Multisymplectic formulation of Yang–Mills equations and Ehresmann connections

Frédéric Hélein

We present a multisymplectic formulation of the Yang–Mills equations. The connections are represented by normalized equivariant 1-forms on the total space of a principal bundle, with values in a Lie algebra. Within the multisymplectic framework we realize that, under reasonable hypotheses, it is not necessary to assume the equivariance condition a priori, since this condition is a consequence of the dynamical equations.

Introduction

The motivation of the following work was at first to provide a Hamiltonian formulation of the Yang–Mills system of equations which would be as much covariant as possible. This means that we look for a formulation which does not depend on choices of space-time coordinates nor on the trivialization of the principal bundle. Among all possible frameworks (covariant phase space, etc.) we favor the multisymplectic formalism which takes automatically into account the locality of fields theories. Following this approach we have been led to discover a new variational formulation of the Yang–Mills equations with nice geometrical features.

The origin of the multisymplectic formalism goes back to the discovery by V. Volterra in 1890 [29, 29] of generalizations of the Hamilton equations for variational problems with several variables. These ideas were first developed mainly around 1930 [4, 7, 24, 31] and in 1950 [6]. After 1968 this theory was geometrized in a way analogous to the construction of symplectic geometry by several mathematical physicists [10, 11, 23] and in particular by a group around W. Tulczyjew in Warsaw [20–22]. This theory has many

recent developments which we cannot report here (see e.g. [1, 9, 12, 15–17, 19, 25, 26] and, for surveys, [3, 8, 13, 14, 27]). Today the Hamilton–Volterra equations are often called the De Donder–Weyl equations for reference to [7, 31], which is inaccurate [14]. However in this paper we name them the HVDW equations for Hamilton–Volterra–De Donder–Weyl.

The basic concept is the notion of a multisymplectic $(n + 1)$-form ω on a smooth manifold $\mathcal{N}$, where n refers to the number of independent variables. The form ω is always *closed* and one often assumes that it is non degenerate, i.e. that the only vector field ξ on the manifold such that $\xi \lrcorner \omega = 0$ is zero. An extra ingredient is a Hamiltonian function $H : \mathcal{N} \longrightarrow \mathbb{R}$. One can then describe the solutions of the HVDW equations by oriented n-dimensional submanifolds Γ of $\mathcal{N}$ which satisfy the condition that, at any point $\text{M} \in \mathcal{N}$, there exists a basis $(X_1, \ldots, X_n)$ of $T_\text{M}\Gamma$ such that $X_1 \wedge \cdots \wedge X_n \lrcorner \omega = (-1)^n dH$. Equivalently one can replace ω by its restriction to the level set $H^{-1}(0)$ and describe the solutions as the submanifolds Γ of $H^{-1}(0)$ such that $X_1 \wedge \cdots \wedge X_n \lrcorner \omega = 0$ everywhere (plus some independence conditions, see e.g. [14]).

All that have led to elegant formulations of most variational problems in mathematical physics involving e.g. maps and sections of bundles. However the multisymplectic formulation of the Yang–Mills raises difficulties [18, 28], because the dynamical field is a connection and is subject to gauge invariance, hence their geometrical description is delicate. A possible approach consists in building a suitable reduction of the geometry of connections on a $\mathfrak{G}$-principal bundle as for instance in [2]. We follow another approach, which is based on ideas which are now quite standard since Élie Cartan: we lift the connection defined on some manifold $\mathcal{M}$ representing the space-time to the principal bundle $\mathcal{P}$ over $\mathcal{M}$ with structure group $\mathfrak{G}$. The connection is then represented by a $\mathfrak{g}$-valued 1-form $\boldsymbol{\eta}$ on $\mathcal{P}$ which satisfies a *normalization* (3) and an *equivariance* (4) hypothesis. Although a priori necessary the equivariance condition has the drawback of being a constraint on the first order derivatives of the field, which, to our opinion, is not a natural condition.

In the following we compute the Legendre transform for the Yang–Mills action by treating connections as normalized and equivariant $\mathfrak{g}$-valued 1-forms on $\mathcal{P}$. We find that the natural multisymplectic manifold can be built from the vector bundles $\mathfrak{g} \otimes T^*\mathcal{P}$ and $\mathfrak{g}^* \otimes \Lambda^{n+r-2}T^*\mathcal{P}$ over $\mathcal{P}$, where $n + r$ is the dimension of $\mathcal{P}$, $\mathfrak{g}$ is the structure Lie algebra and $\mathfrak{g}^*$ its dual vector space. These vector bundles are endowed with a canonical $\mathfrak{g}$-valued 1-form η and a canonical $\mathfrak{g}^*$-valued $(n + r - 2)$-form p respectively. Inside $\mathfrak{g} \otimes T^*\mathcal{P}$ we consider the subbundle $\mathfrak{g} \otimes^{\text{N}} T^*\mathcal{P}$ of *normalized* forms. Then the multisymplectic manifold corresponds more or less to the total space of the vector

bundle $\mathbb{R} \oplus_{\mathcal{P}} (\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}) \oplus_{\mathcal{P}} (g^* \otimes \Lambda^{n+r-2}T^*\mathcal{P})$, equipped with the $(n+r)$-form $\theta = \epsilon\beta \wedge \gamma + p \wedge (d\eta + \eta \wedge \eta)$, where ϵ a coordinate on $\mathbb{R}$ and $\beta \wedge \gamma$ is the volume form on $\mathcal{P}$. The Hamiltonian function H is (up to a factor $-\frac{1}{4}$) the squared norm of the coefficients $p^{\mu\nu}$ such that $p \wedge dx^\mu \wedge dx^\nu + p^{\mu\nu}\beta \wedge \gamma = 0$. Once this is done, we will see that we may **remove** the unnatural equivariance constraint and derive the corresponding generalized Hamilton equations without this assumption; then we discover that, if the structure group of the gauge theory is unimodular and compact (which is the case for $U(1)$ and all $SU(k)$'s), the dynamical Hamilton equations *force the $\mathfrak{g}$-valued 1-forms to be equivariant* and give us back the Yang–Mills equations.

What are the byproducts of this approach? The fact that we obtain a first order formulation of the Yang–Mills equations is not new. But, most importantly, this formulation works on the space of normalized $\mathfrak{g}$-valued 1-forms on a principal bundle which are not equivariant, i.e. which don't correspond to a connection in the usual sense. Instead these 1-forms correspond to Ehresmann connections on the total space of the bundle $\mathcal{P}$. However the classical Euler–Lagrange equations contains conditions which, under some hypotheses on the structure group, forces these fields to be equivariant on-shell and hence to correspond to a connection, which turns out to be also a solution of the Yang–Mills equation. Hence, although it is different from the standard Yang–Mills variational problem, this problem has the same classical solutions. We also note that our problem is invariant under an action of the standard gauge group of the usual Yang–Mills action, plus the action of another gauge group, which is Abelian and acts additively on the momentum variables.

It is interesting to note that our new Lagrangian density in (65) is not that mysterious and could have been invented out of the blue. The merit of the multisymplectic approach here is to provide a conceptual way to build it from the standard Yang–Mills action. In particular, performing the Legendre transform *by respecting the equivariance constraint* produces automatically the extra fields p^{aj} which play the role of Lagrange multipliers for this constraint. A more miraculous fact is however that these extra fields which may not be equivariant themselves are dynamically decoupled from the other fields if the gauge structure group is compact unimodular.

Various interesting questions can be set. It seems interesting to study the quantization of this model and, in particular, to explore the mass gap question [18] in this setting. Indeed one could expect that the elastic mechanism which replaces the usual equivariance constraint could induce a mass at the quantum level. Another point is that our formulation has some flavor of a Kaluza–Klein theory, so it would be interesting to study gravitational

theories by following similar ideas and to build a Kaluza–Klein gravitational theory where the mechanism of spontaneaous dynamical reduction that we observed here could be useful.

1. Geometric preliminaries

1.1. Yang–Mills gauge fields

We are interested in the critical points of a Yang–Mills action functional on an n-dimensional manifold $\mathcal{M}$ with coordinates $(x^1, \ldots, x^n)$. We fix some Lie group $\mathfrak{G}$, which will be the structure group of our gauge theory, and we denote by $\mathfrak{g}$ its Lie algebra. The fields are then $\mathfrak{g}$-valued 1-forms $\boldsymbol{A} = \boldsymbol{A}_\mu dx^\mu$ on $\mathcal{M}$. The curvature 2-form of $\boldsymbol{A}$ is $\boldsymbol{F} = d\boldsymbol{A} + \frac{1}{2}[\boldsymbol{A} \wedge \boldsymbol{A}]$. We will assume for simplicity that $\mathfrak{G}$ is a group of matrices and write equivalently $\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A}$. We have in local coordinates $\boldsymbol{F} = \frac{1}{2}\boldsymbol{F}_{\mu\nu}dx^\mu \wedge dx^\nu$, where $\boldsymbol{F}_{\mu\nu} := \frac{\partial \boldsymbol{A}_\nu}{\partial x^\mu} - \frac{\partial \boldsymbol{A}_\mu}{\partial x^\nu} + [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]$. We fix a pseudo-Riemannian metric $\mathsf{g}_{\mu\nu}$ on $\mathcal{M}$ and a metric h_{ij} on $\mathfrak{g}$ which is invariant under the adjoint action of $\mathfrak{G}$. Then the Yang–Mills action of $\boldsymbol{A}$ is

$$(1) \qquad \mathcal{YM}[\boldsymbol{A}] := \int_{\mathcal{M}} d\mathrm{vol}_{\mathsf{g}} \left(-\frac{1}{4}|\boldsymbol{F}|^2 \right)$$

where $|\boldsymbol{F}|^2 = \mathsf{g}^{\lambda\nu}(x)\mathsf{g}^{\mu\sigma}(x)\mathsf{h}_{ij}\boldsymbol{F}^i_{\lambda\mu}\boldsymbol{F}^j_{\nu\sigma}$ and $d\mathrm{vol}_{\mathsf{g}}$ is the pseudo-Riemannian measure on $\mathcal{M}$. This action is invariant by gauge transformations $\boldsymbol{A} \longmapsto f^{-1}df + f^{-1}\boldsymbol{A}f$, for any map f from $\mathcal{M}$ to $\mathfrak{G}$. It is well-known that one interprets geometrically a gauge field $\boldsymbol{A}$ as a connection on a principal bundle with structure group $\mathfrak{G}$ over $\mathcal{M}$. Our first task will be to recast this problem by replacing the gauge fields $\boldsymbol{A}$ by $\mathfrak{g}$-valued 1-forms defined on the total space of the principal bundle, which satisfy suitable conditions.

1.2. Working on the principal bundle

Let $\mathcal{P}$ be the total space of a principal bundle which is fibered over $\mathcal{M}$ and with structure group $\mathfrak{G}$. We denote by $\pi_{\mathcal{M}} : \mathcal{P} \longrightarrow \mathcal{M}$ the fibration map. We assume that $\mathfrak{G}$ is acting on the right on $\mathcal{P}$:

$$\begin{array}{ccc} \mathcal{P} \times \mathfrak{G} & \longrightarrow & \mathcal{P} \\ (z, g) & \longmapsto & z \cdot g = R_g(z) \end{array}$$

This induces an infinitesimal action of $\mathfrak{g}$: to any $\xi \in \mathfrak{g}$, we associate the vector field ρ_ξ on $\mathcal{P}$ defined by: $\forall z \in \mathcal{P}, \forall \xi \in \mathfrak{g}, \rho_\xi(z) := \frac{d}{dt}(z \cdot e^{t\xi})|_{t=0}$; we

also set $\rho_\xi(z) = z \cdot \xi$. For any $z \in \mathcal{P}$ the orbit of the $\mathfrak{G}$ action containing z is the fiber $\mathcal{P}_x$, where $x = \pi_{\mathcal{M}}(z)$; the tangent vector subspace to $\mathcal{P}_x$ at z is called the *vertical* subspace and is denoted by $V_z := \ker d(\pi_{\mathcal{M}})_z$. Since the map $\mathfrak{G} \ni g \longmapsto z \cdot g \in \mathcal{P}_x$ is a diffeomorphism, V_z is isomorphic to the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ through the differential of the latter diffeomorphism:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & T_z\mathcal{P}_x = V_z \\ \xi & \longmapsto & z \cdot \xi \end{array}$$

We denote by $\alpha_z : V_z \longrightarrow \mathfrak{g}$ the inverse map. Then $\alpha|_{\mathcal{P}_x}$ is a $\mathfrak{g}$-valued 1-form on $\mathcal{P}_x$ (the Maurer–Cartan form) and is characterized by one of the two following conditions: $\forall z \in \mathcal{P}_x$,

$$(2) \qquad [z \cdot \alpha_z(v) = v, \quad \forall v \in V_z] \quad \Longleftrightarrow \quad [\alpha_z(z \cdot \xi) = \xi, \quad \forall \xi \in \mathfrak{g}].$$

An *Ehresmann connection* on $\mathcal{P}$ is a distribution of 'horizontal' vector subspaces $(H_z)_{z \in \mathcal{P}}$, where $\forall z \in \mathcal{P}$, $H_z \subset T_z\mathcal{P}$ and:

$$\forall z \in \mathcal{P}, \quad H_z \oplus V_z = T_z\mathcal{P}.$$

Ehresmann connections can be described by using the space $\Gamma(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ of sections of the bundle $\mathfrak{g} \otimes T^*\mathcal{P}$ over $\mathcal{P}$, i.e. of $\mathfrak{g}$-valued 1-forms on $\mathcal{P}$. Indeed any Ehresmann connection $(H_z)_{z \in \mathcal{P}}$ can be defined by some $\boldsymbol{\eta} \in \Gamma(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ such that $\ker \boldsymbol{\eta}_z = H_z$, $\forall z \in \mathcal{P}$. The 1-form $\boldsymbol{\eta}$ is unique if, furthermore, it satisfies the *normalization* condition

$$(3) \qquad\qquad\qquad \boldsymbol{\eta}_z|_{V_z} = \alpha_z, \quad \forall z \in \mathcal{P}.$$

We denote by $\Gamma_{\mathrm{N}}(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ the subspace of $\boldsymbol{\eta} \in \Gamma(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ which satisfy (3). Alternatively we define the 'normalized' affine subbundle of the bundle $\mathfrak{g} \otimes T^*\mathcal{P}$ to be:

$$\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P} := \{(z, \eta) \in \mathfrak{g} \otimes T^*\mathcal{P}; \forall \xi \in \mathfrak{g}, \eta(z \cdot \xi) = \xi\}$$

and observe that $\Gamma_{\mathrm{N}}(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ is the space of sections of $\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$.

Among the forms in $\Gamma_{\mathrm{N}}(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ the ones which lift gauge fields in the sense of Paragraph 1.1 are characterized by the additional *equivariance* condition:

$$(4) \qquad\qquad \forall (g, z) \in \mathfrak{G} \times \mathcal{P}, \quad \left(R_g^*\boldsymbol{\eta}\right)_z = Ad_{g^{-1}} \circ \boldsymbol{\eta}_z = g^{-1}\boldsymbol{\eta}_z g,$$

where R_g^* is the pull-back by the right action mapping R_g. We denote by $\Gamma_{\mathrm{N}}^{\mathfrak{g}}(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ the subspace of *normalized* and *equivariant* $\mathfrak{g}$-valued 1-forms

on $\mathcal{P}$. Assuming that $\mathfrak{G}$ is connected, Condition (4) is equivalent to its Lie algebraic analogue:

$$(5) \qquad L_{\rho_\xi}\boldsymbol{\eta} + [\xi, \boldsymbol{\eta}] = 0, \quad \forall \xi \in \mathfrak{g},$$

where L_{ρ_ξ} is the Lie derivative. Lastly if $\boldsymbol{\eta} \in \Gamma_N^{\mathfrak{g}}(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ the quantity $d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta}$ represents the *curvature* of the connection defined by $\boldsymbol{\eta}$ on $\mathcal{M}$.

All that is made clear through a trivialization of $\mathcal{P}$. Let $\boldsymbol{\sigma} : \mathcal{M} \longrightarrow \mathcal{P}$ be a section of $\mathcal{P}$. Then

$$\begin{array}{ccc} \mathcal{M} \times \mathfrak{G} & \longrightarrow & \mathcal{P} \\ (x, g) & \longmapsto & \boldsymbol{\sigma}(x) \cdot g \end{array}$$

is a local diffeomorphism. Its inverse map:

$$\begin{array}{cccc} \varphi : & \mathcal{P} & \longrightarrow & \mathcal{M} \times \mathfrak{G} \\ & z & \longmapsto & (x, g) \end{array} \quad \text{where} \quad x = \pi_{\mathcal{M}}(z) \text{ and } \boldsymbol{\sigma}(x) \cdot g = z,$$

provides us with a coordinate system. In this setting (2) reads $\alpha|_{\mathcal{P}_x} = g^{-1}dg$. Using local coordinates $(x^1, \ldots, x^n)$ on $\mathcal{M}$ and the identification $\boldsymbol{\eta} \simeq \varphi^*\boldsymbol{\eta}$, we can translate the normalization condition (3) by:

$$(6) \qquad \boldsymbol{\eta}_{(x,g)} = g^{-1}dg + \boldsymbol{\eta}_\mu(x, g)dx^\mu.$$

Setting $\boldsymbol{A}_\mu(x, g) := g\boldsymbol{\eta}_\mu(x, g)g^{-1}$ and $\boldsymbol{A}_{(x,g)} := \boldsymbol{A}_\mu(x, g)dx^\mu$, (6) reads

$$(7) \qquad \boldsymbol{\eta}_{(x,g)} = g^{-1}dg + g^{-1}\boldsymbol{A}_{(x,g)}g.$$

We then have the identity

$$(8) \qquad d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta} = g^{-1}(d\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A})g.$$

Still assuming (6) the extra equivariance condition (4) then translates as $\boldsymbol{A}_\mu(x, g) = \boldsymbol{A}_\mu(x)$, i.e. that $\boldsymbol{A}_\mu$ does not depend on $g \in \mathfrak{G}$. If so,

$$(9) \qquad \boldsymbol{\eta}_{(x,g)} = g^{-1}dg + g^{-1}\boldsymbol{A}_x g, \quad \text{where} \quad \boldsymbol{A}_x := \boldsymbol{A}_\mu(x)dx^\mu.$$

In particular the pull-back of $\boldsymbol{\eta}$ by $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}^*\boldsymbol{\eta} = \boldsymbol{A}$ and, if $\boldsymbol{\sigma}' : \mathcal{M} \longrightarrow \mathcal{P}$ is another section, then there exists $\boldsymbol{\gamma} : \mathcal{M} \longrightarrow \mathfrak{G}$ such that $\boldsymbol{\sigma}'(x) = \boldsymbol{\sigma}(x) \cdot \boldsymbol{\gamma}(x)$, $\forall x \in \mathcal{M}$ and the pull-back of $\boldsymbol{\eta}$ by $\boldsymbol{\sigma}'$ is: $(\boldsymbol{\sigma}')^*\boldsymbol{\eta} = \boldsymbol{\gamma}^{-1}d\boldsymbol{\gamma} + \boldsymbol{\gamma}^{-1}\boldsymbol{A}\boldsymbol{\gamma}$. This shows the correspondence between the normalized and equivariant $\mathfrak{g}$-valued 1-forms on $\mathcal{P}$ on the one hand, and the connection 1-forms on the corresponding principal bundle up to gauge transformations on the other hand.

1.3. Coframe on the total space $\mathcal{P}$

Let $(t_1, \ldots, t_r)$ be a basis of $\mathfrak{g}$ and, for $i = 1, \ldots, r$, set $\rho_i := \rho_{t_i}$. We hence obtain a rank r family of tangent vector fields on $\mathcal{P}$ which, at every point $z \in \mathcal{P}$, spans the vertical subspace V_z. We also choose a local orthonormal moving frame $(\underline{e}_1, \ldots, \underline{e}_n)$ on $\mathcal{M}$. This means that we are given a reference pseudo-Riemannian metric $\mathbf{g}_{ab}$ with constant coefficients on $\mathbb{R}^n$ and that $\langle \underline{e}_a, \underline{e}_b \rangle = \mathbf{g}_{ab}$, $\forall a, b = 1, \ldots, n$. In order to obtain a moving frame on $\mathcal{P}$ we choose a section $\boldsymbol{\sigma} : \mathcal{M} \longrightarrow \mathcal{P}$ which induces a trivialization $z = \boldsymbol{\sigma}(x) \cdot g \simeq (x, g)$ and we set

$$e_a(z) := d(R_g \circ \boldsymbol{\sigma})_x(\underline{e}_a(x)) \simeq \underline{e}_a(x) \cdot g, \quad \text{for } a = 1, \ldots, n.$$

Then $(e_1, \ldots, e_n, \rho_1, \ldots, \rho_r)$ is a moving frame on $\mathcal{P}$. We define its dual coframe

$$(\beta^1, \ldots, \beta^n, \gamma^1, \ldots, \gamma^r),$$

i.e. the family of sections of $T^*\mathcal{P}$ such that $\beta^a(e_b) = \delta_b^a$, $\gamma^i(\rho_j) = \delta_j^i$ and $\beta^a(\rho_j) = \gamma^i(e_b) = 0$. This provides us with coordinates on $\mathfrak{g} \otimes T^*\mathcal{P}$: a point (z, η) in $\mathfrak{g} \otimes T^*\mathcal{P}$ (where $z \in \mathcal{P}$ and $\eta \in \mathfrak{g} \otimes T_z^*\mathcal{P}$) has the coordinates $(x, g, \eta_a^i, \eta_j^i)$, where $z = \boldsymbol{\sigma}(x) \cdot g$ and $\eta = t_i(\eta_a^i \beta^a + \eta_j^i \gamma^j)$.

Let $\underline{\nabla}$ be the Levi-Civita connection on $T\mathcal{M}$ for the metric $\mathbf{g}_{\mu\nu}$ on $\mathcal{M}$ and let $\omega_a^b \in \Omega^1(\mathcal{M})$ be the connection 1-forms of $\underline{\nabla}$ in the moving frame $(\underline{e}_1, \ldots, \underline{e}_n)$, i.e. such that $\underline{\nabla}_{\underline{e}_a} \underline{e}_b = \omega_b^c(\underline{e}_a)\underline{e}_c$. Using $\underline{\nabla}$ and the the choice of a section $\boldsymbol{\sigma}$, we construct a connection ∇ on $T\mathcal{P}$: we extend ω_a^b on $\mathcal{P}$ by letting $\omega_a^b \simeq (\pi_{\mathcal{M}})^*\omega_a^b$ and we set

$$\nabla_{e_a} e_b = \omega_b^c(e_a)e_c; \quad \nabla_{\rho_i} e_b = 0;$$
$$\nabla_{e_a} \rho_j = 0; \qquad \nabla_{\rho_i} \rho_j = 0.$$

This connection acts on sections $\boldsymbol{\eta}$ of $\Gamma(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$: if $\boldsymbol{\eta} = \boldsymbol{\eta}_a \beta^a + \boldsymbol{\eta}_i \gamma^i$, where $\boldsymbol{\eta}_a$ and $\boldsymbol{\eta}_i$ are functions on $\mathcal{P}$ with values in $\mathfrak{g}$, then $\forall v \in T_z\mathcal{P}$, $\nabla_v \boldsymbol{\eta} = \left(d\boldsymbol{\eta}_a(v) - \omega_a^b(v)\boldsymbol{\eta}_b\right)\beta^a + d\boldsymbol{\eta}_i(v)\gamma^i$. Moreover, because of the torsion free conditions

$$(10) \qquad\qquad\qquad d\beta^a + \omega_b^a \wedge \beta^b = 0,$$

we have the following expression for the exterior differential of $\boldsymbol{\eta}$,

$$(11) \qquad\qquad d\boldsymbol{\eta} = d\boldsymbol{\eta}_a \wedge \beta^a - \boldsymbol{\eta}_c \omega_b^c \wedge \beta^b + d\boldsymbol{\eta}_i \wedge \gamma^i + \boldsymbol{\eta}_i d\gamma^i.$$

Hence in particular the $\beta^a \wedge \beta^b$ component of the curvature $d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta}$ is

$$(12) \qquad \boldsymbol{F}_{ab} = (d\boldsymbol{\eta}_b - \boldsymbol{\eta}_c \omega_b^c)(e_a) - (d\boldsymbol{\eta}_a - \boldsymbol{\eta}_c \omega_a^c)(e_b) + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]$$
$$= (\nabla_{e_a} \boldsymbol{\eta})_b - (\nabla_{e_b} \boldsymbol{\eta})_a + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b].$$

In the decomposition $\boldsymbol{\eta} = \boldsymbol{\eta}_a \beta^a + \boldsymbol{\eta}_i \gamma^i$ of some $\boldsymbol{\eta} \in \Gamma(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$, conditions (3) and (5) can be expressed as follows. The normalization condition means that $\boldsymbol{\eta}_i = t_i$, so that (3) reads

$$(13) \qquad\qquad\qquad \boldsymbol{\eta} = \boldsymbol{\eta}_a \beta^a + t_i \gamma^i$$

and, since $L_{\rho_\xi} \beta^a = 0$, $\forall \xi \in \mathfrak{g}$, the equivariance condition (5) reads

$$(14) \qquad\qquad d\boldsymbol{\eta}_a(\rho_i) + [t_i, \boldsymbol{\eta}_a] = 0, \quad \forall i = 1, \dots, r.$$

Let us denote by c_{ij}^k the constants such that

$$(15) \qquad\qquad\qquad [t_i, t_j] = c_{ij}^k t_k,$$

where the summation over repeated indices is assumed. Then from the decomposition $g^{-1}dg = t_i \gamma^i$ and the zero curvature condition $d(g^{-1}dg) + (g^{-1}dg) \wedge (g^{-1}dg) = 0$ we deduce the relation

$$(16) \qquad\qquad\qquad d\gamma^i + \frac{1}{2} c_{jk}^i \gamma^j \wedge \gamma^k = 0.$$

To conclude, we define $\beta := \beta^1 \wedge \cdots \wedge \beta^n$ and $\gamma = \gamma^1 \wedge \cdots \wedge \gamma^r$ and, for $1 \leq a, b \leq n$, $1 \leq i, j \leq r$, we set (the symbol $\lrcorner$ denotes the interior product)

$$\beta_a := e_a \lrcorner \beta, \quad \beta_{ab} := e_b \lrcorner (e_a \lrcorner \beta), \quad \gamma_i := \rho_i \lrcorner \gamma, \quad \gamma_{ij} := \rho_j \lrcorner (\rho_j \lrcorner \gamma).$$

We note the following useful relations

$$(17) \qquad\qquad \beta^a \wedge \beta_b = \delta_b^a \beta, \quad \beta^a \wedge \beta^b \wedge \beta_{cd} = (\delta_c^a \delta_d^b - \delta_d^a \delta_c^b)\beta$$

and similarly

$$(18) \qquad\qquad \gamma^i \wedge \gamma_j = \delta_j^i \gamma, \quad \gamma^i \wedge \gamma^j \wedge \gamma_{kl} = (\delta_k^i \delta_l^j - \delta_l^i \delta_k^j)\gamma.$$

The following result will be helpful later on. We recall that, if $\xi \in \mathfrak{g}$, then $\mathrm{ad}_\xi : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the linear map defined by $\mathrm{ad}_\xi(\eta) = [\xi, \eta]$, $\forall \eta \in \mathfrak{g}$.

Lemma 1.1. *The following identities holds*

(i) For any $i = 1, \ldots, r$,

$$d\gamma_i + tr(ad_{t_i})\gamma = 0. \tag{19}$$

(ii) For any $a, b = 1, \ldots, n$,

$$d\beta_a = \omega_a^b \wedge \beta_b, \tag{20}$$

$$d\beta_{ab} = \omega_a^c \wedge \beta_{cb} + \omega_b^c \wedge \omega_{ac}. \tag{21}$$

Proof. The proof of (19) follows from the following computation, where we assume a summation over each repeated index and we use (16) and (18),

$$d\gamma_i = d\gamma^j \wedge \gamma_{ij} = -\frac{1}{2}c_{kl}^j\gamma^k \wedge \gamma^l \wedge \gamma_{ij} = -c_{ij}^j\gamma = -\mathrm{tr}(\mathrm{ad}_{t_i})\gamma;$$

(20) and (21) are obtained by similar computations, by using (10) and $\omega_b^a + \omega_a^b = 0$:

$$d\beta_a = d\beta^b \wedge \beta_{ab} = -\omega_c^b \wedge \beta^c \wedge \beta_{ab} = -\omega_b^b \wedge \beta_a + \omega_a^b \wedge \beta_b,$$
$$d\beta_{ab} = d\beta^c \wedge \beta_{abc} = -\omega_d^c \wedge \beta^d \wedge \beta_{abc} = -\omega_c^c \wedge \beta_{ac} + \omega_b^c \wedge \beta_{ac} - \omega_a^c \wedge \beta_{bc}.$$

$$\square$$

Identity (19) has the following straightforward consequence. We recall that a Lie algebra is unimodular iff $\mathrm{tr}(\mathrm{ad}_\xi) = 0$, $\forall \xi \in \mathfrak{g}$. Note that $U(1)$ and all $SU(k)$'s are unimodular.

Corollary 1.1. *Assume that $\mathfrak{g}$ is unimodular, then $d\gamma_i = 0$, $\forall i = 1, \ldots, r$.*

2. Towards the multisymplectic formulation

2.1. The multisymplectic framework

In order to set the multisymplectic framework it is simpler to start with an abstract general description: let $\mathcal{Z}$ be a m-dimensional manifold and consider the fiber bundle $\Lambda^m T^*\mathcal{Z}$ of m-forms over $\mathcal{Z}$. By using the fibration $\pi_{\mathcal{Z}} : \Lambda^m T^*\mathcal{Z} \longrightarrow \mathcal{Z}$ we define a canonical m-form $\theta^{\mathcal{Z}}$ on $\Lambda^m T^*\mathcal{Z}$ by $\theta^{\mathcal{Z}}_{(Z,\varpi)}(X_1, \ldots, X_m) := \varpi(\pi_{\mathcal{Z}}^* X_1, \ldots, \pi_{\mathcal{Z}}^* X_m)$, $\forall Z \in \mathcal{Z}$, $\forall \varpi \in \Lambda^m T_Z^*\mathcal{Z}$,

$\forall X_1, \ldots, X_m \in T_{(Z,\varpi)}(\Lambda^m T^*\mathcal{Z})$. If $\mathcal{Z}$ is itself fibered over a manifold $\mathcal{X}$ by a projection map $\pi_{\mathcal{X}} : \mathcal{Z} \longrightarrow \mathcal{X}$, this defines in each tangent space $T_Z\mathcal{Z}$ a vertical subspace V_Z which is the kernel of $\pi_{\mathcal{X}}^*$. We then define the subbundle of so-called $(m-1)$-horizontal forms (see [9])

$$\Lambda_1^m T^*\mathcal{Z} := \{(\varpi, Z) \in \Lambda^m T^*\mathcal{Z}; \forall v_1, v_2 \in V_Z, v_1 \wedge v_2 \lrcorner \varpi = 0\}.$$

This corresponds to assuming that each m-multilinear map $\varpi \in \Lambda_1^m T_Z^*\mathcal{Z}$ has a degree at most one in the vertical coordinates of vectors in $T_Z\mathcal{Z}$. Then $\Lambda_1^m T^*\mathcal{Z}$ is the geometrical framework for the so-called 'De Donder–Weyl' theory for sections of $\mathcal{Z}$ over $\mathcal{X}$ which are critical points of a first order variational problem [15]. We will denote by $\theta_1^{\mathcal{Z}}$ the restriction of $\theta^{\mathcal{Z}}$ to $\Lambda_1^m T^*\mathcal{Z}$.

We use this setting for $m = n + r$, $\mathcal{Z} = \mathfrak{g} \otimes T^*\mathcal{P}$ and $\mathcal{X} = \mathcal{P}$. Coordinate functions on $\Lambda_1^m T^*(\mathfrak{g} \otimes T^*\mathcal{P})$ are (x^μ, g) for a point $z \in \mathcal{P}$, (η_a^i, η_j^i) for the components of $\eta \in \mathfrak{g} \otimes T_z^*\mathcal{P}$ in the basis $(t_i \otimes \beta^a, t_i \otimes \gamma^j)$ and $(e, p_i^{ab}, p_i^{jb}, p_i^{aj}, p_i^{jk})$ for the components of $\varpi \in \Lambda_1^m T_{(z,\eta)}^*(\mathfrak{g} \otimes T^*\mathcal{P})$ in the basis $(\beta \wedge \gamma, d\eta_a^i \wedge \beta_b \wedge \gamma, d\eta_j^i \wedge \beta_b \wedge \gamma, (-1)^n d\eta_a^i \wedge \beta \wedge \gamma_j, (-1)^n d\eta_j^i \wedge \beta \wedge \gamma_k)$. The Poincaré–Cartan form $\theta_1^{\mathcal{Z}}$ then reads

$$(22) \qquad \theta_1^{\mathcal{Z}} = e\beta \wedge \gamma + p_i^{ab} d\eta_a^i \wedge \beta_b \wedge \gamma + p_i^{jb} d\eta_j^i \wedge \beta_b \wedge \gamma$$
$$+ (-1)^n p_i^{aj} d\eta_a^i \wedge \beta \wedge \gamma_j + (-1)^n p_i^{jk} d\eta_j^i \wedge \beta \wedge \gamma_k.$$

Since we are interested in normalized sections of $\mathfrak{g} \otimes T^*\mathcal{P}$, i.e. satisfying (3), we must actually work on $\Lambda_1^{n+r} T^*(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$. The latter space is a bundle over $\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$ and can actually be constructed through a reduction of $\Lambda_1^{n+r} T^*(\mathfrak{g} \otimes T^*\mathcal{P})$: we restrict ourself on $(\pi_{\mathfrak{g} \otimes T^*\mathcal{P}})^{-1}(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$ and for any $(z, \eta) \in \mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$, we replace the fiber $\Lambda_1^{n+r} T_{(z,\eta)}^*(\mathfrak{g} \otimes T^*\mathcal{P})$ by its quotient by the annihilator of $T_{(z,\eta)}(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$, i.e. the space of forms ϖ in $\Lambda_1^{n+r} T_{(z,\eta)}^*(\mathfrak{g} \otimes T^*\mathcal{P})$ such that $v \lrcorner \varpi = 0$, $\forall v \in T_{(z,\eta)}(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$. This amounts to impose (see also (13))

$$(23) \qquad\qquad\qquad\qquad \eta_j^i = \delta_j^i$$

and to assume that $(\tilde{e}, \tilde{p}_i^{ab}, \tilde{p}_i^{aj}, \tilde{p}_i^{jb}, \tilde{p}_i^{jk}) \sim (e, p_i^{ab}, p_i^{aj}, p_i^{jb}, p_i^{jk})$ whenever $(\tilde{e}, \tilde{p}_i^{ab}, \tilde{p}_i^{aj}) = (e, p_i^{ab}, p_i^{aj})$, so that we may forget about coordinates (p_i^{jb}, p_i^{jk}). Denoting simply by θ the restriction to $\Lambda_1^{n+r} T^*(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$ of $\theta_1^{\mathcal{Z}}$ given in (22), this leads to the simplification

$$(24) \qquad \theta = e\beta \wedge \gamma + p_i^{ab} d\eta_a^i \wedge \beta_b \wedge \gamma + (-1)^n p_i^{aj} d\eta_a^i \wedge \beta \wedge \gamma_j.$$

2.2. The Legendre correspondence

A Lagrangian for a gauge theory is a real valued function L defined on the bundle $T^*\mathcal{P} \otimes_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} (T(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T\mathcal{P})$ over $\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$, whose fiber at $(z,\eta) \in \mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$ is the space of linear maps $\lambda : T_z\mathcal{P} \longrightarrow T_{(z,\eta)}(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$ such that $d(\pi_{\mathcal{P}})_{(z,\eta)} \circ \lambda = \mathrm{Id}_{T_z\mathcal{P}}$ (this vector space can be canonically identified with $T^*\mathcal{P}_z \otimes (T_{(z,\eta)}(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T_z\mathcal{P}))$. We define coordinates $(x, g, \eta_a^i, \lambda_{a;b}^i, \lambda_{a;j}^i)$ on $T^*\mathcal{P} \otimes_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} (T(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T\mathcal{P})$ in a natural way from the ones on $\mathfrak{g} \otimes T^*\mathcal{P}$: for any $(z, \eta, \lambda) \in T^*\mathcal{P} \otimes_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} (T(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T\mathcal{P})$, take a section $\boldsymbol{\eta} \in \Gamma(\mathcal{P}, \mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$ such that $\boldsymbol{\eta}(z) = \eta$ and (viewing $\boldsymbol{\eta}$ as a map from $\mathcal{P}$ to the total space of the bundle $\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$) the differential of $\boldsymbol{\eta}$ at z is λ. Then λ has the coordinates $\lambda_{a;b}^i(z, \eta, \lambda) := (\nabla_{e_b}\boldsymbol{\eta}^i)_a = (d(\boldsymbol{\eta}_a^i)_z - \boldsymbol{\eta}_c^i\omega_a^c)(e_b)$ and $\lambda_{a;j}^i(z, \eta, \lambda) := (\nabla_{\rho_j}\boldsymbol{\eta}^i)_a = d(\boldsymbol{\eta}_a^i)_z(\rho_j)$.

However we have to take into account the following important fact. The problem we start with concerns gauge fields on a space-time manifold $\mathcal{M}$ but *not all* normalized $\mathfrak{g}$-valued 1-forms $\boldsymbol{\eta}$ on $\mathcal{P}$, so that we actually need to compute the Legendre correspondence along *equivariant* 1-forms $\boldsymbol{\eta}$. In view of (14) this means that we must impose the extra constraint on λ

$$(25) \qquad \lambda_{a;j}^i = [\eta_a, t_j]^i.$$

We denote by $T^*\mathcal{P} \otimes_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} (T(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T\mathcal{P})^{\mathfrak{g}}$ the submanifold of points $(z, \eta, \lambda) \in T^*\mathcal{P} \otimes_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} (T(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T\mathcal{P})$ which satisfy Condition (25).

The standard Yang–Mills Lagrangian in (1) has the following expression by using the moving frame (e_a, ρ_i):

$$(26) \qquad L(z, \eta, \lambda) = -\frac{1}{4}\mathsf{g}^{ac}\mathsf{g}^{bd}\mathsf{h}_{ij}F_{ab}^i F_{cd}^j,$$

where (see (12))

$$F_{ab}^i = \lambda_{b;a}^i - \lambda_{a;b}^i + [\eta_a, \eta_b]^i.$$

Such a Lagrangian induces a correspondence between $T^*\mathcal{P} \otimes_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} (T(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T\mathcal{P})^{\mathfrak{g}}$ and a submanifold of $\Lambda_1^{n+r}T^*(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$ as follows (see [15]).

Assume as in the previous section that the coframe (β^a, γ^i) is orthonormal, then the volume element $d\mathrm{vol}_{\mathsf{g}}$ in (1) is equal to $\beta \wedge \gamma$. We define function W on $(T^*\mathcal{P} \otimes_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} (T(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})/T\mathcal{P})^{\mathfrak{g}}) \times_{\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}} \Lambda_1^{n+r}T^*(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P})$ (sorry for the notation) by:

$$W(z, \eta, \lambda, \varpi) := \theta_{(z,\eta,\varpi)}(\lambda(e_1), \dots, \lambda(e_n), \lambda(\rho_1), \dots, \lambda(\rho_r)) - L(z, \eta, \lambda)$$

and we say that (z, η, λ) *is in correspondence with* (z, η, ϖ) *if* $\frac{\partial W}{\partial \lambda}(z, \eta, \lambda, \varpi)$ $= 0$. (This condition amounts to say that $(z, \eta, \lambda, \varpi)$ is a critical point of the restriction of W to the fiber of the map $(z, \eta, \lambda, \varpi) \longmapsto (z, \eta, \varpi)$.) If so the value of W at $(z, \eta, \lambda, \varpi)$ defines a function H of (z, η, ϖ), which is the Hamiltonian.

We now need to compute $\theta_{(z,\eta,\varpi)}(\lambda(e_1), \ldots, \lambda(e_n), \lambda(\rho_1), \ldots, \lambda(\rho_r))$. In order to avoid a messy computation we use the following trick: choose the right coframe (as we learned from Cartan). Here given some $(z, \eta, \lambda, \varpi)$, we replace the coframe $(\beta^a, \gamma^i, d\eta_a^i)$ by $(\beta^a, \gamma^i, \delta\eta_a^i)$ in the expression of $\theta_{(z,\eta,\varpi)}$, where

$$\delta\eta_a^i = d\eta_a^i - \lambda_{a;b}^i \beta^b - \eta_c^i \omega_a^c - \lambda_{a;j}^i \gamma^j,$$

so that

(27) $\forall v \in T_z \mathcal{P}, \quad \delta\eta_a^i(\lambda(v)) = 0.$

Hence by using (25),

(28) $d\eta_a^i = \delta\eta_a^i + \lambda_{a;b}^i \beta^b + \eta_c^i \omega_a^c + [\eta_a, t_j]^i \gamma^j$

This gives us by using (17) and (18)

$$\theta = e\beta \wedge \gamma + p_i^{ab}(\delta\eta_a^i + \lambda_{a;c}^i \beta^c + \eta_c^i \omega_a^c + [\eta_a, t_k]^i \gamma^k) \wedge \beta_b \wedge \gamma$$
$$+ (-1)^n p_i^{aj}(\delta\eta_a^i + \lambda_{a;c}^i \beta^c + \eta_c^i \omega_a^c + [\eta_a, t_k]^i \gamma^k) \wedge \beta \wedge \gamma_j$$

and, noting $\Gamma_{ab}^c := \omega_b^c(e_a)$ (so that $\omega_b^c = \Gamma_{ab}^c \beta^a$),

$$= e\beta \wedge \gamma + p_i^{ab}(\lambda_{a;b}^i + \eta_c^i \Gamma_{ba}^c)\beta \wedge \gamma + p_i^{aj}[\eta_a, t_j]^i \beta \wedge \gamma$$
$$+ p_i^{ab}\delta\eta_a^i \wedge \beta_b \wedge \gamma + (-1)^n p_i^{aj}\delta\eta_a^i \wedge \beta \wedge \gamma_j.$$

Hence by using (27) it follows that

$$\theta_{(z,\eta,\varpi)}(\lambda(e_1), \ldots, \lambda(e_n), \lambda(\rho_1), \ldots, \lambda(\rho_r))$$
$$= e + p_i^{ab}(\lambda_{a;b}^i + \eta_c^i \Gamma_{ba}^c) + p_i^{aj}[\eta_a, t_k]^i$$

and thus

$$W(z, \eta, \lambda, \varpi) = e + p_i^{ab}(\lambda_{a;b}^i + \eta_c^i \Gamma_{ba}^c) + p_i^{aj}[\eta_a, t_j]^i - L(z, \eta, \lambda).$$

We hence find immediately that the condition $\frac{\partial W}{\partial \lambda} = 0$ reads

(29) $p_i^{ab} = \frac{\partial L}{\partial \lambda_{a;b}^i}(z, \eta, \lambda).$

We apply this relation with the standard Yang–Mills action (26) and find

$$(30) \qquad p_i^{ab} = \mathsf{h}_{ij}\mathsf{g}^{ac}\mathsf{g}^{bd}F_{cd}^j = \mathsf{h}_{ij}\mathsf{g}^{ac}\mathsf{g}^{bd}\left(\lambda_{d;c}^j - \lambda_{c;d}^j + [\eta_c, \eta_d]^j\right).$$

We observe that ϖ is subject to the constraints

$$(31) \qquad\qquad\qquad\qquad p_i^{ab} + p_i^{ba} = 0.$$

We thus define the image of the Legendre correspondence:

$$\mathcal{N} := \{(z, \eta, \varpi) \in \Lambda_1^{n+r}T^*(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}); p_i^{ab} + p_i^{ba} = 0\}.$$

We still denote by θ the restriction of θ on $\mathcal{N}$ and set $\omega := d\theta$, $(\mathcal{N}, \omega)$ is the multisymplectic manifold we will work with. Assuming (30) we deduce from (26) that $L(z, \eta, \lambda) = -\frac{1}{4}\mathsf{h}^{ij}\mathsf{g}_{ac}\mathsf{g}_{bd}p_i^{ab}p_j^{cd}$ and, by using (31),

$$
\begin{aligned}
p_i^{ab}(\lambda_{a;b}^i + \eta_c^i\Gamma_{ba}^c) &= -\frac{1}{2}p_i^{ab}(\lambda_{b;a}^i + \eta_c^i\Gamma_{ab}^c - \lambda_{a;b}^i - \eta_c^i\Gamma_{ba}^c)\\
&= -\frac{1}{2}p_i^{ab}(\lambda_{b;a}^i - \lambda_{a;b}^i + [\eta_a, \eta_b]^i)\\
&\quad - \frac{1}{2}p_i^{ab}\eta_c^i(\Gamma_{ab}^c - \Gamma_{ba}^c) + \frac{1}{2}p_i^{ab}[\eta_a, \eta_b]^i\\
&= -\frac{1}{2}p_i^{ab}F_{ab}^i - \frac{1}{2}p_i^{ab}\eta_c^i(\Gamma_{ab}^c - \Gamma_{ba}^c) + \frac{1}{2}p_i^{ab}[\eta_a, \eta_b]^i\\
&= -\frac{1}{2}p_i^{ab}\mathsf{h}^{ij}\mathsf{g}_{ac}\mathsf{g}_{bd}p_i^{bd} - \frac{1}{2}p_i^{ab}\eta_c^i(\Gamma_{ab}^c - \Gamma_{ba}^c) + \frac{1}{2}p_i^{ab}[\eta_a, \eta_b]^i.
\end{aligned}
$$

We hence deduce the expression for the Hamiltonian function H

$$(32) \qquad H(z, \eta, \varpi) = e - \frac{1}{4}\mathsf{h}^{ij}\mathsf{g}_{ac}\mathsf{g}_{bd}p_i^{ab}p_j^{cd} - \frac{1}{2}p_i^{ab}\eta_c^i(\Gamma_{ab}^c - \Gamma_{ba}^c)$$
$$+ \frac{1}{2}p_i^{ab}[\eta_a, \eta_b]^i + p_i^{aj}[\eta_a, t_j]^i.$$

2.3. Change of coordinates

We will change the coordinates on $\mathcal{N}$ in order to simplify the expression of the Hamiltonian function and in such a way that θ depends on η uniquely through the quantity $d\eta + \eta \wedge \eta$. We set

$$\epsilon := e - \frac{1}{2}p_i^{ab}\eta_c^i(\Gamma_{ab}^c - \Gamma_{ba}^c) + \frac{1}{2}p_i^{ab}[\eta_a, \eta_b]^i + p_i^{aj}[\eta_a, t_j]^i.$$

We note then that

$$(33) \qquad H(z, \eta, \varpi) = \epsilon - \frac{1}{4} \mathsf{h}^{ij} \mathsf{g}_{ac} \mathsf{g}_{bd} p_i^{ab} p_j^{cd}.$$

Moreover, from (24),

$$(34) \quad \theta = \epsilon \beta \wedge \gamma + p_i^{ab} \left(d\eta_a^i \wedge \beta_b \wedge \gamma + \frac{1}{2} \eta_c^i (\Gamma_{ab}^c - \Gamma_{ba}^c) - \frac{1}{2} [\eta_a, \eta_b]^i \beta \wedge \gamma \right)$$
$$+ p_i^{aj} \left((-1)^n d\eta_a^i \wedge \beta \wedge \gamma_j - [\eta_a, t_j]^i \beta \wedge \gamma \right).$$

In order to transform this expression, we need some preliminaries. First by setting $\eta_a = t_i \eta_a^i$ and $\eta = \eta_a \beta^a + t_i \gamma_i$ (the canonical $\mathfrak{g}$-valued 1-form on $\mathfrak{g} \otimes^{\mathrm{N}} T^* \mathcal{P}$), we get by using (11):

$$d\eta = d\eta_a \wedge \beta^a - \eta_c \omega_b^c \wedge \beta^b - \frac{1}{2} [t_j, t_k] \gamma^j \wedge \gamma^k.$$

Since on the other hand

$$\eta \wedge \eta = \frac{1}{2} [\eta_a, \eta_b] \beta^a \wedge \beta^b + \frac{1}{2} [t_i, t_j] \gamma^i \wedge \gamma^j + [\eta_a, t_i] \beta^a \wedge \gamma^i,$$

we deduce that

$$(35) \quad d\eta + \eta \wedge \eta = d\eta_a \wedge \beta^a + \left(\frac{1}{2} [\eta_a, \eta_b] - \eta_c \Gamma_{ab}^c \right) \beta^a \wedge \beta^b + [\eta_a, t_i] \beta^a \wedge \gamma^i.$$

This implies by using (17) that

$$(36) \qquad (d\eta + \eta \wedge \eta) \wedge \beta_{ab} \wedge \gamma$$
$$= (d\eta_b \wedge \beta_a - d\eta_a \wedge \beta_b - \eta_c (\Gamma_{ab}^c - \Gamma_{ba}^c) \beta + [\eta_a, \eta_b] \beta) \wedge \gamma$$

and by using (18):

$$(37) \quad (d\eta + \eta \wedge \eta) \wedge \beta_a \wedge \gamma_j = (-1)^n \left((-1)^n d\eta_a \wedge \beta \wedge \gamma_j + [t_j, \eta_a] \beta \wedge \gamma \right).$$

Hence we deduce from (36) and (31) the second r.h.s. term of (34):

$$(38) \qquad p_i^{ab} \left(d\eta_a^i \wedge \beta_b + \frac{1}{2} \eta_c^i (\Gamma_{ab}^c - \Gamma_{ba}^c) \beta - \frac{1}{2} [\eta_a, \eta_b]^i \beta \right) \wedge \gamma$$
$$= -\frac{1}{2} p_i^{ab} (d\eta + \eta \wedge \eta)^i \wedge \beta_{ab} \wedge \gamma$$

and from (37) the last r.h.s. term of (34):

$$(39) \qquad p_i^{aj} \left((-1)^n d\eta_a^i \wedge \beta \wedge \gamma_j - [\eta_a, t_j]^i \beta \wedge \gamma \right)$$
$$= (-1)^n p_i^{aj} (d\eta + \eta \wedge \eta)^i \wedge \beta_a \wedge \gamma_j.$$

We thus deduce by summarizing (34), (38) and (39):

Proposition 2.1. *The Poincaré–Cartan form θ on $\mathcal{N}$ reads:*

$$(40) \qquad \theta = \epsilon\beta \wedge \gamma + p_i \wedge (d\eta + \eta \wedge \eta)^i,$$

where

$$(41) \qquad p_i := -\frac{1}{2} p_i^{ab} \beta_{ab} \wedge \gamma + (-1)^n p_i^{aj} \beta_a \wedge \gamma_j.$$

An alternative expression is $\theta = \epsilon\beta \wedge \gamma + p \wedge (d\eta + \eta \wedge \eta)$, where in the r.h.s a duality pairing between the $\mathfrak{g}^*$-valued coefficients of p and the $\mathfrak{g}$-valued coefficients of $d\eta + \eta \wedge \eta$ is assumed.

2.4. Re-interpretation of the previous result

Let us rephrase the previous result. We see a posteriori that the multisymplectic manifold $(\mathcal{N}, \omega)$, where $\omega = d\theta$ and θ is given by (40) and (41), has a simple alternative construction. We consider the pair of vector bundles $\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$ and $\mathfrak{g}^* \otimes \Lambda^{n+r-2} T^*\mathcal{P}$ over $\mathcal{P}$ (where $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$) and their fibered direct sum over $\mathcal{P}$ with $\mathbb{R}$:

$$\widetilde{\mathcal{N}} := \mathbb{R} \oplus_{\mathcal{P}} (\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}) \oplus_{\mathcal{P}} \left(\mathfrak{g}^* \otimes \Lambda^{n+r-2} T^*\mathcal{P} \right).$$

The base $\mathcal{P}$ is equipped with the volume form $\beta \wedge \gamma$ and ϵ is a coordinate on $\mathbb{R}$. Denote by (p^{ab}, p^{aj}, p^{jk}) the $\mathfrak{g}^*$-valued coordinates on the fibers of $\mathfrak{g}^* \otimes \Lambda^{n+r-2} T^*\mathcal{P}$ in the basis $(-\beta_{ab} \wedge \gamma, (-1)^n \beta_a \wedge \gamma_j, \beta \wedge \gamma_{jk})$. The bundle $\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}$ is equipped with the canonical $\mathfrak{g}$-valued 1-form η (which reads $\eta_a \beta^a + t_i \gamma^i$ in $\mathfrak{g}$-valued coordinates) and $\mathfrak{g}^* \otimes \Lambda^{n+r-2} T^*\mathcal{P}$ with the canonical $\mathfrak{g}^*$-valued $(n+r-2)$-form p (which reads $-\frac{1}{2} p^{ab} \beta_{ab} \wedge \gamma + (-1)^n p^{aj} \beta_a \wedge \gamma_j + \frac{1}{2} p^{jk} \beta \wedge \gamma_{jk}$ in $\mathfrak{g}^*$-valued coordinates). We also define the vector subbundles $\mathfrak{g}^* \otimes \Lambda_0^{n+r-2} T^*\mathcal{P} := \{(x, p \in \mathfrak{g}^* \otimes \Lambda^{n+r-2} T^*\mathcal{P}; \forall a, \beta^a \wedge p = 0\}$ and $\mathfrak{g}^* \otimes \Lambda_1^{n+r-2} T^*\mathcal{P} := \{(x, p \in \mathfrak{g}^* \otimes \Lambda^{n+r-2} T^*\mathcal{P}; \forall a, b, \beta^a \wedge \beta^b \wedge p = 0\}$. In coordinates $\mathfrak{g}^* \otimes \Lambda_0^{n+r-2} T^*\mathcal{P}$ is defined by the equations $p^{ab} = p^{aj} = 0$ and $\mathfrak{g}^* \otimes$

$\Lambda_1^{n+r-2}T^*\mathcal{P}$ by $p^{ab} = 0$. We have the obvious inclusions

$$\mathfrak{g}^* \otimes \Lambda_0^{n+r-2}T^*\mathcal{P} \ \subset \ \mathfrak{g}^* \otimes \Lambda_1^{n+r-2}T^*\mathcal{P} \ \subset \ \mathfrak{g}^* \otimes \Lambda^{n+r-2}T^*\mathcal{P}.$$

By setting $\theta := \epsilon\beta \wedge \gamma + p_i \wedge (d\eta + \eta \wedge \eta)^i$, we obtain the same expression as (40), because, in view of (35), all terms involving p_i^{jk} cancel. Hence $(\mathcal{N}, \theta)$ is recovered by quotienting out $\mathfrak{g}^* \otimes \Lambda^{n+r-2}T^*\mathcal{P}$ by $\mathfrak{g}^* \otimes \Lambda_0^{n+r-2}T^*\mathcal{P}$.

In this setting the Hamiltonian function H has also an intrinsic characterization: up to a factor $-\frac{1}{4}$, it is the squared norm of all quantities p^{ab} such that $p^{ab}\beta \wedge \gamma + \beta^a \wedge \beta^b \wedge p = 0$.

3. The HVDW equations

The multisymplectic form $\omega = d\theta$ on $\mathcal{N}$ is

$$(42) \qquad \omega = d\epsilon \wedge \beta \wedge \gamma + dp_i \wedge (d\eta + \eta \wedge \eta)^i + (d\eta \wedge \eta - \eta \wedge d\eta)^i \wedge p_i.$$

3.1. What do we want to do and how to proceed?

The geometrical expression of the *HVDW equations* in $(\mathcal{N}, \omega)$ for the Hamiltonian function H consists in a condition on an oriented submanifold $\mathbf{\Gamma}$ of $\mathcal{N}$ of dimension $n + r$ (representing the graph of a solution), which says that, for any point M of coordinates $(x, g, \eta_a^i, p_i^{ab}, p_i^{aj})$ of $\mathbf{\Gamma}$, if $(X_1, \ldots, X_n, Y_1, \ldots, Y_r)$ is a basis of the tangent space to $\mathbf{\Gamma}$ at M such that $\beta \wedge \gamma(X_1, \ldots, X_n, Y_1, \ldots, Y_r) = 1$, then

$$(43) \qquad (X_1 \wedge \cdots \wedge X_n \wedge Y_1 \wedge \cdots \wedge Y_r) \lrcorner \, \omega = (-1)^{n+r}dH$$

(see [15]). However for the Yang–Mills problem we started from a variational problem on *equivariant* $\mathfrak{g}$-valued 1-forms. But is is not clear a priori whether we should impose a similar constraint in the Hamiltonian version. In the following we will derive the HVDW equations in the most general case, i.e. without assuming any equivariance hypothesis a priori. The HVDW equations with an equivariance constraint will be simply obtained by adding this extra constraint to the dynamical equations. We will see however that both approaches work and that, under some reasonable hypotheses, they lead to the Yang–Mills system.

Any fixed $(n + r)$-dimensional submanifold $\mathbf{\Gamma}$ which is a graph can be represented as the image of an unique embedding of $\mathcal{P}$ in $\mathbb{R} \oplus_\mathcal{P} (\mathfrak{g} \otimes^N T^*\mathcal{P}) \oplus_\mathcal{P} (\mathfrak{g}^* \otimes \Lambda^{n+r-2}T^*\mathcal{P})$ of the form $\boldsymbol{u} : z \longmapsto (z, \boldsymbol{\epsilon}(z), \boldsymbol{\eta}(z), \boldsymbol{p}(z))$. It suffices to

estimate the l.h.s. of (43) when replacing $(X_1, \ldots, X_n, Y_1, \ldots, Y_r)$ by $(\boldsymbol{u}_* e_1, \ldots, \boldsymbol{u}_* e_n, \boldsymbol{u}_* \rho_1, \ldots, \boldsymbol{u}_* \rho_r)$. However a direct computation of this quantity can be very messy. So again we use the same trick as for the Legendre transform and, given some point M of $\boldsymbol{\Gamma}$ of coordinates $(z, \boldsymbol{\epsilon}(z), \boldsymbol{\eta}(z), \boldsymbol{p}(z))$, we replace the coframe $(\beta^a, \gamma^i, d\epsilon, d\eta^i_a, dp^{ab}_i, dp^{aj}_i)$ at M by $(\beta^a, \gamma^i, \delta\epsilon, \delta\eta^i_a, \delta p^{ab}_i, \delta p^{aj}_i)$, where

(44)
$$\begin{aligned}
\delta\epsilon &:= d\boldsymbol{\epsilon} - d\boldsymbol{\epsilon}(e_a)\beta^a - d\boldsymbol{\epsilon}(\rho_j)\gamma^i \\
\delta\eta^i_a &:= d\eta^i_a - d\boldsymbol{\eta}^i_a(e_b)\beta^b - d\boldsymbol{\eta}^i_a(\rho_j)\gamma^j \\
\delta p^{ab}_i &:= dp^{ab}_i - d\boldsymbol{p}^{ab}_i(e_c)\beta^c - d\boldsymbol{p}^{ab}_i(\rho_j)\gamma^j \\
\delta p^{aj}_i &:= dp^{aj}_i - d\boldsymbol{p}^{aj}_i(e_b)\beta^b - d\boldsymbol{p}^{aj}_i(\rho_j)\gamma^j,
\end{aligned}$$

It follows in particular that

(45)
$$\delta\epsilon \circ d\boldsymbol{u}_z = \delta\eta^i_a \circ d\boldsymbol{u}_z = \delta p^{ab}_i \circ d\boldsymbol{u}_z = \delta p^{aj}_i \circ d\boldsymbol{u}_z = 0.$$

It will be useful to introduce the covariant derivatives at M $\epsilon_{;a} := \nabla_{e_a}\boldsymbol{\epsilon} = d\boldsymbol{\epsilon}(e_a)$, $\epsilon_{;i} := \nabla_{\rho_i}\boldsymbol{\epsilon} = d\boldsymbol{\epsilon}(\rho_i)$, $\eta^i_{b;a} := (\nabla_{e_a}\boldsymbol{\eta}^i)_b = (d\boldsymbol{\eta}^i_b - \boldsymbol{\eta}^i_c\omega^c_b)(e_a)$, $\boldsymbol{p}^{ab}_{i;c} := (\nabla_{e_c}\boldsymbol{p}_i)^{ab} = (d\boldsymbol{p}^{ab}_i + \boldsymbol{p}^{db}_i\omega^a_d + \boldsymbol{p}^{ad}_i\omega^b_d)(e_c)$, etc., so that:

(46)
$$\begin{aligned}
d\epsilon &= \delta\epsilon + \boldsymbol{\epsilon}_{;a}\beta^a + \boldsymbol{\epsilon}_{;i}\gamma^i \\
d\eta^i_a &= \delta\eta^i_a + \boldsymbol{\eta}^i_{a;b}\beta^b + \boldsymbol{\eta}^i_c\omega^c_a + \boldsymbol{\eta}^i_{a;j}\gamma^j \\
dp^{ab}_i &= \delta p^{ab}_i + \boldsymbol{p}^{ab}_{i;c}\beta^c - \boldsymbol{p}^{cb}_i\omega^a_c - \boldsymbol{p}^{ac}_i\omega^b_c + \boldsymbol{p}^{ab}_{i;j}\gamma^j \\
dp^{aj}_i &= \delta p^{aj}_i + \boldsymbol{p}^{aj}_{i;b}\beta^b - \boldsymbol{p}^{cj}_i\omega^a_c + \boldsymbol{p}^{aj}_{i;j}\gamma^j,
\end{aligned}$$

where we assume implicitly that the symbols in bold characters denotes components of $\boldsymbol{u}$ at z such that $\boldsymbol{u}(z) = $ M. In the following we evaluate separately the terms in (42) in view of finding the HVDW equations.

3.2. The computation of $d\boldsymbol{p}_i \wedge (d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta})^i$

To enlight the notations we drop here the upper indices in $\boldsymbol{\eta}$, coefficients are thus $\mathfrak{g}$-valued. Substituting the expression for $d\eta_a$ in (46) and using (10) and (16) we obtain

$$\begin{aligned}
d\eta &= d\eta_a \wedge \beta^a + \boldsymbol{\eta}_a d\beta^a + t_i d\gamma^i \\
&= \left(\delta\eta_a + \boldsymbol{\eta}_{a;b}\beta^b + \boldsymbol{\eta}_c\omega^c_a + \boldsymbol{\eta}_{a;j}\gamma^j\right) \wedge \beta^a - \boldsymbol{\eta}_a\omega^a_b \wedge \beta^b - \frac{1}{2}[t_i, t_j]\gamma^i \wedge \gamma^j
\end{aligned}$$

hence

(47) $d\eta = \delta\eta_a \wedge \beta^a + \frac{1}{2}(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b})\beta^a \wedge \beta^b - \boldsymbol{\eta}_{a;j}\beta^a \wedge \gamma^j - \frac{1}{2}[t_i, t_j]\gamma^i \wedge \gamma^j.$

On the other hand

$$\begin{aligned} \eta \wedge \eta &= (\boldsymbol{\eta}_a\beta^a + t_i\gamma^i) \wedge (\boldsymbol{\eta}_b\beta^b + t_j\gamma^j) \\ &= \frac{1}{2}[\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]\beta^a \wedge \beta^b + [\boldsymbol{\eta}_a, t_j]\beta^a \wedge \gamma^j + \frac{1}{2}[t_i, t_j]\gamma^i \wedge \gamma^j. \end{aligned}$$

Hence

(48) $\qquad d\eta + \eta \wedge \eta = \delta\eta_a \wedge \beta^a + \frac{1}{2}(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b])\beta^a \wedge \beta^b$
$$- (\boldsymbol{\eta}_{a;j} - [\boldsymbol{\eta}_a, t_j])\beta^a \wedge \gamma^j.$$

On the other hand we need also to compute dp. We drop the lower indices, so that coefficients are now $\mathfrak{g}^*$-valued. This quantity splits in two terms:

(49) $\qquad\qquad dp = -\frac{1}{2}d(p^{ab}\beta_{ab} \wedge \gamma) + (-1)^n d(p^{ai}\beta_a \wedge \gamma_i).$

Substituting the expression for dp^{ab} given by (46) and using (21) we obtain for the first r.h.s. term of (49)

$$\begin{aligned} d(p^{ab}\beta_{ab} \wedge \gamma) &= dp^{ab} \wedge \beta_{ab} \wedge \gamma + \boldsymbol{p}^{ab}\omega_a^c \wedge \beta_{cb} \wedge \gamma + \boldsymbol{p}^{ab}\omega_b^c \wedge \beta_{ac} \wedge \gamma \\ &= (\delta p^{ab} + \boldsymbol{p}_{;c}^{ab}\beta^c - \boldsymbol{p}^{cb}\omega_c^a - \boldsymbol{p}^{ac}\omega_c^b + \boldsymbol{p}_{;i}^{ab}\gamma^i) \wedge \beta_{ab} \wedge \gamma \\ &\quad + (\boldsymbol{p}^{cb}\omega_c^a \wedge \beta_{ab} + \boldsymbol{p}^{ac}\omega_c^b \wedge \beta_{ab}) \wedge \gamma \\ &= \delta p^{ab} \wedge \beta_{ab} \wedge \gamma + (\boldsymbol{p}_{;b}^{ab}\beta_a - \boldsymbol{p}_{;a}^{ab}\beta_b) \wedge \gamma \end{aligned}$$

and for the second r.h.s. term of (49) we substitute the expression for dp^{ai} given by (46) and we use (20) and (19)

$$\begin{aligned} d(p^{ai}\beta_a \wedge \gamma_i) &= dp^{ai} \wedge \beta_a \wedge \gamma_i + \boldsymbol{p}^{ai}\omega_a^b \wedge \beta_b \wedge \gamma_i + (-1)^n \boldsymbol{p}^{ai}\beta_a \wedge \mathrm{tr}(\mathrm{ad}_{t_i})\gamma \\ &= (\delta p^{ai} + \boldsymbol{p}_{;b}^{ai}\beta^b - \boldsymbol{p}^{bi}\omega_b^a + \boldsymbol{p}_{;j}^{ai}\gamma^j) \wedge \beta_a \wedge \gamma_i \\ &\quad + \boldsymbol{p}^{bi}\omega_b^a \wedge \beta_a \wedge \gamma_i + (-1)^n\mathrm{tr}(\mathrm{ad}_{t_i})\boldsymbol{p}^{ai}\beta_a \wedge \gamma \\ &= \delta p^{ai} \wedge \beta_a \wedge \gamma_i + \boldsymbol{p}_{;a}^{ai}\beta \wedge \gamma_i - (-1)^n(\boldsymbol{p}_{;i}^{ai} - \mathrm{tr}(\mathrm{ad}_{t_i})\boldsymbol{p}^{ai})\beta_a \wedge \gamma. \end{aligned}$$

Hence

$$\begin{aligned} dp = &-\frac{1}{2}\delta p^{ab} \wedge \beta_{ab} \wedge \gamma - \boldsymbol{p}_{;b}^{ab}\beta_a \wedge \gamma \\ &+ (-1)^n\delta p^{ai} \wedge \beta_a \wedge \gamma_i + (-1)^n\boldsymbol{p}_{;a}^{ai}\beta \wedge \gamma_i - (\boldsymbol{p}_{;i}^{ai} - \mathrm{tr}(\mathrm{ad}_{t_i})\boldsymbol{p}^{ai})\beta_a \wedge \gamma \end{aligned}$$

or

$$(50) \qquad dp = -\frac{1}{2}\delta p^{ab} \wedge \beta_{ab} \wedge \gamma + (-1)^n \delta p^{ai} \wedge \beta_a \wedge \gamma_i$$
$$- (\boldsymbol{p}^{ab}_{;b} + \boldsymbol{p}^{ai}_{;i} - \text{tr}(\text{ad}_{t_i})\boldsymbol{p}^{ai})\beta_a \wedge \gamma + (-1)^n \boldsymbol{p}^{ai}_{;a}\beta \wedge \gamma_i.$$

The last step consists in computing the product $dp_i \wedge (d\eta + \eta \wedge \eta)^i$. For that purpose we split $dp = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4$, where

$$\Pi_1 := -\frac{1}{2}\delta p^{ab} \wedge \beta_{ab} \wedge \gamma; \qquad\qquad \Pi_2 := (-1)^n \delta p^{ai} \wedge \beta_a \wedge \gamma_i$$

$$\Pi_3 := -(\boldsymbol{p}^{ab}_{;b} + \boldsymbol{p}^{ai}_{;i} - \text{tr}(\text{ad}_{t_i})\boldsymbol{p}^{ai})\beta_a \wedge \gamma; \qquad \Pi_4 := (-1)^n \boldsymbol{p}^{ai}_{;a}\beta \wedge \gamma_i.$$

Similarly we split $d\eta + \eta \wedge \eta = H_1 + H_2 + H_3$, where

$$H_1 := \delta \eta_a \wedge \beta^a; \qquad\qquad H_2 := \frac{1}{2}(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b])\beta^a \wedge \beta^b$$

$$H_3 := -(\boldsymbol{\eta}_{a;j} - [\boldsymbol{\eta}_a, t_j])\beta^a \wedge \gamma^j.$$

All products $\Pi_J \wedge H_K$ vanish, except the following ones:

$$\Pi_1 \wedge H_1 = \frac{1}{2}(\delta \eta^i_b \wedge \delta p^{ab}_i \wedge \beta_a - \delta \eta^i_a \wedge \delta p^{ab}_i \wedge \beta_b) \wedge \gamma,$$

$$\Pi_2 \wedge H_1 = -(-1)^n \delta \eta^i_a \wedge \delta p^{aj}_i \wedge \beta \wedge \gamma_j,$$

$$\Pi_3 \wedge H_1 = -(\boldsymbol{p}^{ab}_{i;b} + \boldsymbol{p}^{aj}_{i;j} - \text{tr}(\text{ad}_{t_j})\boldsymbol{p}^{aj})\delta \eta^i_a \wedge \beta \wedge \gamma,$$

$$\Pi_1 \wedge H_2 = -\frac{1}{2}(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b])^i \delta p^{ab}_i \wedge \beta \wedge \gamma,$$

$$\Pi_2 \wedge H_3 = (\boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a])^i \delta p^{aj}_i \wedge \beta \wedge \gamma.$$

Hence using (31)

$$(51) \quad dp_i \wedge (d\eta + \eta \wedge \eta)^i = \delta \eta^i_b \wedge \delta p^{ab}_i \wedge \beta_a \wedge \gamma - (-1)^n \delta \eta^i_a \wedge \delta p^{aj}_i \wedge \beta \wedge \gamma_j$$
$$- \Big[(\boldsymbol{p}^{ab}_{i;b} + \boldsymbol{p}^{aj}_{i;j} - \text{tr}(\text{ad}_{t_j})\boldsymbol{p}^{aj})\delta \eta^i_a$$
$$+ \frac{1}{2}(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b])^i \delta p^{ab}_i$$
$$- (\boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a])^i \delta p^{aj}_i\Big] \wedge \beta \wedge \gamma.$$

3.3. The computation of $(d\eta \wedge \eta - \eta \wedge d\eta)^i \wedge p_i$

In the following we note $[d\eta \wedge \eta] := d\eta \wedge \eta - \eta \wedge d\eta$. From (47) we know that:

$$(52) \qquad d\eta = (\delta\eta_a + \boldsymbol{\eta}_{a;b}\beta^b + \boldsymbol{\eta}_{a;j}\gamma^j) \wedge \beta^a - \frac{1}{2}[t_i, t_j]\gamma^i \wedge \gamma^j.$$

On the other hand, we have $[d\eta \wedge \eta] = [d\eta, \boldsymbol{\eta}_a] \wedge \beta^a + [d\eta, t_j] \wedge \gamma^j$ and thus

$$
\begin{aligned}
&[d\eta \wedge \eta]^i \wedge p_i \\
&= \left([d\eta, \boldsymbol{\eta}_b]^i \wedge \beta^b + [d\eta, t_j]^i \wedge \gamma^j\right) \wedge \left(-\frac{1}{2}\boldsymbol{p}_i^{ac}\beta_{ac} \wedge \gamma + (-1)^n \boldsymbol{p}_i^{ak}\beta_a \wedge \gamma_k\right) \\
&= -\frac{1}{2}[d\eta, \boldsymbol{\eta}_b]^i \wedge \boldsymbol{p}_i^{ac}(\delta_c^b\beta_a - \delta_a^b\beta_c) \wedge \gamma + (-1)^n[d\eta, \boldsymbol{\eta}_b]^i \wedge \boldsymbol{p}_i^{ak}\delta_a^b\beta \wedge \gamma_k \\
&\quad + (-1)^n[d\eta, t_j]^i \wedge \boldsymbol{p}_i^{ak}(-1)^{n-1}\delta_k^j\beta_a \wedge \gamma,
\end{aligned}
$$

which gives us

$$(53) \quad
\begin{aligned}
[d\eta \wedge \eta]^i \wedge p_i = &-[d\eta, \boldsymbol{\eta}_b]^i \wedge \boldsymbol{p}_i^{ab}\beta_a \wedge \gamma + (-1)^n[d\eta, \boldsymbol{\eta}_a]^i \wedge \boldsymbol{p}_i^{ak}\beta \wedge \gamma_k \\
&- [d\eta, t_j]^i \wedge \boldsymbol{p}_i^{aj}\beta_a \wedge \gamma.
\end{aligned}
$$

The r.h.s. of (53) is the sum of the three quantities $M_1 := -[d\eta, \boldsymbol{\eta}_b]^i \wedge \boldsymbol{p}_i^{ab}\beta_a \wedge \gamma$, $M_2 := (-1)^n[d\eta, \boldsymbol{\eta}_a]^i \wedge \boldsymbol{p}_i^{ak}\beta \wedge \gamma_k$ and $M_3 := -[d\eta, t_j]^i \wedge \boldsymbol{p}_i^{aj}\beta_a \wedge \gamma$. When substituting the value of $d\eta$ given by (52) in (53), we see that M_2 vanishes and we just have

$$(54) \qquad [d\eta \wedge \eta]^i \wedge p_i = \boldsymbol{p}_i^{ab}[\eta_b, \delta\eta_a]^i \wedge \beta \wedge \gamma + \boldsymbol{p}_i^{aj}[t_j, \delta\eta_a]^i \wedge \beta \wedge \gamma.$$

It is here useful to note that the summation over i of the quantities $\boldsymbol{p}_i^{ab}[\eta_b, \delta\eta_a]^i$ is a duality product between $\boldsymbol{p}^{ab} \in \mathfrak{g}^*$ and $[\boldsymbol{\eta}_b, \delta\eta_a] = \mathrm{ad}_{\boldsymbol{\eta}_b}(\delta\eta_a) \in \mathfrak{g}$. It thus coincides with the duality product between $\mathrm{ad}_{\boldsymbol{\eta}_b}^*(\boldsymbol{p}^{ab}) \in \mathfrak{g}^*$ and $\delta\eta_a \in \mathfrak{g}$, i.e. with $\left(\mathrm{ad}_{\boldsymbol{\eta}_b}^*(\boldsymbol{p}^{ab})\right)_i\delta\eta_a^i$, where $\mathrm{ad}_{\boldsymbol{\eta}_b}^*$ is the adjoint of $\mathrm{ad}_{\boldsymbol{\eta}_b}$. Similarly we have $\boldsymbol{p}_i^{aj}[t_j, \delta\eta_a]^i = \left(\mathrm{ad}_{t_j}^*(\boldsymbol{p}^{aj})\right)_i\delta\eta_a^i$. Hence (54) reads

$$(55) \qquad [d\eta \wedge \eta]^i \wedge p_i = \left(\left(\mathrm{ad}_{\boldsymbol{\eta}_b}^*(\boldsymbol{p}^{ab})\right)_i + \left(\mathrm{ad}_{t_j}^*(\boldsymbol{p}^{aj})\right)_i\right)\delta\eta_a^i \wedge \beta \wedge \gamma.$$

3.4. Conclusion

Collecting (46), (51) and (55) and substituting in (42), we obtain

$$
(56) \quad \omega = \delta\eta_b^i \wedge \delta p_i^{ab} \wedge \beta_a \wedge \gamma - (-1)^n \delta\eta_a^i \wedge \delta p_i^{aj} \wedge \beta \wedge \gamma_j + \delta\epsilon \wedge \beta \wedge \gamma
$$
$$
- \left(p_{i;b}^{ab} - \left(\mathrm{ad}_{\boldsymbol{\eta}_b}^* (\boldsymbol{p}^{ab}) \right)_i + \boldsymbol{p}_{i;j}^{aj} - \left(\mathrm{ad}_{t_j}^* (\boldsymbol{p}^{aj}) \right)_i \right.
$$
$$
\left. - \mathrm{tr}(\mathrm{ad}_{t_j}) \boldsymbol{p}_i^{aj} \right) \delta\eta_a^i \wedge \beta \wedge \gamma
$$
$$
- \frac{1}{2} \left(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b] \right)^i \delta p_i^{ab} \wedge \beta \wedge \gamma
$$
$$
+ \left(\boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a] \right)^i \delta p_i^{aj} \wedge \beta \wedge \gamma.
$$

We can now come back to the considerations of Section 3.1 and write Equation (43) with $(X_1, \ldots, X_n, Y_1, \ldots, Y_r)$ equal to $(\boldsymbol{u}_* e_1, \ldots, \boldsymbol{u}_* e_n, \boldsymbol{u}_* \rho_1, \ldots, \boldsymbol{u}_* \rho_r)$. Writing $\boldsymbol{U} = X_1 \wedge \cdots \wedge X_n \wedge Y_1 \wedge \cdots \wedge Y_r$ for short, we deduce from (45) that, up to the factor $(-1)^{n+r}$, the l.h.s. of (43) reduces to:

$$
(57) \quad (-1)^{n+r} \boldsymbol{U} \lrcorner \omega = \delta\epsilon - \frac{1}{2} \left(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b] \right)^i \delta p_i^{ab}
$$
$$
+ (\boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a])^i \delta p_i^{aj} - \left(\boldsymbol{p}_{i;b}^{ab} - \left(\mathrm{ad}_{\boldsymbol{\eta}_b}^* (\boldsymbol{p}^{ab}) \right)_i \right.
$$
$$
\left. + \boldsymbol{p}_{i;j}^{aj} - \left(\mathrm{ad}_{t_j}^* (\boldsymbol{p}^{aj}) \right)_i - \mathrm{tr}(\mathrm{ad}_{t_j}) \boldsymbol{p}_i^{aj} \right) \delta\eta_a^i.
$$

We observe that the first line in the r.h.s. of (56) does not contribute because it contains terms quadratic in $\delta(\cdot)$.

On the other hand we also need to estimate dH. In the following we use the metric g_{ab} and its inverse g^{ab} to respectively lower and lift indices. We set e.g. $\boldsymbol{p}_{ab} := \mathrm{g}_{ac} \mathrm{g}_{bd} \boldsymbol{p}^{cd}$, etc.

$$
dH = d\epsilon - \frac{1}{2} \mathrm{h}^{ij} p_{abj} dp_i^{ab}
$$
$$
= \delta\epsilon - \frac{1}{2} \mathrm{h}^{ij} p_{abj} \delta p_i^{ab} + \left(\epsilon_{;e} - \frac{1}{2} \mathrm{h}^{ij} p_{abj} \boldsymbol{p}_{i;e}^{ab} \right) \beta^e + \left(\epsilon_{;k} - \frac{1}{2} \mathrm{h}^{ij} p_{abj} \boldsymbol{p}_{i;k}^{ab} \right) \gamma^k
$$
$$
= \delta\epsilon - \frac{1}{2} \mathrm{h}^{ij} p_{abj} \delta p_i^{ab} + \boldsymbol{H}_{;e} \beta^e + \boldsymbol{H}_{;k} \gamma^k,
$$

where we wrote $\boldsymbol{H} := \epsilon - \frac{1}{4} \mathrm{h}^{ij} \boldsymbol{p}_{abj} \boldsymbol{p}_i^{ab}$ for short.

Let us pose $T := (-1)^{n+r} \boldsymbol{U} \lrcorner \omega - dH$, so that (43) reads $T = 0$. The previous computation shows that

$$(58) \quad T = - \boldsymbol{H}_{;a} \beta^a - \boldsymbol{H}_{;i} \gamma^i$$
$$- \frac{1}{2} \left(\boldsymbol{\eta}_{b;a}^i - \boldsymbol{\eta}_{a;b}^i + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]^i - \mathsf{h}^{ij} \boldsymbol{p}_{abj} \right) \delta p_i^{ab}$$
$$+ \left(\boldsymbol{\eta}_{a;j}^i + [t_j, \boldsymbol{\eta}_a]^i \right) \delta p_i^{aj}$$
$$- \left(\boldsymbol{p}_{i;b}^{ab} - \left(\mathrm{ad}_{\boldsymbol{\eta}_b}^*(\boldsymbol{p}^{ab}) \right)_i + \boldsymbol{p}_{i;j}^{aj} - \left(\mathrm{ad}_{t_j}^*(\boldsymbol{p}^{aj}) \right)_i - \mathrm{tr}(\mathrm{ad}_{t_j}) \boldsymbol{p}_i^{aj} \right) \delta \boldsymbol{\eta}_a^i.$$

4. Classical solutions of the HVDW equations

We study here the solutions of the HVDW system of equations. We first note that the vanishing of the coefficients of β^a and γ^i in (58) means that the solution $\boldsymbol{\Gamma}$ is contained in a level set of H, a general feature in multi-symplectic geometry. In the following we look more carefully at the other equations.

As a preliminary we introduce some notations. We denote by $\mathsf{h}_* : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ the vector isomorphisme s.t. $(\mathsf{h}_*\xi)(\zeta) = \mathsf{h}_{ij}\xi^i\zeta^j$, $\forall \xi, \zeta \in \mathfrak{g}$ and by $\mathsf{h}^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ the inverse mapping. Note that, since the metric h is invariant by the adjoint action of $\mathfrak{G}$ on $\mathfrak{g}$, the following relations hold

$$(59) \quad \mathsf{h}_*[\xi, \zeta] = -\mathrm{ad}_\xi^*(\mathsf{h}_*\zeta) \quad \text{and} \quad [\xi, \mathsf{h}^*\ell] = -\mathsf{h}^*(\mathrm{ad}_\xi^*\ell), \quad \forall \xi, \zeta \in \mathfrak{g}, \forall \ell \in \mathfrak{g}^*$$

4.1. The HVDW equations with the equivariance assumption

We consider here a system of HVDW equations on fields which are assumed to be *equivariant* a priori. The equivariance condition on $\boldsymbol{\eta}$ automatically implies that the coefficients of δp_i^{aj} in (58) vanishes. Hence it turns out that the field p_i^{aj} is unuseful and that one can set it to be equal to zero a priori. This leads to the simplification

$$T = - \boldsymbol{H}_{;a} \beta^a - \boldsymbol{H}_{;i} \gamma^i - \frac{1}{2} \left(\boldsymbol{\eta}_{b;a}^i - \boldsymbol{\eta}_{a;b}^i + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]^i - \mathsf{h}^{ij} \boldsymbol{p}_{abj} \right) \delta p_i^{ab}$$
$$- \left(\boldsymbol{p}_{i;b}^{ab} - \left(\mathrm{ad}_{\boldsymbol{\eta}_b}^*(\boldsymbol{p}^{ab}) \right)_i \right) \delta \boldsymbol{\eta}_a^i.$$

Hence equation $T = 0$ is equivalent to the condition that H is constant along $\boldsymbol{\Gamma}$ and that $\boldsymbol{u}$ satisfies the system of equations

$$\begin{cases} \boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b] = \mathsf{h}^* \boldsymbol{p}_{ab} \\ \boldsymbol{p}_{;b}^{ab} - \mathrm{ad}_{\boldsymbol{\eta}_b}^*(\boldsymbol{p}^{ab}) = 0. \end{cases}$$

By (59) one sees that the second equation is equivalent to $\left(\mathsf{h}^* \boldsymbol{p}^{ab}\right)_{;b} + [\boldsymbol{\eta}_b, \mathsf{h}^* \boldsymbol{p}^{ab}] = 0$, i.e. the Yang–Mills equation.

We note that the first equation implies also

$$\boldsymbol{p}^{ab}_{;i} = \mathsf{h}_* \left(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]\right)_{;i} = -\mathsf{h}_* \left(\mathrm{ad}_{t_i}\left(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]\right)\right)$$
$$= \mathrm{ad}^*_{t_i} \left(\mathsf{h}_* \left(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]\right)\right) = \mathrm{ad}^*_{t_i} \boldsymbol{p}^{ab},$$

where we used first (14), then (59). Hence this implies that $\boldsymbol{p}^{ab}$ is equivariant (we may assume it a priori or not, it does not change the result).

4.2. The HVDW equations without assuming the equivariance a priori

Beside the condition that H is constant along a solution $\boldsymbol{\Gamma}$, the equation $T = 0$ gives us the system:

(60)
$$\begin{cases} \boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b] = \mathsf{h}^* \boldsymbol{p}_{ab} \\ \boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a] = 0 \\ \boldsymbol{p}^{ab}_{;b} - \mathrm{ad}^*_{\boldsymbol{\eta}_b}(\boldsymbol{p}^{ab}) + \boldsymbol{p}^{aj}_{;j} - \mathrm{ad}^*_{t_j}(\boldsymbol{p}^{aj}) - \mathrm{tr}(\mathrm{ad}_{t_j})\boldsymbol{p}^{aj} = 0. \end{cases}$$

(i) The first equation in (60) is the same as in the previous paragraph.

(ii) The second equation in (60) is just the equivariance condition (14) for the 1-form $\boldsymbol{\eta}$: here this condition is not assumed a priori but is obtained as one of the dynamical equations ! This is due to the fact that the fields $\boldsymbol{p}^{aj}_i$ plays the role of a Lagrange multiplier for this constraint. This condition reads also:

$$0 = \boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a] = g^{-1} \left(g\boldsymbol{\eta}_a g^{-1}\right)_{;j} g.$$

It is equivalent to say that there exists $\mathfrak{g}$-valued functions $\boldsymbol{A}_a$, for $a = 1, \ldots, n$, which depends only on x (and not on g) such that

$$\boldsymbol{\eta}_a(x, g) = g^{-1} \boldsymbol{A}_a(x) g, \quad \forall x \in M, \forall g \in \mathfrak{G}.$$

Plugging this expression in the first equation in System (60) it has the consequence that

$$\mathsf{h}^* \boldsymbol{p}_{ab} = g^{-1} \boldsymbol{\Phi}_{ab} g,$$

where $\boldsymbol{\Phi}_{ab} := \boldsymbol{A}_{b;a} - \boldsymbol{A}_{a;b} + [\boldsymbol{A}_a, \boldsymbol{A}_b]$ does not depend on g. We then observe that

$$(61) \qquad \left(\mathsf{h}^*\boldsymbol{p}^{ab}\right)_{;b} + [\boldsymbol{\eta}_b, \mathsf{h}^*\boldsymbol{p}^{ab}] = g^{-1}\left(\boldsymbol{\Phi}^{ab}_{;b} + [\boldsymbol{A}_b, \boldsymbol{\Phi}^{ab}]\right)g.$$

(iii) The third equation in (60) can be translated by using (59) to the form:

$$(62) \quad \left(\mathsf{h}^*\boldsymbol{p}^{ab}\right)_{;b} + [\boldsymbol{\eta}_b, \mathsf{h}^*\boldsymbol{p}^{ab}] + \left(\mathsf{h}^*\boldsymbol{p}^{aj}\right)_{;j} + [t_j, \mathsf{h}^*\boldsymbol{p}^{aj}] - \mathrm{tr}(\mathrm{ad}_{t_j})\mathsf{h}^*\boldsymbol{p}^{aj} = 0,$$

Let us set $\boldsymbol{\Phi}^{aj} := g(\mathsf{h}^*\boldsymbol{p}^{aj})g^{-1}$, so that

$$(63) \qquad\qquad \left(\mathsf{h}^*\boldsymbol{p}^{aj}\right)_{;j} + [t_j, \mathsf{h}^*\boldsymbol{p}^{aj}] = g^{-1}\boldsymbol{\Phi}^{aj}_{;j}g.$$

In view of (61) and (63), (62) is equivalent to

$$(64) \qquad\qquad \boldsymbol{\Phi}^{ab}_{;b} + [\boldsymbol{A}_b, \boldsymbol{\Phi}^{ab}] + \boldsymbol{\Phi}^{aj}_{;j} - \mathrm{tr}(\mathrm{ad}_{t_j})\boldsymbol{\Phi}^{aj} = 0.$$

We then have the result:

Theorem 4.1. *Assume that $\mathfrak{g}$ is unimodular and that $\mathfrak{G}$ is compact, then for any solution to (60), the 1-form $\boldsymbol{\eta}$ is a solution of the classical Yang–Mills equations.*

Proof. The assumption that $\mathfrak{g}$ is unimodular leads to the simplification of (64):

$$\boldsymbol{\Phi}^{ab}_{;b} + [\boldsymbol{A}_b, \boldsymbol{\Phi}^{ab}] = -\boldsymbol{\Phi}^{aj}_{;j}.$$

We observe that the left hand side of this relation does not depend on $g \in \mathfrak{G}$ (because $\boldsymbol{A}_a$ and hence $\boldsymbol{\Phi}^{ab}$ are constant along the fibers of $\mathcal{P}$). Hence the same is true for $\boldsymbol{\Phi}^{aj}_{;j}$.

For any $x \in \mathcal{M}$, consider the restriction of the $\mathfrak{g}$-valued $(r-1)$-form $\boldsymbol{\Phi}^{aj}\gamma_j$ on the fiber $\mathcal{P}_x$. Corollary 1.1 implies that

$$d\left(\boldsymbol{\Phi}^{aj}\gamma_j|_{\mathcal{P}_x}\right) = d\boldsymbol{\Phi}^{aj} \wedge \gamma_j|_{\mathcal{P}_x} = \boldsymbol{\Phi}^{aj}_{;j}\gamma|_{\mathcal{P}_x}.$$

Hence, since the fiber $\mathcal{P}_x$ is compact and $\boldsymbol{\Phi}^{aj}_{;j}$ is constant on $\mathcal{P}_x$,

$$\boldsymbol{\Phi}^{aj}_{;j}\mathrm{Vol}(\mathcal{P}_x) = \boldsymbol{\Phi}^{aj}_{;j}\int_{\mathcal{P}_x}\gamma = \int_{\mathcal{P}_x}\boldsymbol{\Phi}^{aj}_{;j}\gamma = \int_{\mathcal{P}_x}d\left(\boldsymbol{\Phi}^{aj}\gamma_j\right) = 0,$$

thus $\boldsymbol{\Phi}^{aj}_{;j} = 0$. Hence Equation (64) gives us

$$\boldsymbol{\Phi}^{ab}_{;b} + [\boldsymbol{A}_b, \boldsymbol{\Phi}^{ab}] = 0,$$

i.e. the Yang–Mills system. □

5. The Lagrangian action and gauge symmetries

5.1. The Lagrangian action

It is easy to deduce from our Hamiltonian multisymplectic model a Lagrangian formulation (see e.g. [14]). We first restrict the multisymplectic manifold to the level set $H^{-1}(0)$. In coordinates this amounts to eliminate the coordinate ϵ through the relation $\epsilon = \frac{1}{4}\mathsf{h}^{ij}p^{ab}_i p_{abj}$. Any submanifold $\boldsymbol{\Gamma}$ in $H^{-1}(0)$ which is a graph over $\mathcal{P}$ of a map $\boldsymbol{u}$ is then given by the collection of $\mathfrak{g}$-valued functions $\boldsymbol{\eta}_a$ and of $\mathfrak{g}^*$-valued functions $\boldsymbol{p}^{ab}$ and $\boldsymbol{p}^{aj}$. We define the value of the Lagrangian density L at $(\boldsymbol{\eta}, \boldsymbol{p}) = (\boldsymbol{\eta}_a, \boldsymbol{p}^{ab}, \boldsymbol{p}^{aj})$ by

$$L(\boldsymbol{\eta}, \boldsymbol{p})\beta \wedge \gamma = \boldsymbol{u}^*\theta.$$

The computation of $L(\boldsymbol{\eta}_a, \boldsymbol{p}^{ab}, \boldsymbol{p}^{aj})$ is relatively easy: one deduces from (48) that $\boldsymbol{u}^*(d\eta + \eta \wedge \eta) = \frac{1}{2}(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b])\beta^a \wedge \beta^b - (\boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a])\beta^a \wedge \gamma^j$ and obviously we have $\boldsymbol{u}^*(\epsilon\beta \wedge \gamma) = \epsilon\beta \wedge \gamma$ and $\boldsymbol{u}^*p = -\frac{1}{2}\boldsymbol{p}^{ab}\beta_{ab} \wedge \gamma + (-1)^n\boldsymbol{p}^{aj}\beta_a \wedge \gamma_j$. A straightforward computation thus gives us:

$$
\begin{aligned}
(65) \qquad L(\boldsymbol{\eta}, \boldsymbol{p}) = {} & \frac{1}{4}\mathsf{h}^{ij}\boldsymbol{p}^{ab}_i \boldsymbol{p}_{abj} - \frac{1}{2}\boldsymbol{p}^{ab}(\boldsymbol{\eta}_{b;a} - \boldsymbol{\eta}_{a;b} + [\boldsymbol{\eta}_a, \boldsymbol{\eta}_b]) \\
& + \boldsymbol{p}^{aj}(\boldsymbol{\eta}_{a;j} + [t_j, \boldsymbol{\eta}_a]).
\end{aligned}
$$

Critical points of the functional $\int_{\mathcal{P}} L(\boldsymbol{\eta}_a, \boldsymbol{p}^{ab}, \boldsymbol{p}^{aj})\beta \wedge \gamma$ are the solutions of the HVDW system of equations (60).

Alternatively we may decompose $\boldsymbol{\eta} = g^{-1}dg + g^{-1}\boldsymbol{A}g$ as in (7) and replace the dual variables $\boldsymbol{p}$ by the $\mathfrak{g}$-valued $(n+r-2)$-form $\boldsymbol{\Phi}$ such that $\mathsf{h}^*\boldsymbol{p} = g^{-1}\boldsymbol{\Phi}g$. Then our action functional reads

$$L(\boldsymbol{A}, \boldsymbol{\Phi}) = \frac{1}{4}\mathsf{h}\left(\boldsymbol{\Phi}^{ab}, \boldsymbol{\Phi}_{ab}\right) - \frac{1}{2}\mathsf{h}_*\boldsymbol{\Phi}^{ab}\left(\boldsymbol{A}_{b;a} - \boldsymbol{A}_{a;b} + [\boldsymbol{A}_a, \boldsymbol{A}_b]\right) + \mathsf{h}_*\boldsymbol{\Phi}^{aj}\boldsymbol{A}_{a;j}.$$

5.2. Gauge symmetries

Our variational problem is invariant under the action of the gauge group of the standard Yang–Mills action. We set this gauge group to be:

$$\mathcal{G} := \{\boldsymbol{\gamma} : \mathcal{P} \longrightarrow \mathfrak{G}; \forall z \in \mathcal{P}, \forall g \in \mathfrak{G}, \boldsymbol{\gamma}(z \cdot g) = g^{-1}\boldsymbol{\gamma}(z)g\}.$$

Note that, through a local trivialization of $\mathcal{P}$ induced by a section $\boldsymbol{\sigma} : \mathcal{M} \longrightarrow \mathcal{P}$, we can represent all maps $\boldsymbol{\gamma} \in \mathcal{G}$ in the form

$$(66) \qquad\qquad \boldsymbol{\gamma}(z) = \boldsymbol{\gamma}(\boldsymbol{\sigma}(x) \cdot g) = g^{-1}\boldsymbol{f}(x)g,$$

where $\boldsymbol{f} : \mathcal{M} \longrightarrow \mathfrak{G}$ is an arbitrary map. The gauge group $\mathcal{G}$ acts on $\Gamma_{\mathrm{N}}(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$ through the transformation

$$(67) \qquad\qquad \boldsymbol{\eta} \longmapsto \widetilde{\boldsymbol{\eta}} := \boldsymbol{\gamma}^{-1}d\boldsymbol{\gamma} + \boldsymbol{\gamma}^{-1}\boldsymbol{\eta}\boldsymbol{\gamma}.$$

Indeed in the decomposition $\boldsymbol{\eta} = g^{-1}dg + \boldsymbol{\eta}_a\beta^a$, we compute that

$$\widetilde{\boldsymbol{\eta}} = g^{-1}dg + \left[g^{-1}(\boldsymbol{f}^{-1}d\boldsymbol{f})g + \boldsymbol{\gamma}^{-1}\boldsymbol{\eta}_a\beta^a\boldsymbol{\gamma}\right],$$

confirming that $\widetilde{\boldsymbol{\eta}}$ is still normalized. Alternatively if we write $\boldsymbol{\eta} = g^{-1}dg + g^{-1}\boldsymbol{A}g$, we then obtain $\widetilde{\boldsymbol{\eta}} = g^{-1}dg + g^{-1}\widetilde{\boldsymbol{A}}g$, where $\widetilde{\boldsymbol{A}} := \boldsymbol{f}^{-1}d\boldsymbol{f} + \boldsymbol{f}^{-1}\boldsymbol{A}\boldsymbol{f}$. This shows also that, if $\boldsymbol{\eta}$ is *normalized* and *equivariant*, i.e. if $\boldsymbol{A}$ does not depend on $g \in \mathfrak{G}$, then $\widetilde{\boldsymbol{\eta}}$ is also normalized and equivariant. We also observe that

$$(68) \qquad d\widetilde{\boldsymbol{\eta}} + \widetilde{\boldsymbol{\eta}} \wedge \widetilde{\boldsymbol{\eta}} = \boldsymbol{\gamma}^{-1}(d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta})\boldsymbol{\gamma} = \mathrm{Ad}_{\gamma^{-1}}(d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta}).$$

We extend this action of $\mathcal{G}$ on sections of

$$\mathbb{R} \oplus_{\mathcal{P}} \left(\mathfrak{g} \otimes^{\mathrm{N}} T^*\mathcal{P}\right) \oplus_{\mathcal{P}} \left(\mathfrak{g}^* \otimes \Lambda^{n+r-2}T^*\mathcal{P}\right)$$

over $\mathcal{P}$ by letting

$$(69) \qquad\qquad \boldsymbol{p} \longmapsto \widetilde{\boldsymbol{p}} := \mathrm{Ad}^*_{\gamma}\boldsymbol{p}.$$

Then $\boldsymbol{p} \wedge (d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta})$ is transformed as follows

$$\boldsymbol{p} \wedge (d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta}) \longmapsto \mathrm{Ad}^*_{\gamma}\boldsymbol{p} \wedge \mathrm{Ad}_{\gamma^{-1}}(d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta})$$
$$= \boldsymbol{p} \wedge [\mathrm{Ad}_{\gamma} \circ \mathrm{Ad}_{\gamma^{-1}}(d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta})] = \boldsymbol{p} \wedge (\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta}),$$

i.e. is invariant by the gauge action. Hence θ is invariant by the gauge action and, obviously the Hamiltonian function H also.

5.3. An alternative action of the gauge group

The gauge group $\mathcal{G}$ has a different action on $\Gamma_N(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P})$. First we observe that any $\boldsymbol{\gamma} \in \mathcal{G}$ acts on $\mathcal{P}$ by the map $\varphi : z \longmapsto z \cdot \boldsymbol{\gamma}(z)$. This induces the action by pull-back $\boldsymbol{\eta} \longmapsto \varphi^*\boldsymbol{\eta}$ on sections of $\mathfrak{g} \otimes T^*\mathcal{P}$. If $\boldsymbol{\eta}$ is normalized and has the form $\boldsymbol{\eta}_{(x,g)} = g^{-1}dg + \boldsymbol{\eta}_a(x, g)\beta^a$ in a local trivialization, then $\varphi^*\boldsymbol{\eta}_{(x,g)} = g^{-1}dg + [\boldsymbol{\eta}_a(x, \boldsymbol{f}(x)g)\beta^a + g^{-1}(\boldsymbol{f}^{-1}d\boldsymbol{f})g]$, which shows in particular that $\varphi^*\boldsymbol{\eta}$ is still normalized. If furthermore $\boldsymbol{\eta}$ is equivariant and reads $\boldsymbol{\eta} = g^{-1}dg + g^{-1}\boldsymbol{A}_a(x)\beta^a g$, then $\varphi^*\boldsymbol{\eta} = g^{-1}dg + g^{-1}\widetilde{\boldsymbol{A}}_a(x)\beta^a g$, where $\widetilde{\boldsymbol{A}}_a = \boldsymbol{f}^{-1}d\boldsymbol{f} + \boldsymbol{f}^{-1}\boldsymbol{A}_a\boldsymbol{f}$. Hence this action coincides with the previous one on the *equivariant* normalized sections of $\mathfrak{g} \otimes T^*\mathcal{P}$. However it differs from the previous one on non equivariant nomalized sections. In particular the Lagrangian given by (65) is not invariant off-shell by this gauge action. It is however an on-shell symmetry if $\mathfrak{G}$ is unimodular and compact, since then any solution of the HVDW is equivariant.

5.4. Gauge symmetries on dual fields

Our action functional is also invariant under the action of another group, which is additive (and hence Abelian). This group is parametrized by the space $\mathcal{G}^\star$ of sections $\boldsymbol{U}$ of the bundle $\mathfrak{g}^* \otimes_\mathcal{P} \pi^*_\mathcal{M}T\mathcal{M} \otimes_\mathcal{P} \Lambda^{r-1}T^*\mathcal{P}$ over $\mathcal{P}$ which satisfy

$$(70) \qquad (d\boldsymbol{U} - \mathrm{ad}^*_\alpha \wedge \boldsymbol{U}) |_{\mathcal{P}_x} = 0, \quad \forall x \in \mathcal{M},$$

where α is given by (2). This definition requires some comments: for any $z \in \mathcal{P}$, the value of $\boldsymbol{U}$ at z is a $(r-1)$-form with coefficients in $\mathfrak{g}^* \otimes T_x\mathcal{M}$, where $x = \pi_\mathcal{M}(z)$, hence we can write $\boldsymbol{U} = \boldsymbol{U}^a\underline{e}_a$, where $(\underline{e}_1, \ldots, \underline{e}_n)$ is a basis of $T_x\mathcal{M}$ and each $\boldsymbol{U}^a$ is a $\mathfrak{g}$*-valued $(r-1)$-form. Then Condition (70) means that $\left(d\boldsymbol{U}^a - \gamma^i \wedge \mathrm{ad}^*_{t_i}\boldsymbol{U}^a\right) |_{\mathcal{P}_x} = 0$, for any a. If we set $\boldsymbol{U}^a = \mathsf{h}_*\boldsymbol{\psi}^a \Longleftrightarrow \boldsymbol{\psi}^a = \mathsf{h}^*\boldsymbol{U}^a$, where $\boldsymbol{\psi}^a$ is a $\mathfrak{g}$-valued $(r-1)$-form, then the latter condition reads $\left(d\boldsymbol{\psi}^a + [g^{-1}dg, \boldsymbol{\psi}^a]\right) |_{\mathcal{P}_x} = 0$ or equivalently $\beta \wedge \left(d\boldsymbol{\psi}^a + [g^{-1}dg, \boldsymbol{\psi}^a]\right) = 0$. Solutions $\boldsymbol{\psi}^a$ of this equation are of the form $\boldsymbol{\psi}^a = g^{-1}\boldsymbol{\varphi}^a g$, where $\boldsymbol{\varphi}^a \in \mathfrak{g} \otimes \Omega^{r-2}\mathcal{P}$ is *closed*. In conclusion $\boldsymbol{U} = \underline{e}_a\mathsf{h}_*\left(\mathrm{Ad}_{g^{-1}}\boldsymbol{\varphi}^a\right)$, where $d\boldsymbol{\varphi}^a = 0$.

The action of any $\boldsymbol{U} \in \mathcal{G}^\star$ is defined by $(\eta, p) \longmapsto (\eta, p + (-1)^n\beta_a \wedge \boldsymbol{U}^a)$. Since components p^{ab} are left unchanged, the Hamiltonian function H is obviously invariant. Moreover under this gauge action θ is changed into

$$\theta + \beta \wedge \left(d\eta_a + [g^{-1}dg, \eta_a]\right) \wedge \boldsymbol{U}^a = \theta + (-1)^n d\left(\beta \wedge \mathrm{Ad}_g\eta_a \wedge \mathsf{h}_*\boldsymbol{\varphi}^a\right),$$

so that we see that θ is affected by the addition of an exact form and, in particular, $\omega = d\theta$ is left unchanged. An alternative description of this gauge group is that it coincides with sections $\boldsymbol{V}$ of $\mathfrak{g}^* \otimes \Lambda_1^{n+r-2} T^* \mathcal{P}$ mod $\mathfrak{g}^* \otimes \Lambda_0^{n+r-2} T^* \mathcal{P}$ (see Paragraph 2.4) which satisfy $d\boldsymbol{V} - \mathrm{ad}_\alpha^* \wedge \boldsymbol{V} = 0$, since any such section has the form $\beta_a \wedge \boldsymbol{U}^a$, where $\boldsymbol{U} \in \mathcal{G}^\star$.

Using the variables $\boldsymbol{\Phi}$ as in Paragraph 5.1, the $\mathcal{G}^\star$ gauge action reads $(\boldsymbol{A}, \boldsymbol{\Phi}) \longmapsto (\boldsymbol{A}, \boldsymbol{\Phi} + \boldsymbol{\chi})$, where $\boldsymbol{\chi}^{ab} = 0$ and $\boldsymbol{\chi}^{aj}$ satisfies $\boldsymbol{\chi}_{;j}^{aj} = 0$ or equivalently $\beta \wedge d(\boldsymbol{\chi}^{aj} \gamma_j) = 0$.

5.5. Gauge fixing

We can fix the action of $\mathcal{G}$ by choosing a critical point (with respect to $\mathcal{G}$ deformations) of the functional $\int_\mathcal{P} \frac{1}{2} h(\boldsymbol{\eta}_a, \boldsymbol{\eta}^a) \beta \wedge \gamma = \int_\mathcal{P} \frac{1}{2} h(\boldsymbol{A}_a, \boldsymbol{A}^a) \beta \wedge \gamma$. It leads to the condition $\int_{\mathcal{P}_x} \mathrm{Ad}_g(\boldsymbol{\eta}_{;a}^a) \gamma = \int_{\mathcal{P}_x} \boldsymbol{A}_{;a}^a \gamma = 0$, $\forall x \in \mathcal{M}$.

Similarly the action of $\mathcal{G}^*$ can fixed by using, for each $x \in \mathcal{M}$, a Hodge decomposition of the $\mathfrak{g}$-valued $(r-1)$-form $\boldsymbol{\Phi}^{aj} \gamma_j|_{\mathcal{P}_x}$. This leads to choose $\boldsymbol{\Phi}^{aj}$ such that, for any $x \in \mathcal{M}$, $\forall a$, $\boldsymbol{\Phi}^{aj} \gamma_j|_{\mathcal{P}_x} = \boldsymbol{h}^a + *d\boldsymbol{V}^a$, where $\boldsymbol{V}^a$ is a function from $\mathcal{P}_x$ to $\mathfrak{g}$ and $\boldsymbol{h}^a|_{\mathcal{P}_x}$ is a harmonic $\mathfrak{g}$-valued $(r-1)$-form on $\mathcal{P}_x$ (note that $\boldsymbol{h}^a = 0$ if the de Rham cohomology group $H^{r-1}(\mathfrak{G})$ is trivial).

Putting these gauge fixing conditions together with equations (60) then leads to a well-posed system, which, if $H^{r-1}(\mathfrak{G}) = \{0\}$, reduces to the standard Yang–Mills system in the Lorentz gauge.

Acknowledgements

References

[1] J. C. Baez and C. L. Rogers, *Categorified Symplectic Geometry and the String Lie 2-algebra*. Homology, Homotopy and Applications, **12**(1): 221–236, 2010.

[2] D. Bruno, R. Cianci and S. Vignolo, *On the Hamiltonian formulation of YangMills gauge theories*. International Journal of Geometric Methods in Modern Physics, **2**(06):1115–1131, 2005.

[3] F. Cantrijn, A. Ibort and M. de Leon, *On the geometry of multisymplectic manifolds*. J. Austral. Math. Soc. Ser. A, **66**(3):303–330, 1999.

[4] C. Carathéodory, *Variationsrechnung und partielle Differentialgleichungen erster Ordnung.* Teubner, Leipzig (reprinted by Chelsea, New York, 1982); Acta litt. ac scient. univers. Hungaricae, Szeged, Sect. Math. 4, p. 193, 1929.

[5] E. Cartan, *Leçons sur les invariants intégraux.* Hermann, 1922.

[6] P. Dedecker, *Calcul des variations, formes différentielles et champs géodésiques.* In: Géométrie différentielle, Colloq. Intern. du CNRS LII, Strasbourg 1953, Publ. du CNRS, Paris, 1953, pp. 17–34; *On the generalization of symplectic geometry to multiple integrals in the calculus of variations.* In: Differential Geometrical Methods in Mathematical Physics (K. Bleuler and A. Reetz, eds.), Lect. Notes Maths., vol. 570, Springer-Verlag, Berlin, 1977, pp. 395–456.

[7] T. De Donder, *Théorie invariantive du calcul des variations.* Gauthiers-Villars, Paris, 1930.

[8] M. Forger and S. V. Romero, *Covariant Poisson bracket in geometric field theory.* Commun. Math. Phys., **256**:375–410, 2005.

[9] M. Forger and L. Gomes, *Multisymplectic and polysymplectic structures on fiber bundles.* Rev. Math. Phys., **25**(9), 2013.

[10] P. L. García, *Geometría simplética en la teoria de campos.* Collect. Math., **19**(1–2), 1968.

[11] H. Goldschmidt and S. Sternberg, *The Hamilton–Cartan formalism in the calculus of variations.* Ann. Inst. Fourier Grenoble, **23**(1):203–267, 1973.

[12] M. J. Gotay, J. Isenberg and J. E. Marsden (with the collaboraton of R. Montgomery, J. Śnyatycki and P. B. Yasskin), *Momentum maps and classical relativistic fields, Part I / covariant field theory.* Preprint, `arXiv:physics/9801019`.

[13] F. Hélein, *Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory.* In: Noncompact problems at the intersection of geometry, analysis, and topology, Contemp. Math., **350**: 127–147, 2004.

[14] F. Hélein, *Multisymplectic formalism and the covariant phase space.* In: Variational Problems in Differential Geometry (Roger Bielawski, Kevin Houston and Martin Speight, eds.), London Mathematical Society Lecture Note Series 394, Cambridge University Press, pp. 94–126, 2012.

[15] F. Hélein and J. Kouneiher, *Covariant Hamiltonian formalism for the calculus of variations with several variables: Lepage–Dedecker versus De Donder–Weyl.* Adv. Theor. Math. Phys., **8**:565–601, 2004.

[16] F. Hélein and J. Kouneiher, *The notion of observable in the covariant Hamiltonian formalism for the calculus of variations with several variables.* Adv. Theor. Math. Phys., **8**:735–777, 2004.

[17] I. V. Kanatchikov, *Canonical structure of classical field theory in the polymomentum phase space.* Rep. Math. Phys., **41**(1), 1998.

[18] I. V. Kanatchikov, *Precanonical quantization of Yang-Mills fields and the functional Schrdinger representation.* Rep. Math. Phys., **53**:181–193, 2004. `arXiv:hep-th/0301001`.

[19] I. V. Kanatchikov, *On the precanonical structure of the Schrödinger wave functional.* `arXiv:1312.4518`.

[20] J. Kijowski, *A finite dimensional canonical formalism in the classical field theory.* Comm. Math. Phys., **30**, 99–128, 1973.

[21] J. Kijowski, *Multiphase spaces and gauge in the calculus of variations.* Bull. de l'Acad. Polon. des Sci., Série sci. Math., Astr. et Phys. XXII, pp. 1219–1225, 1974.

[22] J. Kijowski and W. Szczyrba, *A canonical structure for classical field theories.* Commun. Math Phys., **46**:183–206, 1976.

[23] D. Krupka, *A geometric theory of ordinary first order variational problems in fibered manifolds*, *I. Critical sections.* J. Math. Anal. Appl., **49**: 180–206, 1975; *II. Invariance.* J. Math. Anal. Appl., **49**:469–476, 1975.

[24] T. Lepage, *Sur les champs géodésiques du calcul des variations.* Bull. Acad. Roy. Belg., Cl. Sci., **27**:716–729, 1036–1046, 1936.

[25] M. C. López and J. E. Marsden, *Some remarks on Lagrangian and Poisson reduction for eld theories.* J. Geom. Phys., **48**:52–83, 2003.

[26] C. Rovelli, *A note on the foundation of relativistic mechanics — II: Covariant Hamiltonian general relativity.* `arXiv:gr-qc/0202079`.

[27] J. Śnyatycki, *Geometric quantization and quantum Mechanics.* Appl. Math. Sci. 30, Springer-Verlag 1980.

[28] L. Vitagliano, *Partial Differential Hamiltonian Systems.* Canad. J. Math., **65**:1164–1200, 2013.

[29] V. Volterra, *Sulle equazioni differenziali che provengono da questiono di calcolo delle variazioni.* Rend. Cont. Acad. Lincei, ser. IV, vol. VI, pp. 42–54, 1890.

[30] V. Volterra, *Sopra una estensione della teoria Jacobi–Hamilton del calcolo delle varizioni.* Rend. Cont. Acad. Lincei, ser. IV, vol. VI, pp. 127–138, 1890.

[31] H. Weyl, *Geodesic fields in the calculus of variation for multiple integrals.* Ann. Math. (3), **36**:607–629, 1935.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU–PARIS RIVE GAUCHE
UMR CNRS 7586 UNIVERSITÉ DENIS DIDEROT — PARIS 7
UFR DE MATHÉMATIQUES, CASE 7012
BÂTIMENT SOPHIE GERMAIN
75205 PARIS CEDEX 13, FRANCE
E-mail address: `helein@math.jussieu.fr`