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Supersymmetric gauge theories, quantization of $\mathcal{M}_{\text{flat}}$, and conformal field theory

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We will propose a derivation of the correspondence between certain gauge theories with $N = 2$ supersymmetry and conformal field theory discovered by Alday, Gaiotto and Tachikawa in the spirit of Seiberg-Witten theory. Based on certain results from the literature we argue that the quantum theory of the moduli spaces of flat $SL(2,\mathbb{R})$-connections represents a non-perturbative "skeleton" of the gauge theory, protected by supersymmetry. It follows that instanton partition functions can be characterized as solutions to a Riemann-Hilbert type problem. In order to solve it, we describe the quantization of the moduli spaces of flat connections explicitly in terms of two natural sets of Darboux coordinates. The kernel describing the relation between the two pictures represents the solution to the Riemann Hilbert problem, and is naturally identified with the Liouville conformal blocks.

1. Introduction

This work is motivated by the discovery [AGT] of remarkable relations between certain $N = 2$ supersymmetric gauge theories and conformal field theories. The defining data for the relevant class of gauge theories, nowadays often called class $\mathcal{S}$, can be encoded in certain geometrical structures associated to Riemann surfaces C of genus g with n punctures [G09]. We will restrict attention to the case where the gauge group is $[SU(2)]^{3g-3+n}$, for which the corresponding conformal field theory is the Liouville theory. The gauge theory corresponding to a Riemann surface C will be denoted $\mathcal{G}_C$.

The authors of [AGT] discovered relations between the instanton partition functions $\mathcal{Z}^{\text{inst}}(a, m, \tau, \epsilon_1, \epsilon_2)$ defined in [N][1] for some gauge theories $\mathcal{G}_C$ of class $\mathcal{S}$ on the one hand, and the conformal blocks [BPZ] of the Liouville conformal field theory [T01] on the other hand. Using this observation one may furthermore use the variant of the localization technique developed in [Pe] to find relations between expectation values of Wilson loops in $\mathcal{G}_C$ and certain Liouville correlation functions on C. The results of [Pe, AGT] were further developed and generalized in particular in [GOP, HH], and the results of [AFLT] prove the validity of these relations for the cases where the Riemann surface C has genus zero or one, and arbitrary number of punctures.

This correspondence can be used as a powerful tool for the study of non-perturbative effects in N=2 gauge theories. As an example let us note that techniques from the study of Liouville theory [T01] can be used to effectively resum the instanton expansions, leading to highly nontrivial quantitative checks of the strong-weak coupling conjectures formulated in [G09] for gauge theories of class $\mathcal{S}$. However, gaining a deeper understanding of the origin of the relations between N=2 gauge theories and conformal field theories discovered in [AGT] seems highly desirable.

We will propose a derivation of the relations discovered in [AGT] based on certain physically motivated assumptions. We will in particular make the following assumptions:

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[1]Based in parts on earlier work [MNS1, MNS2, LNS] in this direction.

- The instanton partition functions $\mathcal{Z}^{\text{inst}}(a, m, \tau, \epsilon_1, \epsilon_2)$ are holomorphic in the UV gauge couplings τ, and can be analytically continued over the gauge theory coupling constant space. Singularities are in one-to-one correspondence with weakly-coupled Lagrangian descriptions of $\mathcal{G}_C$.
- Electric-magnetic duality exchanges Wilson- and 't Hooft loops.

Our approach works for all g and n. One may observe an analogy with the reasoning used by Seiberg and Witten in their derivations of the prepotentials for certain examples of gauge theories from this class [SW1, SW2]. This is not completely surprising, as the prepotential can be recovered from the instanton partition functions $\mathcal{Z}^{\text{inst}}(a, m, \tau, \epsilon_1, \epsilon_2)$ in the limit $\epsilon_1, \epsilon_2 \to 0$.

A basic observation underlying our approach is that the instanton partition functions $\mathcal{Z}^{\text{inst}}$ can be interpreted as certain wave-functions $\Psi_\tau(a)$ representing states in subspaces $\mathcal{H}_0$ of the Hilbert spaces $\mathcal{H}$ defined by studying $\mathcal{G}_C$ on suitable four-manifolds. Indeed, the localization methods used in [Pe, HH] show that the path integrals representing Wilson loop expectation values, for example, localize to the quantum mechanics of the scalar zero modes of $\mathcal{G}_C$. The instanton partition functions represent certain wave-functions in the zero mode quantum mechanics the path integral localizes to.

Supersymmetric versions of the Wilson- and 't Hooft loop operators act naturally on the zero mode Hilbert space $\mathcal{H}_0$, generating a sub-algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ of the algebra of operators. A key information needed as input for our approach is contained in the statement that the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ is isomorphic to the quantized algebra of functions on the moduli space $\mathcal{M}_{\text{flat}}(C)$ of flat $SL(2,\mathbb{R})$-connections on C. A derivation of this fact, applicable to all theories $\mathcal{G}_C$, was proposed in [NW]. It is strongly supported by the explicit calculations performed for certain theories from class $\mathcal{S}$ in [Pe, AGT, GOP, IOT]. A more direct way to understand why the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ is related to the quantization of the moduli spaces $\mathcal{M}_{\text{flat}}(C)$ can propbably be based on the work [GMN3] which relates the algebra of the loop operators to the quantization of the Darboux coordinates from [GMN1].

We view the algebra of supersymmetric loop operators $\mathcal{A}_{\epsilon_1\epsilon_2}$ and its representation on $\mathcal{H}_0$ as a non-perturbative "skeleton" of the gauge theory $\mathcal{G}_C$ which is protected by some unbroken supersymmetry. This structure determines the low-energy physics of $\mathcal{G}_C$ and its finite-size corrections on certain supersymmetric backgrounds, as follows from the localization of the path integral studied in [Pe, GOP, HH]

The instanton partition functions $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon_1,\epsilon_2)$ may then be characterized as wave-functions of joint eigenstates of the Wilson loop operators whose eigenvalues are given by the Coulomb branch parameters a. It follows from our assumptions above that the instanton partition functions $\mathcal{Z}_2^{\rm inst}(a_2,m,\tau_2,\epsilon_1,\epsilon_2)$ and $\mathcal{Z}_1^{\rm inst}(a_1,m,\tau_1,\epsilon_1,\epsilon_2)$ associated to two different weakly-coupled Lagrangian descriptions must be related linearly as

$$
\begin{aligned}
(1.1)\quad & \mathcal{Z}_2^{\rm inst}(a_2,m,\tau_2,\epsilon_1,\epsilon_2) \\
&= f(m,\tau_2,\epsilon_1,\epsilon_2)\int da_1\; K(a_2,a_1;m;\epsilon_1,\epsilon_2)\mathcal{Z}_1^{\rm inst}(a_1,m,\tau_1(\tau_2),\epsilon_1,\epsilon_2)\,.
\end{aligned}
$$

The a_2-independent prefactor $f(m,\tau_2,\epsilon_1,\epsilon_2)$ describes a possible change of regularization scheme used in the definition of the instanton partition functions. Knowing the relation between the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ and the quantum theory of $\mathcal{M}_{\rm flat}(C)$ will allow us to determine the kernels $K(a_2,a_1;m;\epsilon_1,\epsilon_2)$ in (1.1) explicitly. These are the main pieces of data needed for the formulation of a generalized Riemann-Hilbert problem characterizing the instanton partition functions.

The resulting mathematical problem is not of standard Riemann-Hilbert type in two respects: One is, on the one hand, dealing with infinite dimensional representations of the relevant monodromy groups, here the mapping class groups of the Riemann surfaces C. We will, on the other hand, find that the a_2-independent prefactors $f(m,\tau_2,\epsilon_1,\epsilon_2)$ in (1.1) can not be eliminated in general[2]. Their appearance is closely related to the fact that the representation of the mapping class group of C described by the kernels $K(a_2,a_1;m;\epsilon_1,\epsilon_2)$ is found to be *projective*. Without prefactors $f(m,\tau_2,\epsilon_1,\epsilon_2)$ which, roughly speaking, cancel the projectiveness there could not exist any solution to our generalized Riemann-Hilbert problem.

Working out the kernels $K(a_2,a_1;m;\epsilon_1,\epsilon_2)$ is the content of Part II of this paper, containing a detailed study of the quantum theory of the relevant connected component $\mathcal{M}^0_{\rm flat}(C)$ of $\mathcal{M}_{\rm flat}(C)$. In Part III we describe how the Riemann-Hilbert problem for $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon_1,\epsilon_2)$ is solved by Liouville theory. We explain how Liouville theory is related to the quantum theory of $\mathcal{M}^0_{\rm flat}(C)$, which is equivalent to the quantum theory of the Teichmüller spaces $\mathcal{T}(C)$. The relation between Liouville theory and the quantization of $\mathcal{M}^0_{\rm flat}(C)$, combined with the connection between instanton partition functions $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon_1,\epsilon_2)$ and wave-functions in the quantum theory of

[2]This is the case for surfaces of higher genus. The prefactors could be eliminated for the cases studied in [AGT], and some generalizations like the so-called linear quiver theories.

$\mathcal{M}^0_{\rm flat}(C)$ yields a way to derive the correspondence found in [AGT]. One of the main technical problems addressed in Part III is the proper characterization of the prefactors $f(m,\tau_2,\epsilon_1,\epsilon_2)$ in (1.1) which are related to the projective line bundle whose importance for conformal field theory was emphasized by Friedan and Shenker [FS]. The results obtained in this paper have interesting connections to the work of Nekrasov, Rosly and Shatashvili [NRS] devoted to the case $\epsilon_2 = 0$.

There is an alternative approach towards proving the AGT-correspondence, which relates the series expansion of $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon_1,\epsilon_2)$ defined from the equivariant cohomology of instanton moduli spaces more directly to the definition of the conformal blocks of Liouville theory obtained from the representation theory of the Virasoro algebra. Important progress has been made along these lines. A first proof of the AGT-correspondence for a subset of gauge theories $\mathcal{G}_C$ from class $\mathcal{S}$ was obtained in [AFLT] by finding closed formulae for the coefficients appearing in the series expansions of the Liouville conformal blocks that directly match the formulae known for the expansion coefficients of $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon_1,\epsilon_2)$ from the instanton calculus. An important step towards a more conceptual explanation was taken by identifying the Virasoro algebra as a symmetry of the equivariant cohomology of the instanton moduli spaces [SchV, MO]. A physical approach to these results was described in [Tan].

This approach may be seen as complementary to the one used in this paper: It elucidates the mathematical structure of the perturbative expansion of $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon_1,\epsilon_2)$ as defined from a given Lagrangian description for $\mathcal{G}_C$. The arguments presented here relate the non-perturbative "skeleton" of $\mathcal{G}_C$ to global objects on C instead.

The results in Parts II and III of this paper are of independent interest. Part II describes the quantization of $\mathcal{M}^0_{\rm flat}(C)$ using the Darboux variables which were recently used in a related context in [NRS].[3] These results give an alternative representation for the quantum theory of the Teichmüller spaces which is based on pants decompositions instead of triangulations of C, as is important for understanding the relation to Liouville theory. Our approach is related to the one pioneered in [F97, CF1, Ka1] by a nontrivial unitary transformation that we construct explicitly.

In Part III we extend the relation between quantization of the Teichmüller spaces and Liouville theory found in [T03] for surfaces of genus 0 to arbitrary genus. An important subtlety is to properly take into account the projective line bundle over moduli space whose relevance for conformal field

[3] Partial results in this direction were previously obtained in [DG].

theory was first emphasized in [FS]. This allows us to find the appropriate way to cancel the central extension of the canonical connection on the space of conformal blocks defined by the energy-momentum tensor. Doing this is crucial for having a solution of the Riemann-Hilbert problem of our interest at all.

The results of Part III also seem to be interesting from a purely mathematical perspective. They amount to an interpretation of conformal field theory in terms of the harmonic analysis on the Teichmüller spaces, which can be seen as symmetric spaces for the group $\mathrm{Diff}_0(S^1)$.

Our work realizes part of a larger picture outlined in [T10] relating the quantization of the Hitchin moduli spaces, integrable models and conformal field theory. In order to get a connection to supersymmetric gauge theories extending the connections discussed here one needs to consider insertions of surface operators on the gauge theory side. This is currently under investigation [FGT].

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2. Riemann surfaces: Some basic definitions and results

Let us introduce some basic definitions concerning Riemann surfaces that will be used throughout the paper.

2.1. Complex analytic gluing construction

A convenient family of particular choices for coordinates on $\mathcal{T}(C)$ is produced from the complex-analytic gluing construction of Riemann surfaces C from three punctured spheres [Ma, HV]. Let us briefly review this construction.

Let C be a (possibly disconnected) Riemann surface. Fix a complex number q with $|q| < 1$, and pick two points Q_1 and Q_2 on C together with coordinates $z_i(P)$ in a neighborhood of Q_i, $i = 1, 2$, such that $z_i(Q_i) = 0$, and such that the discs D_i,

$$D_i := \{ P_i \in C_i \, ; \, |z_i(P_i)| < |q|^{-\frac{1}{2}} \} ,$$

do not intersect. One may define the annuli A_i,

$$A_i := \{ P_i \in C_i \, ; \, |q|^{\frac{1}{2}} < |z_i(P_i)| < |q|^{-\frac{1}{2}} \} .$$

To glue A_1 to A_2 let us identify two points P_1 and P_2 on A_1 and A_2, respectively, iff the coordinates of these two points satisfy the equation

$$z_1(P_1)z_2(P_2) = q\,. \tag{2.1}$$

If C is connected one creates an additional handle, and if $C = C_1 \sqcup C_2$ has two connected components one gets a single connected component after performing the gluing operation. In the limiting case where $q = 0$ one gets a nodal surface which represents a component of the boundary $\partial\mathcal{M}(C)$ defined by the Deligne-Mumford compactification $\overline{\mathcal{M}}(C)$.

By iterating the gluing operation one may build any Riemann surface C of genus g with n punctures from three-punctured spheres $C_{0,3}$. Embedded into C we naturally get a collection of annuli $A_1, \dots, A_h$, where

$$h := 3g - 3 + n\,, \tag{2.2}$$

The construction above can be used to define an $3g - 3 + n$-parametric family or Riemann surfaces, parameterized by a collection $q = (q_1, \dots, q_h)$ of complex parameters. These parameters can be taken as complex-analytic coordinates for a neighborhood of a component in the boundary $\partial\mathcal{M}(C)$ with respect to its natural complex structure [Ma].

Conversely, assume given a Riemann surface C and a cut system, a collection $\mathcal{C} = \{\gamma_1, \dots, \gamma_h\}$ of homotopy classes of non-intersecting simple closed curves on C. Cutting along all the curves in $\mathcal{C}$ produces a pants decompostion, $C \setminus \mathcal{C} \simeq \bigsqcup_v C_{0,3}^v$, where the $C_{0,3}^v$ are three-holed spheres.

Having glued C from three-punctured spheres defines a distinguished cut system, defined by a collection of simple closed curves $\mathcal{C} = \{\gamma_1, \dots, \gamma_h\}$ such that γ_r can be embedded into the annulus A_r for $r = 1, \dots, h$.

An important deformation of the complex structure of C is the Dehn-twist: It corresponds to rotating one end of an annulus A_r by 2π before regluing, and can be described by a change of the local coordinates used in the gluing construction. The coordinate q_r can not distinguish complex structures related by a Dehn twist in A_r. It is often useful to replace the coordinates q_r by logarithmic coordinates τ_r such that $q_r = e^{2\pi i \tau_r}$. This corresponds to replacing the gluing identification (2.1) by its logarithm. In order to define the logarithms of the coordinates z_i used in (2.1), one needs to introduce branch cuts on the three-punctured spheres, an example being depicted in Figure 1.

By imposing the requirement that the branch cuts chosen on each three-punctured sphere glue to a connected three-valent graph Γ on C, one gets an unambiguous definition of the coordinates τ_r. We see that the logarithmic

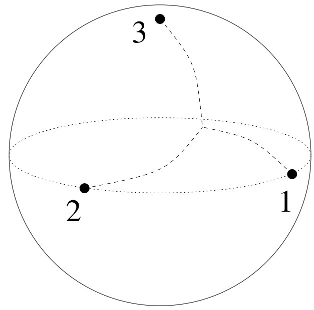


Figure 1: A sphere with three punctures, and a choice of branch cuts for the definition of the logarithms of local coordinates around the punctures.

versions of the gluing construction that define the coordinates τ_r are parameterized by the pair of data $\sigma = (\mathcal{C}_\sigma, \Gamma_\sigma)$, where $\mathcal{C}_\sigma$ is the cut system defined by the gluing construction, and Γ_σ is the three-valent graph specifying the choices of branch cuts. In order to have a handy terminology we will call the pair of data $\sigma = (\mathcal{C}_\sigma, \Gamma_\sigma)$ a *pants decomposition*, and the three-valent graph Γ_σ will be called the Moore-Seiberg graph, or MS-graph associated to a pants decomposition σ.

The gluing construction depends on the choices of coordinates around the punctures Q_i. There exists an ample supply of choices for the coordinates z_i such that the union of the neighborhoods $\mathcal{U}_\sigma$ produces a cover of $\mathcal{M}(C)$ [HV]. For a fixed choice of these coordinates one produces families of Riemann surfaces fibred over the multi-discs $\mathcal{U}_\sigma$ with coordinates q. Changing the coordinates z_i around q_i produces a family of Riemann surfaces which is locally biholomorphic to the initial one [RS].

2.2. The Moore-Seiberg groupoid

Let us note [MS, BK] that any two different pants decompositions σ_2, σ_1 can be connected by a sequence of elementary moves localized in subsurfaces of $C_{g,n}$ of type $C_{0,3}$, $C_{0,4}$ and $C_{1,1}$. These will be called the B, F, Z and S-moves, respectively. Graphical representations for the elementary moves B, Z, F, and S are given in Figures 2, 3, 4, and 5, respectively.

One may formalize the resulting structure by introducing a two-dimensional CW complex $\mathcal{M}(C)$ with set of vertices $\mathcal{M}_0(C)$ given by the pants decompositions σ, and a set of edges $\mathcal{M}_1(C)$ associated to the elementary moves.

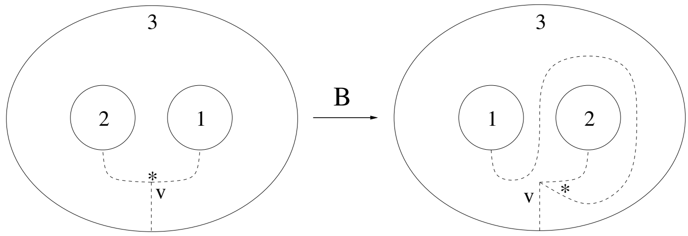


Figure 2: The B-move

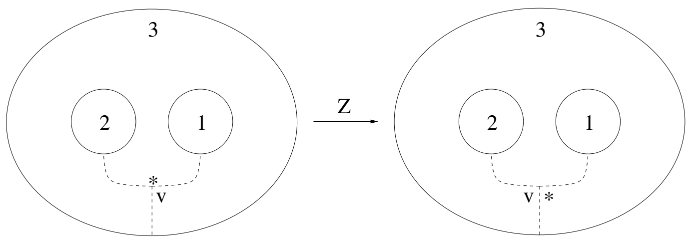


Figure 3: The Z-move

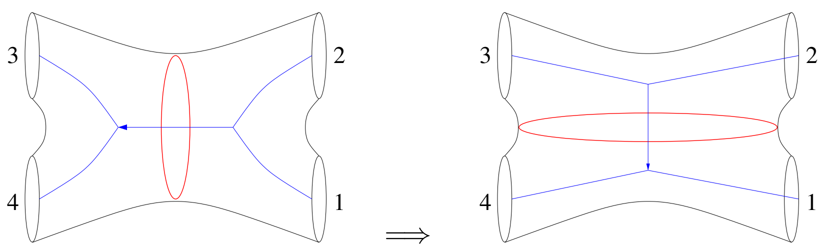


Figure 4: The F-move

The Moore-Seiberg groupoid is defined to be the path groupoid of $\mathcal{M}(C)$. It can be described in terms of generators and relations, the generators being associated with the edges of $\mathcal{M}(C)$, and the relations associated with the faces of $\mathcal{M}(C)$. The classification of the relations was first presented in [MS], and rigorous mathematical proofs have been presented in [FG, BK]. The relations are all represented by sequences of moves localized in subsurfaces $C_{g,n}$ with genus $g = 0$ and $n = 3, 4, 5$ punctures, as well as $g = 1$, $n = 1, 2$. Graphical representations of the relations can be found in [MS, FG, BK].

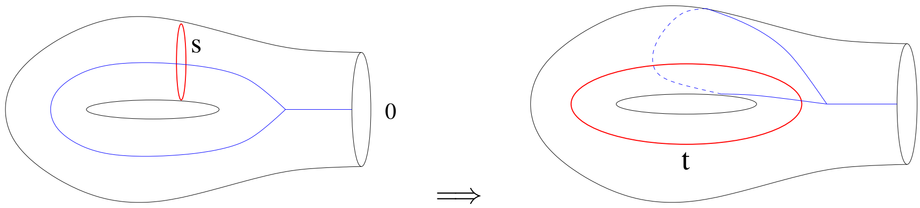


Figure 5: The S-move

2.3. Hyperbolic metrics vs. flat connections

The classical uniformization theorem ensures existence and uniqueness of a hyperbolic metric, a metric of constant negative curvature, on a Riemann surface C. In a local chart with complex analytic coordinates y one may represent this metric in the form $ds^2 = e^{2\varphi} dy d\bar{y}$, with φ being a solution to the Liouville equation $\partial\bar{\partial}\varphi = \mu e^{2\varphi} dy d\bar{y}$.

There is a well-known relation between the Teichmüller space $\mathcal{T}(C)$ and a connected component of the moduli space $\mathcal{M}_{\rm flat}(C)$ of flat $\mathrm{PSL}(2,\mathbb{R})$-connections on C. The relevant component will be denoted as $\mathcal{M}^0_{\rm flat}(C)$. The relation between $\mathcal{T}(C)$ and $\mathcal{M}^0_{\rm flat}(C)$ may be desrcribed as follows.

To a hyperbolic metric $ds^2 = e^{2\varphi} dy d\bar{y}$ let us associate the connection $\nabla = \nabla' + \nabla''$, and

$$\nabla'' = \bar{\partial}\,, \qquad \nabla' = \partial + M(y)dy\,, \qquad M(y) = \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

with t constructed from $\varphi(y,\bar{y})$ as

$$t := -(\partial_y\varphi)^2 + \partial_y^2\varphi\,. \tag{2.4}$$

This connection is flat since $\partial_y\bar{\partial}_{\bar{y}}\varphi = \mu e^{2\varphi}$ implies $\bar{\partial}t = 0$. The form (2.3) of ∇ is preserved by changes of local coordinates if $t = t(y)$ transforms as

$$t(y) \;\mapsto\; (y'(w))^2 t(y(w)) + \frac{1}{2}\{y,w\}\,, \tag{2.5}$$

where the Schwarzian derivative $\{y,w\}$ is defined as

$$\{y,w\} \;\equiv\; \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 . \tag{2.6}$$

Equation (2.5) is the transformation law characteristic for *projective* connections, which are also called $\mathfrak{sl}_2$-opers, or opers for short.

The hyperbolic metric $ds^2 = e^{2\varphi}dyd\bar{y}$ can be constructed from the solutions to $\nabla s = 0$ which implies that the component χ of $s = (\eta, \chi)$ solves a second order differential equation of the form

$$(\partial_y^2 + t(y))\chi = 0\,. \tag{2.7}$$

Picking two linearly independent solutions $\chi_\pm$ of (2.7) with $\chi'_+\chi_- - \chi'_-\chi_+ = 1$ allows us to represent $e^{2\varphi}$ as $e^{2\varphi} = -(\chi_+\bar{\chi}_- - \chi_-\bar{\chi}_+)^{-2}$. The hyperbolic metric $ds^2 = e^{2\varphi}dyd\bar{y}$ may now be written in terms of the quotient $A(y) := \chi_+/\chi_-$ as

$$ds^2 = e^{2\varphi}dyd\bar{y} = \frac{\partial A \bar{\partial}\bar{A}}{(\mathrm{Im}(A))^2}\,. \tag{2.8}$$

It follows that $A(y)$ represents a conformal mapping from C to a domain Ω in the upper half plane $\mathbb{U}$ with its standard constant curvature metric. C is therefore conformal to $\mathbb{U}/\Gamma$, where the Fuchsian group Γ is the monodromy group of the connection ∇.

2.4. Hyperbolic pants decomposition and Fenchel-Nielsen coordinates

Let us consider hyperbolic surfaces C of genus g with n holes. We will assume that the holes are represented by geodesics in the hyperbolic metric. A pants decomposition of a hyperbolic surface C is defined, as before, by a cut system which in this context may be represented by a collection $\mathcal{C} = \{\gamma_1, \dots, \gamma_h\}$ of non-intersecting simple closed geodesics on C. The complement $C \setminus \mathcal{C}$ is a disjoint union $\bigsqcup_v C_{0,3}^v$ of three-holed spheres (trinions). One may reconstruct C from the resulting collection of trinions by pairwise gluing of boundary components.

For given lengths of the three boundary geodesics there is a unique hyperbolic metric on each trinion $C_{0,3}^v$. Introducing a numbering of the boundary geodesics $\gamma_i(v)$, $i = 1, 2, 3$, one gets three distinguished geodesic arcs $\gamma_{ij}(v)$, $i, j = 1, 2, 3$ which connect the boundary components pairwise. Up to homotopy there are exactly two tri-valent graphs $\Gamma_\pm^v$ on $C_{0,3}^v$ that do not intersect any $\gamma_{ij}(v)$. We may assume that these graphs glue to two connected graphs $\Gamma_\pm$ on C. The pair of data $\sigma = (\mathcal{C}_\sigma, \Gamma_\sigma)$, where Γ_σ is one of the MS graphs $\Gamma_\pm$ associated to a hyperbolic pants decomposition, can be used to distinguish different pants decompositions in hyperbolic geometry.

The data $\sigma = (\mathcal{C}_\sigma, \Gamma_\sigma)$ can also be used to define the classical Fenchel-Nielsen coordinates for $\mathcal{T}(C)$ as follows. Note that the edges e of Γ_σ are in one-to-one correspondence with the curves γ_e in $\mathcal{C}_\sigma$. To each edge e let us first associate the length l_e of the geodesic γ_e.

In order to define the Fenchel-Nielsen twist variables we need to consider two basic cases: Either a given $\gamma_e \in \mathcal{C}$ separates two different trinions $C_{0,3}^{v_1}$ and $C_{0,3}^{v_2}$, or it is the result of the identification of two boundary components of a single trinion. In order to fix a precise prescription in the first case let us assume that C and the edge e are oriented. One may then define a numbering of the boundary components of the four-holed sphere $C_{0,4}^{v_{12}}$ obtained by gluing $C_{0,3}^{v_1}$ and $C_{0,3}^{v_2}$: Number 1 is assigned to the boundary component intersecting the next edge of Γ_σ on the right of the tail of the edge e, number 4 to the boundary component intersecting the next edge of Γ_σ to the left of the tip of e. There are geodesic arcs $\gamma_{4e}(v_2)$ and $\gamma_{1e}(v_1)$ on $C_{0,3}^{v_1}$ and $C_{0,3}^{v_2}$ that intersect γ_e in points P_1, and P_2, respectively. This set-up is drawn in Figure 6.

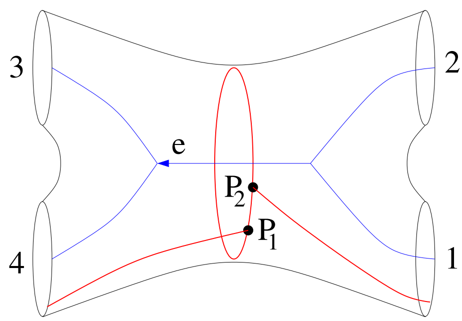


Figure 6: A four-holed sphere with MS graph (blue) and the geodesics used in the definition of the Fenchel-Nielsen coordinates (red).

The twist variable k_e is then defined to be the geodesic distance between P_1 and P_2, and the twist angle $\theta_e = 2\pi k_e/l_e$. The second case (gluing of two holes in one trinion gives sub-surface C_e of type $C_{1,1}$) is treated similarly.

We see that the role of the MS-graph Γ_σ is to distinguish pants decompositions related by Dehn-twists, corresponding to $\theta_e \to \theta_e + 2\pi$.

2.5. Trace coordinates

Given a flat $\mathrm{SL}(2,\mathbb{C})$-connection $\nabla = d - A$, one may define its holonomy $\rho(\gamma)$ along a closed loop γ as $\rho(\gamma) = \mathcal{P}\exp(\int_\gamma A)$. The assignment $\gamma \mapsto \rho(\gamma)$ defines a representation of $\pi_1(C)$ in $\mathrm{SL}(2,\mathbb{C})$, defining a point in the so-called character variety

$$\mathcal{M}_{\text{char}}^{\mathbb{C}}(C) := \text{Hom}(\pi_1(C), \text{PSL}(2,\mathbb{C}))/\text{PSL}(2,\mathbb{C})\,. \tag{2.9}$$

The Fuchsian groups Γ represent a connected component $\mathcal{M}_{\text{char}}^{\mathbb{R},0}(C) \simeq \mathcal{T}(C)$ in the *real* character variety

$$\mathcal{M}_{\text{char}}^{\mathbb{R}}(C) := \text{Hom}(\pi_1(C), \text{PSL}(2,\mathbb{R}))/\text{PSL}(2,\mathbb{R})\,. \tag{2.10}$$

which will be of main interest here. $\mathcal{M}_{\text{char}}^{\mathbb{R}}(C)$ is naturally identified with the moduli space $\mathcal{M}_{\text{flat}}(C)$ of flat $\text{PSL}(2,\mathbb{R})$ connections on C, and $\mathcal{M}_{\text{char}}^{\mathbb{R},0}(C)$ represents the so-called Teichmüller component $\mathcal{M}_{\text{flat}}^{0}(C)$ within $\mathcal{M}_{\text{flat}}(C)$.

2.5.1. Topological classification of closed loops. With the help of pants decompositions one may conveniently classify all non-selfintersecting closed loops on C up to homotopy. To a loop γ let us associate the collection of integers (r_e, s_e) associated to all edges e of Γ_σ which are defined as follows. Recall that there is a unique curve $\gamma_e \in \mathcal{C}_\sigma$ that intersects a given edge e on Γ_σ exactly once, and which does not intersect any other edge. The integer r_e is defined as the number of intersections between γ and the curve γ_e. Having chosen an orientation for the edge e_r we will define s_e to be the intersection index between e and γ.

Dehn's theorem (see [DMO] for a nice discussion) ensures that the curve γ is up to homotopy uniquely classified by the collection of integers (r,s), subject to the restrictions

$$\begin{aligned} &\text{(i) } r_e \geq 0\,, \\ &\text{(ii) if } \; r_e = 0 \;\Rightarrow\; s_e \geq 0\,, \\ &\text{(iii) } r_{e_1} + r_{e_2} + r_{e_3} \in 2\mathbb{Z} \text{ whenever } \gamma_{e_1}, \gamma_{e_2}, \gamma_{e_3} \text{ bound the same trinion.} \end{aligned} \tag{2.11}$$

We will use the notation $\gamma_{(r,s)}$ for the geodesic which has parameters $(r,s): e \mapsto (r_e, s_e)$.

2.5.2. Trace functions. The trace functions

$$L_\gamma := \nu_\gamma \text{tr}(\rho(\gamma))\,, \tag{2.12}$$

represent useful coordinate functions for $\mathcal{M}_{\text{char}}^{\mathbb{C}}(C)$. The signs $\nu_\gamma \in \{+1,-1\}$ in the definition (2.12) will be specified shortly. Real values of the trace functions L_γ characterize $\mathcal{M}_{\text{char}}^{\mathbb{R}}(C)$.

If the representation ρ is the one coming from the uniformization of C, it is an elementary exercise in hyperbolic geometry to show that the length l_γ of the geodesic γ is related to L_γ by

$$|L_\gamma| = 2\cosh(l_\gamma/2)\,. \tag{2.13}$$

Representing the points in $\mathcal{M}^{\mathbb{R},0}_{\rm char}(C)$ by representations $\rho : \pi_1(C) \to \mathrm{SL}(2, \mathbb{R})$, we will always choose the sign ν_γ in (2.12) such that $L_\gamma = 2\cosh(l_\gamma/2)$.

We may then analytically continue the trace functions L_γ defined thereby to coordinates on the natural complexification $\mathcal{M}^{\mathbb{C},0}_{\rm char}(C) \subset \mathcal{M}^{\mathbb{C}}_{\rm char}(C)$ of $\mathcal{M}^{\mathbb{R},0}_{\rm char}(C)$. The representations $\rho : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ that are parameterized by $\mathcal{M}^{\mathbb{C},0}_{\rm char}(C)$ are called quasi-Fuchsian. It is going to be important for us to have coordinates L_γ that are complex analytic on $\mathcal{M}^{\mathbb{C},0}_{\rm char}(C)$ on the one hand, but *positive* (and larger than two) when restricted to the real slice $\mathcal{M}^{\mathbb{R},0}_{\rm char}(C)$ on the other hand.

2.5.3. Skein algebra. The well-known relation $\mathrm{tr}(g)\mathrm{tr}(h) = \mathrm{tr}(gh) + \mathrm{tr}(gh^{-1})$ valid for any pair of $SL(2)$-matrices g, h implies that the geodesic length functions satisfy the so-called skein relations,

$$L_{\gamma_1} L_{\gamma_2} = L_{S(\gamma_1,\gamma_2)}\,, \tag{2.14}$$

where $S(\gamma_1,\gamma_2)$ is the loop obtained from γ_1, γ_2 by means of the smoothing operation, defined as follows. The application of S to a single intersection point of γ_1, γ_2 is depicted in Figure 7 below. The general result is obtained

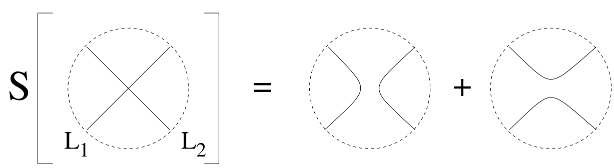


Figure 7: The symmetric smoothing operation

by applying this rule at each intersection point, and summing the results.

The coordinate functions L_γ generate the commutative algebra $\mathcal{A}(C) \simeq \mathrm{Fun}^{\rm alg}(\mathcal{M}_{\rm flat}(C))$ of functions on $\mathcal{M}_{\rm flat}(C)$. As set of generators one may take the functions $L_{(r,s)} \equiv L_{\gamma_{(r,s)}}$. The skein relations imply various relations among the $L_{(r,s)}$. It is not hard to see that these relations allow one to express arbitrary $L_{(r,s)}$ in terms of a finite subset of the set of $L_{(r,s)}$.

2.5.4. Generators and relations. The pants decompositions allow us to describe $\mathcal{A}(C)$ in terms of generators and relations. Let us note that to each internal[4] edge e of the MS-graph Γ_σ of σ there corresponds a unique curve γ_e in the cut system $\mathcal{C}_\sigma$. There is a unique subsurface $C_e \hookrightarrow C$ isomorphic to either $C_{0,4}$ or $C_{1,1}$ that contains γ_e in the interior of C_e. The subsurface C_e has boundary components labeled by numbers $1,2,3,4$ according to the convention introduced in Subsection 2.4 if $C_e \simeq C_{0,4}$, and if $C_e \simeq C_{1,1}$ we will assign to the single boundary component the number 0.

For each edge e let us introduce the geodesics γ_t^e which have Dehn parameters $(r^e, 0)$, where $r^e_{e'} = 2\delta_{e,e'}$ if $C_e \simeq C_{0,4}$ and $r^e_{e'} = \delta_{e,e'}$ if $C_e \simeq C_{1,1}$. These geodesics are depicted as red curves on the right halfs of Figures 4 and 5, respectively. There furthermore exist unique geodesics γ_u^e with Dehn parameters (r^e, s^e), where $s^e_{e'} = \delta_{e,e'}$. We will denote $L^e_k \equiv |\mathrm{tr}(\gamma^e_k)|$, where $k \in \{s,t,u\}$ and $\gamma^e_s \equiv \gamma_e$. The set $\{L^e_s, L^e_t, L^e_u\,;\,\gamma_e \in \mathcal{C}_\sigma\}$ generates $\mathcal{A}(C)$.

These coordinates are not independent, though. Further relations follow from the relations in $\pi_1(C)$. It can be shown (see e.g. [Go09] for a review) that any triple of coordinate functions L^e_s, L^e_t and L^e_u satisfies an algebraic relation of the form

$$P_e(L^e_s, L^e_t, L^e_u) = 0\,. \tag{2.15}$$

The polynomial P_e in (2.15) is for $C_e \simeq C_{0,4}$ explicitly given as[5]

$$\begin{aligned} &P_e(L_s, L_t, L_u) \\ &:= - L_sL_tL_u + L_s^2 + L_t^2 + L_u^2 \\ &\quad + L_s(L_3L_4 + L_1L_2) + L_t(L_2L_3 + L_1L_4) + L_u(L_1L_3 + L_2L_4) \\ &\quad - 4 + L_1^2 + L_2^2 + L_3^2 + L_4^2 + L_1L_2L_3L_4\,, \end{aligned} \tag{2.16}$$

while for $C_e \simeq C_{1,1}$ we take P to be

$$P_e(L_s, L_t, L_u) := L_s^2 + L_t^2 + L_u^2 - L_sL_tL_u + L_0 - 2\,. \tag{2.17}$$

In the expressions above we have denoted $L_i := |\mathrm{Tr}(\rho(\gamma_i))|$, $i = 0,1,2,3,4$, where γ_0 is the geodesic representing the boundary of $C_{1,1}$, while γ_i, $i = 1,2,3,4$ represent the boundary components of $C_{0,4}$, labelled according to the convention above.

[4]An internal edge does not end in a boundary component of C.

[5]Comparing to [Go09] note that some signs were absorbed by a suitable choice of the signs ν_γ in (2.12).

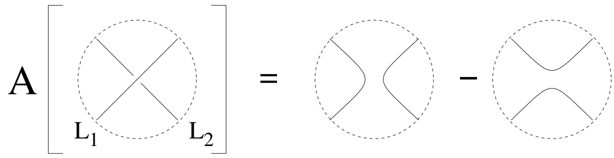


Figure 8: The anti-symmetric smoothing operation

2.5.5. Poisson structure. There is also a natural Poisson bracket on $\mathcal{A}(C)$ [Go86], defined such that

$$\{ L_{\gamma_1} , L_{\gamma_2} \} = L_{A(\gamma_1,\gamma_2)} \,, \tag{2.18}$$

where $A(\gamma_1, \gamma_2)$ is the loop obtained from γ_1, γ_2 by means of the anti-symmetric smoothing operation, defined as above, but replacing the rule depicted in Figure 7 by the one depicted in Figure 8.

The resulting expression for the Poisson bracket $\{ L_s^e , L_t^e \}$ can be written elegantly in the form

$$\{ L_s^e , L_t^e \} = \frac{\partial}{\partial L_u^e} P_e(L_s^e, L_t^e, L_u^e) \,. \tag{2.19}$$

It is remarkable that the same polynomial appears both in (2.15) and in (2.19), which indicates that the symplectic structure on $\mathcal{M}_{\rm flat}$ is compatible with its structure as algebraic variety.

This Poisson structure coincides with the Poisson structure coming from the natural symplectic structure on $\mathcal{M}_{\rm flat}(C)$ which was introduced by Atiyah and Bott.

2.6. Darboux coordinates for $\mathcal{M}_{\rm flat}(C)$

One may express L_s^e, L_t^e and L_u^e in terms of the Fenchel-Nielsen coordinates l_e and k_e [Go09]. The expressions are

$$L_s^e = 2\cosh(l_e/2) \,, \tag{2.20a}$$

and for $C_e \simeq C_{1,1}$,

$$L_t^e \big((L_s^e)^2 - 4\big)^{\frac{1}{2}} = 2\cosh(k_e/2)\sqrt{(L_s^e)^2 + L_0^e - 2} \tag{2.20b}$$

$$L_u^e \big((L_s^e)^2 - 4\big)^{\frac{1}{2}} = 2\cosh((l_e + k_e)/2)\sqrt{(L_s^e)^2 + L_0^e - 2} \,, \tag{2.20c}$$

while for $C_e \simeq C_{0,4}$,

$$L_t^e\big((L_s^e)^2-4\big) = 2(L_2^eL_3^e+L_1^eL_4^e)+L_s^e(L_1^eL_3^e+L_2^eL_4^e) + 2\cosh(k_e)\sqrt{c_{12}(L_s^e)c_{34}(L_s^e)}\,, \tag{2.20d}$$

$$L_u^e\big((L_s^e)^2-4\big) = L_s^e(L_2^eL_3^e+L_1^eL_4^e)+2(L_1^eL_3^e+L_2^eL_4^e) + 2\cosh((2k_e-l_e)/2)\sqrt{c_{12}(L_s^e)c_{34}(L_s^e)}\,, \tag{2.20e}$$

where $L_i^e = 2\cosh\frac{l_i^e}{2}$, and $c_{ij}(L_s)$ is defined as

$$\begin{aligned} c_{ij}(L_s) &= L_s^2+L_i^2+L_j^2+L_sL_iL_j-4 \\ &= 2\cosh\tfrac{l_s+l_i+l_j}{4}2\cosh\tfrac{l_s+l_i-l_j}{4}2\cosh\tfrac{l_s-l_i+l_j}{4}2\cosh\tfrac{l_s-l_i-l_j}{4}. \end{aligned} \tag{2.21}$$

These expressions ensure that the algebraic relations $P_e(L_s,L_t,L_t)=0$ are satisfied.

The coordinates l_e and k_e are known to be Darboux-coordinates for $\mathcal{M}_{\text{flat}}(C)$, having the Poisson bracket

$$\{\, l_e\,,\, k_{e'}\,\} = 2\delta_{e,e'}\,. \tag{2.22}$$

This was recently observed and exploited in a related context in [NRS].

Other natural sets of Darboux-coordinates (l_e,k_e) can be obtained by means of canonical transformations $k_e' = k_e + f(l)$. By a suitable choice of $f(l)$, one gets Darboux coordinates (l_e,k_e) in which, for example, the expression for L_t^e in (2.20) is replaced by

$$\begin{aligned} &L_t^e((L_s^e)^2-4) \\ &= 2(L_2^eL_3^e+L_1^eL_4^e)+L_s^e(L_1^eL_3^e+L_2^eL_4^e) \\ &\quad + 2\cosh\tfrac{l_s^e+l_1^e-l_2^e}{4}2\cosh\tfrac{l_s^e+l_2^e-l_1^e}{4}2\cosh\tfrac{l_s^e+l_3^e-l_4^e}{4}2\cosh\tfrac{l_s^e+l_4^e-l_3^e}{4}\;e^{+k_s'} \\ &\quad + 2\cosh\tfrac{l_s^e+l_1^e+l_2^e}{4}2\cosh\tfrac{l_s^e-l_1^e-l_2^e}{4}2\cosh\tfrac{l_s^e+l_3^e+l_4^e}{4}2\cosh\tfrac{l_s^e-l_3^e-l_4^e}{4}\;e^{-k_s'}\,. \end{aligned} \tag{2.23}$$

The Darboux coordinates (l_e,k_e) are equally good to represent the Poisson structure of $\mathcal{M}_G(C_{0,4})$, but they have the advantage that the expressions for L_κ^e do not contain square-roots. This remark will later turn out to be useful.

Part I. Supersymmetric gauge theories

Summary:

- Review of SUSY gauge theories $\mathcal{G}_C$ of class $\mathcal{S}$ on $4d$ ellipsoids.
- The path integrals representing supersymmetric observables on $4d$ ellipsoids localize to the quantum mechanics of the scalar zero modes of $\mathcal{G}_C$.
- The instanton partition functions can be interpreted as certain wave-functions $\Psi_\tau(a)$ in the zero mode quantum mechanics.
- The Wilson and 't Hooft loops act nontrivially on the wave-functions $\Psi_\tau(a)$.
- Algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ generated by supersymmetric Wilson and 't Hooft loops is isomorphic to the quantized algebra of functions on a component of $\mathcal{M}_{\rm flat}(C)$.
- Physical reality properties of Wilson and 't Hooft loops $\Rightarrow$ Relevant for $\mathcal{G}_C$ is the component $\mathcal{M}^0_{\rm flat}(C) \subset \mathcal{M}_{\rm flat}(C)$ isomorphic to the Teichmüller space $\mathcal{T}(C)$.
- Analyticity + behavior under S-duality $\Rightarrow$ Instanton partition functions can be characterized as solutions to a Riemann-Hilbert type problem.

3. Quantization of $\mathcal{M}_{\rm flat}(C)$ from gauge theory

To a Riemann surface C of genus g and n punctures one may associate [G09] a four-dimensional gauge theory $\mathcal{G}_C$ with $\mathcal{N}=2$ supersymmetry, gauge group $(\mathrm{SU}(2))^{3g-3+n}$ and flavor symmetry $(\mathrm{SU}(2))^n$. In the cases where $(g,n)=(0,4)$ and $(g,n)=(1,1)$ one would get the supersymmetric gauge theories commonly referred to as $N_f=4$ and $N=2^*$-theory, respectively. The aim of this introductory section is to review the relation between C and $\mathcal{G}_C$ along with recent exact results on expectation values of certain supersymmetric observables in $\mathcal{G}_C$.

3.1. Supersymmetric gauge theories of class $\mathcal{S}$

The gauge theory $\mathcal{G}_C$ has a Lagrangian description for each choice of a pants decomposition σ. We will now describe the relevant parts of the mapping between geometric structures on C and the defining data of $\mathcal{G}_C$.

The field content of $\mathcal{G}_C$ is determined as follows. To each internal edge $e \in \Gamma_\sigma$ there is an associated $\mathcal{N} = 2$ vector multiplet containing a vector field A^e_μ, two fermions λ_e, $\bar{\lambda}_e$, and two real scalars ϕ_e, $\bar{\phi}_e$. Matter fields are represented by (half-)hypermultiplets associated to the vertices v of Γ_σ. They couple only to the gauge fields associated to the edges that meet at the vertex v. There are n mass parameters associated to the boundary components of C. We refer to [HKS2] for a description of the necessary building blocks for building the Lagrangian of $\mathcal{G}_C$ associated to a pants decomposition σ.

The Lagrangian for $\mathcal{G}_C$ will include kinetic terms for the gauge fields A^e_μ with gauge coupling constants g_e, and it may include topological terms with theta angles θ_e. These parameters are related to the gluing parameters q_e as

$$q_e := e^{2\pi i \tau_e}\,, \qquad \tau_e := \frac{4\pi i}{g_e^2} + \frac{\theta_e}{2\pi}\,. \tag{3.1}$$

In order to define UV couplings constants like g_e^2 one generically needs to fix a particular scheme for calculating amplitudes or expectation values. Using a different scheme will lead to equivalent results related by analytic redefinitions of the coupling constants. This ambiguity will be mapped to the dependence of the coordinates q_e for $\mathcal{T}(C)$ on the choices of local coordinates around the punctures. Equation (3.1) describes the relation which holds for a particular scheme in $\mathcal{G}_C$, and a particular choice of local coordinates around the punctures of $C_{0,3}$.

Different Lagrangian descriptions are related by S-duality. It follows from the description of the gauge theories $\mathcal{G}_C$ from class $\mathcal{S}$ given in [G09] that the groupoid of S-duality transformations coincides with the Moore-Seiberg groupoid for the gauge theories of class $\mathcal{S}$.

3.2. Supersymmetric gauge theories on ellipsoids

It may be extremely useful to study quantum field theories on compact Euclidean space-times or on compact spaces rather than flat $\mathbb{R}^4$. Physical quantities get finite size corrections which encode deep information on the quantum field theory we study. The zero modes of the fields become dynamical, and have to be treated quantum-mechanically.

In the case of supersymmetric quantum field theories there are not many compact background space-times that allow us to preserve part of the supersymmetry. A particularly interesting family of examples was studied in [HH], generalizing the seminal work of Pestun [Pe].

3.2.1. The four-dimensional ellipsoid. Let us consider gauge theories $\mathcal{G}_C$ on the four-dimensional ellipsoid

$$E^4_{\epsilon_1,\epsilon_2} := \{\,(x_0,\dots,x_4)\,|\,x_0^2+\epsilon_1^2(x_1^2+x_2^2)+\epsilon_2^2(x_3^2+x_4^2)=1\,\}\,. \tag{3.2}$$

Useful polar coordinates for $E^4_{\epsilon_1,\epsilon_2}$ are defined as

$$x_0=\sin\rho\,,\qquad \begin{aligned} x_1&=\epsilon_1^{-1}\cos\rho\,\cos\theta\,\cos\varphi\,,\\ x_2&=\epsilon_1^{-1}\cos\rho\,\cos\theta\,\sin\varphi\,,\\ x_3&=\epsilon_2^{-1}\cos\rho\,\sin\theta\,\cos\chi\,,\\ x_4&=\epsilon_2^{-1}\cos\rho\,\sin\theta\,\sin\chi\,. \end{aligned} \tag{3.3}$$

It was shown in [Pe, HH] for some examples of gauge theories $\mathcal{G}_C$ that one of the supersymmetries Q is preserved on $E^4_{\epsilon_1,\epsilon_2}$. It should be possible to generalize the proof of existence of an unbroken supersymmetry Q to all gauge theories $\mathcal{G}_C$ of class $\mathcal{S}$.

3.2.2. Supersymmetric loop operators. Supersymmetric Wilson loops can be defined as[6]

$$W_{e,1} := \mathrm{Tr}_F\,\mathcal{P}\exp\left(i\int_{S^1_1} d\varphi\,\left(A^e_\varphi-\frac{1}{\epsilon_1}(\phi_e+\bar\phi_e)\right)\right), \tag{3.4a}$$

$$W_{e,2} := \mathrm{Tr}_F\,\mathcal{P}\exp\left(i\int_{S^1_2} d\chi\,\left(A^e_\chi-\frac{1}{\epsilon_2}(\phi_e+\bar\phi_e)\right)\right), \tag{3.4b}$$

with traces taken in the fundamental representation of $SU(2)$, and contours of integration being

$$S^1_1 := \{\,(x_0,\dots,x_4)=(\pi/2,\epsilon_1^{-1}\cos\varphi,\epsilon_1^{-1}\sin\varphi,0,0)\,,\ \varphi\in[0,2\pi)\,\}\,, \tag{3.5}$$

$$S^1_2 := \{\,(x_0,\dots,x_4)=(\pi/2,0,0,\epsilon_2^{-1}\cos\chi,\epsilon_2^{-1}\sin\chi)\,,\ \chi\in[0,2\pi)\,\}\,. \tag{3.6}$$

The 't Hooft loop observables $T_{e,i}$, $i=1,2$, can be defined semiclassically for vanishing theta-angles $\theta_e=0$ by the boundary condition

$$F_e \sim \frac{B_e}{4}\epsilon_{ijk}\frac{x^i}{|\vec{x}|^3}dx^k\wedge dx^j\,, \tag{3.7}$$

[6] We adopt the conventiones used in [HH].

near the contours S_i^1, $i = 1, 2$. The coordinates x^i are local coordinates for the space transverse to S_i^1, and B_e is an element of the Cartan subalgebra of $SU(2)_e$. In order to get supersymmetric observables one needs to have a corresponding singularity at S_i^1 for the scalar fields ϕ_e, $\bar{\phi}_e$. For the details of the definition and the generalization to $\theta_e \neq 0$ we refer to [GOP].

It is shown in [Pe, HH, GOP] that these observables are left invariant by the supersymmetry Q preserved on $E^4_{\epsilon_1,\epsilon_2}$.

3.2.3. Expectation values on the ellipsoid. Interesting physical quantities include the partition function $\mathcal{Z}_{\mathcal{G}_C}$, or more generally expectation values of supersymmetric loop operators $\mathcal{L}_\gamma$ such as the Wilson- and 't Hooft loops. Such quantities are formally defined by the path integral over all fields on $E^4_{\epsilon_1,\epsilon_2}$. It was shown in a few examples for gauge theories from class $\mathcal{S}$ in [Pe, HH] how to evaluate this path integral by means of the localization technique. A variant of the localization argument was used to show that the integral over all fields actually reduces to an integral over the locus in field space where the scalars ϕ_e take constant values $\phi_e = \bar{\phi}_e \equiv \frac{i}{2} a_e \sigma_3 = \text{const}$, and all other fields vanish. This immediately implies that the path integral reduces to an ordinary integral over the variables a_e. It seems clear that this argument can be generalized to all theories of class $\mathcal{S}$.

A more detailed study [Pe, HH] then leads to the conclusion that the Wilson loop expectation values have expressions of the form

$$\big\langle\, W_{e,i} \,\big\rangle_{E^4_{\epsilon_1\epsilon_2}} = \int d\mu(a)\; |\mathcal{Z}_{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)|^2\, 2\cosh(2\pi a_e/\epsilon_i)\,, \tag{3.8}$$

where $i = 1, 2$. $\mathcal{Z}_{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)$ is the so-called instanton partition function. It depends on Coulomb branch moduli $a = (a_1, \ldots, a_h)$, hypermultiplett mass parameters $m = (m_1, \ldots, m_n)$, UV gauge coupling constants $\tau = (\tau_1, \ldots, \tau_h)$, and two parameters ϵ_1, ϵ_2. We will briefly summarize some relevant issues concerning its definition in Subsection 3.3 below.

A rather nontrivial extension of the method from [Pe] allows one to treat the case of 't Hooft loops [GOP] as well, in which case a result of the following form is found

$$\big\langle\, T_{e,i} \,\big\rangle_{E^4_{\epsilon_1,\epsilon_2}} = \int d\mu(a)\; (\mathcal{Z}_{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2))^*\, \mathcal{D}_{e,i} \cdot \mathcal{Z}_{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)\,, \tag{3.9}$$

with $\mathcal{D}_{e,i}$ being a certain difference operator acting only on the variable a_e, which has coefficients that depend only on a, m and ϵ_i, in general.

3.3. Instanton partition functions - scheme dependence

Let us briefly discuss some relevant aspects concerning the definition of $\mathcal{Z}^{\rm inst}(a,m,\tau;\epsilon_1,\epsilon_2)$. This function is defined in [N] as a partition function of a two-parametric deformation $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ of $\mathcal{G}_C$ on $\mathbb{R}^4$. The theory $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ is defined by deforming the Lagrangian of $\mathcal{G}_C$ by (ϵ_1,ϵ_2)-dependent terms which break four-dimensional Lorentz invariance, but preserve one of the supersymmetries of $\mathcal{G}_C$ on $\mathbb{R}^4$. The unbroken supersymmetry allows one to localize the path integral defining $\mathcal{Z}_{\rm inst}(a,m,\tau;\epsilon_1,\epsilon_2)$ to a sum over integrals over the instanton moduli spaces.

Subsequent generalizations to gauge theories from class $\mathcal{S}$ [AGT, HKS1, HKS2] of linear quiver type lead to expressions of the following form,

$$\mathcal{Z}^{\rm inst}(a,m,\tau;\epsilon_1,\epsilon_2)=\mathcal{Z}^{\rm pert}\sum_{\mathbf{k}\in(\mathbb{Z}^{\geq 0})^h} q_1^{k_1}\cdots q_h^{k_h}\,\mathcal{Z}^{\rm inst}_{\mathbf{k}}(a,m;\epsilon_1,\epsilon_2)\,. \tag{3.10}$$

Let us first discuss the terms $\mathcal{Z}^{\rm inst}_{\mathbf{k}}(a,m;\epsilon_1,\epsilon_2)$ summed in (3.10). These terms can be represented as multiple (h-fold) integrals over the moduli spaces $\mathcal{M}^{\rm inst}_{k,2}$ of $SU(2)$-instantons of charge k.

3.3.1. UV issues in the instanton corrections. It is important to bear in mind that the integrals defining $\mathcal{Z}^{\rm inst}_{\mathbf{k}}(a,m;\epsilon_1,\epsilon_2)$ are UV divergent due to singularities caused by pointlike, and possibly colliding instantons, see e.g. [DHKM]. Possible IR divergencies are regularized by the above-mentioned (ϵ_1,ϵ_2)-dependent deformation of the Lagrangian [N].

The explicit formulae for $\mathcal{Z}^{\rm inst}_{\mathbf{k}}(a,m;\epsilon_1,\epsilon_2)$ that were used in the calculations of expectation values $\langle\, \mathcal{L}_\gamma \,\rangle_{E^4_{\epsilon_1,\epsilon_2}}$ performed in [Pe, AGT, GOP, HH] have been obtained using particular prescriptions for regularizing the UV-divergencies which were introduced in [N, NO] and [NS04]. The approach of [N, NO] uses a non-commutative deformation of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ which is known to yield a smooth resolution of the instanton moduli spaces $\mathcal{M}^{\rm inst}_{k,2}$ [NS98]. Another approach, presented in [NS04], uses a representation of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ as the limit of a five-dimensional gauge theory on $\mathbb{R}^4\times S^1$ when the radius of the factor S^1 vanishes. It was shown in [NS04] that both prescriptions yield identical results.

These approaches work most straightforwardly for gauge theories with gauge group $(U(2))^h$ rather than $(SU(2))^h$. In order to use the known results for $(U(2))^h$, the authors of [AGT] proposed that the instanton partition functions $\mathcal{Z}^{\rm inst}(a,m,\tau;\epsilon_1,\epsilon_2)$ for gauge group $(SU(2))^h$ are related to their counterparts $\mathcal{Z}^{\rm inst}_{U(2)}(a,m,\tau;\epsilon_1,\epsilon_2)$ defined in theories with gauge group $(U(2))^h$

by splitting off a "$U(1)$-factor",

$$\mathcal{Z}^{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2) = \mathcal{Z}^{\text{spur}}_{U(1)}(m,\tau;\epsilon_1,\epsilon_2)\,\mathcal{Z}^{\text{inst}}_{U(2)}(a,m,\tau;\epsilon_1,\epsilon_2)\,. \tag{3.11}$$

Note that the $U(1)$-factor $\mathcal{Z}^{\text{spur}}_{U(1)}(m,\tau;\epsilon_1,\epsilon_2)$ does not depend on the Coulomb branch moduli a. However, the precise form of the factor proposed in [AGT] was so far mainly motivated by the relations with conformal field theory discorvered there.

3.3.2. Non-perturbative scheme dependence ?. One would expect that there should be other possibilities for regularizing the UV divergencies in general. Some examples were explicitly discussed in [HKS1, HKS2]. One may, for example, use that $Sp(1) \simeq SU(2)$ in order to set up an alternative scheme for the definition of the instanton partition functions. It was found to give an answer $\widetilde{\mathcal{Z}}^{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)$ that differs from $\mathcal{Z}^{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)$ by factors that do not depend on the Coulomb branch moduli a,

$$\widetilde{\mathcal{Z}}^{\text{inst}}(a,m,\tilde{\tau};\epsilon_1,\epsilon_2) = \mathcal{Z}^{\text{spur}}(m,\tau;\epsilon_1,\epsilon_2)\,\mathcal{Z}^{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)\,, \tag{3.12}$$

together with a redefinition $\tilde{\tau} = \tilde{\tau}(\tau)$ of the UV gauge coupling constants. The possibility to have redefinitions of the UV gauge couplings in general is suggested by the structure of the Uhlenbeck-compactification $\overline{\mathcal{M}}^{\text{inst}}_{k,2}$ of $\mathcal{M}^{\text{inst}}_{k,2}$,

$$\overline{\mathcal{M}}^{\text{inst}}_{k,2} = \mathcal{M}^{\text{inst}}_{k,2} \cup \big[\mathcal{M}^{\text{inst}}_{k-1,2} \times \mathbb{R}^4\big] \cup \cdots \cup \big[\text{Sym}^k(\mathbb{R}^4)\big]\,. \tag{3.13}$$

The factors $\mathcal{Z}^{\text{spur}}(m,\tau;\epsilon_1,\epsilon_2)$ in (3.12) will be called spurious following [HKS1, HKS2]. One way to justify this terminology is to note that such factors will drop out in *normalized* expectation values defined as

$$\langle\!\langle\, \mathcal{L}_\gamma \,\rangle\!\rangle_{E^4_{\epsilon_1,\epsilon_2}} := \big(\langle\, 1 \,\rangle_{E^4_{\epsilon_1,\epsilon_2}}\big)^{-1} \langle\, \mathcal{L}_\gamma \,\rangle_{E^4_{\epsilon_1,\epsilon_2}}\,, \tag{3.14}$$

as follows immediately from the general form of the results for the expectation values quoted in (3.8) and (3.9). The scheme dependence contained in the spurious factors $\mathcal{Z}^{\text{spur}}(m,\tau;\epsilon_1,\epsilon_2)$ should therefore be considered as unphysical.

It would be very interesting to understand the issue of the scheme dependence, the freedom in the choice of UV regularization used to define $\mathcal{Z}^{\text{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)$, more systematically. We will later arrive at a precise description of the freedom left by the approach taken in this paper.

3.3.3. Perturbative part. The perturbative part $\mathcal{Z}^{\rm pert}$ in (3.10) factorizes as $\mathcal{Z}^{\rm pert} = \mathcal{Z}^{\rm tree}\,\mathcal{Z}^{1-\rm loop}$.

The factor $\mathcal{Z}^{\rm inst}_{\rm tree}$ represents the tree-level contribution. It is given by a simple expression proportional (up to spurious factors) to

$$\mathcal{Z}^{\rm tree} = \prod_{e\in\sigma_1} q_e^{a_e^2/\epsilon_1\epsilon_2}\,, \tag{3.15}$$

where σ_1 is the set of edges of the MS graph Γ_σ associated to the pants decomposition σ defining the Lagrangian of $\mathcal{G}_C$.

The factor $\mathcal{Z}^{\rm inst}_{1-\rm loop}$ is given by certain determinants of differential operators. It has the following form

$$\mathcal{Z}^{1-\rm loop} = \prod_{v\in\sigma_0} \mathcal{Z}_v^{1-\rm loop}(a_{e_1(v)}, a_{e_2(v)}, a_{e_3(v)}; \epsilon_1, \epsilon_2)\,, \tag{3.16}$$

where σ_0 is the set of vertices of the MS graph Γ_σ associated to the pants decomposition σ, and $e_1(v), e_2(v), e_3(v)$ are the edges of Γ_σ that emanate from v. If an edge $e_i(v)$ ends in a boundary component of C, then $a_{e_i(v)}$ will be identified with the mass parameter associated to that boundary component.

It should be noted that there is a certain freedom in the definition of $\mathcal{Z}_v^{1-\rm loop}$ due to the regularization of divergencies in the infinite products defining $\mathcal{Z}_v^{1-\rm loop}$. This issue has a natural resolution in the case of partition functions or expectation values on $E^4_{\epsilon_1,\epsilon_2}$ going back to [Pe]: what enters into these quantities is the absolute value squared $|\mathcal{Z}_v^{1-\rm loop}(a_{e_1(v)}, a_{e_2(v)}, a_{e_3(v)}; \epsilon_1, \epsilon_2)|^2$ which is unambigously defined [Pe, HH]. There does not seem to be a preferred prescription to fix the phase of $\mathcal{Z}_v^{1-\rm loop}(a_{e_1(v)}, a_{e_2(v)}, a_{e_3(v)}; \epsilon_1, \epsilon_2)$, in general, which can be seen as a part of the perturbative scheme dependence.

3.4. Reduction to zero mode quantum mechanics

We may assign to the expecation values $\langle\mathcal{L}_\gamma\rangle$ an interpretation in terms of expectation values of operators L_γ which act on the Hilbert space obtained by canonical quantization of the gauge theory $\mathcal{G}_C$ on the space-time $\mathbb{R}\times E^3_{\epsilon_1,\epsilon_2}$, where $E^3_{\epsilon_1,\epsilon_2}$ is the three-dimensional ellipsoid defined as

$$E^3_{\epsilon_1,\epsilon_2} := \{\,(x_1,\dots,x_4)\,|\,\epsilon_1^2(x_1^2+x_2^2)+\epsilon_2^2(x_3^2+x_4^2) = 1\,\}\,. \tag{3.17}$$

This is done by interpreting the coordinate x_0 for $E^4_{\epsilon_1,\epsilon_2}$ as Euclidean time. Noting that $E^4_{\epsilon_1,\epsilon_2}$ looks near $x_0 = 0$ as $\mathbb{R}\times E^3_{\epsilon_1,\epsilon_2}$, we expect to be able to represent partition functions $\mathcal{Z}_{\mathcal{G}_C}(E^4_{\epsilon_1,\epsilon_2})$ or expectation values $\langle\, \mathcal{L}_\gamma \,\rangle_{\mathcal{G}_C(E^4_{\epsilon_1,\epsilon_2})}$ as matrix elements of states in the Hilbert space $\mathcal{H}_{\mathcal{G}_C}$ defined by canonical quantization of $\mathcal{G}_C$ on $\mathbb{R}\times E^3_{\epsilon_1,\epsilon_2}$. More precisely

$$\mathcal{Z}_{\mathcal{G}_C}(E^4_{\epsilon_1,\epsilon_2}) = \langle\,\tau\,|\,\tau\,\rangle\,, \qquad \langle\, \mathcal{L}_\gamma \,\rangle_{E^4_{\epsilon_1,\epsilon_2}} = \langle\,\tau\,|\,\mathsf{L}_\gamma\,|\,\tau\,\rangle\,, \tag{3.18}$$

where $\langle\,\tau\,|$ and $|\,\tau\,\rangle$ are the states created by performing the path integral over the upper/lower half-ellipsoid

$$\begin{aligned} E^{4,\pm}_{\epsilon_1,\epsilon_2} := \{\,(x_0,\dots,x_4)\,|\,x_0^2+\epsilon_1^2(x_1^2+x_2^2) \\ +\,\epsilon_2^2(x_3^2+x_4^2)=1\,,\ \pm x_0>0\,\}\,, \end{aligned} \tag{3.19}$$

respectively, and L_γ is the operator that represents the observable $\mathcal{L}_\gamma$ in the Hilbert space $\mathcal{H}_{\mathcal{G}_C}(E^3_{\epsilon_1,\epsilon_2})$.

3.4.1. Localization – Interpretation in the functional Schrödinger picture. The form (3.8), (3.9) of the loop operator expectation values is naturally interpreted in the Hamiltonian framework as follows. In the functional Schroedinger picture one would represent the expecation values $\langle\, \mathcal{L}_\gamma \,\rangle_{E^4_{\epsilon_1,\epsilon_2}}$ schematically in the following form

$$\langle\, \mathcal{L}_\gamma \,\rangle_{E^4_{\epsilon_1,\epsilon_2}} = \int [\mathcal{D}\Phi]\ (\Psi[\Phi])^*\,\mathsf{L}_\gamma\Psi[\Phi]\,, \tag{3.20}$$

the integral being extended over all field configuration on the three-ellipsoid $E^3_{\epsilon_1,\epsilon_2}$ at $x_0=0$. The wave-functional $\Psi[\Phi]$ is defined by means of the path integral over the lower half-ellipsoid $E^{4,-}_{\epsilon_1,\epsilon_2}$ with Dirichlet-type boundary conditions defined by a field configuration Φ on the boundary $E^3_{\epsilon_1,\epsilon_2}$ of $E^{4,-}_{\epsilon_1,\epsilon_2}$.

The fact that the path integral localizes to the locus Loc_C defined by constant values of the scalars, and zero values for all other fields implies that the path integral in (3.20) can be reduced to an ordinary integral of the form

$$\langle\, \mathcal{L}_\gamma \,\rangle_{E^4_{\epsilon_1,\epsilon_2}} = \int da\ (\Psi_\tau(a))^*\,\mathsf{L}'_\gamma\Psi_\tau(a)\,, \tag{3.21}$$

with L'_γ being the restriction of L_γ to Loc_C, and $\Psi(a)$ defined by means of the path integral over the lower half-ellipsoid $E^{4,-}_{\epsilon_1,\epsilon_2}$ with Dirichlet boundary conditions $\Phi\in\mathsf{Loc}_C$, $\phi_e=\bar{\phi}_e=\frac{i}{2}a_e\sigma_3$. The form of the results for expectation values of loop observables quoted in (3.8), (3.9) is thereby naturally explained.

Comparing the results (3.8) and (3.9) with (3.21) leads to the conclusion that the wave-functions $\Psi_\tau(a)$ appearing in (3.21) are represented by the instanton partition functions,

$$\Psi_\tau(a) = \mathcal{Z}^{\rm inst}(a, m, \tau; \epsilon_1, \epsilon_2)\,. \tag{3.22}$$

Our goal will be to find an alternative way to characterize the wave-functions $\Psi_\tau(a)$, based on their transformation properties under electric-magnetic duality.

3.4.2. Reduction to a subspace of the Hilbert space. The Dirichlet boundary condition $\Phi \in \mathsf{Loc}_C$, $\phi_e \propto a_e$ is naturally interpreted as defining a Hilbert subspace $\mathcal{H}_0$ within $\mathcal{H}_{\mathcal{G}_C}$. States in $\mathcal{H}_0$ can, by definition, be represented by wave-functions $\Psi(a)$, $a = (a_1, \ldots, a_h)$.

Note that field configurations that satisfy the boundary condition $\Phi \in \mathsf{Loc}_C$ are annhilated by the supercharge Q used in the localization calculations of [Pe, GOP, HH] – that's just what defined the locus Loc_C in the first place. This indicates that the Hilbert subspace $\mathcal{H}_0$ represents the cohomology of Q within $\mathcal{H}_{\mathcal{G}_C}$.

The algebra of observables acting on $\mathcal{H}_0$ should contain the supersymmetric Wilson- and 't Hooft loop observables. The Wilson loops $W_{e,1}$ and $W_{e,2}$ act diagonally as operators of multiplication by $2\cosh(2\pi a_e/\epsilon_1)$ and $2\cosh(2\pi a_e/\epsilon_2)$, respectively. The 't Hooft loops will act as certain difference operators.

Let us denote the non-commutative algebra of operators generated by polynomial functions of the loop operators $\mathsf{W}_{e,i}$ and $\mathsf{T}_{e,i}$ by $\mathcal{A}_{\epsilon_i}$, where $i = 1, 2$. We will denote the algebra generated by all such supersymmetric loop operators by $\mathcal{A}_{\epsilon_1\epsilon_2} \equiv \mathcal{A}_{\epsilon_1} \times \mathcal{A}_{\epsilon_2}$.

4. Riemann-Hilbert problem for instanton partition functions

The main result of this paper may be summarized in the statement that, up to spurious factors, the wave-functions $\Psi_\tau(a)$ in the quantum mechanics of the zero modes of $\mathcal{G}_C$ coincide with the Liouville conformal blocks $\mathcal{Z}^{\rm Liou}(\beta, \alpha, q; b)$,

$$\Psi_\tau(a) \,\simeq\, \mathcal{Z}^{\rm Liou}(\beta, \alpha, q; b)\,. \tag{4.1}$$

The definition of $\mathcal{Z}^{\rm Liou}(\beta, \alpha, q; b)$ will be reviewed and generalized in Part III below, where we will also spell out the dictionary between the variables involved. Combined with (3.22), we arrive at the relation $\mathcal{Z}^{\rm inst}(a,m,\tau,\epsilon_1,\epsilon_2) \simeq \mathcal{Z}^{\rm Liou}(\beta,\alpha,q;b)$ proposed in [AGT].

In this paper we will characterize the wave-functions $\Psi_\tau(a)$ using the relation between the algebra $\mathcal{A}_{\epsilon_1\epsilon_2}$ of supersymmetric loop observables to the quantized algebras of functions on moduli spaces of flat connections. These quantized algebras of functions are deeply related to Liouville theory, as will be explained in Part III of this paper. Taking into account these relations will lead to the relation (4.1) of $\Psi_\tau(a)$ with the Liouville conformal blocks.

Before we continue to discuss our approach to the relation (4.1) let us briefly review some of the known evidence for (4.1), mainly coming from its relation with the observations of [AGT].

4.1. Available evidence

The authors of [AGT] observed in some examples of theories from class $\mathcal{S}$ that one has (up to spurious factors) an equality of instanton partition functions to the conformal blocks $\mathcal{Z}^{\rm Liou}(\beta,\alpha,\tau;b)$ of Liouville theory,

$$\mathcal{Z}^{\rm inst}(a,m,\tau;\epsilon_1,\epsilon_2) \simeq \mathcal{Z}^{\rm Liou}(\beta,\alpha,\tau;b)\,, \tag{4.2}$$

assuming a suitable dictionary between the variable involved. The results of [AGT] can be generalized to a subset of the family of theories from class $\mathcal{S}$ called the linear quiver theories corresponding to surfaces C of genus 0 or 1 [AFLT].

For surfaces C of genus 0 we know, on the other hand, that the Liouville conformal blocks coincide with certain wave-functions in the quantum theory of the Teichmüller spaces $\mathcal{T}(C)$ of Riemann surfaces ([T03], see also Part III of this paper),

$$\mathcal{Z}^{\rm Liou}(\beta,\alpha,\tau;b) = \Psi_\tau^{\mathcal{T}}(a) \equiv \langle\, a\,|\,\tau\,\rangle_{\mathcal{T}(C)}\,. \tag{4.3}$$

The state $\langle\, a\,|$ is an eigenstate of a maximal family of commuting geodesic length operators, while $|\,\tau\,\rangle_{\mathcal{T}(C)}$ is defined as an eigenstate of the operators obtained in the quantization of certain complex-analytic coordinates on $\mathcal{T}(C)$. The definition of $\Psi_\tau^{\mathcal{T}}(a)$ and the derivation of (4.3) will be reviewed and generalized to surfaces C of higher genus in Part III of our paper.

Combining the observations (3.22), (4.2) and (4.3) suggests that the quantum mechanics of the zero modes of $\mathcal{G}_C$ is equivalent to the quantum theory of the Teichmüller spaces, and that we have in particular

$$\Psi_\tau(a) \simeq \Psi_\tau^{\mathcal{T}}(a)\,. \tag{4.4}$$

This conclusion was anticipated in [DGOT], where it was noted that the existing results on Wilson loop observables can be rewritten in the form

$$\big\langle\, \mathcal{L}_\gamma \,\big\rangle_{E^4_{\epsilon_1,\epsilon_2}} = \langle\, \tau \,|\, \mathsf{L}_\gamma \,|\, \tau \,\rangle_{\mathcal{T}(C)}\,, \tag{4.5}$$

using the observations (4.2) and (4.3) quoted above. The gauge theoretical calculations leading to (4.5) were later generalized to the case of 't Hooft loops in [GOP]. These results confirmed the earlier proposals made in [AGGTV, DGOT] that the supersymmetric loop operators in gauge theories $\mathcal{G}_C$ are related to the analogs of the Verlinde loop operators in Liouville theory. The Verlinde loop operators are further mapped to the geodesic length operators by the correspondence between Liouville theory and the quantum Teichmüller theory [T03, DGOT].

One should keep in mind that the Teichmüller spaces $\mathcal{T}(C)$ are naturally isomorphic to the connected components $\mathcal{M}^0_{\rm flat}(C)$ of $\mathcal{M}_{\rm flat}(C)$. Combining all these observations we may conclude that for surfaces C of genus 0 the expectation values of supersymmetric loop operators in $\mathcal{G}_C$ can be represented as expectation values of certain operators in the quantum mechanics obtained by quantizing $\mathcal{M}^0_{\rm flat}(C)$.

Our goal is to understand more directly why this is so, and to generalize this result to all theories from class $\mathcal{S}$.

4.2. Assumptions

Our approach for deriving (4.1) is based on physically motivated assumptions. We will first formulate the underlying assumptions concisely, and later dicuss the underlying motivations.

(a) $\Psi_\tau(a)$ can be analytically continued with respect to the variables τ to define a multi-valued analytic function on the coupling constant space $\mathcal{M}(\mathcal{G}_C)$. The boundaries of $\mathcal{M}(\mathcal{G}_C)$, labelled by pants decompositions σ correspond to weakly-coupled Lagrangian descriptions for $\mathcal{G}_C$.

(b) The transitions between any two different weakly-coupled Lagrangian descriptions for $\mathcal{G}_C$ are generated from the elementary electric-magnetic duality transformations of the $N_f = 4$ and the $\mathcal{N} = 2^*$-theories. The electric-magnetic duality transformations exchange the respective Wilson- and 't Hooft loop observables.

(c) The algebra $\mathcal{A}_{\epsilon_1\epsilon_2}$ generated by the supersymmetric loop observables is isomorphic to the algebra $\mathrm{Fun}_{\epsilon_1}(\mathcal{M}_{\mathrm{flat}}(C)) \times \mathrm{Fun}_{\epsilon_2}(\mathcal{M}_{\mathrm{flat}}(C))$, where $\mathrm{Fun}_{\epsilon}(\mathcal{M}_{\mathrm{flat}}(C))$ is the quantized algebra of functions on $\mathcal{M}^0_{\mathrm{flat}}(C) \simeq \mathcal{T}(C)$.

Assumptions (a) and (b) can be motivated by noting that the theories of class $\mathcal{S}$ are all quiver gauge theories [G09]. This combinatorial structure reduces the S-duality transformations to those of the building blocks, the $N_f = 4$ and the $\mathcal{N} = 2^*$-theories. The realization of electric-magnetic duality in these theories has been discussed extensively in the literature, going back to the works of Seiberg and Witten [SW1, SW2].

Of particular importance for us is assumption (c). Let us first note that this assumption is strongly supported by the explicit calculation of the 't Hooft loop operator expectation values in the $\mathcal{N} = 2^*$-theory carried out in [GOP]. One finds a precise correspondence between the difference operator $\mathcal{D}_{e,i}$ in (3.9) and operator L_t representing the trace coordinate L_t in the quantum theory of $\mathcal{M}_{\mathrm{flat}}(C)$ (see Equation (6.16a) below).

It should be possible to verify assumption (c) directly by studying the algebra of Wilson-'t Hooft loop operators in the theories $\mathcal{G}_C$ in more generality. It was proposed in [IOT] that in order to study the *algebra* of supersymmetric loop operators one may replace the background space-time $E^4_{\epsilon_1,\epsilon_2}$ by the local model $S^1 \times \mathbb{R}^3$ for the vicinity of the loop operators, taking into account the relevant effects of the curvature by a simple twist in the boundary conditions. This has been used in [IOT] to calculate expectation values of supersymmetric loop operators in several cases. The results give additional support for the validity of assumption (c). Further development of this approach may well lead to a derivation of (c) purely within four-dimensional gauge theory.

As also pointed out in [IOT], the twisted boundary conditions on $S^1 \times \mathbb{R}^3$ used in this paper are essentially equivalent to the deformation of $\mathcal{G}_C$ studied in [GMN3]. Specializing the results of [GMN3] to the A_1 theories of class $\mathcal{S}$ considered here, one gets a non-commutative algebra of observables with generators $\mathbb{L}_\gamma$ which can be represented in the form

$$\mathbb{L}_\gamma = \sum_\eta \overline{\underline{\Omega}}(\gamma, \eta; y) X_\eta \,, \tag{4.6}$$

where X_η are generators of the non-commutative algebra obtained by canonical quantization of the Darboux-coordinates studied in [GMN1, GMN2], and the coefficients $\underline{\Omega}(\gamma,\eta;y)$ are indices for certain BPS states extensively studied in [GMN3]. It is pointed out in this paper, on the one hand, that there is a simple physical reason for getting a non-commutative deformation of the algebra of the Darboux-coordinates generated by the X_η. On the other hand it is argued in [GMN3] that (4.6) coincides with the decomposition of geodesic length operators into the (quantized) coordinates for the Teichmüller spaces introduced by Fock [F97]. It follows that the algebra generated by the $\mathbb{L}_\gamma$ is isomorphic to the algebra of geodesic length operators in the quantum Teichmüller theory. This is exactly the algebra $\mathrm{Fun}_\epsilon(\mathcal{M}_{\mathrm{flat}}(C))$ studied in this paper. We believe that this line of thoughts can lead to an insightful derivation of our assumption (c), but it seems desirable to have a more detailed discussion of the applicability of the results of [GMN3] to our set-up.

Yet another approach towards understanding assumption (c) starts from a modified set-up in which the gauge theory $\mathcal{G}_C$ is replaced by its Omega-deformed version $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ [N, NW]. In the Omega-deformed theory one may define analogs of the loop observables L_γ and wave-functions $\Psi_\tau^{\mathrm{top}}(a)$ in a very similar way as above, and one has $\Psi_\tau^{\mathrm{top}}(a)=\mathcal{Z}^{\mathrm{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)$. Combined with the observation (3.22) made above we see that

$$\Psi_\tau(a)=\mathcal{Z}^{\mathrm{inst}}(a,m,\tau;\epsilon_1,\epsilon_2)=\Psi_\tau^{\mathrm{top}}(a)\,. \tag{4.7}$$

This strongly indicates that we may use the results on the Omega-deformed theory $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ from [NW] for the study of the gauge theory on $E^4_{\epsilon_1,\epsilon_2}$. In the following Section 5 we will briefly review the argument for (c) in the Omega-deformed theory $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ which was given by Nekrasov and Witten in [NW].

4.3. The Riemann-Hilbert problem

The strategy for deriving (4.1) may now be outlined as follows.

Assumption (b) implies that the S-duality transformations induce a change of representation for the Hilbert space $\mathcal{H}_{\mathrm{top}}$. Recall that $\Psi_\tau^{\sigma_1}(a)$ is defined to be a joint eigenfunction of the Wilson loop operators constructed using the weakly coupled Lagrangian description associated to a pants decomposition σ_1. Considering another pants decomposition σ_2 one defines in a similar manner eigenfunctions $\Psi_\tau^{\sigma_2}(a)$ of another family of operators which are not commuting with the Wilson loop operators defined from pants decomposition σ_1, but can be constructed as Wilson loop observables using the fields used in the Lagrangian description of $\mathcal{G}_C$ associated to pants decompostion σ_2. The eigenfunctions $\Psi_\tau^{\sigma_1}(a)$ and $\Psi_\tau^{\sigma_2}(a)$ must therefore be related by an integral transformations of the form

$$\Psi_\tau^{\sigma_2}(a_2) = f_{\sigma_2\sigma_1}(\tau) \int da_1 \; K_{\sigma_2\sigma_1}(a_2, a_1) \, \Psi_\tau^{\sigma_1}(a_1) \,. \tag{4.8}$$

We allow for a spurious prefactor $f_{\sigma_2\sigma_1}(\tau)$ in the sense explained in Subsection 3.3, as it will turn out that we can not eliminate such prefectors by choosing an appropriate scheme in general.

Given that we know the data $K_{\sigma_2\sigma_1}(a_2, a_1)$ and $f_{\sigma_2\sigma_1}(\tau)$, the assumptions (a) - (c) completely describe of the analytic properties of $\mathcal{Z}^{\text{inst}}(a, m, \tau, \epsilon_1, \epsilon_2)$ as function on the coupling constant space $\mathcal{M}(\mathcal{G}_C)$. This means that $\mathcal{Z}^{\text{inst}}(a, m, \tau, \epsilon_1, \epsilon_2)$ can be characterized as the solution to a Riemann-Hilbert type problem.

A detailed construction of the representation of $\mathcal{A}_{\epsilon_1\epsilon_2}$ on $\mathcal{H}_0$ will be given in Part II of this paper. The main result for our purposes is to show that the kernels $K_{\sigma_2\sigma_1}(a_2, a_1)$ appearing in (4.8) can be characterized by the requirement that this transformation correctly exchanges the Wilson- and 't Hooft loops defined in the two Lagrangian descriptions associated to σ_1 and σ_2, respectively. The technically hardest part is to ensure that the Moore-Seiberg groupoid of transformations from one Lagrangian description to another is correctly realized by the transformations (4.8).

In Part III we will then show that this Riemann-Hilbert type problem has a solution that is unique up to spurious factors as encountered in (3.12), and given by the Liouville conformal blocks appearing on the right hand side of (4.1). A precise mathematical charcterization of the possible spurious factors is obtained.

It may be instructive to compare this type of reasoning to the derivations of exact results for prepotentials in supersymmetric gauge theories pioneered by Seiberg and Witten. The key assumptions made in these derivations were the analyticity of the prepotential, and assumptions on the physical interpretation of its singularities. Well-motivated assumptions on effective descriptions near the singularities of the prepotential $\mathcal{F}$ lead Seiberg and Witten to a characterization of this quantity in terms of a Riemann-Hilbert problem. A key assumption was that the transition between any two singularities of the prepotential corresponds to electric-magnetic duality.

5. The approach of Nekrasov and Witten

An approach towards understanding the link between the gauge theories $\mathcal{G}_C$ and Liouville theory expressed in formula (4.2) was proposed in the work [NW] of Nekrasov and Witten. This work considers the gauge theory $\mathcal{G}_C$

on four-manifolds M^4 that have $(U(1))^2$-isometries and therefore allow to define the Omega-deformation $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ of $\mathcal{G}_C$. The result may imprecisely be summarized by saying that the topological sector of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ is represented by the quantum mechanics obtained from the quantization of $\mathcal{M}_{\rm flat}(C)$. The arguments presented in [NW] do not quite suffice to derive the AGT-correspondence in the strong form (4.2).

We will argue that one may take the arguments of [NW] as a starting point to reach the more precise result (4.2): Certain wave-functions in the topologically twisted version of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ considered by [NW] coincide with the conformal blocks of Liouville theory. As the wave-functions in question also coincide with the instanton partition functions (almost by definition), we will thereby get a derivation of the AGT-correspondence which is somewhat in the spirit of the characterization of the prepotentials that was pioneered by Seiberg and Witten.

5.1. The basic ideas

The approach of Nekrasov and Witten is based on three main ideas:

(i) The instanton partition functions are defined in [N] as partition functions of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on $\mathbb{R}^4$. The deformation of $\mathcal{G}_C$ into $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ preserves a supersymmetry which can be used to define a topologically twisted version $\mathcal{G}_C^{\rm top}$ of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$. The partition function of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on $\mathbb{R}^4$ coincides with the partition function of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on any four-manifold B^4 with the same topology as $\mathbb{R}^4$ that has the $(S^1)^2$-isometries needed to define the Omega-deformation $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ of $\mathcal{G}_C$ [NW].

(ii) The four-manifold B^4 may be assumed to have a boundary M^3, and the metric near the boundary may be assumed to be the metric on $R\times M^3$. Canonical quantization on $\mathbb{R}\times M^3$ yields a quantum theory with Hilbert space $\mathcal{H}_{M^3}(\mathcal{G}_C^{\epsilon_1\epsilon_2})$. The partition function on B^4 can then be interpreted as a wave-function of the state created by performing the path integral over B^4.

(iii) Viewing S^3 as a fibration of $(S^1)^2$ over an interval I, one may represent $\mathcal{G}_C^{\rm top}$ on $R\times S^3$ in terms of a topologically twisted two-dimensional non-linear sigma model on the world-sheet $R\times I$ with target space $\mathcal{M}_H$, the Hitchin moduli space. This means that the instanton partition function gets re-interpreted as a wave-function of a certain state in the two-dimensional sigma model on the strip.

Let us consider the topologically twisted theory $\mathcal{G}_C^{\rm top}$ on $\mathbb{R}\times M^3$. The topological twist preserves two super-charges Q and $\mathsf{Q}^\dagger$. Choosing Q to be the preferred super-charge, one may identify the Hilbert-space $\mathcal{H}_{\rm top}\equiv\mathcal{H}_{M^3}^{\rm top}(\mathcal{G}_C^{\epsilon_1\epsilon_2})$ of $\mathcal{G}_C^{\rm top}$ with the Q-cohomology within $\mathcal{H}_{M^3}(\mathcal{G}_C^{\epsilon_1\epsilon_2})$.

A few points are clear. The Hilbert space $\mathcal{H}_{\rm top}$ is acted on by the chiral ring operactors

$$\mathsf{u}_e := \mathrm{Tr}(\phi_e^2)\,. \tag{5.1}$$

These operators generate a commutative ring of operators acting on $\mathcal{H}_{\rm top}$. It is furthermore argued in [NW, Section 4.9.1] that analogs of the Wilson- and 't Hooft loop operators can be be defined within the gauge theory $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on $\mathbb{R}\times M^3$ which commute with Q, and therefore define Wilson- and 't Hooft loop operators $\mathsf{W}_{e,i}$ and $\mathsf{T}_{e,i}$ acting on $\mathcal{H}_{\rm top}$. We will denote the algebra generated by all such supersymmetric loop operators by $\mathcal{A}^{\rm top}_{\epsilon_1\epsilon_2}\equiv\mathcal{A}^{\rm top}_{\epsilon_1}\times\mathcal{A}^{\rm top}_{\epsilon_2}$.

And indeed, one of the main results of [NW] were the isomorphisms

$$\mathcal{A}^{\rm top}_{\epsilon_1}\times\mathcal{A}^{\rm top}_{\epsilon_2}\simeq \mathrm{Fun}_{\epsilon_1}(\mathcal{M}_{\rm flat}(C))\times \mathrm{Fun}_{\epsilon_2}(\mathcal{M}_{\rm flat}(C))\,, \tag{5.2}$$

where $\mathrm{Fun}_q(\mathcal{M}_{\rm flat}(C))$ is the quantized algebra of functions on $\mathcal{M}_{\rm flat}(C)$ that will be defined precisely in Part II, together with

$$\mathcal{H}_{M^3}^{\rm top}(\mathcal{G}_C^{\epsilon_1,\epsilon_2})\;\simeq\;\mathcal{H}(\mathcal{M}^0_{\rm flat}(C))\,, \tag{5.3}$$

both sides being understood as module of $\mathrm{Fun}_{\epsilon_1}(\mathcal{M}_{\rm flat}(C))\times \mathrm{Fun}_{\epsilon_2}(\mathcal{M}_{\rm flat}(C))$.

5.2. The effective sigma model description

It may be instructive to briefly outline the approach that lead to the results (5.2) and (5.3), see [NW] for more details.

In order to get a useful effective representation for $\mathcal{G}_C^{\epsilon_1\epsilon_2}$, let us note that we may view three manifolds M^3 with the necessary $(U(1))^2$-isometries as a circle fibration $S^1\times S^1\to I$, where the base I is an interval. It was argued in [NW] that the low energy physics of $\mathcal{G}_C$ can be represented by a $(4,4)$-supersymmetric sigma model with world-sheet $\mathbb{R}\times I$ and target space being the Hitchin moduli space $\mathcal{M}_{\rm H}(C)$. This sigma model can be thought of as being obtained by compactifying $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on $S^1\times S^1$. Due to topological invariance one expects that supersymmetric observables of $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ get represented within the quantum theory of the sigma model.

An elegant argument for why the sigma model has target space $\mathcal{M}_{\rm H}(C)$ can be based on the description of $\mathcal{G}_C$ as compactification of the six-dimensional $(0,2)$-superconformal theories of the A_1-type on spaces of the form $M^4 \times C$. If M^4 has the structure of a circle fibration, one expects that the result of compactifying first on C, then on the circle fibers should be equivalent to the result of first compactifying on the circle fibers, and then on C, as far as the resulting topological subsector is concerned. If one compactifies the six-dimensional $(0,2)$-superconformal theory on a circle S^1, or on $S^1 \times S^1$, the result is a maximally supersymmetric Yang-Mills theory with gauge group $SU(2)$ on a five-, or four-dimensional space-time, respectively. Minimal energy configurations in the resulting theories on space-times of the form $M \times C$ are represented by solutions of Hitchin's equations on C [BJSV], see also [GMN2, Subsection 3.1.6]. It follows that the low-energy physics can be effectively represented by a sigma model on M which has $\mathcal{M}_H(C)$ as a target space. This argument has been used in [NW], see also [NRS] for a similar discussion.

The effect of the Ω-deformation is represented within the sigma-model description by boundary conditions $\mathcal{B}_{\epsilon_1}$ and $\mathcal{B}_{\epsilon_2}$ imposed on the sigma model at the two ends of the interval I. It is shown that the boundary conditions are represented by the so-called canonical co-isotropic branes, see [NW] for the definition and further references. The Hilbert space $\mathcal{H}^{\rm top}_{M^3}(\mathcal{G}_C^{\epsilon_1\epsilon_2})$ thereby gets identified with the space of states $\mathrm{Hom}(\mathcal{B}_{\epsilon_1},\mathcal{B}_{\epsilon_2})$ of this open two-dimensional sigma model.

It was furthermore argued in [NW] that the action of the algebra $\mathcal{A}^{\rm top}_{\epsilon_1,\epsilon_2}$ of supersymmetric loop operators on $\mathcal{H}^{\rm top}_{M^3}(\mathcal{G}_C^{\epsilon_1,\epsilon_2})$ gets represented in the sigma model as the action of the quantized algebra of functions on the canonical coisotropic branes via the joining of open strings, which defines a natural left (resp. right) action of $\mathcal{A}_{\epsilon_1}(C) \simeq \mathrm{Hom}(\mathcal{B}_{\epsilon_1},\mathcal{B}_{\epsilon_1})$ (resp. $\mathcal{A}_{\epsilon_2}(C) \simeq \mathrm{Hom}(\mathcal{B}_{\epsilon_2},\mathcal{B}_{\epsilon_2})$) on $\mathrm{Hom}(\mathcal{B}_{\epsilon_1},\mathcal{B}_{\epsilon_2})$. The key result obtained in [NW] is then that the algebras $\mathrm{Hom}(\mathcal{B}_{\epsilon_i},\mathcal{B}_{\epsilon_i})$, $i = 1,2$, with multiplication naturally defined by the joining of strips, are isomorphic to the quantized algebras of functions $\mathrm{Fun}_{\epsilon_i}(\mathcal{M}_{\rm flat}(C))$ on $\mathcal{M}_{\rm flat}(C)$. The method by which this conclusion is obtained can be seen as special case of a more general framework for producing quantizations of algebras of functions on hyper-Kähler manifolds from the canonical coisotropic branes of the sigma models on such manifolds [GV].

5.3. Instanton partition functions as wave-functions

Let us extract from [NW] some implications that will be relevant for us.

Recall that the algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ is generated by the quantized counterparts of Wilson- and 't Hooft loop operators. Using localisation [Pe] one may show that the Wilson loop operators $\mathsf{W}_{e,i}$ are positive self-adjoint, and mutually commutative $[\mathsf{W}_{e,i}, \mathsf{W}_{e',i}] = 0$ for $i = 1, 2$. It follows that there exists a representation for $\mathcal{H}_{\rm top}$ in which the states are realized by wave-functions $\Psi(a) \equiv \langle a|\Psi\rangle_{\rm top}$, where $a = (a_1, \dots, a_h)$.

As S-duality exchanges Wilson- and 't Hooft loops, the 't Hooft loops must also be positive self-adjoint. What is relevant for us is therefore the subspace of the space of functions on $\mathcal{M}_{\rm flat}(C)$ characterized by the positivity of all loop observables. This subspace is isomorphic to the space of functions on the Teichmüller space $\mathcal{T}(C)$, and will be denoted $\mathcal{M}^0_{\rm flat}(C)$.

Considering the gauge theory $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on $\mathbb{R} \times M^3$ one may naturally consider a state $|\,\tau\,\rangle \in \mathcal{H}_{\mathcal{G}_C}(M^3)$ created by performing the path integral over the a Euclidean four-manifold $B^{4,-}$ with boundary M^3, and its projection $|\,\tau\,\rangle_{\rm top}$ to $\mathcal{H}^{\rm top}$. We may represent $|\,\tau\,\rangle_{\rm top}$ by its wave-function

$$\Psi^{\rm top}_\tau(a) := \langle\, a\,|\,\tau\,\rangle_{\rm top}\,. \tag{5.4}$$

Note that the overlap between an eigenstate $\langle\, a\,|$ of all the Wilson loop operators with the state $|\,\tau\,\rangle_{\rm top}$ should be related to the instanton partition function by means of the metric-independence of the path integrals for $\mathcal{G}_C^{\rm top}$. This should relate $\langle\, a\,|\,\tau\,\rangle_{\rm top}$, given by the path integral for $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on $B^{4,-}$ to $\mathcal{Z}^{\rm inst}(a, m, \tau, \epsilon_1, \epsilon_2)$ which is defined by a path integral on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$,

$$\Psi^{\rm top}_\tau(a) = \mathcal{Z}^{\rm inst}(a, m, \tau, \epsilon_1, \epsilon_2)\,. \tag{5.5}$$

The projection onto an eigenstate $\langle\, a\,|$ of the Wilson loop operators is traded for the boundary condition to have fixed scalar expectation values at the infinity of $\mathbb{R}^4$.

We conclude that the instanton partition functions $\mathcal{Z}^{\rm inst}(a, m, \tau, \epsilon_1, \epsilon_2)$ represent particular wave-functions within the quantum theory of $\mathcal{M}^0_{\rm flat}(C)$. The isomorphisms (5.2) and (5.3) established in [NW] can be taken as the basis for a characterization of the wave-functions $\Psi^{\rm top}_\tau(a)$ in terms of a Riemann-Hilbert type problem which will coincide with the one discussed in our previous Section 4. This leads to yet another way to find the relation $\Psi_\tau(a) = \Psi^{\rm top}_\tau(a)$ that we had pointed out above in (4.7). This relation can be understood in a more phyical way by combining the following two observations: On the one hand one may note that both in the case of $\mathcal{G}_C$ on $E^{4,-}_{\epsilon_1,\epsilon_2}$, and in the case of the Omega-deformed theory $\mathcal{G}_C^{\epsilon_1\epsilon_2}$ on $\mathbb{R}^4$ the instanton corrections get localized to the fixed points of the relevant $U(1) \times U(1)$ actions. The two cases are then linked by the key observation from [Pe] that the residual effect of the curvature of $E^4_{\epsilon_1,\epsilon_2}$ in the vicinity of the poles can be modeled by the Omega-deformation of [N].

Part II. Quantization of $\mathcal{M}^0_{\text{flat}}$

We are now going to describe the quantum theory of $\mathcal{M}^0_{\text{flat}}(C) \simeq \mathcal{T}(C)$ in a way that is suitable for the gauge theoretical applications. This will in particular lead to a precise description of the kernels $K_{\sigma_2\sigma_1}(a_2, a_1)$ that define the Riemann-Hilbert problem for the instanton partition functions.

In Section 6 we will explain how the use of pants decompositions reduces the task to the specification of a finite set of data. In order to characterize the relevant representations of the algebra $\text{Fun}_b(\mathcal{M}_{\text{flat}}(C))$ it suffices to define the counterparts fo the Wilson- and 't Hooft loop operators, and to describe the relations in $\text{Fun}_b(\mathcal{M}_{\text{flat}}(C))$. Transitions between pants decompositions (corresponding to the S-duality transformations) can be composed from elementary moves associated to surfaces of type $C_{0,3}$, $C_{0,4}$ and $C_{1,1}$. This section summarizes our main results by listing the explicit formulae for the defining data.

The rest of Part II of this paper (Sections 7 and 8) explains how the results summarized in Section 6 can be derived. Our starting point is the quantization of the Teichmüller spaces constructed in [F97, Ka1, CF1, CF2] which is briefly reviewed in the beginning of Section 7. The main technical problem is to diagonalize a maximal commuting set of geodesic length operators which in our context correspond to the set of Wilson loop operators [T05]. The relevant results from [T05] are summarized in Section 7.

Section 8 describes what remains to be done to complete the derivation of the results listed in Section 6. An important step, the explicit calculation of the generators associated to surfaces of genus 0, has recently been taken in [NT]. A new result of particular importance for us is the explicit calculation of the central extension of the representation of the Moore-Seiberg groupoid that is canonically associated to the quantum theory of $\mathcal{M}^0_{\text{flat}}(C) \simeq \mathcal{T}(C)$.

Another approach to the quantization of moduli spaces of flat connections for noncompact groups is described in particular in [Gu], and the case of one-holed tori was previously discussed in [DG].

6. Construction of the quantization of $\mathcal{M}^0_{\text{flat}}(C)$

An important feature of the description of $\mathcal{M}_{\text{flat}}$ summarised in Section 2.6 is the fact that it exhibits a form of locality in the sense that the description can be reconstructed from the local pieces isomorphic to $C_{0,4}$ or $C_{1,1}$ appearing in pants decompositions. In the relation with gauge theory one may view this locality as a consequence of the description of the A_1 theories in class $\mathcal{S}$ as quiver gauge theories [G09]. The Lagrangian includes only couplings between neighbouring parts of the MS graph. We are now going to describe in more detail how this locality is reflected in the quantum theory, and introduce the main data that characterize the quantum theory in such a description.

6.1. Algebra

Our aim is to construct a one-parameter family of non-commutative deformations $\mathcal{A}_b(C) \equiv \mathrm{Fun}_b^{\mathrm{alg}}(\mathcal{M}_{\mathrm{flat}}(C))$ of the Poisson-algebra of algebraic functions on $\mathcal{M}_{\mathrm{flat}}(C)$.

For a chosen pants decompostion defined by a cut system $\mathcal{C}$ we will choose as set of generators $\{(L_s^e, L_t^e, L_u^e); \gamma \in \mathcal{C}\} \cup \{L_r; r = 1, \dots, n\}$. The generators L_s^e, L_t^e, and L_u^e are associated to the simple closed curves γ_s^e, γ_t^e, and γ_u^e introduced in Subsection 2.5.4, respectively. The generators L_r $r = 1, \dots, n$ are associated to the n boundary components of $C \simeq C_{g,n}$. They will be central elements in $\mathcal{A}_b(C)$.

For each subsurface $C_e \subset C$ associated to a curve γ_e in the cut system $\mathcal{C}$ there will be two types of relations: Quadratic relations of the general form

$$\mathcal{Q}_e(L_s^e, L_t^e, L_u^e) = 0\,, \tag{6.1}$$

and cubic relations

$$\mathcal{P}_e(L_s^e, L_t^e, L_u^e) = 0\,. \tag{6.2}$$

We have not indicated in the notations that the polynomials $\mathcal{Q}_e$ and $\mathcal{P}_e$ may depend also on the loop variables associated to the boundary components of C_e in a way that is similar to the classical case described in Subsection 2.5.4. In order to describe the relations it therefore suffices to specify the polynomials $\mathcal{Q}_e$ and $\mathcal{P}_e$ for the two cases $C_e \simeq C_{0,4}$ and $C_e \simeq C_{1,1}$.

6.1.1. Case $C_e \simeq C_{0,4}$:.

Quadratic relation:

$$\begin{aligned}\mathcal{Q}_e(L_s, L_t, L_u) := {}& e^{\pi \mathrm{i} b^2} L_s L_t - e^{-\pi \mathrm{i} b^2} L_t L_s - (e^{2\pi \mathrm{i} b^2} - e^{-2\pi \mathrm{i} b^2}) L_u \\ & - (e^{\pi \mathrm{i} b^2} - e^{-\pi \mathrm{i} b^2})(L_1 L_3 + L_2 L_4)\,.\end{aligned} \tag{6.3}$$

Cubic relation:

$$
\begin{aligned}
\mathcal{P}_e(L_s,L_t,L_u) = & -e^{\pi i b^2}L_sL_tL_u + e^{2\pi i b^2}L_s^2 + e^{-2\pi i b^2}L_t^2 + e^{2\pi i b^2}L_u^2 \\
& + e^{\pi i b^2}L_s(L_3L_4+L_1L_2) + e^{-\pi i b^2}L_t(L_2L_3+L_1L_4) \\
& + e^{\pi i b^2}L_u(L_1L_3+L_2L_4) + L_1^2+L_2^2+L_3^2+L_4^2 \\
& + L_1L_2L_3L_4 - \big(2\cos\pi b^2\big)^2 .
\end{aligned} \tag{6.4}
$$

In the limit $b\to 0$ it matches (2.16).

6.1.2. Case $C_e \simeq C_{1,1}$:. Quadratic relation:

$$
\mathcal{Q}_e(L_s,L_t,L_u) := e^{\frac{\pi i}{2}b^2}L_sL_t - e^{-\frac{\pi i}{2}b^2}L_tL_s \; - \; (e^{\pi i b^2}-e^{-\pi i b^2})L_u . \tag{6.5}
$$

Cubic relation:

$$
\begin{aligned}
\mathcal{P}_e(L_s,L_t,L_u) = & \, e^{\pi i b^2}L_s^2 + e^{-\pi i b^2}L_t^2 + e^{\pi i b^2}L_u^2 - e^{\frac{\pi i}{2}b^2}L_sL_tL_u \\
& + L_0 - 2\cos\pi b^2 .
\end{aligned} \tag{6.6}
$$

The quadratic relations represent the deformation of the Poisson bracket (2.19), while the cubic relations will be deformations of the relations (2.15).

6.2. Quantization of the Darboux coordinates

Natural representations π_σ, of $\mathcal{A}_b(C)$ by operators on suitable spaces of functions can be constructed in terms of the quantum counterparts l_e, k_e of the Darboux variables l_e, k_e. The algebra $\mathcal{A}_b(C)$ will be represented on functions $\psi_\sigma(l)$ of the tuple l of $h=3g-3+n$ variables l_e associated to the edges of Γ_σ. The representations π_σ will be constructed from operators l_e, k_e which are defined as

$$
\mathsf{l}_e\,\psi_\sigma(l) := l_e\,\psi_\sigma(l)\,, \qquad \mathsf{k}_e\,\psi_\sigma(l) := 4\pi b^2\frac{1}{i}\frac{\partial}{\partial l_e}\psi_\sigma(l)\,. \tag{6.7}
$$

We are using the notation b^2 for the quantization parameter $\hbar$.

The construction of the representations will reflect the locality properties emphasized above. In order to make this visible in the notations let us introduce the one-dimensional Hilbert space $\mathcal{H}^{l_3}_{l_2l_1}$ associated to a hyperbolic three-holed sphere $C_{0,3}$ with boundary lengths l_i, $i=1,2,3$. We may then identify the Hilbert space $\mathcal{H}_\sigma$ of square-integrable functions $\psi_\sigma(l)$ on $\mathbb{R}^h_+$

with the direct integral of Hilbert spaces

$$\mathcal{H}_\sigma \simeq \int_{\mathbb{R}_+^h}^{\oplus} \prod_{e\in\sigma_1} dl_e \bigotimes_{v\in\sigma_0} \mathcal{H}^{l_3(v)}_{l_2(v),l_1(v)} . \tag{6.8}$$

We denoted the set of internal edges of the MS graph σ by σ_1, and the set of vertices by σ_0.

For $C \simeq C_{0,4}$ we may consider, in particular, that pants decomposition $\sigma = \sigma_s$ depicted on the left of Figure 4. We then have

$$\mathcal{H}^{0,4}_s := \mathcal{H}_{\sigma_s} \simeq \int^{\oplus} dl_e \, \mathcal{H}^{l_4}_{l_3 l_e} \otimes \mathcal{H}^{l_e}_{l_2 l_1} . \tag{6.9}$$

For $C = C_{1,1}$, using the pants decomposition on the left of Figure 5,

$$\mathcal{H}^{1,1}_s \simeq \int^{\oplus} dl_e \, \mathcal{H}^{l_e}_{l_0 l_e} . \tag{6.10}$$

For each edge e of the MS graph Γ_σ associated to a pants decomposition σ one has a corresponding subsurface C_e that can be embedded into C. For any given operator O on $\mathcal{H}^{0,4}_s$ and any edge e of Γ_σ such that $C_e \simeq C_{0,4}$ there is a natural way to define an operator O^e on $\mathcal{H}_\sigma$ acting "locally" only on the tensor factors in (6.8) associated to C_e.

More formally one may define O^e as follows. Let $\mathsf{O} \equiv \mathsf{O}_{l_4 l_3 l_2 l_1}$ be a family of operators on $\mathcal{H}^{0,4}_s$. It can be considered as a function $\mathsf{O}(\mathsf{l}, \mathsf{k}; l_1, l_2, l_3, l_4)$ of the operators l, k that depends parametrically on l_1, l_2, l_3, l_4. Let Γ_σ be an MS graph on C. To an edge e of Γ_σ such that $C_e \simeq C_{0,4}$ let us associate the neighboring edges $f_i(e)$, $i = 1, 2, 3, 4$ numbered according to the convention defined in Subsection 2.4. We may then use $\mathsf{O}_{l_4 l_3 l_2 l_1}$ to define an operator O^e on $\mathcal{H}_\sigma$ as

$$\mathsf{O}^e := \mathsf{O}(\mathsf{l}_e, \mathsf{k}_e; \mathsf{l}_{f_1(e)}, \mathsf{l}_{f_2(e)}, \mathsf{l}_{f_3(e)}, \mathsf{l}_{f_4(e)}) . \tag{6.11}$$

We are using the notation l_f for the operators defined above if f is an internal edge, and we identify $\mathsf{l}_f \equiv l_f$ if f is an edge that ends in a boundary component of C. If $C_e \simeq C_{1,1}$ one may associate in a similar fashion operators O^e to families $\mathsf{O} \equiv \mathsf{O}_{l_0}$ of operators on $\mathcal{H}^{1,1}_s$.

It will sometimes be useful to introduce "basis vectors" $\langle\, l \,|$ for $\mathcal{H}_\sigma$, more precisely distributions on dense subspaces of $\mathcal{H}_\sigma$ such that the wave-function $\psi(l)$ of a state $|\,\psi\,\rangle$ is represented as $\psi(l) = \langle\, l \,|\, \psi \,\rangle$. Representing $\mathcal{H}_\sigma$ as in (6.8) one may identify

$$\langle\, l \,| \simeq \bigotimes_{v\in\sigma_0} v^{l_3(v)}_{l_2(v),l_1(v)}\,, \tag{6.12}$$

where $v^{l_3}_{l_2,l_1}$ is understood as an element of the dual $\big(\mathcal{H}^{l_3}_{l_2,l_1}\big)^t$ of the one-dimensional Hilbert space $\mathcal{H}^{l_3}_{l_2,l_1}$.

6.3. Representations of the trace coordinates

It suffices to define the operators $\mathsf{L}_i \equiv \pi_{\sigma_s}(L_i)$, $i = s,t,u$, for the two cases $C \simeq C_{0,4}$ and $C \simeq C_{1,1}$. For these cases we don't need the labelling by edges e. In both cases we will have

$$\mathsf{L}_s := 2\cosh(\mathsf{l}/2)\,. \tag{6.13}$$

The operators L_i, $i = t,u$ will be represented as finite difference operators. Considering the operator L_t representing the 't Hooft loop operator, for example, we will find that it can be represented in the form

$$\mathsf{L}_t \,\equiv\, \pi_\sigma(L_t)\psi_\sigma(l) = \big[D_+(\mathsf{l})e^{+\mathsf{k}} + D_0(\mathsf{l}) + D_-(\mathsf{l})e^{-\mathsf{k}}\big]\psi_\sigma(l)\,, \tag{6.14}$$

with coefficients $D_\epsilon(l)$ that may depend on l_1, l_2, l_3, l_4 for $C \simeq C_{0,4}$, and on l_0 for $C \simeq C_{1,1}$.

6.3.1. Case $C_e \simeq C_{0,4}$:. The operators L_t and L_u are constructed out of the quantized Darboux coordinates k and l as follows

(6.15a)

$$\begin{aligned}\mathsf{L}_t = \;&\frac{1}{2(\cosh\mathsf{l} - \cos 2\pi b^2)}\Big(2\cos\pi b^2(L_2L_3 + L_1L_4) + \mathsf{L}_s(L_1L_3 + L_2L_4)\Big)\\ &+ \sum_{\epsilon=\pm1}\frac{1}{\sqrt{2\sinh(\mathsf{l}/2)}}e^{\epsilon\mathsf{k}/2}\,\frac{\sqrt{c_{12}(\mathsf{L}_s)c_{34}(\mathsf{L}_s)}}{2\sinh(\mathsf{l}/2)}e^{\epsilon\mathsf{k}/2}\frac{1}{\sqrt{2\sinh(\mathsf{l}/2)}}\end{aligned}$$

where the notation $c_{ij}(L_s)$ was introduced in (2.21). The operator L_u is then obtained from L_t my means of a simple unitary operator

$$\mathsf{L}_u = \big[\,\mathsf{B}^{-1}\cdot\mathsf{L}_t\cdot\mathsf{B}\,\big]_{L_1\leftrightarrow L_2}\,, \tag{6.15b}$$

where we are using the notations $L_i := 2\cosh(l_i/2)$, and

$$\mathsf{B} := e^{\pi\mathrm{i}(\Delta(\mathsf{l}) - \Delta(l_2) - \Delta(l_1))}\,, \qquad \Delta(l) := \frac{l^2}{(4\pi b)^2} + \frac{1+b^2}{4b}\,.$$

The operator B will later be recognized as representing the braiding of holes 1 and 2.

6.3.2. Case $C_e \simeq C_{1,1}$:. We now find the following expressions for the operators L_t and L_u:

$$\text{(6.16a)}\quad \mathsf{L}_t = \sum_{\epsilon=\pm 1} \frac{1}{\sqrt{\sinh(\mathsf{l}/2)}} e^{\epsilon \mathsf{k}/4} \sqrt{\cosh \tfrac{2\mathsf{l}+l_0}{4} \cosh \tfrac{2\mathsf{l}-l_0}{4}}\; e^{\epsilon \mathsf{k}/4} \frac{1}{\sqrt{\sinh(\mathsf{l}/2)}}$$

The operator L_u can be obtained from L_t by means of a unitary operator T,

$$\text{(6.16b)}\qquad \mathsf{L}_u = \mathsf{T}^{-1} \cdot \mathsf{L}_t \cdot \mathsf{T}\,,$$

which is explicitly constructed as

$$\text{(6.17)}\qquad \mathsf{T} := e^{-2\pi i \Delta(\mathsf{l})}\,.$$

This operator will later be found to represent the Dehn twist.

It is straightforward to check by explicit calculations that the relations of $\mathcal{A}_b(C)$ are satisfied. It can furthermore be shown that the representations above are unique, see Appendix A for some details.

We furthermore observe that the operators L_i, are positive self-adjoint, but unbounded. There is a maximal dense subset $\mathcal{S}_\sigma$ inside of $\mathcal{H}_\sigma$ on which the whole algebra $\mathcal{A}_b(C)$ of algebraic functions on $\mathcal{M}_{\text{flat}}$ is realized.

6.4. Transitions between representation

For each MS graph σ one will get a representation π_σ of the quantized algebra of $\mathcal{A}_b(C)$ of functions on $\mathcal{M}_{\text{flat}}(C)$. A natural requirement is that the resulting quantum theory does not depend on the choice of σ in an essential way. This can be ensured if there exist unitary operators $\mathsf{U}_{\sigma_2\sigma_1}$ intertwining between the different representations in the sense that

$$\text{(6.18)}\qquad \pi_{\sigma_2}(L_\gamma) \cdot \mathsf{U}_{\sigma_2\sigma_1} = \mathsf{U}_{\sigma_2\sigma_1} \cdot \pi_{\sigma_1}(L_\gamma)\,.$$

Having such intertwining operators allows one to identify the operators $\pi_\sigma(L_\gamma)$ as different representatives of one and the same abstract element L_γ of the quantized algebra of functions $\mathcal{A}_b(C)$. The intertwining property (6.18) turn out to determine the operators $\mathsf{U}_{\sigma_2\sigma_1}$ essentially uniquely.

It will be found that the operators $\mathsf{U}_{\sigma_2\sigma_1}$ can be represented as integral operators

$$(\mathsf{U}_{\sigma_2\sigma_1}\psi_{\sigma_1})(l_2) = \int dl_1\, A_{\sigma_2\sigma_1}(l_2,l_1)\,\psi_{\sigma_1}(l_1)\,. \tag{6.19}$$

This intertwining relation (6.18) is then equivalent to a system of difference equations for the kernels $A_{\sigma_2\sigma_1}(l_2,l_1)$,

$$\pi_{\sigma_2}(L_\gamma)\cdot A_{\sigma_2\sigma_1}(l_2,l_1), = A_{\sigma_2\sigma_1}(l_2,l_1)\cdot \overleftarrow{\pi}_{\sigma_1}(L_\gamma)^t\,. \tag{6.20}$$

The notation $\overleftarrow{\pi}_{\sigma_1}(L_\gamma)^t$ indicates that the transpose of the difference operator $\pi_{\sigma_1}(L_\gamma)$ acts on the variables l_1 from the right. $\pi_{\sigma_2}(L_\gamma)$ acts only on the variables l_2. The equations (6.20) represent a system of difference equations which constrain the kernels $A_{\sigma_2\sigma_1}(l_2,l_1)$ severely. They will determine the kernels $A_{\sigma_2\sigma_1}(l_2,l_1)$ essentially uniquely once the representations π_σ have been fixed.

6.5. Kernels of the unitary operators between different representations

We now want to list the explicit representations for the generators of the Moore-Seiberg groupoid in the quantization of $\mathcal{M}^0_{\rm flat}(C)$.

For many of the following considerations we will find it useful to replace the variables l_e by

$$\alpha_e := \frac{Q}{2} + \mathrm{i}\frac{l_e}{4\pi b}\,. \tag{6.21}$$

Using the variables α_e instead of l_e will in particular help to compare with Liouville theory.

6.5.1. B-move.

$$\mathsf{B}\cdot v^{\alpha_3}_{\alpha_2\alpha_1} = B^{\alpha_3}_{\alpha_2\alpha_1} v^{\alpha_3}_{\alpha_1\alpha_2}\,, \tag{6.22}$$

where

$$B^{\alpha_3}_{\alpha_2\alpha_1} = e^{\pi i(\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1})}\,. \tag{6.23}$$

6.5.2. Z-move.

$$\mathsf{Z}\cdot v^{\alpha_3}_{\alpha_2\alpha_1} = v^{\alpha_2}_{\alpha_1\alpha_3}\,. \tag{6.24}$$

6.5.3. F-move.

$$\mathsf{F}\cdot v^{\alpha_4}_{\alpha_3\alpha_s}\otimes v^{\alpha_s}_{\alpha_2\alpha_1} = \int_{\mathbb{S}}^{\oplus} d\beta_t\; F_{\beta_s\beta_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] v^{\alpha_4}_{\beta_t\alpha_1}\otimes v^{\beta_t}_{\alpha_3\alpha_2}\,, \tag{6.25}$$

where $\mathbb{S}=\frac{Q}{2}+i\mathbb{R}^+$, $Q:=b+b^{-1}$. The kernel describing the transition between representation π_s and π_t is given as

$$F_{\beta_s\beta_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = (M_{\beta_s}M_{\beta_t})^{\frac{1}{2}}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\right\}_b\,, \tag{6.26}$$

where

$$M_\beta := |S_b(2\beta)|^2 = -4\sin\pi(b(2\beta-Q))\sin\pi(b^{-1}(2\beta-Q))\,, \tag{6.27}$$

and the b-6j symbols $\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\right\}_b$ are defined as [PT1, PT2, TeVa]

$$\begin{aligned}&\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\right\}_b\\ &= \Delta(\alpha_s,\alpha_2,\alpha_1)\Delta(\alpha_4,\alpha_3,\alpha_s)\Delta(\alpha_t,\alpha_3,\alpha_2)\Delta(\alpha_4,\alpha_t,\alpha_1)\\ &\quad\times\int_{\mathcal{C}} du\; S_b(u-\alpha_{12s})S_b(u-\alpha_{s34})S_b(u-\alpha_{23t})S_b(u-\alpha_{1t4})\\ &\qquad\times S_b(\alpha_{1234}-u)S_b(\alpha_{st13}-u)S_b(\alpha_{st24}-u)S_b(2Q-u)\,.\end{aligned} \tag{6.28}$$

The expression involves the following ingredients:

- We have used the notations $\alpha_{ijk}=\alpha_i+\alpha_j+\alpha_k$, $\alpha_{ijkl}=\alpha_i+\alpha_j+\alpha_k+\alpha_l$.
- The special function $S_b(x)$ is a variant of the non-compact quantum dilogarithm, definition and properties being collected in Appendix B.
- $\Delta(\alpha_3,\alpha_2,\alpha_1)$ is defined as

$$\Delta(\alpha_3,\alpha_2,\alpha_1) = \left(\frac{S_b(\alpha_1+\alpha_2+\alpha_3-Q)}{S_b(\alpha_1+\alpha_2-\alpha_3)S_b(\alpha_1+\alpha_3-\alpha_2)S_b(\alpha_2+\alpha_3-\alpha_1)}\right)^{\frac{1}{2}}.$$

- The integral is defined in the cases that $\alpha_k\in Q/2+\mathrm{i}\mathbb{R}$ by a contour $\mathcal{C}$ which approaches $2Q+\mathrm{i}\mathbb{R}$ near infinity, and passes the real axis in the interval $(3Q/2,2Q)$.

6.5.4. S-move.

$$\mathsf{S}\cdot v^{\beta_1}_{\alpha,\beta_1} \;=\int_{\mathbb{S}}^{\oplus} d\beta_2\; S_{\beta_1\beta_2}(\alpha)\; v^{\beta_2}_{\alpha,\beta_2}\,, \tag{6.29}$$

where

(6.30)

$$\begin{aligned}S_{\beta_1\beta_2}(\alpha_0) = \sqrt{2}\,\frac{\Delta(\beta_1,\alpha_0,\beta_1)}{\Delta(\beta_2,\alpha_0,\beta_2)}\,(M_{\beta_1}M_{\beta_2})^{\frac{1}{2}}\,\frac{e^{\frac{\pi i}{2}\Delta_{\alpha_0}}}{S_b(\alpha_0)}\\ \times\int_{\mathbb{R}} dt\; e^{2\pi t(2\beta_1-Q)}\,\frac{S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)+it\big)S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)-it\big)}{S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)+it\big)S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)-it\big)}\,.\end{aligned}$$

This ends our list of operators representing the generators of the Moore-Seiberg groupoid.

6.6. Representation of the Moore-Seiberg groupoid

A projective unitary representation of the Moore-Seiberg groupoid is defined by the family of unitary operators $\mathsf{U}_{\sigma_2\sigma_1}:\mathcal{H}_{\sigma_1}\to\mathcal{H}_{\sigma_2}$, $\sigma_2,\sigma_1\in\mathcal{M}_0(\Sigma)$ which satisfy the composition law projectively

$$\mathsf{U}_{\sigma_3\sigma_2}\cdot\mathsf{U}_{\sigma_2\sigma_1} = \zeta_{\sigma_3,\sigma_2,\sigma_1}\,\mathsf{U}_{\sigma_3\sigma_2}\,, \tag{6.31}$$

where $\zeta_{\sigma_3,\sigma_2,\sigma_1}\in\mathbb{C}$, $|\zeta_{\sigma_3,\sigma_2,\sigma_1}|=1$. The operators $\mathsf{U}_{\sigma_2\sigma_1}$ which intertwine the representations π_σ according to (6.18) will generate a representation of the Moore-Seiberg groupoid.

6.6.1. Moore-Seiberg equations. Let us next list the explicit representations for the relations of the Moore-Seiberg groupoid in the quantization of $\mathcal{M}^0_{\rm flat}(C)$. In order to state some of them it will be convenient to introduce the operator T representing the Dehn twist such that

$$\mathsf{T}\cdot v^{\alpha_3}_{\alpha_2\alpha_1} = T_{\alpha_3}v^{\alpha_3}_{\alpha_2\alpha_1}\,, \tag{6.32}$$

where

$$T_{\alpha_2} := B^{\alpha_3}_{\alpha_2\alpha_1}\,B^{\alpha_1}_{\alpha_3\alpha_2} = e^{-2\pi i\Delta_{\alpha_2}} \tag{6.33}$$

We claim that the kernels of the operators B, F, S and Z defined above satisfy the Moore-Seiberg equations in the following form:

Genus zero, four punctures

(6.34a)
$$\int_{\mathbb{S}} d\beta_t \; F_{\beta_s\beta_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] \, B^{\alpha_4}_{\beta_t\alpha_1} \, F_{\beta_t\beta_u}\big[\begin{smallmatrix}\alpha_1 & \alpha_3\\ \alpha_4 & \alpha_2\end{smallmatrix}\big] = B^{\beta_s}_{\alpha_2\alpha_1} \, F_{\beta_s\beta_u}\big[\begin{smallmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{smallmatrix}\big] \, B^{\beta_u}_{\alpha_3\alpha_1} \,,$$

(6.34b)
$$\int_{\mathbb{S}} d\beta_2 \; F_{\beta_1\beta_2}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] F_{\beta_2\beta_3}\big[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_4 & \alpha_3\end{smallmatrix}\big] = \delta_{\mathbb{S}}(\beta_1-\beta_3)\,.$$

Genus zero, five punctures

(6.34c)
$$\int_{\mathbb{S}} d\beta_5 \; F_{\beta_1\beta_5}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \beta_2 & \alpha_1\end{smallmatrix}\big] F_{\beta_2\beta_4}\big[\begin{smallmatrix}\alpha_4 & \beta_5\\ \alpha_5 & \alpha_1\end{smallmatrix}\big] F_{\beta_5\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \beta_4 & \alpha_2\end{smallmatrix}\big] = F_{\beta_1\beta_4}\big[\begin{smallmatrix}\beta_3 & \alpha_2\\ \alpha_5 & \alpha_1\end{smallmatrix}\big] F_{\beta_2\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \alpha_5 & \beta_1\end{smallmatrix}\big]\,.$$

Genus one, one puncture

(6.34d)
$$\int_{\mathbb{S}} d\beta_2 \; S_{\beta_1\beta_2}(\alpha) S_{\beta_2\beta_3}(\alpha) = \delta_{\mathbb{S}}(\beta_1-\beta_3)\,(B^{\beta_1}_{\beta_1\alpha})^{-1}\,,$$

(6.34e)
$$\int_{\mathbb{S}} d\beta_2 \; S_{\beta_1\beta_2}(\alpha)\, T_{\beta_2}\, S_{\beta_2\beta_3}(\alpha) \;= e^{6\pi i\chi_b}\, T^{-1}_{\beta_1}\, S_{\beta_1\beta_3}(\alpha)\, T^{-1}_{\beta_3}\;.$$

Genus one, two punctures

(6.34f)
$$S_{\beta_1\beta_2}(\beta_3) \int_{\mathbb{S}} d\beta_4 \; F_{\beta_3\beta_4}\big[\begin{smallmatrix}\beta_2 & \alpha_1\\ \beta_2 & \alpha_2\end{smallmatrix}\big] \, T_{\beta_4}\, T^{-1}_{\beta_2} \, F_{\beta_4\beta_5}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \beta_2 & \beta_2\end{smallmatrix}\big]$$
$$= \int_{\mathbb{S}} d\beta_6 \; F_{\beta_3\beta_6}\big[\begin{smallmatrix}\beta_1 & \alpha_1\\ \beta_1 & \alpha_2\end{smallmatrix}\big] F_{\beta_1\beta_5}\big[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \beta_6 & \beta_6\end{smallmatrix}\big] S_{\beta_6\beta_2}(\beta_5)\, e^{\pi i(\Delta_{\alpha_1}+\Delta_{\alpha_2}-\Delta_{\beta_5})}\,.$$

The delta-distribution $\delta_{\mathbb{S}}(\beta_1-\beta_2)$ is defined by the ordinary delta-distribution on the real positive half-line $-i(\mathbb{S}-Q/2)$.

6.6.2. Mapping class group action. Having a representation of the Moore-Seiberg groupoid automatically produces a representation of the mapping class group. An element of the mapping class group μ represents a diffeomorphism of the surface C, and therefore maps any MS graph σ to another one denoted $\mu.\sigma$. Note that the Hilbert spaces $\mathcal{H}_\sigma$ and $\mathcal{H}_{\mu.\sigma}$ are canonically isomorphic. Indeed, the Hilbert spaces $\mathcal{H}_\sigma$, described more explicitly in (6.8), depend only on the combinatorics of the graphs σ, but not on their embedding into C. We may therefore define an operator $\mathsf{M}_\sigma(\mu)$:

$\mathcal{H}_\sigma \to \mathcal{H}_\sigma$ as

$$\mathsf{M}_\sigma(\mu) := \mathsf{U}_{\mu.\sigma,\sigma} \,. \tag{6.35}$$

It is automatic that the operators $\mathsf{M}(\mu)$ define a projective unitary representation of the mapping class group $\mathrm{MCG}(C)$ on $\mathcal{H}_\sigma$.

The operators $\mathsf{U}_{\sigma_2,\sigma_1}$ intertwine the actions defined thereby, as follows from (6.31), which implies

$$\mathsf{M}_{\sigma_2}(\mu)\cdot\mathsf{U}_{\sigma_2,\sigma_1} = \eta_{\sigma_2\sigma_1}\mathsf{U}_{\mu.\sigma_2,\,\mu.\sigma_1}\cdot\mathsf{M}_{\sigma_1}(\mu) \equiv \eta_{\sigma_2\sigma_1}\mathsf{U}_{\sigma_2,\,\sigma_1}\cdot\mathsf{M}_{\sigma_1}(\mu), \tag{6.36}$$

where $\eta_{\sigma_2\sigma_1} = \zeta_{\mu.\sigma_2,\sigma_2,\sigma_1}/\zeta_{\mu.\sigma_2,\mu.\sigma_1,\sigma_1}$. We may therefore naturally identify the mapping class group actions defined on the various $\mathcal{H}_\sigma$.

6.7. Self-duality

For the application to gauge theory we are looking for a representation of *two* copies of $\mathrm{Fun}_{\epsilon_i}(\mathcal{M}_{\text{flat}}(C))$, $i=1,2$, generated from the two sets of supersymmetric Wilson- and 't Hooft loop operators $\mathsf{T}_{e,i}$, $\mathsf{W}_{e,i}$ one can define of the four-ellipsoid. The eigenvalues of the Wilson loop operators $\mathsf{W}_{e,i}$ are $2\cosh(2\pi a_e/\epsilon_i)$, for $i=1,2$, respectively. This can be incorporated into the quantum theory of $\mathcal{M}^0_{\text{flat}}(C)$ as follows.

Let us identify the quantization parameter b^2 with the ratio of the parameters ϵ_1, ϵ_2,

$$b^2 = \epsilon_1\,/\,\epsilon_2\,. \tag{6.37}$$

Let us furthermore introduce the rescaled variables

$$a_e \,:= \epsilon_2\frac{l_e}{4\pi}\,. \tag{6.38}$$

The representations π_σ on functions $\psi_\sigma(l)$ are equivalent to representations on functions $\phi_\sigma(a)$, defined by

$$\mathsf{l}_e\,\phi_\sigma(a) := \frac{4\pi a_e}{\epsilon_2}\,\psi_\sigma(l)\,, \qquad \mathsf{k}_e\,\psi_\sigma(l) := \frac{2}{\mathrm{i}}\epsilon_1\frac{\partial}{\partial a_e}\phi_\sigma(a)\,. \tag{6.39}$$

Let us introduce a second pair of operators

$$\tilde{\mathsf{l}}_e := \frac{\epsilon_2}{\epsilon_1}\,\mathsf{l}_e\,, \qquad \tilde{\mathsf{k}}_e := \frac{\epsilon_2}{\epsilon_1}\,\mathsf{k}_e\,. \tag{6.40}$$

Replacing in the construction of the operators L_γ all operators l_e by $\tilde{\mathsf{l}}_e$, all k_e by $\tilde{\mathsf{k}}_e$, and all variables l_i by $\frac{\epsilon_2}{\epsilon_1} l_i$ defines operators $\tilde{\mathsf{L}}_\gamma$. The operators $\tilde{\mathsf{L}}_\gamma$ generate a representation of the algebra $\mathrm{Fun}_{b^{-1}}(\mathcal{M}_{\mathrm{flat}}(C))$. It can be checked that the operatos L_γ (anti-)commute with the operators $\tilde{\mathsf{L}}_\gamma$. Taken together we thereby get a representation of $\mathrm{Fun}_b(\mathcal{M}_{\mathrm{flat}}(C)) \times \mathrm{Fun}_{b^{-1}}(\mathcal{M}_{\mathrm{flat}}(C))$. The operators L_s^e and $\tilde{\mathsf{L}}_s^e$ correspond to the Wilson loop operators $\mathsf{W}_{e,1}$ and $\mathsf{W}_{e,2}$, respectively.

6.8. Gauge transformations

Note that the requirement that the $\pi_\sigma(L_s^e)$ act as multiplication operators leaves a large freedom. A gauge transformation

(6.41) $$\psi_\sigma(l) = e^{\mathrm{i}\chi(l)} \psi'_\sigma(l)\,,$$

would lead to a representation π'_σ of the form (6.14) with k_e replaced by

(6.42) $$\mathsf{k}'_e := \mathsf{k}_e + 4\pi b^2\, \partial_{l_e} \chi(l)\,.$$

This is nothing but the quantum version of a canonical transformation $(l,k) \to (l,k')$ with $k'_e = k_e + f_e(l)$. The representation $\pi'_{\sigma_s}(L_t)$ may then be obtained from (6.14) by replacing $D_\epsilon(l) \to D'_\epsilon(l)$ with

(6.43) $$D'_\epsilon(\mathsf{l}_s) = e^{-i\chi(\mathsf{l}_s)}\, e^{\epsilon \mathsf{k}_s} e^{i\chi(\mathsf{l}_s)} e^{-\epsilon \mathsf{k}_s}\, D_\epsilon(\mathsf{l}_s)\,, \quad \epsilon = -1, 0, 1\,.$$

Locality leads to an important restriction on the form of allowed gauge transformations $\chi(l)$. They should preserve the local nature of the representation π_σ^χ. This means that function $\nu \equiv e^{i\chi}$ must have the form of a product

(6.44) $$\nu(l) = \prod_{v\in\sigma_0} \nu(l_3(v), l_2(v), l_1(v))\,,$$

over functions ν which depend only on the variables associated to the vertices v of σ. This corresponds to replacing the basis vectors $v_{l_2,l_1}^{l_3}$ in (6.12) by $v_{l_2,l_1}^{l_3} = \nu(l_3,l_2,l_1) v'^{l_3}_{l_2,l_1}$. We then have, more explicitly,

(6.45) $$D'_\epsilon(l) = d_{43}^\epsilon(l) d_{21}^\epsilon(l)\, D_\epsilon(l)\,,$$

where

(6.46) $$d_{43}^\epsilon(l) = \frac{\nu(l_4, l_3, l - 4\epsilon\pi \mathrm{i} b^2)}{\nu(l_4,l_3,l)}\,, \qquad d_{21}^\epsilon(l) = \frac{\nu(l - 4\epsilon\pi\mathrm{i}b^2, l_2, l_1)}{\nu(l,l_2,l_1)}\,,$$

It is manifest that the property of the coefficients $D_\epsilon(l)$ to depend only on the variables l_f assigned to the nearest neighbours f of an edge e is preserved by the gauge transformations.

The freedom to change the representations of $\mathcal{A}_b(C)$ by gauge transformations reflects the perturbative scheme dependence mentioned in Subsection 3.3.3.

7. Relation to the quantum Teichmüller theory

The Teichmüller spaces had previously been quantized using other sets of coordinates associated to triangulations of C rather than pants decompositions [F97, CF1, Ka1]. This quantization yields geodesic length operators quantizing the geodesic length functions on $\mathcal{T}(C)$ [CF2, T05]. By diagonalizing the commutative subalgebra generated by the geodesic length operators associated to a cut system one may construct a representation of the Moore-Seiberg groupoid [T05]. We will show that this representation is equivalent to the one defined in Section 6.

This section starts by presenting the definitions and results from the quantum Teichmüller theory that will be needed in this paper. We will use the formulation introduced by R. Kashaev [Ka1], see also [T05] for a more detailed exposition and a discussion of its relation to the framework of Fock [F97] and Chekhov and Fock [CF1]. We then review the results from [T05] on the diagonalization of maximal sets of commuting length operators and the corresponding representation of the Moore-Seiberg groupoid.

7.1. Algebra of operators and its representations

The formulation from [Ka1] starts from the quantization of a somewhat enlarged space $\hat{\mathcal{T}}(C)$. The usual Teichmüller space $\mathcal{T}(C)$ can then be characterized as subspace of $\hat{\mathcal{T}}(C)$ using certain linear constraints. This is motivated by the observation that the spaces $\hat{\mathcal{T}}(C)$ have natural polarizations, which is not obvious in the formulation of [F97, CF1].

For a given surface C with constant negative curvature metric and at least one puncture one considers ideal triangulations τ. Such ideal triangulations are defined by maximal collection of non-intersecting open geodesics which start and end at the punctures of C. We will assume that the triangulations are decorated, which means that a distinguished corner is chosen in each triangle.

We will find it convenient to parameterise triangulations τ by their dual graphs which are called fat graphs φ_τ. The vertices of φ_τ are in one-to-one correspondence with the triangles of τ, and the edges of φ_τ are in one-to-one correspondence with the edges of τ. The relation between a triangle t in τ and the fat graph φ_τ is depicted in Figure 9. φ_τ inherits a natural decoration of its vertices from τ, as is also indicated in Figure 9.

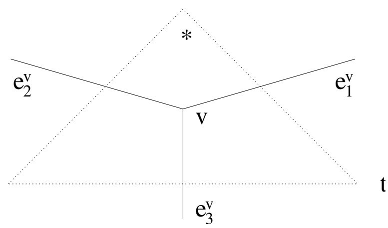


Figure 9: Graphical representation of the vertex v dual to a triangle t. The marked corner defines a corresponding numbering of the edges that emanate at v.

The quantum theory associated to the Teichmüller space $\mathcal{T}(C)$ is defined on the kinematical level by associating to each vertex $v \in \varphi_0$, $\varphi_0 = \{\text{vertices of } \varphi\}$, of φ a pair of generators p_v, q_v which are supposed to satisfy the relations

$$[\, p_v \,,\, q_{v'} \,] = \frac{\delta_{vv'}}{2\pi i}\,. \tag{7.1}$$

There is a natural representation of this algebra on the Schwarz space $\hat{\mathcal{S}}_\varphi(C)$ of rapidly decaying smooth functions $\psi(q)$, $q : \varphi_0 \ni v \to q_v$, generated from $\pi_\varphi(q_v) := \mathsf{q}_v$, $\pi_\varphi(p_v) := \mathsf{p}_v$, where

$$\mathsf{q}_v\, \psi(q) := q_v \psi(q)\,, \qquad \mathsf{p}_v\, \psi(q) := \frac{1}{2\pi i}\frac{\partial}{\partial q_v}\psi(q)\,. \tag{7.2}$$

For each surface C we have thereby defined an algebra $\hat{\mathcal{A}}(C)$ together with a family of representations π_φ of $\hat{\mathcal{A}}(C)$ on the Schwarz spaces $\hat{\mathcal{S}}_\varphi(C)$ which are dense subspaces of the Hilbert space $\mathcal{K}(\varphi) \simeq L^2(\mathbb{R}^{4g-4+2n})$.

The quantized algebra of functions $\mathcal{A}_{\mathcal{T}(C)}$ on the Teichmüller spaces is then defined by the quantum version of the Hamiltonian reduction with respect to a certain set of constraints. To each element $[\gamma]$ of the first homology $H_1(C, \mathbb{R})$ of C one may associate an operator $\mathsf{z}_{\varphi,\gamma}$ that is constructed as a linear combination of the operators p_v and q_v, $v \in \varphi_0$, see [Ka1, T05] for details. The operators $\mathsf{z}_{\varphi,\gamma}$ represent the constraints which can be used to characterize the subspace associated to the quantum Teichmüller theory within $\mathcal{K}(\varphi)$.

7.2. The projective representation of the Ptolemy groupoid on $\mathcal{K}(\varphi)$

The next step is to show that the choice of fat graph φ is inessential by constructing unitary operators $\mathsf{W}_{\varphi_2\varphi_1} : \mathcal{K}(\varphi_1) \to \mathcal{K}(\varphi_2)$ intertwining the representations π_{φ_1} and π_{φ_2}.

The groupoid generated by the transitions $[\varphi', \varphi]$ from a fat graph φ to φ' is called the Ptolemy groupoid. It can be described in terms of generators ω_{uv}, ρ_u, (uv) and certain relations. The generator ω_{uv} is the elementary change of diagonal in a quadrangle, ρ_u is the clockwise rotation of the decoration, and (uv) is the exchange of the numbers associated to the vertices u and v. More details and further references can be found in [T05, Section 3].

Following [Ka3] closely we shall define a projective unitary representation of the Ptolemy groupoid in terms of the following set of unitary operators

$$(7.3)\qquad \begin{aligned} \mathsf{A}_v &\equiv e^{\frac{\pi i}{3}} e^{-\pi i(\mathsf{p}_v+\mathsf{q}_v)^2} e^{-3\pi i \mathsf{q}_v^2} \\ \mathsf{T}_{vw} &\equiv e_b(\mathsf{q}_v + \mathsf{p}_w - \mathsf{q}_w) e^{-2\pi i \mathsf{p}_v \mathsf{q}_w}, \end{aligned} \qquad \text{where } \; v, w \in \varphi_{\mathsf{o}}\,.$$

The special function $e_b(z)$ can be defined in the strip $|\Im z| < |\Im c_b|$, $c_b \equiv i(b + b^{-1})/2$ by means of the integral representation

$$(7.4)\qquad \log e_b(z) \;\equiv\; \frac{1}{4} \int\limits_{i0-\infty}^{i0+\infty} \frac{dw}{w} \frac{e^{-2\mathrm{i}zw}}{\sinh(bw)\sinh(b^{-1}w)}.$$

These operators are unitary for $(1 - |b|)\Im b = 0$. They satisfy the following relations [Ka3]

$$(7.5\text{a})\qquad \text{(i)} \quad \mathsf{T}_{vw}\mathsf{T}_{uw}\mathsf{T}_{uv} = \mathsf{T}_{uv}\mathsf{T}_{vw},$$

$$(7.5\text{b})\qquad \text{(ii)} \quad \mathsf{A}_v\mathsf{T}_{uv}\mathsf{A}_u = \mathsf{A}_u\mathsf{T}_{vu}\mathsf{A}_v,$$

$$(7.5\text{c})\qquad \text{(iii)} \quad \mathsf{T}_{vu}\mathsf{A}_u\mathsf{T}_{uv} = \zeta\mathsf{A}_u\mathsf{A}_v\mathsf{P}_{uv},$$

$$(7.5\text{d})\qquad \text{(iv)} \quad \mathsf{A}_u^3 = \mathrm{id},$$

where $\zeta = e^{-\pi i c_b^2/3}$, $c_b \equiv \frac{i}{2}(b + b^{-1})$. The relations (7.5a) to (7.5d) allow us to define a projective representation of the Ptolemy groupoid as follows.

- Assume that $\omega_{uv} \in [\varphi', \varphi]$. To ω_{uv} let us associate the operator

$$\mathsf{u}(\omega_{uv}) \;\equiv\; \mathsf{T}_{uv} \;:\; \mathcal{K}(\varphi) \ni \mathfrak{v} \;\to\; \mathsf{T}_{uv}\mathfrak{v} \in \mathcal{K}(\varphi').$$

- For each fat graph φ and vertices $u, v \in \varphi_\circ$ let us define the following operators

$$\begin{aligned} \mathsf{A}_u^{\varphi} &: \mathcal{K}(\varphi) \ni \mathfrak{v} \;\rightarrow\; \mathsf{A}_u \mathfrak{v} \in \mathcal{K}(\rho_u \circ \varphi). \\ \mathsf{P}_{uv}^{\varphi} &: \mathcal{K}(\varphi) \ni \mathfrak{v} \;\rightarrow\; \mathsf{P}_{uv} \mathfrak{v} \in \mathcal{K}((uv) \circ \varphi). \end{aligned}$$

It follows from (7.5a)-(7.5d) that the operators T_{uv}, A_u and P_{uv} can be used to generate a unitary projective representation of the Ptolemy groupoid.

The corresponding automorphisms of the algebra $\mathcal{A}(C)$ are

$$\mathsf{a}_{\varphi_2\varphi_1}(\mathsf{O}) := \mathsf{ad}[\mathsf{W}_{\varphi_2\varphi_1}](\mathsf{O}) := \mathsf{W}_{\varphi_2\varphi_1} \cdot \mathsf{O} \cdot \mathsf{W}_{\varphi_2\varphi_1} \,. \tag{7.6}$$

The automorphism $\mathsf{a}_{\varphi_2\varphi_1}$ generate the canonical quantization of the changes of coordinates for $\hat{\mathcal{T}}(C)$ from one fat graph to another [Ka1]. Let us note that the constraints transform under a change of fat graph as $\mathsf{a}_{\varphi_2\varphi_1}(\mathsf{z}_{\varphi_1,\gamma}) = \mathsf{z}_{\varphi_2,\gamma}$.

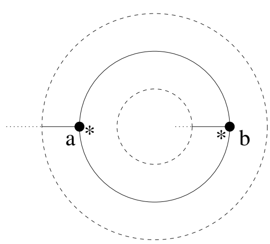
7.3. Length operators

A particularly important class of coordinate functions on the Teichmüller spaces are the geodesic length functions. The quantization of these observables was studied in [CF1, CF2, T05].

Such length operators can be constructed in general as follows [T05]. We will first define the length operators for two special cases in which the choice of fat graph φ simplifies the representation of the curve γ. We then explain how to generalize the definition to all other cases.

(i) Let A_γ be an annulus embedded in the surface C containing the curve γ, and let φ be a fat graph which looks inside of A_γ as depicted in Figure 7.7.

(7.7)



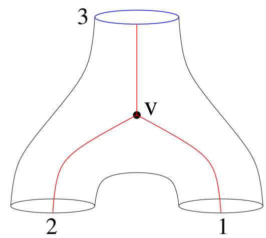
Annulus A_γ: Region bounded by the two dashed circles, and part of φ contained in A_γ.

Let us define the length operators

$$\begin{aligned} &\mathsf{L}_{\varphi,\gamma} := 2\cosh 2\pi b \mathsf{p}_\gamma + e^{-2\pi b \mathsf{q}_\gamma}\,, \quad \text{where} \\ &\mathsf{p}_\gamma := \frac{1}{2}(\mathsf{p}_a - \mathsf{q}_a - \mathsf{p}_b)\,, \quad \mathsf{q}_\gamma := \frac{1}{2}(\mathsf{q}_a + \mathsf{p}_a + \mathsf{p}_b - 2\mathsf{q}_b)\,. \end{aligned} \tag{7.8}$$

(ii) Assume that the curve $\gamma \equiv \gamma_3$ is the boundary component labelled by number 3 of a trinion P_γ embedded in C within which the fat graph φ looks as follows:

(7.9)



.

Let γ_ϵ, $\epsilon = 1, 2$ be the curves which represent the other boundary components of P_γ as indicated in Figure 7.9. Assume that L_{γ_1} and L_{γ_2} are already defined and define $\mathsf{L}_{\varphi,\gamma} \equiv \mathsf{L}_{\gamma_3}$ by

$$\mathsf{L}_{\varphi,\gamma} = 2\cosh(\mathsf{y}_v^2 + \mathsf{y}_v^1) + e^{-\mathsf{y}_v^2}\mathsf{L}_{\gamma_1} + e^{\mathsf{y}_v^1}\mathsf{L}_{\gamma_2} + e^{\mathsf{y}_v^1 - \mathsf{y}_v^2}\,, \tag{7.10}$$

where y_v^ϵ, $\epsilon = 1, 2$ are defined as $\mathsf{y}_v^2 = 2\pi b(\mathsf{q}_v + \mathsf{z}_{\gamma_2})$, $\mathsf{y}_v^1 = -2\pi b(\mathsf{p}_v - \mathsf{z}_{\gamma_1})$.

In practise it may be necessary to use part (ii) of the definition recursively. In all remaining cases we will define the length operator $\mathsf{L}_{\varphi,\gamma}$ as follows: There always exists a fat graph φ_0 for which one of the two definitions above can be used to define $\mathsf{L}_{\varphi_0,\gamma}$. Let then

$$\mathsf{L}_{\varphi,\gamma} := \mathsf{a}_{\varphi,\varphi_0}(\mathsf{L}_{\varphi_0,\gamma})\,. \tag{7.11}$$

It was explicitly verified in [NT] that the definition given above is consistent. The length operators $\mathsf{L}_{\varphi,\gamma}$ are unambigously defined by (i), (ii) and (7.11) above, and we have $\mathsf{L}_{\varphi',\gamma} = \mathsf{a}_{\varphi',\varphi}(\mathsf{L}_{\varphi,\gamma})$ if $[\varphi', \varphi]$ represents an element of the Ptolemy groupoid.

The length operators satisfy the following properties:

(a) **Spectrum:** $\mathsf{L}_{\varphi,\gamma}$ is self-adjoint. The spectrum of $\mathsf{L}_{\varphi,\gamma}$ is simple and equal to $[2, \infty)$ [Ka4]. This ensures that there exists an operator $\mathsf{l}_{\varphi,\gamma}$ — the *geodesic length operator* — such that $\mathsf{L}_{\varphi,\gamma} = 2\cosh \frac{1}{2}\mathsf{l}_{\varphi,\gamma}$.

(b) **Commutativity:**

$$\big[\,\mathsf{L}_{\varphi,\gamma}\,,\,\mathsf{L}_{\varphi,\gamma'}\,\big] = 0 \quad \text{if} \;\; \gamma\cap\gamma' = \emptyset.$$

(c) **Mapping class group invariance:**

$$\mathsf{a}_\mu(\mathsf{L}_{\varphi,\gamma}) = \mathsf{L}_{\mu.\varphi,\gamma}, \quad \mathsf{a}_\mu \equiv \mathsf{a}_{[\mu.\varphi,\varphi]}, \quad \text{for all} \;\; \mu\in \mathrm{MC}(\Sigma).$$

It can furthermore be shown that this definition reproduces the classical geodesic length functions on $\mathcal{T}(C)$ in the classical limit.

7.4. The Teichmüller theory of the annulus

As a basic building block let us develop the quantum Teichmüller theory of an annulus in some detail. To the simple closed curve γ that can be embedded into A we associate

- the constraint

$$\mathsf{z}\equiv \mathsf{z}_{\varphi,\gamma} := \frac{1}{2}(\mathsf{p}_a - \mathsf{q}_a + \mathsf{p}_b)\,, \tag{7.12}$$

- the length operator $\mathsf{L}\equiv\mathsf{L}_{\varphi,\gamma}$, defined as in (7.8).

The operator L is positive-self-adjoint, The functions

$$\phi_s(p) := \langle\, p\,|\, s\,\rangle = \frac{s_b(s+p+c_b-i0)}{s_b(s-p-c_b+i0)}\,. \tag{7.13}$$

represent the eigenfunctions of the operator L with eigenvalue $2\cosh 2\pi bs$ in the representation where $\mathsf{p}\equiv\mathsf{p}_\gamma$ is diagonal with eigenvalue p. It was shown in [Ka4] that the family of eigenfunctions $\phi_s(p)$, $s\in\mathbb{R}^+$, is delta-function orthonormalized and complete in $L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} dp\;\langle\, s\,|\,p\,\rangle\langle\, p\,|\,s'\,\rangle = \delta(s-s')\,. \tag{7.14a}$$

$$\int_{\mathbb{R}_+} d\mu(s)\;\langle\, p\,|\,s\,\rangle\langle\, s\,|\,p'\,\rangle = \delta(p-p')\,, \tag{7.14b}$$

where the Plancherel measure $\mu(s)$ is defined as $d\mu(s) = 2\sinh(2\pi bs)\cdot 2\sinh(2\pi b^{-1}s)ds$.

For later use let us construct the change of representation from the representation in which p_a and p_b are diagonal to a representation where z

and L are diagonal. To this aim let us introduce $\mathsf{d} := \frac{1}{2}(\mathsf{q}_a + \mathsf{p}_a - \mathsf{p}_b + 2\mathsf{q}_b)$. We have

$$
\begin{aligned}
&[\mathsf{z}, \mathsf{d}] = (2\pi i)^{-1}\,, \qquad && [\mathsf{z}, \mathsf{p}] = 0\,, \quad && [\mathsf{z}, \mathsf{q}] = 0\,,\\
&[\mathsf{p}, \mathsf{q}] = (2\pi i)^{-1}\,, \qquad && [\mathsf{d}, \mathsf{p}] = 0\,, \quad && [\mathsf{d}, \mathsf{q}] = 0\,.
\end{aligned}
$$

Let $\langle\, p, z\,|$ be an eigenvector of p and z with eigenvalues p and z, and $|\, p_a, p_b\,\rangle$ an eigenvector of p_a and p_b with eigenvalues p_a and p_b, respectively. It follows easily that

$$
\langle\, p, z\,|\, p_a, p_b\,\rangle = \delta(p_b - z + p)e^{\pi i(p+z-p_a)^2}\,. \tag{7.15}
$$

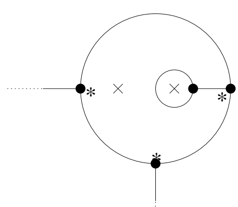
The transformation

$$
\psi(s, z) = \int_{\mathbb{R}^2} dp dp_a\, \frac{s_b(s - p + c_b - i0)}{s_b(s + p - c_b + i0)} e^{\pi i(p+z-p_a)^2}\Psi(p_a, z - p)\,, \tag{7.16}
$$

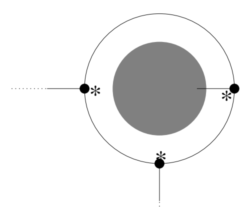
will then map a wave function $\Psi(p_a, p_b)$ in the representation which diagonalizes p_a, p_b to the corresponding wave function $\psi(s, z)$ in the representation which diagonalizes L and z.

7.5. Teichmüller theory for surfaces with holes

The formulation of quantum Teichmüller theory introduced above has only punctures (holes with vanishing geodesic circumference) as boundary components. In order to generalize to holes of non-vanishing geodesic circumference one may represent each hole as the result of cutting along a geodesic surrounding a pair of punctures.



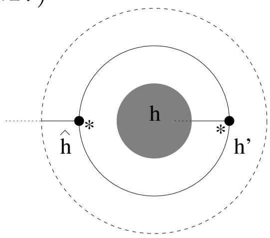
Example for a fat graph in the vicinity of two punctures (crosses)



The same fat graph after cutting out the hole

On a surface C with n holes one may choose φ to have the following simple standard form near at most $n - 1$ of the holes, which will be called “incoming” in the following:

(7.17)



Incoming boundary component:
Hole h (shaded), together with an annular neighborhood A_h of h inside C, and the part of φ contained in A_h.

The price to pay is a more complicated representation of the closed curves which surround the remaining holes.

The simple form of the fat graph near the incoming boundary components allows us to use the transformation (7.16) to pass to a representation where the length operators and constraints associated to these holes are diagonal. In order to describe the resulting hybrid representation let us denote by s_b and z_b the assignments of values s_h and z_h to each incoming hole h, while p assigns real numbers p_v to all vertices v of φ which do not coincide with any vertex $\hat{h}$ or h' associated to an incoming hole h. The states will then be described by wave-functions $\psi(p; s_b, z_b)$ on which the operators L_h and z_h act as operators of multiplication by $2\cosh 2\pi b s_h$ and z_h, respectively.

For a given hole h one may define a projection $\hat{\mathcal{H}}(C_{h(s,z)})$ of $\hat{\mathcal{H}}(C)$ to the eigenspace with fixed eigenvalues $2\cosh 2\pi bs$ and z of L_h and z_h. States in $\hat{\mathcal{H}}(C_{h(s,z)})$ can be represented by wave-functions $\psi_h(p_h)$, where p_h assigns real values to all vertices in $\varphi_0 \setminus \{\hat{h}, h'\}$. The mapping class action on $\hat{\mathcal{H}}(C)$ commutes with L_h and z_h. It follows that the operators $\mathsf{M}_\mu \equiv \mathsf{M}_{\mu.\varphi,\varphi}$ representing the mapping class group action on $\hat{\mathcal{H}}(C)$ project to operators $\mathsf{M}_{s,z}(\mu)$ generating an action of $\mathrm{MCG}(C)$ on $\hat{\mathcal{H}}(C_{h(s,z)})$.

7.6. Passage to the length representation

Following [T05], we will now describe how to map a maximal commuting family of length operators to diagonal form. We will start from the hybrid representation described above in which the length operators and constraints associated to the incoming holes are diagonal. Recall that states are represented by wave-functions $\psi(p; s_b, z_b)$ in such a representation, where $p : \tilde{\varphi}_0 \mapsto \mathbb{R}$, and $\tilde{\varphi}_0$ is the subset of φ_0 that does not contain $\hat{h}$ nor h' for any incoming hole h. A maximal commuting family of length operators is associated to a family of simple closed curves which define a pants decomposition.

7.6.1. Adapted fat graphs. Let us consider a decorated Moore-Seiberg graph σ on C, the decoration being the choice of an distinguished boundary component in each trinion of the pants decomposition defined by σ. The distinguished boundary component will be called outgoing, all others incoming.

Such a graph σ allows us to define a cutting of C into annuli and trinions. If cutting along a curve γ in the cut system $\mathcal{C}_\sigma$ produces two incoming boundary components, let $\gamma^\pm$ be two curves bounding a sufficiently small annular neighborhood A_γ of γ in C. Replacing γ by $\{\gamma^+, \gamma^-\}$ for all such curves γ produces an extended cut system $\hat{\mathcal{C}}_\sigma$ which decomposes C into trinions and annuli.

Let us call a pants decomposition σ admissible if no curve $\gamma_e \in \mathcal{C}_\sigma$ is an outgoing boundary component for both of the two trinions it may separate. To admissible pants decompostions σ we may associate a natural fat graph φ_σ defined by gluing the following pieces:

- Annuli: See Figure (7.7).
- Trinions: See Figure (7.9).
- Holes: See Figure (7.17).

Gluing these pieces in the obvious way will produce the connected graph φ_σ adapted to the Moore-Seiberg graph σ we started from. The restriction to admissible fat graphs turns out to be inessential [NT].

7.6.2. The unitary map to the length representation. To each vertex $v \in \varphi_{\sigma,0}$ assign the length operator L_v^{2} and L_v^{1} to the incoming and L_v to the outgoing boundary components of the pair of pants P_v containing v. The main ingredient will be an operator C_v which maps L_v to a simple standard form,

$$\mathsf{C}_v \cdot \mathsf{L}_v \cdot (\mathsf{C}_v)^{-1} = 2\cosh 2\pi b \mathsf{p}_v + e^{-2\pi b \mathsf{q}_v} \,. \tag{7.18}$$

Such an operator can be constructed explicitly as [T05]

$$\mathsf{C}_v := e^{-2\pi i s_{2} \mathsf{q}_v} \frac{e_b(\mathsf{s}_v^{1} + \mathsf{p}_v)}{e_b(\mathsf{s}_v^{1} - \mathsf{p}_v)} \, e^{-2\pi i \mathsf{s}_v^{1} \mathsf{p}_v} \, (e_b(\mathsf{q}_v - \mathsf{s}_v^{2}))^{-1} \, e^{-2\pi i (\mathsf{z}_v^{2} \mathsf{p}_v + \mathsf{z}_v^{1} \mathsf{q}_v)}, \tag{7.19}$$

where $\mathsf{s}_v^{\imath}$, $\imath = 1, 2$ are the positive self-adjoint operators defined by $\mathsf{L}_v^{\imath} = 2\cosh 2\pi b \mathsf{s}_v^{\imath}$, and z_v^{2}, z_v^{1} are the constraints associated to the incoming boundary components of P_v.

The map to the length representation is then constructed as follows. Let us first apply the product of the transformations (7.16) that diagonalizes the length operators associated to all incoming holes and embedded annuli. The resulting hybrid representation has states represented by wave-functions $\psi(p; s_{\rm A}, z_{\rm A})$, where p assigns a real number p_v to each vertex v of Γ_σ, whereas $s_{\rm A}$ (resp. $z_{\rm A}$) assigns real positive numbers (resp. real numbers) to all[7] annuli A, respectively.

In order to diagonalize all length operators associated to all edges of the MS graph Γ_σ it remains to apply an ordered product the operators C_v. The resulting operator may be represented as the following explicit integral transformation: Let s be the assignment of real positive numbers s_e to all edges e of Γ_σ. Define

$$\Phi(s, z_{\rm A}) = \int\limits_{\mathbb{R}^h} \Big(\prod\nolimits_{v\in\tilde{\varphi}_0} dp_v \; K_{s_v^2 s_v^1}^{z_v^2 z_v^1}(s_v, p_v)\Big)\, \psi(p; s_{\rm A}, z_{\rm A})\,. \tag{7.20}$$

The kernel $K_{s_2 s_1}^{z_2 z_1}(s,p)$ has the following explicit form [NT]

$$K_{s_2 s_1}^{z_2 z_1}(s,p) = \zeta_0 \int_{\mathbb{R}} dp' \; e^{-2\pi i (s_2 - c_b)(s_2 + p' - p + z_1)} e_b(p - z_1 - s_2 - p' + c_b) \\ \times \frac{s_b(s_1 - p' - s_2)}{s_b(s_1 + p' + s_2)} \frac{s_b(s + p' - c_b)}{s_b(s - p' + c_b)} e^{-2\pi i z_2 (2p - z_1)}\,.$$

The explicit integral transformation (7.20) defines an operator $\hat{\mathsf{C}}_\sigma$. In order to get an operator C_σ which maps the representation $\pi_{\varphi_\sigma}^{\mathcal{T}}$ for the quantum Teichmüller theory based on the Penner-Fock coordinates to the representation π_σ defined in this paper it suffices to compose $\hat{\mathsf{V}}_\sigma$ with the projection Π defined as $\phi(s) \equiv (\Pi\Phi)(s) := \Phi(s, 0)$. This corresponds to imposing the constraints $\mathsf{z}_{\varphi,\gamma} \simeq 0$.

7.6.3. Changes of MS-graph. The construction above canonically defines operators $\mathsf{U}_{\sigma_2\sigma_1}$ intertwining between the representations π_{σ_1} and π_{σ_2} as

$$\mathsf{U}_{\sigma_2\sigma_1} := \hat{\mathsf{C}}_{\sigma_2} \cdot \mathsf{W}_{\varphi_{\sigma_2}\varphi_{\sigma_1}} \cdot \hat{\mathsf{C}}_{\sigma_1}^{-1}\,, \tag{7.21}$$

where $\mathsf{W}_{\varphi_{\sigma_2}\varphi_{\sigma_1}}$ is any operator representing the move $[\varphi_{\sigma_2}, \varphi_{\sigma_1}]$ between the fat graph associated to σ_1 and σ_2, respectively. In this way one defines operators B, F, Z and S associated to the elementary moves between different MS-graphs. These operators satisfy operatorial versions of the Moore-Seiberg consistency conditions [T05, NT], which follow from the relations of the Ptolemy groupoid (7.5) using (7.21).

[7] both embedded annuli and annuli representing incoming boundary components

8. Completing the proofs

In order to prove the consistency of the quantization of $\mathcal{M}^0_{\text{flat}}(C)$ defined in Section 6 we will take the results of [T05] reviewed in the previous Section 7 as a starting point. It remains to

(i) calculate the kernels of the operators F, B, Z and S,

(ii) calculate the explicit form of the difference operators L^e_t in this representation, and

(iii) calculate the central extension of the Moore-Seiberg groupoid.

The solution of these tasks will be described in this section.

8.1. The Moore-Seiberg groupoid for surfaces of genus 0

To begin, let us note that the kernels of the operators F, B and Z have been calculated in [NT], giving the results stated in Section 6.

The key observation [NT] leading to the explicit calculation of the kernels of F, Z and B is the fact that the operators C_v defined in (7.19) are closely related to the Clebsch-Gordan maps of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ [PT2]. This observation implies directly that the matrix elements of the operator F must coincide with the b-$6j$ symbols of [PT2]. Fixing a suitable normalization and using the alternative integral representation found in [TeVa] one gets precisely formula (6.26).

One may furthermore use the results of [BT1] to prove that the operator B acts diagonally with eigenvalue given in (6.23). For more details we may refer to [NT].

8.2. Preparation I – Alternative normalizations

The representation for $\mathcal{A}_b(C)$ constructed in Section 6 has a severe drawback: The appearance of square-roots in the expressions for the loop operators and for the kernels of $\mathsf{U}_{\sigma_2\sigma_1}$ obscures some beautiful and profound analytic properties that will later be found to have important consequences. We shall therefore now introduce useful alternative normalizations obtained by writing

$$v^{\alpha_3}_{\alpha_2\alpha_1} = \varrho(\alpha_3,\alpha_2,\alpha_1)\tilde{v}^{\alpha_3}_{\alpha_2\alpha_1}\,, \tag{8.1}$$

and taking $\tilde{v}^{\alpha_3}_{\alpha_2\alpha_1}$ as the new basis vector for $\mathcal{H}^{\alpha_3}_{\alpha_2\alpha_1}$. It will be useful to consider vectors $\tilde{v}^{\alpha_3}_{\alpha_2\alpha_1}$ that may have a norm different from unity. It will be useful to consider, in particular,

$$\varrho(\alpha_3,\alpha_2,\alpha_1) = \sqrt{C(\bar{\alpha}_3,\alpha_2,\alpha_1)}\,, \tag{8.2}$$

where $\bar{\alpha}_3 = Q-\alpha_3$, and $C(\alpha_3,\alpha_2,\alpha_1)$ is the function defined as

(8.3)

$$C(\alpha_1,\alpha_2,\alpha_3) = \Big[\pi\mu\gamma(b^2)b^{2-2b^2}\Big]^{(Q-\sum_{i=1}^3\alpha_i)/b}$$
$$\times\,\frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)}.$$

The expression on the right hand side of (8.3) is constructed out of the special function $\Upsilon(x)$ which is related to the Barnes double Gamma function $\Gamma_b(x)$ as $\Upsilon(x) = (\Gamma_b(x)\Gamma_b(Q-x))^{-1}$. The function $C(\alpha_1,\alpha_2,\alpha_3)$ is known to be the expression for the three-point function in Liouville theory, as was conjectured in [DO, ZZ95], and derived in [T01].

Note that the gauge transformation defined by (8.1) will modify the kernels representing the elementary moves of the MS groupoid. In the representation defined via (8.1) one may represent the F-move, for example, by the kernel

$$F^{\rm L}_{\beta_1\beta_2}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] = \frac{\varrho(\alpha_4,\alpha_t,\alpha_1)\varrho(\alpha_t,\alpha_3,\alpha_2)}{\varrho(\alpha_4,\alpha_3,\alpha_s)\varrho(\alpha_s,\alpha_2,\alpha_1)}F_{\beta_1\beta_2}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]\,. \tag{8.4}$$

We'd like to stress that the appearance of the function $C(\alpha_1,\alpha_2,\alpha_3)$ can be motivated without any reference to Liouville theory by the intention to make important analytic properties of the kernels representing F and S more easily visible. One may note, in particular, that $S_b(x) = \Gamma_b(x)/\Gamma_b(Q-x)$, from which it is easily seen that the change of normalization removes all square-roots from the expressions for $F_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$. The kernel $F^{\rm L}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ is then found to be meromorphic in all of its arguments. A more complete summary of the relevant analytic properties will be given in the following Subsection 8.3 below.

8.3. Preparation II – Analytic properties

The kernels representing the operators F and S have remarkable analytic properties which will later be shown to have profound consequences. The origin of the analytic properties can be found in the structure of $\mathcal{M}_{\text{flat}}^{\mathbb{C}}(C)$ as an algebraic variety. The simple form of the relations describing $\mathcal{M}_{\text{flat}}^{\mathbb{C}}(C)$ as an algebraic variety implies nice analytic properties of the expressions for the loop operators in terms of the Darboux coordinates, and this leads to nice analytic properties of the kernels $A_{\sigma_2\sigma_1}(l_2, l_1)$ via (6.20).

We will here summarize some of the most important properties.

8.3.1. Symmetries. The kernel representing F has a large group of symmetries. We will state them in the normalization which makes the realization of the respective symmetries most manifest.

- *Tetrahedral symmetries*: The coefficients $\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t \end{smallmatrix} \right\}_b$ satisfy the tetrahedral symmetries

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \alpha_2 & \alpha_1 & \alpha_s \\ \alpha_4 & \alpha_3 & \alpha_t \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \alpha_2 & \alpha_s & \alpha_1 \\ \alpha_4 & \alpha_t & \alpha_3 \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \alpha_3 & \alpha_4 & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_t \end{smallmatrix} \right\}_b . \tag{8.5}$$

- *Weyl symmetries*: The kernel $F^{\mathrm{L}}_{\alpha_s\alpha_t}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right]$ is symmetric under all reflections $\alpha_i \to Q - \alpha_i$, $i \in \{1, 2, 3, 4, s, t\}$.

The tetrahedral symmetries are easily read off from the integral representation (6.28). The derivation of the Weyl symmetries can be done with the help of the alternative integral representation (D.22).

Similar properties hold for the kernel representing the operator S.

- *Permutation symmetry*: The coefficients $S_{\alpha_1,\alpha_2}(\alpha_0)$ satisfy

$$\Delta(\alpha_1, \alpha_0, \alpha_1) S_{\alpha_1\alpha_2}(\alpha_0) = \Delta(\alpha_2, \alpha_0, \alpha_2) S_{\alpha_2\alpha_1}(\alpha_0) . \tag{8.6}$$

- *Weyl symmetries*: The kernel $S^{C}_{\alpha_1,\alpha_2}(\alpha_0)$ is symmetric under all reflections $\alpha_i \to Q - \alpha_i$, $i \in \{0, 1, 2\}$.

In other normalizations one will of course find a slightly more complicated realization of these symmetries.

8.3.2. Resonances and degenerate values. We will now summarize some of the most important facts concerning poles, residues and special values of the intertwining kernels. Proofs of the statements below are given in Appendix C.

Important simplifications are found for particular values of the arguments. Each α_i, $i \in \{1, 2, 3, 4\}$ is member of two out of the four triples $T_{12s} := (\alpha_1, \alpha_2, \alpha_s)$, $T_{34s} := (\alpha_3, \alpha_4, \alpha_s)$, $T_{23t} := (\alpha_2, \alpha_3, \alpha_t)$, $T_{14t} := (\alpha_1, \alpha_4, \alpha_t)$. We will say that a triple T_{ijk} is resonant if there exist $\epsilon_i \in \{+1, -1\}$ and $k, l \in$

$\mathbb{Z}^{\geq 0}$ such that

$$\epsilon_1\big(\alpha_3 - \tfrac{Q}{2}\big) = \epsilon_2\big(\alpha_2 - \tfrac{Q}{2}\big) + \epsilon_3\big(\alpha_1 - \tfrac{Q}{2}\big) + \tfrac{Q}{2} + kb + lb^{-1}\,. \tag{8.7}$$

Poles in the variables α_i, $i \in \{1,2,3,4\}$ will occur only if one of the triples T_{12s}, T_{34s}, T_{23t}, T_{14t} is resonant. The location of poles is simplest to describe in the case of F^C, which has poles in α_i, $i \in \{1,2,3,4\}$ if and only if either T_{t32} or T_{4t1} are resonant.

Of particular importance will be the cases where one of α_i, $i \in \{1,2,3,4\}$ takes one of the so-called degenerate values

$$\alpha_i \in \mathbb{D}\,, \quad \mathbb{D} := \{\, \alpha_{nm}\,, n, m \in \mathbb{Z}^{\geq 0} \,\}\,, \quad \alpha_{nm} := -nb/2 - mb/2\,. \tag{8.8}$$

Something remarkable may happen under this condition if the triple containing both α_i and α_s becomes resonant: Let us assume that $\alpha_s \in \mathbb{F}_{nm}(\alpha_j)$, where

$$\begin{aligned}\mathbb{F}_{nm}(\alpha_j) = \big\{\, & \alpha_j - (n-k)\tfrac{b}{2} - (m-l)\tfrac{1}{2b}\,,\\ & k = 0, 2, \dots, 2n\,, l = 0, 2, \dots, 2m \,\big\}\,.\end{aligned} \tag{8.9}$$

The kernel F^C becomes proportional to a sum of delta-distributions supported on resonances of the triple containing both α_t and α_i as expressed in the formulae

(8.10a)
$$\lim_{\alpha_1 \to \alpha_{nm}} F^{\rm L}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]_{\alpha_s \in \mathbb{F}_{nm}(\alpha_2)} = \sum_{\beta_t \in \mathbb{F}_{nm}(\alpha_4)} \delta(\alpha_t - \beta_t) f_{\alpha_s\beta_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \underline{\alpha}_1\end{smallmatrix}\big]$$

(8.10b)
$$\lim_{\alpha_2 \to \alpha_{mn}} F^{\rm L}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]_{\alpha_s \in \mathbb{F}_{nm}(\alpha_1)} = \sum_{\beta_t \in \mathbb{F}_{nm}(\alpha_3)} \delta(\alpha_t - \beta_t) f_{\alpha_s\beta_t}\big[\begin{smallmatrix}\alpha_3 & \underline{\alpha}_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]\,,$$

and similarly for α_3 and α_4. The delta-distributions $\delta(\alpha_t - \beta_t)$ on the right of (8.10) are to be understood as complexified versions of the usual delta-distributions. $\delta(\alpha - \beta)$ is defined to be the linear functional defined on spaces $\mathcal{T}$ of entire analytic test function $t(\alpha)$ as

$$\big\langle\, \delta(\alpha - \beta)\,,\, t \,\big\rangle = t(\beta)\,, \tag{8.11}$$

with $\langle .,. \rangle : \mathcal{T}' \times \mathcal{T} \to \mathbb{C}$ being the pairing between $\mathcal{T}$ and its dual $\mathcal{T}'$. The identities (8.10) are likewise understood as identities between distributions on $\mathcal{T}$.

8.4. Intertwining property

In this subsection we are going to describe a quick way to prove that the unitary operators defined in Subsection 6.5 correctly map the representations π_{σ_1} to π_{σ_2}, as expressed in Equations (6.18). The proof we will give here exploits the remarkable analytic properties of the fusion coefficients $F_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ summarized in Subsection 8.3.

One may, in particular, use the relations (8.10a) in order to derive from the pentagon relation (6.34c) systems of difference equations relating $F_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ to the residues of its poles, like for example

$$
\begin{aligned}
\sum_{\beta_5\in\mathbb{F}_{nm}(\beta_2)} & f_{\beta_1\beta_5}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \beta_2 & \underline{\alpha}_1\end{smallmatrix}\big] f_{\beta_2\beta_4}\big[\begin{smallmatrix}\alpha_4 & \beta_5\\ \alpha_5 & \underline{\alpha}_1\end{smallmatrix}\big] F_{\beta_5\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \beta_4 & \alpha_2\end{smallmatrix}\big] \\
&= f_{\beta_1\beta_4}\big[\begin{smallmatrix}\beta_3 & \alpha_2\\ \alpha_5 & \underline{\alpha}_1\end{smallmatrix}\big] F_{\beta_2\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \alpha_5 & \beta_1\end{smallmatrix}\big],
\end{aligned} \tag{8.12}
$$

valid for $\alpha_1=\alpha_{nm}$, $\beta_1\in\mathbb{F}_{nm}(\alpha_2)$ and $\beta_4\in\mathbb{F}_{nm}(\alpha_5)$. Similar equations can be derived for $\alpha_i=\alpha_{nm}$, $i=2,3,4,5$.

Further specializing (8.12) to $\alpha_1=-b$ and $\beta_1=\alpha_2$, $\beta_4=\alpha_5$, for example, yields a somewhat simpler difference equation of the form

$$
\sum_{k=-1}^{1} d^{(k)}_{\beta_2\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \alpha_5 & \alpha_2\end{smallmatrix}\big]\, F_{\beta_2+kb,\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \alpha_5 & \alpha_2\end{smallmatrix}\big] = 0\,. \tag{8.13}
$$

The coefficients $d^{(k)}_{\beta_2\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \alpha_5 & \alpha_2\end{smallmatrix}\big]$ are given as

$$
d^{(k)}_{\beta_2\beta_3}\big[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \alpha_5 & \alpha_2\end{smallmatrix}\big] = f_{\alpha_2,\beta_2+kb}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \beta_2 & -b\end{smallmatrix}\big] f_{\beta_2\alpha_5}\big[\begin{smallmatrix}\alpha_4 & \beta_2+kb\\ \alpha_5 & -b\end{smallmatrix}\big] - \delta_{k,0} f_{\alpha_2\alpha_5}\big[\begin{smallmatrix}\beta_3 & \alpha_2\\ \alpha_5 & -b\end{smallmatrix}\big]\,. \tag{8.14}
$$

By carefully evaluating the relevant residues of $F_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ (see Appendix C.4 for a list of the relevant results) one may show that (8.13) is equivalent to the statement (6.20) that the fusion transformation correctly intertwines the representation π_{σ_s} and π_{σ_t} of L_t associated to the pants decompositions σ_s and σ_t, respectively. Alternatively one may use this argument in order to *compute* the explicit form of the operator L_t in the representation where L_s is diagonal.

8.5. The S-kernel and the central extension

It remains to calculate the kernel of S and the central extension, as parameterized by the real number χ_b depending on the deformation parameter

b in (6.34f). One way to calculate the kernel of S directly within quantum Teichmüller theory is described in Appendix D.1. We conclude that the operators B, F, S are all represented by kernels that depend *meromorphically* on all their variables.

We had noted above that the operators B, F, Z and S satisfy the operatorial form of the Moore-Seiberg consistency equations up to projective phases [T05]. Being represented by meromorphic kernels, this implies the validity of (6.34) up to projective phases. One may then use special cases of (8.10), like

$$F^{\rm L}_{\alpha_1\beta_t}\left[\begin{smallmatrix}\alpha_3 & 0\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \delta(\beta_t-\alpha_3) \tag{8.15}$$

in order to check that the relations (6.34a), (6.34c) and (6.34f) have to hold *identically*, not just up to a phase. All but one of the remaining projective phases can be eliminated by a redefinition of the generators. We have chosen to parameterize the remaining phase by means of the real number χ_b which appears in the relation (6.34e). This is of course conventional, redefining the the kernels by a phase would allow one to move the phase from relation (6.34e) to other relations. Our convention will turn out to be natural in Part III of this paper. The explicit formula for the phase χ_b will be determined below.

In order to derive a formula for $S_{\beta_1\beta_2}(\alpha_0)$ we may then consider the relation (6.34f) in the special case $\alpha_1=\alpha_2$ and take the limit where β_1 and β_3 are sent to zero. The details are somewhat delicate. We will here give an outline of the argument, with more details given in Appendix D. It turns out to be necessary to send β_1 and β_3 to zero simultaneously. One will find a simplification of relation (6.34f) in this limit due to the relation

$$\lim_{\epsilon\downarrow 0} {\rm F}^{\rm L}_{\epsilon,\alpha_3}\left[\begin{smallmatrix}\epsilon & \alpha_1\\ \epsilon & \alpha_1\end{smallmatrix}\right] = \delta(\alpha_3-\alpha_1)\,. \tag{8.16}$$

Using Equation (8.16) it becomes straightforward to take $\beta_1=\beta_3=\epsilon$ and send $\epsilon\to 0$ in the relation (6.34f), leading to

$$\begin{aligned} & F^{\rm L}_{0\alpha}\left[\begin{smallmatrix}\beta_1 & \beta_1\\ \beta_1 & \beta_1\end{smallmatrix}\right] S^{\rm L}_{\beta_1\beta_2}(\alpha) \\ & = S^{\rm L}_{0\beta_2}\int_{\mathbb{S}} d\beta_3\; e^{-\pi i(2\Delta_{\beta_2}+2\Delta_{\beta_1}-2\Delta_{\beta_3}-\Delta_\alpha)}\; {\rm F}^{\rm L}_{0\beta_3}\left[\begin{smallmatrix}\beta_2 & \beta_1\\ \beta_2 & \beta_1\end{smallmatrix}\right] {\rm F}^{\rm L}_{\beta_3\alpha}\left[\begin{smallmatrix}\beta_1 & \beta_1\\ \beta_2 & \beta_2\end{smallmatrix}\right], \end{aligned} \tag{8.17}$$

where $S^{\rm L}_{0\beta} := \lim_{\epsilon\to 0} S^{\rm L}_{\epsilon\beta}(\epsilon)$. This formula determines $S^{\rm L}_{\beta_1\beta_2}(\alpha)/S^{\rm L}_{0\beta_2}$ in terms of ${\rm F}^{\rm L}_{\beta_s\beta_t}\left[\begin{smallmatrix}\beta_3 & \beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right]$. In Appendix D.3 it is shown that one may evaluate the integral in (8.17) explicitly, leading to the formula

(8.18)

$$S^{\mathrm{L}}_{\beta_1\beta_2}(\alpha_0) = S^{\mathrm{L}}_{0\beta_2}\,\frac{N(\beta_1,\alpha_0,\beta_1)}{N(\beta_2,\alpha_0,\beta_2)}\,\frac{e^{\frac{\pi i}{2}\Delta_{\alpha_0}}}{S_b(\alpha_0)} \times \int_{\mathbb{R}} dt\; e^{2\pi t(2\beta_1-Q)}\,\frac{S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)+it\big)S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)-it\big)}{S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)+it\big)S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)-it\big)}\,,$$

where $N(\alpha_3,\alpha_2,\alpha_1)$ is defined as

$$N(\alpha_3,\alpha_2,\alpha_1) = \frac{1}{\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_3)\Gamma_b(Q-\alpha_1-\alpha_2+\alpha_3)} \cdot \frac{\Gamma_b(2Q-2\alpha_3)\Gamma_b(2\alpha_2)\Gamma_b(2\alpha_1)\Gamma_b(Q)}{\Gamma_b(\alpha_1+\alpha_3-\alpha_2)\Gamma_b(\alpha_2+\alpha_3-\alpha_1)}\,. \tag{8.19}$$

It remains to determine $S^{\mathrm{L}}_{0\beta_2}$. In order to do this, let us note (using formula (D.34c) in Appendix D.4) that the expression (8.18) simplifies for $\alpha_0 \to 0$ to an expression of the form

$$S^{\mathrm{L}}_{\beta_1\beta_2} := \lim_{\alpha_0\to 0} S^{\mathrm{L}}_{\beta_1\beta_2}(\alpha_0) = \frac{S^{\mathrm{L}}_{0\beta_2}}{|S_b(2\beta_2)|^2}\, 2\cos\left(\pi(2\beta_1-Q)(2\beta_2-Q)\right)\,. \tag{8.20}$$

It then follows from (6.34d) that we must have

$$S^{\mathrm{L}}_{0\beta} = \sqrt{2}|S_b(2\beta)|^2 = -2^{\frac{5}{2}}\,\sin\pi b(2\beta-Q)\,\sin\pi b^{-1}(2\beta-Q)\,. \tag{8.21}$$

One may observe an interesting phenomenon: The analytic continuation of $S^{\mathrm{L}}_{\beta_1\beta_2}$ to the value $\beta_1 = 0$ does not coincide with the limit $S^{\mathrm{L}}_{0\beta} := \lim_{\epsilon\to 0} S^{\mathrm{L}}_{\epsilon\beta}(\epsilon)$. This can also be shown directly using the integral representation (8.18), see Appendix D.4.

Direct calculation using relation (6.34e) in the special case $\alpha = 0$ then shows that χ_b is equal to

$$\boxed{\chi_b = \frac{\mathbf{c}}{24}\,, \qquad \mathbf{c} = 1 + 6(b+b^{-1})^2\,.} \tag{8.22}$$

We conclude that the quantization of Teichmüller space produces a projective representation of the Moore-Seiberg groupoid with central extension given in terms of the Liouville central charge $\mathbf{c}$, as is necessary for the relation between Liouville theory and the quantum Teichmüller theory to hold in higher genus.

8.6. General remarks

It should be possible to verify the consistency of the quantum theory of $\mathcal{M}^0_{\text{flat}}(C) \simeq \mathcal{T}(C)$ defined in Section 6 without using the relation with the quantum Teichmüller theory described in Section 7 above. However, the most difficult statements to prove would then be the consistency conditions (6.34). We may note, however, that the relations (6.34a)-(6.34c) can be proven by using the relation between the fusion coefficients $F_{\beta_s\beta_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ and the 6j-symbols of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ [PT2, NT], or with the fusion and braiding coefficients of quantum Liouville theory [PT1, T01, T03a].

Any proof that the operators defined in Section 6 satisfy the full set of consistency conditions (6.34) could be taken as the basis for an alternative approach to the quantum Teichmüller theory that is entirely based on the loop coordinates associated to pants decompositions rather than triangulations of the Riemann surfaces.

A more direct way to prove the consistency conditions (6.34) could probably start by demonstrating the fact that the operators $\mathsf{U}_{\sigma_2\sigma_1}$ correctly intertwine the representations π_σ according to (6.18). It follows that any operator intertwining a representation π_{σ_1} with itself like $\mathsf{U}_{\sigma_1\sigma_3}\mathsf{U}_{\sigma_3\sigma_2}\mathsf{U}_{\sigma_2\sigma_1}$ acts trivially on all generators $\mathsf{L}_{\sigma_1,\gamma}$. This should imply that such operators must be proportional to the identity, from which the validity of the consistency conditions (6.34) up to projective phases would follow.

However, such an approach would lead into difficulties of functional-analytic nature that we have not tried to solve. One would need to show, in particular, that any operator commuting with $\pi_\sigma(\mathcal{A}_b(C))$ has to be proportional to the identity.

The proof of (6.34) using the quantum Teichmüller theory described in Section 7 seems to be the most elegant for the time being.

Part III. Conformal field theory

We are now going to describe an alternative approach to the quantization of $\mathcal{M}^0_{\text{flat}}(C)$, and explain why it is intimately related to the Liouville theory. It will be shown that the conformal blocks, naturally identified with certain wave-functions in the quantum theory of $\mathcal{M}^0_{\text{flat}}(C)$, represent solutions to the Riemann-Hilbert type problem formulated in Subsection 4.3 above.

This will in particular clarify why we need to have the spurious prefactors $f_{\sigma_2\sigma_1}(\tau)$ in the S-duality transformations (4.8), in general. They will be identified with transition functions of the projective line bundle which plays an important role in the geometric approach to conformal field theory going back to [FS]. This observation will lead us to the proper geometric characterization of the non-perturbative scheme dependence observed in Subsection 3.3.2, and will allow us to define natural prescriptions fixing the resulting ambiguities.

9. Classical theory

9.1. Complex analytic Darboux coordinates

In order to establish the relation with conformal field theory it will be useful to consider an alternative quantization scheme for $\mathcal{M}^0_{\rm flat}(C) \simeq \mathcal{T}(C)$ which makes explicit use of the complex structure on these spaces. In order to do this, it will first be convenient to identify a natural complexification of the spaces of interest by representing $\mathcal{M}^0_{\rm flat}(C)$ as a connected component of the real slice $\mathcal{M}_{\rm flat}(C)$ within $\mathcal{M}^{\mathbb{C}}_{\rm flat}(C)$.

Let us begin by recalling that natural Darboux coordinates for an important component of the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections can be defined in terms of a special class of local systems called *opers*.

9.1.1. Opers. In the case $\mathfrak{g} = \mathfrak{sl}_2$ one may define opers as bundles admitting a connection that locally can be represented as

$$(9.1) \qquad \nabla' = \frac{\partial}{\partial y} + \frac{1}{\epsilon} M(y)\,, \qquad M(y) = \begin{pmatrix} 0 & -t(y) \\ 1 & 0 \end{pmatrix}.$$

The equation $\nabla' h = 0$ for horizontal sections $s = (s_1, s_2)^t$ implies the second order differential equation $(\epsilon^2 \partial_y^2 + t(y))s_2 = 0$. Under holomorphic changes of the local coordinates on C, $t(y)$ transforms as

$$(9.2) \qquad t(y) \;\mapsto\; (y'(w))^2 t(y(w)) + \frac{c}{12}\{y, w\}\,,$$

where $c \equiv c_{\rm cl} := 6\epsilon^2$, and the Schwarzian derivative $\{y, w\}$ is defined as

$$(9.3) \qquad \{y, w\} \;\equiv\; \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 .$$

Equation (9.2) is the transformation law characteristic for *projective* c-connections, which are also called $\mathfrak{sl}_2$-opers, or opers for short.

Let Op(C) the space of $\mathfrak{sl}_2$-opers on a Riemann surface C. Two opers represented by t and t', respecticely, differ by a holomorphic quadratic differentials $\vartheta = (t-t')(dy)^2$. This implies that the space Op($C_{g,n}$) of $\mathfrak{sl}_2$-opers on a fixed surface $C_{g,n}$ of genus g with n marked points is $h = 3g-3+n$-dimensional. Complex analytic coordinates for Op($C_{g,n}$) are obtained by picking a reference oper t_0, a basis $\vartheta_1,\dots,\vartheta_h$ for the vector space of quadratic differentials, and writing any other oper as

$$t(dy)^2 = t_0(dy)^2 + \sum_{r=1}^{h} h_r\,\vartheta_r\,. \tag{9.4}$$

The space of opers forms an affine bundle $\mathcal{P}(C)$ over the Teichmüller space of deformations of the complex structure of C. The monodromy representations $\rho_P : \pi_1(C_{g,n}) \to \mathrm{SL}(2,\mathbb{C})$ of the connections ∇' will generate a $3g-3+n$-dimensional subspace in the character variety $\mathcal{M}_{\rm char}^{\mathbb{C}}(C)$ of surface group representations. Varying the complex structure of the underlying surface C, too, we get a subspace of $\mathcal{M}_{\rm char}^{\mathbb{C}}(C)$ of complex dimension $6g-6+2n$. It is important that the mapping $\mathcal{P}(C) \to \mathcal{M}_{\rm char}^{\mathbb{C}}(C)$ defined by associating to the family of opers $\epsilon^2\partial_y^2 + t(y;q)$ its monodromy representation ρ_t is locally biholomorphic [He, Ea, Hu].

9.1.2. Projective structures. A projective structure is a particular atlas of complex-analytic coordinates on C which is such that the transition functions are all given by Moebius transformations

$$y'(y) = \frac{ay+b}{cy+d}\,. \tag{9.5}$$

It will be useful to note that there is a natural one-to-one correspondence between projective structures and opers. Given an oper, in a patch $\mathcal{U} \subset C$ locally represented by the differential operator $\epsilon^2\partial_y^2 + t(y)$, one may construct a projective structure by taking the ratio

$$w(y) := f_1(y)/f_2(y)\,, \tag{9.6}$$

of two linearly independent solutions f_1, f_2 of the differential equation $(\epsilon^2\partial_y^2 + t(y))f(y) = 0$ as the new coordinate in $\mathcal{U}$. The oper will be represented by the differential operator ∂_w^2 in the coordinate w, as follows from (9.2) observing that

$$t(y) = \frac{c}{2}\{w, y\}\,. \tag{9.7}$$

The bundle $\mathcal{P}(C)$ may therefore be identified with the space of projective structures on C.

9.1.3. Complex structure on $\mathcal{P}(C)$. The space $\mathcal{P}(C)$ is isomorphic as a complex manifold to the holomorphic cotangent bundle $T^*\mathcal{T}(C)$ over the Teichmüller space $\mathcal{T}(C)$. In order to indicate how this isomorphism comes about, let us recall some basic results from the complex analytic theory of the Teichmüller spaces.[8]

Let $\mathcal{Q}(C)$ be the vector space of meromorphic quadratic differentials on C which are allowed to have poles only at the punctures of C. The poles are required to be of second order, with fixed leading coefficient. A Beltrami differential μ is a $(-1,1)$-tensor, locally written as $\mu_{\bar{z}}^{z} d\bar{z}/dz$. Let $\mathcal{B}(C)$ be the space of all measurable Beltrami differentials such that $\int_C |\mu\vartheta| < \infty$ for all $\vartheta \in \mathcal{Q}(C)$. There is a natural pairing between $\mathcal{Q}(C)$ and $\mathcal{B}(C)$ defined as

$$\langle\, \vartheta\,,\, \mu\,\rangle := \int_C \mu\vartheta\,. \tag{9.8}$$

Standard Teichmüller theory establishes the basic isomorphisms of vector spaces

$$T\mathcal{T}(C) \,\simeq\, \mathcal{B}(C)/\mathcal{Q}(C)^{\perp}\,, \tag{9.9}$$

$$T^*\mathcal{T}(C) \,\simeq\, \mathcal{Q}(C)\,, \tag{9.10}$$

where $\mathcal{Q}(C)^{\perp}$ is the subspace in $\mathcal{B}(C)$ on which all linear forms f_ϑ, $\vartheta \in \mathcal{Q}(C)$, defined by $f_\vartheta(\mu) \equiv \langle\vartheta, \mu\rangle$ vanish identically.

The relation between $\mathcal{P}(C)$ and $T^*\mathcal{T}(C)$ follows from the relation between $\mathrm{Op}(C)$ and the space $\mathcal{Q}(C)$ of quadratic differentials explained above. What's not immediately obvious is the fact there is a natural complex structure on $\mathcal{P}(C)$ that makes the isomorphism $\mathcal{P}(C) \simeq T^*\mathcal{T}(C)$ an isomorphism of complex manifolds.

To see this, the key ingredient is the existence of a *holomorphic section* of the bundle $\mathcal{P}(C) \to \mathcal{T}(C)$, locally represented by opers $\epsilon^2\partial_y^2 + t(y;q)$

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[8] A standard reference is [Na]. A useful summary and further references to the original literature can be found in [TT03]. The results that are relevant for us are very concisely summarized in [BMW, Section 1].

that depend holomorphically on q. Such a section is provided by the Bers double uniformization. Given two Riemann surfaces C_1 and C_2 there exists a subgroup $\Gamma(C_1, C_2)$ of $\mathrm{PSL}(2,\mathbb{C})$ that uniformizes C_1 and C_2 simultaneously in the following sense: Considering the natural action of $\Gamma(C_1, C_2)$ on $\mathbb{C}$ by Moebius transformations, the group Γ will have a domain of discontinuity of the form $\Omega(C_1, C_2) = \Omega_1 \sqcup \Omega_2$ such that $\Omega_1/\Gamma(C_1, C_2) \simeq C_1$, $\Omega_2/\Gamma(C_1, C_2) \simeq \bar{C}_2$, where $\bar{C}_2$ is obtained from C_2 by orientation reversal. Let $\pi_1 : \Omega_1 \to C_1$ be the corresponding covering map. The Schwarzian derivatives $\mathcal{S}(\pi_1^{-1})$ and $\mathcal{S}(\pi_2^{-1})$ then define a families of opers on C_1 and $\bar{C}_2$, respectively. The family of opers defined by $\mathcal{S}(\pi_1^{-1})$ depends holomorphically on the complex structure moduli q_2 of C_2.

9.1.4. Symplectic structure on $\mathcal{P}(C)$. Note furthermore that the corresponding mapping $\mathcal{P}(C) \simeq T^*\mathcal{T}(C) \to \mathcal{M}^{\mathbb{C}}_{\mathrm{char}}(C)$ is symplectic in the sense that the canonical cotangent bundle symplectic structure is mapped to the Atiyah-Bott symplectic structure Ω_J on the space of flat complex connections [Kaw]. We may, therefore, choose a set of local coordinates $q = (q_1, \dots, q_h)$ on $\mathcal{T}(C_{g,n})$ which are conjugate to the coordinates h_r defined above in the sense that the Poisson brackets coming from this symplectic structure are

$$\{ q_r \, , q_s \} = 0 \, , \qquad \{ h_r \, , q_s \} = \delta_{r,s} \, , \qquad \{ h_r \, , h_s \} = 0 \, . \tag{9.11}$$

Let us note that one may also use non-holomorphic sections $t'(y; q, \bar{q})$ in $\mathcal{P}(C) \to \mathcal{T}(C)$ in order to get such Darboux coordinates (q, h). This amounts to a shift of the variables h_r by a function of the variables q,

$$h'_r = h_r + \nu_r(q, \bar{q}) \, , \qquad r = 1, \dots, h \, ,$$

which clearly preserves the canonical form of the Poisson brackets (9.11).

9.2. Twisted cotangent bundle $T^*_c\mathcal{M}(C)$

The affine bundle $\mathcal{P}(C)$ over $\mathcal{T}(C)$ descends to a twisted cotangent bundle over the moduli space $\mathcal{M}(C)$ of complex structures on C. To explain what this means let us use a covering $\{\mathcal{U}_\imath; \imath \in \mathcal{I}\}$ of $\mathcal{M}(C)$. Within each patch $\mathcal{U}_\imath$ we may consider local coordinates $q = (q_1, \dots, q_h)$ for $\mathcal{M}(C)$, which may be completed to a set of local Darboux coordinates (q, h) for $\mathcal{P}(C)$ such that

$$\Omega = \sum_{r=1}^{h} dh_r \wedge dq_r \, .$$

$\mathcal{P}(C)$ is a twisted holomorphic cotangent bundle over $\mathcal{M}(C)$ if the Darboux coordinates transform as

$$\Lambda = \sum_r h_r^\imath \, dq_r^\imath = \sum_r h_r^\jmath \, dq_r^\jmath - \chi^{\imath\jmath} \,, \tag{9.12}$$

with $\chi^{\imath\jmath}$ being locally defined holomorphic one-forms on $\mathcal{U}_{\imath\jmath} \equiv \mathcal{U}_\imath \cap \mathcal{U}_\jmath$. The collection of one-forms defines a 1-cocycle with values in the sheaf of holomorphic one-forms. We may always write $\chi^{\imath\jmath} = \partial g_{\imath\jmath}$ for locally defined holomorphic functions $g_{\imath\jmath}$ on $\mathcal{U}_{\imath\jmath}$. The functions $f_{\imath\jmath} := e^{2\pi i g_{\imath\jmath}}$ will then satisfy relations of the form

$$f_{\imath_3\imath_2} \, f_{\imath_2\imath_1} = \sigma_{\imath_3\imath_2\imath_1} \, f_{\imath_3\imath_1} \,, \tag{9.13}$$

where $\sigma_{\imath_3\imath_2\imath_1}$ is constant on the triple overlaps $\mathcal{U}_{\imath_1\imath_2\imath_3} \equiv \mathcal{U}_{\imath_1} \cap \mathcal{U}_{\imath_2} \cap \mathcal{U}_{\imath_3}$. A collection of functions $f_{\imath\jmath}$ on $\mathcal{U}_{\imath\jmath}$ that satisfy (9.13) defines a so-called *projective line bundle* [FS]. The obstruction to represent it as an ordinary line bundle is represented by a class in $\check{H}^2(\mathcal{M}(C), \mathbb{C}^*)$.

It was pointed out in [FS] that any holomorphic section of $\mathcal{P}(C) \to \mathcal{T}(C)$, represented by a family of opers $\epsilon^2 \partial_y^2 + t(y;q)$, can be considered as a connection on a certain holomorphic projective line bundle $\mathcal{E}_c$. The connection is locally represented by the one-forms $(\partial_r + A_r) dq_r$ on $\mathcal{T}(C)$ such that

$$A_r(\tau) = \int_C t\mu_r \,, \tag{9.14}$$

for a collection of Beltrami differentials μ_r which represent a basis to the tangent space $T\mathcal{T}(C)$ dual to the chosen set of coordinates q_r. One may define a family of local sections $\mathcal{F}_\imath$ of $\mathcal{E}_c$ which are horizontal with respect to the connection A_t as solutions to the differential equations

$$\partial_r \ln \mathcal{F}_\imath = - \int_C t\mu_r \,. \tag{9.15}$$

The transition functions $f_{\imath\jmath}^c$ of $\mathcal{E}_c$ are then defined by $f_{\imath\jmath}^c := \mathcal{F}_\imath^{-1} \mathcal{F}_\jmath$. In general it will not be possible to choose the integration constants in the solution of (9.15) in such a way that in (9.13) we find $\sigma_{\imath_3\imath_2\imath_1} = 1$ for all nontrivial triple intersections $\mathcal{U}_{\imath_3\imath_2\imath_1}$.

The resulting projective line bundle $\mathcal{E}_c$ is uniquely characterized by the real number c if the family t is regular at the boundary of $\mathcal{M}(C)$. It was shown in [FS] that

$$\mathcal{E}_c = (\lambda_{\rm H})^{\frac{c}{2}}\,, \tag{9.16}$$

where $\lambda_{\rm H}$ is the so-called Hodge line bundle, the determinant bundle $\det\Omega_{\rm H} \equiv \bigwedge^g \Omega_{\rm H}$ of the bundle of rank g over $\mathcal{M}(C)$ whose fiber over a point of $\mathcal{M}(C)$ is the space of abelian differentials of first kind on C.

9.3. Projective structures from the gluing construction

In Subsection 2.1 we have described how to construct local patches of coordinates $q = (q_1, \dots, q_h)$ for $\mathcal{T}(C)$ by means of the gluing construction. There is a corresponding natural choice of coordinates $H = (h_1, \dots, h_h)$ for $T^*\mathcal{T}(C)$ defined as follows. The choice of the coordinates q defines a basis for $T\mathcal{T}(C)$ generated by the tangent vectors ∂_{q_r} which can be represented by Beltrami differentials μ_r via (9.9). The dual basis of quadratic differentials ϑ_r is then defined by the condition $\langle \vartheta_r,\, \mu_s \rangle = \delta_{r,s}$. This defines coordinates h_r for $T^*\mathcal{T}(C)$.

In order to make the coordinates (q, h) for $T^*\mathcal{T}(C)$ into coordinates for $\mathcal{P}(C)$, one needs to choose a section $\mathcal{S} : \mathcal{T}(C) \to \mathcal{P}(C)$. It will be important to note that the gluing construction allows one to define natural choices for local sections of $\mathcal{P}(C)$ as follows.

Let us represent the three-punctured spheres used in the gluing construction as $C_{0,3} \sim \mathbb{P}^1 \setminus \{0, 1, \infty\} \sim \mathbb{C} \setminus \{0, 1\}$. A natural choice of coordinate on $C_{0,3}$ is then coming from the coordinate y on the complex plane $\mathbb{C}$. Let us choose the coordinates around the punctures 0, 1 and ∞ to be y, $1 - y$ and $1/y$, respectively. The surfaces C obtained from the gluing construction will then automatically come with an atlas of local coordinates which has transition functions always represented by Moebius transformations (9.5). It follows that the gluing construction naturally defines families of projective structures over the multi-discs $\mathcal{U}_\sigma$ with coordinates q, or equivalently according to Subsection 9.1.2 a section $\mathcal{S}_\sigma : \mathcal{U}_\sigma \to \mathcal{P}(C)$. One could replace the representation of $C_{0,3}$ as $C_{0,3} \sim \mathbb{P}^1 \setminus \{0, 1, \infty\}$ by $C_{0,3} \sim \mathbb{P}^1 \setminus \{z_1, z_2, z_3\}$, leading to other sections $\mathcal{S} : \mathcal{U}_\sigma \to \mathcal{P}(C)$.

We may define such a section $\mathcal{S}_\sigma$ for any pants decomposition σ. The sections $\mathcal{S}_\sigma$ define corresponding local trivializations of the projective line bundle $\mathcal{E}_c$ according to our discussion in Subsection 9.2. The trivializations coming from pants decompositions lead to a particularly simple representation for the transition functions $f^c_{\sigma_2\sigma_1}$ defining $\mathcal{E}_c$, which will be calculated explicitly in the following.

9.3.1. Transition functions. It is enough to calculate the resulting transition functions for the elementary moves B, F and S generating the MS groupoid. In the case of B and F it suffices to note that the gluing of two three-punctured spheres produces a four-punctured sphere that may be represented as $C_{0,4} \sim \mathbb{P}^1 \setminus \{0, 1, q, \infty\}$, with q being the gluing parameter. The B-move corresponds to the Moebius transformation $y' = q - y$ which exchanges 0 and q. Being related by a Moebius transformation, the projective structures associated to two pants decompositions σ_1 and σ_2 related by a B-move must coincide. We may therefore assume that $g_{\sigma_2\sigma_1} = 1$ if σ_1 and σ_2 differ by a B-move. The F-move corresponds to $y' = 1 - y$, so that $g_{\sigma_2\sigma_1} = 1$ if σ_1 and σ_2 differ by a F-move.

The only nontrivial case is the S-move. We assume that $C_{1,1}$ is obtained from a three-punctured sphere $C_{0,3} \sim \mathbb{P}^1 \setminus \{0, 1, \infty\}$ by gluing annular neighborhoods of 0 and ∞. The resulting coordinate y_σ on $C_{1,1}$ is coming from the coordinate y on $C_{0,3} \sim \mathbb{P}^1 \setminus \{0, 1, \infty\}$. A nontrivial transition function $g_{\sigma_2\sigma_1}$ will be found if σ_1 and σ_2 differ by a S-move since the coordinates y_{σ_1} and y_{σ_2} are *not* related by a Moebius transformation.

In order to see this, it is convenient to introduce the coordinate w_σ related to the coordinate y_σ on the complex plane by $y_\sigma = e^{w_\sigma}$. The coordinate w_σ would be the natural coordinate if we had represented $C_{1,1}$ as

$$C_{1,1} \sim \big\{\, w \in \mathbb{C};\ w \sim w + n\pi + m\pi\tau;\ n, m \in \mathbb{Z} \,\big\} \setminus \{0\}\,.$$

This corresponds to representing $C_{1,1}$ by gluing the two infinite ends of the punctured cylinder $\{w \in \mathbb{C};\ w \sim w + n\pi; n \in \mathbb{Z}\} \setminus \{0\}$. The corresponding alternative pants decomposition of $C_{1,1}$ will be denoted $\tilde{\sigma}$.

The transition function $g_{\sigma_2\sigma_1}$ defined by our conventions for the gluing construction will then be nontrivial since the relation $y_\sigma = e^{w_\sigma}$ is *not* a Moebius transformation. The relation between the projective structures associated to pants decompositions σ and $\tilde{\sigma}$ can be calculated from (9.2),

$$\tilde{t}(w) = e^{2w}\, t(e^w) - \frac{c}{24}\,. \tag{9.17}$$

We thereby get a nontrivial transition function $g_{\tilde{\sigma}\sigma}$ between the trivializations of $\mathcal{E}_c$ associated to σ and $\tilde{\sigma}$ equal to $\frac{c}{24}\tau$ up to an additive constant.

Let us assume that σ_2 is obtained from σ_1 by an S-move. The projective structures associated to the coordinates w_{σ_1} and w_{σ_2} will coincide since the S-move is represented by the Moebius transformation $w_{\sigma_2} = -w_{\sigma_1}/\tau$. The resulting transition function $g_{\tilde{\sigma}_2\tilde{\sigma}_1} = 1$ is trivial. Taken together we conclude that

$$g_{\sigma_2\sigma_1} = g_{\sigma_2\tilde{\sigma}_2} + g_{\tilde{\sigma}_2\tilde{\sigma}_1} + g_{\tilde{\sigma}_1\sigma_1} = \frac{c}{24}\Big(\tau + \frac{1}{\tau}\Big) + h_{\sigma_2\sigma_1}\,, \tag{9.18}$$

with $h_{\sigma_2\sigma_1}$ being constant, if σ_1 and σ_2 are related by an S-move. These are the only nontrivial transition functions of $\mathcal{E}_c$ in the representation associated to pants decompositions defined above. The argument above determines $g_{\sigma_2\sigma_1}$ up to an additive ambiguitiy $h_{\sigma_2\sigma_1}$. Precise normalizations fixing this ambiguity will be defined next.

10. The generating functions $\mathcal{W}$

In the following we will set $\epsilon = 1$. The parameter ϵ can easily be restored by rescaling $t(y)$.

10.1. Definition

We have used two radically different representations for the space $\mathcal{P}(C)$: As cotangent bundle $T^*\mathcal{T}(C)$, on the one hand, and as character variety $\mathcal{M}^{\mathbb{C}}_{\rm char}(C)$ on the other hand. In Section 2.6 we had introduced systems of Darboux coordinates (l,k) associated to MS-graphs σ for the character variety $\mathcal{M}^{\mathbb{C}}_{\rm char}(C)$. We had previously introduced Darboux coordinates (q,h) with the help of the isomorphism $\mathcal{P}(C) \simeq T^*\mathcal{T}(C)$. Important objects are the generating functions $\mathcal{W}(l,q)$ that characterize the transitions between these sets of coordinates.

Let us briefly explain how the functions $\mathcal{W}(l,q)$ are defined. The locally defined one-forms $\sum_r k_r dl_r - \sum_r h_r dq_r$ are ∂-closed since $\sum_r dk_r \wedge dl_r = \sum_r dh_r \wedge dq_r$ [Kaw], therefore locally exact,

$$\sum_r k_r dl_r - \sum_r h_r dq_r = \partial \mathcal{W}\,. \tag{10.1}$$

It follows that the change of coordinates $(l,k) \to (q,h)$ can locally be described in terms of a generating function $\mathcal{W}$. Let us start, for example, with the coordinates (q,h). For fixed values of l, let us define the functions $h_r(l,q)$ as the solutions to the system of equations

$$2\cosh(l_r/2) = \mathrm{tr}(\rho_{q,h}(\gamma_r))\,, \tag{10.2}$$

where $\rho_{q,h}$ is the monodromy representation of the oper $\partial_y^2 + t_0(y;q) + \sum_r h_r \vartheta_r(y)$. Equation (10.1) ensures integrability of the equations

$$h_r(l,q) = -\frac{\partial}{\partial q_r}\mathcal{W}(l,q)\,, \tag{10.3}$$

which define $\mathcal{W}(q,l)$ up to a function of l. This ambiguity is fixed by the equations

$$k_r(l,q) = \frac{\partial}{\partial l_r}\mathcal{W}(l,q)\,, \qquad k_r(l,q) \equiv k_r(\rho_{q,h(l,q)})\,, \tag{10.4}$$

following from (10.1), where $k_r(\rho)$ is the value of the coordinate k_r on the monondromy ρ as defined in Section 2.6.

Comparing (10.3) with (9.15) we realize $\mathcal{F}_{\rm cl}(l,q) \equiv e^{\mathcal{W}(l,q)}\mathcal{F}_0(q)$ as the local section of the projective holomorphic line bundle $\mathcal{E}_c$ that is horizontal with respect to the connection defined by the family of opers $\partial_y^2 + t_0(y;q) + \sum_r h_r(q,l)\vartheta_r(y)$.

10.2. Changes of coordinates

We have introduced systems of coordinates (l,k) and (q,h) that both depend on the choice of a pants decomposition σ. In order to indicate the dependence on the choices of pants decompositions underlying the defininitions of the coordinates we shall use the notation $\mathcal{W}_{\sigma,\sigma'}(l,q)$ if coordinates (l,k) were defined using the pants decomposition σ and if coordinates (q,h) were define using the pants decomposition σ'.

10.2.1. Changes of coordinates (l,k). Let us compare the functions $\mathcal{W}_{\sigma_2,\sigma'}(l,q)$ and $\mathcal{W}_{\sigma_1,\sigma'}(l,q)$ associated to two different choices of pants decompositions σ_2 and σ_1, respectively. It is clear that there must exist a relation of the form

$$\mathcal{W}_{\sigma_2,\sigma'}(l_2,q) = \mathcal{F}_{\sigma_2\sigma_1}(l_2,l_1(l_2,q)) + \mathcal{W}_{\sigma_1,\sigma'}(l_1(l_2,q),q)\,, \tag{10.5}$$

where $\mathcal{F}_{\sigma_2\sigma_1}(l_2,l_1)$ is the generating function for the change of Darboux coordinates (k_2,l_2) associated to σ_2 to (k_1,l_1) associated to σ_1, respectively.

The generating function $\mathcal{F}_{\sigma_2\sigma_1}(l_2,l_1)$ can be represented up to an additive constant by choosing a path $\varpi_{\sigma_2\sigma_1} \in [\sigma_2,\sigma_1]$ connecting σ_1 and σ_2, representing it as sequence of Moore-Seiberg moves $[m_N \circ m_{N-1} \circ \cdots \circ m_1]$, and adding the generating functions $\mathcal{F}_{m_i}$ representing the changes of Darboux variables associated to the moves m_i. Changes of the path $\varpi_{\sigma_2\sigma_1} \in [\sigma_2, \sigma_1]$ will change the result by an additive constant.

The generating functions $\mathcal{F}_{\sigma_2\sigma_1}(l_2, l_1)$ can be identified as the semiclassical limits of $b^2 \log A_{\sigma_2\sigma_1}(l_2, l_1)$, with $A_{\sigma_2\sigma_1}(l_2, l_1)$ being the kernels of the operators generating the representation of the Moore-Seiberg groupoid constructed in Part II.

10.2.2. Changes of coordinates (q, h). It turns out that $\mathcal{W}_{\sigma,\sigma'}(l, q)$, considered as function of q, can be extended to functions on all of $\mathcal{T}(C)$ by analytic continuation[9]. We will use the same notation $\mathcal{W}_{\sigma,\sigma'}(l, q)$ for the result of the analytic continuation.

Comparing the transformation (9.12) of the coordinates h_r with (10.3), we see that the functions $\mathcal{W}_{\sigma,\sigma'_2}(l, q)$ and $\mathcal{W}_{\sigma,\sigma'_1}(l, q)$ defined by using different pants decompositions for the definition of coordinates (q, h) are related by the transition functions in the projective line bundle $\mathcal{E}_c$,

$$\mathcal{W}_{\sigma,\sigma'_2}(l, q) = g_{\sigma'_2,\sigma'_1}(q) + \mathcal{W}_{\sigma,\sigma'_1}(l, q)\,. \tag{10.6}$$

This reflects the changes of coordinates h_r induced by changes of the sections $\mathcal{P}(C) \to \mathcal{T}(C)$ associated to transitions between different pants decompositions.

By combining (10.5) and (10.6) one gets, in particular,

$$\mathcal{W}_{\sigma_2,\sigma_2}(l_2, q) = g_{\sigma_2,\sigma_1}(q) + \mathcal{F}_{\sigma_2\sigma_1}(l_2, l_1(l_2, q)) + \mathcal{W}_{\sigma_1,\sigma_1}(l_1(l_2, q), q)\,. \tag{10.7}$$

In order to define $\mathcal{F}_{\sigma_2\sigma_1}(l_2, l_1(l_2, q))$ and $g_{\sigma_2,\sigma_1}(q)$ unambigously one would need to fix a normalization prescription for $\mathcal{W}_{\sigma,\sigma'}(l, q)$.

10.2.3. Mapping class group action. Note that in the case $\sigma'_2 = \mu.\sigma'$, $\sigma'_1 \equiv \sigma'$ we get from (10.6)

$$\mathcal{W}_{\sigma,\mu.\sigma'}(l, q) = g_\mu(q) + \mathcal{W}_{\sigma,\sigma'}(l, q)\,. \tag{10.8}$$

We have used the shortened notation

$$g_\mu(q) := g_{\mu.\sigma,\sigma}(q)\,. \tag{10.9}$$

[9]We don't have a direct proof of this fact at the moment, but we may infer it indirectly from the corresponding statement about the Liouville conformal blocks $\mathcal{Z}^{\mathrm{L}}$ together with the fact that the $\mathcal{W}_{\sigma,\sigma}(l, q)$ coincide with the semiclassical limit $b \to 0$ of $b^2 \log \mathcal{Z}^{\mathrm{L}}$.

Taken together we find, in the particular case $\sigma = \sigma'$

$$\mathcal{W}_{\mu.\sigma,\mu.\sigma}(l,q) = g_\mu(q) + F_{\mu.\sigma,\sigma}(l,\tilde{l}(l,q)) + \mathcal{W}_{\sigma,\sigma}(\tilde{l}(l,q),q)\,. \tag{10.10}$$

Below we will fix a specific normalization for $\mathcal{W}_{\sigma,\sigma}(l,q)$. Thanks to the uniqueness of analytic continuation the sum of terms $F_{\mu.\sigma,\sigma}(l,\tilde{l}(l,q)) + g_\mu(q)$ appearing in (10.10) will then be uniquely defined.

10.3. Behavior at the boundaries of $\mathcal{T}(C)$

It will be important for us to understand the behavior of the generating functions $\mathcal{W}(l,q)$ at the boundaries of the Teichmüller spaces $\mathcal{T}(C)$. This will in particular allow us to define a natural choice for the precise normalization of the functions $\mathcal{W}_{\sigma,\sigma}(l,q)$.

By means of pants decompositions one may reduce the problem to the cases of the four-punctured sphere $C = C_{0,4}$, and the one-punctured torus $C = C_{1,1}$.

10.3.1. Genus zero, four punctures, singular term. Let us first consider $C = C_{0,4} = \mathbb{P}^1 \setminus \{z_1, z_2, z_3, z_4\}$. We may assume that $z_1 = 0$, $z_3 = 1$, $z_4 = \infty$, and identify the complex structure parameter q with z_2. The opers on C can be represented in the form $\partial_y^2 + t(y)$, where

$$t(y) = \frac{\delta_3}{(y-1)^2} + \frac{\delta_1}{y^2} + \frac{\delta_2}{(y-q)^2} + \frac{\upsilon}{y(y-1)} + \frac{q(q-1)}{y(y-1)}\frac{H}{y-q}, \tag{10.11}$$

where $\upsilon = \delta_4 - \delta_1 - \delta_2 - \delta_3$. The relation (10.3) becomes simply

$$H(l,q) = -\frac{\partial}{\partial q}\mathcal{W}(l,q)\,. \tag{10.12}$$

This relation determines $\mathcal{W}(l,q)$ up to q-independent functions of l. For $q \to 0$ it may be shown that $\mathcal{W}(l,q)$ behaves as

$$\mathcal{W}(l,q) = (\delta(l_1) + \delta(l_2) - \delta(l))\log q + \mathcal{W}_0(l) + \mathcal{O}(q)\,, \tag{10.13}$$

where $\delta(l) = \frac{1}{4} + \left(\frac{l}{4\pi}\right)^2$. Indeed, this is equivalent to the statement that $H(l,q)$ behaves as

$$H(l,q) \sim \frac{\delta(l) - \delta(l_1) - \delta(l_2)}{q} + \mathcal{O}(q^0)\,, \tag{10.14}$$

for $q \to 0$. To prove this, let us first calculate the monodromy of $\partial_y^2 + t(y)$ around the pair of points z_1 and z_2 as function of the parameters q and

$D := qH$. It is straightforward to show that the differential equation $(\partial_y^2 + t(y))g(y) = 0$ will have a solution of the form

$$g(y) = y^\nu \sum_{l=0}^{\infty} y^l \, g_l + \mathcal{O}(q)\,, \tag{10.15}$$

provided that ν is one of the two solutions of

$$\nu(\nu - 1) + \delta(l_1) + \delta(l_2) + D = 0 + \mathcal{O}(q)\,. \tag{10.16}$$

The solution (10.15) has diagonal monodromy $e^{2\pi i\nu}$ around $(z_1, z_2) \equiv (0, q)$. Note that ν and l are related as $\nu = \frac{1}{2} + i\frac{l}{4\pi}$. The Equation (10.14) follows.

A more detailed analysis of the solutions to the differential equation $\partial_y^2 + t(y)$ shows that the expansion of the function $\mathcal{W}(l, q)$ in powers of q is fully defined by (10.12) combined with the boundary condition (10.13) once $\mathcal{W}_0(l)$ is specified.

10.3.2. Genus zero, four punctures, constant term. In order to determine $\mathcal{W}_0(l)$ let us recall that the Darboux variable k conjugate to l is obtained from $\mathcal{W}(l, q)$ as

$$k = 4\pi i \frac{\partial}{\partial l}\mathcal{W}(l, q)\,. \tag{10.17}$$

Having fixed a definition for the coordinate k by means of (2.20), we should therefore be able to determine $\mathcal{W}(l, q)$ up to a constant, including the precise form of $\mathcal{W}_0(l)$. The result is the following:

Claim 1. *The function $\mathcal{W}_0(l)$ characterizing the asymptotics (10.13) of $\mathcal{W}_0(l, q)$ is explicitly given as*

$$\mathcal{W}_0(l) = \frac{1}{2}(C^{\rm cl}(l_4, l_3, l) + C^{\rm cl}(-l, l_2, l_1))\,, \tag{10.18}$$

where $C^{\rm cl}(l_3, l_2, l_1)$ is explicitly given as

$$\begin{aligned} C^{\rm cl}(l_3, l_2, l_1) = {} & \left(\frac{1}{2} + \frac{i}{4\pi}(l_3 + l_2 + l_1)\right)\log(\pi\mu) - \sum_{i=1}^{3} \Upsilon_{\rm cl}\big(1 + \tfrac{i}{2\pi} l_i\big) \\ & + \sum_{s_1, s_2 = \pm} \Upsilon_{\rm cl}\big(\tfrac{1}{2} + \tfrac{i}{4\pi}(l + s_1 l_1 + s_2 l_2 + l_3)\big) \end{aligned} \tag{10.19}$$

with function $\Upsilon_{\rm cl}(x)$ *defined as*

$$\Upsilon_{\rm cl}(x) = \int_{1/2}^{x} du\ \log\frac{\Gamma(u)}{\Gamma(1-u)}\,. \tag{10.20}$$

The proof is described in Appendix E. A formula for $\mathcal{W}_0(l)$ that is very similar (but not quite identical) to (10.18) was previously proposed in [NRS].

Let us note that the function $C^{\rm cl}(l_3, l_2, l_1)$ coincides with the classical Liouville action for the three-punctured sphere [ZZ95].

10.3.3. Genus one, one puncture. It remains to discuss the case $C = C_{1,1}$. The discussion is similar, the results are the following. The opers on $C_{1,1}$ can be represented in the form $\partial_y^2 + t(y)$, where

$$t(y) = \delta(l_0)\,\wp(\ln y) + H(l,q)\,, \tag{10.21}$$

with $\wp(w)$ being the Weierstrass elliptic function

$$\wp(w) = \frac{1}{w^2} + \sum_{(n,m)\neq(0,0)}\left(\frac{1}{(w-\pi n - m\pi\tau)^2} - \frac{1}{(\pi n + m\pi\tau)^2}\right). \tag{10.22}$$

$\mathcal{W}(l,q)$ behaves as

$$\mathcal{W}(l,q) = -\delta\log q + \mathcal{W}_0(l) + \mathcal{O}(q)\,, \tag{10.23}$$

where

$$\mathcal{W}_0(l) = \frac{1}{2}C^{\rm cl}(l,-l,l_0)\,. \tag{10.24}$$

As before we note that (10.3), (10.4) determine $\mathcal{W}(l,q)$ only up to a constant, Equation (10.24) holds for a particular convention fixing this constant.

10.4. The real slice

We had pointed out earlier that the monodromy map induces a map $\rho : \mathcal{P}(C) \to \mathcal{M}^{\mathbb{C}}_{\rm char}(C)$ that is locally biholomorphic. A natural real slice in $\mathcal{M}^{\mathbb{C}}_{\rm char}(C)$ is $\mathcal{M}^{\mathbb{R}}_{\rm char}(C)$, which contains a connected component isomorphic to $\mathcal{M}^{0}_{\rm flat}(C)$. The corresponding slice in $\mathcal{P}(C)$ can locally be described by a family of opers $t(y;q,\bar{q})$ that is real analytic in $q,\bar{q}$.

Let us consider coordinates $q,\bar{q}$ introduced using a pants decomposition σ. We will furthermore assume that the local coordinates y are coming from the projective structure naturally associated to the pants decomposition σ.

There exists a real analytic function $S_\sigma(q,\bar{q})$ on $\mathcal{T}(C)$ such that

$$(10.25)\qquad t(y) = \sum_{r=1}^{h} h_r\,\vartheta_r\,, \qquad h_r = -\frac{\partial}{\partial q_r} S_\sigma(q,\bar{q})\,.$$

The function $S_\sigma(q,\bar{q})$ is related to the generating function $\mathcal{W}_{\sigma,\sigma}(l,q)$ as

$$(10.26)\qquad S_\sigma(q,\bar{q}) = 2\,\mathrm{Re}(\mathcal{W}_{\sigma,\sigma}(l(q,\bar{q}),q))\,,$$

where $l_e(q,\bar{q})$ is the length of the geodesic γ_e in the hyperbolic metric which corresponds to the complex structure specified by $q,\bar{q}$.

It is clear that the function $S_\sigma(q,\bar{q})$ represents a hermitian metric in a (generically) projective line bundle $\mathcal{E}_c$. This means more concretely that the mapping class group acts on $S_\sigma(q,\bar{q})$ as follows

$$(10.27)\qquad S_\sigma(\mu.q,\mu.\bar{q}) = |f_\mu^c(q)|^2\, S_\sigma(q,\bar{q})\,, \qquad \mu\in\mathrm{MCG}(C)\,.$$

The functions $f_\mu^c(q)$ are transition functions of the projective line bundle $\mathcal{E}_c$.

The function $S_\sigma(q,\bar{q})$ is nothing but the classical Liouville action. It should be possible to give a direct proof of this claim along the lines of [ZT87a, ZT87b, TT03]. It will follow indirectly from the relations with quantum Liouville theory to be described later.

10.5. Scheme dependence

In the above we have given an unambiguous definition of the generating functions $\mathcal{W}_{\sigma,\sigma}(l,q)$. One should keep in mind that the definition was based on the use of the projective structures that were defined using the gluing constructions of Riemann surfaces C. This corresponds to choosing particular local sections $t_0(y,q)$ of $\mathcal{P}(C)$ in the definition of the coordinates h_r via (9.4).

One may, of course, consider other choices for the local sections $t_0(y,q)$ than the one chosen for convenience above. This would modify the coordinates h_r by functions of q, leading to a modification of $\mathcal{W}(l,q)$ by some function $\mathcal{W}_0(q)$ that depends on q and parameterically on c. The dependence of $\mathcal{W}(l,q)$ on the variables l would be unaffected.

11. Quantization

Summary:

- Functions on $\mathcal{P}(C) \rightsquigarrow$ Ring of holomorphic differential operators on $\mathcal{T}_{g,n}$.
- Quantization of twisted cotangent bundle $T_c^*\mathcal{M}(C) \rightsquigarrow$ Eigenstates v_q of operators q_e: Section of holomorphic vector bundle $\mathcal{W}(C) \otimes \mathcal{E}_c$, where $\mathcal{W}(C)$: flat projective vector bundle defined from repr. of MCG(C) defined in Part II.
- Quantization of generating functions $\mathcal{W}_\sigma(l,q) \rightsquigarrow$ matrix elements $\mathcal{F}_l^\sigma(q) \equiv {}_\sigma\langle\, v_q\,,\, \delta_l^\sigma\,\rangle_\sigma$.
- Results of Parts II and III $\rightsquigarrow$ Riemann-Hilbert type problem for $\mathcal{F}_l^\sigma(q)$.

11.1. Algebra of functions - representations

11.1.1. We want to describe the quantization of the spaces $\mathcal{M}^0_{\rm flat}(C) \simeq \mathcal{T}(C)$ in a way that makes explicit use of the complex structure on these spaces. In order to do this, we find it convenient to represent $\mathcal{M}^0_{\rm flat}(C)$ as a connected component of the real slice $\mathcal{M}_{\rm flat}(C)$ within $\mathcal{M}^{\mathbb{C}}_{\rm flat}(C)$. As a preliminary, we are going to explain how such a description works in a simple example.

Let us consider $\mathbb{R}^2$ with real coordinates x and p and Poisson bracket $\{x,p\} = 1$. Canonical quantization will produce operators p and x with commutation relations $[\mathsf{p},\mathsf{x}] = -i\hbar$, which can be realized on a space of functions $\psi(x)$ of a real variable x. This is a simple analog of the quantization scheme discussed in Part II.

We now want to use a quantization scheme that makes explicit use of the complex structure of $\mathbb{R}^2 \simeq \mathbb{C}$. In order to do this let us consider $\mathbb{R}^2$ as a real slice of the space $\mathbb{C}^2$. One could, of course, use complex coordinates x and p for $\mathbb{C}^2$ with Poisson bracket $\{x,p\} = 1$, and describe the real slice $\mathbb{R}^2$ by the requirement $x^* = x$, $p^* = p$. Alternatively one may use the complex analytic coordinates $a = x + ip$ and $a' = x - ip$ for $\mathbb{C}^2$ which have Poisson bracket $\{a,a'\} = -2i$. The real slice $\mathbb{R}^2$ is then described by the equation $a' = a^*$ which expresses a' as a non-holomorphic function of the complex analytic coordinate a on the real slice $\mathbb{R}^2$.

Quantization of the Poisson bracket $\{a,a'\} = -2i$ gives operators a, a' which satisfy $[\mathsf{a},\mathsf{a}'] = 2\hbar$. This algebra can be represented on functions $\Psi(a)$ in terms of the holomorphic differential operator $\frac{\partial}{\partial a}$. If a and a' were independent variables, we could also realize the algebra $[\bar{\mathsf{a}}, \bar{\mathsf{a}}'] = -2\hbar$ generated by the hermitian conjugate operators on non-holomorphic functions $\Psi(a) \equiv \Psi(a, \bar{a})$.

But in the case of interest, a' is a non-holomorphic function of a by restriction to the real slice. We want to point out that it is then natural to realize $[\mathsf{a}, \mathsf{a}'] = 2\hbar$ on *holomorphic* functions $\Psi(a)$, thereby making explicit use of the complex structure on the phase space $\mathbb{R}^2$. There is a natural isomorphism with the representation defined on functions $\psi(x)$ of a real variable x which can be described as an integral transformation of the form

$$\Psi(a) = \int dx \; \langle a|x\rangle \Psi(x)\,, \tag{11.1}$$

where the kernel $\langle a|x\rangle$ is the complex conjugate of the wave-function $\psi_a(x) = \langle x|a\rangle$ of an eigenstate of the operator $\mathsf{a} = \mathsf{x} + i\mathsf{p}$ with eigenvalue a.

The representation of the Hilbert space using *holomorphic* functions $\Psi(a)$ is known as the coherent state representation in quantum mechanics.

11.1.2. In the present case we regard the Darboux coordinates (l, k) as analogs of the coordinates (x, p), while the coordinates (q, h) take the role of (a, a^*). Both (k, l) and (q, h) form systems of Darboux coordinates for $\mathcal{T}(C)$. The coordinates q_r alone are complex analytic coordinates for $\mathcal{Q}(C)$, and the coordinates h_r are non-holomorphic functions $h_r = h_r(q, \bar{q})$ – this is in exact analogy to the case of (a, a^*). Important differences will follow from the fact that the relation between (q, h) and (l, k) is much more complicated than the relation between (x, p) and (a, a^*). It is no longer true that h_r is the complex conjugate of q_r.

Quantization is canonical on a purely algebraic level: We introduce a noncommutative algebra with generators $\hat{q} = (\hat{q}_1, \dots, \hat{q}_h)$ and $\hat{h} = (\hat{h}_1, \dots, \hat{h}_h)$ and relations

$$[\,\hat{h}_r\,,\,\hat{q}_s\,] = b^2\,\delta_{r,s}\,. \tag{11.2}$$

The resulting algebra is the natural quantization of the algebra of holomorphic functions on the cotangent bundle $T^*\mathcal{T}(C)$ which will be denoted as $\mathrm{Fun}_b(T^*\mathcal{T}(C))$.

There is an obvious realization of the algebra $\mathrm{Fun}_b(T^*\mathcal{T}(C))$ on functions $\Psi(q)$ locally defined on subsets of $\mathcal{T}(C)$. The generators $\hat{\mathsf{q}}_r$ corresponding to the coordinate q_r introduced in Section 9.1 are represented as operators of multiplication by q_r, and the generators $\hat{h}_r$ associated to the conjugate "momenta" h_r should be represented by the differential operators $\mathsf{h}_r \equiv b^2\partial_{q_r}$

in such a representation,

$$\mathsf{q}_r\Psi(q) = q_r\Psi(q)\,, \qquad \mathsf{h}_r\Psi(q) = b^2\,\frac{\partial}{\partial q_r}\,\Psi(q)\,. \tag{11.3}$$

The resulting representation should be seen as an analog of the coherent state representation of quantum mechanics.

11.1.3. As both (k,l) and (q,h) form systems of Darboux coordinates for $\mathcal{T}(C)$, we expect that there exists a unitary equivalence between the representations on functions $\psi(l)$ defined in Part II, and the representation on holomorphic functions $\Psi(q)$ we are constructing here. This means in particular that there should ultimately be a representation of the scalar product in $\mathcal{H}(C)$ within each of these representations

$$\langle\,\Psi\,,\,\Psi\,\rangle = \int d\mu(l)\;|\psi(l)|^2 = \int_{\mathcal{T}(C)} d\mu(q,\bar{q})\;|\Psi(q)|^2\,. \tag{11.4}$$

Normalizability of the wave-functions $\psi(q)$ will restrict both the appearance of singularities in the analytic continuation of $\psi(q)$ over all of $\mathcal{T}(C)$, and the behavior of $\psi(q)$ at the boundaries of $\mathcal{T}(C)$. In our case it is not apriori obvious how to identify a natural domain for the action of the operators (q,h) which represent $\mathrm{Fun}_b(T^*\mathcal{T}(C))$ on *holomorphic* wave-functions $\Psi(q)$. However, it is certainly natural to expect that $\Psi(q)$ has to be analytic on all of $\mathcal{T}(C)$. It will furthermore be necessary to demand that the behavior of $\Psi(q)$ at the boundaries of $\mathcal{T}(C)$ is "regular" in a sense that needs to be made more precise. A more precise description of the space of wave-functions that is relevant here will eventually follow from the results to be described below.

It is natural to introduce eigenstates v_q of the position operators q_r such that

$$\Psi(q) = \langle\,v_q\,,\,\Psi\,\rangle\,. \tag{11.5}$$

The definition of the coordinates q will in general require the consideration of a local patch $\mathcal{U}_\imath \subset \mathcal{T}(C)$. The corresponding wave-functions will be denoted as $\Psi_\imath(q) \equiv \langle v_q^\imath, \Psi\rangle$. When the coordinates q come from the gluing construction we will use the index σ instead of $\imath$.

11.1.4. Important further requirements are motivated by the fact that the cotangent bundle $T^*\mathcal{T}(C)$ descends to a twisted cotangent bundle over $T^*_c\mathcal{M}(C)$ for which coordinates like (q,h) represent local systems of coordinates. Recall that the coordinates $H^\imath$ and $H^\jmath$ associated to different patches

$\mathcal{U}_\imath$ and $\mathcal{U}_\jmath$ are related via (9.12), where

$$\chi_{\imath\jmath} = \frac{1}{2\pi i}\partial \log f_{\imath\jmath}^{c_{\rm cl}} \,. \tag{11.6}$$

The relation (9.12) has a natural quantum counterpart,

$$\sum_r dq_r^\imath \frac{\partial}{\partial q_r^\imath}\Psi_\imath(q) = \sum_r dq_r^\jmath \frac{\partial}{\partial q_r^\jmath}\Psi_\jmath(q) - \frac{1}{b^2}\,\chi^{\imath\jmath}\,, \tag{11.7}$$

which leads us to require that

$$\Psi_\imath(q) = f_{\imath\jmath}^{\mathbf{c}}(q)\Psi_\jmath(q)\,, \tag{11.8}$$

where the parameter $\mathbf{c}$ will be given by $c_{\rm cl}/b^2$ up to corrections of order b^2 that will be determined later.

The mapping class group MCG(C) acts by holomorphic transformations on $\mathcal{T}(C)$. We will use the notation $\mu.\tau$ for the image of a point $\tau \in \mathcal{T}(C)$ under $\mu \in$ MCG(C). We require that there is a representation of MCG(C) on $\mathcal{H}(C)$ which is represented on the wave-functions $\Psi_\sigma(q)$ naturally as

$$(\mathsf{M}_\mu\Psi)_{\mu.\sigma}(q) = \Psi_\sigma(\mu.q)\,, \quad \text{or} \quad \mathsf{M}_\mu^{-1}v_q^{\mu.\sigma} = v_{\mu.q}^\sigma\,. \tag{11.9}$$

This requirement should be understood as one of the properties defining the representations $\Psi_\sigma(q)$, or equivalently the eigenstates v_q^σ.

11.2. Relation between length representation and Kähler quantization

There should exist expansions of the form

$$\Psi_{\sigma,\sigma'}(q) = \int dl\,\langle\, v_q^\sigma\,,\, \delta_l^{\sigma'}\,\rangle\langle\, \delta_l^{\sigma'}\,,\, \Psi\,\rangle \equiv \int dl\; \mathcal{F}_{\sigma,\sigma'}(l,q)\,\psi_{\sigma'}(l)\,. \tag{11.10}$$

The requirement (11.10) introduces key objects, the eigenfunctions $\Psi_l^{\sigma,\sigma'}(q)$ $\equiv \mathcal{F}_{\sigma,\sigma'}(l,q)$ of the length operators. We will mostly restrict attention to the diagonal case $\sigma \equiv \sigma'$ in the following, and denote $\Psi_l^\sigma(q) \equiv \Psi_l^{\sigma,\sigma}(q)$.

The wave-functions $\Psi_l^{\sigma_1}(q)$ and $\Psi_l^{\sigma_2}(q)$ associated to different patches $\mathcal{U}_{\sigma_1}$ and $\mathcal{U}_{\sigma_2}$ are related by an integral transformation of the following form:

$$\Psi_{l_1}^{\sigma_1}(q) = f_{\sigma_1\sigma_2}^{\mathbf{c}}(q)\int dl_2\; U_{\sigma_1\sigma_2}(l_1,l_2)\Psi_{l_2}^{\sigma_2}(q)\,, \tag{11.11}$$

as follows from

$$\begin{aligned}\langle\, v_q^{\sigma_1}\,,\,\delta_{l_1}^{\sigma_1}\,\rangle &= f^{\mathbf{c}}_{\sigma_1\sigma_2}(q)\,\langle\, v_q^{\sigma_2}\,,\,\delta_{l_1}^{\sigma_1}\,\rangle \\ &= f^{\mathbf{c}}_{\sigma_1\sigma_2}(q)\,\langle\, v_q^{\sigma_2}\,,\,\mathsf{U}_{\sigma_1\sigma_2}\delta_{l_1}^{\sigma_2}\,\rangle\,.\end{aligned} \tag{11.12}$$

Let us now consider the wave-function $\Psi_l^{\mu.\sigma}(q) := \langle\, v_q^{\mu.\sigma}\,,\,\delta_l^{\mu.\sigma}\,\rangle$, where $\mu \in \mathrm{MCG}(C)$. On the one hand,

$$\begin{aligned}\Psi_l^{\mu.\sigma}(q) &= \langle\, v_q^{\mu.\sigma}\,,\,\delta_l^{\mu.\sigma}\,\rangle = \langle\, v_q^{\mu.\sigma}\,,\,\mathsf{M}_\mu\delta_l^{\sigma}\,\rangle \\ &\overset{(11.9)}{=} \langle\, v_{\mu.q}^{\sigma}\,,\,\delta_l^{\sigma}\,\rangle = \Psi_l^{\sigma}(\mu.q)\,.\end{aligned}$$

In the first line we have beeen using that $\mathsf{M}_\mu = \mathsf{U}_{\mu.\sigma,\sigma}$, in passing to the second the unitarity of M_μ and our requirement (11.9). Another way of representing the wave-function $\langle\, v_q^{\mu.\sigma}\,,\,\delta_l^{\mu.\sigma}\,\rangle$ is found by specializing (11.11) to the case that $\sigma_1 = \mu.\sigma$, and $\sigma_2 = \sigma$. Taken together we find

$$\Psi_{l_1}^{\sigma}(\mu.q) = f^{\mathbf{c}}_{\mu.\sigma,\sigma}(q)\int dl_2\; M_\mu(l_1,l_2)\Psi_{l_2}^{\sigma}(q)\,. \tag{11.13}$$

Note that one may read (11.13) as expression of the fact that the wave-functions $\Psi_l^\sigma(q)$ represent sections of the holomorphic vector bundle $\mathcal{V}(C) := \mathcal{W}(C)\otimes\mathcal{E}_{\mathbf{c}}$ over $\mathcal{M}(C)$, where $\mathcal{W}(C)$ is the projective local system defined by the projective representation of the mapping class group constructed in Part II. For the reader's convenience we have reviewed the notion of a projective local system in Appendix F. It is important that the holomorphic bundle $\mathcal{V}(C)$ of Hilbert spaces over $\mathcal{M}(C)$ is an ordinary vector bundle as opposed to a projective one, as the latter can not have any section.

The kernels $M_\mu(l_1,l_2)$ in (11.13) have been defined in Part II. The classical limits of $-b^2\log M_\mu(l_1,l_2)$ may be identified with the generating functions $F_{\mu.\sigma,\sigma}(l_1,l_2)$ that appear in (10.10). The transition functions $f^{\mathbf{c}}_{\mu.\sigma,\sigma}(q)$ in (11.13) may then be identified with $e^{2\pi i g^{\mathbf{c}}_{\mu.\sigma,\sigma}(q)}$, with $g^{c}_{\sigma_2\sigma_1}(q)$ being the transition function of $\mathcal{E}_c$ defined via (10.10).

Having specified the data $M_\mu(l_1,l_2)$ and $f^{\mathbf{c}}_{\mu.\sigma,\sigma}(q)$ defining the vector bundle $\mathcal{V}(C)$, one may regard (11.11) as definition of a Riemann-Hilbert type problem for the wave-functions $\Psi_l^\sigma(q)$. If $\mathcal{V}(C)$ were a projective vector bundle, the Riemann-Hilbert problem (11.13) would not have any solution. The fact that it has a solution for

$$\mathbf{c} = \frac{c}{b^2}\,, \qquad c = c_{\mathrm{cl}} + 13b^2 + 6b^4\,, \tag{11.14}$$

will immediately follow from the relation with Liouville theory to be exhibited in the next section. Note that

$$\mathbf{c} = 13 + 6\left(b^2 + b^{-2}\right) , \tag{11.15}$$

coincides with the expression of the central extension found in Eq. (8.22) above.

11.3. Uniqueness and asymptotics

Uniqueness of the solution to the Riemann-Hilbert problem defined above can then be shown by a variant of the argument used in [T03b]: Any two solutions of the Riemann-Hilbert problem differ by multiplication with a meromorphic function with possible poles at the boundary $\partial\mathcal{M}(C)$ of $\mathcal{M}(C)$. In order to fix this ambiguity one needs to fix the asymptotic behavior at $\partial\mathcal{M}(C)$. Let us consider the component of $\partial\mathcal{M}(C)$ where the gluing parameter q_e vanishes. We need to distinguish the cases $C_e \simeq C_{0,4}$ and $C_e \simeq C_{1,1}$, as before.

Let us consider the case $C_e \simeq C_{0,4}$. Note that the functions $\mathcal{F}_\sigma(l,q) \equiv \Psi_l^\sigma(q)$ represent the quantum counterparts of $e^{-\frac{1}{b^2}\mathcal{W}_\sigma(l,q)}$,

$$\mathcal{W}_\sigma(l,q) = -\lim_{b\to 0} b^2 \log \mathcal{F}_\sigma(l,q)\,. \tag{11.16}$$

In view of the asymptotic behavior (10.13) and (10.23) of $\mathcal{W}_\sigma(l,q)$ it is therefore natural to require that the functions $\mathcal{F}_\sigma(l,q)$ should have asymptotics of the form

$$\log \mathcal{F}_\sigma(l,q) = (\Delta(l_e) - \Delta(l_1) - \Delta(l_2)) \log q_e + \mathcal{F}_{0,\sigma}(l) + \mathcal{O}(q_e)\,. \tag{11.17}$$

The functions $\Delta(l)$ should coincide with $\frac{1}{b^2}\delta(l)$ up to possible quantum corrections, $b^2\Delta(l) = \delta(l) + \mathcal{O}(b^2)$. The form (11.17) of the asymptotic behavior is equivalent to the validity of a quantized version of the relation (10.14) which takes the following form

$$\Big(b^2\big[(1-\nu)q_e\partial_{q_e} + \nu\partial_{q_e}q_e\big] + \delta(l_1) + \delta(l_2) - \delta(l)\Big)\mathcal{F}_\sigma(l,q) = 0\,. \tag{11.18}$$

On the left hand side we have parameterized the ambiguity in the operator ordering using the parameter $\nu \in [0,1]$. Consistency with the realization of the B-move, given in (6.23), requires that $\nu = \frac{1}{2} + \frac{b^2}{4}$. This determines the possible quantum corrections in the definition of the function $\Delta(l)$ to be

$\Delta(l) = \frac{1}{b^2}\delta(l) + \nu$, which gives

$$\Delta(l) = \left(\frac{l}{4\pi b}\right)^2 + \frac{Q^2}{4}\,, \qquad Q = b + b^{-1}\,. \tag{11.19}$$

In a very similar way one may treat the case $C_e \simeq C_{1,1}$. Having fixed the asymptotics, the solution to the Riemann-Hilbert problem is unique up to multiplication by a constant.

11.4. Scheme dependence

We had noted above in Subsection 10.5 that the definition of the observables h_r depends on the choice of a projective structure. A similar issue must therefore be found in the quantum theory concerning the definition of the operators h_r. We have to allow for redefinitions of the operators h_r that correspond to redefinitions of the eigenstates v_q by multiplicative factors which may depend on q.

This freedom is physically irrelevant in the following sense. What is physically relevant are normalized expectation values of observable like

$$\langle\!\langle\, \mathsf{O}\, \rangle\!\rangle_q := \frac{\langle\, v_q\,, \mathsf{O}\, v_q\,\rangle}{\langle\, v_q\,, v_q\,\rangle}\,. \tag{11.20}$$

It is clear that such expectation values are unaffected by redefinitions of the eigenstates v_q by multiplicative, q-dependent factors. This is how the scheme dependence discussed in Section 3.3 manifests itself in the quantum theory of $\mathcal{M}^0_{\rm flat}(C)$.

12. Relation to quantum Liouville theory

We will now argue that the conformal field theory called Liouville theory is mathematically best interpreted as the harmonic analysis on Teichmüller spaces, which is another name for the quantum theory defined in the previous section. This will partly explain why the Riemann-Hilbert type problems defined in Sections 4 and 11 are solved by Liouville theory.

12.1. Virasoro conformal blocks

12.1.1. Definition of the conformal blocks. The Virasoro algebra Vir_c has generators L_n, $n \in \mathbb{Z}$, and relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\mathbf{c}}{12}n(n^2-1)\delta_{n+m,0}. \tag{12.1}$$

Let C be a Riemann surface C with n marked points $P_1, \ldots, P_n$. At each of the marked points P_r, $r = 1, \ldots, n$, we choose local coordinates w_r, which vanish at P_r. We will fix a projective structure on C and assume that the patches around the points P_r are part of an atlas defining the projective structure. We associate highest weight representations $\mathcal{V}_r$, of $\mathrm{Vir}_{\mathbf{c}}$ to P_r, $r = 1, \ldots, n$. The representations $\mathcal{V}_r$ are generated from highest weight vectors e_r with weights Δ_r.

The conformal blocks are then defined to be the linear functionals $\mathcal{F}$: $\mathcal{V}_{[n]} \equiv \otimes_{r=1}^n \mathcal{V}_r \to \mathbb{C}$ that satisfy the invariance property

$$\mathcal{F}_C(T[\chi] \cdot v) = 0 \quad \forall v \in \mathcal{R}_{[n]}, \quad \forall \chi \in \mathfrak{V}_{\mathrm{out}}, \tag{12.2}$$

where $\mathfrak{V}_{\mathrm{out}}$ is the Lie algebra of meromorphic differential operators on C which may have poles only at $P_1, \ldots, P_n$. The action of $T[\chi]$ on $\otimes_{r=1}^n \mathcal{R}_r \to \mathbb{C}$ is defined as

$$\begin{aligned} T[\chi] &= \sum_{r=1}^{n} \mathrm{id} \otimes \cdots \otimes \underset{\text{(r−th)}}{L[\chi^{(r)}]} \otimes \cdots \otimes \mathrm{id}, \\ L[\chi^{(r)}] &:= \sum_{k \in \mathbb{Z}} L_k \chi_k^{(r)} \in \mathrm{Vir}_{\mathbf{c}}, \end{aligned} \tag{12.3}$$

where $\chi_k^{(r)}$ are the coefficients of the Laurent expansions of χ at the points $P_1, \ldots P_n$,

$$\chi(w_r) = \sum_{k \in \mathbb{Z}} \chi_k^{(r)} \, w_r^{k+1} \, \partial_{w_r} \in \mathbb{C}((w_r))\partial_{w_r} \,. \tag{12.4}$$

It can be shown that the central extension vanishes on the image of the Lie algebra $\mathfrak{V}_{\mathrm{out}}$ in $\bigoplus_{r=1}^n \mathrm{Vir}_{\mathbf{c}}$, making the definition consistent. We may refer to [AGMV, W88] for early discussions of this definition in the physics literature, and to [BF] for a mathematically rigorous treatment.

The vector space of conformal blocks associated to the Riemann surface C with representations $\mathcal{V}_r$ associated to the marked points P_r, $r = 1, \ldots, n$ will be denoted as $\mathsf{CB}(\mathcal{V}_{[n]}, C)$. It is the space of solutions to the defining invariance conditions (12.2). The space $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ is infinite-dimensional in general. Considering the case $n = 1$, $\Delta_1 = 0$ and $g > 1$, for example, one may see this more explicitly by noting that for P_1 in generic position[10] one may find a basis for $\mathfrak{V}_{\rm out}$ generated by vector fields which have a pole at P_1 of order higher than $3g - 3$. This follows from the Weierstrass gap theorem. The conditions (12.2) will then allow us to express the values of $\mathcal{F}$ on arbitrary vectors in $\mathcal{V}_1$ in terms of the values

$$\mathcal{F}\big(L_{3-3g}^{k_{3g-3}} \cdots L_{-1}^{k_1} e_1\big)\,, \qquad k_1, \dots, k_{3g-3} \in \mathbb{Z}^{>0}\,, \tag{12.5}$$

were e_1 is the highest weight vector of $\mathcal{V}_1$. We note that $\mathcal{F}$ is completely defined by the values (12.5). $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ is therefore isomorphic as a vector space to the space of *formal* power series in $3g - 3$ variables.

12.1.2. Conformal blocks as expectation values of chiral vertex operators. Let us also introduce the notation

$$\mathcal{Z}^{\rm L}(\mathcal{F}, C) = \mathcal{F}(e_1 \otimes \cdots \otimes e_n)\,, \tag{12.6}$$

for the value of $\mathcal{F}$ on the product of highest weight vectors. $\mathcal{Z}^{\rm L}(\mathcal{F}, C)$ can be interpreted as a chiral "partition function" from a physicist's point of view. It may alternatively be interpreted as an expectation value of a product of n chiral primary fields inserted into a Riemann surface C. This interpretation may be expressed using the notation

$$\mathcal{Z}^{\rm L}(\mathcal{F}, C) = \langle\, \Phi_n(z_n) \cdots \Psi_1(z_1)\, \rangle_{\mathcal{F}}\,, \tag{12.7}$$

with z_r being (local) coordinates of the points P_r. The state-operator correspondence associates chiral vertex operators $\Phi_r(v_r|z_r)$ to arbitrary vectors $v_r \in \mathcal{V}_r$. The vertex operators $\Phi_r(v_r|z_r)$ are called the descendants of $\Phi_r(z_r)$. The value $\mathcal{F}(v_1 \otimes \cdots \otimes v_n)$ is therefore identified with the expectation value

$$\mathcal{F}(v_1 \otimes \cdots \otimes v_n) = \langle\, \Phi_n(v_n|z_n) \cdots \Psi_1(v_1|z_1)\, \rangle_{\mathcal{F}}\,. \tag{12.8}$$

There are generically many different ways to "compose" chiral vertex operators. The necessary choices are encoded in the choice of $\mathcal{F}$ in a way that will become more clear in the following.

[10]We assume that P_1 is not a Weierstrass point.

12.1.3. Deformations of the complex structure of C. A key point that needs to be understood about spaces of conformal blocks is the dependence on the complex structure of C. There is a canonical way to represent infinitesimal variations of the complex structure on the spaces of conformal blocks. By combining the definition of conformal blocks with the so-called "Virasoro uniformization" of the moduli space $\mathcal{M}_{g,n}$ of complex structures on $C = C_{g,n}$ one may construct a representation of infinitesimal motions on $\mathcal{M}_{g,n}$ on the space of conformal blocks.

The "Virasoro uniformization" of the moduli space $\mathcal{M}_{g,n}$ may be formulated as the statement that the tangent space $T\mathcal{M}_{g,n}$ to $\mathcal{M}_{g,n}$ at C can be identified with the double quotient

(12.9)
$$T\mathcal{M}_{g,n} = \Gamma(C \setminus \{P_1, \ldots, P_n\}, \Theta_C) \Big\backslash \bigoplus_{k=1}^{n} \mathbb{C}(\!(w_k)\!)\partial_k \Big/ \bigoplus_{k=1}^{n} w_k \mathbb{C}[[w_k]]\partial_k,$$

where $\mathbb{C}(\!(w_k)\!)$ and $\mathbb{C}[[w_k]]$ are the spaces of formal Laurent and Taylor series respectively, and $\Gamma(C \setminus \{P_1, \ldots, P_n\}, \Theta_C)$ is the space of vector fields that are holomorphic on $C \setminus \{P_1, \ldots, P_n\}$.

Given a tangent vector $\vartheta \in T\mathcal{M}_{g,n}$, it follows from the Virasoro uniformization (12.9) that we may find an element η_ϑ of $\bigoplus_{k=1}^{n} \mathbb{C}(\!(w_k)\!)\partial_k$, which represents ϑ via (12.9). Let us then consider $\mathcal{F}(T[\eta_\vartheta]v)$ with $T[\eta]$ being defined in (12.3) in the case that the vectors v_k are the highest weight vectors e_k for all $k = 1, \ldots, n$. (12.9) suggests to define the derivative $\delta_\vartheta \mathcal{F}(v)$ of $\mathcal{F}(v)$ in the direction of $\vartheta \in T\mathcal{M}_{g,n}$ as

$$\delta_\vartheta \mathcal{F}(v) := \mathcal{F}(T[\eta_\vartheta]v), \tag{12.10}$$

Dropping the condition that v is a product of highest weight vectors one may use (12.10) to define $\delta_\vartheta \mathcal{F}$ in general. Indeed, it is well-known that (12.10) leads to the definition of a canonical connection on the space $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ of conformal blocks which is projectively flat, see e.g. [BF] for more details.

There is no hope to integrate the canonical connection on $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ to produce a bundle over $\mathcal{M}(C)$ with fiber at a Riemann surface C being $\mathsf{CB}(\mathcal{V}_{[n]}, C)$, in general.

The first problem is that the connection defined by (12.10) is not flat, but only projectively flat. It can only define a connection on the projectivized space $\mathbb{P}\mathsf{CB}(\mathcal{V}_{[n]}, C)$, in general. For the readers convenience we have gathered some basic material on connections on bundles of projective spaces in Appendix F. As we will see in a little more detail later, one may trivialize the curvature at least locally, opening the possibility to integrate (12.10) at least in some local patches $\mathcal{U} \subset \mathcal{M}(C)$.

The other problem is simply that $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ is way too big, as no growth conditions whatsoever are imposed on the values (12.5) for general elements $\mathcal{F} \in \mathsf{CB}(\mathcal{V}_{[n]}, C)$. One needs to find interesting subspaces of $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ which admit useful topologies.

We will later even be able to identify natural Hilbert-subspaces $\mathsf{HCB}(\mathcal{V}_{[n]}, C)$ of $\mathsf{CB}(\mathcal{V}_{[n]}, C)$. The Hilbert-subspaces $\mathsf{HCB}(\mathcal{V}_{[n]}, C)$ will be found to glue into a bundle of projective vector spaces $\mathcal{W}(\mathcal{V}_{[n]}, C)$ over $\mathcal{M}(C)$ with connection defined via (12.10) – this is the best possible situation one can hope for in cases where the spaces of conformal blocks are infinite-dimensional.

12.1.4. Propagation of vacua. The vacuum representation $\mathcal{V}_0$ which corresponds to $\Delta_r = 0$ plays a distinguished role. If $\Phi_0(v_0|w_0)$ is the vertex operator associated to the vacuum representation, we have

$$\Phi_0(e_0|w_0) = \mathrm{id}, \quad \Phi_0(L_{-2}e_0|w_0) = T(w_0), \tag{12.11}$$

where $T(z)$ is the energy-momentum tensor. It can be shown that the spaces of conformal blocks with and without insertions of the vacuum representation are canonically isomorphic, see e.g. [BF] for a proof. The isomorphism between $\mathsf{CB}(\mathcal{V}_0 \otimes \mathcal{V}_{[n]}, C_{g,n+1})$ and $\mathsf{CB}(\mathcal{V}_{[n]}, C_{g,n})$ is simply given by evaluation at the vacuum vector $e_0 \in \mathcal{V}_0$

$$\mathcal{F}'(e_0 \otimes v) \equiv \mathcal{F}(v), \qquad v \in \mathcal{V}_{[n]}\,, \tag{12.12}$$

as is also suggested by (12.11). This fact is often referred to as the "propagation of vacua".

One may then define the expectation value of the energy momentum tensor defined by a fixed element $\mathcal{F}$ as follows

$$T_{\mathcal{F}}(w_0) \equiv \langle\!\langle\, T(w_0)\,\rangle\!\rangle_{\mathcal{F}} := \mathcal{F}'(L_{-2}e_0 \otimes v)\,/\,\mathcal{F}(v)\,. \tag{12.13}$$

We are assuming that the local coordinate w_0 is part of an atlas defining the chosen projective structure on C. It follows that $T_{\mathcal{F}}(w_0)$ transforms like a quadratic differential when going from one patch of this atlas to another.

The invariance property (12.2) allows us to rewrite $\mathcal{F}'(L_{-2}e_0 \otimes v)$ in the form

$$\mathcal{F}'(L_{-2}e_0 \otimes v) = \mathcal{F}'(e_0 \otimes \vartheta_{w_0} v)\,, \tag{12.14}$$

with $\vartheta_{w_0} = T[\chi_{w_0}]$, for a vector field χ_{w_0} that has a pole at w_0. We may then use (12.12) to write $\mathcal{F}'(e_0 \otimes \vartheta_{w_0} v) = \mathcal{F}(\vartheta_{w_0} v)$. It follows that $T_{\mathcal{F}}(w_0)$ can be expressed in terms of $\mathcal{F}$ as

$$T_{\mathcal{F}}(w_0) = \mathcal{F}(\vartheta_{w_0} v)/\mathcal{F}(v)\,. \tag{12.15}$$

Recalling the definition (12.10), we observe that that the canonical connection can be characterized in terms of the expectation value $T_{\mathcal{F}}(w_0)$.

12.1.5. Parallel transport. Note that the value $\mathcal{F}(\vartheta_{w_0} v)$ in (12.15), by definition, represents the action of a differential operator $\mathcal{T}_{w_0}$ corresponding to a tangent vector to $\mathcal{M}(C)$ on $\mathcal{F}$. This statement may be expressed in the form of a differential equation for $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, C)$

$$\mathcal{T}_{w_0}\, \mathcal{Z}^{\mathrm{L}}(\mathcal{F}, C) = T_{\mathcal{F}}(w_0)\, \mathcal{Z}^{\mathrm{L}}(\mathcal{F}, C)\,. \tag{12.16}$$

The differential equation (12.16) may be re-written using local coordinates $q = (q_1, \dots, q_h)$ for $\mathcal{T}(C)$ whose variation is described by means of Beltrami-differentials $(\mu_1, \dots, \mu_h)$ as

$$\big[\partial_{q_r} + \mathcal{A}_r(\mathcal{F}, q)\big]\, \mathcal{Z}^{\mathrm{L}}(\mathcal{F}, C) = 0\,, \qquad \mathcal{A}_r(\mathcal{F}, q) := \int_C \mu_r T_{\mathcal{F}}\,. \tag{12.17}$$

Our aim is to use (12.17) to construct a family $\mathcal{F}_q$ of conformal blocks over a neighborhood $\mathcal{U}$ of $\mathcal{M}(C)$. We first need to ensure that the partial derivatives $\frac{\partial}{\partial q_r}$ whose action is defined via (12.17) do indeed commute. This amounts to the trivialization of the curvature of the canonical connection within $\mathcal{U}$.

One way do this concretely uses the atlas of local coordinates produced by the gluing construction of Riemann surfaces. One may consider Beltrami-differentials μ_r which are compactly supported in non-intersecting annular regions A_r on C. Equation (12.17) then describes the variations of the conformal blocks with respect to the coordinates q_r for $\mathcal{T}(C)$ defined by the gluing construction.

Let us assume that $\mathcal{F}$ is such that (12.17) can be integrated to define a function $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, q)$ in a neighborhood of a point in $\mathcal{M}$ represented by the surface C. Note that the Taylor expansion of $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, q)$ is completely defined by the conformal block $\mathcal{F} \in \mathsf{CB}(\mathcal{V}_{[n]}, C)$. Derivatives of $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, q)$ are related to the values $\mathcal{F}(T[\eta_\vartheta]v)$ via (12.10). These values can be computed in terms of the values (12.5) which characterize $\mathcal{F}$ by using the defining invariance condition (12.2). Conversely let us note that the values (12.5) characterizing a conformal block can be computed from the derivatives of $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, q)$ via (12.10).

It may not be possible to integrate (12.17) for arbitrary $\mathcal{F} \in \mathsf{CB}(\mathcal{V}_{[n]}, C)$ as the numbers (12.5) which characterize $\mathcal{F}$ may grow too quickly. We will denote the subspace of $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ spanned by the conformal blocks $\mathcal{F}$ for which (12.17) can locally be integrated to an analytic function $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, q)$ by $\mathsf{CB}^{\mathrm{an}}_{\mathrm{loc}}(\mathcal{V}_{[n]}, C)$.

Let us stress that for any given function $\mathcal{Z}^{\mathrm{L}}(q)$ which is analytic in a neighborhood of a point q_0 in $\mathcal{M}$ represented by the surface C one may define a family of conformal blocks $\mathcal{F}_q \in \mathsf{CB}^{\mathrm{an}}_{\mathrm{loc}}(\mathcal{V}_{[n]}, C_q)$ by using the Taylor expansion of $\mathcal{Z}^{\mathrm{L}}(q)$ around q to define the values (12.5) which characterize the elements $\mathcal{F}_q \in \mathsf{CB}(\mathcal{V}_{[n]}, C_q)$. The conformal blocks $\mathcal{F}$ in $\mathsf{CB}^{\mathrm{an}}_{\mathrm{loc}}(\mathcal{V}_{[n]}, C)$ are therefore in one-to-one correspondence with analytic functions $\mathcal{Z}^{\mathrm{L}}(q)$ defined locally in open subsets $\mathcal{U} \subset \mathcal{M}$.

12.1.6. Scheme dependence. In the definition of the conformal blocks we assumed that a projective structure on C had been chosen. This allows us in particular to define an expectation value $T_{\mathcal{F}}(w_0)$ of the energy-momentum tensor which transforms as a quadratic differential when going from one local coordinate patch on C to another. In order to define families of conformal blocks using the canonical connection one needs to have *families* of projective structures over local patches $\mathcal{U} \subset \mathcal{M}(C)$ that allow one to trivialize the curvature of the canonical connection locally in $\mathcal{U}$. Such families certainly exist, we had pointed out earlier that the families of projective structures defined by the gluing construction described in Subsection 9.3 do the job.

One may describe changes of the underlying projective structure by considering the corresponding oper $\partial_y^2 + t_0(y)$, and modifying $t_0(y)$ by addition of a quadratic differential $\sum_{r=1}^{h} h_r \,\vartheta_r$. The parallel transport defined using the modified projective structure will remain integrable if there exists a potential $\mathcal{Z}_0(q)$ on $\mathcal{U}$ such that $h_r = -\partial_{q_r} \mathcal{Z}_0(q)$. The result will be a modification of the partition functions $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, q)$ by a universal factor, a function $\mathcal{Z}_0(q)$ of q independent of the choice of $\mathcal{F}$. This may be regarded as the conformal field theory counterpart of the scheme dependence discussed in Subsections 3.3 and 10.5.

12.1.7. Mapping class group action. Let $\mathsf{CB}^{\mathrm{an}}(\mathcal{V}_{[n]}, C)$ be the subspace of $\mathsf{CB}^{\mathrm{an}}_{\mathrm{loc}}(\mathcal{V}_{[n]}, C)$ which can be analytically continued over all of $\mathcal{T}(C)$. Note that $T_{\mathcal{F}}(w_0)$ defines a projective **c**-connection on C. Given a family of conformal blocks $\mathcal{F}_q$ defined in a subset $\mathcal{U} \subset \mathcal{M}$ one gets a corresponding family of projective connections $T_{\mathcal{F}_q}(w_0)$. If $\mathcal{F} \in \mathsf{CB}^{an}(\mathcal{V}_{[n]}, C)$ one may analytically continue the family of projective connections $T_{\mathcal{F}_q}(w_0)$ over all of $\mathcal{T}(C)$. The resulting section of $\mathcal{P}(C) \to \mathcal{T}(C)$ may then be used to define a family of local sections of the projective line bundle $\mathcal{E}_{\mathbf{c}}$ as explained in Subsection 9.2. It is defined by the Equations (9.15) which coincide with (12.17) in the present case.

Analytic continuation along closed curves in $\mathcal{M}(C)$ defines an action of the mapping class group on $\mathsf{CB}^{\mathrm{an}}(\mathcal{V}_{[n]}, C)$. We will later define a subspace $\mathsf{CB}^{\mathrm{temp}}(\mathcal{V}_{[n]}, C)$ of $\mathsf{CB}^{\mathrm{an}}(\mathcal{V}_{[n]}, C)$ which is closed under this action. It may be characterized by the condition that the partition functions $\mathcal{Z}^{\mathrm{L}}(\mathcal{F}, q)$ are "tempered" in a sense that will be made more precise. The spaces $\mathsf{CB}^{\mathrm{temp}}(\mathcal{V}_{[n]}, C_q)$ associated to local families C_q of Riemann surfaces glue into a projective local system $\mathcal{W}_{\mathrm{L}}(C)$ over $\mathcal{M}(C)$.

A vector bundle that is not projective is [FS]

$$\mathcal{V}_{\mathrm{L}}(C) := \mathcal{W}_{\mathrm{L}}(C) \otimes \mathcal{E}_{\mathbf{c}}\,. \tag{12.18}$$

Picking a basis for $\mathsf{CB}^{\mathrm{temp}}(\mathcal{V}_{[n]}, C_q)$ in some $\mathcal{U} \subset \mathcal{M}(C)$ one may define a section of $\mathcal{V}_{\mathrm{L}}(C)$ by means of analytic continuation. Natural bases for $\mathsf{CB}^{\mathrm{temp}}(\mathcal{V}_{[n]}, C_q)$ can be defined by means of the gluing construction, as will be explained next.

12.2. Gluing construction of conformal blocks

12.2.1. Gluing boundary components. Let us first consider a Riemann surface C_{21} that was obtained by gluing two surfaces C_2 and C_1 with n_2+1 and n_1+1 boundary components, respectively. Given an integer n, let sets I_1 and I_2 be such that $I_1 \cup I_2 = \{1, \dots, n\}$. Let us consider conformal blocks $\mathcal{F}_{C_i} \in \mathsf{CB}(\mathcal{V}_i^{[n_i]}, C_i)$ where $\mathcal{V}_2^{[n_2]} = (\otimes_{r\in I_2}\mathcal{V}_r) \otimes \mathcal{V}_0$ and $\mathcal{V}_1^{[n_1]} = \mathcal{V}_0 \otimes (\otimes_{r\in I_1}\mathcal{V}_r)$ with the same representation $\mathcal{V}_0$ assigned to $P_{0,1}$ and $P_{0,2}$, respectively. Let $\langle\, .\,, .\,\rangle_{\mathcal{V}_0}$ be the invariant bilinear form on $\mathcal{V}_0$. For given $v_2 \in \otimes_{r\in I_2}\mathcal{V}_r$ let W_{v_2} be the linear form on $\mathcal{V}_0$ defined by

$$W_{v_2}(w) := \mathcal{F}_{C_2}(v_2 \otimes w), \quad \forall w \in \mathcal{V}_0, \tag{12.19}$$

and let $\mathsf{C}_1(q)$ be the family of linear operators $\mathcal{V}_1^{[n_1]} \to \mathcal{V}_0$ defined as

$$\mathsf{C}_1(q) \cdot v_1 := \sum_{e\in B(\mathcal{V}_0)} q^{L_0} e\; \mathcal{F}_{C_1}(\check{e} \otimes v_1), \tag{12.20}$$

where we have used the notation $B(\mathcal{V}_0)$ for a basis of the representation $\mathcal{V}_0$ and $\check{e}$ for the dual of an element e of $B(\mathcal{V}_0)$ defined by $\langle\, \check{e}, e'\,\rangle_{\mathcal{V}_0} = \delta_{e,e'}$. We may then consider the expression

$$\mathcal{F}_{C_{21}}(v_2 \otimes v_1) := W_{v_2}(\mathsf{C}_1(q) \cdot v_1). \tag{12.21}$$

We have thereby defined a new conformal block associated to the glued surface C_{21}, see [T08] for more discussion. The insertion of the operator q^{L_0} plays the role of a regularization. It is not a priori clear that the linear form W_{v_2} is defined on infinite linear combinations such as $\mathsf{C}_1(q) \cdot v_1$. Assuming $|q| < 1$, the factor q^{L_0} will produce an suppression of the contributions with large L_0-eigenvalue, which renders the infinite series produced by the definitions (12.21) and (12.20) convergent.

An operation representing the gluing of two boundary components of a single Riemann surface can be defined in a very similar way.

12.2.2. Gluing from pairs of pants. One can produce any Riemann surface C by gluing pairs of pants. The different ways to obtain C in this way are labeled by pants decompositions σ. The elementary building blocks are the conformal blocks associated to three-punctured spheres $C_{0,3}$, which are well-known to be uniquely defined up to normalization by the invariance property (12.2). We fix the normalization such that the value of $\mathcal{F}_{C_{0,3}}$ on the product of highest weight vectors is

$$\mathcal{F}_{C_{0,3}}(e_3 \otimes e_2 \otimes e_1) = \sqrt{C(Q-\alpha_3, \alpha_2, \alpha_1)}\,, \tag{12.22}$$

where $C(\alpha_3, \alpha_2, \alpha_1)$ is the function defined in (8.3).

Using the gluing construction recursively leads to the definition of a family of conformal blocks $\mathcal{F}^{\sigma}_{\beta,q}$ depending on the following set of data:

- σ is a pants decomposition.
- q is the coordinate for $\mathcal{U}_\sigma \subset \mathcal{T}(C)$ defined by the gluing construction.
- β is an assignment $e \mapsto \beta_e \in \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}$, defined for all edges on Γ_σ.

The parameters β_e determine the Virasoro representations $\mathcal{V}_{\Delta_e}$ to be used in the gluing construction of the conformal blocks from pairs of pants via

$$\Delta_e = \beta_e(Q - \beta_e)\,, \qquad \mathbf{c} = 1 + 6Q^2\,. \tag{12.23}$$

The partition functions $\mathcal{Z}^{\mathrm{L}}_{\sigma}(\beta, q)$ defined from $\mathcal{F}^{\sigma}_{\beta,q}$ via (12.6) are entire analytic with respect to the variables α_r, meromorphic in the variables β_e, with poles at the zeros of the Kac determinant, and it can be argued that the dependence on the gluing parameters q is analytic in a open multi-disc $\mathcal{U}_\sigma$ of full dimension $3g - 3 + n$ [T03a, T08].

12.2.3. Change of pants decomposition. It turns out that the conformal blocks $\mathcal{Z}^{\rm L}_{\sigma_1}(\beta,q)$ constructed by the gluing construction in a neighborhood of the asymptotic region of $\mathcal{T}(C)$ that is determined by σ_1 have an analytic continuation to the asymptotic region of $\mathcal{T}(C)$ determined by a second pants decomposition σ_2. A fact [T01, T03a, T08][11] of foundational importance for the subject is that the analytically continued conformal blocks $\mathcal{Z}^{\rm L}_{\sigma_2}(\beta_2,q)$ can be represented as a linear combination of the conformal blocks $\mathcal{Z}^{\rm L}_{\sigma_1}(\beta_1,q)$, which takes the form

$$\mathcal{Z}^{\rm L}_{\sigma_2}(\beta_2,q) = E_{\sigma_2\sigma_1}(q)\int d\mu(\beta_1)\; W_{\sigma_2\sigma_1}(\beta_2,\beta_1)\,\mathcal{Z}^{\rm L}_{\sigma_1}(\beta_1,q)\,. \tag{12.24}$$

The mapping class group acts naturally,

$$\mathcal{Z}^{\rm L}_{\mu.\sigma}(\beta,q) = \mathcal{Z}^{\rm L}_{\sigma}(\beta,\mu.q)\,. \tag{12.25}$$

Combining (12.24) and (12.25) yields a relation of the form

$$\mathcal{Z}^{\rm L}_{\sigma}(\beta_2,\mu.q) = E_{\mu.\sigma,\sigma}(q)\int d\mu(\beta_1)\; W_{\mu.\sigma,\sigma}(\beta_2,\beta_1)\,\mathcal{Z}^{\rm L}_{\sigma}(\beta_1,q)\,. \tag{12.26}$$

The transformations (12.26) define the infinite-dimensional vector bundle $\mathcal{V}_{\rm L}(C)=\mathcal{E}_{\mathbf{c}}\otimes\mathcal{W}_{\rm L}(C)$. The constant kernels $W_{\sigma_2\sigma_1}(\beta_2,\beta_1)$ represent the transition functions of $\mathcal{W}_{\rm L}(C)$, while the prefactors $E_{\sigma_2\sigma_1}(q)$ can be identified as transition functions of the projective line bundle $\mathcal{E}_{\mathbf{c}}$.

It suffices to calculate the relations (12.24) in the cases of surfaces $C=C_{0,4}$, and $C=C_{1,1}$. This was done in [T01] for $C=C_{0,4}$, where a relation of the form

$$\mathcal{Z}^{\rm L}_{\sigma_s}(\beta_1,q) = \int_{\mathbb{S}} d\beta_2\; F_{\beta_1\beta_2}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]\,\mathcal{Z}^{\rm L}_{\sigma_t}(\beta_2,q)\,, \tag{12.27}$$

was found. The pants decompositions σ_s and σ_t are depicted on the left and right half of Figure 4, respectively. Using this result, the case $C=C_{1,1}$ was

[11] A full proof of the statements made here does not appear in the literature yet. It can, however, be assembled from building blocks that are published. By using the groupoid of changes of the pants decompositions it is sufficient to verify the claim for the cases $g=0,n=4$ and $g=1,n=1$, respectively. For $g=0,n=4$ this was done in [T01], see also [T03a]. The case of $g=1,n=1$ was recently reduced to the case $g=0,n=4$ in [HJS].

treated in [HJS], the result being

$$\mathcal{Z}^{\rm L}_{\sigma_s}(\beta_1,q) = e^{\pi i \frac{\mathbf{c}}{12}(\tau+1/\tau)} \int_{\mathbb{S}} d\beta_2 \; S_{\beta_1\beta_2}(\alpha_0)\, \mathcal{Z}^{\rm L}_{\sigma_t}(\beta_2,q)\,, \tag{12.28}$$

where $q=e^{2\pi i\tau}$, as usual. The pants decompositions σ_s and σ_t are depicted in Figure 5. The prefactor is due to the fact that the conformal blocks defined according to the gluing construction differ by a factor of $q^{\frac{\mathbf{c}}{24}}$ from the conformal blocks considered in [HJS]. It represents the only non-trivial transition functions of $\mathcal{E}_{\mathbf{c}}$ according to our discussion in Subsection 9.3.1.

We should again remember that the definition of the partition functions $\mathcal{Z}^{\rm L}_{\sigma}(\beta,q)$ was based on a particular scheme, the choice of the projective structure coming from the gluing construction described above. Using a diffent scheme would modify the partition functions by β-independent functions of q.

12.3. Comparison with the Kähler quantization of $\mathcal{T}(C)$

We had previously identified the space of conformal blocks $\mathsf{CB}^{\rm an}_{\rm loc}(\mathcal{V}_{[n]},C)$ with the space of functions $\mathcal{Z}(q)$ locally defined on patches $\mathcal{U}\subset\mathcal{T}(C)$. This space is naturally acted on by the algebra of differential operators $\mathsf{DO}(\mathcal{T}(C))$, which is directly related to the action of $\mathsf{DO}(\mathcal{T}(C))$ on spaces of conformal blocks defined by means of the Virasoro algebra via (12.9). These observations already indicate that the space of wave-functions $\Psi(q)$ that represent the Hilbert space $\mathcal{H}(C)$ in the representation coming from the Kähler quantization scheme should coincide with a suitable Hilbert-subspace $\mathsf{HCB}(\mathcal{V}_{[n]},C)$ of $\mathsf{CB}(\mathcal{V}_{[n]},C)$.

The direct calculations of the kernels $W_{\sigma_2\sigma_1}(\beta_2,\beta_1)$ carried out for the generators Z,B,F in [T01, T03a], and for S in [HJS] yield results that coincide with the kernels defined in Subsection 6.5. It follows that $\mathcal{W}_{\rm L}$ coincides with the projective local system from the quantization of $\mathcal{M}^0_{\rm flat}(C)$,

$$\mathcal{W}_{\rm L}(C) = \mathcal{W}(C)\,. \tag{12.29}$$

This implies immediately that the conformal blocks $\mathcal{Z}^{\rm L}_{\sigma}(\beta,q)$ represent the solution to the Riemann-Hilbert problem that was found to characterize the wave-functions $\Psi^{\sigma}_{l}(q)$ which describe the relation between length representation and Kähler quantization.

These results imply furthermore that there is a natural Hilbert space structure on the spaces of conformal blocks which is such that the mapping class group action becomes unitary. The Hilbert spaces $\mathsf{HCB}(\mathcal{V}_{[n]},C)$

of conformal blocks are isomorphic as representations of the Moore-Seiberg groupoid to the Hilbert spaces of states constructed in the quantization of $\mathcal{M}^0_{\rm flat}(C)$ in Part II.

Within $\mathsf{HCB}(\mathcal{V}_{[n]}, C)$ one may consider the maximal domains of definition of the algebras $\mathcal{A}_b(C)$ of quantized trace functions, which can be seen as natural analogs $\mathsf{SCB}(\mathcal{V}_{[n]}, C)$ of the Schwarz spaces of test functions in distribution theory. The spaces $\mathsf{SCB}(\mathcal{V}_{[n]}, C)$ are Fréchet spaces with topology given by the family of semi-norms defined from the expectation values of the operators representing the elements of $\mathcal{A}_b(C)$ on $\mathsf{SCB}(\mathcal{V}_{[n]}, C)$. The (topological) dual of $\mathsf{SCB}(\mathcal{V}_{[n]}, C)$ is the space of "tempered" distributions on $\mathsf{SCB}(\mathcal{V}_{[n]}, C)$, which will be identified with the subspace $\mathsf{CB}^{\rm temp}(\mathcal{V}_{[n]}, C_q)$ of $\mathsf{CB}(\mathcal{V}_{[n]}, C)$ spanned by "tempered" conformal blocks.

13. Relation to gauge theory

13.1. The solution to the Riemann-Hilbert problem

We have seen that the kernels representing S-duality transformations in the gauge theory coincide with the kernels representing the changes of pants decomposition in Liouville theory. Taken together we conclude that

$$\mathcal{Z}^{\rm inst}_\sigma(a, m, \tau; \epsilon_1, \epsilon_2) = \mathcal{Z}^{\rm spur}_\sigma(\alpha, \tau; b)\, \mathcal{Z}^{\rm L}_\sigma(\beta, \alpha, q; b)\,, \tag{13.1}$$

where the following identifications of parameters have been used,

$$b^2 = \frac{\epsilon_1}{\epsilon_2}\,, \qquad \hbar^2 = \epsilon_1\epsilon_2\,, \qquad q = e^{2\pi i \tau}\,, \tag{13.2a}$$

$$\beta_e = \frac{Q}{2} + i\frac{a_e}{\hbar}\,, \qquad \alpha_r = \frac{Q}{2} + i\frac{m_r}{\hbar}, \qquad Q := b + b^{-1}\,. \tag{13.2b}$$

The factors $\mathcal{Z}^{\rm spur}_\sigma(\alpha, \tau; b)$ represents the scheme dependence discussed previously. We expect that the possibility to have such factors is related to the issues raised by the necessity to introduce a UV regularization in the study of the gauge theories $\mathcal{G}_C$ mentioned in Subsection 3.3.

13.2. Chiral ring

Let us recall that there are further supersymmetric observables which should be realized on $\mathcal{H}_0$ or $\mathcal{H}_{\rm top}$, respectively: the chiral ring operators $\mathsf{u}_r := \mathrm{Tr}(\phi_r^2)$. We are going to propose that the operators u_r are directly related to the operators h_r arising in the quantum theory of $\mathcal{M}^0_{\text{flat}}(C)$,

$$\mathsf{u}_r \simeq \epsilon_2^2 \mathsf{h}_r\,. \tag{13.3}$$

This is nontrivially supported by the calculations of certain examples in [LMN, FFMP, FMPT].

The existence of a relation of the form (13.3) is natural in view of the fact that the prepotential

$$\mathcal{F}(a,m,\tau) := -\lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \mathcal{Z}^{\text{inst}}_\sigma(a,m,\tau;\epsilon_1,\epsilon_2)\,, \tag{13.4}$$

satisfies Matone type relations of the general form

$$u_r = \frac{\partial}{\partial \tau_r}\mathcal{F}(a,\tau)\,. \tag{13.5}$$

A proof of the relations (13.5) that is valid for all theories of class $\mathcal{S}$ was given in [GT]. It was based on the observation that both the coordinates (a, a^D) describing the special geometry underlying Seiberg-Witten theory, and the coordinates (τ, h) introduced above can be seen as systems of Darboux coordinates for the same space $T^*\mathcal{T}(C)$. The prepotential $\mathcal{F}(a,m,\tau)$ is the generating function of the change of variables between (a, a^D) and (τ, h) [GT].

This observation can be obtained in the limit for $\epsilon_2 \to 0$ from the fact that

$$\mathcal{W}(a,m,\tau;\epsilon_2) := -\lim_{\epsilon_1\to 0} \epsilon_1 \mathcal{Z}^{\text{inst}}_\sigma(a,m,\tau;\epsilon_1,\epsilon_2)\,, \tag{13.6}$$

coincides with the generating function $\mathcal{W}(l,\tau)$ defined above, taking into account the identifications (13.2). Passing to the limit $\epsilon_2 \to 0$, we may observe that

$$\epsilon_2^2(\partial_y^2 + t(y)) \equiv \epsilon_2^2\partial_y^2 + \vartheta(y)$$

turns into the quadratic differential $\vartheta(y)$ when ϵ_2 is sent to zero keeping $\vartheta(y)$ finite. Using $\vartheta(y)$ we define the Seiberg-Witten curve Σ as usual by

$$\Sigma = \{\,(v,u)\,|\, v^2 = \vartheta(u)\,\}\,. \tag{13.7}$$

It follows by WKB analysis of the differential equation $(\epsilon_2^2\partial_y^2 + \vartheta(y))\chi = 0$ that the coordinates l_e have asymptotics that can be expressed in terms of the Seiberg-Witten differential Λ on Σ defined such that $\Lambda^2 = \vartheta(u)(du)^2$. We find

$$(13.8) \qquad \frac{l_e}{2} \sim \frac{2\pi}{\epsilon_2} a_e \,, \qquad \frac{\kappa_e}{2} \sim \frac{2\pi}{\epsilon_2} a_e^{\rm D} \,,$$

where a_e and $a_e^{\rm D}$ are periods of the Seiberg-Witten differential Λ defined as

$$(13.9) \qquad a_e := \int_{\hat{\gamma}_s^e} \Lambda \,, \qquad a_e^{\rm D} := \int_{\hat{\gamma}_t^e} \Lambda \,,$$

with $\hat{\gamma}_s^e$, $\hat{\gamma}_t^e$ being cycles on Σ that project to γ_s^e and γ_t^e, respectively.

It may also be interesting to note that the relation (13.3) relates the scheme dependence in the definition of the conformal blocks to a possible quantum-field theoretical scheme-dependence in the definition of the chiral ring operators u_r.

We thereby realize that the quantum theory of $\mathcal{M}^0_{\rm flat}(C)$ studied in this paper can also be interpreted as the quantization of the geometrical structure encoding the low energy physics of the A_1 gauge theories of class $\mathcal{S}$: Recall that the prepotential can be characterized as the generating function for the change of Darboux coordinates $(a, a^D) \leftrightarrow (\tau, h)$ for $T^*\mathcal{T}(C)$ [GT]. Turning on ϵ_2 "deforms" (a, a^D) into (k, l), see (13.8). The wave-functions $\Psi_\tau(a)$ studied in this paper represent the change of coordinates $(k, l) \leftrightarrow (\tau, h)$ on the quantum level. By combining these observations we realize that the quantum mechanics of scalar zero modes that represents the non-perturbative skeleton of $\mathcal{G}_C$ can be obtained from the Seiberg-Witten theory of $\mathcal{G}_C$ in two steps: The first is the deformation of the cotangent bundle $T^*\mathcal{T}(C)$ representing the Seiberg-Witten theory of $\mathcal{G}_C$ into the twisted cotangent bundle $T^*_{\epsilon_2}\mathcal{T}(C)$ which is isomorphic to $\mathcal{M}^0_{\rm flat}(C)$. The second step is the quantization of $T^*_{\epsilon_2}\mathcal{T}(C) \simeq \mathcal{M}^0_{\rm flat}(C)$. The parameter ϵ_1 of the Omega-deformation plays the role of Planck's constant in the second step. The combination of the two steps may be interpreted as the quantization of the Seiberg-Witten theory of $\mathcal{G}_C$, with quantization parameter $\hbar = \epsilon_1\epsilon_2$. One has a certain freedom in quantizing $T^*\mathcal{T}(C)$ which is parameterized by the "refinement parameter" $b^2 = \epsilon_1/\epsilon_2$.

Part IV. Appendices

Appendix A. Uniqueness of the representations

Let us look at the question of uniqueness of representation for the algebra (6.3) with the constraint (6.2). Let us write the operators L_t and L_u in the following form

$$L_t = D_+ e^{+\mathsf{k}} + D_0 + D_- e^{-\mathsf{k}}, \qquad \text{(A.1)}$$
$$L_u = E_+ e^{+\mathsf{k}} + E_0 + E_- e^{-\mathsf{k}},$$

and substitute these operators into the relation (6.3). Considering the coefficient corresponding to different difference operators $e^{+\mathsf{k}}, I, e^{-\mathsf{k}}$ one finds the following relation between the coefficients $\underline{E} = \{E_+, E_0, E_-\}$ and $\underline{D} = \{D_+, D_0, D_-\}$, respectively,

$$\begin{aligned} E_+ &= e^{-l_s/2} e^{-\pi \mathrm{i} b^2} D_+ \\ E_0 &= \frac{1}{e^{\pi \mathrm{i} b^2} + e^{-\pi \mathrm{i} b^2}} \left(L_s D_0 - L_1 L_3 - L_2 L_4 \right) \\ E_- &= e^{l_s/2} e^{-\pi \mathrm{i} b^2} D_-, \end{aligned} \qquad \text{(A.2)}$$

which is true for the set of coefficients $\underline{D}$ and $\underline{E}$ defined in the main text.

Let us now check which constraints we obtain from (6.2). Again combining the coefficients corresponding to the shift operators $e^{+2\mathsf{k}}, e^{+\mathsf{k}}, I, e^{-\mathsf{k}}, e^{-2\mathsf{k}}$ we see that coefficients of the shift operators $e^{+2\mathsf{k}}$ and $e^{-2\mathsf{k}}$ are trivially zero while the conditions for the coefficients of $e^{+\mathsf{k}}$ and $e^{-\mathsf{k}}$ to be zero are equivalent and take the following form

$$\begin{aligned} &\frac{e^{-\pi \mathrm{i} b^2} e^{l} - e^{-3\pi \mathrm{i} b^2}}{e^{\pi \mathrm{i} b^2} + e^{-\pi \mathrm{i} b^2}} D_0 + \frac{e^{3\pi \mathrm{i} b^2} e^{-l} - e^{-3\pi \mathrm{i} b^2}}{e^{\pi \mathrm{i} b^2} + e^{-\pi \mathrm{i} b^2}} e^{-\mathsf{k}} D_0 e^{+\mathsf{k}} \\ &= \frac{e^{-\pi \mathrm{i} b^2} e^{l/2} - e^{\pi \mathrm{i} b^2} e^{-l/2}}{e^{\pi \mathrm{i} b^2} + e^{-\pi \mathrm{i} b^2}} (L_1 L_3 + L_2 L_4) + e^{-\pi \mathrm{i} b^2} (L_2 L_3 + L_1 L_4), \end{aligned} \qquad \text{(A.3)}$$

which is satisfied for D_0 presented in the main text. Let us now write the constraint appearing from the trivial shift operator

$$\begin{aligned}(A.4)\quad &(e^{-2\pi ib^2}-e^{2\pi ib^2}e^{l})D_+e^{+\mathsf{k}}D_-e^{-\mathsf{k}}+(e^{-2\pi ib^2}-e^{2\pi ib^2}e^{-l})D_-e^{-\mathsf{k}}D_+e^{+\mathsf{k}}\\ &+\frac{e^{-4\pi ib^2}+2e^{-2\pi ib^2}-1-e^{l}-e^{-l}}{(e^{\pi ib^2}+e^{-\pi ib^2})^2}D_0^2-\frac{(L_1L_3+L_2L_4)^2}{(e^{\pi ib^2}+e^{-\pi ib^2})^2}\\ &+\left(2\frac{(e^{l/2}+e^{-l/2})(L_1L_3+L_2L_4)}{(e^{\pi ib^2}+e^{-\pi ib^2})^2}+e^{-\pi ib^2}(L_2L_3+L_1L_4)\right)D_0\\ &+e^{2\pi ib^2}(e^{l/2}+e^{-l/2})^2-(e^{\pi ib^2}+e^{-\pi ib^2})^2\\ &+e^{\pi ib^2}(e^{l/2}+e^{-l/2})(L_3L_4+L_1L_2)\\ &+L_1^2+L_2^2+L_3^2+L_4^2+L_1L_2L_3L_4=0,\end{aligned}$$

which is satisfied by (6.15).

Let us look more closely at the constraint (A.3). We already know that there exists one solution D_0 but it might happen that there are additional solutions. Imagine that the solution we have could be modified as follows,

$$D_0=D_0^{(0)}+D_0^{\mathrm{add}},$$

where $D_0^{(0)}$ is coefficient in (6.15). D_0^{add} would have to be a solution to the following equation

$$\frac{e^{-\pi ib^2}e^{l}-e^{-3\pi ib^2}}{e^{\pi ib^2}+e^{-\pi ib^2}}D_0^{\mathrm{add}}+\frac{e^{3\pi ib^2}e^{-l}-e^{-3\pi ib^2}}{e^{\pi ib^2}+e^{-\pi ib^2}}e^{-\mathsf{k}}D_0^{\mathrm{add}}e^{+\mathsf{k}}=0.$$

A solution exists and is equal to

$$\begin{aligned}(A.5)\qquad D_0^{\mathrm{add}}&=\tilde{D}_0\,e^{-\frac{l^2}{8\pi ib^2}}\frac{S_b(-\frac{l}{2\pi ib}+b)S_b(-\frac{l}{2\pi ib}-b)}{S_b(-\frac{l}{2\pi ib})S_b(-\frac{l}{2\pi ib}+2b)}\\ &=e^{-\frac{l^2}{8\pi ib^2}}\frac{\sinh(\frac{l}{2}+\pi ib^2)}{\sinh(\frac{l}{2}-\pi ib^2)},\end{aligned}$$

with $\tilde{D}_0$ being an $4\pi ib^2$-periodic functions of l. However, any non-vanishing modification of this kind would spoil the reality of the solution.

For analysing the constraint (A.4) we introduce

$$(A.6)\qquad E_{-+}\;=\;D_-e^{-k}D_+e^{+k},$$

and observe that

$$D_+e^{+k}D_-e^{-k}=e^{+k}\big(e^{-k}D_+e^{+k}D_-e^{-k}\big)=e^{+k}E_{-+}e^{-k},$$

which allows us to rewrite (A.4) as

$$
\begin{aligned}
(\mathrm{A.7})\qquad & (e^{-2\pi ib^2}-e^{2\pi ib^2}e^{l})e^{+\mathsf{k}}E_{-+}e^{-\mathsf{k}}+(e^{-2\pi ib^2}-e^{2\pi ib^2}e^{-l})E_{-+}\\
&+\frac{e^{-4\pi ib^2}+2e^{-2\pi ib^2}-1-e^{l}-e^{-l}}{(e^{\pi ib^2}+e^{-\pi ib^2})^2}D_0^2-\frac{(L_1L_3+L_2L_4)^2}{(e^{\pi ib^2}+e^{-\pi ib^2})^2}\\
&+\left(2\frac{(e^{l/2}+e^{-l/2})(L_1L_3+L_2L_4)}{(e^{\pi ib^2}+e^{-\pi ib^2})^2}+e^{-\pi ib^2}(L_2L_3+L_1L_4)\right)D_0\\
&+e^{2\pi ib^2}(e^{l/2}+e^{-l/2})^2-(e^{\pi ib^2}+e^{-\pi ib^2})^2\\
&+e^{\pi ib^2}(e^{l/2}+e^{-l/2})(L_3L_4+L_1L_2)\\
&+L_1^2+L_2^2+L_3^2+L_4^2+L_1L_2L_3L_4=0.
\end{aligned}
$$

As in the case of constraint for D_0 we consider an additive deviation to $E_{-+}^{(0)}$,

$$E_{-+}=E_{-+}^{(0)}+E_{-+}^{\mathrm{add}},$$

and find the following equation for E_{-+}^{add}:

$$(\mathrm{A.8})\qquad \left(e^{-2\pi ib^2}-e^{2\pi ib^2}e^{l}\right)e^{+\mathsf{k}}E_{-+}^{\mathrm{add}}e^{-\mathsf{k}}+\left(e^{-2\pi ib^2}-e^{2\pi ib^2}e^{-l}\right)E_{-+}^{\mathrm{add}}=0,$$

whose solution is

$$(\mathrm{A.9})\qquad E_{-+}^{\mathrm{add}}=\tilde{E}_0\,\frac{e^{-\frac{l^2}{8\pi ib^2}-\frac{l}{2}}}{\sinh\frac{l}{2}\sinh(\frac{l}{2}-2\pi ib^2)},$$

with $\tilde{E}_0$ being $4\pi ib^2$-periodic. Again one sees that solution (A.9) would spoil the reality of the solution.

The only freedom we are left with is the gauge transformation since (A.7) fixes only the product (up to the shift) of D_- and D_+. To see more clearly the conclusion above let us take the classical limit of constraints (A.3) and (A.4) which become

$$(\mathrm{A.10})\qquad (e^{l/2}-e^{-l/2})^2D_0=L_s(L_1L_3+L_2L_4)+2(L_2L_3+L_1L_4),$$

which defines D_0 unambiguously. Let us now consider the condition obtained by comparing coefficients of the trivial shift operator

$$
\begin{aligned}
\text{(A.11)}\quad & -(e^{l/2}-e^{-l/2})^2D_+D_- - \frac{1}{4}(e^{l/2}-e^{-l/2})^2D_0^2 - \frac{1}{4}(L_1L_3+L_2L_4)^2 \\
& + \left(\frac{1}{4}L_s(L_1L_3+L_2L_4)+(L_2L_3+L_1L_4)\right)D_0 + L_s^2 - 4 \\
& + L_s(L_3L_4+L_1L_2) + L_1^2+L_2^2+L_3^2+L_4^2+L_1L_2L_3L_4 = 0,
\end{aligned}
$$

from which one finds unambiguously D_+D_-. The only freedom is to multiply D_+ by $e^{\pi \mathrm{i}\chi(l)}$ and D_- by $e^{-\pi\mathrm{i}\chi(l)}$, i. e. the gauge freedom.

Let us finally remark that assuming the cyclic symmetry for algebra of loop operators under permutations of two points on a sphere

$$
\text{(A.12)}\qquad L_s \rightarrow_{2\leftrightarrow 3} L_t \rightarrow_{1\leftrightarrow 2} L_u \rightarrow_{2\leftrightarrow 4} L_s
$$

one gets the cyclic symmetry for the cubic relation (6.2), so in a sense the two first lines in (6.2) are fixed by cyclic symmetry.

Appendix B. Special functions

B.1. The function $\Gamma_b(x)$

The function $\Gamma_b(x)$ is a close relative of the double Gamma function studied in [Br]. It can be defined by means of the integral representation

$$
\text{(B.1)}\quad \log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t}\right) .
$$

Important properties of $\Gamma_b(x)$ are

(B.2) functional equation $\Gamma_b(x+b) = \sqrt{2\pi}b^{bx-\frac{1}{2}}\Gamma^{-1}(bx)\Gamma_b(x)$.

(B.3) analyticity $\Gamma_b(x)$ is meromorphic, it has poles only at $x=-nb-mb^{-1}, n,m\in\mathbb{Z}^{\geq 0}$.

A useful reference for further properties is [Sp].

B.2. Double Sine function

The special functions denoted $e_b(x)$ was introduced under the name of *quantum dilogarithm* in [FK2]. These special functions are simply related to the

Barnes double Gamma function [Br], and were also introduced in studies of quantum groups and integrable models in [F2, Ru, Wo, V].

In the strip $|\mathrm{Im}(x)| < \frac{Q}{2}$, function $e_b(x)$ has the following integral representation

$$e_b(x) = \exp\left\{ -\int_{\mathbb{R}+\mathrm{i}0} \frac{dt}{4\,t} \frac{e^{-2\mathrm{i}tx}}{\sinh bt \, \sinh \frac{t}{b}} \right\}, \tag{B.4}$$

where the integration contour goes around the pole $t = 0$ in the upper half–plane. The function $s_b(x)$ is then related to $e_b(x)$ as follows

$$s_b(x) = e^{\frac{\mathrm{i}\pi}{2}x^2 + \frac{\mathrm{i}\pi}{24}(b^2+b^{-2})} e_b(x)\,. \tag{B.5}$$

The analytic continuation of $s_b(x)$ to the entire complex plane is a meromorphic function with the following properties

$$\text{functional equation} \quad \frac{s_b(x + \frac{\mathrm{i}}{2}b^{\pm 1})}{s_b(x - \frac{\mathrm{i}}{2}b^{\pm 1})} = 2\,\cosh(\pi b^{\pm 1} x)\,, \tag{B.6}$$

$$\text{reflection property} \quad s_b(x)\, s_b(-x) = 1\,, \tag{B.7}$$

$$\text{complex conjugation} \quad \overline{s_b(x)} = s_b(-\bar{x})\,, \tag{B.8}$$

$$\text{zeros / poles} \quad (s_b(x))^{\pm 1} = 0 \;\Leftrightarrow\; \pm x \in \left\{ \mathrm{i}\tfrac{Q}{2} + nb + mb^{-1} ; n, m \in \mathbb{Z}^{\geq 0} \right\}, \tag{B.9}$$

$$\text{residue} \quad \operatorname*{Res}_{x=-\mathrm{i}\frac{Q}{2}} s_b(x) = \frac{\mathrm{i}}{2\pi}\,, \tag{B.10}$$

(B.11)

$$\text{asymptotics} \quad s_b(x) \sim \begin{cases} e^{-\frac{\mathrm{i}\pi}{2}(x^2 + \frac{1}{12}(b^2+b^{-2}))} & \text{for } |x| \to \infty, \ |\arg(x)| < \frac{\pi}{2}\,, \\ e^{+\frac{\mathrm{i}\pi}{2}(x^2 + \frac{1}{12}(b^2+b^{-2}))} & \text{for } |x| \to \infty, \ |\arg(x)| > \frac{\pi}{2}\,. \end{cases}$$

The behavior for $b \to 0$ is given as

$$e_b\left(\frac{v}{2\pi b}\right) = \exp\left(-\frac{1}{2\pi b^2}\mathrm{Li}_2(-e^v)\right)\left(1 + \mathcal{O}(b^2)\right). \tag{B.12}$$

In our paper we often use the special function $S_b(x)$ defined by

$$S_b(x) := s_b(\mathrm{i}x - \tfrac{\mathrm{i}}{2}Q)\,. \tag{B.13}$$

In terms of $\Gamma_b(x)$ the double Sine-function is given as

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}.$$

We will use the properties

$$\text{self–duality} \quad S_b(x) = S_{b^{-1}}(x)\,, \tag{B.14}$$

$$\text{functional equation} \quad S_b(x+b^{\pm 1}) = 2\,\sin(\pi b^{\pm 1}x)\,S_b(x)\,, \tag{B.15}$$

$$\text{reflection property} \quad S_b(x)\,S_b(Q-x) = 1\,. \tag{B.16}$$

B.3. Integral identities

We will use the following set of integral identities.

Proposition 1.

$$\int_{i\mathbb{R}} dz \prod_{i=1}^{3} S_b(\mu_i - z) S_b(\nu_i + z) = \prod_{i,j=1}^{3} S_b(\mu_i + \nu_j), \tag{B.17}$$

where the balancing condition is $\sum_{i=1}^{3}(\mu_i + \nu_i) = Q$.

This identity was recently understood as a pentagon identity in [KLV].

Proposition 2.

$$\frac{1}{2}\int_{i\mathbb{R}} dz\, \frac{S_b(\mu \pm z) S_b(\nu \pm z)}{S_b(\pm 2z)} e^{-2\pi \mathrm{i} z^2} = S_b(\mu+\nu) e^{-\frac{1}{2}\pi\mathrm{i}(\mu-\nu)^2 + \frac{1}{2}\pi\mathrm{i}Q(\mu+\nu)}. \tag{B.18}$$

The following notation has been used $S_b(\alpha \pm u) := S_b(\alpha+u)S_b(\alpha-u)$.

Proposition 3.

$$\int_{i\mathbb{R}} dy \prod_{i=1}^{3} S_b(\mu_i - y) \prod_{i=1}^{2} S_b(\nu_i + y) e^{\pi \mathrm{i}\lambda y} e^{-\frac{1}{2}\pi \mathrm{i} y^2} \tag{B.19}$$
$$= \prod_{i=1}^{3} S_b(\mu_i + \nu_2) e^{\frac{1}{2}\pi \mathrm{i}\lambda^2} e^{\frac{1}{8}\pi \mathrm{i} Q^2} e^{-\frac{1}{2}\pi \mathrm{i} Q(\lambda+\nu_1)}$$
$$\times \frac{1}{2} \int_{i\mathbb{R}} dy \, \frac{\prod_{i=1}^{3} S_b(\mu_i + \sigma \pm y) S_b(\nu_1 - \sigma \pm y)}{S_b(\pm 2y)} e^{-2\pi \mathrm{i} y^2},$$

where

$$2\sigma = Q - \sum_{i=1}^{3} \mu_i - \nu_2,$$

and the following balancing condition is satisfied

$$\sum_{i=1}^{3} \mu_i + \sum_{i=1}^{2} \nu_i = \lambda + \frac{Q}{2}.$$

The proof of the above Propositions is easily obtained from the reduction of elliptic hypergeometric integrals to the hyperbolic level [DS] (the details can be found in [Bu] or in [SV11]). Identity B.17, B.18 and B.19 are equivalent to Theorem 5.6.7, Theorem 5.6.6 and Theorem 5.6.17 in [Bu], respectively.

Appendix C. Analytic properties of intertwining kernels

C.1. Preparations

It will be convenient to factorize the expression for $F^{\mathrm{L}}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ as

$$F^{\mathrm{L}}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] = \frac{\Gamma_b(2Q-2\alpha_s)\Gamma_b(2\alpha_s)}{\Gamma_b(2Q-2\alpha_t)\Gamma_b(2\alpha_t)} \times \big\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\big\}_b^C \tag{C.1}$$

with b-6j symbols $\big\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\big\}_b^C$ in the normalization from Subsection 8.2 given by the formula

$$\big\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\big\}_b^C := \frac{T(\alpha_t,\alpha_3,\alpha_2)T(\alpha_4,\alpha_t,\alpha_1)}{S(\alpha_s,\alpha_2,\alpha_1)S(\alpha_4,\alpha_3,\alpha_s)} \times \mathcal{J}\,, \tag{C.2}$$

where

$$\mathcal{J} := \int_{\mathcal{C}} du\, S_b(u-\alpha_{12s})S_b(u-\alpha_{s34})S_b(u-\alpha_{23t})S_b(u-\alpha_{1t4}) \\ \times S_b(\alpha_{1234}-u)S_b(\alpha_{st13}-u)S_b(\alpha_{st24}-u)S_b(2Q-u)\,,$$

and

$$S(\alpha_3,\alpha_2,\alpha_1) = \Gamma_b(2Q-\alpha_{123})\Gamma_b(\alpha_{12}^3)\Gamma_b(\alpha_{23}^1)\Gamma_b(\alpha_{31}^2) \tag{C.3}$$

$$T(\alpha_3,\alpha_2,\alpha_1) = \Gamma_b(\alpha_{123}-Q)\Gamma_b(Q-\alpha_{12}^3)\Gamma_b(Q-\alpha_{23}^1)\Gamma_b(Q-\alpha_{31}^2)\,. \tag{C.4}$$

We are using the notations $\alpha_{ijk} = \alpha_i+\alpha_j+\alpha_k$, $\alpha_{ij}^k = \alpha_i+\alpha_j-\alpha_k$.

C.2. Resonant values

Singular behavior of the integral $\mathcal{J}$ could be caused by the behavior of the integrand at infinity, or by the pinching of the contour $\mathcal{C}$ between poles of the integrand. It is not hard to check that the integral converges for $u\to\infty$ for all values of the variables α_i. It is furthermore straightforward to check that the pinching of the contour of integration in (C.3) only occurs when at least one of the triples T_{s12}, T_{43s}, T_{t32}, T_{4t1} is resonant, using the terminology from Subsection 8.3.2. Taking into account the poles and zeros of the prefactors in (C.1) one easily verifies that the b-6j symbols $\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\right\}_b^C$ are entire in α_s, and have poles iff one of T_{t32}, T_{4t1} is resonant.

We are going to consider the b-6j symbols $\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\right\}_b^C$ as distribution on a space $\mathcal{T}$ of functions $f(\alpha_t)$ which are (i) entire, (ii) decay faster than any exponential for $\alpha_r\to\infty$ along the axis $Q/2+i\mathbb{R}$, and (iii) Weyl-symmetric $f(\alpha_t)=f(Q-\alpha_t)$. For $\alpha_i\in Q/2+i\mathbb{R}$, $i=1,2,3,4,s$ one defines

$$D_{\alpha_s}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\right\}(f) := \frac{1}{2}\int_{Q/2+i\mathbb{R}} d\alpha_t\,\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\right\}_b^C f(\alpha_t)\,. \tag{C.5}$$

Assuming $\alpha_i\in Q/2+i\mathbb{R}$, $i=1,2,3,4$, one easily checks that $\tilde{f}(\alpha_s) := D_{\alpha_s}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\right\}(f)$ has the properties (i)-(iii) above. This means that the operator F maps $\mathcal{T}$ to itself.

Consider now the analytic continuation of $D_{\alpha_s}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\right\}$ with respect to the parameter α_2. It can always be represented in the form (C.5), but the contour of integration may need to be deformed. The result can generically be represented as an integral over the original contour $Q/2+i\mathbb{R}$ plus a finite sum over residue terms. The residue terms define generalized delta-distributions as introduced in (8.11).

C.3. Degenerate values

We are particularly interested in the case where takes one of the degenerate values

$$\alpha_2 = -kb/2 - lb^{-1}/2\,. \tag{C.6}$$

Note that this is a necessary condition for having a double resonance,

$$\alpha_{12}^s = -k'b - l'b^{-1} \quad \wedge \quad \alpha_{s2}^1 = -k''b - l''b^{-1}\,, \tag{C.7}$$

where $k = k' + k''$, $l = l' + l''$. The prefactor in (C.2) proportional to $(S(\alpha_s, \alpha_2, \alpha_1))^{-1}$ vanishes in the case of a double resonance. It follows that only residue terms can appear in the expression for $D_{\alpha_s}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\right\}$ at double resonance (C.7).

So let us look at the residue terms that become relevant in the analytic continuation from $\Re(\alpha_2) = Q/2$ to the values (C.6). Relevant are the poles from the triple T_{t32}, in particular the poles at

$$\alpha_{32}^t = -k_1 b - l_1 b^{-1}\,, \tag{C.8}$$

$$\alpha_{t2}^3 = -k_2 b - l_2 b^{-1}\,, \tag{C.9}$$

where $k = k_1 + k_2$, $l = l_1 + l_2$. It is for some considerations convenient to assume that $\Re(\alpha_3) = Q/2 - \epsilon + iP_3$ for some small real number $0 < \epsilon < b/2$. It follows that the poles

$$\alpha_t = \tfrac{Q}{2} - \epsilon + iP_3 + (k_1 - k_2)\tfrac{b}{2} + (l_1 - l_2)\tfrac{1}{2b}\,, \tag{C.10}$$

with $k_1 - k_2 \leq 0$ and $l_1 - l_2 \leq 0$ will have crossed the contour of integration from the right, and the poles

$$\alpha_t = \tfrac{Q}{2} - \epsilon + iP_3 - (k_2 - k_1)\tfrac{b}{2} - (l_2 - l_1)\tfrac{1}{2b}\,, \tag{C.11}$$

with $k_2 - k_1 < 0$ and $l_2 - l_1 < 0$ will have crossed the contour of integration from the left. The form of the distribution given in (8.10b) follows easily from these observations.

C.4. Residues

We list here some relevant residues.

(C.12)

$$f_{10}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & -b\end{smallmatrix}\right] = \frac{1}{\Gamma(-2b^2)} \times \frac{\Gamma(1-2b\alpha_4)\Gamma(-Qb-b^2+2b\alpha_4)\Gamma(2-2b\alpha_2)\Gamma(2Qb-2b\alpha_2-b^2)}{\Gamma(2Qb-b^2-b\alpha_{234}))\Gamma(-b^2-b\alpha_{34}^2)\Gamma(1-b\alpha_{24}^3)\Gamma(1-b\alpha_{23}^4)};$$

(C.13)

$$f_{-10}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & -b\end{smallmatrix}\right] = \frac{1}{\Gamma(-2b^2)} \times \frac{\Gamma(1-2b\alpha_4)\Gamma(-Qb-b^2+2b\alpha_4)\Gamma(2b\alpha_2-2b^2)\Gamma(2b\alpha_2-b^2)}{\Gamma(-b^2+b\alpha_{23}^4)\Gamma(-b^2+b\alpha_{24}^3))\Gamma(1-b\alpha_{34}^2)\Gamma(-Qb-b^2+b\alpha_{234})};$$

(C.14)

$$f_{01}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & -b\end{smallmatrix}\right] = 2\cos(\pi b^2)\frac{\Gamma(-2b^2)}{\Gamma(-b^2)^2} \times \frac{\Gamma(-Qb+2b\alpha_4)\Gamma(-Qb+2b\alpha_4+b^2)\Gamma(2Qb-2b\alpha_2)\Gamma(2b\alpha_2)}{\Gamma(b\alpha_{34}^2)\Gamma(b\alpha_{24}^3)\Gamma(Qb-b\alpha_{23}^4)\Gamma(-Qb+b\alpha_{234})};$$

(C.15)

$$f_{0-1}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & -b\end{smallmatrix}\right] = 2\cos(\pi b^2)\frac{\Gamma(-2b^2)}{\Gamma(-b^2)^2} \times \frac{\Gamma(Qb-2b\alpha_4)\Gamma(Qb-2b\alpha_4+b^2)\Gamma(2Qb-2b\alpha_2)\Gamma(2b\alpha_2)}{\Gamma(2Qb-b\alpha_{234})\Gamma(Qb-b\alpha_{24}^3)\Gamma(Qb-b\alpha_{34}^2)\Gamma(b\alpha_{23}^4)};$$

(C.16)

$$f_{00}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & -b\end{smallmatrix}\right] = \frac{\Gamma(-Qb+2b\alpha_4-b^2)\Gamma(2Qb-2b\alpha_2)}{\Gamma(-b^2)\Gamma(2b\alpha_4)\Gamma(1-2b\alpha_2)} \times \left\{1+2\cos\pi b^2\frac{\sin[\pi b(\alpha_2-\alpha_3+\alpha_4)]\sin[\pi b(-Q+\alpha_2+\alpha_3+\alpha_4)]}{\sin[2\pi b\alpha_2]\sin[2\pi b\alpha_4]}\right\},$$

where the notation $\alpha_{ij}^k = \alpha_i + \alpha_j - \alpha_k$ was used. From the above fusion matrices one can derive the 't Hooft–Wilson loop intertwining relation.

Appendix D. The kernel for the S-move

We here describe in more detail our derivation of formula (6.30) for the kernel representing the S-move. As outlined in the main text, we are using the following strategy:

- Definition (7.21) defines operators B, F, Z and S within the quantum Teichmüller theory which satisfy operatorial versions of the Moore-Seiberg consistency conditions [T05].
- The direct calculation of the kernel of the operator S presented in Subsection D.1 below shows that this operator is represented by a kernel which depends meromorphically on its arguments.
- It was explained in Subsection 8.5 that this allows us to use the Moore-Seiberg equation

$$\text{(D.1)}\qquad S_{\beta_1\beta_2}(\beta_3)\int_{\mathbb{S}} d\beta_4\; F_{\beta_3\beta_4}\big[\begin{smallmatrix}\beta_2 & \alpha_1\\ \beta_2 & \alpha_2\end{smallmatrix}\big]\, T_{\beta_4}\, T_{\beta_2}^{-1}\, F_{\beta_4\beta_5}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \beta_2 & \beta_2\end{smallmatrix}\big] = \int_{\mathbb{S}} d\beta_6\; F_{\beta_3\beta_6}\big[\begin{smallmatrix}\beta_1 & \alpha_1\\ \beta_1 & \alpha_2\end{smallmatrix}\big] F_{\beta_1\beta_5}\big[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \beta_6 & \beta_6\end{smallmatrix}\big] S_{\beta_6\beta_2}(\beta_5)\, e^{\pi i(\Delta_{\alpha_1}+\Delta_{\alpha_2}-\Delta_{\beta_5})}\,,$$

  to derive a formula for $S_{\beta_1\beta_2}(\beta_3)$ in terms of the kernel for F. More details are given in Subsection D.2 below.
- The integrals in the resulting formula for $S_{\beta_1\beta_2}(\beta_3)$ will be calculated explicitly in Subsection D.3, leading to our formula (6.30).

A faster way to find the formula (6.30) would be to use the intertwining property (6.20) to derive an difference equation for the kernel $S_{\beta_1\beta_2}(\alpha)$. The problem would then be to show that the resulting formula solves the Moore-Seiberg equations. This is manifest in our approach.

D.1. Calculation using Teichmüller theory

We shall work within the representation for quantum Teichmüller theory associated to the fat graph drawn in Figure D1. The representation associated to the annulus A_s in Figure D1 is taken to be the one defined in Subsection 7.4.

For the following it will suffice to work in a reduced representation defined by setting the constraint z to zero. The length operator L_s is then defined by using (7.8). In order to define the operator L_0 representing the length of the hole of $C_{1,1}$ we may use formula (7.10). The length operator L_t has to be calculated using (7.11) by finding a fat graph φ_0 which allows one to use the definition (7.8). The resulting formulae for the relevant length

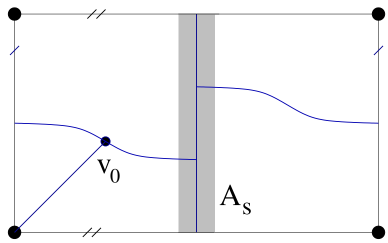


Figure D1: Fat graph on a one-holed torus $C_{1,1}$, represented as rectangle with opposite sides identified. The hole sits at the corners of the rectangle. The annulus A_s (grey) contains the geodesic γ_s defining a pants decomposition of $C_{1,1}$.

operators are

(D.2) $$\mathsf{L}_s = 2\cosh 2\pi b \mathsf{p}_s + e^{2\pi b \mathsf{q}_s}\,,$$

(D.3) $$\mathsf{L}_t = 2\cosh 2\pi b \mathsf{p}_t + e^{2\pi b \mathsf{q}_t} + e^{-2\pi b \mathsf{q}_0} e^{\pi b(\mathsf{q}_t - \mathsf{p}_t)}\,,$$

(D.4) $$\mathsf{L}_0 = 2\cosh 2\pi b \mathsf{p}_0 + 2\cosh(\pi b \mathsf{p}_0)\mathsf{L}_s e^{-2\pi b \mathsf{q}_0} + e^{-4\pi b \mathsf{q}_0}\,.$$

In the expression for L_t we have been using the notations

(D.5) $$\mathsf{p}_t := \tfrac{1}{2}(\mathsf{q}_s - \mathsf{p}_s - \mathsf{p}_0)\,, \qquad \mathsf{q}_t := -\tfrac{1}{2}(3\mathsf{p}_s + \mathsf{q}_s + \mathsf{p}_0)\,.$$

Let us consider eigenstates $|\, a, m\,\rangle_s$ and $|\, a, m\,\rangle_t$ to the pairs of mutually commuting operators $(\mathsf{L}_s, \mathsf{L}_0)$ and $(\mathsf{L}_t, \mathsf{L}_0)$, respectively

(D.6)
$$\mathsf{L}_s\,|\, a, m\,\rangle_s = 2\cosh 2\pi b a\,|\, a, m\,\rangle_s\,, \qquad \mathsf{L}_0\,|\, a, m\,\rangle_s = 2\cosh 2\pi b m\,|\, a, m\,\rangle_s\,,$$
$$\mathsf{L}_t\,|\, a, m\,\rangle_t = 2\cosh 2\pi b a\,|\, a, m\,\rangle_t\,, \qquad \mathsf{L}_0\,|\, a, m\,\rangle_t = 2\cosh 2\pi b m\,|\, a, m\,\rangle_t\,.$$

We shall work in a representation where the operators p_s and q_0 are diagonal. States are represented by wave-functions $\phi^s_{a,m}(p_s, q_0) := \langle\, p_s, q_0\,|\, a, m\,\rangle_s$ and $\phi^t_{a,m}(p_s, q_0) := \langle\, p_s, q_0\,|\, a, m\,\rangle_t$.

These wave-functions are related by an integral transformation of the form

(D.7) $$\phi^t_{a_t,m}(p_s, q_0) = \int da_s\; S_{a_t a_s}(m)\, \phi^s_{a_s,m}(p_s, q_0)\,.$$

In order to simplify the calculation it helps to consider the limit $q_0 \to \infty$. Note that L_0 can be approximately be represented by $2\cosh 2\pi b \mathsf{p}_0$ in this limit. Both $\phi^s_{a,m}(p_s,q_0)$ and $\phi^t_{a,m}(p_s,q_0)$ can be normalized to have a leading asymptotic behavior for $q_0\to\infty$ of the form

$$\phi^s_{a,m}(p_s,q_0)\,\sim\,(e^{2\pi imq_0}+R^s_m e^{-2\pi imq_0})\psi^s_a(p_s)\,, \tag{D.8}$$

$$\phi^t_{a,m}(p_t,q_0)\,\sim\,(e^{2\pi imq_0}+R^t_m e^{-2\pi imq_0})\psi^t_a(p_s)\,, \tag{D.9}$$

where $\psi^s_a(p_s)$ and $\psi^t_a(p_t)$ must be eigenfunctions of the operators L'_s and L'_t obtained from L_s and L_t by sending $q_0\to\infty$ and considering a representations of $(\mathsf{p}_s,\mathsf{q}_s)$ on functions $\psi(p_s)$ of a single variable on which p_s acts as multiplication operator. Equation (D.7) implies

$$\psi^t_{a_t,m}(p_s)=\int da_s\;S_{a_ta_s}(m)\,\psi^s_{a_s,m}(p_s)\,. \tag{D.10}$$

The calculation of the kernel $S_{a_ta_s}(m)$ is now straightforward. Recall that a complete set of orthonormalized eigenfunction of L_s is given by the functions defined in (7.13). Note furthermore that

$$\mathsf{L}'_t=2\cosh 2\pi b\mathsf{p}_t+e^{2\pi b\mathsf{q}_t}\,. \tag{D.11}$$

The eigenfunctions of L'_t in a representation in which p_t is diagonal are therefore obtained from (7.13) by obvious substitutions. We finally need that $\langle\,p_s\,|\,p_t\,\rangle=e^{\pi\mathrm{i}(p_s^2+p_t^2)}\,e^{4\pi\mathrm{i}p_sp_t}e^{-2\pi\mathrm{i}m(p_s+p_t)}$. The kernel representing the modular transformation S is then given as

$$\begin{aligned}S_{a_sa_t}(m_0)&=\langle\,a_s\,|\,a_t\,\rangle\\&=\int dp_sdp_t\;\langle\,a_s\,|\,p_s\,\rangle\,\langle\,p_s\,|\,p_t\,\rangle\,\langle\,p_t\,|\,a_t\,\rangle\\&=\int dp_s\;e^{\pi\mathrm{i}(p_s-2m)p_s}\,\frac{s_b(a_s-p_s+c_b-i0)}{s_b(a_s+p_s-c_b+i0)}\\&\quad\cdot\int dp_t\;e^{\pi\mathrm{i}(p_t-2m)p_t}\,\frac{s_b(a_t+p_t+c_b-i0)}{s_b(a_t-p_t-c_b+i0)}e^{4\pi\mathrm{i}p_sp_t}\,.\end{aligned} \tag{D.12}$$

It is easy to see that $S_{a_sa_t}(m_0)$ is meromorphic in m_0, a_s and a_t.

D.2. Solving the Moore-Seiberg relations for the S-kernel

We now want to explain how to derive the formula

$$\begin{aligned}(\text{D.13}) \qquad & F^{\mathrm{L}}_{0\alpha}\left[\begin{smallmatrix}\beta_1 & \beta_1\\ \beta_1 & \beta_1\end{smallmatrix}\right] S^{\mathrm{L}}_{\beta_1\beta_2}(\alpha) \\ & = S^{\mathrm{L}}_{0\beta_2}\int d\beta_3\, e^{-\pi i(2\Delta_{\beta_2}+2\Delta_{\beta_1}-2\Delta_{\beta_3}-\Delta_\alpha)}\, \mathrm{F}^{\mathrm{L}}_{0\beta_3}\left[\begin{smallmatrix}\beta_2 & \beta_1\\ \beta_2 & \beta_1\end{smallmatrix}\right] \mathrm{F}^{\mathrm{L}}_{\beta_3\alpha}\left[\begin{smallmatrix}\beta_1 & \beta_1\\ \beta_2 & \beta_2\end{smallmatrix}\right].\end{aligned}$$

for $S^{\mathrm{L}}_{\beta_1\beta_2}(\alpha)$ from Equation (D.1). As explained in the main text, we mainly need the identity

$$(\text{D.14}) \qquad \lim_{\epsilon\downarrow 0} \mathrm{F}^{\mathrm{L}}_{\epsilon,\alpha_3}\left[\begin{smallmatrix}\epsilon & \alpha_1\\ \epsilon & \alpha_1\end{smallmatrix}\right] = \delta(\alpha_3-\alpha_1)\,.$$

Setting $\alpha_1=\alpha_2$ and taking $\beta_1=\epsilon, \beta_3=\epsilon$, $\epsilon\to 0$ using (D.14) yields (D.13).

One might be tempted to take $\beta_1\to 0$ first. This turns out not to be straightforward, as the convergence of the integrals in (D.1) would then be lost. Doing this naively would seem to lead to an equation similar to (D.13), but with $S^{\mathrm{L}}_{0\beta_2}$ replaced by $\tilde{S}^{\mathrm{L}}_{0\beta_2} := \lim_{\beta_1\to 0} S^{\mathrm{L}}_{\beta_1\beta_2}$, which is not the same as $S^{\mathrm{L}}_{0\beta_2} := \lim_{\epsilon\to 0} S^{\mathrm{L}}_{\epsilon,\beta_2}(\epsilon)$. The fact that $S^{\mathrm{L}}_{0\beta_2}\neq\tilde{S}^{\mathrm{L}}_{0\beta_2}$ can be verified explicitly using Equations (D.34c), (D.34d) below.

It remains to prove (D.14). In order to do this, we will show that

$$(\text{D.15}) \qquad \mathrm{F}^{\mathrm{L}}_{\epsilon,\alpha_3}\left[\begin{smallmatrix}\epsilon & \alpha_1\\ \epsilon & \alpha_1\end{smallmatrix}\right] = \mathrm{F}^{\mathrm{L}}_{0,\alpha_3}\left[\begin{smallmatrix}\epsilon & \alpha_1\\ \epsilon & \alpha_1\end{smallmatrix}\right] + \mathcal{O}(\epsilon)\,,$$

and use the remarkable identity [T08, Sa]

$$(\text{D.16}) \qquad \mathrm{F}^{\mathrm{L}}_{0\alpha_3}\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_2 & \alpha_1\end{smallmatrix}\right] = \frac{1}{2\pi}\frac{Z(0)\;Z(\alpha_3)}{Z(\alpha_2)Z(\alpha_1)} C(Q-\alpha_3,\alpha_2,\alpha_1)\,,$$

proven below. The function $C(\alpha_3,\alpha_2,\alpha_1)$ was defined in (8.3), and $Z(\alpha)$ is explicitly given as

$$(\text{D.17}) \qquad Z(\alpha) = \frac{(\pi\mu\gamma(b^2))^{\frac{1}{2b}(Q-2\alpha)}\, 2\pi(Q-2\alpha)}{\Gamma(1+b(Q-2\alpha))\Gamma(1+b^{-1}(Q-2\alpha))}\,.$$

The normalization factor $Z(\alpha)$ is closely related to the Liouville one-point function on the unit disc [ZZ01]. Note furthermore that [T01, Section 4.4]

$$(\text{D.18}) \qquad \lim_{\alpha_2\to 0} C(Q-\alpha_3,\alpha_2,\alpha_1) = 2\pi\delta(\alpha_3-\alpha_1)\,.$$

The identity (D.14) follows from the combination of (D.16) and (D.18).

For the calculations necessary to prove (D.15) and (D.16) we will find it convenient to use a further gauge transformation defined by writing

$$\tilde{v}^{\alpha_3}_{\alpha_2\alpha_1} = N(\alpha_3,\alpha_2,\alpha_1) w^{\alpha_3}_{\alpha_2\alpha_1}\,, \tag{D.19}$$

with $N(\alpha_3,\alpha_2,\alpha_1)$ being defined in (8.19). The kernels representing the F- and S-moves in the corresponding representation will be denoted as $F^{\rm PT}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ and $S^{\rm PT}_{\beta_1\beta_2}(\alpha_0)$, respectively. We have

$$F^{\rm L}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] = \frac{N(\alpha_s,\alpha_2,\alpha_1)N(\alpha_4,\alpha_3,\alpha_s)}{N(\alpha_t,\alpha_3,\alpha_2)N(\alpha_4,\alpha_t,\alpha_1)} F^{\rm PT}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]\,, \tag{D.20}$$

$$S^{\rm L}_{\beta_1\beta_2}(\alpha_0) = \frac{N(\beta_1,\alpha_0,\beta_1)}{N(\beta_2,\alpha_0,\beta_2)} S^{\rm PT}_{\beta_1\beta_2}(\alpha_0)\,. \tag{D.21}$$

The kernel $F^{\rm PT}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ can be expressed using the formula first derived in in [PT2][12],

$$\begin{aligned}
&F^{\rm PT}_{\alpha_s\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\big] \\
&= \frac{S_b(\alpha_2+\alpha_s-\alpha_1)S_b(\alpha_t+\alpha_1-\alpha_4)}{S_b(\alpha_2+\alpha_t-\alpha_3)S_b(\alpha_s+\alpha_3-\alpha_4)}|S_b(2\alpha_t)|^2 \\
&\quad\times\int_{\mathcal{C}} du\; S_b(-\alpha_2\pm(\alpha_1-Q/2)+u)S_b(-\alpha_4\pm(\alpha_3-Q/2)+u) \\
&\qquad\times S_b(\alpha_2+\alpha_4\pm(\alpha_t-Q/2)-u)S_b(Q\pm(\alpha_s-Q/2)-u)\,.
\end{aligned} \tag{D.22}$$

The following notation has been used $S_b(\alpha\pm u) := S_b(\alpha+u)S_b(\alpha-u)$. The integral in (D.22) will be defined for $\alpha_k \in Q/2 + i\mathbb{R}$ by using a contour $\mathcal{C}$ that approaches $Q+i\mathbb{R}$ near infinity, and passes the real axis in $(Q/2,Q)$, and for other values of α_k by analytic continuation. The equivalence between the two different integral representations of the b-6j symbols was proven in [TeVa] using methods from [DSV].

Using the the representation (D.22) and the integral identity (B.17) it becomes easy to find that

$$\mathrm{F}^{\rm PT}_{\epsilon\alpha_3}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_2 & \alpha_1\end{smallmatrix}\big] = |S_b(2\alpha_3)|^2\frac{S_b(\alpha_1+\alpha_2+\alpha_3-Q)}{S_b(\alpha_2+\alpha_3-\alpha_1)S_b(2\alpha_1)}(S_b(\epsilon))^2(1+\mathcal{O}(\epsilon)), \tag{D.23}$$

from which Equation (D.15) and identity (D.16) follow straightforwardly.

[12]The formula below coincides with Equation (228) in [T01] after shifting $s \to u-\alpha_s-Q/2$.

D.3. Evaluating the integral

We start from Equation (D.13). Considering the right hand side, let us represent $\mathrm{F}^{\mathrm{PT}}_{0\beta_3}\left[\begin{smallmatrix}\beta_2 & \beta_1\\ \beta_2 & \beta_1\end{smallmatrix}\right]$ as

$$\mathrm{F}^{\mathrm{PT}}_{0\beta_3}\left[\begin{smallmatrix}\beta_2 & \beta_1\\ \beta_2 & \beta_1\end{smallmatrix}\right] = \lim_{\delta\to 0}\mathrm{F}^{\mathrm{PT}}_{\delta\beta_3}\left[\begin{smallmatrix}\beta_2 & \beta_1\\ \beta_2 & \beta_1\end{smallmatrix}\right]. \tag{D.24}$$

One may then represent the right hand side of (D.13) as the limit $\delta \to 0$ of an expression proportional to the following integral:

$$\begin{aligned} I = C\int \frac{e^{-2\pi\mathrm{i}(\frac{Q}{2}-y)^2}d\gamma}{S_b(\pm 2(\frac{Q}{2}-\gamma))} &\cdot \int dx \frac{S_b(-\beta_1 \pm (\frac{Q}{2}-\beta_1)+x)S_b(-Q+\beta_2 \pm (\frac{Q}{2}-\beta_2)+x)}{S_b(\frac{Q}{2}+\delta+x)S_b(-\frac{Q}{2}+x)S_b(\beta_2-\beta_1 \pm (\frac{Q}{2}-\gamma)+x)} \\ &\times \int dy \frac{S_b(-\beta_1 \pm (\frac{Q}{2}-\beta_2)+y)S_b(-Q+\beta_2 \pm (\frac{Q}{2}-\beta_1)+y)}{S_b(\pm(\frac{Q}{2}-\gamma)+y)S_b(\beta_2-\beta_1 \pm (\frac{Q}{2}-\alpha_0)+y)}, \end{aligned} \tag{D.25}$$

where

$$C \simeq e^{\frac{\pi\mathrm{i}Q^2}{2}-\pi\mathrm{i}(2\Delta_{\beta_2}+2\Delta_{\beta_1}-\Delta_{\alpha_0})}\frac{S_b(-Q+\alpha_0+2\beta_2)}{S_b(-Q+2\beta_2)S_b(\alpha_0)}S_b(\delta).$$

We use the notation $\simeq$ to indicate equality up to terms that are less singular when $\delta \to 0$. The divergent factor $S_b(\delta)$ will be cancelled by zeros in the prefactors, see (C.2), so that we only need to consider the leading singular behavior of the integral I when $\delta \to 0$.

Simplifying the above expression one gets

$$\begin{aligned} I \simeq C\int \frac{e^{-2\pi\mathrm{i}(\frac{Q}{2}-y)^2}d\gamma}{S_b(\pm 2(\frac{Q}{2}-\gamma))} &\cdot \int dx \frac{S_b(\frac{Q}{2}-2\beta_1+x)S_b(-\frac{Q}{2}+x)S_b(-\frac{3}{2}Q+2\beta_2+x)}{S_b(\frac{Q}{2}+x+\delta)S_b(\beta_2-\beta_1 \pm (\frac{Q}{2}-\gamma)+x)} \\ &\times \int dy \frac{S_b(-\beta_1 \pm (\frac{Q}{2}-\beta_2)+y)S_b(-Q+\beta_2 \pm (\frac{Q}{2}-\beta_1)+y)}{S_b(\pm(\frac{Q}{2}-\gamma)+y)S_b(\beta_2-\beta_1 \pm (\frac{Q}{2}-\alpha_0)+y)}. \end{aligned} \tag{D.26}$$

We may take the integral over the variable x in (D.26) using identity (B.17) and get

$$I \simeq C_1 \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{e^{-2\pi\mathrm{i}(\frac{Q}{2}-y)^2}d\gamma}{S_b(\pm 2(\frac{Q}{2}-\gamma))} S_b(\tfrac{Q}{2}-\beta_2+\beta_1\pm(\tfrac{Q}{2}-\gamma))$$
$$\times \int dy \frac{S_b(-\beta_1\pm(\frac{Q}{2}-\beta_2)+y)S_b(-Q+\beta_2\pm(\frac{Q}{2}-\beta_1)+y)}{S_b(\pm(\frac{Q}{2}-\gamma)+y)S_b(\beta_2-\beta_1\pm(\frac{Q}{2}-\alpha_0)+y)}, \tag{D.27}$$

where

$$C_1 \simeq C\, S_b(Q-2\beta_1)S_b(-Q+2\beta_2)S_b(-\delta).$$

Next we take the integral over γ using identity (B.18) with taking $\nu_2 = Q-\beta_2+\beta_1+(\frac{Q}{2}-\alpha_0)$ (and then apply change of variables $y\to -y$)

$$I \simeq C_2 \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dy\; S_b(\tfrac{Q}{2}-\beta_1-\beta_2-y)$$
$$\cdot S_b(-\tfrac{Q}{2}+\beta_2-\beta_1-y)S_b(-\tfrac{3}{2}Q+\beta_1+\beta_2-y)$$
$$\times S_b(Q-\beta_2+\beta_1\pm(\tfrac{Q}{2}-\alpha_0)+y)e^{\pi\mathrm{i}y(\beta_1-\beta_2)}e^{-\frac{\pi\mathrm{i}y^2}{2}}, \tag{D.28}$$

where

$$C_2 \simeq e^{-\frac{1}{2}\pi\mathrm{i}(\frac{Q}{2}+\beta_2-\beta_1)^2}e^{\frac{1}{2}\pi\mathrm{i}Q(\frac{3}{2}Q-\beta_2+\beta_1)}C_1.$$

As a final step we use (B.19) taking $\nu_2 = Q-\beta_2+\beta_1+(\frac{Q}{2}-\alpha_0)$,

$$I \;\simeq\; C_3\frac{1}{2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dy \frac{S_b(\frac{\alpha_0}{2}\pm(\frac{Q}{2}-\beta_1)\pm(\frac{Q}{2}-\beta_2)\pm y)}{S_b(\pm 2y)}e^{-2\pi\mathrm{i}y^2}dy, \tag{D.29}$$

with

$$C_3 \simeq C_2\, e^{\pi\mathrm{i}\frac{Q^2}{8}}e^{\pi\mathrm{i}(\beta_1-\beta_2)^2}e^{-\frac{1}{2}\pi\mathrm{i}Q(\frac{Q}{2}+\alpha_0)}$$
$$\cdot S_b(2Q-2\beta_2-\alpha_0)S_b(Q-\alpha_0)S_b(2\beta_1-\alpha_0)\,.$$

We also need $F^{\rm PT}_{\epsilon\alpha_0}\left[\begin{smallmatrix}\beta_1 & \beta_1\\ \beta_1 & \beta_1\end{smallmatrix}\right]$ for $\epsilon\to 0$. Formula (D.23) gives

$$F^{\rm PT}_{\epsilon\alpha_0}\left[\begin{smallmatrix}\beta_1 & \beta_1\\ \beta_1 & \beta_1\end{smallmatrix}\right] \simeq (S_b(\epsilon))^2\,\frac{S_b(Q-2\beta_1)S_b(-Q+2\beta_1+\alpha_0)}{S_b(\alpha_0)}. \tag{D.30}$$

By assembling the pieces we come to the following relation

$$
\begin{aligned}
S^{\mathrm{PT}}_{\beta_1\beta_2}(\alpha_0) = \frac{1}{2} S^{\mathrm{PT}}_{0\beta_2} & \frac{S_b(2\beta_1-\alpha_0)S_b(2Q-2\beta_1-\alpha_0)}{S_b(\alpha_0)} \\
& \cdot e^{2\pi \mathrm{i}(\beta_1-\frac{Q}{2})^2} e^{2\pi \mathrm{i}(\beta_2-\frac{Q}{2})^2} e^{-\pi \mathrm{i}(\alpha_0^2-2\frac{Q}{4}\alpha_0)} \\
& \times \int \frac{S_b(Q-\beta_1-\beta_2+\frac{\alpha_0}{2}\pm y)S_b(-Q+\beta_1+\beta_2+\frac{\alpha_0}{2}\pm y)}{S_b(\pm 2y)} \\
& \quad \cdot e^{-2\pi \mathrm{i} y^2} dy \times S_b(\beta_1-\beta_2+\tfrac{\alpha_0}{2}\pm y) S_b(-\beta_1+\beta_2+\tfrac{\alpha_0}{2}\pm y)\,.
\end{aligned} \tag{D.31}
$$

It remains to apply the following formula [SV11],

$$
\begin{aligned}
& \int_{i\mathbb{R}} dz\, \frac{S_b(Q/4-\mu+m/2\pm z)}{S_b(3Q/4-\mu-m/2\pm z)} e^{4\pi \mathrm{i}\xi z} \\
& = \frac{1}{2} e^{2\pi \mathrm{i}(\xi^2-(\frac{Q}{4}+\frac{m}{2})^2+\mu^2)} S_b(Q/2-m\pm 2\xi) \\
& \quad \cdot \int_{i\mathbb{R}} dy\, \frac{S_b(\frac{Q}{4}+\frac{m}{2}\pm\mu\pm\xi\pm y)}{S_b(\pm 2y)} e^{-2\pi \mathrm{i} y^2}\,,
\end{aligned} \tag{D.32}
$$

which had been used in this form in [SV11], in order to get the desired result,

$$
\begin{aligned}
S^{\mathrm{PT}}_{\beta_1\beta_2}(\alpha_0) = S^{\mathrm{PT}}_{0\beta_2} \frac{e^{\frac{\pi i}{2}\Delta_{\alpha_0}}}{S_b(\alpha_0)} \int_{\mathbb{R}} dt\, e^{2\pi t(2\beta_1-Q)} & \frac{S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)+it\big)}{S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)+it\big)} \\
& \cdot \frac{S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)-it\big)}{S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)-it\big)}\,.
\end{aligned} \tag{D.33}
$$

This is equivalent to formula (6.30), taking into account (D.21).

D.4. Properties of $S_{\beta_1\beta_2}(\alpha_0)$

In order to derive the key properties of $S_{\beta_1\beta_2}(\alpha_0)$ let us define the integral

$$
\begin{aligned}
I^{\alpha_0}_{\beta_1\beta_2} := \frac{1}{S_b(\alpha_0)} \int_{\mathbb{R}} dt\, e^{2\pi t(2\beta_1-Q)} & \\
\cdot \frac{S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)+it\big) S_b\big(\frac{1}{2}(2\beta_2+\alpha_0-Q)-it\big)}{S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)+it\big) S_b\big(\frac{1}{2}(2\beta_2-\alpha_0+Q)-it\big)}&\,.
\end{aligned}
$$

$I^{\alpha_0}_{\beta_1\beta_2}$ has the following properties:

(D.34a) $$I^{\alpha_0}_{\beta_1\beta_2} = I^{Q-\alpha_0}_{\beta_2\beta_1}\,,$$

(D.34b) $$I^{\alpha_0}_{\beta_1,\beta_2} = I^{\alpha_0}_{\beta_1,Q-\beta_1} = I^{\alpha_0}_{Q-\beta_1,\beta_1}\,,$$

(D.34c) $$\lim_{\epsilon\downarrow 0} I^{\epsilon}_{\beta_1\beta_2} = \frac{1}{M_{\beta_2}}\,2\cos(\pi(2\beta_1 - Q)(2\beta_2 - Q))\,,$$

(D.34d) $$\lim_{\epsilon\downarrow 0} I^{\epsilon}_{\epsilon\beta} = 1\,, \qquad \lim_{\epsilon\downarrow 0} I^{Q-\epsilon}_{\epsilon\beta} = \frac{M_\beta}{M_0}\,,$$

recalling that $M_\beta = |S_b(2\beta)|^2 = -4\sin(\pi b(2\beta - Q))\sin(\pi b^{-1}(2\beta - Q))$. Identity (D.34a) follows easily from Equation (A.31) in [BT2]. (D.34b) is an easy consequence of the symmetry properties of the integrand under $t =\!\!\to -t$ and (D.34a).

In order to derive (D.34c) note that the zero of the prefactor in the definition of $I^{\alpha_0}_{\beta_1\beta_2}$ is canceled by a pole of the integral. This pole results from the fact that the contour of integration gets pinched between the poles at $it = \pm\frac{1}{2}(2\beta_2 \pm \alpha_0 - Q)$ in the limit $\alpha_0 \to 0$. The residue may be evaluated by deforming the contour into the sum of two small circles around $it = \pm\frac{1}{2}(2\beta_2 - Q) + \alpha_0$ plus some residual contour that does not get pinched when $\alpha_0 \to 0$.

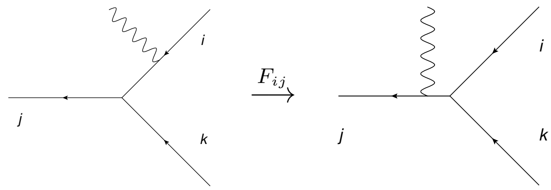
In order to prove (D.34d), one may first use (D.34a), and then similar arguments as used to prove (D.34c).

Appendix E. Asymptotics of the generating function $\mathcal{W}$

E.1. Monodromy on nodal surfaces

We need to calculate the monodromy of the oper $\partial_y^2 + t(y)$ on the nodal surface representing the boundary component of $\mathcal{T}(C)$ corresponding to an pants decomposition σ. We will need the result to leading order in the gluing parameters q_r. Using the gluing construction one may represent the nodal surface as union of punctured spheres and long thin cylinders. Parallel transport along a closed curve γ breaks up into a sequence $M_1, \ldots, M_N$ of moves which represent either the transition F_{ij} from puncture i to puncture j of a three-punctured sphere, the braiding B_i of puncture i on a three-punctured sphere with the additional puncture at y, or the propagation T_e along the long thin tube containing the edge e of Γ_σ. To each moves M_k let us associate a 2x2 matrix m_k according to the following rules:

- Moves F_{ij}:



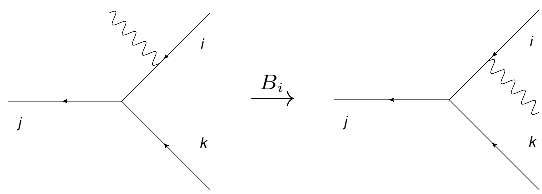
are represented by the matrix f^{ij} with elements

(E.1a) $$f^{ij}_{s_1 s_2} \equiv f_{s_1 s_2}(l_k; l_j, l_i)\,,$$

where

(E.1b) $$f_{s_1 s_2}(l_3; l_2, l_1) = \frac{\Gamma(1 + \mathrm{i}s_1 \frac{l_1}{2\pi})\Gamma_b(-\mathrm{i}s_2 \frac{l_2}{2\pi})}{\prod_{s_3=\pm} \Gamma\big(\frac{1}{2} + \frac{\mathrm{i}}{4\pi}(s_1 l_1 - s_2 l_2 + s_3 l_3)\big)}\,.$$

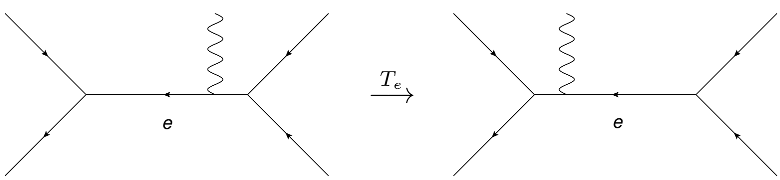
- Moves B_i:



are represented by the matrix b^i with elements

(E.1c) $$b^i_{ss'} = \delta_{ss'}\, e^{\frac{\pi \mathrm{i}}{2} - s\frac{l_i}{2}}\,.$$

- Moves T_e:



are represented by the matrix t^e with elements

(E.1d) $$t^e_{ss'} = \delta_{ss'}\, q_e^{ik_e/4\pi}\,.$$

If the curve γ is decribed as the composition of segments $M_1 \circ M_2 \circ \cdots \circ M_N$, the trace function L_γ is calculated as

$$L_\gamma = \mathrm{Tr}(m_1 \cdot m_2 \cdot \cdots \cdot m_N)\,, \tag{E.2}$$

where m_k are the 2×2-matrices representing the moves along the segments M_k.

It is easy to see that the rules above are nothing but the limit $b \to 0$ of the rules defining the Verlinde loop operators from conformal field theory [AGGTV, DGOT]. This is of course no accident. The comparison of the explicit expressions for Verlinde loop operators found in [AGGTV, DGOT] with the expressions for the expressions quoted in Subsection 6.3 shows that the Verlinde loop operators coincide with the quantized trace coordinates, the respective representations differing only by gauge transformations. A more direct explanation of this fact will be given elsewhere.

E.2. Calculation of the constant term

We may therefore use the results of references [AGGTV, DGOT]. This yields, in particular, an expression for L_t of the form

$$L_t = D_+^{\mathrm{cl}}(l)\, e^{+k_0} + D_0^{\mathrm{cl}}(l) + D_-^{\mathrm{cl}}(l)\, e^{-k_0}\,, \tag{E.3a}$$

where $k_0 = -\frac{\mathrm{i}}{2\pi} l \log(q)$, and the coefficients $D_\pm^{\mathrm{cl}}(l)$ are explicitly given as

$$\begin{aligned}
D_\pm^{\mathrm{cl}}(l) &= (2\pi)^4 \frac{(\Gamma(1 \pm \frac{\mathrm{i}}{2\pi} l)\Gamma(\pm\frac{\mathrm{i}}{2\pi} l))^2}{\prod_{s,s'=\pm} \Gamma\big(\frac{1}{2} \pm \frac{\mathrm{i}}{4\pi}(l + sl_1 + s'l_2)\big)\Gamma\big(\frac{1}{2} \pm \frac{\mathrm{i}}{4\pi} b(l + sl_3 + s'l_4)\big)}\,,\\
D_0^{\mathrm{cl}}(l) &= \frac{4}{\cosh l - 1}(\cosh(l_2/2)\cosh(l_3/2) + \cosh(l_1/2)\cosh(l_4/2))\\
&\quad + \frac{4\cosh(l/2)}{\cosh l - 1}(\cosh(l_1/2)\cosh(l_3/2) + \cosh(l_2/2)\cosh(l_4/2))\,.
\end{aligned}$$

This should be compared to (2.20d). In the degeneration limit we may use (10.13) to represent the leading behavior of k in the form

$$k = 4\pi\mathrm{i}\frac{\partial}{\partial l}\mathcal{W}(l,q) = k_0 + 4\pi\mathrm{i}\frac{\partial}{\partial l}\mathcal{W}_0(l) + \mathcal{O}(q)\,. \tag{E.4}$$

It follows that we must have

$$\log D_\pm^{\mathrm{cl}}(l) = \log\sqrt{c_{12}(L_s)c_{34}(L_s)} \pm 4\pi\mathrm{i}\frac{\partial}{\partial l}\mathcal{W}_0(l)\,. \tag{E.5}$$

This is a differential equation for $\mathcal{W}_0(l)$, solved by (10.18). □

Appendix F. Projectively flat connections

For the reader's convenience we will collect here some generalities on connections on bundles of projective spaces and projective line bundles. We follow in parts the discussions in [FS, Fe].

F.1. Connections on bundles of projective spaces

Given a holomorphic vector bundle $\mathcal{E}$ over a complex manifold X, let $\mathbb{P}(\mathcal{E})$ be its projectivization, the bundle of projective spaces with fiber at $x \in X$ being the projectivization $\mathbb{P}(\mathcal{E}_x)$ of the fiber $\mathcal{E}_x$ of $\mathcal{E}$. A connection on $\mathbb{P}(\mathcal{E})$ is an equivalence class of locally defined connections $\nabla_\imath$ on $\mathcal{E}|_{\mathcal{U}_\imath}$, where $\{\mathcal{U}_\imath; \imath \in \mathcal{I}\}$ is a covering of X, subject to the condition that $a_{\imath\jmath} := \nabla_\imath - \nabla_\jmath$ is a scalar holomorphic one-form on the overlaps $\mathcal{U}_{\imath\jmath} = \mathcal{U}_\imath \cap \mathcal{U}_\jmath$. Two such families of connections are identified in $\nabla_\imath - \nabla'_\imath$ is a scalar holomorphic one-form for all $\imath \in \mathcal{I}$.

The curvature $F_\imath = \nabla_\imath^2$ is a two-form with values in $\mathrm{End}(\mathcal{E})$ that satisfies $F_\imath - F_\jmath = da_{\imath\jmath}$ on overlaps $\mathcal{U}_{\imath\jmath}$. A connection is (projectively) flat if $F_\imath$ is a scalar, i.e. proportional to the identity in all patches $\mathcal{U}_\imath$. As the curvature $F_\imath$ of a flat connection is locally exact, we may always choose a representative $\nabla_\imath$ for the equivalence class such that $F_\imath = 0$ in $\mathcal{U}_\imath$. Alternatively one may trivialize the scalar one-forms $a_{\imath\jmath} := \nabla_\imath - \nabla_\jmath$ by choosing smooth scalar one-forms $c_\imath$ such that $a_{\imath\jmath} = c_\imath - c_\jmath$, and considering $\nabla'_\imath := \nabla_\imath + c_\imath$ as the preferred representative for a given equivalence class. The connection $\nabla'_\imath$ is globally defined, but it has non-trivial scalar curvature.

The representation in terms of locally defined flat connections, is sometimes referred to as the *Čech point of view.* This point of view will make it clear that the deviation from being a vector bundle with an ordinary flat connection is controlled by a *projective holomorphic line bundle.* Such a line bundle L is defined by transitions functions $f_{\imath\jmath}$ defined on overlaps $\mathcal{U}_{\imath\jmath}$ that satisfy

$$f_{\imath_3\imath_2}\, f_{\imath_2\imath_1} = \sigma_{\imath_3\imath_2\imath_1}\, f_{\imath_3\imath_1}\,,$$

on the triple overlaps $\mathcal{U}_{\imath_1\imath_2\imath_3} \equiv \mathcal{U}_{\imath_1} \cap \mathcal{U}_{\imath_2} \cap \mathcal{U}_{\imath_3}$. The 1-cochain $f_{\imath\jmath}$, $\imath, \jmath \in \mathcal{I}$, defines a class in $\check{H}^2(\Omega^0)$. The collection of $f_{\imath\jmath}$ will be called a *section* of L. Being one level higher in the Čech-degree than in the case of ordinary line bundles makes it seem natural to identify sections with 1-cochains rather than 0-cochains in the rest of this appendix.

Having a projectively flat vector bundle one gets a projective *line* bundle by setting $f_{\imath\jmath} \equiv e^{2\pi i g_{\imath\jmath}}$, where the $f_{\imath\jmath}$ are any solutions of $\partial g_{\imath\jmath} = \frac{1}{2\pi i} a_{\imath\jmath}$. The collection of one-forms $a_{\imath\jmath}$ defines a Čech-cohomology class in $\check{H}^1(\Omega^1)$, which corresponds to a globally defined $(1,1)$-form ϖ by the Čech-Dolbeault isomorphism. This $(1,1)$-form represents the first Chern class of the projective line bundle defined by the transition functions $f_{\imath\jmath}$.

F.2. Projective local systems

Recall the natural correspondence between

(i) vector bundles $\mathcal{V}$ with flat connections ∇,

(ii) *local systems* – vector bundles with *constant* transition functions,

(iii) representations of the fundamental group $\rho : \pi_1(X) \to \mathrm{End}(V)$ modulo overall conjugation.

Indeed, any flat connection ∇ in a vector bundle $\mathcal{V}$ may be trivialized locally in the patches $\mathcal{U}_\imath$ by means of gauge transformations. This defines a system of preferred trivializations for ∇ with constant transition functions. Parallel transport w.r.t. to ∇ defines a representation of the fundamental group. Conversely, given a representation of the fundamental group one gets a local system as $(\tilde{X}, V)/\sim$, where $\tilde{X}$ is the universal cover of X, and $\sim$ is the equivalence relation

$$(\tilde{x}\,,\,v\,) \sim (\,\gamma\tilde{x}\,,\,\rho(\gamma)v\,)\,, \quad \forall(\,\tilde{x}\,,\,v\,) \in (\,\tilde{X}\,,\,V\,)\,, \quad \forall\gamma \in \pi_1(X)\,. \tag{F.1}$$

This vector bundle has a canonical flat connection – the trivial one.

Parallel transport w.r.t. a projectively flat connection defines projective representations of the fundamental group $\pi_1(X)$, a map $\rho : \pi_1(X)$, which assigns to each element γ of $\pi(X)$ an operator $\rho(\gamma) \in \mathrm{End}(E)$ such that

$$\rho(\gamma_1) \cdot \rho(\gamma_2) = e^{2\pi i \chi(\gamma_1,\gamma_2)}\,\rho(\gamma_1 \circ \gamma_2)\,. \tag{F.2}$$

The notation anticipates that we will ultimately be interested in unitary connections leading to unitary representations of the fundamental group.

It is easy to see that there are equally natural correspondences between

(i) projective vector bundles $\mathbb{P}(\mathcal{V})$ with projectively flat connections ∇,

(ii) *projective local systems* – projective vector bundles with *constant* transition functions,

(iii) projective representations of the fundamental group $\rho : \pi_1(X) \to \mathrm{End}(V)$.

One needs, in particular, to replace the vector space V in the equivalence relation (F.1) by its projectivization. The resulting equivalence relation makes perfect sense thanks to (F.2).

F.3. Riemann-Hilbert type problems

It directly follows from the definition of a projectively flat vector bundle that an ordinary vector bundle can be obtained by tensoring with a projective line bundle. This makes clear how to formulate a suitable generalization of the Riemann-Hilbert correspondence in case of projectively flat vector bundles. We need *two* pieces of data:

(a) a projective representation of the fundamental group $\rho : \pi_1(X) \to \mathrm{End}(V)$, and:

(b) a holomorphic section of the projective line bundle canonically associated to ρ.

We may then ask for V-valued holomorphic functions $F(\tilde{x})$ on $\tilde{X}$ that satisfy

$$F(\gamma\tilde{x}) = f_\gamma(\tilde{x})\,(\rho(\gamma)F)(\tilde{x})\,, \tag{F.3}$$

where the functions $f_\gamma(\tilde{x})$ represent the holomorphic section of the projective line bundle $\mathcal{P}_\rho$ associated to ρ.

There is of course an inevitable ambiguity in the solution of this generalized Riemann Hilbert problem, represented by the choice of a section of the projective line bundle $\mathcal{P}_\rho$. This is closely related to the issue called scheme dependence in the main text. A natural point of view is to consider classes of solutions to the generalized Riemann-Hilbert problem which differ by the choice of a section of $\mathcal{P}_\rho$. In our concrete application we will be able to do slightly better by identifying natural choices for the sections of $\mathcal{P}_\rho$.

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