\left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] \in \mathrm{BH}(\mathbb{F}),$$
then $C \in \mathrm{BH}(\mathbb{F})$. We call $A \in \mathrm{BH}_n(\mathbb{F})$ *irreducible* if it cannot be written as $B \oplus C$ with $B, C \in \bigcup_{m \leq n} \mathrm{BH}_m(\mathbb{F})$.
3) Let $\mathcal{W}_n \subseteq \mathrm{GL}_n(\mathbb{Z}_{\geq 0})$ be the Weyl group. Given $S \in \mathcal{W}_n$ and $A \in \mathrm{BH}_n(\mathbb{F})$, then $SA, AS \in \mathrm{BH}_n(\mathbb{F})$. Moreover,
$$W_{SA}(x) = W_A(x) \quad \text{and} \quad W_{AS}(x) = W_A(x) \cdot S \,,$$
where $\cdot$ denotes the right action of $\mathcal{W}_n$ on $\mathbb{F}[x]$ by permutation of the variables.

Remark 2.2. Let $A \in \mathrm{BH}_n(\mathbb{F})$ and suppose that $\mathbb{F}$ contains a primitive $(\det A)$-th root of unity ζ. We define the *group of scaling symmetries* of $A \in \mathrm{BH}_n(\mathbb{F})$ to be $G_A = \mathbb{Z}^n/(\mathbb{Z}^n A^T)$. The terminology is justified by the following observation. The group $\mathbb{Z}^n/(\det(A)\mathbb{Z}^n)$ acts on $\mathbb{F}[x]$ by

$$\mu \cdot x^\gamma = \zeta^{\gamma \mu^T} x^\gamma. \tag{2.1}$$

Under this action $\mu \cdot W_A(x) = W_A(x)$ if and only if $\mu A^T = (\det A)\lambda$ for some $\lambda \in \mathbb{Z}^n$. Let $\nu : G_A \to \mathbb{Z}^n/(\det(A)\mathbb{Z}^n)$ be such that $\nu(\lambda) = \lambda \det(A) A^{-T}$ for every $\lambda \in G_A$. Then ν provides a canonical identification between G_A and the stabilizer of $W_A(x)$ under the action defined by (2.1). In the rest of the paper we slightly abuse notation and identify each equivalence class $\lambda + \mathbb{Z}^n A^T \in G_A$ with its unique representative $\lambda \in \mathbb{Z}^n$ such that $0 \leq (\lambda A^{-T})_i < 1$ for all $i = 1, \ldots, n$. Using this identification, to $\lambda \in G_A$ we attach the subset $J_\lambda \subseteq \{1, \dots, n\}$ such that $i \in J_\lambda$ if and only if $(\lambda A^{-T})_i \in \mathbb{Q} \setminus \mathbb{Z}$. We define the submatrix A^λ of A such that $W_{A^\lambda}(x)$ is obtained from $W_A(x)$ by setting $x_i = 0$ whenever $i \in J_\lambda$.

Proposition 2.3 ([6]). *Let $A \in \mathrm{BH}_n(\mathbb{F})$ be irreducible. Then there exists $S \in \mathcal{W}_n$ such that $W_{AS}(x)$ is in one of the following canonical forms:*

1) a loop,

$$x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 \,,$$

2) a chain,

$$x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} \,.$$

Corollary 2.4. *Let $A \in \mathrm{BH}_n(\mathbb{F})$. Then*

1) $A^T \in \mathrm{BH}_n(\mathbb{F})$,

2) for each $\lambda \in G_A$, we have $A^\lambda \in \mathrm{BH}_{n-|J_\lambda|}(\mathbb{F})$, and

3) the matrix defined by

$$A^{\mathrm{orb}} := \bigoplus_{\lambda \in G_A} A^\lambda$$

is in $\mathrm{BH}_{n|G_A| - \sum |J_\lambda|}(\mathbb{F})$.

Corollary 2.5. *Let $A \in \mathrm{BH}_\mathrm{n}(\mathbb{F})$ and let $\beta \in \mathbb{Z}^n$ such that $(\beta A^{-1})_i \in \mathbb{Q} \setminus \mathbb{Z}$.*

1) If A is a chain, then $(\beta A^{-1})_j$, $(\beta A^{-T})_k \in \mathbb{Q} \setminus \mathbb{Z}$ for all $1 \le j \le i \le k \le n$.

2) If A is a loop, then $(\beta A^{-1})_j, (\beta A^{-1})_k \in \mathbb{Q} \setminus \mathbb{Z}$ for all $1 \le j, k \le n$.

Proof. Both statements follow from

$$A^T_{ii} (\beta A^{-T})_i + (\beta A^{-T})_{i+1} = \beta_i = (\beta A^{-1})_{i-1} + A_{ii} (\beta A^{-1})_i \,,$$

where i is considered modulo n in the case of loops. □

3. Exterior operators

Let $e_1, \dots, e_n$ be the standard generators of $\mathbb{F}^n$. We denote by $\bigwedge(\mathbb{F}^n)$ the exterior algebra $\bigwedge(\mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_n)$ viewed as a representation of the Clifford algebra $\mathrm{Cl}_n(\mathbb{F})$ with generators e_i (multiplication) and $e_i^\vee$ (contraction), and (odd) commutators $[e_i, e_j^\vee] = \delta_{ij}$ for all $1 \le i, j \le n$. As an $\mathbb{F}$-module, $\bigwedge(\mathbb{F}^n)$ is generated by monomials $e^I = e_1^{I_1} \cdots e_n^{I_n}$, where $I = (I_1, \ldots, I_n) \in \mathbb{Z}^n_{\ge 0}$. In particular, $e^I = 0$ if and only if $I_i \ge 2$ for some i. Given $A \in \mathrm{BH}_n(\mathbb{F})$ and $\pi \in \mathbb{F}^*$, for $1 \le i \le n$ we also consider

$$E_{A,i} = \pi \sum_{j=1}^n e_j A^T_{ji} \quad \text{and} \quad E^\vee_{A,i} = \frac{1}{\pi} \sum_{j=1}^n e_j^\vee (A^{-1})_{ji}\,,$$

so that

$$[E_{A,i} E^\vee_{A,j}] = \sum_{k,m} A^T_{ki} (A^{-1})_{mj} [e_k, e_m^\vee] = \sum_k A_{ik} (A^{-1})_{kj} = \delta_{ij}\,.$$

Lemma 3.1. *If* $*^A \in \mathrm{GL}\left(\bigwedge(\mathbb{F}^n)\right)$ *is defined by*

$$*^A(e_{i_1} \cdots e_{i_k}) = E^\vee_{A^T,i_1} E^\vee_{A^T,i_2} \cdots E^\vee_{A^T,i_k} \left(E_{A^T,1} E_{A^T,2} \cdots E_{A^T,n}\right),$$

then

1) $*^A E_{A,i} = e_i^\vee *^A$, $*^A E^\vee_{A,i} = e_i *^A$, *and*
2) $*^{A^T} *^A$ *commutes with the action of* $\mathrm{Cl}_n(\mathbb{F})$ *on* $\bigwedge(\mathbb{F}^n)$.

Proof. By definition,

$$*^A e_i = E^\vee_{A^T,i} *^A \quad \text{and} \quad *^A e_i^\vee = E_{A^T,i} *^A\,.$$

Therefore,

$$*^A E_{A,i} = *^A \pi \sum_j e_j A^T_{ji} = \pi \sum_j E^\vee_{A^T,j} A^T_{ji} *^A = \sum_{k,j} e_k^\vee (A^{-T})_{kj} A^T_{ji} *^A e_i^\vee *^A\ .$$

Similarly, $*^A E^\vee_{A,i} = e_i *^A$. This proves part (1). Part (2) follows from

$$*^{A^T} *^A e_i = *^{A^T} E^\vee_{A^T,i} *^A = e_i *^{A^T} *^A$$

and

$$*^{A^T} *^A e_i^\vee = *^{A^T} E_{A,i} *^A = e_i^\vee *^{A^T} *^A\,.$$

□

Remark 3.2. The operator

$$\mathrm{ext} = \sum_{i=1}^{n} e_i e_i^{\vee} = \sum_{i=1}^{n} E_{A,i} E_{A,i}^{\vee}$$

is diagonal on $\bigwedge(\mathbb{F}^n)$. If $\operatorname{char}\mathbb{F} = 0$, its eigenvalues count the total exterior degree. Moreover,

$$*^A \,\mathrm{ext} = \sum_{i=1}^{n} e_i^{\vee} e_i *^A = (n\,\mathrm{Id} - \mathrm{ext})\, *^A .$$

4. The basic complex

Given a graded vector space V endowed with a differential d of degree 1, we denote by (V, d) the corresponding chain complex and by $H(V, d)$ its cohomology. If V is bigraded and d, d' are differentials of bidegree $(1, 0)$ and $(0, 1)$ respectively, we denote the corresponding bicomplex by (V, d, d') and by $H(V, d, d')$ its total cohomology. If V is a vector space acted upon by a collection of commuting endomorphisms $\phi_1, \ldots, \phi_n$, we denote the corresponding Koszul complex by $\mathrm{Kos}(V, \phi_1, \ldots, \phi_n)$.

Given $A \in \mathrm{BH}_n(\mathbb{F})$, consider the subring $\widetilde{\mathcal{R}_A}(\mathbb{F})$ of $\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by monomials $x^\gamma y^\lambda$ such that $(\lambda A^{-T})_i \geq 0$ for all $1 \leq i \leq n$. We define $\mathcal{R}_A(\mathbb{F})$ to be the quotient of $\widetilde{\mathcal{R}_A}(\mathbb{F})$ by the ideal generated by monomials $x^\gamma y^\lambda$ for which $\gamma A^{-1} \lambda^T > 0$. Given $\pi \in \mathbb{F}^*$, we define $\theta_{A,i}, T_{A,i}^{\vee}, \psi_{A,i}^{\vee}, \varphi_{A,i}, \in \mathrm{End}_{\mathbb{F}}(\mathcal{R}_A(\mathbb{F}))$ by the formulas

$$\begin{aligned}
\theta_{A,i}(x^\gamma y^\lambda) &= \gamma_i\, x^\gamma y^\lambda\,; \\
T_{A,i}^{\vee}(x^\gamma y^\lambda) &= \pi^{-1} (\lambda A^{-T})_i\, x^\gamma y^\lambda\,; \\
\psi_{A,i}^{\vee}(x^\gamma y^\lambda) &= x^\gamma y^{\lambda + e_i A^T}\,; \\
\varphi_{A,i}(x^\gamma y^\lambda) &= \pi\,(\theta_i W_A(x))\, x^\gamma y^\lambda = \pi \sum_{j=1}^{n} A_{ji}\, x^{\gamma + e_j A} y^\lambda\,.
\end{aligned}$$

We also define the odd linear endomorphisms of $\mathcal{R}_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)$

$$d_{A,i} = (\theta_{A,i} + \varphi_{A,i})\, e_i\,, \quad d_A = \sum_{i=1}^{n} d_{A,i}$$

and

$$d^\vee_{A,i} = \left(T^\vee_{A,i} + \psi^\vee_{A,i}\right) e^\vee_i\,, \quad d^\vee_A = \sum_{i=1}^n d^\vee_{A,i}\,.$$

Lemma 4.1. $\mathcal{B}_A(\mathbb{F}) = (\mathcal{R}_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n), d_A + d^\vee_A)$ *is a chain complex.*

Proof. The morphism d_A is the Koszul differential for the sequence

$$(\theta_{A,1} + \varphi_{A,1}, \theta_{A,2} + \varphi_{A,2}, \ldots, \theta_{A,n} + \varphi_{A,n})$$

of commuting operators acting on $\mathcal{R}_A(\mathbb{F})$. Therefore, $[d_A, d_A] = 0$ and similarly $[d^\vee_A, d^\vee_A] = 0$. Moreover, since $(\theta_{A,i} + \varphi_{A,i})$ and $\left(T^\vee_{A,j} + \psi^\vee_{A,j}\right)$ commute,

$$\begin{aligned}[d_{A,i}, d^\vee_{A,j}] &= \left[(\theta_{A,i} + \varphi_{A,i})\, e_i, \left(T^\vee_{A,j} + \psi^\vee_{A,j}\right) e^\vee_j\right] \\ &= (\theta_{A,i} + \varphi_{A,i}) \left(T^\vee_{A,j} + \psi^\vee_{A,j}\right) \left[e_i, e^\vee_j\right] \\ &= (\theta_{A,i} + \varphi_{A,i}) \left(T^\vee_{A,j} + \psi^\vee_{A,j}\right) \delta_{ij}\,.\end{aligned}$$

If

$$0 \neq \left(\theta_{A,i}\, T^\vee_{A,i}\right)(x^\gamma y^\lambda) = \gamma_i\, (A^{-1}\lambda^T)_i\, x^\gamma y^\lambda\,,$$

then $x^\gamma y^\lambda = 0$ in $\mathcal{R}_A(\mathbb{F})$ and thus $(\theta_{A,i} + \varphi_{A,i})\left(T^\vee_{A,i} + \psi^\vee_{A,i}\right) = 0$. For

$$\left(\varphi_{A,i}\, T^\vee_{A,i}\right)(x^\gamma y^\lambda) = \sum_{j=1}^n A_{ji}(\lambda A^{-T})_i\, x^{\gamma + e_j A} y^\lambda$$

we note that if for some j we have $A_{ji}, (\lambda A^{-T})_i > 0$, then

$$(\gamma + e_j A)A^{-1}\lambda^T \geq (e_j A)(A^{-1}\lambda^T) = \sum_{m=1}^n A_{jm}(A^{-1}\lambda^T)_m \geq A_{ji}(A^{-1}\lambda^T)_i > 0$$

and conclude as before that $x^{\gamma+e_jA}y^\lambda = 0$ in $\mathcal{R}_A(\mathbb{F})$. It is similarly shown that $\varphi_{A,i}\,\psi^\vee_{A,i} = 0$ and $\theta_{A,i}\,\psi^\vee_{A,i} = 0$. Therefore, $[d_A, d^\vee_A] = 0$. □

Remark 4.2. Since for $\lambda \in G_A$ we take $0 \leq (\lambda A^{-T})_i < 1$ for each i by Remark 2.2, the condition $\gamma A^{-1}\lambda^T = 0$ imposed on a monomial $x^\gamma y^\lambda$ means that $\gamma_i = 0$ if λ acts non-trivially on x_i.

5. Bigrading

Let $P^{\vee}_{A,i} \in \mathrm{End}_{\mathbb{F}}\left(\mathcal{R}_A(\mathbb{F})\right)$ be given by

$$P^{\vee}_{A,i}(x^{\gamma}y^{\lambda}) = \begin{cases} 0, & \text{if } (\lambda A^{-T})_i = 0\,; \\ x^{\gamma}y^{\lambda}, & \text{otherwise}\,. \end{cases}$$

Lemma 5.1. *Let $Q_{A,i}, Q^{\vee}_{A,i}, Q_A$ and $Q^{\vee}_A$ be linear endomorphisms of $\mathcal{R}_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)$ defined by*

$$Q^{\vee}_{A,i} = P^{\vee}_{A,i}\, e^{\vee}_i e_i \quad \textit{and} \quad Q_{A,i} = e_i e^{\vee}_i + Q^{\vee}_{A,i}$$

as well as

$$Q_A = \sum_{i=1}^{n} Q_{A,i} \quad \textit{and} \quad Q^{\vee}_A = \sum_{i=1}^{n} Q^{\vee}_{A,i}\,.$$

Then for each $1 \le i, j \le n$,

1) $[Q_{A,i}, Q^{\vee}_{A,j}] = 0$*,*

2) $[Q^{\vee}_{A,i}, d_{A,j}] = 0$ *and* $[Q_{A,i}, d_{A,j}] = \delta_{ij} d_{A,j}$*,*

3) $[Q_{A,i}, d^{\vee}_{A,j}] = 0$ *and* $[Q^{\vee}_{A,i}, d^{\vee}_{A,j}] = \delta_{ij} d^{\vee}_{A,j}$*.*

Proof. The operators $Q_{A,i}$ and $Q^{\vee}_{A,j}$ commute because they have monomials of the form $x^{\gamma}y^{\lambda}e^{I}$ as a common basis of eigenvectors, which proves (1). For (2),

$$\begin{aligned} [Q^{\vee}_{A,i}, d_{A,j}] &= \left[P^{\vee}_{A,i}\, e^{\vee}_i e_i, (\theta_{A,j} + \varphi_{A,j})\, e_j\right] \\ &= P^{\vee}_{A,i}\, (\theta_{A,j} + \varphi_{A,j})\, [e^{\vee}_i e_i, e_j] \\ &= \delta_{ij} P^{\vee}_{A,i}\, (\theta_{A,j} + \varphi_{A,j})\, e^{\vee}_j\,. \end{aligned}$$

The proof of Lemma 4.1 shows that $P^{\vee}_{A,i}\, \varphi_{A,i} = 0$. Similarly, $P^{\vee}_{A,i}\, \theta_{A,i}(x^{\gamma}y^{\lambda}) \neq 0$ implies that $\gamma_i(A^{-1}\lambda^{T})_i > 0$ so that the corresponding term is 0 in $\mathcal{R}_A(\mathbb{F})$. Therefore, $[Q^{\vee}_{A,i}, d_{A,j}] = 0$, which in turn implies that

$$[Q_{A,i}, d_{A,j}] = (\theta_{A,j} + \varphi_{A,j})\, [e_i e^{\vee}_i, e_j] = \delta_{ij} d_{A,j}\,.$$

For part (3), if $i \neq j$

$$[P^{\vee}_{A,i}, T^{\vee}_{A,j} + \psi^{\vee}_{A,j}] = 0 = [e^{\vee}_i e_i, e^{\vee}_j]\,.$$

If $i=j$, then $e_i^\vee e_i^\vee e_i = 0$ and $e_i^\vee e_i e_i^\vee = e_i^\vee$, which means that

$$[Q^\vee_{A,i}, d^\vee_{A,j}] = \delta_{ij} P^\vee_{A,j}\left(T^\vee_{A,j} + \psi^\vee_{A,j}\right) e_i^\vee = \delta_{ij} d^\vee_{A,j}\,.$$

Similarly,

$$\begin{aligned}[Q_{A,i}, d^\vee_{A,j}] &= \left[e_i e_i^\vee, \left(T^\vee_{A,j} + \psi^\vee_{A,j}\right) e_j^\vee\right] + \delta_{ij} d^\vee_{A,j} \\ &= \delta_{ij}\left(-\left(T^\vee_{A,j} + \psi^\vee_{A,j}\right) e_j^\vee e_j e_j^\vee + d^\vee_{A,j}\right) = 0\,. \qquad \square\end{aligned}$$

Remark 5.2. In particular, with respect to the $\mathrm{Spec}(Q_A) \times \mathrm{Spec}(Q_A^\vee)$ bigrading, $\mathcal{B}_A(\mathbb{F})$ is the total complex of the bicomplex $\big(\mathcal{R}_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n), d_A, d_A^\vee\big)$.

6. Unprojected orbifold de Rham cohomology

Given $A \in \mathrm{BH}_n(\mathbb{F})$ and $\pi \in \mathbb{F}^*$, let

$$M_A(\mathbb{F}) = \mathbb{F}[x] e^{\pi W_A(x)}$$

be the module over the Weyl algebra $\mathbb{F}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ on which x_i acts by multiplication and ∂_i acts according to the formula

$$\partial_i \cdot P(x) = \partial_i P(x) + \pi(\partial_i W_A(x)) P(x) \tag{6.1}$$

for each $1 \le i \le n$ and $P(x) \in M_A(\mathbb{F})$. Note that $e^{\pi W_A(x)}$ is a formal symbol serving as a reminder of the Weyl algebra action. We denote by $\mathrm{DR}_A(\mathbb{F})$ the *de Rham complex* of $M_A(\mathbb{F})$, which is by definition the Koszul complex

$$\mathrm{Kos}\left(M_A(\mathbb{F}),\, \partial_1, \partial_2, \ldots, \partial_n\right),$$

where each ∂_i acts as in Equation (6.1). Given $\lambda \in \mathbb{Z}^n_{\ge 0}$ such that $(\lambda A^{-T})_i \ge 0$ for all $1 \le i \le n$, let $\mathcal{R}^\lambda_A(\mathbb{F}) \subseteq \mathcal{R}_A(\mathbb{F})$ be generated by monomials of the form $x^\gamma y^{\lambda + \mu A^T}$ for some $\gamma, \mu \in \mathbb{Z}^n_{\ge 0}$. Then $\mathcal{R}^\lambda_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)$ is closed under $d_A + d_A^\vee$ and we denote by $\mathcal{B}^\lambda_A(\mathbb{F}) \subseteq \mathcal{B}_A(\mathbb{F})$ the corresponding subcomplex.

Lemma 6.1. *If $A \in \mathrm{BH}_n(\mathbb{F})$, then*

1) $\mathcal{B}_A(\mathbb{F}) \cong \bigoplus_{\lambda \in G_A} \mathcal{B}^\lambda_A(\mathbb{F})$, *and*
2) $\mathcal{B}^\lambda_A(\mathbb{F})$ *is quasi-isomorphic to* $\mathcal{B}^0_{A^\lambda}(\mathbb{F})$.

Proof. Part (1) holds because $G_A = \mathbb{Z}^n/\mathbb{Z}^n A^T$ and $(\lambda A^{-T})_i \geq 0$ for $1 \leq i \leq n$. For (2), note that since $x_i y^\lambda = 0$ in $\mathcal{R}_A^\lambda(\mathbb{F})$ for any $i \in J_\lambda$

$$\mathcal{R}_A^\lambda(\mathbb{F}) \cong \mathcal{R}_{A^\lambda}^0(\mathbb{F}) \otimes \mathbb{F}[\psi_{A,i}^\vee]_{\{i\in J_\lambda\}}\, y^\lambda\,,$$

and thus

$$\mathcal{B}_A^\lambda(\mathbb{F}) \cong \mathcal{B}_{A^\lambda}^0(\mathbb{F}) \otimes \operatorname{Kos}\left(\mathbb{F}[\psi_{A,i}^\vee]_{\{i\in J_\lambda\}}\, y^\lambda, (T_{A,i}^\vee + \psi_{A,i}^\vee)_{\{i\in J_\lambda\}}\right)\,. \tag{6.2}$$

The cohomology of the second factor is isomorphic to $\mathbb{F}y^\lambda$, making the inclusion $\mathcal{B}_{A^\lambda}^0(\mathbb{F}) \hookrightarrow \mathcal{B}_A^\lambda(\mathbb{F})$ a quasi-isomorphism. □

Proposition 6.2. *The complex $\mathcal{B}_A^0(\mathbb{F})$ is canonically quasi-isomorphic to* $\mathrm{DR}_A(\mathbb{F})$.

Proof. The map $\Theta\colon M_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n) \to \mathcal{R}_A^0(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)$ defined by $\Theta(x^\gamma e^I) = x^{\gamma+I} e^I$ gives rise to an embedding

$$\mathrm{DR}_A(\mathbb{F}) \hookrightarrow \mathcal{B}_A^0(\mathbb{F})$$

of complexes. For each $\gamma \in \mathbb{Z}_{\geq 0}^n$, let $\bigwedge_\gamma = \bigwedge\left(\bigoplus_{\gamma_i=0} \mathbb{F}e_i\right)$. Then

$$\left(\mathcal{R}_A^0(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n), d_A^\vee\right) = \bigoplus_{\gamma,I} \mathcal{C}_{\gamma,I}\,,$$

where

$$\mathcal{C}_{\gamma,I} = \left(x^{\gamma+I}\mathbb{F}[y^{e_i A^T}]_{(\gamma+I)_i=0} \otimes e^I \bigwedge\nolimits_{\gamma+I}, \sum_{(\gamma+I)_i=0} d_{A,i}^\vee\right)$$

is a Koszul complex with cohomology $\mathbb{F}x^{\gamma+I}e^I$. This implies that

$$H\left(\frac{\mathcal{R}_A^0(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)}{\operatorname{Im}\Theta}, d_A^\vee\right) = 0\,,$$

and using the spectral sequence of first quadrant bicomplexes we conclude that

$$H\left(\mathcal{B}_A^0(\mathbb{F})/\mathrm{DR}_A(\mathbb{F})\right) = 0\,.$$

Therefore, the inclusion $\mathrm{DR}_A(\mathbb{F}) \hookrightarrow \mathcal{B}_A^0(\mathbb{F})$ is a quasi-isomorphism. □

Corollary 6.3. *Let $A \in \mathrm{BH}_n(\mathbb{F})$ and let $S(A)$ be the collection of monomials $x^\gamma e^I$ such that $|I| = n$ and $1 \leq \gamma_i \leq a_i = A_{ii}$ for all $i = 1, \ldots, n$. Then*

1) *$H(\mathcal{B}^0_A(\mathbb{F}))$ is isomorphic to the Milnor ring $\mathbb{F}[x]/dW_A(x)$.*
2) *If $W_A(x)$ is a loop, then $H(\mathcal{B}^0_A(\mathbb{F}))$ is generated by monomials in $S(A)$.*
3) *If $W_A(x)$ is a chain, then $H(\mathcal{B}^0_A(\mathbb{F}))$ is generated by those monomials in $S(A)$ of the form*

$$x_1^{a_1} x_2 x_3^{a_2} x_4 \cdots x_{2m-1}^{a_{2m-1}} x_{2m} x_{2m+1}^{\gamma_{2m+1}} \cdots x_n^{\gamma_n}$$

where $m \geq 0$ is such that $\gamma_{2m+1} < a_{2m+1}$.

Proof. Part (1) follows from Proposition 6.2 and the fact (see e.g. [9]) that there is a linear map from $\mathbb{F}[x]/dW_A(x)$ to $H(\mathrm{DR}_A(\mathbb{F}))$ sending monomials to monomials. Comparison with the standard monomial basis for the Milnor ring of chains and loops (see e.g. [6]) establishes (2) and (3). □

Proposition 6.4. *The natural inclusion of $\mathrm{DR}_{A^{\mathrm{orb}}}(\mathbb{F})$ into $\mathcal{B}_A(\mathbb{F})$ is a quasi-isomorphism.*

Proof. The proposition follows from Lemma 6.1 and Proposition 6.2. □

7. Unprojected duality

Given $A \in \mathrm{BH}_n(\mathbb{F})$ and $\pi \in \mathbb{F}^*$, let $\psi_{A,i}, T_{A,i} \in \mathrm{End}_{\mathbb{F}}(\mathcal{R}_A(\mathbb{F}))$ for $1 \leq i \leq n$ be defined by

$$\psi_{A,i}(x^\gamma y^\lambda) = x^{\gamma + e_i A} y^\lambda$$

and

$$T_{A,i}(x^\gamma y^\lambda) = \pi^{-1}(\gamma A^{-1})_i \, x^\gamma y^\lambda,$$

so that $d_A = \sum_{i=1}^n \hat{d}_{A,i}$, where

$$\hat{d}_{A,i} = (T_{A,i} + \psi_{A,i}) E_{A,i} \, .$$

Remark 7.1. Since we are using logarithmic differentials, e_i can be naturally interpreted as dx_i/x_i. One motivation for the change of basis to $E_{A,i}$ is the Shioda map $x^\gamma \mapsto z^{\gamma A^{-1} \det(A)}$ which sends W_A to $z^{e_1 \det(A)} + \cdots + z^{e_n \det(A)}$. If we interpret $E_{A,i}$ as dz_i/z_i, its definition is simply the chain rule.

Let $\mathcal{S}_A(\mathbb{F}) \subseteq \mathcal{R}_A(\mathbb{F})$ be generated by monomials $x^\gamma y^\lambda$ such that $(\gamma A^{-1})_i \geq 0$ for all $1 \leq i \leq n$. Then $\mathcal{S}_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)$ is closed under $d_A + d_A^\vee$. Let $\mathcal{C}_A(\mathbb{F}) \subseteq \mathcal{B}_A(\mathbb{F})$ denote the corresponding subcomplex.

Lemma 7.2. *The inclusion $\mathcal{C}_A(\mathbb{F}) \hookrightarrow \mathcal{B}_A(\mathbb{F})$ is a quasi-isomorphism.*

Proof. Consider the filtration

$$\mathcal{S}_A(\mathbb{F}) \subseteq F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^1 = \mathcal{R}_A(\mathbb{F}),$$

where F^i is spanned by monomials $x^\gamma y^\lambda$ such that $(\gamma A^{-1})_j \geq 0$ for all $j < i$. In particular, F^i/F^{i+1} is canonically identified with the space of monomials $x^\gamma y^\lambda$ such that $(\gamma A^{-1})_i < 0$. Consider the filtration $G^\bullet(\mathbb{F}) = F^\bullet \otimes \bigwedge(\mathbb{F}^n)$ of $\mathcal{R}_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)$. Notice that

$$\left(\frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})}; \, d_A, d_A^\vee\right)$$

is a bicomplex with respect to the $\mathrm{Spec}(Q_A) \times \mathrm{Spec}(Q_A^\vee)$ bigrading, while

$$\left(\frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})}; \, \hat{d}_{A,i}, d_A - \hat{d}_{A,i}\right)$$

is a bicomplex with respect to the

$$\mathrm{Spec}(E_{A,i}\, E_{A,i}^\vee) \times (\mathrm{ext} - E_{A,i}\, E_{A,i}^\vee)$$

bigrading. Therefore, in order to prove that

$$H\left(\frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})}, d_A + d_A^\vee\right) = 0\,, \tag{7.1}$$

it is sufficient to show that

$$H\left(\frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})}, \hat{d}_{A,i}\right) = 0\,. \tag{7.2}$$

If this is the case, the result then follows from the spectral sequence of the filtered complex $(\mathcal{B}_A(\mathbb{F}), G^\bullet(\mathbb{F}))$. To prove (7.2), we distinguish the following two cases.

First, suppose that $\operatorname{char}\mathbb{F} = 0$. In this case, $T_{A,i}$ acts by nonzero eigenvalues on F^i/F^{i+1}. By looking at the filtration of F^i/F^{i+1} by $\mathrm{Spec}(T_{A,i})$,

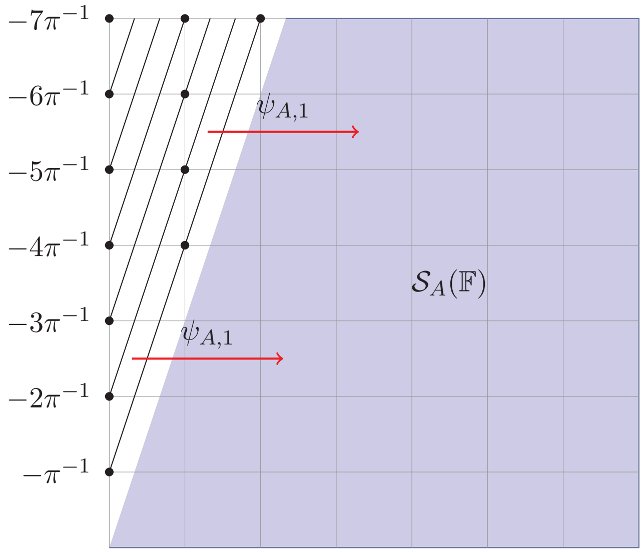

Figure 1: Eigenspaces of $T_{A,2}$ in F^2 for $W_A = x_1^2 + x_1x_2^3$, with eigenvalues designated along the left. Each point represents γ in x^γ.

we conclude that $T_{A,i} + \psi_{A,i}$ is injective. Therefore, $H\left(\frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})}, \hat{d}_{A,i}\right)$ is concentrated in top $\mathrm{Spec}(E_{A,i}E^\vee_{A,i})$-degree and isomorphic to the quotient

$$F^i/\left(F^{i+1} + \mathrm{Im}(T_{A,i} + \psi_{A,i})\right).$$

On the other hand, for each $f \in F^i$ there exists $N \in \mathbb{N}$ such that $\psi^n_{A,i} f \in F^{i+1}$, which implies (7.2). See Figure 1 for an illustration.

Second, suppose that $\mathrm{char}\,\mathbb{F} = p > \det A$. Let $\mathbb{K}$ be a field such that $\mathrm{char}\,\mathbb{K} = 0$ and $\mathbb{F} = \mathbb{K}/p\mathbb{K}$. Consider the short exact sequence of complexes

$$0 \longrightarrow \left(\frac{G^i(\mathbb{K})}{G^{i+1}(\mathbb{K})}, \hat{d}_{A,i}\right) \xrightarrow{\ p\ } \left(\frac{G^i(\mathbb{K})}{G^{i+1}(\mathbb{K})}, \hat{d}_{A,i}\right) \longrightarrow \left(\frac{G^i(\mathbb{F})}{G^{i+1}(\mathbb{F})}, \hat{d}_{A,i}\right) \longrightarrow 0\,.$$

Taking the long exact sequence and using the characteristic 0 case established above yields (7.2). □

Proposition 7.3. *Let* $D^A\colon \mathcal{S}_A(\mathbb{F}) \to \mathcal{S}_{A^T}(\mathbb{F})$ *be defined by* $D^A(x^\gamma y^\lambda) = x^\lambda y^\gamma$. *Then,*

$$\Delta^A = D^A \otimes *^A\colon \mathcal{S}_A(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n) \to \mathcal{S}_{A^T}(\mathbb{F}) \otimes \bigwedge(\mathbb{F}^n)$$

induces an isomorphism of complexes

$$\mathcal{C}_A(\mathbb{F}) \xrightarrow{\cong} \mathcal{C}_{A^T}(\mathbb{F})\,.$$

Proof. Since $D^{A^T} D^A = \mathrm{Id}$ and $*^A \in \mathrm{GL}\left(\bigwedge(\mathbb{F}^n)\right)$, we only need to prove that Δ^A is a chain map. Using

$$D^A \psi_{A,i} = \psi^\vee_{A^T,i} D^A \quad \text{and} \quad D^A T_{A,i} = T^\vee_{A^T,i} D^A\,,$$

it follows that

$$\Delta^A\left((T_{A,i} + \psi_{A,i})E_{A,i}\right) = \left((T^\vee_{A^T,i} + \psi^\vee_{A^T,i})e^\vee_i\right)\Delta^A$$

and

$$\Delta^A\left((T^\vee_{A,i} + \psi^\vee_{A,i})e^\vee_i\right) = \left((T_{A^T,i} + \psi_{A^T,i})E_{A^T,i}\right)\Delta^A. \qquad \square$$

Theorem 7.4 (Unprojected Berglund-Hübsch Duality). *The complexes* $\mathrm{DR}_{A^{\mathrm{orb}}}(\mathbb{F})$ *and* $\mathrm{DR}_{(A^T)^{\mathrm{orb}}}(\mathbb{F})$ *are canonically quasi-isomorphic.*

Proof. The theorem follows from Proposition 7.3, Lemma 7.2, and Proposition 6.4. $\square$

8. Overconvergent power series

Let $p \in \mathbb{Z}_{\geq 0}$ be a prime, $\mathbb{K} = \mathbb{C}_p$, $\mathbb{F} = \mathbb{K}/p\mathbb{K}$, $A \in \mathrm{BH}_n(\mathbb{F})$, and $\pi \in \mathbb{K}$ such that $\pi^{p-1} = -p$. Let $\widetilde{\mathcal{R}_A}^\dagger(\mathbb{K})$ be the ring of *overconvergent power series*

$$\sum_{\gamma,\lambda \in \mathbb{Z}^n_{\geq 0}} a_{\gamma,\lambda}\, x^\gamma y^\lambda$$

such that $(\lambda A^{-T})_i \geq 0$ for all $1 \leq i < n$, and such that there exists $M > 0$ for which

$$\mathrm{ord}_p(a_{\gamma,\lambda}) \geq M(|\gamma| + |\lambda|) \tag{8.1}$$

for all but finitely many γ, λ. Similarly, define $\mathcal{R}^\dagger_A(\mathbb{K})$, $\mathcal{S}^\dagger_A(\mathbb{K})$, $\mathcal{B}^\dagger_A(\mathbb{K})$, and $\mathcal{C}^\dagger_A(\mathbb{K})$ as before, by replacing polynomials with overconvergent power series.

Lemma 8.1. *1) The inclusions*

$$\begin{array}{ccc} \mathcal{B}_A(\mathbb{K}) & \longrightarrow & \mathcal{B}^\dagger_A(\mathbb{K}) \\ \uparrow & & \uparrow \\ \mathcal{C}_A(\mathbb{K}) & \longrightarrow & \mathcal{C}^\dagger_A(\mathbb{K}) \end{array}$$

are quasi-isomorphisms.

2) Δ^A *extends to an isomorphism of complexes*

$$\mathcal{C}^\dagger_A(\mathbb{K}) \xrightarrow{\cong} \mathcal{C}^\dagger_{A^T}(\mathbb{K}).$$

Proof. To prove (1), let $f = \sum a_{\gamma,\lambda}\, x^\gamma y^\lambda$ be an overconvergent power series. Since $\mathrm{ord}_p(a_{\gamma,\lambda}) \geq 1$ for all but finitely many γ, λ, by reducing modulo p we obtain a polynomial $\overline{f} \in \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Therefore, $\mathcal{R}_A(\mathbb{K})$ and $\mathcal{R}^\dagger_A(\mathbb{K})$ both reduce modulo p to $\mathcal{R}_A(\mathbb{F})$. Since $\mathcal{B}_A(\mathbb{F})$ and $\mathcal{C}_A(\mathbb{F})$ decompose into subcomplexes with cohomology concentrated in top degree, the statement follows from the long exact sequences of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}^\dagger_A(\mathbb{K}) & \xrightarrow{p} & \mathcal{B}^\dagger_A(\mathbb{K}) & \longrightarrow & \mathcal{B}_A(\mathbb{F}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \cong\uparrow & & \\ 0 & \longrightarrow & \mathcal{B}_A(\mathbb{K}) & \xrightarrow{p} & \mathcal{B}_A(\mathbb{K}) & \longrightarrow & \mathcal{B}_A(\mathbb{F}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

as in [9, Theorem 8.5]. Therefore $\mathcal{B}_A(\mathbb{K}) \hookrightarrow \mathcal{B}^\dagger_A(\mathbb{K})$ is quasi-isomorphism. Similarly, $\mathcal{C}_A(\mathbb{K}) \hookrightarrow \mathcal{C}^\dagger_A(\mathbb{K})$ is a quasi-isomorphism. The rest of the statement follows from Lemma 7.2.

Part (2) follows as in Proposition 7.3 after noticing that the overconvergence property is preserved by D^A. □

9. The Frobenius endomorphism

Let $p^{Q_A}, p^{Q_A^\vee} \in \mathrm{End}\left(\mathcal{R}^\dagger_A(\mathbb{K}) \otimes \bigwedge(\mathbb{K}^n)\right)$ be defined by

$$p^{Q_A}(x^\gamma y^\lambda e^I) = p^\xi x^\gamma y^\lambda e^I \quad \text{and} \quad p^{Q_A^\vee}(x^\gamma y^\lambda e^I) = p^{\xi^\vee} x^\gamma y^\lambda e^I,$$

where $Q_A(x^\gamma y^\lambda e^I) = \xi x^\gamma y^\lambda e^I$ and $Q_A^\vee(x^\gamma y^\lambda e^I) = \xi^\vee x^\gamma y^\lambda e^I$.

Lemma 9.1. *If* $\Theta'_A\colon \mathcal{R}^\dagger_A(\mathbb{K}) \to \mathcal{R}^\dagger_{pA}(\mathbb{K})$ *is defined by* $\Theta'_A(x^\gamma y^\lambda) = x^{p\gamma}y^{p\lambda}$, *then*

$$\mathrm{Fr}'_A = \left(\Theta'_A \otimes \mathrm{Id}_{\bigwedge(\mathbb{K}^n)}\right) p^{Q_A}$$

defines a chain map $\mathcal{B}^\dagger_A(\mathbb{K}) \to \mathcal{B}^\dagger_{pA}(\mathbb{K})$.

Proof. This follows from

$$\begin{aligned}\mathrm{Fr}'_A d_{A,i} &= \Theta'_A(\theta_{A,i} + \varphi_{A,i})e_i\, p^{Q_A+1} = d_{pA,i}\mathrm{Fr}'_A\,;\\ \mathrm{Fr}'_A d^\vee_{A,i} &= \Theta'_A(T^\vee_{A,i} + \psi^\vee_{A,i})e^\vee_i\, p^{Q_A} = d^\vee_{pA,i}\mathrm{Fr}'_A\,.\end{aligned}$$

□

Lemma 9.2. *If* $\Theta''_A\colon \mathcal{R}^\dagger_{pA}(\mathbb{K}) \to \mathcal{R}^\dagger_A(\mathbb{K})$ *is defined by*

$$\Theta''_A(x^\gamma y^\lambda) = Z_A(x)Z_{A^T}(y)x^\gamma y^\lambda,$$

where $Z_A(x) = e^{\pi(W_{pA}(x)-W_A(x))}$, *then*

$$\mathrm{Fr}''_A = \left(\Theta''_A \otimes \mathrm{Id}_{\bigwedge(\mathbb{K}^n)}\right) p^{Q^\vee_{pA}}$$

defines a chain map $\mathcal{B}^\dagger_{pA}(\mathbb{K}) \to \mathcal{B}^\dagger_A(\mathbb{K})$.

Proof. It is well known (see e.g. [9]) that $Z_A(x)$ satisfies (8.1). Therefore, Θ''_A is well defined. We compute

$$\begin{aligned}(\theta_{A,i} + \varphi_{A,i})\,\Theta''_A &= \theta_{A,i}\,(W_{pA}(x) - W_A(x))\,\Theta''_A + \Theta''_A\,\theta_{pA,i} + \varphi_{A,i}\,\Theta''_A\\ &= \Theta''_A\,\varphi_{pA,i} - \varphi_{A,i}\,\Theta''_A + \Theta''_A\,\theta_{pA,i} + \varphi_{A,i}\,\Theta''_A\\ &= \Theta''_A\,(\theta_{pA,i} + \varphi_{pA,i})\,,\end{aligned}$$

from which we see that

$$\mathrm{Fr}''_A\, d_{pA,i} = \Theta''_A\,(\theta_{pA,i} + \varphi_{pA,i})\, e_i\, p^{Q^\vee_{pA}} = d_{A,i}\,\mathrm{Fr}''_A\,.$$

To see that $\mathrm{Fr}''_A d^\vee_{pA,i} = d^\vee_{A,i}\mathrm{Fr}''_A$, note that for each $1 \le i \le n$, $T^\vee_{A,i}$ satisfies

$$\begin{aligned}&T^\vee_{A,i}\left(y^\lambda Z_{A^T}(y)\right)\\ &= T^\vee_{A,i}\left(y^\lambda e^{\pi\left(W_{pA^T}(y)-W_{A^T}(y)\right)}\right)\\ &= \pi\left(T^\vee_{A,i}W_{pA^T}(y) - T^\vee_{A,i}W_{A^T}(y)\right)y^\lambda Z_{A^T}(y) + Z_{A^T}(y)T^\vee_{A,i}y^\lambda\\ &= y^\lambda Z_{A^T}(y)\left(py^{pe_iA^T} - y^{e_iA^T}\right) + Z_{A^T}(y)\,p\pi^{-1}(\lambda\frac{1}{p}A^{-T})_i\,y^\lambda\\ &= y^\lambda Z_{A^T}(y)\left(p\psi^\vee_{pA,i} - \psi^\vee_{A,i}\right) + Z_{A^T}(y)\,pT^\vee_{pA,i}y^\lambda\,.\end{aligned}$$

We therefore have that

$$\begin{aligned}\left(T^\vee_{A,i}+\psi^\vee_{A,i}\right)\Theta''_A &= p\Theta''_A\psi^\vee_{pA,i}-\Theta''_A\psi^\vee_{A,i}+p\Theta''_A T^\vee_{pA,i}+\Theta''_A\psi^\vee_{A,i}\\ &= p\,\Theta''_A\left(T^\vee_{pA,i}+\psi^\vee_{pA,i}\right),\end{aligned}$$

and so

$$\mathrm{Fr}''_A\, d^\vee_{pA,i}=\Theta''_A\left(T^\vee_{pA,i}+\psi^\vee_{pA,i}\right)e_i\,p^{Q^\vee_A-1}=d^\vee_{A,i}\,\mathrm{Fr}''_A. \qquad \square$$

Lemma 9.3. *Let* $\widehat{P}_{A,i}\in\mathrm{End}_{\mathbb{K}}\left(\mathcal{S}^\dagger_A(\mathbb{K})\right)$ *be defined by*

$$\widehat{P}_{A,i}(x^\gamma y^\lambda)=\begin{cases}0 & \text{if } (\gamma A^{-1})_i=0\,;\\ x^\gamma y^\lambda & \text{otherwise}\,.\end{cases}$$

1) If we define

$$\begin{aligned}\widehat{Q}_{A,i}&=\widehat{P}_{A,i}\,E_{A,i}\,E^\vee_{A,i}\,;\\ \widehat{Q}^\vee_{A,i}&=E^\vee_{A,i}\,E_{A,i}+\widehat{Q}_{A,i}\,,\end{aligned}$$

then

$$\Delta^A\,Q^\vee_{A,i}=\widehat{Q}_{A^T,i}\,\Delta^A \quad \textit{and} \quad \Delta^A\,Q_{A,i}=\widehat{Q}^\vee_{A^T,i}\,\Delta^A\,.$$

2) If we define

$$\widehat{d}_{A,i}=(T_{A,i}+\psi_{A,i})\,E_{A,i} \quad \textit{and} \quad \widehat{D}^\vee_{A,i}=\left(\theta^\vee_{A,i}+\psi^\vee_{A,i}\right)E^\vee_{A,i}\,,$$

then

$$[\widehat{Q}_{A,i},\widehat{d}^\vee_{A,j}]=0=[\widehat{Q}^\vee_{A,i},\widehat{d}_{A,j}]$$

and

$$[\widehat{Q}_{A,i},\widehat{d}_{A,j}]=\delta_{ij}\,\widehat{d}_{A,j}\,;\quad [\widehat{Q}^\vee_{A,i},\widehat{d}^\vee_{A,j}]=\delta_{ij}\,\widehat{D}^\vee_{A,j}\,.$$

Proof. We compute

$$\begin{aligned}[\widehat{Q}_{A,i},\widehat{d}_{A,k}]&=[\widehat{P}_{A,i}\,E_{A,i}\,E^\vee_{A,i},(T_{A,j}+\psi_{A,j})\,E_{A,j}]\\ &=\delta_{ij}\,\widehat{P}_{A,i}\,(T_{A,j}+\psi_{A,j})\,E_{A,j}\\ &=\delta_{ij}\,\widehat{d}_{A,j},\end{aligned}$$

from which we see that

$$[\widehat{Q}^\vee_{A,i},\widehat{d}_{A,j}]=[E^\vee_{A,i}\,E_{A,i},\widehat{d}_{A,j}]+\delta_{ij}\,\widehat{d}_{A,j}=\delta_{ij}\left(-\widehat{d}_{A,j}+\widehat{d}_{A,j}\right)=0\,.$$

Similarly,

$$\begin{aligned}[\widehat{Q}_{A,i}, \widehat{d}^{\vee}_{A,j}] &= \widehat{P}_{A,i}\left(\theta^{\vee}_{A,j} + \psi^{\vee}_{A,j}\right)[E_{A,i}\, E^{\vee}_{A,i}, E^{\vee}_{A,j}] \\ &= -\delta_{ij}\, \widehat{P}_{A,i}\left(\theta^{\vee}_{A,i} + \varphi^{\vee}_{A,i}\right) E^{\vee}_{A,i}\,.\end{aligned}$$

Since $\widehat{P}_{A,i}\, \theta^{\vee}_{A,i}(x^{\gamma} y^{\lambda}) \neq 0$ implies $(\gamma A^{-1})_i (\lambda A^{-T})_i > 0$, then

$$(\gamma A^{-1})_i A_{ii} (\lambda A^{-T})_i > 0$$

and thus $\gamma A^{-1} \lambda^T > 0$. Therefore, $[\widehat{Q}_{A,i}, \widehat{d}^{\vee}_{A,j}] = 0$. As a consequence,

$$[\widehat{Q}^{\vee}_{A,i}, \widehat{d}^{\vee}_{A,j}] = [E^{\vee}_{A,i}\, E_{A,i}, \widehat{d}^{\vee}_{A,j}] = \delta_{ij} \widehat{d}^{\vee}_{A,j},$$

which concludes the proof of (2). For (1), we compute

$$\begin{aligned}&\Delta^A\, Q^{\vee}_{A,i} = D^A\, P^{\vee}_{A,i} \otimes *^A\, e^{\vee}_i\, e_i = \widehat{P}_{A^T,i}\, D^A \otimes E_{A^T,i}\, E^{\vee}_{A^T,i} *^A = \widehat{Q}_{A^T,i}\, \Delta^A\,; \\ &\Delta^A\, Q_{A,i} = \Delta\left(e_i\, e^{\vee}_i + Q^{\vee}_{A,i}\right) = \left(E^{\vee}_{A^T,i}\, E_{A^T,i} + \widehat{Q}_{A^T,i}\right) \Delta^A = \widehat{Q}^{\vee}_{A^T,i}\, \Delta^A\,.\end{aligned}$$

□

Proposition 9.4. *For each A in* $\mathrm{BH}_n(\mathbb{F})$ *the* Frobenius endomorphism *defined by*

$$\mathrm{Fr}_A = \left((\Theta''_A \Theta'_A) \otimes \mathrm{Id}_{\bigwedge(\mathbb{K}^n)}\right) p^{Q_A + Q^{\vee}_A}$$

is a chain map and

$$\Delta^A\, \mathrm{Fr}_A = \mathrm{Fr}_{A^T}\, \Delta^A\, p^{2\,\mathrm{ext}-n}\, p^{-2\widehat{Q}_A}\, p^{2Q^{\vee}_A}.$$

Proof. Since $p^{Q^{\vee}_{pA}}\, (\Theta'_A \otimes \mathrm{Id}) = (\Theta'_A \otimes \mathrm{Id})\, p^{Q^{\vee}_A}$, then $\mathrm{Fr}_A = \mathrm{Fr}''_A \mathrm{Fr}'_A$ is a chain map. For the second statement, $D^A\, \Theta''_A\, \Theta'_A = \Theta''_{A^T}\, \Theta'_{A^T}\, D^A$ implies

$$\begin{aligned}\Delta^A\, \mathrm{Fr}_A &= \Delta^A\left(\Theta''_A\, \Theta'_A \otimes \mathrm{Id}\right) p^{Q^{\vee}_A + Q_A} \\ &= \mathrm{Fr}_{A^T}\, p^{-Q_{A^T} - Q^{\vee}_{A^T}}\, \Delta^A\, p^{Q^{\vee}_A + Q_A} \\ &= \mathrm{Fr}_{A^T}\, \Delta^A\, p^{-\widehat{Q}_A - \widehat{Q}^{\vee}_A}\, p^{Q^{\vee}_A + Q_A} \\ &= \mathrm{Fr}_{A^T}\, \Delta^A\, p^{2\,\mathrm{ext}-n}\, p^{-2\widehat{Q}_A}\, p^{2Q^{\vee}_A}.\end{aligned}$$

□

Theorem 9.5. *Let $\#_A$ (respectively $\#^{\vee}_A$) be the operator on $\mathcal{S}_A(\mathbb{K})$ diagonalized by monomials and such that the eigenvalue of $x^{\gamma} y^{\lambda}$ is the number of non-integer entries of γA^{-1} (respectively λA^{-T}). If κ is such that*

$(\kappa\pi)^{p-1} = p$, *then the* twisted Frobenius endomorphism

$$\mathrm{TFr}_A = \mathrm{Fr}_A\,(\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}$$

is a chain map, and

$$H(\Delta^A)H(\mathrm{TFr}_A) = H(\mathrm{TFr}_{A^T})H(\Delta^A)\,.$$

Proof. Since $(\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}$ is diagonalized by monomials and acts trivially on $\bigwedge(\mathbb{K}^n)$, it commutes with $d_A + d_A^\vee$. Therefore, TFr_A is a chain map. Using Proposition 9.4, we calculate

$$\begin{aligned}\Delta^A\,\mathrm{TFr}_A &= \Delta^A\,\mathrm{Fr}_A\,(\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}\\ &= \mathrm{Fr}_{A^T}\,\Delta^A\,p^{2\,\mathrm{ext}-n}\,p^{-2\widehat{Q}_A}\,p^{2Q_A^\vee}\,(\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}\\ &= \mathrm{TFr}_{A^T}\,\Delta^A\,p^{2\,\mathrm{ext}-n}\,p^{-2\widehat{Q}_A}\,p^{2Q_A^\vee}\,(\kappa\pi)p^{(p-1)(\#_A - \#_A^\vee)},\end{aligned}$$

where the last step follows from

$$(\kappa\pi)^{-(p-1)(\#_{A^T} - \#_{A^T}^\vee)/2}\,\Delta^A = \Delta^A\,(\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}.$$

Therefore, the theorem is proven if the eigenvalues of

$$2\,\mathrm{ext} - n - 2\widehat{Q}_A + 2Q_A^\vee \quad \text{and} \quad -(\#_A - \#_A^\vee) \tag{9.1}$$

agree on a monomial basis $x^\gamma y^\lambda e^I$ for $H(\mathcal{B}_A^\lambda(\mathbb{K}))$ for each $\lambda \in G_A$. By Lemma 6.1 and Corollary 6.3 one can choose generators of the form $x^{\gamma+I}y^\lambda e^I$, where $|I| = n - |J_\lambda|$ and $0 \le (\lambda A^{-1})_i < 1$ for all $i = 1,\ldots,n$. In particular, the eigenvalue of $2\,\mathrm{ext} - n + 2Q_A^\vee - \#_A^\vee$ on $x^{\gamma+I}y^\lambda e^I \in S(A)$ is $|I|$. On the other hand, inspection of the bases for the cohomology of chains and loops given in Corollary 6.3 shows that $(2\widehat{Q}_A - \#_A) = \mathrm{ext}$ on $S(A)$, which concludes the proof. □

10. Examples

Example 10.1. Let $n = 1$ and $A_{11} = 2$. Then $W_A(x) = W_A^T(x) = x_1^2$ and $G_A = G_{A^T} = \mathbb{Z}/2\mathbb{Z}$. The exterior operators are $E_{A,1} = 2\pi e_1$ and $E_{A,1}^\vee =$

$\frac{1}{2\pi}e_1^\vee$. Moreover, $\mathcal{R}_A^0(\mathbb{F}) = \mathbb{F}[x_1] \oplus y_1^2\mathbb{F}[y_1^2]$ and $\mathcal{R}_A^1(\mathbb{F}) = y_1\mathbb{F}[y_1^2]$. The differentials are

$$d(x_1^{\gamma_1}) = \gamma_1 x_1^{\gamma_1} e_1 + 2\pi x_1^{\gamma_1+2} e_1;$$
$$d^\vee(y_1^{\lambda_1} e_1) = \frac{1}{2\pi}\lambda_1 y_1^{\lambda_1} + y_1^{\lambda_1+2}.$$

It follows that $H(\mathcal{B}_A^0(\mathbb{F})) = \mathbb{F}x_1e_1$ and $H(\mathcal{B}_A^1(\mathbb{F})) = \mathbb{F}y_1$ are mapped one into the other by Δ^A. The relations in cohomology are

$$x_1^{2k+1}e_1 = (-2\pi)^{-1}(2k-1)x_1^{2(k-1)+1}e_1 = \cdots = (-2\pi)^{-k}(2k-1)!!\, x_1e_1;$$
$$y_1^{2k+1} = (-2\pi)^{-1}(2k-1)y_1^{2(k-1)+1} = \cdots = (-2\pi)^{-k}(2k-1)!!\, y_1.$$

Let (c_m) be the sequence of rational numbers defined by

$$e^{\pi(t^p-t)} = \sum_{m\geq 0} c_m(-\pi)^m t^m.$$

The action of the twisted Frobenius map in cohomology is thus

$$\begin{aligned}
&H(\mathrm{TFr}_A)(x_1e_1)\\
&= p(\kappa\pi)^{(p-1)/2} e^{\pi(x_1^{2p}-x_1^2)} x_1^p e_1\\
&= p(\kappa\pi)^{(p-1)/2} \sum_{m\geq 0} c_m(-\pi)^m x_1^{2(m+\frac{p-1}{2})+1} e_1\\
&= p(\kappa\pi)^{(p-1)/2} \left(\sum_{m\geq 0} c_m(-\pi)^{-\frac{p-1}{2}} 2^{-(m+\frac{p-1}{2})}(2(m-1)+p)!!\right) x_1e_1\\
&= p\kappa^{(p-1)/2}\left(\left(\frac{p-1}{2}\right)! + \mathcal{O}(p)\right) x_1e_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&H(\mathrm{TFr}_A)(y_1)\\
&= p^2(\kappa\pi)^{-(p-1)/2}\left(\sum c_m(-\pi)^{-\frac{p-1}{2}} 2^{-(m+\frac{p-1}{2})}(2(m-1)+p)!!\right) y_1\\
&= p\kappa^{(p-1)/2}\left(\left(\frac{p-1}{2}\right)! + \mathcal{O}(p)\right) y_1.
\end{aligned}$$

Comparison with the non-commutative Weil conjectures of Kontsevich [5] seems to suggest a further overall rescaling of TFr_A. This is likely to be

G_A	λ	$(0,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
	λA^{-T}	$(0,0)$	$(\frac{1}{2},0)$	$(\frac{1}{3},\frac{1}{3})$	$(\frac{1}{6},\frac{2}{3})$	$(\frac{5}{6},\frac{1}{3})$	$(\frac{2}{3},\frac{2}{3})$
G_{A^T}	λ	$(0,0)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,2)$	$(1,3)$
	λA^{-1}	$(0,0)$	$(0,\frac{1}{3})$	$(0,\frac{2}{3})$	$(\frac{1}{2},\frac{1}{6})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{5}{6})$

Table 1: Elements of G_A and G_{A^T} for $W_A(x) = x_1^2x_2 + x_2^3$.

relevant for arithmetic applications. We hope to come back to this point in future work.

Example 10.2. Consider the dual chains $W_A(x) = x_1^2x_2 + x_2^3$ and $W_{A^T}(x) = x_1^2 + x_1x_2^3$. The elements of $G_A \cong \mathbb{Z}^2/\mathbb{Z}^2A^T$ and $G_{A^T} \cong \mathbb{Z}^2/\mathbb{Z}^2A$ are given in Table 1. We can find basis elements $x^\gamma y^\lambda e^I$ of $\mathcal{C}_A$ and $\mathcal{C}_{A^T}$ as described in the proof of Theorem 9.5. Each row of Table 2 contains a pair of elements dual under Δ^A (up to constants), as well as the eigenvalues of

$$Q_A + Q_A^\vee \quad \text{and} \quad (\#_A - \#_A^\vee)/2$$

applied to $x^\gamma y^\lambda e^I$. Here we are using $*^A(e_1e_2) = 1$, $*^A(e_2) = -E_{A^T,1} = -2\pi e_1$ and

$$*^A(1) = E_{A^T,1}E_{A^T,2} = (2\pi e_1)(\pi e_1 + 3\pi e_2) = 6\pi^2 e_1e_2.$$

Note also that

$$\Delta^A(x_1^2x_2e_1e_2) = y_1^2y_2 \equiv 3\pi x_1x_2^3e_1e_2,$$

since $(d_{A^T} + d_{A^T}^\vee)(e_1) = 3\pi x_1x_2^3e_1e_2 + y_1^2y_2$.

We now turn to writing $\mathrm{TFr}_A(x^\gamma y^\lambda e^I)$ in terms of this basis for a few elements. Since for any x^γ,

$$(\theta_{A,1} + \varphi_{A,1})(x^{\gamma+e_1A}) = \gamma_1 x^\gamma + \pi\left(2x^{\gamma+e_1A}\right);$$
$$(\theta_{A,2} + \varphi_{A,2})(x^{\gamma+e_2A}) = \gamma_2 x^\gamma + \pi\left(x^{\gamma+e_1A} + 3x^{\gamma+e_2A}\right),$$

in $H\left(\mathcal{B}_A^\lambda(\mathbb{F})\right)$ we have the relation

$$\gamma x^\gamma y^\lambda e^I = (-\pi)(x^{\gamma+e_1A}y^\lambda e^I, x^{\gamma+e_2A}y^\lambda e^I)A,$$

$\mathcal{C}_A$	$Q_A + Q_A^\vee$	$(\#_A - \#_A^\vee)/2$	$\mathcal{C}_{A^T}$	$Q_{A^T} + Q_{A^T}^\vee$	$(\#_{A^T} - \#_{A^T}^\vee)/2$
$x_1x_2e_1e_2$	2	1	y_1y_2	4	-1
$x_1x_2^2e_1e_2$	2	1	$y_1y_2^2$	4	-1
$x_1x_2^3e_1e_2$	2	1	$y_1y_2^3$	4	-1
$x_1^2x_2e_1e_2$	2	0	$x_1x_2^3e_1e_2$	2	0
$x_2y_1e_2$	3	0	$x_1y_2e_1$	3	0
$x_2^2y_1e_2$	3	0	$x_1y_2^2e_1$	3	0
y_1y_2	4	-1	$x_1x_2e_1e_2$	2	1
$y_1y_2^2$	4	-1	$x_1x_2^2e_1e_2$	2	1
$y_1^2y_2$	4	-1	$x_1^2x_2e_1e_2$	2	1
$y_1^2y_2^2$	4	-1	$x_1^2x_2^2e_1e_2$	2	1

Table 2: Duality between $\mathcal{C}_A$ and $\mathcal{C}_{A^T}$ for $W_A(x) = x_1^2x_2 + x_2^3$.

which implies for $i = 1, 2$ that

$$x^{\gamma+e_iA}y^\lambda e^I = (-\pi)^{-1}\left(\gamma A^{-1}\right)_i x^\gamma y^\lambda e^I.$$

Therefore, for $i = 1, 2$,

$$\begin{aligned}(10.1)\qquad x^{\gamma+k_ie_iA}y^\lambda e^I &= (-\pi)^{-1}\left((\gamma+(k_i-1)e_iA)A^{-1}\right)_i x^{\gamma+(k_i-1)e_iA}y^\lambda e^I \\ &= (-\pi)^{-2}\left((\gamma A^{-1})_i + (k_i-1)\right) \\ &\quad \left((\gamma A^{-1})_i + (k_i-2)\right) x^{\gamma+(k_i-2)e_iA}y^\lambda e^I \\ &= (-\pi)^{-k_i}\left((\gamma A^{-1})_i\right)_{(k_i)} x^\gamma y^\lambda e^I.\end{aligned}$$

Take $x_1x_2\,e_1e_2$ so that $\gamma = (1,1)$ and $\gamma A^{-1} = (\frac{1}{2}, \frac{1}{6})$. Suppose that p is a prime such that $6 \mid (p-1)$. Then we can write

$$(p,p) = (1,1) + \left(\frac{p-1}{2}, \frac{p-1}{6}\right)A,$$

which using Equation (10.1) gives

$$
\begin{aligned}
&\mathrm{TFr}_A(x_1x_2\,e_1e_2)\\
&= p^2(\kappa\pi)^{p-1}x_1^px_2^pZ_A(x)e_1e_2\\
&= p^3x_1^px_2^p\left(\sum_{k_1\geq 0}(-\pi)^{k_1}c_{k_1}x^{k_1e_1A}\right)\left(\sum_{k_2\geq 0}(-\pi)^{k_2}c_{k_2}x^{k_2e_2A}\right)e_1e_2\\
&= p^3(-\pi)^{-\frac{2(p-1)}{3}}\left(\sum_{k_1,k_2\geq 0}c_{k_1}c_{k_2}\left(\frac{1}{2}\right)_{(k_1+\frac{p-1}{2})}\left(\frac{1}{6}\right)_{(k_2+\frac{p-1}{6})}\right)x_1^1x_2^1\,e_1e_2,
\end{aligned}
$$

where we have used the fact that $Z_{A^T}(y) = 1 + \mathcal{O}(y_1, y_2)$. Next, consider

$$
\mathrm{TFr}_A(x_2^2y_1e_2) = p^3e^{\pi\left(x_2^{3p}-x_2^3\right)}e^{\pi\left(y_1^{2p}-y_1^p\right)}x_2^{2p}y_1^p\,e_2.
$$

By Equation (6.2), in cohomology we have the relation

$$
y^{\lambda+k_1'e_1A^T} = (-\pi)^{-k_1'}\left((\lambda A^{-T})_1\right)_{(k_1')}y^\lambda = (-\pi)^{k_1'}\left(\frac{3\lambda_1-\lambda_2}{6}\right)_{(k_1')}y^\lambda,
$$

which if $6 \mid (p-1)$ implies that

$$
\begin{aligned}
&\mathrm{TFr}_A(x_2^2y_2e_2)\\
&= p^3(-\pi)^{-\frac{7(p-1)}{6}}\left(\sum_{k_1',k_2\geq 0}c_{k_1'}c_{k_2}\left(\frac{1}{2}\right)_{(k_1'+\frac{p-1}{2})}\left(\frac{2}{3}\right)_{\left(k_2+\frac{2(p-1)}{3}\right)}\right)x_2^2y_1e_2.
\end{aligned}
$$

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