A McKay-like correspondence for $(0,2)$-deformations

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We present a local computation of deformations of the tangent bundle for a resolved orbifold singularity $\mathbb{C}^d/G$. These correspond to $(0,2)$-deformations of $(2,2)$-theories. A McKay-like correspondence is found predicting the dimension of the space of first-order deformations from simple calculations involving the group. This is confirmed in two dimensions using the Kronheimer–Nakajima quiver construction. In higher dimensions such a computation is subject to nontrivial worldsheet instanton corrections and some examples are given where this happens. However, we conjecture that the special crepant resolution given by the G-Hilbert scheme is never subject to such corrections, and show this is true in an infinite number of cases. Amusingly, for three-dimensional examples where G is abelian, the moduli space is associated to a quiver given by the toric fan of the blow-up. It is shown that an orbifold of the form $\mathbb{C}^3/\mathbb{Z}_7$ has a nontrivial superpotential and thus an obstructed moduli space.

1. Introduction

Perhaps the first result in string theory to excite interest from geometers arose from orbifolds. In [1] it was shown that one could straight-forwardly analyze string theory on $\mathbb{C}^d/G$, where G is some finite subgroup of $\mathrm{SL}(d,\mathbb{C})$, despite the fact that this orbifold is a singular space. It was shown that certain string states are in one-to-one correspondence with conjugacy classes of G. If one were to consider string theory on a smooth space X, such string states would correspond to even-dimensional homology classes. Thus, if $\mathbb{C}^d/G$ is deformed into some smooth space X, and there is reason to expect the string spectrum is unchanged by such a resolution, one predicts a relationship between the even-dimensional cohomology of X and the conjugacy classes of G.

Such a statement is a consequence of the McKay correspondence in two dimensions [2]. This has been extended to dimension three in papers such as [3, 4].

It is interesting to note, however, that the original orbifold paper [1] also contained another numerical prediction for orbifolds which has received a good deal less attention. This concerns the deformation of $N = (2, 2)$ theories to $(0, 2)$ theories. This is phrased in the language of heterotic string as follows. One may compactify[1] an $E_8 \times E_8$ heterotic string on a Calabi–Yau n-fold X together with a principal $E_8 \times E_8$ bundle $E \to X$. The easiest way to do this is to use the "standard embedding" which was very much in vogue in the early days of string theory. In this case, one uses the tangent bundle T for E by embedding $SU(d)$ into $E_8 \times E_8$.

Such a model has an underlying $N = (2, 2)$ superconformal field theory. There are deformations of this theory which preserve this supersymmetry. If X is smooth such deformations can be interpreted as deforming the complex structure and complexified Kähler form of X. Such marginal deformations are always truly marginal thanks to the extended supersymmetry [5]. This is exactly the same statement as the unobstructedness of the moduli space as in [6, 7].

More interestingly, as far as this paper is concerned, one may deform to a conformal field theory with only $(0, 2)$ supersymmetry. This corresponds to a deformation of the compactification of the heterotic string by deforming the tangent bundle to another bundle E. In the context of the analysis of this paper, E will always be a holomorphic bundle with structure group $SU(d)$.

From a phenomenological point of view, it is clearly absurd these days to use the standard embedding. However, for a better understanding of $(0, 2)$ theories, it makes sense to begin with the much better-understood $(2, 2)$-theories and venture into the world of $(0, 2)$-theories through deformations. In this way, the standard embedding becomes a very important idea.

String theory would appear to imply that there is a connection between counting certain massless states in an orbifold conformal field theory and the number of deformations of the tangent bundle on a crepant resolution of the orbifold. We will dub this the "(0,2)-McKay Correspondence".

The unobstructedness theorems of $(2, 2)$-deformations are no longer valid for $(0, 2)$-deformations when one deals with spaces of more than two complex dimensions. In particular, three possibilities are of interest:

1) The deformations of E itself may be geometrically obstructed.

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[1]In a loose sense. In this paper X is not compact!

2) The mutual deformations of X and the bundle E may be obstructed.

3) There may be worldsheet instanton effects ruling out certain $E \to X$ as true vacua.

These possibilities all correspond to a nontrivial superpotential for the scalar fields associated to the deformations of the theory. In particular, quadratic terms in the superpotential affect masses and thus change the number of massless states. The analysis of (0,2)-deformations is therefore a good deal more interesting than (2,2)-deformations. Of course, a pessimist would say that the worldsheet instanton effects, which are included in the conformal field theory but not in the geometric nonlinear σ-model, should completely ruin such a (0,2)-McKay correspondence in dimension ≥ 3. In this paper we shall show that a particular class of orbifold resolution, namely the G-Hilbert scheme, appears to be immune from such effects, at least in many cases.

After reviewing the counting of orbifold states in Section 2 we will prove that the (0,2)-McKay correspondence works perfectly in dimension two. This should come as no surprise to a string theorist, but is an interesting mathematical result.

For the remainder of the paper we consider dimension three. In Section 4 we develop machinery to compute the number of deformations of the tangent sheaf on a resolved abelian orbifold. The moduli space of the tangent sheaf is described in terms of a quiver which, in a satisfying coincidence, is given in terms of the toric diagram of the resolution itself. In Section 5 we give various examples and this motivates our conjecture that only a particular resolution, the G-Hilbert scheme, need satisfy the (0,2)-McKay correspondence.

In Section 7 we show how a nontrivial cubic superpotential can appear in a specific example, and finally in Section 8 we present concluding comments.

2. Counting States in an Orbifold

In this section we review how to count the number of massless states in an orbifold theory on $\mathbb{C}^d/G$ that can be used to marginally deform a $N = (2, 2)$ superconformal symmetry to a theory with at least $N = (0, 2)$ supersymmetry. We call such states "singlets" from their rôle in heterotic compactifications.

Let x^i be the bosonic fields corresponding to the holomorphic coordinates in the target space $\mathbb{C}^d$. Their (target space) complex conjugates are $\bar{x}^i$. We have right moving fermions denoted ψ^i and $\bar{\psi}^i$ which are superpartners of

x^i, $\bar{x}^i$ with respect to the right-moving $N = 2$ worldsheet supersymmetry. We have left-moving fermions $\gamma^i, \bar{\gamma}^i$.

Let us consider fields in the twisted sector corresponding to an element $g \in G$. First recall how we would do this computation if we were only concerned with singlets which, when used as marginal operators, preserved the full $N = (2,2)$ supersymmetry. These correspond to (anti)chiral primary operators. Upon twisting the theory to a topological field theory, these states are elements of BRST cohomology. The spectrum of a twisted sector is accordingly given by the Dolbeault or De Rahm (depending on the twisting) cohomology of the fixed point set of g [8].

In our case we have a larger spectrum as we wish only to preserve $N = (0,2)$ supersymmetry. One of the right-moving worldsheet supersymmetry generators, $\bar{Q}$, can be used to play the rôle of a BRST operator. We need only consider fields in representations of the left-moving $N = 2$ supersymmetry which are elements of cohomology of $\bar{Q}$. This was the trick used in[9] and again in [10]. Fixing on $\bar{Q}$-cohomology will again localize information about the twisted sector to the fixed point set. Our interest in this paper is in *isolated* fixed points and thus the $\bar{Q}$-cohomology is trivial. In other words, we ignore all right-moving oscillators. Thus the computation focuses purely on the left-moving sector.

As an element of $\mathrm{SL}(d, \mathbb{C})$ let g have eigenvalues $\exp(2\pi i\nu_i)$, for $0 \leq \nu_i < 1$ and $i = 1 \cdots d$. Following the notation of [10] this implies we have expansions for the left-movers:

$$x^i(z) = \sum_{u \in \mathbb{Z} - \nu_i} x_u^i z^{-u}$$

$$2\partial\bar{x}^i(z) = \sum_{u \in \mathbb{Z} + \nu_i} \rho_u^i z^{-u-1}$$

(1)

$$\gamma^i(z) = \sum_{u \in \mathbb{Z} - \tilde{\nu}_i} \gamma_u^i z^{-u-\frac{1}{2}}$$

$$\bar{\gamma}^i(z) = \sum_{u \in \mathbb{Z} + \tilde{\nu}_i} \bar{\gamma}_u^i z^{-u-\frac{3}{2}}$$

where $\tilde{\nu}_i$ is defined as $\nu_i - \frac{s}{2}$ (mod 1) such that $-1 < \tilde{\nu}_i \leq 0$; with the Ramond sector given by $s = 0$ and the Neveu-Schwarz sector given by $s = 1$.

The energy of the twisted vacuum is [1, 10]

(2)
$$E = \frac{1}{2} \sum_{i=1}^{d} \Big(\nu_i(1 - \nu_i) + \tilde{\nu}_i(1 + \tilde{\nu}_i) \Big) + \frac{d}{8} - 1.$$

The $N = (2,2)$ superconformal field theory has left and right U(1) currents. For a heterotic compactification, the left-moving current is part of the unbroken gauge symmetry whilst the right-moving current corresponds to an R-charge. The vacuum charges are

(3)
$$q = -\frac{d}{2} - \sum_i \tilde{\nu}_i$$

$$\bar{q} = -\frac{d}{2} + \sum_i \nu_i.$$

From the vacuum we build states by applying operators from the expansions (1). The lowest modes we denote by

(4) $$x_i \equiv x^i_{-\nu_i}, \quad \rho_i \equiv \rho^i_{\nu_i-1}, \quad \gamma_i \equiv \gamma^i_{-1-\tilde{\nu}_i}, \quad \overline{\gamma}_i \equiv \overline{\gamma}^i_{\tilde{\nu}_i}.$$

Thus we have weights and charges given by Table 1.

	x_i	ρ_i	γ_i	$\overline{\gamma}_i$
E	ν_i	$1 - \nu_i$	$1 + \tilde{\nu}_i$	$-\tilde{\nu}_i$
q	0	0	-1	1
$\bar{q}$	0	0	0	0

Table 1: Weights and charges of the fields

To preserve the $N = 2$ right-moving supersymmetry when the singlet is used as a marginal operator, when viewed as a right-moving chiral primary field, it must have $\bar{q} = 1$. We will work in the right-moving Ramond sector. By spectral flow this means we are looking for states with

(5) $$\bar{q} = 1 - \frac{d}{2}.$$

States in the orbifold must be invariant under the action of G. Consider the action of $h \in G$ on a g-twisted state. The result is a state in the $h^{-1}gh$-twisted sector. G-invariant states therefore consist of orbits spanning the elements of each conjugacy class of G. Each member of this orbit must be invariant under elements $h \in G$ which commute with g.

Suppose h commutes with g. We may assume the action of both g and h have been diagonalized on x_i. The action of h on the modes (4) is thus given.

The g-twisted vacuum itself may transform nontrivially under h as analyzed in [10]. We also need to restrict attention to an odd fermion number for the GSO projection. The condition (5) is given by $\sum \nu_i = 1$. In this case we may then state the transformation rules as follows:

- If $\nu_i \leq \frac{1}{2}$ for all i, then the vacuum is invariant and the NS-vacuum has odd fermion number.

- If $\nu_j > \frac{1}{2}$ (which can only be true for a single j) then the vacuum transforms like $\bar{x}_j$ and the NS-vacuum has even fermion number.

This gives a complete algorithm for computing the spectrum of massless singlets on an orbifold. For each conjugacy class of G we enumerate states with $E = q = 0$ and $\bar{q}$ given by (5) by applying polynomials in the modes (4) to the vacuum which are invariant under the centralizer and have odd fermion number.

3. Dimension 2

In this section we will prove the (0,2)-McKay correspondence in dimension two by checking each possible case of $\mathbb{C}^2/G$ for G a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. As is well-known, these finite groups are in one-to-one correspondence with the Dynkin diagrams of A_n, D_n or E_n and we label the groups accordingly.

The moduli space of vector bundles is unobstructed in dimension two [11]. In other words, there is too much supersymmetry for an "interesting" superpotential. This also implies that there can be no worldsheet instanton corrections to the dimension of the moduli space. It follows that string theory implies the (0,2)-McKay correspondence must work.

In dimension two the moduli space has a quaternionic Kähler structure. We do not use this fact but it motivates the counting of states in terms of quaternionic degrees of freedom.

3.1. The Orbifold Spectrum of States

First we do the orbifold computation.

Theorem 1. *For the orbifold $\mathbb{C}^2/G$, where G corresponds to A_n, D_n or E_n as a subgroup of $\mathrm{SL}(2, \mathbb{C})$, we have $3n + 1$ quaternionic singlets.*

Before we prove this theorem, let us note that we know from the standard McKay correspondence that n singlets correspond to blowing up the

singularity since the finite groups A_n, D_n or E_n each have n nontrivial conjugacy classes. Thus the theorem indicates that we should expect there to be $2n + 1$ singlets associated with deforming the tangent bundle. We now prove the theorem individually in each case of A_n, D_n or E_n. From (5) we look for $\bar{q} = 0$ states.

Any nontrivial element $g \in G$ corresponds to $\nu_1 + \nu_2 = 1$. First we consider the left-moving Ramond sector for which $\tilde{\nu}_1 = -\nu_2$ and $\tilde{\nu}_2 = -\nu_1$. One easily computes $E = q = \bar{q} = 0$ but the vacuum has even fermion number. There are thus no singlets intrinsic to this sector.[2]

We can therefore focus purely on the NS-sector. We now consider each case.

3.1.1. $\mathbb{C}^2/\mathbb{Z}_2$. There is only one twisted sector. One computes for the vacuum

(6)
$$\nu_i = \tfrac{1}{2}, \quad \tilde{\nu}_i = 0$$
$$q = -\sum(\tilde{\nu}_i + \tfrac{1}{2}) = -1$$
$$\bar{q} = \sum(\nu_i - \tfrac{1}{2}) = 0$$
$$E = -\tfrac{1}{2}.$$

The modes have weights and charges

(7)

	E	q	$\bar{q}$
x_i	$\frac{1}{2}$	0	0
ρ_i	$\frac{1}{2}$	0	0
γ_i	1	-1	0
$\bar{\gamma}_i$	0	1	0

This yields the following states with $E = q = \bar{q} = 0$:

- $x_i \bar{\gamma}_j |g\rangle$ giving 4 singlets.

- $\rho_i \bar{\gamma}_j |g\rangle$ giving 4 more singlets.

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[2]A heterotic string compactification on a complex surface with the standard embedding gives a six-dimensional theory with an $E_8 \times E_7$ gauge symmetry. In this context we have left-moving fermions ψ_α, $\alpha = 1, \ldots, 6$ from the SO(12) remnant of E_8 not involved in the conformal field theory. Odd products of these may be applied to the twisted vacuum to build a spinor **32** of SO(12). Other massless states build this up into a **56** of E_7 for each conjugacy class of G. This is again a statement of the conventional McKay correspondence.

All of these are invariant under g and have odd fermion number. These states are complex. Therefore this counts as 4 quaternionic singlets in total. This proves the theorem for A_1.

3.1.2. $\mathbb{C}^2/\mathbb{Z}_q$. Let $g \in \mathbb{C}^2/\mathbb{Z}_q$ have eigenvalues $\exp(2\pi i \nu_i)$ for $= 1, 2$. If $\nu_1 = \nu_2 = \frac{1}{2}$ we reduce to the $\mathbb{Z}_2$ case above. Otherwise, by relabeling if necessary, we let $\nu_1 = p/q$, where $2p < q$.

The vacuum $|p, q\rangle$ then has

$$q = -\sum(\tilde{\nu}_i + \tfrac{1}{2})$$
(8)
$$\bar{q} = \sum(\nu_i - \tfrac{1}{2}) = 0$$
$$E = \tfrac{p}{q} - 1$$

while the excitation modes have

(9)

	E	q	$\bar{q}$
x_i	$\frac{p}{q}, 1 - \frac{p}{q}$	0	0
ρ_i	$1 - \frac{p}{q}, \frac{p}{q}$	0	0
γ_i	$\frac{p}{q} + \frac{1}{2}, \frac{1}{2} - \frac{p}{q}$	-1	0
$\bar{\gamma}_i$	$\frac{1}{2} - \frac{p}{q}, \frac{p}{q} + \frac{1}{2}$	1	0

This yields the following states

- $x_2|p, q\rangle, \rho_1|p, q\rangle, \bar{\gamma}_1\bar{\gamma}_2 x_1|p, q\rangle, \bar{\gamma}_1\bar{\gamma}_2\rho_2|p, q\rangle$. Always invariant.

- $x_1^a\rho_2^b|p, q\rangle$ where $a + b = q - 1$, but invariant only if $p = 1$.

This counts as $4 + q$ half-quaternionic singlets in total if $p = 1$, and 4 otherwise. Combining all sectors in the $\mathbb{Z}_q$-orbifold we then have $3q - 2$ complete quaternionic singlets. That is, an A_n singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$ contributes $3n + 1$ singlets.

3.1.3. $\mathbb{C}^2/D_n$. Let g generate the group $\mathbb{Z}_{2(n-2)}$ and h be such that $g^{n-2} = h^2 = -1$ and $ghg = h$.

- The (-1)-twisted sector needs to be invariant under all of D_n. Only a single quaternionic dimension remains.

- The g-twisted sector (including the g^{-1} conjugate) contributes n as above.

- Each $g^2, \ldots, g^{n-3}$ sector contributes 2 as above.

- There remain two conjugacy classes given by h and gh. Each is like the $\mathbb{Z}_4$-twist and each gives 4 quaternionic singlets.

This gives a total of $3n + 1$.

3.1.4. $\mathbb{C}^2/E_6$. There are 7 conjugacy classes. One is the identity. Then:

- The (-1)-twisted sector contributes 1.
- A single $\mathbb{Z}_4$ sector gives 4.
- Two sectors are twisted by $\mathbb{Z}_6$. Each gives 5.
- The squares of the above each give 2.

The total is 19.

3.1.5. $\mathbb{C}^2/E_7$. There are 8 conjugacy classes. One is the identity. Then:

- The (-1)-twisted sector contributes 1.
- A single independent $\mathbb{Z}_4$ sector gives 4.
- One sector is twisted by $\mathbb{Z}_8$. This gives 6. The square of this sector and cube each give 2 more.
- One sector is twisted by $\mathbb{Z}_6$. This gives 5. The square of this gives 2.

The total is 22.

3.1.6. $\mathbb{C}^2/E_8$. There are 9 conjugacy classes. One is the identity. Then:

- The (-1)-twisted sector contributes 1.
- A single independent $\mathbb{Z}_4$ sector gives 4.
- One sector is twisted by $\mathbb{Z}_{10}$. This gives 7. The square of this sector and cube and 4th power each give 2 more.
- One sector is twisted by $\mathbb{Z}_6$. This gives 5. The square of this gives 2.

The total is 25.

This concludes the proof of Theorem 1.　　　　　　　　　　$\square$

3.2. Geometry of the Resolution

The quotient singularity $\mathbb{C}^2/G$ is resolved by an ALE space which we denote X. We want to count the number of deformations of the tangent bundle of this space. This problem has been solved by Kronheimer and Nakajima [12].
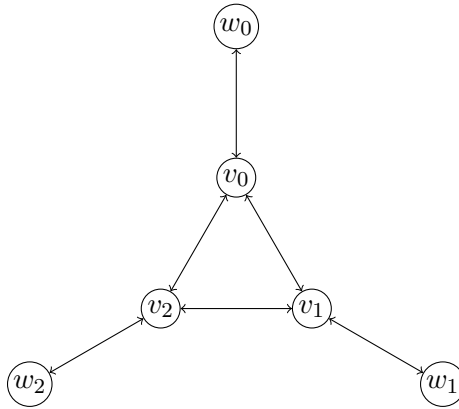
Figure 1: The Kronheimer–Nakajima quiver for A_2.

The moduli space of (Yang–Mills connections on) the bundle is given by the moduli space of a quiver representation. To construct this quiver one begins with the McKay quiver for G and then, for each node, one adds a new node and a path each way between these two nodes. For example, for A_2 we obtain Figure 1 where each double-headed arrow represents an arrow in both directions.

Let V and W be two finite-dimensional representations of G. Let R_i be the irreducible representations of G. The decompositions of V and W are then

$$V = \bigoplus_i R_i^{\oplus v_i}, \quad W = \bigoplus_i R_i^{\oplus w_i}. \tag{10}$$

The dimension vector for the quiver representation is given by v_i and w_i. The v_i's are the dimensions for the McKay quiver and the w_i's correspond to the added nodes as in Figure 1.

Let Q be the two-dimensional representation given by the embedding $G \subset \mathrm{SL}(2, \mathbb{C})$. Kronheimer and Nakajima then construct the desired bundle E as the cohomology of a complex

$$(11) \quad (V \otimes \mathscr{R})^G \longrightarrow (Q \otimes V \otimes \mathscr{R})^G \oplus (W \otimes \mathscr{R})^G \longrightarrow (\Lambda^2 Q \otimes V \otimes \mathscr{R})^G,$$

where $\mathscr{R}$ is a naturally defined bundle on X which breaks up as a sum of so-called tautological bundles $\mathscr{R}_i$. We refer to [12] for details of the construction. The data giving the maps in (11) come from the data of the representation of the quiver subject to constraints coming from the ADHM equation.

The representation W yields the asymptotic behaviour of the bundle $E \to X$. Since we are looking at the tangent bundle this is simply the representation Q. To determine the representation V we need to compute the Chern classes of the tangent bundle. Obviously $c_1(E) = 0$.

To compute c_2 of the tangent bundle we use the Gauss-Bonnet formula for a manifold with boundary given by Chern [13]:

$$(12) \qquad \chi(M) = \int_M c_{\dim M} + \int_{\partial M} \Pi,$$

where Π is the Chern–Simons form for the tangent bundle. For M given by a 4-dimensional ball we immediately obtain

$$(13) \qquad \int_{S^3} \Pi = 1.$$

Thus, if X is an ALE space with asymptotic holonomy G we have

$$(14) \qquad \chi(X) = \int_X c_2(T_X) + \frac{1}{|G|}.$$

This Euler characteristic is given by the Euler characteristic of the exceptional set and is therefore $n + 1$ for A_n, D_n or E_n.

Following [12] we introduce four $(n + 1)$-dimensional vectors indexed by $i = 0, \ldots, n$ where the 0 index is associated to the trivial representation. $\mathbf{v}$ is defined with components v_i, $\mathbf{w}$ with components w_i and $\mathbf{d}$ (denoted $\mathbf{n}$ in [12]) has components given by the dimensions of the irreducible representations of G. Finally $\mathbf{u}$ is defined by the components u_i given by

$$(15) \qquad \begin{aligned} c_1(E) &= \sum_{i \neq 0} u_i c_1(\mathscr{R}_i) \\ u_0 &= \mathbf{d}.\mathbf{w} - \sum_{i \neq 0} d_i u_i. \end{aligned}$$

In the case of the tangent bundle $W = Q$ and thus $\mathbf{u}$ has components $(2, 0, 0, \ldots)$. There is a relationship

$$(16) \qquad C\mathbf{v} = \mathbf{w} - \mathbf{u},$$

where C is the Cartan matrix for the extended Dynkin diagram of G. This determines $\mathbf{v}$ up to the kernel of C, which is given by $\mathbf{d}$. The relation

$$(17) \qquad -\int_X ch_2(E) = -\sum u_i \int_X ch_2(\mathscr{R}_i) + \frac{\dim V}{|G|},$$

fixes this ambiguity. In our case, using (14), this equation becomes

$$\dim V = (n+1)|G| - 1. \tag{18}$$

Using $|G| = \sum d_i^2$, (16) and (18) fixes

$$\mathbf{v} = (n+1)\mathbf{d} - (1,0,0,\dots). \tag{19}$$

The dimension of the moduli space of a quiver is easy to determine. The dimension vectors v_i and w_i determine the sizes of the matrices associated to each arrow. From the total number of matrix entries one then subtracts the number of relations and finally one subtracts the dimensions of the $\mathrm{GL}(-,\mathbb{C})$ actions on each node.

Since X is a noncompact space we need to worry about boundary conditions on the bundle. A natural choice is to consider a "framed bundle" by fixing the asymptotic behaviour. The Kronheimer–Nakajima quiver makes the framing picture very clean. The asymptotic form of the bundle is determined purely by W. The GL transformations on the nodes associated with W rotate this framing. In order to provide a fixed framing for the bundle we therefore *ignore* the GL action on these nodes. So, when computing the dimension of the moduli space, we only consider the GL action on the nodes associated to V.

At the end of Section 9 of [12] the quaternionic dimension for framed bundles is computed as

$$\dim \mathscr{M} = \tfrac{1}{2}\mathbf{v}.(\mathbf{w} + \mathbf{u}). \tag{20}$$

Plugging in the values for the tangent bundle obtained above this yields

$$\dim \mathscr{M} = 2n + 1. \tag{21}$$

Comparing to Theorem 1, this proves the (0,2)-McKay correspondence for dimension two.

3.3. Compact Examples

A K3 surface can be constructed as the resolution of an orbifold S/G where S is a 4-torus or another K3 surface. We should check that the (0,2)-McKay correspondence works in these compact examples. The naïve statement would be that the number of deformations of the tangent bundle on

K3 should equal the number of G-invariant deformations on S plus a contribution of $2n + 1$ from each ADE-singularity.

This is not quite true — there is a correction as observed in [14]. We may compute the dimension of the moduli space of the tangent bundle of a K3 surface easily by index theory:

$$\sum_j (-1)^j \dim H^j(\mathrm{End}(T)) = 2 - 2 \dim \mathscr{M}$$

(22)

$$= \int_X ch(T) \wedge ch(T) \wedge td(T) = -88.$$

It follows that $\dim \mathscr{M} = 45$.

Let S/G have global holonomy H (i.e., $H \cong G$ if S is a 4-torus and $H \cong \mathrm{SU}(2)$ if S is a K3 surface) and let $Z(H)$ be the centralizer of $H \subset \mathrm{SU}(2)$. A heterotic string compactified on S/G will have gauge group $E_8 \times E_7 \times Z(H)$. The process of blowing up the singularities will Higgs away $Z(H)$ which will eat up $\dim Z(H)$ scalars. The correct prediction is therefore

(23) $$45 = \dim \mathscr{M}(T_S)^G + \sum_j (2n_j + 1) - \dim Z(H),$$

where the sum is over the A_n, D_n, E_n singularities of S/G.

Consider, for example, the case where S is a 4-torus and $G \cong \mathbb{Z}_3$. Let S have holomorphic coordinates (z^1, z^2) and let G be generated by the action $g : (z^1, z^2) \mapsto (\omega z^1, \omega^2 z^2)$. $H^1(\mathrm{End}(T_S))$ is spanned by differential forms

(24) $$d\bar{z}^{\bar{i}} \otimes dz^j \otimes \frac{\partial}{\partial z^k}.$$

It is easy to check that two of these are invariant under g. This amounts to $\mathscr{M}(T_S)^G$ having one quaternionic dimension. The centralizer of G in $\mathrm{SU}(2)$ is $\mathrm{U}(1)$ and the orbifold has nine A_2 singularities. Equation (23) is indeed satisfied.

There are only seven possible orbifolds T^4/G where G fixes a point in T^4. These are listed in Table 2. All satisfy (23).

It is interesting to ask how one would prove (23) mathematically without using the Higgsing argument from string theory. It seems[3] that the correction from $Z(H)$ arises from the bundle framing issue. For the resolution of $T^4/\mathbb{Z}_2$, for example, there must be some $\mathrm{SU}(2)$ symmetry of the bundle that amounts to a simultaneous frame rotation of each of the 16 local pictures of

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[3]The author thanks M. Douglas for suggesting this.

G	Singularities	dim $\mathscr{M}(T_S)^G$	$\sum_j (2n_j + 1)$	$Z(H)$
$\mathbb{Z}_2$	$16A_1$	0	48	SU(2)
$\mathbb{Z}_3$	$9A_2$	1	45	U(1)
$\mathbb{Z}_4$	$4A_3 + 6A_1$	0	46	U(1)
$\mathbb{Z}_6$	$A_5 + 4A_2 + 5A_1$	0	46	U(1)
D_4	$2D_4 + 3A_3 + 2A_1$	0	45	
D_5	$D_5 + 2A_2 + 3A_3 + A_1$	0	45	
E_6	$E_6 + D_4 + 4A_2 + A_1$	0	45	

Table 2: Quotients of the form T^4/G.

the tangent bundle on the ALE space associated to A_1. Thus we over-count by 3 if we simply add up the local contributions from each ALE space. The author does not know how to make this argument rigorous.

One may also check that the prediction works for the cases where S is a K3 surface. These are listed in [15]. Let $c = \sum_j n_j$ denote the contribution to the Picard number of the exceptional set as in Table 2 of [15]. Let there be N_{sing} singularities in the K3 orbifold. Then (23) implies

$$(25) \qquad N_{\text{sing}} = 45 - 2c - \dim \mathscr{M}(T_S)^G.$$

If G is a large group it is reasonable to expect that dim $\mathscr{M}(T_S)^G = 0$. Indeed, one can check that $N_{\text{sing}} = 45 - 2c$ for most of the examples that appear late in Table 2 of [15].

4. Abelian Quotients in Dimension Three

The moduli space of vector bundles in dimension three can be obstructed. Accordingly, there is also the possibility of worldsheet instanton corrections to the superpotential and hence the dimension of the moduli space. Therefore, there is no reason to expect the number of first order deformations counted by massless states at the orbifold should agree with the moduli count in the resolution. Nevertheless we will find perfect agreement in many cases.

4.1. Toric Geometry

We will restrict attention to abelian quotients so that we may use the tools of toric geometry. As usual, we use the homogeneous coordinate ring [16] as based on a short exact sequence:

$$(26) \qquad 0 \longrightarrow \mathsf{M} \overset{\mathscr{A}}{\longrightarrow} \mathbb{Z}^{\oplus N} \overset{\Phi}{\longrightarrow} \mathsf{D} \longrightarrow 0$$

M is a lattice of rank d. $\mathscr{A}$ is an $N \times d$ matrix. The rows of $\mathscr{A}$ give the coordinates of a set of N points in the lattice N dual to M. We will use the same symbol $\mathscr{A}$ to denote this point set. Each point is associated to a homogeneous coordinate x_i, $i = 1, \ldots, N$. We then have a homogeneous coordinate ring $S = \mathbb{C}[x_1, \ldots, x_N]$.

D is an abelian group which induces an action on the homogeneous coordinates as follows. Let $r = N - d$ denote the rank of D. Then $\mathsf{D} \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^r$ acts on the homogeneous coordinates with charges given by the matrix Φ. If D has a torsion part G, then, in addition to $(\mathbb{C}^*)^r$, we have a finite group action of G on the homogeneous coordinates the charges of which are given by the kernel of the matrix $\mathscr{A}^t$ acting on $(\mathbb{Q}/\mathbb{Z})^{\oplus N}$.

The fan Σ is defined as a fan over a simplicial complex with vertices $\mathscr{A}$. This combinatorial information defines an ideal $B \subset S$ as explained in [16]. The toric variety X_Σ is then given by the quotient

$$(27) \qquad X_\Sigma = \frac{\operatorname{Spec} S - V(B)}{(\mathbb{C}^*)^r \times G}.$$

For example, we may build $\mathbb{C}^3/\mathbb{Z}_5$, where the generator of $\mathbb{Z}_5$ acts as $\exp\left(2\pi i(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})\right)$ on (x_1, x_2, x_3) as follows. We set $N = n = 3$ and thus require a 3×3 integral matrix $\mathscr{A}$ whose kernel from a right-action on $(\mathbb{Q}/\mathbb{Z})^{\oplus 3}$ is generated by $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$. Clearly

$$(28) \qquad \mathscr{A} = \begin{pmatrix} 5 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

suffices. In this case Σ is comprised of a single cone over a triangle and $B = S$.

In the case X is a Calabi–Yau variety, all the points $\mathscr{A}$ lie in a hyperplane in N. As is well-known, the orbifold singularity $\mathbb{C}^3/G$ can be completely resolved crepantly by expanding the set $\mathscr{A}$ to contain all the points of N in
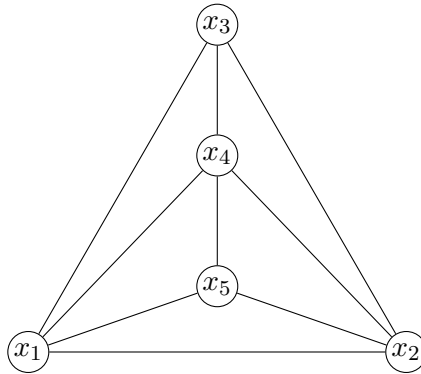
Figure 2: Toric Resolution of $\mathbb{C}^3/\mathbb{Z}_5$.

its interior hull and taking Σ to be a fan over a simplicial complex including all the new points of $\mathscr{A}$. Let Δ denote this simplicial complex. Each triangle in Δ has area $\frac{1}{2}$, D is torsion-free and X_Σ is smooth. For the above example of $\mathbb{C}^3/\mathbb{Z}_5$ we show the only choice of Δ in Figure 2.

For a smooth toric variety, the homogeneous coordinate ring $S = \mathbb{C}[x_1, \ldots, x_N]$ is multigraded by the free module D:

$$(29) \qquad S = \bigoplus_{\mathbf{d} \in \mathsf{D}} S_{\mathbf{d}}.$$

Let $S(\mathbf{q})$ be the free S-module with grades shifted so that $S(\mathbf{q})_{\mathbf{d}} = S_{\mathbf{q}+\mathbf{d}}$ as usual. Line bundles or invertible sheaves $\mathscr{O}(\mathbf{q})$ on X_Σ are then associated to modules $S(\mathbf{q})$ for any $\mathbf{q} \in \mathsf{D}$.

Let $\mathbf{q}_i$ denote the multi-degree of x_i. It was shown in [17] that the tangent sheaf $\mathscr{T}$ of X_Σ is given by

$$(30) \qquad 0 \longrightarrow \mathscr{O}^{\oplus r} \overset{E}{\longrightarrow} \bigoplus_{i=1}^N \mathscr{O}(\mathbf{q}_i) \longrightarrow \mathscr{T} \longrightarrow 0,$$

where E is an $N \times r$ matrix whose (i,j)-th entry is $\Phi_{ji} x_i$.

We want to analyze the moduli space of the tangent sheaf. The first order deformations are given by the vector space $\operatorname{Ext}^1(\mathscr{T}, \mathscr{T})$ but there may be obstructions. We refer to [18] for a thorough review of these facts.

In analogy with the Kronheimer–Nakajima construction we need to identify the asymptotic form of the tangent bundle away from the exceptional set. For the time being let us assume that $\mathbb{C}^3/G$ has an isolated singularity at the origin. Since (x_1, x_2, x_3) were coordinates on the orbifold prior to

resolution it is natural to define the sheaf

$$(31) \qquad \mathscr{W} = \mathscr{O}(\mathbf{q}_1) \oplus \mathscr{O}(\mathbf{q}_2) \oplus \mathscr{O}(\mathbf{q}_3),$$

which asymptotically resembles the tangent sheaf away from the origin.

The inclusion map $\mathscr{W} \to \bigoplus_{i=1}^{N} \mathscr{O}(\mathbf{q}_i)$ induces a map $f : \mathscr{W} \to \mathscr{T}$. The cokernel of this map is isomorphic to the cokernel of the map

$$(32) \qquad \mathscr{O}^{\oplus r} \xrightarrow{\;E'\;} \bigoplus_{\alpha=4}^{N} \mathscr{O}(\mathbf{q}_\alpha),$$

where the matrix E' is the matrix E with the first three rows removed. From now on, the index α will always be in the set $4, \ldots, N$, and $N = r + 3$.

Let Φ' be the matrix Φ with the first three columns removed. One can show the square matrix Φ' must be invertible as follows. Suppose it were not. Then there would be set of row operations which would render a row all zero. The same row operations applied to Φ would imply that there is a linear relation between the coordinates of x_1, x_2 and x_3. This cannot be true as they are the vertices of a triangle. Since Φ' is invertible, by a change of basis we may replace E' by the diagonal matrix $\mathrm{diag}(x_4, x_5, \ldots, x_{r+3})$.

Let D_i be the toric divisor given by $x_i = 0$, $i = 1, \ldots, N$. The exceptional divisor then has r irreducible components given by D_α, $\alpha = 4, \ldots, N$. The short exact sequences

$$(33) \qquad 0 \longrightarrow \mathscr{O}(-\mathbf{q}_\alpha) \xrightarrow{\;x_\alpha\;} \mathscr{O} \longrightarrow \mathscr{O}_{D_\alpha} \longrightarrow 0,$$

show that the cokernel of the map in (32) is isomorphic to $\bigoplus_\alpha \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)$. This proves

Theorem 2. *The tangent sheaf $\mathscr{T}$ of the toric resolution of $\mathbb{C}^3/G$ is given by an extension*

$$(34) \qquad 0 \longrightarrow \bigoplus_{i=1}^{3} \mathscr{O}(\mathbf{q}_i) \longrightarrow \mathscr{T} \longrightarrow \bigoplus_{\alpha=4}^{N} \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha) \longrightarrow 0.$$

4.2. Quivers

From Theorem 2 the tangent bundle is a deformation of the direct sum

$$(35) \qquad \bigoplus_{i=1}^{3} \mathscr{O}(\mathbf{q}_i) \oplus \bigoplus_{\alpha=4}^{N} \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha).$$

A quiver is associated with this sum in a standard way [19]. That is, each summand $\mathscr{V}_i$ is associated with a node for $i = 1, \ldots, N$. Then $\dim \mathrm{Ext}^1(\mathscr{V}_i, \mathscr{V}_j)$ arrows are drawn from node i to node j.

The tangent bundle then corresponds to a representation of this quiver. According to the precise form of the extension (34), we associate nonzero values to certain arrows representing $\mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}(\mathbf{q}_j))$. We now explicitly construct this quiver.

Theorem 3. *The groups* $\mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta))$ *and* $\mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha),$ $\mathscr{O}(\mathbf{q}_j))$ *are determined by the normal bundle of the curve* $C_{\alpha\beta}$ *or* $C_{\alpha j}$ *respectively, where* $C_{ij} = D_i \cap D_j$. *If* N_i *is the normal bundle of the embedding* $C_{ij} \subset D_i$ *then*

(36)
$$\dim \mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta)) = h^0(C_{\alpha\beta}, N_\alpha)$$
$$\dim \mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}(\mathbf{q}_j)) = 1 + h^0(C_{\alpha j}, N_\alpha).$$
$$\dim \mathrm{Ext}^1(\mathscr{O}(\mathbf{q}_j), \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)) = 0$$

The proof is facilitated by the language of the derived category. In what follows, an underline represents position zero in a complex and we freely identify a sheaf with a complex of sheaves with an entry only in position zero. A shift of a complex n places left is denoted $[n]$.

First note that $\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)$ is $\mathscr{O} \xrightarrow{x_\alpha} \underline{\mathscr{O}(\mathbf{q}_\alpha)}$. So $\mathscr{H}om(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta))$ is

(37)
$$\mathscr{O}(-\mathbf{q}_\alpha) \xrightarrow{\begin{pmatrix} -x_\alpha \\ x_\beta \end{pmatrix}} \begin{matrix} \mathscr{O} \\ \oplus \\ \underline{\mathscr{O}(\mathbf{q}_\beta - \mathbf{q}_\alpha)} \end{matrix} \xrightarrow{(\, x_\beta \ \ x_\alpha \,)} \mathscr{O}(\mathbf{q}_\beta).$$

But this is exactly $\mathscr{O}_{C_{\alpha\beta}}(\mathbf{q}_\beta)[-1]$, where $C_{\alpha\beta} = D_\alpha \cap D_\beta$. So

(38)
$$\mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta)) = H^0(C_{\alpha\beta}, \mathscr{O}(\mathbf{q}_\beta))$$
$$= H^0(C_{\alpha\beta}, N_\alpha),$$

where N_α is the normal bundle of $C_{\alpha\beta} \subset D_\alpha$. It should be noted that if $C_{\alpha\beta}$ is the empty set then (37) is an exact complex and thus the above Ext group vanishes.

Similarly $\mathscr{H}om(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}(\mathbf{q}_j))$ is

(39)
$$\underline{\mathscr{O}(\mathbf{q}_j - \mathbf{q}_\alpha)} \xrightarrow{x_\alpha} \mathscr{O}(\mathbf{q}_j)$$

which equals $\mathscr{O}_{D_\alpha}(\mathbf{q}_j)[-1]$. So

(40) $$\mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}(\mathbf{q}_j)) = H^0(\mathscr{O}_{D_\alpha}(\mathbf{q}_j)).$$

But we have an exact sequence

(41) $$0 \longrightarrow \mathscr{O}_{D_\alpha} \overset{x_j}{\longrightarrow} \mathscr{O}_{D_\alpha}(\mathbf{q}_j) \longrightarrow \mathscr{O}_{C_{\alpha j}}(\mathbf{q}_j) \longrightarrow 0$$

which, since $H^1(\mathscr{O}_{D_\alpha}) = 0$ (as D_α is rational), implies

(42) $$\dim \mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}(\mathbf{q}_j)) = 1 + \dim H^0(C_{\alpha j}, N_\alpha).$$

Next, $\mathrm{Ext}^1(\mathscr{O}(\mathbf{q}_j), \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)) = H^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha - \mathbf{q}_j))$. We have a short exact sequence

(43) $$0 \longrightarrow \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha - \mathbf{q}_j) \overset{x_j}{\longrightarrow} \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha) \longrightarrow \mathscr{O}_{C_{\alpha j}}(\mathbf{q}_\alpha) \longrightarrow 0.$$

Noting that $\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)$ is the canonical sheaf $\mathscr{K}_{D_\alpha}$ of D_α, we have by Serre duality that $H^0(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)) = H^2(\mathscr{O}_{D_\alpha}) = 0$ and $H^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)) = H^1(\mathscr{O}_{D_\alpha}) = 0$. The long exact sequence then gives

(44) $$\mathrm{Ext}^1(\mathscr{O}(\mathbf{q}_j), \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)) = H^0(\mathscr{O}_{C_{\alpha j}}(N_j)).$$

To compute the proof of Theorem 3 we need to show that $H^0(\mathscr{O}_{C_{\alpha j}}(N_j)) = 0$. The dimensions of the groups $H^0(\mathscr{O}_{C_{ij}}(N_j))$ can be read directly from the toric data, i.e., the triangulation of the pointset $\mathscr{A}$ as in Figure 2. The following result is standard in toric geometry. See, for example, [20].

Let D_i and D_j correspond to two points a_i and a_j in $\mathscr{A}$. If D_i and D_j intersect at all it is along a $\mathbb{P}^1$ corresponding to a line joining a_i and a_j in the triangulation. Choose integral coordinates in the plane containing $\mathscr{A} \subset \mathsf{N}$ such that a_i becomes the origin and a_j has coordinates (x_1, y_1). Then $C = D_i \cap D_j$ is described by the toric subfan in Figure 3. The normal bundle for C is $\mathscr{O}(m) \oplus \mathscr{O}(-2 - m)$ where $m = x_2 y_3 - x_3 y_2$ and, in particular, the normal bundle for $C \subset D_i$ is $\mathscr{O}(m)$.

If $\theta > \pi$ then $m < 0$ and so $H^0(C, N_i) = 0$. This is is the case for (44) since D_j is a vertex of the triangle forming the convex hull of the points in $\mathscr{A}$. This completes the proof. $\square$

We should also note that Figure 3 gives a very quick method of computing the dimension of the required Ext groups. These numbers can read off the toric diagram describing the resolution. This first trivial observation is that C_{ij} is only non-empty if the nodes a_i and a_j are connected by a line.
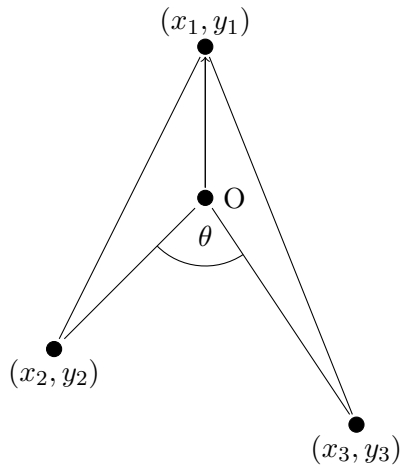
Figure 3: Toric diagram for $C_{ij} = D_i \cap D_j$.

This means that the toric diagram itself becomes a quiver. The rules are as follows.

- For each pair of points a_α, a_β in the interior of the triangle joined by a line compute $m = x_2 y_3 - x_3 y_2$ as in Figure 3. If $m = -1$ there is no contribution, otherwise we have an arrow of multiplicity $m + 1$ in the direction shown.

- For each pair of points a_α in the interior and α_j as a vertex we do the following. If these points are joined by a line this becomes an arrow from a_α to a_j of multiplicity $m + 2$. If the points are not joined we have an arrow of multiplicity one from a_α to a_j.

The form of the extension corresponding to the tangent sheaf in Theorem 2 tells us to give nonzero values to the arrows in the quiver associated to the maps in $\mathrm{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}(\mathbf{q}_j))$ associated to multiplication by x_j. From (41) this is the arrow associated with the "1" in the right-hand side of (42). All other arrows in the quiver will be associated with a zero map for the quiver representation yielding the tangent sheaf.

Now we have constructed the quiver, we need to compute the relations. The obstruction theory of sheaves on a Calabi–Yau threefold has been explored in the context of string theory in the D-brane literature and the same methods apply here. After all, both cases are concerned with $N = 1$ supersymmetry in four dimensions. We refer to [21, 22] for further details.

The key idea is that the obstruction theory is governed by a superpotential that corresponds to *loops* in the quiver. If there are no loops in the quiver then the superpotential must be zero. It follows that the moduli space is obstruction free. Most of the examples we consider have no loops. We we consider the case of a loop in Section 7.

Finally, to compute the dimension of the moduli space we total the number of degrees of freedom in the arrows of the quiver and subtract the dimension of the $GL(-, \mathbb{C})$ actions on each node. As in the case of Kronheimer–Nakajima quiver, we do not include the $\mathbb{C}^*$-actions on the three vertices of the triangle in order to fix a framing from the bundle. Each of the interior points corresponds to a $\mathbb{C}^*$-action that should be include. Thus the dimension of the moduli space is equal to the number of arrows (with multiplicities) minus the number of interior vertices.

Note that the number of interior points is equal to the number of irreducible components of the exceptional set which, in turn, equals the number of Kähler form deformations of the resolution. The total number of singlets for the resolved isolated singularity equals this number plus the number of deformations of the tangent sheaf. Thus we have a statement for the geometric construction:

> The number of singlets is given by the number of arrows in the quiver (including multiplicities) constructed from the toric diagram of the resolution as above.

This count corresponds to the non-linear σ-model computation and, as such, can be corrected by worldsheet instanton effects.

It should be noted that computations of moduli spaces of quivers require knowledge of stability conditions. Indeed, for the (0,2)-theory we should concern ourselves with μ-stable holomorphic bundles. However, it is clear that the tangent bundle on a Calabi–Yau manifold that is not locally a product is stable and lives in the interior of the space of stability conditions. This follows since the Hermitian connection on the tangent bundle is Hermitian-Yang-Mills. Furthermore, the bundle is not a direct sum and so we are not on the boundary of the space of polystable connections. Thus we may ignore issues of stability when dealing only with first order deformations of the tangent bundle.

5. Examples

5.1. $\mathbb{C}^3/\mathbb{Z}_{2m+1}$

In complex dimension three, from (5) we demand $\bar{q} = -\frac{1}{2}$.

We will first consider the quotient $\mathbb{Z}_{2m+1}$ generated by the weights

$$
(45) \qquad \left(\frac{1}{2m+1}, \frac{1}{2m+1}, \frac{2m-1}{2m+1} \right).
$$

For explicitness we begin with the case $m = 2$ corresponding to to $\mathbb{C}^3/\mathbb{Z}_5$ considered in Figure 2. Here we have four twisted sectors, twisted by $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ or $\frac{4}{5}$ respectively. The $\frac{1}{5}$ and $\frac{2}{5}$-twisted sectors contribute only $\bar{q} = -\frac{1}{2}$ states while the $\frac{3}{5}$ and $\frac{4}{5}$-twisted sectors contribute only $\bar{q} = \frac{1}{2}$ states.

In the $\frac{1}{5}$-sector we have, for the vacuum

$$
(46) \qquad
\begin{aligned}
\nu_i &= (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}), \quad \tilde{\nu}_i = (-\tfrac{3}{10}, -\tfrac{3}{10}, -\tfrac{9}{10}) \\
q &= -\sum (\tilde{\nu}_i + \tfrac{1}{2}) = 0 \\
\bar{q} &= \sum (\nu_i - \tfrac{1}{2}) = -\tfrac{1}{2} \\
E &= -\tfrac{5}{8} + \tfrac{1}{2} \sum \left(\nu_i(1-\nu_i) + \tilde{\nu}_i(1+\tilde{\nu}_i) \right) \\
&= -\tfrac{3}{5}
\end{aligned}
$$

The eigenvalues for the excitations are shown in Table 3. This gives 11 singlets:

- $x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_3, \rho_3 x_i, \gamma_3 \bar{\gamma}_i x_j$, for $i, j = 1, 2$.

$$(47)$$

$\frac{1}{5}$	E	q	$\bar{q}$
x_i	$\frac{1}{5}, \frac{1}{5}, \frac{3}{5}$	0	0
ρ_i	$\frac{4}{5}, \frac{4}{5}, \frac{2}{5}$	0	0
γ_i	$\frac{7}{10}, \frac{7}{10}, \frac{1}{10}$	-1	0
$\bar{\gamma}_i$	$\frac{3}{10}, \frac{3}{10}, \frac{9}{10}$	1	0

$\frac{2}{5}$	E	q	$\bar{q}$
x_i	$\frac{2}{5}, \frac{2}{5}, \frac{1}{5}$	0	0
ρ_i	$\frac{3}{5}, \frac{3}{5}, \frac{4}{5}$	0	0
γ_i	$\frac{9}{10}, \frac{9}{10}, \frac{7}{10}$	-1	0
$\bar{\gamma}_i$	$\frac{1}{10}, \frac{1}{10}, \frac{3}{10}$	1	0

Table 3: Excited modes in $\mathbb{C}^3/\mathbb{Z}_5$ twisted sectors.

In the $\frac{2}{5}$-sector we have

$$(48) \qquad \begin{aligned} \nu_i &= (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}), \quad \tilde{\nu}_i = (-\tfrac{1}{10}, -\tfrac{1}{10}, -\tfrac{3}{10}) \\ q &= -1 \\ \bar{q} &= -\tfrac{1}{2} \\ E &= -\tfrac{1}{2}. \end{aligned}$$

This yields 7 singlets:

- $x_i \bar{\gamma}_j, x_3 \bar{\gamma}_3, x_3^2 \bar{\gamma}_i$, for $i, j = 1, 2$.

This gives a total of 18 singlets.

Applying the rules of Section 4.2, the tangent sheaf corresponds to the quiver shown in Figure 4. Adding up the multiplicities on the arrows gives a total of 18. Thus the (0,2)-McKay correspondence works for this example.

The general case $\mathbb{C}^3/\mathbb{Z}_{2m+1}$ is very similar. Assuming $m > 1$ one can compute the number of singlets in each sector:

- $\frac{1}{2m+1}$: $5 + m(m+1)$
- $\frac{2}{2m+1}$: 5
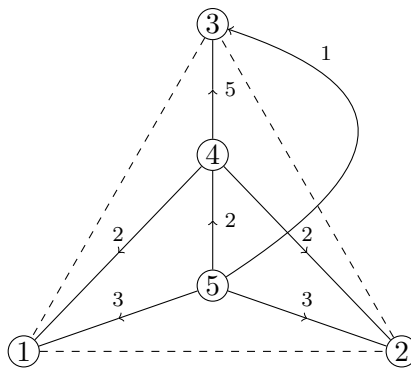 $\vdots$
- $\frac{m-1}{2m+1}$: 5
- $\frac{m}{2m+1}$: 7

Figure 4: Quiver for $\mathbb{C}^3/\mathbb{Z}_5$.
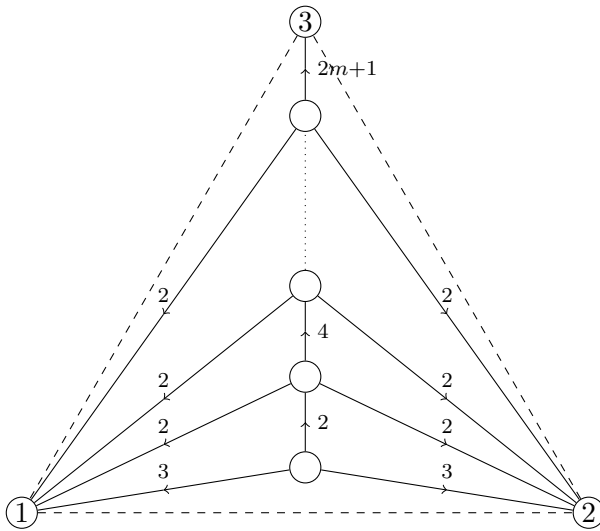
Figure 5: Quiver for $\mathbb{C}^3/\mathbb{Z}_{2m+1}$.

giving a total of $m^2 + 6m + 2$. This formula is also valid for $m = 1$.

The quiver is shown in Figure 5. Note that for clarity we have omitted the arrows of multiplicity one from interior points to non-adjacent vertices. **_We will always do this from now on._** Adding up all the multiplicities we again get $m^2 + 6m + 2$.

5.2. $\mathbb{C}^3/\mathbb{Z}_{11}$

The example of $\mathbb{C}^3/\mathbb{Z}_{2m+1}$ had no ambiguities in the resolution. The simplest case of an isolated singularity with ambiguities is $\mathbb{C}^3/\mathbb{Z}_{11}$ where the generator acts with weights $(\frac{1}{11}, \frac{2}{11}, \frac{8}{11})$.

The conformal field theory computation yields a count of singlets with $\bar{q} = -\frac{1}{2}$ as follows:

- $\frac{1}{11}$: 14
- $\frac{2}{11}$: 4
- $\frac{3}{11}$: 5
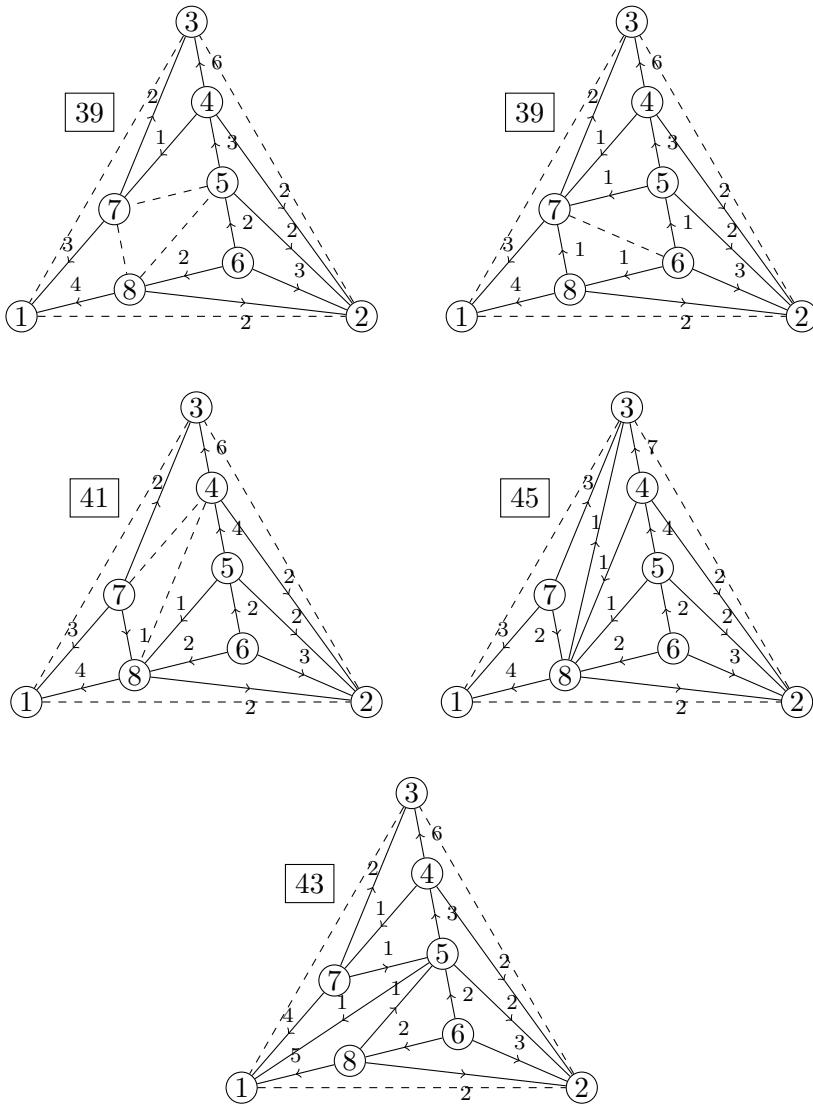- $\frac{6}{11}$: 7
- $\frac{7}{11}$: 9

for a total of 39.

Figure 6: Five quivers for the five resolutions of $\mathbb{C}^3/\mathbb{Z}_{11}$.

There are five possible resolutions of this orbifold given by five different triangulations of the point set $\mathscr{A}$. The quivers are shown in Figure 6. The number of singlets is shown in the square box next to each diagram.

Only two of the five possible resolutions give the same number of singlets as the conformal field theory. Thus, the (0,2)-McKay correspondence is not

true in a naïve sense. While the conformal field theory is expected to give a precise count for the number of singlets, the nonlinear sigma model may suffer from instanton corrections which may decrease the number of singlets. Thus it should come as no surprise that the geometrical computation may yield a higher number than the conformal field theory. What is perhaps surprising is that one always seems to need to work quite hard to find an example where there really are instanton corrections [14, 23].[4]

It must therefore be that the last three diagrams in Figure 6 contain rational curves which give corrections to the superpotential along the lines described in [24, 25].

6. The G-Hilbert Scheme

For an orbifold $\mathbb{C}^3/G$, each triangulation of the point set $\mathscr{A}$ leads to resolution of the singularity. There is, however, a distinguished resolution called the G-Hilbert scheme [26, 27]. This particular resolution has played a rôle in the McKay correspondence [4] but, in the context of (2,2)-models it appears to have no distinguished rôle. Indeed, one of the motivations of the analysis of topology change in string theory [28] was the fact that all possible crepant resolutions should somehow be equal. We have just seen above, however, that such egalitarianism does not extend to the (0,2) case.

Let $R = \mathbb{C}[x_1, x_2, x_3]$ and let $I \subset R$ be an ideal such that R/I is isomorphic to $\mathbb{C}^{\oplus|G|}$ as a vector space. Furthermore let us demand that the action of G on R makes R/I appear as the regular representation of G. An obvious example of such an ideal is

$$(49) \qquad (x_1 - a_1, x_2 - a_2, x_3 - a_3)(x_1 - b_1, x_2 - b_2, x_3 - b_3) \cdots ,$$

where $(a_1, a_2, a_3), (b_1, b_2, b_3), \ldots$ are the coordinates of a free orbit of G in $\mathbb{C}^3$. Less obvious examples of I are associated to orbits of G with fixed points. The G-Hilbert scheme parametrizes such ideals and it is shown in [29] that it provides a crepant resolution of the orbifold.

---

[4]It is conceivable that there are instanton corrections that kill moduli even when we do find an agreement in the count of singlets. That is, the superpotential obstructs deformations using massless modes away from the orbifold point. Then the same number of modes that were obstructed magically reappear at large radius when we ignore instanton corrections. We assume this is not the case but it would be nice to confirm this.

In the case where G is abelian, the G-Hilbert scheme is toric and so must correspond to some particular triangulation of the point set $\mathscr{A}$. This is determined as follows [30–32].

Begin with the toric description of $\mathbb{C}^d$. M is then a d-dimensional lattice. M may be viewed as the lattice of characters for the $(\mathbb{C}^*)^d$-action on $\mathbb{C}^d$ [20]. G is a subgroup of this $(\mathbb{C}^*)^d$-action. Let χ be a particular character of G. The embedding $G \subset (\mathbb{C}^*)^d$ yields a subset $\mathsf{M}_\chi \subset \mathsf{M}$ of characters corresponding to χ. Define M_χ^+ as the intersection of M_χ with the non-negative orthant.

For example, suppose G is isomorphic to the cyclic group $\mathbb{Z}_n$ generated by the action $\exp\left(\frac{2\pi i}{n}(a_1, a_2, \ldots, a_d)\right)$ on $\mathbb{C}^d$, where the a_i are integers. The characters of $\mathbb{Z}_n$ correspond to integers $j = 0, \ldots, n-1$. We then define

$$(50) \qquad \mathsf{M}_j^+ = \left\{ \mathbf{m} \in (\mathbb{Z}_{\geq 0})^d \mid \mathbf{m} \cdot \mathbf{a} \equiv j \pmod{n} \right\}.$$

Next define Σ_χ as the fan dual to the convex hull of M_χ^+ and let $\Sigma_{G-\mathrm{Hilb}}$ be the common refinement of all the Σ_χ's as χ varies over all characters of G. If 0 is the trivial character, let N_0 be dual to the lattice M_0. The fan $\Sigma_{G-\mathrm{Hilb}}$ and the lattice N_0 then provide the toric data corresponding to the G-Hilbert scheme.
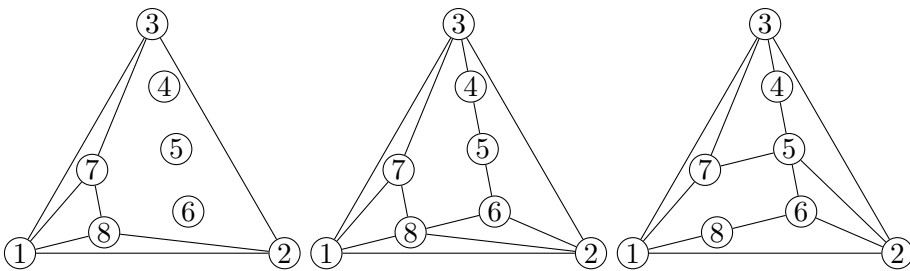

Figure 7: Slices of the fans Σ_j for $j = 1, 3, 7$ in the case of $\mathbb{C}^3/\mathbb{Z}_{11}$.

As an example let us consider the $\mathbb{C}^3/\mathbb{Z}_{11}$ case of Section 5.2. We need to construct the fans $\Sigma_0, \ldots, \Sigma_{10}$ for the 11 conjugacy classes. Each fan is a fan over a triangulation of the point set $\mathscr{A}$. In Figure 7 we show these triangulations in three cases. It is a simple matter to compute these fans using a computer package such as "polymake".

All said, when we combine these 11 fans together we obtain the triangulation given by the first case in Figure 6. This was one of the two triangulations for which the (0,2)-McKay correspondence did not require instanton corrections.

6.1. Non-Isolated Singularities

Suppose the orbifold $\mathbb{C}^3/G$ is not isolated for $G \subset \mathrm{SL}(3, \mathbb{C})$. Then there are fixed lines of singularities passing through the origin. In both the conformal field and the geometric picture the number of singlets is infinite.

In the conformal field theory description we will have bosonic excitations of zero mass. Thus we may add arbitrary such excitations to obtain massless singlets. In the geometric description there are points in $\mathscr{A}$ on the edges of the triangle forming the convex hull. The Ext^1 groups associated to arrows along such edges of the triangle are infinite dimensional.

The lines of singularities emanating from the origin are locally of the form $(\mathbb{C}^2/H) \times \mathbb{C}$, for some $H \subset \mathrm{SL}(2, \mathbb{C})$. We have already proved the (0,2)-McKay correspondence for dimension two in Section 3. We should therefore be able to systematically ignore the infinite number of states associated to two dimensions leaving a finite number intrinsically associated with the three-dimensional singularity at the origin. Let N_0 denote this finite number computed from the geometric picture.

For example, consider $\mathbb{C}^3/\mathbb{Z}_4$ where $\mathbb{Z}_4$ is generated by (the exponential of) $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. The $\frac{1}{4}$-twisted sector contains 9 states with $\bar{q} = -\frac{1}{2}$ while the $\frac{3}{4}$-twisted sector contains their $\bar{q} = \frac{1}{2}$ partners. The $\frac{1}{2}$-twisted sector has an infinite number of states because we may have arbitrary powers of x_3. Accordingly we ignore this sector. Thus we predict $N_0 = 9$ if there are no instanton corrections.
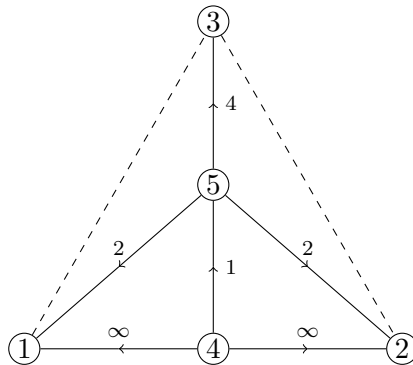

Figure 8: Quiver for $\mathbb{C}^3/\mathbb{Z}_4$.

The quiver for this case is shown in Figure 8. Node 4 in this figure represents a line of singularities of the form $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$. The arrows along the edges of the triangle associated to this vertex are correspondingly infinite.

We identify this line of singularities as associated to the $\frac{1}{2}$-twisted sector in the conformal field theory. Adding up the finite arrow multiplicities gives $N_0 = 9$ in agreement with the CFT count.

This allows us to extend our conjectured (0,2)-McKay correspondence to cases where the singularities are non-isolated.

6.1.1. $\mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_m)$. A particularly symmetric case concerns the quotient $\mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_m)$. Here the G-Hilbert scheme is given torically by equilateral triangles forming "isometric graph paper". We show the case for $m = 7$ in Figure 9.
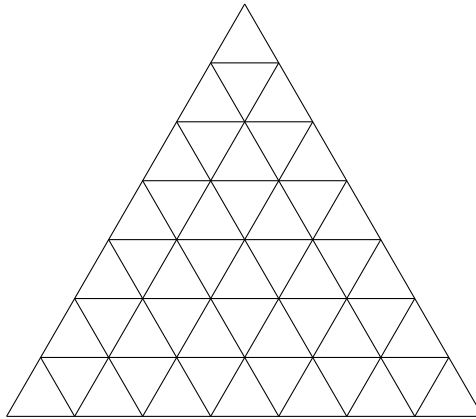

Figure 9: G-Hilbert scheme for $\mathbb{C}^3/(\mathbb{Z}_7 \times \mathbb{Z}_7)$.

This singularity is not isolated; there are 3 lines of $\mathbb{C}^2/\mathbb{Z}_m$ emanating from the origin. These correspond to the three edges of the triangle in Figure 9. We will therefore ignore points on these edges.

From the rules of Section 4.2 we see that the *only* arrows in this diagram are the ones we have been omitting — namely the ones from interior points to non-adjacent vertices. Thus, the number of deformations equals three times the number of strictly interior points. That is, $\frac{3}{2}(m-1)(m-2)$.

In the conformal field theory let us consider states in the sector twisted by $g^p h^q$ where g acts as $\exp 2\pi i \left(\frac{1}{m}, \frac{m-1}{m}, 0\right)$ and h acts as $\exp 2\pi i \left(0, \frac{1}{m}, \frac{m-1}{m}\right)$ on $\mathbb{C}^3$. If $p = 0$, $q = 0$ or $p = q$ then we have an infinite number of massless states. These correspond to the lines of singularities. Otherwise, if $p < q$ then each sector has 3 massless states with $\bar{q} = -\frac{1}{2}$. Similarly, if $p > q$ then we have 3 states with $\bar{q} = \frac{1}{2}$. This gives again a total of $\frac{3}{2}(m-1)(m-2)$ states.

Actually, we have a stronger result:

Proposition 1. *Amongst all crepant resolutions of $\mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_m)$, N_0 is minimized at $\frac{3}{2}(m-1)(m-2)$ for the G-Hilbert scheme. All other resolutions give a greater number. Thus, only the G-Hilbert scheme is free from instanton corrections.*

To see this first note that the contribution to $\operatorname{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}(\mathbf{q}_j))$ in (36) is at least $\frac{3}{2}(m-1)(m-2)$ and that this lower bound is only achieved if no interior point is connected to a vertex. Furthermore, from Figure 3, the contribution to $\operatorname{Ext}^1(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta))$ is only zero if all neighbouring pairs of triangles form strictly convex quadrilaterals. Starting from one corner of the big outer triangle and working inwards, one can then see that these two conditions force the triangulation to look like isometric graph paper.

7. Superpotentials

For three-dimensional examples above we have ignored the possibility of relations. In the case of three dimensions such relations are manifested in the form of a superpotential.

The ADHM relations on the Kronheimer–Nakajima quivers in the case of two dimensions were very important to get the counting correct. The source of such relations may be traced to the fact that every arrow is paired with an arrow in the opposite direction because of Serre duality. In fact, Serre duality itself is enough to derive the superpotential (or its equivalent content) and thus the ADHM equations but we do not include the details here.

In this paper we have thus far only been interested in counting the first-order deformations. This count is affected by linear relations or, equivalently, quadratic mass terms in the superpotential. Higher order terms in the superpotential correspond to higher-order obstructions to the first-order deformations. The ADHM relations from the Kronheimer–Nakajima quivers would appear to be quadric but it is important to remember that the tangent bundle corresponds to nonzero values for maps on the arrows. Expanding about such nonzero values makes the relations equivalent to linear relations and thus masses for these deformations. Conversely, when we consider quivers associated with three-dimensional cases, the arrows forming loops to form a superpotential will be associated to arrows strictly in the interior of the quiver. These maps are zero for the tangent bundle from Theorem 2. Thus we are expanding around zero and a cubic or higher superpotential corresponds purely to obstructions.
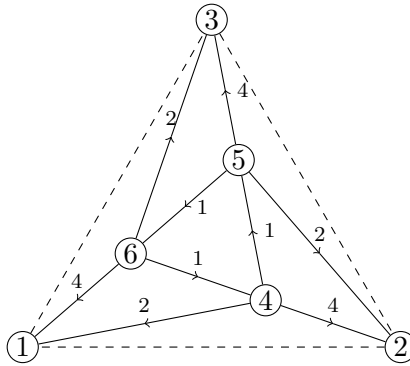
Figure 10: Quiver for $\mathbb{C}^3/\mathbb{Z}_7$.

So far none of the quivers we have drawn have contained an oriented cycle. Figure 10 depicts the case of $\mathbb{C}^3/\mathbb{Z}_7$. Here we do indeed have an oriented cycle. This implies we have a possible superpotential and thus that the moduli space can have obstructions. Our goal in this section is to show that this cubic term is nonzero. Note that we are computing the form of the superpotential *geometrically* which is classical in terms of the non-linear σ-model. So we are not considering worldsheet instanton corrections.

The theory of computing superpotentials in $N = 1$ theories in four dimensions has been explored quite thoroughly in the context of D-brane worldvolumes, see for example, [21]. We may use identical methods here. In particular, the superpotential encodes an A_∞-algebra as described in [22]. This A_∞ structure becomes more apparent for terms in the superpotential higher than cubic. However, in the case at hand we have a loop of length 3 and thus we are just dealing with a cubic term.

A loop around internal nodes α, β, γ corresponds to a cubic term in the superpotential with coefficient given by the Yoneda product

$$
\text{(51)} \quad
\begin{aligned}
\text{Ext}^1 &\big(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta)\big) \times \text{Ext}^1\big(\mathscr{O}_{D_\beta}(\mathbf{q}_\beta), \mathscr{O}_{D_\gamma}(\mathbf{q}_\gamma)\big) \\
&\times \text{Ext}^1\big(\mathscr{O}_{D_\gamma}(\mathbf{q}_\gamma), \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)\big) \to \text{Ext}^3\big(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)\big) \cong \mathbb{C}.
\end{aligned}
$$

The last equality here is obtained from Serre duality which we may use as the sheaves are compactly supported. Using Serre duality again we may rewrite this as

$$
\text{(52)} \quad
\begin{aligned}
\text{Ext}^1 &\big(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta)\big) \times \text{Ext}^1\big(\mathscr{O}_{D_\beta}(\mathbf{q}_\beta), \mathscr{O}_{D_\gamma}(\mathbf{q}_\gamma)\big) \\
&\to \text{Ext}^2\big(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\gamma}(\mathbf{q}_\gamma)\big).
\end{aligned}
$$

This product is explicitly computed using the local cohomology description of sheaf cohomology on toric varieties [33].

For this example, the matrix of charges in (26) is given by

$$
(53) \qquad \Phi = \begin{pmatrix} 0 & 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{pmatrix}
$$

The sheaf $\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha)$ is, in the derived category, equivalent to the complex $\mathscr{O} \xrightarrow{x_\alpha} \mathscr{O}(\mathbf{q}_\alpha)$. The required Ext's can therefore be computed via a spectral sequence in terms of sheaf cohomology of line bundles. To be precise,

$$
(54) \qquad \operatorname{Ext}^n(\mathscr{O}_{D_\alpha}(\mathbf{q}_\alpha), \mathscr{O}_{D_\beta}(\mathbf{q}_\beta)) = \bigoplus_{p+q=n} E_\infty^{p,q},
$$
$$
\text{with } E_1^{p,q} = H^q(\mathscr{E}^p),
$$

and $\mathscr{E}^\bullet$ is the complex given in (37). The cohomology of line bundles is represented by Laurent monomials as explained in [10, 14, 34]. Such computations were explained in gory detail in [14].

For $\operatorname{Ext}^1(\mathscr{O}_{D_4}(\mathbf{q}_4), \mathscr{O}_{D_5}(\mathbf{q}_5))$, the only contribution comes from

$$
\operatorname{Ext}^1(\mathscr{O}(\mathbf{q}_4), \mathscr{O}(\mathbf{q}_5)) \cong H^1(\mathscr{O}(\mathbf{q}_5 - \mathbf{q}_4)) \cong H^1(\mathscr{O}(-3, 1, 2)).
$$

This is represented by the Laurent monomial

$$
(55) \qquad \frac{x_1}{x_3^2 x_4}.
$$

Similarly $\operatorname{Ext}^1(\mathscr{O}_{D_5}(\mathbf{q}_5), \mathscr{O}_{D_6}(\mathbf{q}_6))$ is represented by $x_2/x_1^2 x_5$. The Yoneda product of these two Ext representatives is simply the product of these two monomials.

To compute $\operatorname{Ext}^2(\mathscr{O}_{D_4}(\mathbf{q}_4), \mathscr{O}_{D_6}(\mathbf{q}_6))$ we first note that $\operatorname{Ext}^2(\mathscr{O}(\mathbf{q}_4), \mathscr{O}(\mathbf{q}_6))$ is 2-dimensional and represented by monomials

$$
(56) \qquad \frac{x_2}{x_1 x_3^2 x_4 x_5} \quad \text{and} \quad \frac{1}{x_1 x_2 x_3 x_4^2 x_5}.
$$

However, $\operatorname{Ext}^2(\mathscr{O}, \mathscr{O}(\mathbf{q}_6))$ is one-dimensional and represented by $1/x_1 x_2 x_3 x_4 x_5$. At the E_1 stage of the spectral sequence, the second monomial in (56) is mapped to this by multiplication by x_4 and so the second monomial in (56) is killed.

The result is that $\text{Ext}^2\left(\mathscr{O}_{D_4}(\mathbf{q}_4), \mathscr{O}_{D_6}(\mathbf{q}_6)\right)$ is one-dimensional and is generated by the Yoneda product of the generators of $\text{Ext}^1\left(\mathscr{O}_{D_4}(\mathbf{q}_4), \mathscr{O}_{D_5}(\mathbf{q}_5)\right)$ and $\text{Ext}^1\left(\mathscr{O}_{D_5}(\mathbf{q}_5), \mathscr{O}_{D_6}(\mathbf{q}_6)\right)$. Thus *the superpotential is a nonzero cubic corresponding to the loop in Figure 10.*

$$(57) \qquad\qquad\qquad W = XYZ.$$

The derivatives of this superpotential imply that turning on one of these three deformations obstructs the other two.

Both the geometry and conformal field theory agree that there are 24 singlets associated to $\mathbb{C}^3/\mathbb{Z}_7$. The appearance of a superpotential does not change this count.

8. Discussion

The agreement between the counting of states between the orbifold conformal field theory and the deformations of the tangent bundle on the resolved space clearly motivates the following:

Conjecture 1. *The counting of the number of states on a three-dimensional G-Hilbert scheme corresponding to $(0, 2)$-deformations of an $N = (2, 2)$ theory matches the conformal field theory orbifold count.*

We have proved this conjecture to be true above in an infinite number of cases. Obviously it would be nice to check it in an even larger class, such as all abelian orbifolds.

We have made no attempt in this paper to directly confront the instanton computation. For isolated $\mathbb{P}^1$'s this amounts to computing the splitting type of the bundle $E \to \mathbb{P}^1$ as E is deformed away from the tangent bundle [24, 25]. This is not particularly easy for the following reason. The tangent sheaf has a nice presentation in terms of toric geometry in (30). Deformations of the maps E in this short exact sequence will yield deformations of the tangent sheaf. Unfortunately not all of the deformations can be understood so simply and it is these extra deformations which appear to be volatile under flops between different possible resolutions. Indeed the work of [23] implies that the we should expect all the interesting instanton effects to be associated to these more obscure deformations.

Another obvious unanswered question raised by the conjecture is "Why the G-Hilbert Scheme"? Is there some intrinsic reason why the construction of the G-Hilbert scheme is guaranteed to reproduce the orbifold computation? One thing that seems fairly likely is that, of all the resolutions, the

G-Hilbert scheme minimizes the number of deformations. We proved this for $\mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_m)$. More generally the G-Hilbert scheme tries to get as close to isometric graph paper as it can and thus minimizes the number of deformations. For a more precise statement of this, see [35]. Anyway, assuming the G-Hilbert scheme minimizes the number of deformations, it is therefore the "most instanton free" in some sense.

In this paper we have focused mainly on the instanton effects on mass. We really have the whole superpotential to work with and we showed in Section 7 that there can be nontrivial information here. It would be most interesting to compare conformal field theory computations and geometrical computations of the superpotential beyond the mass term.

Acknowledgments

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