

Multiple M5-branes, String 2-connections, and 7d nonabelian Chern-Simons theory

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The worldvolume theory of coincident M5-branes is expected to contain a nonabelian 2-form/nonabelian gerbe gauge theory that is a higher analog of self-dual Yang-Mills theory. But the precise details — in particular the global moduli / instanton / magnetic charge structure — have remained elusive. Here we deduce from anomaly cancellation a natural candidate for the holographic dual of this nonabelian 2-form field, under $\text{AdS}_7/\text{CFT}_6$ duality. We find this way a 7-dimensional nonabelian Chern-Simons theory of String 2-connection fields, which, in a certain higher gauge, are given locally by non-abelian 2-forms with values in an affine Kac-Moody Lie algebra. We construct the corresponding action functional on the entire smooth moduli 2-stack of field configurations, thereby defining the theory globally, at all levels and with the full instanton structure, which is nontrivial due to the twists imposed by the quantum corrections. Along the way we explain some general phenomena of higher nonabelian gauge theory that we need.

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1. Introduction

The quantum field theory (QFT) on the worldvolume of M5-branes is known [41, 42, 95] to be a 6-dimensional $(0, 2)$ -superconformal theory that contains a 2-form potential field B_2 , whose 3-form field strength H_3 is self-dual (see [53] for a recent survey). Whatever it is precisely and in generality, this QFT has been argued to be the source of deep physical and mathematical phenomena, such as Montonen-Olive S-duality [95], geometric Langlands duality [96], and Khovanov homology [97]. Yet, and despite this interest, a complete description of the precise details of this QFT is still lacking. In particular, as soon as one considers the worldvolume theory of *several coincident* M5-branes, the 2-form appearing locally in this 6d QFT is expected to be *nonabelian* (to take values in a nonabelian Lie algebra). But a description of this *nonabelian gerbe theory* has been elusive (a *gerbe* is a “higher analog” of a gauge bundle, discussed in detail below in Section 3). See [12, 39, 67] for surveys of the problem and recent developments. Here we add another piece to the scenario, by proposing a 7d Chern-Simons theory which appears to be a natural candidate for the holographic dual of the multiple M5-branes 6d QFT via $\text{AdS}_7/\text{CFT}_6$ -duality, and by identifying the nonabelian 2-form fields appearing in the theory as local data of (twisted) String 2-connections.

Namely, for a single M5-brane, the Lagrangian of the theory has been formulated in [1, 46, 57, 58, 82] and in this case there is, due to [91], a *holographic dual* description of the 6d theory by 7-dimensional abelian Chern-Simons theory, as part of $\text{AdS}_7/\text{CFT}_6$ -duality (reviewed for instance in [3]). We give here an argument, following [91, 94] but taking the quantum anomaly cancellation of the M5-brane in 11-dimensional supergravity into account, that in the general case: the $\text{AdS}_7/\text{CFT}_6$ -duality involves a 7-dimensional *nonabelian* Chern-Simons action that is evaluated on higher nonabelian gauge fields which we identify as *twisted 2-connections over the*

String-2-group, as considered in [30, 74]. Then we give a precise description of a certain canonically existing 7-dimensional nonabelian gerbe-theory on boundary values of quantum-corrected supergravity field configurations in terms of nonabelian differential cohomology. We show that this has the properties expected from the quantum anomaly structure of 11-dimensional supergravity. In particular, we discuss that there is a higher gauge in which these field configurations locally involve non-abelian 2-forms with values in the Kac-Moody central extension of the loop Lie algebra of the special orthogonal Lie algebra \mathfrak{so} and of the exceptional Lie algebra \mathfrak{e}_8 . We also describe the global structure of the moduli 2-stack of field configurations, which is more subtle.

Most of the ingredients of the 7d theory that we present are implicit in earlier publications of the authors, notably [74], [76] and [30]. There, however, we focused on viewing the “indecomposable” 7d *Lagrangian* (Section 4.5) as a *differential twist* that controls the magnetic dual heterotic Green-Schwarz anomaly cancellation, in direct analogy of how the ordinary 3-dimensional Chern-Simons Lagrangian serves, as discussed in these references, as a differential twist that controls the direct heterotic Green-Schwarz mechanism. The present article serves to make the corresponding Chern-Simons theory and its role in 11-dimensional supergravity explicit. Its relation to M-branes is also discussed in Section 3.5 of [67]. Further connections between String structures (and their variants) and M-branes are given in [67, 69, 71].

The description of higher degree form fields in string theory via *abelian* differential cohomology has been deeply influenced by the works of Freed [31] and Hopkins-Singer [44]. Such a description is very convenient when the gauge fields involved are, indeed, abelian; this includes Maxwell fields, Kalb-Ramond B -fields, the self-dual field on a single M5-brane (see [9]), and Ramond-Ramond fields. However, when considering systems such as given by multiple M5-branes, the theory becomes nonabelian — in some appropriate sense — and hence one needs to describe in a mathematically precise way the corresponding nonabelian higher degree gauge fields. This requires using *nonabelian* differential cohomology, a generalization not just of the theory of line bundles/circle bundles with connection, but a generalization of the theory of general gauge bundles with connection, hence of general Yang-Mills fields. A theory that accomplishes this has been laid out in [79], based on earlier work that includes [81], [74] and [30]. The use of such a formalism is not only to set up the correct language — which of course is desirable — but also to obtain a machinery that produces the dynamics of the fields, as well as their relation and their consistent coupling to other fields, in a

systematic way, constrained by suitable general principles of higher gauge theory. As a result, couplings and higher order gauge transformations which otherwise have to be guessed may now be derived systematically. Furthermore, the result is typically more subtle than what could have been — and in some cases has been — guessed. The twisted String 2-connections and their canonical 7-dimensional action functional are an example of this, which we will explain in detail.

The systems discussed here are just special cases of an infinite hierarchy of higher nonabelian gauge fields with higher Chern-Simons type action functionals that canonically arise in higher nonabelian differential cohomology in a canonical way that we briefly indicate in Section 4.1 below. The full 11-dimensional Chern-Simons term of 11d-supergravity is another example. In [28] we show that another class of examples is given by globalizations of AKSZ σ -models, such as the Courant σ -model that is induced from generalized Calabi-Yau spaces. There is a whole zoo of further examples; see Section 4.6 of [79]. Moreover, to each such theory in dimension $(n + 1)$ is associated a corresponding generalized higher WZW model in dimension n . In particular there is a 6-dimensional WZW-type theory associated with the boundary of the 7-dimensional String-connection theory discussed here. But details of this are beyond the scope of the present article.

What we do in this article can be summarized in the following main points:

1. As a warm-up, we provide in Sections 3.2 and 3.4 a description of the familiar case of gauge fields on multiple D-branes, but formulated in terms of the nonabelian differential cohomology of stacks of $U(n)$ -bundles in a way that has by direct analogy a generalization to multiple M5-branes considered afterwards.
2. We discuss, in Section 4, a refinement $\hat{\mathbf{I}}_8$ of the anomaly 8-class or one-loop polynomial I_8 of 11-dimensional supergravity to nonabelian differential cohomology by a higher stacky Chern-Weil construction. This refinement is a universal differential characteristic map that is naturally defined on the moduli 2-stack of boundary supergravity C -field configurations constructed in [29]. Note that an elliptic refinement of the one-loop term is given in [70].
3. We consider the 7-dimensional nonabelian Chern-Simons action functional canonically induced by $\hat{\mathbf{I}}_8$ on boundary C -field configurations in Section 4.6. We demonstrate that locally — or globally in the trivial instanton sector — this reduces to the functional that is implied by

the one-loop correction in 11-dimensional supergravity, as discussed in Section 2.3.

4. Throughout the article we discuss various aspects of this 7d theory. We comment on the relation to loop groups and the reason for passing to 2-groups in 2.1, point out the role of Lie n -algebras in 2.2, explain in what sense the 2-form on the M5-brane worldvolume is nonabelian in Sections 3.7 and 4.6.

The discussion is separated into three parts. First, in Section 2 we provide heuristic physical arguments aimed at characterizing the properties that the sought-after mathematical objects should satisfy. This involves looking at the problem from various angles, which we outline below. We present an argument for why the spaces of states of the 7-dimensional nonabelian Chern-Simons theory are a plausible candidate for the conformal blocks of the 6d theory on worldvolume of coincident fivebranes. This argument is necessarily non-rigorous, but it seems to be as trustworthy as the argument in [94], of which it is a direct extension. Then in Section 3 we review aspects of higher nonabelian gauge theory in a way that prepares the ground for our main construction. Finally in Section 4 we give a precise definition and discussion of action functionals of 7-dimensional Chern-Simons theories whose fields are twisted 2-connections with values in the String 2-group. In conjunction with the physical arguments of Section 2, this can be regarded as a proposal for how to make aspects of the physical heuristics involved there precise.

We hope to study the supersymmetric extension of the current constructions in a separate article.

This article makes use of mathematical concepts in the theory of *higher stacks* and *nonabelian (differential) cohomology*. In Section 3 we offer some introduction and explanation that should be sufficient for an appreciation of Section 4. However, the reader who wishes to dig deeper into the mathematics to which we appeal should look at [76], [30] and [79] (perhaps in that order). For ease of reference, in the tables below we list mathematical objects that we will mention frequently, together with their physical meaning. These tables are also useful in the description of the C-field and its dual in [29].

We will use various notions of cohomology, starting with differential forms and working our way up through refinements. These are summarized in the table

cohomological notion	gauge theoretic notion
differential forms Ω_{cl}	field strengths / classical description
cohomology $H(-)$	instanton configurations / magnetic charges
differential cohomology $\hat{H}(-)$	equivalence classes of gauge fields
cocycle ∞ -groupoid $\mathbf{H}(-)$	actual gauge fields with (higher) gauge transformations

We will also use various (higher) *moduli stacks* in order to precisely capture the global nature of (higher) gauge fields and their (higher) gauge transformations. One may think of these as *integrated BRST complexes* or *integrated Lie n -algebroids*, see Section 3.1. To guide the reader through the various stacks, here is a table that should serve to set some notation and also as a dictionary between stacky notions and the corresponding bundle structures appearing in relation to the physics of M5-branes and M-theory. We have

symbol	(higher) moduli stack of...
$\mathbf{BU}(1)$	circle bundles / Dirac magnetic charges
$\mathbf{BU}(1)_{\text{conn}}$	$U(1)$ -connections / abelian Yang-Mills fields
$\mathbf{BSpin}_{\text{conn}}$	Spin connections / field of gravity
\mathbf{BE}_8	E_8 -instanton configurations
$(\mathbf{BE}_8)_{\text{conn}}$	E_8 -Yang-Mills fields
$\mathbf{B}^2\mathbf{U}(1)_{\text{conn}}$	B -field configurations (without twists)
$\mathbf{B}^3\mathbf{U}(1)_{\text{conn}}$	C -field configurations (without twists)
$\mathbf{BString}_{\text{conn}}$	String 2-connections / nonabelian 2-form connections
$\mathbf{BString}^{2\mathbf{DD}_2}$	first Spin characteristic class $\lambda = \frac{1}{2}p_1$ divisible by 2
$\mathbf{BString}^{2\mathbf{a}}$	E_8 -twisted String-2-connections
\mathbf{CField}	bulk configurations of supergravity C -fields (and gravity)
$\mathbf{CField}^{\text{bdr}}$	C -field configurations on (5-brane) boundaries (and E_8 -gauge fields)

These concepts combine to give actual configuration spaces of (higher) gauge fields by evaluating cohomology on spacetime *with coefficients in* a (higher) moduli stack. Let $G = U(1), \text{Spin}, \text{String}, \mathbf{BU}(1), \dots$ be a (higher) gauge group with Lie n -algebra \mathfrak{g} (see [74]), and let X be a (spacetime) manifold. Then we have

symbol	gauge theoretic meaning
$[X, \mathbf{BG}_{\text{conn}}]$	moduli stack of G -gauge fields on X
$\mathbf{H}(X, \mathbf{BG}_{\text{conn}})$	collection of gauge fields with G -gauge transformation on X
$H(X, \mathbf{BG}_{\text{conn}})$	equivalence classes of G -gauge fields on X
$\hat{H}^n(X) \simeq H(X, \mathbf{B}^n\mathbf{U}(1)_{\text{conn}})$	equivalence classes of abelian n -form gauge fields on X
$H(X, \mathbf{BG})$	set of underlying instanton sectors
$\Omega(X, \mathbf{BG})$	\mathfrak{g} -valued (higher) field strengths

2. Evidence for and ingredients of the 7d nonabelian gerbe theory

In this section we present physical arguments for why one should expect the nonabelian gerbe theory to be the right description of the system of multiple fivebranes. Along the way, we provide our own interpretations which help us identify the relevant ingredients in that theory. There are (at least) two aspects to this

1. In one regime, the M5-brane worldvolume is to be thought of as embedded into an ambient 11-dimensional spacetime that carries a supergravity C -field which has a direct restriction to the brane. The restriction at the level of de Rham cohomology and differential forms is discussed in [73] from the point of view of boundary conditions. We provide another description in Section 2.1.
2. In another regime, we have “black” 5-branes identified as the asymptotic boundary [51] of a compactification on S^4 to an AdS₇-solution [59] of 11d supergravity. Here the boundary space of states of the C -field holographically induces the conformal blocks of the 5-brane superconformal theory. This is discussed in Section 2.3.

We should stress that our use of holography, and the particular configuration related to AdS/CFT, serves as a motivation and indeed our constructions will work in full generality.

2.1. The M5-brane worldvolume theory: loop groups and the String group

Various ingredients and aspects of 5-brane physics have been conjectured or argued for before in the literature, see [67] for an outline. Among them are the ones we discuss below, all of which are subsumed by the proposal we make. We will also provide heuristic interpretations and connections to the String 2-group and loop groups, appropriate for the description of M5-branes.

Nonabelian gerbes and 2-gerbes with connections. The notion of a *gerbe* or *2-bundle* with connection is a higher analog of the notion of a *principal bundle* with connection, hence of (instanton) Yang-Mills field configurations, where the term “higher” is as in “higher degree differential forms”: just as a connection on a gauge bundle is locally given by a 1-form (the gauge potential), a connection on a 2-bundle/1-gerbe is locally given

by a 2-form. Several known arguments imply that the worldvolume theory of multiple coincident 5-branes contains a field that is a 2-form connection on a nonabelian gerbe in analogy to how Yang-Mills theory is the theory of a field that is a 1-form connection on a principal bundle.

(i) The first argument invokes the lift of string-D4-brane systems from string theory to M-theory. It is well known that the open string ending on N coincident D-branes couples to a $SU(N)$ -valued 1-form connection field on the Chan-Paton bundle on the D-brane, whose line holonomy along the boundary of the string worldsheet provides the boundary term in the action. To be precise, there is in addition the B -field in the ambient spacetime whose restriction to the D-brane *twists* this 1-form connection field. The lift of this configuration to M-theory through the inverse of the double dimensional reduction is an M2-M5 brane configuration with open membranes [84, 85] ending on N coincident M5-branes. Now, the membrane boundary $\partial M2 \subset M5$ couples to a 2-form connection on the 5-brane and the restriction of the ambient supergravity C -field induces a twist. For this situation to be compatible with its reduction to string theory, the 2-form must in some way take values in a nonabelian Lie algebra. Hence it should be a 2-connection on a nonabelian gerbe on the 5-brane, which is twisted by the 2-gerbe on which the C -field is a 3-connection.

We can deduce precisely this phenomenon also from the nature of the gauge-invariant field strengths on branes that are familiar from the literature: The gauge-invariant 2-form field strength on a D-brane in the background of a B -field is, locally, the combination

$$(2.1.1) \quad \mathcal{F} = B + F,$$

where F is the trace of the curvature of the connection on the worldvolume of the D-brane. This phenomenon (and its global generalization) is explained (as we discuss in Section 3.4) by the fact that the gauge field on the D-brane is a kind of trivialization of the 2-bundle/1-gerbe underlying the restriction of the B -field to the brane: a twisted Chan-Paton bundle is a kind of trivialization of a circle 2-bundle/1-gerbe. This phenomenon has a higher generalization. In general *twisted* n -bundles / $(n - 1)$ -gerbes may serve as a kind of trivialization of an $n + 1$ -bundle. (A detailed mathematical discussion of this is given in Section 1.3.1 and 4.4 of [79].)

Now, it is known that the invariant self dual 3-form field strength on the M5-brane is accordingly locally of the form

$$(2.1.2) \quad \mathcal{H} = H + C,$$

where C is the restriction of the ambient C -field to the brane, and where $H = dB_2$ is the curvature of the 2-form potential on the M5-brane. This can be seen from the M2-brane as follows. Consider the part of the action given by $S_C = \int_{\Sigma_3} C_3$, where Σ_3 is the worldvolume of the M2-brane. This action is not invariant for an open M2-brane unless we introduce a two-form gauge field B coupled to the boundaries of the M2-brane, with $S_B = \int_{\partial\Sigma_3} B_2$ and require that it transforms as $B_2 \rightarrow B_2 - \Lambda_2$ under the C -field gauge transformation $C \rightarrow C + d\Lambda_2$. Here Λ_2 is a two-form field. Then gauge invariance requires considering the combination $\mathcal{H} = C + H$. Therefore this is another reason to expect that the 2-form field on the M5-brane is the local connection on a *twisted 2-bundle* whose twist is given by the C -field.

(ii) The second argument (see [95]) proceeds by a similar dimensional reduction, but now from 6 to 4 dimensions. One finds that compactifying the conformally invariant worldvolume theory of a single fivebrane with its abelian 2-form on a torus yields abelian Yang-Mills theory (electromagnetism) in 4-dimensions, such that the residual conformal transformations on the compactified space becomes the Montonen-Olive electric-magnetic duality of 4-dimensional Yang-Mills theory. Since this gives, in the abelian case, a natural geometric explanation for the otherwise more mysterious S-duality of (super) Yang-Mills theory, it is natural to expect that the same mechanism is the source of electric-magnetic duality also generally in nonabelian (super) Yang-Mills theory. Motivated by these arguments a definition of twisted nonabelian 2-gerbe with connection has been proposed in [5] and argued to be relevant for the description of 5-brane physics. The notion of higher twisted gerbes with higher connections has been fully formalized in [79]. By appealing to theorems about these structures that we provided in [30, 76], as well as to a study of topological effects within AdS/CFT-duality, we argue in Section 2.3 below that indeed these structures on 5-branes are *implied* by quantum anomaly cancellation of M5-branes in M-theory.

There are further connections to other branes which highlight some of the topological and geometric considerations that we consider. We mention that the connection of Fivebrane structures in relation to the NS5-branes in type IIA is given in [72, 76].

Loop group and String 2-group degrees of freedom on the M5-brane. There have been put forward various arguments that mean to identify gauge *loop groups* (see [60] for mathematical background) controlling the gauge theory on M5-branes. We recall these arguments, recasting them on firm mathematical ground within our perspective, and indicating how they will be refined in Sections 3 and 4.

First, consider again the twisted Chan-Paton bundles that appear on the D4-brane in 10-dimensional type IIA string theory. These are controlled topologically by the obstruction theory of lifts through the universal circle extension

$$(2.1.3) \quad \mathrm{U}(1) \rightarrow U(\mathbb{H}) \rightarrow \mathrm{PU}(\mathbb{H})$$

of the group of projective unitary operators $\mathrm{PU}(\mathbb{H})$ on any separable Hilbert space \mathbb{H} (see Section 3.4 for details): they are projective unitary bundles whose obstruction to lift to genuine unitary bundles is the class of the ambient B-field gerbe (related to the third integral Stiefel-Whitney class). Now, under the double dimensional reduction from M-theory to type IIA string theory, the D4-brane in ten dimensions comes from an M5-brane in eleven dimensions. In the spirit of [52] it has, essentially, been argued in [5] that reversing double dimensional reduction goes along with a *delooping* of the above sequence, and that this should involve the M-theory E_8 -degrees of freedom. Notice that the homotopy type of the topological space $\mathrm{PU}(\mathcal{H})$ is that of an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, characterized by the fact that its only nontrivial homotopy group is $\pi_2(\mathrm{PU}(\mathbb{H})) \simeq \mathbb{Z}$, and that the only nonvanishing homotopy group of the topological space underlying the Lie group E_8 in degree < 15 is $\pi_3(E_8) \simeq \mathbb{Z}$, so that the only nonvanishing homotopy group of the *loop group* ΩE_8 in degree < 14 is $\pi_2(\Omega E_8) \simeq \mathbb{Z}$. Therefore, over the manifolds of dimension ≤ 11 that appear in the string theoretical context, projective unitary bundles have the same classification as ΩE_8 -bundles [52], hence in these dimensions the group E_8 is a model for the delooping of $\mathrm{PU}(\mathbb{H})$

$$(2.1.4) \quad \mathrm{PU}(\mathbb{H}) \simeq_{14} \Omega E_8.$$

Therefore, still in these dimensions, the sequence (2.1.3) is homotopy equivalent to a sequence for a central extension $\hat{\Omega} E_8$ of the loop group ΩE_8

$$(2.1.5) \quad \mathrm{U}(1) \rightarrow \hat{\Omega} E_8 \rightarrow \Omega E_8.$$

So far this is the argument from [5, 52]. Note that one has to be a bit careful with this, because it concentrates on homotopy types and ignores the *geometric* (gauge) structure, which is different for projective unitary bundles and for ΩE_8 -principal bundles.

In order to fully deloop the whole sequence, we may observe next that a space of the homotopy type of a $K(\mathbb{Z}, 2)$ is a delooping for $\mathrm{U}(1)$. We will find it useful to make this explicit by writing $\mathrm{BU}(1)$ for $K(\mathbb{Z}, 2)$. Then

a delooping of the sequence (2.1.3) over spaces of dimension < 14 can be written in the form

$$(2.1.6) \quad BU(1) \rightarrow ? \rightarrow E_8.$$

This means that the middle term here is analogous to an ordinary group extension of E_8 by a circle group, only that the circle group $U(1)$ is replaced by a *higher* or *shifted* circle group $BU(1)$. If we keep arguing from the point of view of sufficiently low-dimensional spaces, then we notice the truncated homotopy equivalence $\Omega E_8 \sim_{14} BU(1)$ and observe that for any group G the group LG of *free* loops (without fixed basepoint) forms a split extension

$$(2.1.7) \quad \Omega G \rightarrow LG \rightarrow G.$$

Hence in sufficiently low dimension and ignoring geometry, the above question mark “?” could be filled by the *trivial* extension

$$(2.1.8) \quad \Omega E_8 \rightarrow LE_8 \rightarrow E_8$$

of E_8 by its free loop group. While this may serve as a guide, it is too simplistic, because there is no reason to expect a trivial extension here. At the opposite extreme is the *universal* non-trivial such extension, which is such that every other one is a multiple of it. This universal shifted central extension of E_8 is known as the *String group* of E_8 , denoted

$$(2.1.9) \quad BU(1) \rightarrow \text{String}(E_8) \rightarrow E_8.$$

Its homotopy type is that of a certain topological group, but as a *geometric* (smooth) object it is not a Lie group. Instead it is a *higher* analog of a Lie group called a *smooth 2-group* (technical details of which we review in Section 3.7 below).

In [7] it is shown that $\text{String}(G)$ for any simply connected compact simple Lie group G has a presentation by what is called a *crossed module* of two ordinary Lie groups, namely by the Kac-Moody central extension $\hat{\Omega}G$ of the loop group of G (but now regarded as a genuine Lie group) and the based path group of G . In this sense, the $\text{String}(E_8)$ 2-group remembers the loop group degrees of freedom even after delooping to the 5-brane. Notice, however, that the notion of (gauge) equivalence for higher groups is considerably richer than for ordinary groups, so that one and the same 2-group may be presented by rather different looking constructions, some of which do not manifestly involve loop groups. A review of this phenomenon we append in the Appendix.

There is also a different way to arrive at the conclusion that (twisted) String-2-connections are the right fields in one-loop-corrected supergravity and on 5-branes. This is the perspective of [30, 76], which is at the heart of our development in Sections 3 and 4. The argument requires some background ideas in gauge theory and higher stacks that we survey in 3.1. The reader not familiar with this relation may want to come back to the following argument after having looked at those sections. Here is the argument, central to our main point. The fields of supergravity over an M5-brane boundary Σ locally look like a Spin-connection (gravity) and an E_8 -gauge-field. Crucially, these are subject to a *constraint* [91] which demands that the Pontrjagin class $\frac{1}{2}p_1$ of the former equals twice the canonical 4-class a of the latter. We recall this in more detail below in Section 2.3.

Consider, for the moment, the set of *gauge equivalence classes* of Spin-structures on X , which we write $H(\Sigma, \mathbf{BSpin})$, and the set of gauge equivalence classes of E_8 -instantons, which we write $H(\Sigma, \mathbf{BE}_8)$, and finally the set of degree-4 integral cohomology classes, which we write $H(X, \mathbf{B}^3U(1))$. Then the first Spin class $\frac{1}{2}p_1$ is a map

$$(2.1.10) \quad \frac{1}{2}p_1 : H(\Sigma, \mathbf{BSpin}) \rightarrow H(\Sigma, \mathbf{B}^3U(1))$$

and the E_8 class a is a map

$$(2.1.11) \quad a : H(\Sigma, \mathbf{BE}_8) \rightarrow H(\Sigma, \mathbf{B}^3U(1))$$

and the *set of pairs* $P(\Sigma)$ of gauge equivalence classes of field configurations that satisfy the quantization condition constraint is the *fiber product* (or “pullback”) of these two maps, the set that universally completes a diagram of maps like this:

$$(2.1.12) \quad \begin{array}{ccc} P(\Sigma) & \longrightarrow & H(\Sigma, \mathbf{BE}_8) \\ \downarrow & & \downarrow 2a \\ H(\Sigma, \mathbf{BSpin}) & \xrightarrow{\frac{1}{2}p_1} & H(\Sigma, \mathbf{B}^3U(1)) \end{array} .$$

From this point of view it seems as if the supergravity quantization condition simply restricts the configuration space of fields. However, there is a problem with this argument. For *quantization* of a gauge theory, the set of gauge equivalence classes of field configurations is an invalid starting point. What one instead needs to consider is the *BRST complex* of field configurations, or rather, its integrated version, the *moduli stack* of field

configurations. We write $\mathbf{BSpin}_{\text{conn}}$ for the universal moduli stack of Spin-connections. Then $\mathbf{H}(\Sigma, \mathbf{BSpin}_{\text{conn}})$ denotes the integrated BRST complex, containing the genuine Spin-connection fields on Σ , *and* the gauge transformations between them. This is no longer a set, but is now a *groupoid*. Its set of *connected components* recovers the set $H(X, \mathbf{BSpin}_{\text{conn}})$ of gauge equivalence classes of fields. Similar comments and notation apply to E_8 and $\mathbf{B}^2U(1)$.

These structures are now a valid starting point for quantization. Therefore, the above constraint should be imposed on these structures. The crucial difference now is that when we ask for the structure that universally completes this fiber product diagram

$$(2.1.13) \quad \begin{array}{ccc} ? & \xrightarrow{\quad} & \mathbf{H}(\Sigma, \mathbf{B}E_8) \\ \downarrow & \swarrow \hat{B} & \downarrow 2\hat{a} \\ \mathbf{H}(\Sigma, \mathbf{BSpin}_{\text{conn}}) & \xrightarrow{\quad} & \mathbf{H}(\Sigma, \mathbf{B}^3U(1)) \\ & \searrow \frac{1}{2}\hat{p}_1 & \end{array}$$

then going around the square in the two possible ways no longer needs to yield genuinely equal field configurations. It suffices that the two field configurations obtained are connected by a gauge transformation B , as indicated. This is indeed the only way to make gauge-invariant sense of this diagram. In the following, *all* square diagrams (always of higher smooth stacks) that we display are implicitly filled by a gauge transformation this way, but only sometimes do we display it explicitly.

Such fiber products “up to gauge transformation” are well known in homotopy theory. They are called *homotopy fiber products* or *homotopy pullbacks*. If we compute them in the full context of higher gauge theory, we find two crucial differences to the above naive idea of imposing the quantization constraint.

First, the object in the top left is no longer the simple restriction of the direct product of Spin- and E_8 -connections that satisfy the quantization constraint. The reason is that the *choice* of gauge transformation B on each pair is now part of the field content data. The new field that appears this way is well known in string theory, at least for Hořava-Witten boundaries of 11-d supergravity [45]: it is the field strength of a twisted *2-form* field (“ B -field”) with twisted Bianchi identity

$$(2.1.14) \quad d\mathcal{H}_3 = \langle F_\omega \wedge F_\omega \rangle - 2\langle F_A \wedge F_A \rangle,$$

where F_ω and F_A are the curvatures of the Spin connection ω and the gauge connection A , respectively, and

$$(2.1.15) \quad \mathcal{H}_3 = dB + \text{CS}_3(\omega) - 2\text{CS}_3(A).$$

Equation (2.1.14) manifestly exhibits the structure of the square diagram (2.1.13) presented by de Rham cocycles.

The second effect is that the object denoted “?” above is not itself a groupoid anymore. It turns out to be a higher groupoid, here a *2-groupoid* that contains not just gauge transformations, but *gauge-of-gauge transformations* (coming from *ghosts-of-ghosts* in the corresponding BRST complex). In Section 3.8 below we identify the question mark here with the 2-groupoid $\mathbf{H}(\Sigma, \mathbf{BString}^{2a})$ of *E_8 -twisted String-2-connections*. If the E_8 -twist here vanishes, then this involves the genuine String-2-group which we had motivated already via loop groups in Section 2.1 above. This effect of imposing the relation $\frac{1}{2}p_1 = 2a$ not on gauge equivalence classes but on moduli stacks / integrated BRST complexes of fields is the key step that leads us to nonabelian higher form fields in the following discussion. In the closely related context of anomaly cancellation in heterotic string theory, this very phenomenon has been discussed in some detail in [76].

The theory of such nonabelian 2-form connections has been developed in [30, 79, 81]. We review String 2-connections in Section 3.7 below and discuss twisted String-2-connections in Section 3.8. In Section 4 we systematically derive the *twisted* String 2-connections on M-branes in 11-dimensional supergravity (or M-theory) that were anticipated in [5] in higher analogy with the twisted unitary bundles on boundaries/D-branes in 10-dimensional string theory.

2.2. M-branes and L_∞ -algebras, Lie n -algebras and “3-algebras”

The *higher Lie groups* that appear in higher gauge theory — such as the String-2-group already mentioned - have an infinitesimal approximation by a higher analog of Lie algebras. These higher Lie algebras are known as *Lie-infinity algebras* or *L_∞ -algebras*, for short. (A description in the context that we need here is in [74]). While an ordinary Lie algebra is a vector space equipped with a binary skew bracket that satisfies the Jacobi identity, an L_∞ -algebra is a chain complex of vector spaces, which is equipped with k -ary skew graded brackets for all $k \in \mathbb{N}$, such that these satisfy a certain joint higher analog of the Jacobi identity.

If the underlying chain complex of an L_∞ -algebra is concentrated in the lowest n degrees, then we also speak of an n -term L_∞ -algebra or *Lie n -algebra*. These are the infinitesimal approximations to Lie n -groups. For instance, the smooth 2-group String has a Lie 2-algebra: **string**. This degree “ n ” of Lie n -groups and Lie n -algebras is directly related to the dimension of the branes that can be charged under them. A brane with n -dimensional worldvolume can be charged under a Lie n -group / Lie n -algebra. For instance, there is an abelian Lie 2-algebra $b\mathbb{R}$, given by the chain complex concentrated on \mathbb{R} in degree 1 and having trivial bracket. It is the Lie 2-algebra of the Lie 2-group $\mathbf{BU}(1)$, which is the smooth incarnation of the topological group $BU(1)$ that we encountered before. A 2-connection on a 2-bundle whose gauge 2-group is $\mathbf{BU}(1)$ and whose Lie 2-algebra is $b\mathbb{R}$ is precisely a Kalb-Ramond B -field. Indeed, this has a holonomy over 2-dimensional worldsheets and the string with its 2-dimensional worldvolume is charged under it.

In direct analogy of this situation, there is a Lie 3-algebra $b^2\mathbb{R}$ with Lie 3-group $\mathbf{B}^2U(1)$. A 3-connection with values in this is essentially what the supergravity C -field is (we give the details in Section 4.3). A detailed discussion of Lie 3-algebras related to C -fields and Chern-Simons couplings of 2-branes is in [74], [76] and [30], as are discussions of further higher analogs, such as the Lie 6-algebras related to the magnetic dual C -field and Chern-Simons couplings of the 5-brane, which we will consider in Section 3.9. The description of supergravity theories by D’Auria and Fré [18] can also be formulated in terms of higher gauge field with values in super L_∞ -algebras. Notably there is a super Lie 3-algebra and a super Lie 6-algebra extension of the super Poincaré-Lie algebra such that the action functional of 11-dimensional supergravity is a variant of a higher Chern-Simons action for these. Moreover, the infinitesimal automorphism L_∞ -algebra of these contains in degree 0 the M-theory super Lie algebra ([74] and Section 4.3.2.2 of [79]).

There is therefore ample theory and examples for the role of Lie 2-algebra in string theory, the role of Lie 3-algebras in membrane theory, the role of Lie 6-algebras in 5-brane theory and generally of Lie $(n + 1)$ -algebras in n -brane theory, a large part of which we discussed before (starting in [74]) and some part of which will concern us here.

A different and conjectural proposal for a role of higher Lie algebraic structures in membrane theory has been proposed in [8], motivated from a supersymmetric extension of the M2-brane action. There a certain trilinear term appears, satisfying an invariant condition which the authors called a “3-algebra” structure, a terminology subsequently picked up by

many publications. In the process, the term transmuted sometimes into “3-Lie algebra” and sometimes even into “Lie 3-algebra”. Unfortunately, the Bagger-Lambert “3-algebra” is not a Lie 3-algebra in the established sense of an L_∞ -algebra structure on a graded vector space V . The reason is that for the notion of an L_∞ -algebra it is crucial that V is an \mathbb{N} -graded (or \mathbb{Z} -graded) vector space and that the n -ary brackets respect the degree in a certain way. But in the Bagger-Lambert proposal, V is all concentrated in a single degree (is regarded as ungraded). One immediately finds that in this case the L_∞ -respect of the ternary bracket for the grading would imply that V is taken to be in degree $\frac{1}{2}$. Since this is not in \mathbb{N} , it does not yield an L_∞ -algebra. But the \mathbb{N} -grading (or \mathbb{Z} -grading) of L_∞ -algebras is crucial for the homotopy theoretic interpretation of L_∞ -algebras as higher Lie algebras. None of the good theory of L_∞ -algebras survives when this grading is dropped. This grading has its origin in the Dold-Kan correspondence, which establishes integral graded homological structures as models for structures in homotopy theory (see Section 2.1.7 in [79] for a discussion of this in the context of higher gauge theory). Notably, a higher Lie algebra is supposed to have a Lie integration to a smooth n -groupoid. Under this process, the elements in degree k of the higher Lie algebra become tangents to the space of k -morphisms of this smooth n -groupoid. Clearly, here only integer k make any sense.

On the other hand, it is of course possible to consider the structure of “ L_∞ -algebras without grading”, even if these will not have a good theory. This notion has once been introduced by Filippov [27] under the name “ n -Lie algebra”. The innocent-looking difference between the terms “Lie n -algebra” and “ n -Lie algebra” corresponds, unfortunately, to a major difference in the behavior of the concepts behind these terms. It was argued in [49, 50] that these “3-algebras” might also play a role in the description of multiple M5-branes. For that, a nonabelian generalization of the field content given by the $(0, 2)$ tensor multiplet is proposed; this involves nonabelian versions of the fields in that supermultiplet, namely the scalars, the fermions and the antisymmetric 3-form. In addition, a nonabelian gauge field and a non-propagating vector are introduced. In the construction, however, the nonabelian two-form $B_{\mu\nu}^a$ never appears, which seems to be a problem for the quantum theory.

On general grounds, it is clear from our point of view that 2-brane physics is governed by Lie 3-algebraic structures, but it is not yet clear how the ternary operation highlighted in [8] would be an example. In view of this, it might be noteworthy that the equivalent reformulation and generalization of the BLG model by the ABJM model [2] does not involve any “3-algebras”

at all. On the other hand, comparison with other structures suggests that possibly the trinary operation is indeed a structure in higher Lie theory, but not the trinary bracket on an L_∞ -algebra. Instead, it can be seen to be in analogy with a higher *symplectic structure*, a “2-plectic structure”. This is argued in [62], and this would make sense also in homotopy theory (see the Section 4.5 on higher symplectic geometry in [79]).

In conclusion, the reader expecting to see higher Lie algebraic structures on the M5-brane will find them play a pivotal role in our discussion in Sections 3 and 4. It is not, however, quite the kind of algebraic structure that [8, 49, 50] propose, but one that has a good homotopy-theoretic interpretation.

2.3. Holography and Chern-Simons theory

In this section we give a physical argument that the 7-dimensional non-abelian gauge theory, to be discussed more fully below in Section 4, is the Chern-Simons part of 11-dimensional supergravity on $\text{AdS}_7 \times S^4$ with 4-form flux on the S^4 -factor and with quantum anomaly cancellation conditions taken into account. We, moreover, argue that this implies that the states of this 7-dimensional CS theory over a 7-dimensional manifold encode the conformal blocks of the 6-dimensional worldvolume theory of coincident M5-branes. The argument is based on the available but incomplete knowledge about AdS/CFT-duality, as reviewed in [3], and cohomological effects in M-theory as discussed in [67].

We start in Section 2.3 with some remarks about the relevant compactifications of 11d sugra to AdS_7 . Then in 2.3 we discuss the subtleties of quantum anomaly corrections to the 7-dimensional Chern-Simons theory inside 11-dimensional supergravity.

AdS-compactifications. There are 6d theories with different amount of supersymmetry, whose duals under $\text{AdS}_7/\text{CFT}_6$ have, in particular, different boundary behaviors of the C -field. (We analyze the moduli for different boundary conditions in detail in [29]). While the maximally supersymmetric $(0, 2)$ -theory is dual to supergravity on $\text{AdS}_7 \times S^4$, there is also a $(0, 1)$ -superconformal theory in 6d, and it is dual to a compactification on a $\text{AdS}_7 \times \mathbb{C}^2 // \mathbb{Z}_k$ -orbifold, with the 5-branes sitting at an orbifold fixed point. Whereas in the first case the supergravity fields are otherwise unconstrained, in the second case they satisfy boundary conditions as in Hořava-Witten theory [45].

The minimal supergravity in seven dimensions. The field content of the massless representations of the minimal $D = 7$, $N = 2$ supergravity coupled to vector multiplets, written in terms of 2-form potential is [11] $(g, B_2, A_1^I, \phi^\alpha, \phi, \psi_1^i, \chi^i, \theta^{\alpha i})$, where the first five fields are bosonic and the last three are fermionic. The reduction to six dimensions is as follows:

1. This minimal gauged supergravity compactified on S^1 leads to non-chiral $N = (1, 1)$ 6d supergravity [36].
2. The theory reduces on the orbifold S^1/\mathbb{Z}_2 to 6d, $N = (0, 1)$, chiral theory [6]. Only fields of even \mathbb{Z}_2 -parity survive on the two orbifold planes: $(g, B_2, A^I, \phi^\alpha, \phi, \xi)$, which includes the chiral multiplet (g, B_2^+, ψ_1^{i-}) with $B_2 = B_2^+ + B_2^-$, a sum of self-dual and anti-self-dual parts. This is a Hořava-Witten-like construction in seven dimensions implementing a form of Green-Schwarz anomaly cancellation.

The (0, 1) theory as dual to AdS₇. The AdS₇ vacuum with $N = 2$ supersymmetry is the supergravity dual, in the context of the AdS/CFT correspondence, of the 6d, $N = (0, 1)$, SCFT [26]. The nonabelian gauging of the self-dual tensor fields can be performed by introducing tensor gauge degrees of freedom with p -form gauge parameters, $p = \{0, 1, 2\}$. This is used in [63] to build a 2-form potential which carries a representation of the structure group. This is possible due to the 3-form potential which mediates couplings between the tensor and vector multiplets. The full nonabelian field strengths of the gauge field A and the 2-form gauge potential B are proposed there to be

$$(2.3.1) \quad \mathcal{F}^r = F^r + h_I^r B^I, \quad \mathcal{H}^I = d_A B^I + CS^I + g^{Ir} C_r,$$

where F is the ordinary curvature 2-form of the gauge field, d_A is the A -covariant exterior derivative, CS^I is a Chern-Simons 3-form of A for some bilinear form, and g^{Ir} and h_I^r are couplings of Stückelberg type.

Our approach provides a systematic way of obtaining consistent couplings that include terms of this kind of form.¹ In particular, within the heuristic model presented in Section 2.1, we interpret the appearance of a 3-form potential C as an indication of the presence of a Lie group G , in addition to the appearance of a 2-form potential B_2 which indicates the presence of a based loop group (of that Lie group).

¹See the discussion around Equation (3.7.19) for the case of untwisted String-2-connections, and then Section 4.6 for the general case of twisted String-2-connections.

The anomaly-corrected nonabelian 7d Chern-Simons term. Generally, there are two, seemingly different, realizations of the *holographic principle* in quantum field theory. On the one hand, Chern-Simons theories in dimension $4k + 3$ have spaces of states that can be identified with spaces of correlators of $(4k + 2)$ -dimensional conformal field theories (spaces of “conformal blocks”) on their boundary. For the case $k = 0$ this was discussed in [90], for the case $k = 1$ in [41, 42, 91], and the case $k = 2$ in [9]. On the other hand, AdS/CFT duality (see [3] for a review) identifies correlators of d -dimensional CFTs with states of compactifications of string theory, or M-theory, on asymptotically anti-de Sitter spacetimes of dimension $d + 1$ (see [93]).

However, in [94] it was pointed out that these two mechanisms are in fact closely related. A detailed analysis of the AdS₅/SYM₄-duality shows that the spaces of correlators of the 4-dimensional theory can be identified with the spaces of states obtained by geometric quantization just of the Chern-Simons term in the effective action of type IIB string theory on AdS₅. The relevant part of this action locally reads

$$(2.3.2) \quad (B_{\text{NS}}, B_{\text{RR}}) \mapsto N \int_{\text{AdS}_5} B_{\text{NS}} \wedge dB_{\text{RR}},$$

where B_{NS} is the local Neveu-Schwarz 2-form field, B_{RR} is the local RR 2-form field, and where N is the RR 5-form flux picked up from integration over the (internal) S^5 factor.

The abelian theory. As briefly indicated in [94], the similar form of the Chern-Simons term of 11-dimensional supergravity (M-theory) on AdS₇ suggests that an analogous argument shows that, under AdS₇/CFT₆-duality, the conformal blocks of the $(0, 2)$ -superconformal theory are identified with the geometric quantization of a 7-dimensional Chern-Simons theory. That Chern-Simons action is taken, locally on AdS₇, to be (up to an overall numerical factor)

$$(2.3.3) \quad C_3 \mapsto \int_{\text{AdS}_7 \times S^4} C_3 \wedge G_4 \wedge G_4 = N \int_{\text{AdS}_7} C_3 \wedge dC_3,$$

where now C_3 is the local incarnation of the supergravity C -field, and where G_4 is its curvature 4-form locally equal to dC_3 . This is the $(4 \cdot 1 + 3 = 7)$ -dimensional abelian Chern-Simons theory shown in [91] to induce on its 6-dimensional boundary the self-dual 2-form, in the *abelian* case.

The nonabelian theory. We may notice, however, that there is a term that is missing from (or that can be added to) the above Lagrangian. The quantum

anomaly cancellation via M5-branes in 11-dimensional supergravity is known to require instead a Lagrangian whose Chern-Simons term locally reads

$$(2.3.4) \quad (\omega, C_3) \mapsto \int_{\text{AdS}_7 \times S^4} C_3 \wedge \left(\frac{1}{6} G_4 \wedge G_4 - I_8^{\text{dR}}(\omega) \right),$$

where ω is the Spin connection form, locally, and where $I_8^{\text{dR}}(\omega)$ is a de Rham representative of the integral cohomology class [22, 86]

$$(2.3.5) \quad I_8 = \frac{1}{48} (p_2 - \lambda^2),$$

where

$$(2.3.6) \quad \lambda := \frac{1}{2} p_1$$

with p_1 and p_2 the first and second Pontrjagin classes, respectively, of the given Spin bundle over 11-dimensional spacetime X . This means that after passing to the effective theory on AdS_7 , this corrected Lagrangian picks up another 7-dimensional Chern-Simons term, now one which depends on *nonabelian* fields. Locally, this reads

$$(2.3.7) \quad S_{7d\text{CS}} : (\omega, C_3) \mapsto \frac{N}{6} \int_{\text{AdS}_7} C_3 \wedge dC_3 - N \int_{\text{AdS}_7} \text{CS}_{I_8}(\omega),$$

where $N := \int_{S^4} G_4$ is the C -field flux on the 4-sphere factor and $\text{CS}_{I_8}(\omega)$ is some Chern-Simons form for $I_8^{\text{dR}}(\omega)$, defined locally by (see also [67, 75])

$$(2.3.8) \quad d\text{CS}_{I_8}(\omega) = I_8^{\text{dR}}(\omega).$$

However, the above action functional, which is locally a functional of a 3-form and a Spin connection, cannot globally be of this form, as even the field that looks locally like a Spin connection cannot globally be a Spin connection. To see this, we first notice that there is a quantization condition on the supergravity fields on the 11-dimensional X [92], which in cohomology requires the identity

$$(2.3.9) \quad 2[G_4] = \frac{1}{2} p_1 + 2a \quad \text{in} \quad H^4(X, \mathbb{Z}),$$

where on the left we have the integral class underlying the C -field, and on the right we have the sum of the first fractional Pontrjagin class of the Spin-connection and the canonical class a of an ‘auxiliary’ or ‘topological’ E_8 bundle on the 11-dimensional spacetime X .

Moreover, by the arguments in [68] we expect that the integral class of the C -field vanishes on (a vicinity of) the 5-brane. This means that on an asymptotic neighbourhood of the asymptotic boundary ∂X , the above quantization condition becomes

$$(2.3.10) \quad \frac{1}{2}p_1 + 2a = 0 \quad \text{in} \quad H^4(\partial X, \mathbb{Z}).$$

Notice that requiring $[G_4] = 0$ at the boundary means that the C field is still there, but given by a globally defined differential 3-form C_3 .

As we have indicated around (2.1.13), imposing condition (2.3.10) in a gauge equivariant way involves refining it from an equation between cohomology classes (hence gauge equivalence classes) to a *choice of coboundary* between cocycles for $\frac{1}{2}p_1$ and $2a$. Doing so has two effects.

1. The first is that, according to [30, 68, 76], what locally looks like a Spin connection is globally instead a *2-connection on a twisted String-principal 2-bundle*, or equivalently a *twisted differential String structure*, where the twist is given by the class $2a$. The total space of such a principal 2-bundle may be identified [79] with a (twisted) *non-abelian bundle gerbe*. Therefore, the configuration space of fields of the effective 7-dimensional nonabelian Chern-Simons action above should involve not just Spin connection forms, but also String-*2-connection* form data. By [74] this is locally given by nonabelian 2-form field data.
2. The second effect is that on the space of twisted String-2-connections, the differential 4-form $\text{tr}(F_\omega \wedge F_\omega)$, which under the Chern-Weil homomorphism represents the image of $\frac{1}{2}p_1$, locally satisfies [30, 76]

$$(2.3.11) \quad d\mathcal{H}_3 = \text{tr}(F_\omega \wedge F_\omega) + 2\text{tr}(F_A \wedge F_A) - 2dC_3,$$

where \mathcal{H}_3 is the 3-form curvature component of the twisted String-2-connection, and where F_ω and F_A are the curvatures of the connection ω on the Spin bundle and of a connection A on the auxiliary E_8 bundle, respectively. This is the twisted Bianchi identity of the curvature 3-form, or equivalently the de Rham refinement of Equation (2.3.9), whose form is unaffected by the integral constraint (2.3.10).

Therefore the quantum correction term in the supergravity Lagrangian (2.3.7) now becomes (still for local data)

$$(2.3.12) \quad -N \int \text{CS}_{I_8}(\omega) = \frac{N}{48} \int ((\mathcal{H}_3 + 2C_3 - 2\text{CS}_3(A)) \wedge (d\mathcal{H}_3 + 2dC_3 - 2\langle F_A \wedge F_A \rangle) - \text{CS}_7(\omega)),$$

where $\text{CS}_3(A)$ is the ordinary Chern-Simons term for the E_8 -connection

$$(2.3.13) \quad \text{CS}_3(A) = \text{tr}(A \wedge dA) + \frac{2}{3} \text{tr}(A \wedge A \wedge A),$$

and where $\text{CS}_7(\omega)$ is the degree 7 Chern-Simons term for the Spin-connection, given by

$$(2.3.14) \quad \begin{aligned} \text{CS}_7(\omega) = & \langle \omega \wedge d\omega \wedge d\omega \wedge d\omega \rangle + k_1 \langle \omega \wedge [\omega \wedge \omega] \wedge d\omega \wedge d\omega \rangle \\ & + k_2 \langle \omega \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega] \wedge d\omega \rangle \\ & + k_3 \langle \omega \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega] \wedge [\omega \wedge \omega] \rangle, \end{aligned}$$

for suitable scalar constants k_i (see [75]). Notice that when multiplying out the brackets, a term proportional to $\int C_3 \wedge dC_3$ appears. Since, by the boundary condition, C_3 is here a globally defined form, this term may be rescaled by rescaling the 3-form. Therefore, we can absorb the first summand of (2.3.7) into the quantum correction Lagrangian and take the 7d Chern-Simons action to be a multiple of $\int \text{CS}_{I_8}$.

In [76] we have discussed the local data of such 7-dimensional nonabelian Chern-Simons Lagrangian on String-2-connections. In [30] we have provided the global description of the action functional on the full moduli 2-stack of String-2-connections. There we had concentrated on the role of this 7-dimensional action in the definition of *twisted differential Fivebrane structures*. In Section 4 below we discuss its role as a fully-fledged 7-dimensional Chern-Simons theory on nonabelian 2-form fields.

We therefore see that including the I_8 -correction term and the refined quantization condition in $\text{AdS}_7/\text{CFT}_6$ leads to a 7-dimensional theory that contains orthogonal and unitary nonabelian gauge field degrees of freedom and which is globally controlled by a twisted nonabelian gerbe structure. As we discuss in detail below in Section 3.6, this kind of data has several rather different looking equivalent incarnations, due to the fact that these higher connections have a much richer gauge structure than ordinary connections. In particular, one can pass from descriptions that locally look only slightly nonabelian but have complicated global transformation laws, to equivalent descriptions that have global transformation laws more like ordinary connections but which, in compensation, exhibit a richer nonabelian structure

locally: it is locally given [80][81] by a 1-form $A \in \Omega^1(-, P_*\mathfrak{so})$ with values in the Lie algebra of paths in the Lie algebra \mathfrak{so} of the orthogonal group and a 2-form $B \in \Omega^2(-, \widehat{\Omega}\mathfrak{so})$ with values in the Kac-Moody central extension of the loop Lie algebra of \mathfrak{so} .

The above line of arguments suggests that the Chern-Simons term that governs 11-dimensional supergravity on AdS_7 is an action functional on fields that are twisted String-2-connections such that the action functional is locally given by expression (2.3.7). In Sections 4.2 and 4.5 we discuss a precise formulation of higher Chern-Simons theories satisfying these properties and which extend beyond Anti-de Sitter spaces.

2.4. Which gauge group(s)?

We ask the natural question: Is/are there (a) particular Lie group(s) that is/are associated with the fivebrane theory? We provide several further arguments which favor the exceptional groups, including E_8 .²

First, note that the anomaly cancellation in the ambient seven-dimensional theory with boundary, leading to chiral $N = (0, 1)$ 6d theory, is worked out in [35]. The resulting admissible groups from the anomaly argument are indeed the exceptional groups.

The system of multiple M5-branes, generalizing the system of n D-branes leading to $U(n)$ nonabelian gauge symmetry, can be described by twisted String(G)-gerbes, whose cocycle data involves the universal central extension $\widehat{\Omega}G$ of the based loop group ΩG , where G is any of the Lie groups E_6 , E_7 , E_8 , F_4 , and G_2 , as was also proposed in [5]. Indeed, it has been argued in [78] that classical membrane fields are loops.

The form of the action of the fivebrane suggests working in eight dimensions, where the interpretation of the terms becomes transparent. This also suggests the existence of an E_8 gauge theory on this eight-dimensional extension Z^8 of the worldvolume M^6 . The topological part of the action extended to eight dimensions looks like [91]

$$(2.4.1) \quad \langle [G_4] \cup [G_4] - \lambda \cup [G_4], [Z^8] \rangle,$$

with $[G_4]$ a degree four characteristic class and $[Z^8]$ the fundamental class on which the composite degree eight cohomology class is evaluated. Now we observe that, up to dimension 8, all exceptional Lie groups have the same homotopy type as the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$, so that G_4 can be

²This section expands in part on the discussion in Section 2.1.

viewed essentially as the characteristic class of any bundle with structure group an exceptional Lie group $G_2, F_4, E_i, i = 6, 7, 8$. Therefore, while E_8 is not singled out by this argument, it is certainly the case that it could be assumed to describe the extension of the fivebrane worldvolume to eight dimensions.

The space of conformal blocks, in the sense of Witten [96], has dimension bigger than one for groups other than E_8 . This means that the theory does not have a distinguished partition function. Therefore, this quantum field theory argument favors E_8 .

We hence start with an E_8 bundle on the 8-dimensional extension Z^8 . This extension is the total space of the 2-disk bundle over M^6 (in the sense of [52, 66]), and the process of extension can be done in two different ways. The first way is to take M^6 to be the boundary of W^7 and then take this 7-dimensional space to be the base space of a circle bundle with total space Z^8 . The second way is to take M^6 itself as the base space of a circle bundle with total space a 7-dimensional manifold Y^7 , which we take to be the boundary of Z^8 . So the two-disk bundle $\mathbb{D}^2 \rightarrow Z^8 \rightarrow M^6$ can be viewed as

$$\begin{array}{ccc}
 S^1 \longrightarrow Z^8 & & Z^8 \xleftarrow{j} Y^7 = \partial Z^8 \longleftarrow S^1 \\
 \downarrow & \text{or} & \downarrow \\
 W^7 \xleftarrow{i} M^6 = \partial W^7 & & M^6
 \end{array}$$

Why loop bundles? Proceeding with the second case, we have to some extent a seven-dimensional analog of the eleven-dimensional Hořava-Witten set-up. Here we will consider the M5-brane worldvolume M^6 to be disconnected. That is we will take M^6 to be composed of the two boundary components for the manifold with boundary W^7 . Taking an E_8 bundle on Z^8 gives us an LE_8 bundle on W^7 , in a process which is similar to the one relating M-theory to type IIA string theory (see [24, 52]). Then we consider the reduction of this LE_8 bundle to M^6 , in a process analogous to Horava-Witten, except that it is for loop bundles instead of finite-dimensional bundles. We get this way an LE_8 bundle on each boundary component $M_i^6, i = 1, 2$.

Let us now go with the restriction of the E_8 bundle on Z^8 to its boundary Y^7 . This will give an E_8 bundle, whose reduction over the circle fiber down to M^6 leads again to an LE_8 bundle, in this case in exact analogy to the case of going from Z^{12} to the M-theory boundary and then reducing on the M-theory circle to get type IIA string theory (see [52, 65, 66]). Therefore, we have the following statement

For any exceptional Lie group G , there is a loop G bundle on the five-brane worldvolume. This arises from the dimensional reduction of a G bundle on the extended worldvolume.

Reduction to the D4-brane. Assuming we have a bundle with structure group an exceptional Lie group G or its loop group on the M5-brane worldvolume M^6 , it is natural to ask to what structure this reduces on the D4-brane worldvolume. Since such a reduction would be over a circle, one possibility is to perform a reduction similar to that of going from M-theory to type IIA string theory, as in [21]. For example, the inclusion $E_8 \supset (\mathrm{SU}(5) \times \mathrm{SU}(5)) / \mathbb{Z}_5$ provides a way of getting unitary groups from exceptional groups. Such unitary groups will be in the stable range due to the relatively low dimension of our spaces, so that we would effectively get unitary groups of all ranks. Breaking loop bundles to finite-dimensional (unitary) bundles is considered in a similar context in [66].

Center-of-mass and noncommutativity. In the absence of a B -field, the field theory of N coincident D-branes is a supersymmetric $\mathrm{U}(N)$ gauge theory. The $\mathrm{U}(1)$ part represents the interactions of D-brane open string with the bulk closed strings, representing the supergravity fields. This $\mathrm{U}(1)$ center-of-mass part decouples from the open string dynamics leading to an effective $\mathrm{SU}(N)$ gauge theory. However, we will explain, in Section 3.2 below, that this $\mathrm{U}(1)$ part does have a role to play which is interesting in its own right. We will then generalize this discussion to the case of multiple M5-branes. However, in the presence of a constant B -field background, noncommutativity of the resulting gauge theory [83] makes it impossible to separate the center-of-mass part. The reason is that the B -field makes the left and right mover of the open string sector not to be treated on equal footing. Indeed, gauge transformations on $\mathrm{SU}(N)$ do not close [4]. Noncommutative $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ theories can also be constructed with a similar phenomenon occurring [13]. We anticipate that multiple M5-branes in the presence of a C -field will lead to a similar inability to separate this analog the center-of-mass part. What this means, from the point of view of the description of the String group in Section 2.1, is that we cannot separate the based loop group part ΩG from the finite-dimensional part that corresponds to the underlying Lie group G , as their twisted product is what forms the String group. Recall that in the above-mentioned model: S^1 is replaced by $K(\mathbb{Z}, 2)$, which is homotopy equivalent to a loop group of an exceptional Lie group in our range of dimensions. The S^1 is the center-of-mass in the case of D-branes; analogously, one may interpret the factor $K(\mathbb{Z}, 2) \sim_8 \Omega G$, for G exceptional, as the ‘center-of-mass’ in the case of M5-branes.

2.5. Generalizations: Nontrivial normal bundle and the role of the ADE groups

We used in Section 2.3 the comparatively good available understanding of holography over asymptotically AdS spaces in order to give a plausibility argument that suggests that the 7-dimensional quantum field theory encodes holographically the nonabelian 6-dimensional $(0, 2)$ -theory. Once one accepts this, there are evident generalizations of this 7-dimensional theory to more general setups than covered by AdS/CFT. Here we indicate some of these generalizations and make connections to higher structures.

The M5-brane background in the eleven-manifold X^{11} breaks the structure group $\text{Spin}(10, 1)$ of the Spin bundle to the (twisted) product bundle with structure group $\text{Spin}(5, 1) \times \text{Spin}(5)$ corresponding to the breaking of the tangent bundle $TX^{11}|_{M^6} = TM^6 \oplus \mathcal{N}$, where \mathcal{N} is the normal bundle.

The $N = (0, 2)$ theory associated with arbitrary group G is suggested to have an anomaly of the following general form when coupled to the $\text{Spin}(5)_R$ normal bundle \mathcal{N} [47]

$$(2.5.1) \quad I_8(G) = r(G)I_8(1) + \frac{c(G)}{24}p_2(\mathcal{N}) ,$$

where

$$(2.5.2) \quad I_8(1) = \frac{1}{48} (p_2(\mathcal{N}) - p_2(M) + (\lambda(\mathcal{N}) - \lambda(M))^2)$$

is the anomaly polynomial for a single free $(0, 2)$ supermultiplet [91]. Here $r(G) = \text{rank}(G)$ is the rank of the group and $c_G = \dim G \cdot h_G$, where h_G is the dual Coxeter number of G .

Now, motivated by our discussion on String structure associated to the M5-brane (see the discussion around expression (2.3.10)), we impose our condition leading to a twisted String structure. Inspecting the formulae (2.5.1) and (2.5.2) we see that imposing first the condition $\frac{1}{2}p_1(M) - \frac{1}{2}p_1(\mathcal{N}) = 0$ leads to a simplification. Indeed, with this twisted String condition (with the twist being $\frac{1}{2}p_1(\mathcal{N})$), the general anomaly polynomial (2.5.1) becomes

$$(2.5.3) \quad I_8(G)_{\text{ts}} = -\frac{1}{48} [p_2(M) - (r(G) + 2c(G))p_2(\mathcal{N})] .$$

Assuming the conjectural formula (2.5.1) holds, a couple of remarks are in order:

1. In the case of a twisted String structure, with the twist given by the first Spin characteristic class of the normal bundle, we interpret the

vanishing of the anomaly, given by expression (2.5.3), as saying that we have a twisted Fivebrane structure, with the twist again being due to the normal bundle, that is the twist is given by a fractional second Pontrjagin class of the normal bundle. In order for this to serve as a twist in the sense of [76], it has to be an integral class. This imposes the condition

$$(2.5.4) \quad \frac{1}{6}(r(G) + 2c(G)) \in \mathbb{Z} .$$

This might be considered as a condition analogous to the condition $\frac{1}{6}c(G) \in \mathbb{Z}$, derived in [47].

2. We now would like to take expression (2.5.3) as a basis for the non-abelian Chern-Simons term as in Section 2.3. We can indeed apply the general formula as above (see expression (2.3.12)), but now we have both a Chern-Simons 7-form for the tangent bundle as well as one for the normal bundle. This then resembles the discussion in [76] where the gauge bundle corresponding to the heterotic string plays an analogous topological role — namely a twist — that the normal bundle plays here. Our more general Chern-Simons theory will be given by expression (2.3.12) but with the last term there replaced by the Chern-Simons form corresponding to $I_8(G)_{\text{ts}}$, given in expression (2.5.3).

We summarize what we have as follows:

The Chern-Simons theory can be defined for a general ADE group. An ADE group G induces the corresponding $\text{String}(G)$ -2-group involving the centrally extended loop group $\hat{\Omega}G$, which serves as the structure 2-group for 2-bundles that underly the worldvolume two-form field. For G nonabelian, we have a nonabelian gerbe and hence a theory of multiple M5-branes. A necessary condition for the existence of such theory is the presence of a twisted String structure, with the twist given by the normal bundle. This in turn leads to a twisted Fivebrane structure, with a similar type of twist arising from the normal bundle.

3. Nonabelian higher gauge theory

We review here aspects of higher nonabelian connections and their higher gauge theory, formulated on smooth higher stacks. We do so in a way that is tailored towards our application in Section 4 and should serve as a warmup for that application, but the discussion is of independent relevance for higher nonabelian gauge theory and for other of its applications in string theory.

These structures and their applications have been discussed in earlier work such as [30, 74, 76, 79].³

3.1. Higher smooth moduli stacks — integrated BRST Lie n -algebroids

Physics has for a long time been concerned with and formulated in terms of *geometry*. But ever since the inception of gauge theory, it is secretly also concerned with and formulated in terms of *homotopy theory*. A *gauge transformation* is essentially what mathematically is called an *isomorphism in a groupoid* or *homotopy in a space*. The central idea in physics that “it is bad to quotient out gauge transformations and good to remember them” is equivalently the idea in homotopy theory that “it is bad to force everything to be a discrete set (of equivalence classes) and good to instead retain groupoids with their isomorphisms and spaces with their homotopies”. Therefore, what *really* matters in physics is the combination of both geometry and homotopy theory. This is, then, a theory where we have *geometric families of homotopies*, such as *smooth families of homotopies*. For instance the configuration space of Yang-Mills theory is not just a smooth collection of field configurations, and is not just a groupoid of gauge transformations, but is a combination of both, namely a *smooth groupoid*. In this formulation, both the field configurations may vary smoothly, as do their gauge transformations.

For a *higher gauge theory* such as the 2-form theory of the B -field, we have the same situation, but with even richer structure. The B -field has a *smooth 2-groupoid* of field configurations, where in addition to the gauge transformations there are now smooth families of gauge-of-gauge transformations. Next, for the supergravity C -field the configuration space is correspondingly a *smooth 3-groupoid*, and so on.

Why stacks? For historical reasons, a smooth groupoid is also called a *stack on smooth manifolds*. This terminology is often used in the context of refined *moduli spaces* which are then called *moduli stacks*. For instance, for G any Lie group, there is a topological space BG which is the “moduli space of G -instantons” in that for X any manifold, the homotopy classes of maps $X \rightarrow BG$ correspond to equivalence classes of G -principal bundles (G -instantons) on X . The trouble with this concept is that BG does not know about the smooth gauge transformations given by G -valued functions,

³The reader who feels reasonably comfortable with the general ideas used there might wish to skip ahead to Section 4.

nor does it know about actual gauge fields, namely about connections on G -principal bundles. This is where the moduli *stacks* come in. There is a smooth groupoid / smooth stack which we will write as $\mathbf{B}G$ and which is such that maps of smooth stacks $X \rightarrow \mathbf{B}G$ correspond to G -bundles on X , and smooth homotopies of such maps correspond to smooth gauge transformations of G -bundles. Furthermore, there is a *differential refinement* to a richer smooth stack which we denote $\mathbf{B}G_{\text{conn}}$, and which is such that maps $X \rightarrow \mathbf{B}G_{\text{conn}}$ correspond to G -Yang-Mills gauge fields on X , and homotopies of such maps correspond to smooth gauge transformations. Accordingly then, there is the smooth *mapping stack* $[X, \mathbf{B}G_{\text{conn}}]$ whose elements are gauge fields on X , and whose morphisms are gauge transformations. This is the true “configuration space” of Yang-Mills theory on X . If we forget the smooth structure on this, we write $\mathbf{H}(X, \mathbf{B}G_{\text{conn}})$, the *cocycle groupoid* of nonabelian differential G -cohomology. Its connected components $H(X, \mathbf{B}G_{\text{conn}})$ is the set of gauge equivalence classes of field configurations: the *cohomology set* of nonabelian differential G -cohomology on X .

Importance in (higher) gauge theory. The above stack is in fact a global refinement of an object long familiar in gauge theory, namely the *BRST-complex* for Yang-Mills fields on X . A BRST complex is, in a precise sense, the infinitesimal approximation — the *Lie algebroid* — of a smooth moduli stack of field configurations. The *ghosts* of the BRST complex are the cotangents to the spaces of morphisms / gauge transformations in the stack. For higher gauge theory, the order- n ghosts-of-ghosts in the BRST complex are the cotangents to the space of n -morphisms in the higher moduli stack and exhibit a *Lie n -algebroid* structure. This is one way to understand the use of (higher) moduli stacks in physics, as the natural way to incorporate the BRST quantization of (higher) gauge theories into a powerful ambient mathematical context, and to refine it from infinitesimal (higher) gauge transformations to finite ones.

Therefore, similarly, there is for each natural number n a *higher* moduli stack $\mathbf{B}^n U(1)_{\text{conn}}$ of n -form gauge fields. For instance $[X, \mathbf{B}^2 U(1)_{\text{conn}}]$ is the stacky configuration space of the B -field on X , with its gauge transformations and gauge-of-gauge transformations, whose infinitesimal approximation is the BRST complex for a 2-form field with its ghosts and ghosts-of-ghosts. As opposed to the BRST complex, the full stack of field configurations knows not just about the infinitesimal gauge transformations, but also of the finite gauge transformations. It therefore contains genuinely the full information about the gauge field configurations.

For all $n \in \mathbb{N}$, there is the smooth moduli n -stack $\mathbf{B}^n U(1)_{\text{conn}}$ of n -form fields, discussed in more detail in Section 3.3. Cohomology $H(X, \mathbf{B}^n U(1)_{\text{conn}})$

with coefficients in this is *ordinary differential cohomology*. More generally, for G any higher smooth group, one can consider higher moduli stacks of nonabelian G -connections. The corresponding cohomology $H(X, \mathbf{B}G_{\text{conn}})$ is *nonabelian differential cohomology*.

The Dold-Kan correspondence. Handling higher stacks is a bit more subtle than handling just manifolds or just topological spaces, but there are a handful of simple but powerful tools that allow one to efficiently work with them in a way that is very close to common operations in physics. One such tool is, for instance, the *Dold-Kan correspondence*. In simplified terms this establishes that the homological algebra of *chain complexes of sheaves*, which is familiar in string theory mostly from the study of the topological string, presents a sub-class of higher stacks, namely the “strictly abelian” higher stacks. For instance, for every sheaf A of abelian groups on all smooth manifolds — e.g. the sheaf $A = \mathbf{U}(1)$ of smooth circle-valued functions — there is a chain complex of sheaves

$$(3.1.1) \quad A[n] = (A \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0)$$

concentrated on A in degree n , and the Dold-Kan correspondence identifies this (up to a suitable notion of equivalence) with a moduli n -stack $\mathbf{B}^n A$ that classifies instanton configurations for A -valued gauge fields of higher order n . For instance, $\mathbf{B}^2 \mathbf{U}(1)$ classifies instanton configurations of B -fields and $\mathbf{B}^3 \mathbf{U}(1)$ classifies instanton configurations of C -fields. This we come to in Section 3.3. For technical details on the Dold-Kan correspondence and higher stacks see Section 2.1.7 in [79].

Geometric realizations and smooth refinement. While smooth higher stacks have richer structure than topological spaces, there is a map called *geometric realization* that sends any smooth higher stack to the topological spaces which is the “best approximation” to it, in a precise sense. This is an “ ∞ -functor”⁴

$$(3.1.2) \quad | - | : \text{Smooth}\infty\text{Grpd} \rightarrow \text{Top},$$

For instance the geometric realization of the moduli stack $\mathbf{B}\text{Spin}$ of Spin-principal bundles is the ordinary classifying space $B\text{Spin}$

$$(3.1.3) \quad |\mathbf{B}\text{Spin}| \simeq B\text{Spin}$$

⁴See Sections 3.2.2 and 3.3.3 of [79].

(all up to weak homotopy equivalence). And the geometric realization of the n -stack $\mathbf{B}^n U(1)$ is the Eilenberg-MacLane space $K(\mathbb{Z}, n+1)$ (notice the degree shift) which classifies integral cohomology

$$(3.1.4) \quad |\mathbf{B}^n U(1)| \simeq K(\mathbb{Z}, n+1).$$

Geometric realization necessarily forgets crucial geometric information and information about the nature of gauge transformations. But for a large class of higher moduli stacks (not for all, but for all non-differentially refined stacks that are of interest to us here), it remembers the information about gauge equivalence classes. For instance equivalence classes of morphisms of smooth stacks $X \rightarrow \mathbf{B}E_8$ from a smooth manifold X are in bijection with homotopy classes of continuous maps $X \rightarrow BE_8$. This is important for the present discussion, as very different looking smooth higher moduli stacks may become equivalent after geometric realization. For instance the equivalence

$$(3.1.5) \quad |\mathbf{B}PU(\mathcal{H})| \simeq |\mathbf{B}^2 U(1)|$$

controls the nonabelian cohomology of the restriction of the B -field to D-branes (Section 3.4) and the equivalence

$$(3.1.6) \quad |\mathbf{B}E_8| \simeq_{15} |\mathbf{B}^3 U(1)|$$

controls the higher nonabelian cohomology of the restriction of the C -field to M5-branes (Section 4.3).

Using the notion of geometric realization, we may say that an ordinary universal characteristic class $c \in H^{n+1}(BG, \mathbb{Z})$ has a *smooth refinement* to a morphism of n -stacks

$$(3.1.7) \quad \mathbf{c} : \mathbf{B}G \rightarrow \mathbf{B}^n U(1)$$

if the geometric realization $BG \rightarrow K(\mathbb{Z}, n+1)$ of this morphism represents c . For G a compact Lie group, such smooth lifts exist uniquely, up to equivalence, by Theorem 3.3.29 in [79]. (This is how we obtain the String 2-group in Section 3.7.) Since $\mathbf{B}^n U(1)$ is the moduli stack for circle n -bundles / $(n-1)$ -gerbes, \mathbf{c} also constructs an $(n-1)$ -gerbe on the moduli stack $\mathbf{B}G$. The looping $\Omega \mathbf{c}$ is therefore an $(n-2)$ -bundle gerbe over G itself. For $n=3$ this bundle-gerbe perspective on smooth refinements is spelled out in [88].

There are more such tools for handling higher stacks, but here we will not further dwell on recalling these. The interested reader can find explanations in [30] and in more details in [79].⁵

3.2. Determinant line bundles and c_1 -twists

A central role in our discussion to follow is played by universal characteristic maps refined to smooth stacks, and of their homotopy fibers. As a warmup and general motivation, we expose here the simplest non-trivial example of this general concept, whose component ingredients are all still familiar from traditional theory. This is the example induced by the smooth refinement c_1 of the first Chern class on unitary bundles. In particular we describe how to regard *principal $SU(N)$ -bundles* (with \mathfrak{su}_N -connections) equivalently as *trivially c_1 -twisted principal $U(N)$ -bundles* (with \mathfrak{u}_N -connections).⁶

Recall from the theory of characteristic classes that the obstruction to reducing the structure group of a principal $U(N)$ -bundle to $SU(N)$ is the first Chern class of the bundle. The stacky perspective on this says: the smooth universal first Chern class is the homotopy class of the morphism of stacks

$$(3.2.1) \quad c_1 = \mathbf{Bdet} : \mathbf{BU}(N) \rightarrow \mathbf{BU}(1)$$

from the moduli stack of principal $U(N)$ -bundles to the moduli stack of principal $U(1)$ -bundles induced by the determinant $\det : U(N) \rightarrow U(1)$. Furthermore, the fact that c_1 represents the obstruction to the $U(N)$ -to- $SU(N)$ reduction becomes the following statement: the stack $\mathbf{BSU}(N)$ of principal $SU(N)$ -bundles is the homotopy pullback

$$(3.2.2) \quad \begin{array}{ccc} \mathbf{BSU}(N) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{BU}(N) & \xrightarrow{c_1} & \mathbf{BU}(1) \end{array} .$$

By the definition of homotopy pullbacks (see also the discussion in the introduction around (2.1.13)), this says that a morphism of stacks $X \rightarrow \mathbf{BSU}(N)$,

⁵The reader who does not know yet and who cannot be bothered to go through these details should nevertheless be able to follow the discussion below, if only he or she keeps the intuitive idea of a higher stack as a collection of higher smooth families of higher gauge transformations in mind.

⁶Our discussion of String-2-connections in Section 3.7 will proceed by close analogy with the constructions here.

hence an $SU(N)$ -principal bundle over X , is equivalently a morphism $X \rightarrow \mathbf{BU}(N)$, hence a $U(N)$ -principal bundle P , together with a choice of trivialization of the composite morphism $X \rightarrow \mathbf{BU}(N) \xrightarrow{c_1} \mathbf{BU}(1)$, hence of the determinant bundle $\mathbf{c}_1(P)$. Moreover, the whole groupoid of $SU(N)$ -principal bundles on a manifold X is *equivalent* to the groupoid of $U(N)$ -principal bundles on X that are equipped with a trivialization of their associated determinant $U(1)$ -principal bundle.

Let us describe a morphism of stacks $X \rightarrow \mathbf{BSU}(N)$, with $\mathbf{BSU}(N)$ identified with a homotopy pullback as above, in explicit detail, following [30]. For this we first need to choose an open cover $\mathcal{U} = \bigcup_i U_i$ of X , which is “good”, meaning that all non-empty finite intersections of the U_i are contractible. In terms of this choice, a map of stacks from X into the above homotopy pullback is given by the following data.

- $U(1)$ -valued functions ρ_i on the patch U_i ;
- $U(N)$ -valued functions g_{ij} on the double intersection $U_{ij} := U_i \cap U_j$, with $g_{ii} = 1$,

subject to the constraints

- $\det(g_{ij})\rho_j = \rho_i$ on U_{ij} ;
- $g_{ij}g_{jk}g_{ki} = 1$ on the triple intersection $U_{ijk} := U_i \cap U_j \cap U_k$.

Morphisms between (ρ_i, g_{ij}) and (ρ'_i, g'_{ij}) are the gauge transformations locally given by $U(N)$ -valued functions γ_i on U_i such that $\gamma_i g_{ij} = g'_{ij} \gamma_j$ and $\rho_i \det(\gamma_i) = \rho'_i$.

Note that the classical description of objects in $\mathbf{BSU}(N)$ corresponds to the *gauge fixing* $\rho_i \equiv 1$; at the level of morphisms, imposing this gauge fixing constrains the gauge transformation γ_i to satisfy $\det(\gamma_i) = 1$, i.e. to take values in $SU(N)$. From a categorical point of view, this amounts to saying that the embedding of the groupoid of $SU(N)$ -principal bundles into the homotopy fiber of \mathbf{c}_1 given by $(g_{ij}) \mapsto (1, g_{ij})$ is fully faithful. It is also essentially surjective: use the embedding $U(1) \rightarrow U(N)$ given by $e^{it} \mapsto (e^{it}, 1, 1, \dots, 1)$ to lift ρ_i^{-1} to a $U(N)$ -valued function γ_i with $\det(\gamma_i) = \rho_i^{-1}$; then (γ_i) is an isomorphism between (ρ_i, g_{ij}) and $(1, \gamma_i g_{ij} \gamma_j^{-1})$.

Next we turn to connections. It is a well known fact from Chern-Weil theory that the de Rham image of the first Chern class of a $U(N)$ -principal bundle can be realized as the de Rham cohomology class $[\text{tr}(F_\nabla)]$, where F_∇ is the *curvature* 2-form of a \mathfrak{u}_n -connection ∇ . The cohomology equation

$$(3.2.3) \quad [\text{tr}(F_\nabla)] = 0$$

is equivalent to $\text{tr}(F_{\nabla}) = d\alpha$ for some 1-form α , and we can therefore think of the choice of such an α as the choice of a trivialization of the characteristic form $\text{tr}(F_{\nabla})$. Since the 1-form α is naturally interpreted as a \mathfrak{u}_1 -connection on a *trivial* principal $U(1)$ -bundle, the trivialization of $\text{tr}(F_{\nabla})$ becomes the equation

$$(3.2.4) \quad \text{tr}(F_{\nabla}) = F_{\alpha} ,$$

i.e., we are identifying the characteristic 2-form $\text{tr}(F_{\nabla})$ with the curvature 2-form of a connection on a trivial principal $U(1)$ -bundle. All this has a simple interpretation in terms of stacks: the smooth first Chern class $\mathbf{c}_1 : \mathbf{BU}(N) \rightarrow \mathbf{BU}(1)$ has a *differential refinement* to a morphism

$$(3.2.5) \quad \hat{\mathbf{c}}_1 : \mathbf{BU}(N)_{\text{conn}} \rightarrow \mathbf{BU}(1)_{\text{conn}}$$

from the moduli stack of $U(N)$ -principal bundles with \mathfrak{u}_N -connections to the moduli stack of $U(1)$ -principal bundles with \mathfrak{u}_1 -connections, induced by the Lie algebra morphism

$$(3.2.6) \quad \text{tr} : \mathfrak{u}_N \rightarrow \mathfrak{u}_1 .$$

In terms of local data, the morphism $\hat{\mathbf{c}}_1$ maps the \mathfrak{u}_N -connection 1-form A_i to the Chern-Simons 1-form $\text{CS}_1(A_i) = \text{tr}(A_i)$, and the identity

$$(3.2.7) \quad \text{tr}(F_{A_i}) = d\text{CS}_1(A_i)$$

shows that the curvature characteristic 2-form of the \mathfrak{u}_N -connection (A_i) can be identified with the curvature of the 2-form of the \mathfrak{u}_1 -connection ($\text{CS}_1(A_i)$). This means that, as a morphism of stacks, the traced curvature

$$(3.2.8) \quad \mathbf{BU}(N)_{\text{conn}} \xrightarrow{\text{tr curv}} \Omega_{\text{closed}}^2$$

actually factors as

$$(3.2.9) \quad \mathbf{BU}(N)_{\text{conn}} \xrightarrow{\hat{\mathbf{c}}_1} \mathbf{BU}(1)_{\text{conn}} \xrightarrow{\text{curv}} \Omega_{\text{closed}}^2 ,$$

where Ω_{closed}^2 is the stack whose smooth U -parameterized families of objects are closed 2-forms on U (with trivial morphisms). Moreover, the stack of

$SU(N)$ -principal bundles with \mathfrak{su}_N -connections is the (homotopy) pullback

$$(3.2.10) \quad \begin{array}{ccc} \mathbf{BSU}(N)_{\text{conn}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{BU}(N)_{\text{conn}} & \xrightarrow{\hat{c}_1} & \mathbf{BU}(1)_{\text{conn}} . \end{array}$$

Again, we write out the data for objects and morphisms in the groupoid of $SU(N)$ -bundles with \mathfrak{su}_N -connections over a fixed smooth manifold X , presented as a homotopy pullback this way. For a fixed good open cover \mathcal{U} of X , the objects of this groupoid are

- $U(1)$ -valued functions ρ_i on U_i ;
- \mathfrak{u}_N -valued 1-forms A_i on U_i ;
- $U(N)$ -valued functions g_{ij} on U_{ij} , with $g_{ii} = 1$.

subject to the constraints

- $\text{tr}A_i + d\log\rho_i = 0$ on U_i ;
- $\det(g_{ij})\rho_j = \rho_i$ on U_{ij} ;
- $A_j = g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}dg_{ij}$ on U_{ij} ;
- $g_{ij}g_{jk}g_{ki} = 1$ on U_{ijk} ,

and the classical description of \mathfrak{su}_N -connections on a principal $SU(N)$ -bundle corresponds to the gauge fixing $\rho_i \equiv 1$. This should not be surprising: the data (ρ_i) are the data of the trivialization of the principal $U(1)$ -bundle with \mathfrak{u}_1 -connection induced by \hat{c}_1 ; fixing these data to 1 is equivalent to requiring that this bundle with connection is trivially trivialized.

As far as concerns the morphisms, in the homotopy pullback description, a morphism between (ρ_i, A_i, g_{ij}) and (ρ'_i, A'_i, g'_{ij}) is the datum of

- $U(N)$ -valued functions γ_i on U_i

such that

- $A'_i = \gamma_i^{-1}A_i\gamma_i + \gamma_i^{-1}d\gamma_i$;
- $\rho_i \det(\gamma_i) = \rho'_i$ on U_i ;
- $\gamma_i g_{ij} = g'_{ij} \gamma_j$ on U_{ij} .

Note that the gauge fixing $\rho_i = 1$ imposes $\det(\gamma_i) = 1$, and one recovers the classical description of isomorphisms between principal $SU(N)$ -bundles with \mathfrak{su}_N -connections.

Notice that the curvature characteristic 2-form of a \mathfrak{su}_N -connection (either in the classical or in the homotopy pullback description) is identically zero. This means that we actually went far beyond our original aim that was to kill only the cohomology class of the curvature characteristic form, and not the curvature characteristic 2-form itself. This is not unexpected: morphisms of principal bundles with connections are too narrow to capture the flexible nature of requiring something to be zero only in cohomology. A natural way to remedy this is to consider instead the moduli stack $\mathbf{BU}(N)_{\text{conn}, c_1=0}$ defined as the homotopy fiber of the composite map

$$(3.2.11) \quad \mathbf{c}_1 : \mathbf{BU}(N)_{\text{conn}} \xrightarrow{\hat{\mathbf{c}}_1} \mathbf{BU}(1)_{\text{conn}} \rightarrow \mathbf{BU}(1),$$

where the second morphism forgets the connection. Since we have a homotopy pullback diagram

$$(3.2.12) \quad \begin{array}{ccc} \Omega^1 & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{BU}(1)_{\text{conn}} & \longrightarrow & \mathbf{BU}(1) \end{array}$$

it follows by the pasting law for homotopy pullbacks that the homotopy fiber of (3.2.11) is equivalently described as the homotopy pullback

$$(3.2.13) \quad \begin{array}{ccc} \mathbf{BU}(N)_{\text{conn}, c_1=0} & \longrightarrow & \Omega^1 \\ \downarrow & & \downarrow \\ \mathbf{BU}(N)_{\text{conn}} & \xrightarrow{\hat{\mathbf{c}}_1} & \mathbf{BU}(1)_{\text{conn}} \end{array} .$$

In other words, $\mathbf{BU}(N)_{\text{conn}, c_1=0}$ is the collection of all the homotopy fibers of $\hat{\mathbf{c}}_1 : \mathbf{BU}(N)_{\text{conn}} \rightarrow \mathbf{BU}(1)_{\text{conn}}$, with varying “background field” in Ω^1 , and so local data for a map from a manifold X into this stack are

- $U(1)$ -valued functions ρ_i on U_i ;
- \mathfrak{u}_1 -valued 1-forms \mathcal{H}_i on U_i ;
- \mathfrak{u}_N -valued 1-forms A_i on U_i ;
- $U(N)$ -valued functions g_{ij} on U_{ij} , with $g_{ii} = 1$.

subject to the constraints

- $\mathcal{H}_i = d\log\rho_i + \text{tr}A_i$ on U_i ;
- $\det(g_{ij})\rho_j = \rho_i$ on U_{ij} ;
- $A_j = g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}dg_{ij}$ on U_{ij} ;
- $\mathcal{H}_i = \mathcal{H}_j$ on U_{ij} ;
- $g_{ij}g_{jk}g_{ki} = 1$ on U_{ijk} .

In particular, the local curvature 2-form of a connection classified by $\mathbf{BU}(N)_{\text{conn}, c_1=0}$ is

$$(3.2.14) \quad \text{tr}(F_{A_i}) = dH_i .$$

Notice that since $\mathcal{H}_i = \mathcal{H}_j$ on U_{ij} , the (\mathcal{H}_i) define a global 1-form on X and so $\text{tr}(F_{A_i}) = d\mathcal{H}_i$ precisely says that the cohomology class of the curvature characteristic 2-form vanishes. The stringy analog of this we discuss below, around Equation (3.7.14).

3.3. Higher $\mathbf{U}(1)$ -bundles with connections

We discuss the moduli stack of ordinary circle bundles with connection (abelian Yang-Mill fields) and then the higher analogs, the higher stacks of circle n -bundles with connection (B -fields, C -fields, etc.).

The stack circle-bundles with connections. We start with the moduli stack $\mathbf{BU}(1)_{\text{conn}}$ of ordinary circle bundles with connection in a way that prepares for the generalizations to follow.

The local data for a \mathfrak{u}_1 -connection on a $\mathbf{U}(1)$ -principal bundle over a smooth manifold X , locally trivialized over an open cover $\{U_i \hookrightarrow X\}$, are given by “vertices” (objects) A_i , which are \mathfrak{u}_1 -valued 1-forms on U_i and “edges” (morphisms/gauge transformations) g_{ij} , which are $\mathbf{U}(1)$ -valued functions on the double intersections U_{ij} . This data is subject to two constraints: “an edge g_{ij} has to go from A_i to A_j ”:

$$(3.3.1) \quad d\log g_{ij} = A_j - A_i ;$$

and “going around the boundary of a 2-simplex is a trivial path”:

$$(3.3.2) \quad g_{ij}g_{jk}g_{ik}^{-1} = 1 .$$

$$(3.3.3) \quad \begin{array}{ccc} & A_j & \\ g_{ij} \nearrow & & \searrow g_{jk} \\ A_i & \xrightarrow{g_{ik}} & A_k \end{array} .$$

This can be elegantly stated as follows: the stack $\mathbf{BU}(1)_{\text{conn}}$ is the image under the Dold-Kan correspondence (briefly discussed in Section 3.1) of the 2-term chain complex of sheaves

$$(3.3.4) \quad C^\infty(-; \mathbf{U}(1)) \xrightarrow{d \log} \Omega^1(-) .$$

Here in degree 0 we have the sheaf $\Omega^1(-)$ of 1-forms (which assigns to any smooth manifold U the additively abelian group of 1-forms on U), and in degree 1 similarly the sheaf of smooth $\mathbf{U}(1)$ -valued functions. The differential is the operation that takes a $\mathbf{U}(1)$ -valued function, forms (locally) any \mathbb{R} -valued lift and then produces the differential of that. This is just a rephrasing of the above explicit description of the local data for $\mathbf{BU}(1)_{\text{conn}}$, but it highlights an important aspect. Namely, there is no need to speak Stackish fluently to see the central point of the above sentence: “ $\mathbf{BU}(1)_{\text{conn}}$ is something built from the 2-term complex $C^\infty(-; \mathbf{U}(1)) \xrightarrow{d \log} \Omega^1(-)$.”

The stack of $\mathbf{U}(1)$ -principal n -bundles with connection. The above immediately suggests the following generalization: for every $n \in \mathbb{N}$, the moduli n -stack of *circle n -bundles* or equivalently *$\mathbf{U}(1)$ -bundle $(n - 1)$ -gerbes* is the image $\mathbf{B}^n \mathbf{U}(1)$ under the Dold-Kan map (see Section 3.1) of the chain complex of sheaves (over smooth manifolds)

$$(3.3.5) \quad C^\infty(-, \mathbf{U}(1)) \rightarrow 0 \rightarrow 0 \rightarrow \dots \rightarrow 0 ,$$

which is concentrated in degree n on the sheaf of smooth $\mathbf{U}(1)$ -valued functions. Similarly, the moduli stack $\mathbf{B}^n \mathbf{U}(1)_{\text{conn}}$ of principal $\mathbf{U}(1)$ - n -bundles *with connections* is the n -stack obtained via Dold-Kan from the $(n + 1)$ -term complex of sheaves $\mathbf{U}(1)[n]_{\mathcal{D}}^\infty$ given by

$$(3.3.6) \quad C^\infty(-; \mathbf{U}(1)) \xrightarrow{d \log} \Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(-) ,$$

with the sheaf of n -forms, $\Omega^n(-)$, in degree 0, and the sheaf of $U(1)$ -valued smooth functions in degree n . This is known as the *Beilinson-Deligne complex*. The morphism of chain complexes

$$(3.3.7) \quad \begin{array}{ccccccccc} C^\infty(-; U(1)) & \xrightarrow{d \log} & \Omega^1(-) & \xrightarrow{d} & \Omega^2(-) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n(-) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow d \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \Omega_{\text{closed}}^{n+1}(-) \end{array},$$

where in the top row we have the Deligne complex as before and in the bottom row we have simply the sheaf on n -forms, extended by 0 to a chain complex, induces the *curvature* morphism

$$(3.3.8) \quad \mathbf{B}^n U(1)_{\text{conn}} \rightarrow \Omega_{\text{closed}}^{n+1},$$

mapping a connection on a $U(1)$ -principal n -bundle to its curvature $(n + 1)$ -form. Also, the evident morphism of chain complexes

$$(3.3.9) \quad \begin{array}{ccccccccc} C^\infty(-; U(1)) & \xrightarrow{d \log} & \Omega^1(-) & \xrightarrow{d} & \Omega^2(-) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n(-) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C^\infty(-; U(1)) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \end{array}$$

induces the map on moduli that forgets the connection on a circle n -bundle and induces the natural forgetful morphism

$$(3.3.10) \quad \mathbf{B}^n U(1)_{\text{conn}} \rightarrow \mathbf{B}^n U(1)$$

to the moduli n -stack of principal $U(1)$ - n -bundles.

For X a smooth manifold, homotopy classes of maps of stacks $X \rightarrow \mathbf{B}^n U(1)$ are in bijection with the integral cohomology classes of X in degree $n + 1$. We write

$$(3.3.11) \quad \pi_0 \mathbf{H}(X, \mathbf{B}^n U(1)) \simeq H^{n+1}(X, \mathbb{Z}).$$

Similarly, the homotopy classes of morphisms of smooth higher stacks $X \rightarrow \mathbf{B}^n U(1)_{\text{conn}}$ are in bijection with the *differential cohomology* of X . We write

$$(3.3.12) \quad \pi_0 \mathbf{H}(X, \mathbf{B}^n U(1)_{\text{conn}}) \simeq \hat{H}^{n+1}(X).$$

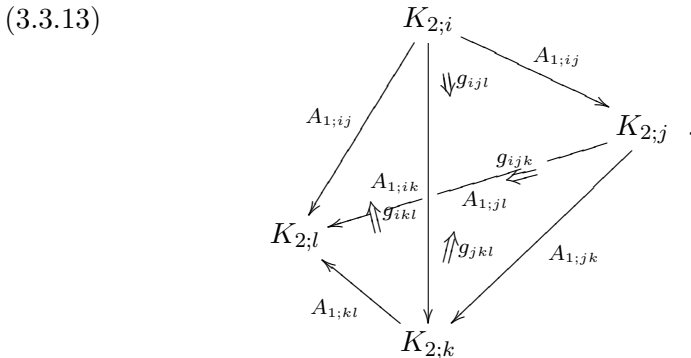
Let us unwind the local data from the above definitions looks like for the case of $n = 2$, hence for circle 2-bundle / bundle gerbes; It is immediate to generalize from this description to higher n 's. In the $n = 2$ case we have:

- 2-forms $K_{2;i}$ on U_i ;
- 1-forms $A_{1;ij}$ on U_{ij} ;
- $U(1)$ -valued functions g_{ijk} on U_{ijk} .

These data are subject to the following constraints:

- $dA_{1;ij} = K_{2;j} - K_{2;i}$ on U_{ij} ;
- $d \log g_{ijk} = A_{1;jk} - A_{1;ik} + A_{1;ij}$ on U_{ijk} ;
- $g_{jkl} g_{ikl}^{-1} g_{ijl} g_{ijk}^{-1} = 1$ on U_{ijkl} .

These data are conveniently depicted on a 3-simplex as follows:



It is evident from the above description that the datum of a trivialization of the $U(1)$ -gerbe underlying an X -point in $\mathbf{B}^2U(1)_{\text{conn}}$ consists of

- 2-forms $\mathcal{H}_{2;i}$ on U_i ;
- 1-forms $\mathcal{H}_{1;ij}$ on U_{ij} ;
- 1-forms $B_{1;i}$ on U_i ;
- $U(1)$ -valued functions ρ_{ij} on U_{ij}

such that

- $\mathcal{H}_{2;i} = dB_{1;i} + K_{2;i}$ on U_i ;
- $\mathcal{H}_{1;ij} = d \log \rho_{ij} + B_{1;j} - B_{1;i} + A_{1;ij}$ on U_{ij} ;

- $\mathcal{H}_{2;i} = dB_{1;i} + K_{2;i}$ on U_i ;
- $d\mathcal{H}_{1;ij} = \mathcal{H}_{2;j} - \mathcal{H}_{2;i}$ on U_{ij} ;
- $g_{ijk} = \rho_{jk} \rho_{ik}^{-1} \rho_{ij}$ on U_{ijk} .

An immediate consequence of these local equations is that $dK_{2;i} - dK_{2;j}$ vanishes on U_{ij} and so $(dK_{2;i})_{i \in I}$ defines a global closed 3-form on X . Moreover, this 3-form is globally exact and so its cohomology class vanishes (this is the triviality on the underlying grebe read in terms of de Rham cohomology). Exhibiting a global primitive for $(dK_{2;i})_{i \in I}$ is an easy exercise in sheaf cohomology. Namely, $\mathcal{H}_{1;jk} - \mathcal{H}_{1;ik} + \mathcal{H}_{1;ij} = 0$ on U_{ink} so $(\mathcal{H}_{1;ij})_{i,j \in I}$ is a Čech 1-cocycle on X with coefficients in the sheaf Ω^1 of smooth 1-forms. Since this sheaf is fine, $(\mathcal{H}_{1;ij})_{i,j \in I}$ is a 1-coboundary, and so there exist 1-forms α_i on U_i with $\mathcal{H}_{1;ij} = \alpha_j - \alpha_i$. Then $(\mathcal{H}_{2;i} - d\alpha_{1;i})_{i \in I}$ is a global 2-form on X which is a primitive of $(dK_{2;i})_{i \in I}$.

3.4. Multiple D-branes in nonabelian differential cohomology

In type II string theory in the presence of D-branes, the background B -field on spacetime X is accompanied by nonabelian gauge fields on the branes satisfying there a compatibility condition with the restriction of the B -field to the branes. This turns out to be analogous, in a precise fashion, to the situation with the C -field in 11-dimensional supergravity, and its restriction to Hořava-Witten boundaries of spacetime. In both cases the total moduli stack of field configurations is given by a nonabelian and twisted version of *relative* (differential) cohomology. For the C -field we discuss this in Section 4.3 (and in full detail in [29]). Here we give the analogous discussion for the B -field in terms of the stacky structures that we have already introduced above. Where for the B -field the trivialization on the brane makes a nonabelian 1-form appear, for the C -field the trivialization on the brane makes a nonabelian 2-form appear.

Let X be a 10-dimensional spacetime. By the discussion in Section 3.3, the B -field on X is given by a morphism of smooth 2-stacks $\hat{B} : X \rightarrow \mathbf{B}^2\mathbf{U}(1)_{\text{conn}}$. Let then $Q \hookrightarrow X$ be a single Spin^c D -brane in X . Freed-Witten anomaly cancellation [33] requires that the restriction of \hat{B} to Q has trivial integral class, hence that there it is, up to a gauge transformation, in the image of the moduli $\Omega^2(-) \rightarrow \mathbf{B}^2\mathbf{U}(1)_{\text{conn}}$. This situation is concisely captured by saying that the field configurations form a homotopy-commuting diagram

$$(3.4.1) \quad \begin{array}{ccc} Q & \xrightarrow{\mathcal{F}} & \Omega^2(-) \\ \downarrow & \swarrow \simeq & \downarrow \\ X & \xrightarrow{\hat{B}} & \mathbf{B}^2\mathbf{U}(1)_{\text{conn}} \end{array}$$

of smooth 2-stacks. The bottom morphism is the B -field on X . Its composite with the left morphism is its restriction to the brane. The top morphism is a globally defined 2-form \mathcal{F} on the brane, and the homotopy in the middle is a gauge transformation from this 2-form, regarded as a connection on a trivial 2-bundle, to the restriction of the B -field. Notice that this means that \hat{A} is locally, on a patch $U_i \hookrightarrow Q$, a 1-form with curvature $F_i = dA_i$, such that Equation (2.1.1) holds

$$(3.4.2) \quad \mathcal{F}_i = B_i + F_i.$$

This 1-form is the Chan-Paton gauge field on the D -brane. Moreover, the collections of all such triples of field configurations naturally form the mapping 2-groupoid, denoted

$$(3.4.3) \quad \mathbf{H}^I(Q \xrightarrow{i} X, \Omega^2(-) \rightarrow \mathbf{B}^2\mathbf{U}(1)_{\text{conn}}),$$

whose cocycles are homotopy-commuting squares as above, and whose co-boundaries are the corresponding relative gauge transformations. We have a variant of this when discussing the boundary C -field configurations in Section 4.3.

More generally, there may be N coincident D -branes with Spin^c -world-volume Q . In this case (see for instance [48] for a clean account) the trivialization \hat{A} in the above is to be replaced by a *twisted* $\mathbf{U}(N)$ -bundle on Q , whose twist is the restriction of \hat{B} to Q . In our context, this is formulated as follows. The short exact sequence of groups (a finite-dimensional counterpart of sequence (2.1.3))

$$(3.4.4) \quad \mathbf{U}(1) \rightarrow \mathbf{U}(N) \rightarrow \mathbf{PU}(N)$$

that exhibits the unitary group as a central extension of the projective unitary gives rise to a long sequence of smooth 2-stacks

$$(3.4.5) \quad \mathbf{BU}(1) \longrightarrow \mathbf{BU}(N) \longrightarrow \mathbf{BPU}(N) \xrightarrow{\mathbf{dd}} \mathbf{B}^2\mathbf{U}(1) .$$

The characteristic map $\mathbf{d}\mathbf{d}$ here may be understood as presenting the universal class on projective bundles that obstructs their lift to genuine unitary bundles. This has an evident differential refinement

$$(3.4.6) \quad \hat{\mathbf{d}}\mathbf{d} : \mathbf{BPU}_{\text{conn}} \rightarrow \mathbf{B}^2\mathbf{U}(1)_{\text{conn}} .$$

where now on the left we have the moduli stack of projective unitary bundles with projective connections. Underlying such a (nonabelian twisted) connection is a globally defined abelian curvature 1-form (induced, locally, by the trace operation, as in Section 3.2). Therefore, we can now consider relative cohomology twisted by $\mathbf{d}\mathbf{d}$ on the brane inclusion $Q \rightarrow X$. Its cocycles are homotopy-commuting diagrams of 2-stacks

$$(3.4.7) \quad \begin{array}{ccc} Q & \xrightarrow{\hat{A}} & \mathbf{BPU}(N)_{\text{conn}} \\ \downarrow & \simeq \swarrow & \downarrow \mathbf{d}\mathbf{d} \\ X & \xrightarrow{\hat{B}} & \mathbf{B}^2\mathbf{U}(1)_{\text{conn}} . \end{array}$$

Here the top morphism now characterizes a twisted Chan-Paton gauge field on N coincident D -branes, whose $\mathbf{d}\mathbf{d}$ -class trivializes the restriction of the spacetime B -field to the branes. When the lower morphism is presented in terms of bundle gerbes, then the top morphism is presented by the corresponding *gerbe modules* as in [16]. A component-discussion of this relative nonabelian differential cohomology describing D-brane gauge fields is for instance in Section 4 of [61]. The C -field analog of this diagram is discussed below as (4.3.2) and (4.3.10).

The groups $\text{PU}(N)$ all embed into the group $\text{PU}(\mathbb{H})$, of projective unitary operators on any separable infinite-dimensional complex Hilbert space \mathbb{H} , and we have a morphism of classifying stacks

$$(3.4.8) \quad \mathbf{d}\mathbf{d} : \mathbf{BPU}(\mathbb{H}) \rightarrow \mathbf{B}^2\mathbf{U}(1)$$

that classifies the $U(1)$ -extension $\mathbf{U}(\mathbb{H})$. In this limit of “arbitrary numbers of D-branes” something interesting happens: under geometric realization (3.1.2) the two moduli stacks $\mathbf{BPU}(\mathbb{H})$ of projective unitary bundles and $\mathbf{B}^2\mathbf{U}(1)$ of circle 2-bundles both become the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$, and the class $\mathbf{d}\mathbf{d}$ simply becomes the identity. This is related to Kuiper’s theorem, which asserts that the topological space underlying $\mathbf{U}(\mathbb{H})$ is a contractible space. This says that, while the geometry and differential geometry of twisted nonabelian 1-form connections is very different from

that of abelian 2-form connections, their instanton sectors may be identified.

We discuss an analog of all these statements for the supergravity C -fields in Section 4.3.

3.5. Higher holonomy

It is a classical fact that a connection on a circle bundle induces a notion of *holonomy* along 1-dimensional curves, and that this holonomy is the gauge coupling action for a charged particle in the background of the gauge field presented by the connection. This fact has a higher analog for the higher circle bundles from Section 3.3. A circle n -bundle with connection, equivalently: an abelian n -form gauge field, has holonomy over n -dimensional trajectories and this holonomy is the gauge coupling action of the $(n - 1)$ -brane charged under the corresponding form field (the Wess-Zumino-Witten term). We recall how this comes about in terms of the Deligne complex and thus as a map on higher stacks.

Let X be a smooth manifold. Then the set of connected components of the n -groupoid $\mathbf{H}(X, \mathbf{B}^n \mathbf{U}(1)_{\text{conn}})$ of $\mathbf{U}(1)$ - n -bundles with connection on X is naturally isomorphic to the $(n + 1)$ -th ordinary differential cohomology group of X :

$$(3.5.1) \quad \pi_0 \mathbf{H}(X, \mathbf{B}^n \mathbf{U}(1)_{\text{conn}}) \cong \hat{H}^{n+1}(X; \mathbb{Z}) .$$

Also, the set of connected components of the n -groupoid $\mathbf{H}(X, \mathbf{B}^n \mathbf{U}(1))$ of $\mathbf{U}(1)$ - n -bundles on X is naturally isomorphic to the $(n + 1)$ -th singular cohomology group of X :

$$(3.5.2) \quad \pi_0 \mathbf{H}(X, \mathbf{B}^n \mathbf{U}(1)) \cong H^{n+1}(X; \mathbb{Z}) .$$

and the forgetful map $\mathbf{B}^n \mathbf{U}(1)_{\text{conn}} \rightarrow \mathbf{B}^n \mathbf{U}(1)$, that “forgets the connection”, induces the natural morphism

$$(3.5.3) \quad \hat{H}^{n+1}(X; \mathbb{Z}) \rightarrow H^{n+1}(X; \mathbb{Z})$$

from differential cohomology to singular cohomology. Moreover, it is well known (see, e.g., [44]) that if X is a smooth oriented manifold, then the above morphisms fit into a short exact sequence

$$(3.5.4) \quad 0 \rightarrow \Omega^n(X) / \Omega_{\text{cl},0}^n(X) \rightarrow \hat{H}^{n+1}(X; \mathbb{Z}) \rightarrow H^{n+1}(X; \mathbb{Z}) \rightarrow 0 ,$$

where $\Omega^n(X)/\Omega_{\text{cl},0}^n(X)$ is the group of differential n -forms on X modulo those n -forms which are closed and have integral periods. For the reader's convenience, let us briefly recall how the above short exact sequence originates. Consider the complex of sheaves $\Omega^{1 \leq \bullet \leq n}[n]$, i.e.,

$$(3.5.5) \quad \Omega^1 \rightarrow \Omega^2 \rightarrow \dots \rightarrow \Omega^n$$

with Ω^n in degree zero. Then we have a short exact sequence of complexes of sheaves

$$(3.5.6) \quad 0 \rightarrow \Omega^{1 \leq \bullet \leq n}[n] \rightarrow \text{U}(1)[n]_D^\infty \rightarrow \text{U}(1)[n] \rightarrow 0$$

inducing a long exact sequence in hypercohomology

$$(3.5.7) \quad \begin{aligned} \dots \rightarrow H^{-1}(X, \text{U}(1)[n]) &\rightarrow H^0(X, \Omega^{1 \leq \bullet \leq n}[n]) \\ &\rightarrow H^0(\text{U}(1)[n]_D^\infty) \rightarrow H^0(X, \text{U}(1)[n]) \rightarrow 0. \end{aligned}$$

Since all the sheaves Ω^i are acyclic, the usual immediate spectral sequence argument shows that $H^0(X, \Omega^{1 \leq \bullet \leq n}[n]) \cong \Omega^n(X)/d\Omega^{n-1}(X)$ and so the above long exact sequence reads

$$(3.5.8) \quad \begin{aligned} \dots \rightarrow H^n(X; \mathbb{Z}) &\rightarrow \Omega^n(X)/d\Omega^{n-1}(X) \\ &\rightarrow \hat{H}^{n+1}(X; \mathbb{Z}) \rightarrow H^{n+1}(X; \mathbb{Z}) \rightarrow 0. \end{aligned}$$

From this we get the short exact sequence

$$(3.5.9) \quad \dots 0 \rightarrow A \rightarrow \hat{H}^{n+1}(X; \mathbb{Z}) \rightarrow H^{n+1}(X; \mathbb{Z}) \rightarrow 0,$$

where A is the image of $\Omega^n(X)/d\Omega^{n-1}(X)$ inside $\hat{H}^{n+1}(X; \mathbb{Z})$, and so is naturally isomorphic to the quotient of $\Omega^n(X)/d\Omega^{n-1}(X)$ by the image of $H^n(X; \mathbb{Z})$ into $\Omega^n(X)/d\Omega^{n-1}(X)$. Since this image is precisely $\Omega_{\text{cl},0}^n(X)/d\Omega^{n-1}(X)$, we have $A \cong \Omega^n(X)/\Omega_{\text{cl},0}^n(X)$.

In particular, if Σ is an n -dimensional smooth oriented manifold, we get a canonical isomorphism

$$(3.5.10) \quad \hat{H}^{n+1}(\Sigma; \mathbb{Z}) \xrightarrow{\sim} \Omega_{\text{cl},0}^n(\Sigma)/\Omega_{\text{cl},0}^n(\Sigma),$$

i.e., each $\text{U}(1)$ - n -bundle with connection on Σ is equivalent to a trivial $\text{U}(1)$ - n -bundle with an n -connection given by a globally defined n -form on Σ . Moreover, this n -form is uniquely determined up to an n -form with integral

periods. By definition of $\Omega_{\text{cl},0}^n(\Sigma)$, the integration over Σ induces a well defined group homomorphism

$$(3.5.11) \quad e^{2\pi i \int_{\Sigma}} : \Omega_{\text{cl},0}^n(\Sigma) / \Omega_{\text{cl},0}^n(\Sigma) \rightarrow \text{U}(1) ,$$

and so finally we get the following result.

Observation 3.5.1. For Σ a compact n -dimensional smooth manifold, there is a canonical morphism

$$(3.5.12) \quad \exp(2\pi i \int_{\Sigma} (-)) : \mathbf{H}(\Sigma, \mathbf{B}^n \text{U}(1)_{\text{conn}}) \rightarrow \text{U}(1) .$$

This is a map that sends n -form gauge fields on Σ to elements in $\text{U}(1)$ and is gauge invariant.

More generally, let be X a (spacetime) manifold of any dimension or, in fact, any orbifold or more general smooth stack or higher smooth stack. Then with Σ as before, there is a canonical morphism

$$(3.5.13) \quad \text{hol}_{\Sigma} : \mathbf{H}(\Sigma, X) \times \mathbf{H}(X, \mathbf{B}^n \text{U}(1)_{\text{conn}}) \rightarrow \text{U}(1) ,$$

where $\mathbf{H}(\Sigma, X)$ denotes the space (or ∞ -groupoid) of maps from Σ to X . This morphism reads in an n -form gauge field ∇ on X as well as a smooth trajectory $\phi : \Sigma \rightarrow X$ and produces the *n-dimensional holonomy*

$$(3.5.14) \quad \text{hol}_{\Sigma}(\phi, \nabla) \in \text{U}(1)$$

of ∇ around Σ under the map ϕ . Formally, the map hol_{Σ} is just the composition

$$(3.5.15) \quad \text{hol}_{\Sigma} : \mathbf{H}(\Sigma, X) \times \mathbf{H}(X, \mathbf{B}^n \text{U}(1)_{\text{conn}}) \xrightarrow{\circ} \mathbf{H}(\Sigma, \mathbf{B}^n \text{U}(1)_{\text{conn}}) \xrightarrow{e^{2\pi i \int_{\Sigma} (-)}} \text{U}(1) .$$

An explicit expression for $\text{hol}_{\Sigma}(\phi, \nabla)$ in terms of local differential forms data can be found in [23, 37, 38].

Holonomy of M-branes. For illustration in the case at hand, we will spell this out for the case of M-branes, and rewrite the above results in this special, but important situation. The M2-brane and the M5-branes correspond to the case $n = 3$ and $n = 6$, respectively. We have

1. Consider the M2-brane with worldvolume a smooth oriented 3-manifold Σ_3 . On the 11-dimensional target space Y^{11} we have a C -field representing a connection on a $U(1)$ -principal 3-bundle. Consider the σ -model for the M2-brane $\phi : \Sigma_3 \rightarrow Y^{11}$ with space of maps $\mathbf{H}(\Sigma_3, Y^{11})$. Then the holonomy of the C -field is given by

$$(3.5.16) \quad \text{hol}_{\text{M2}} : \mathbf{H}(\Sigma_3, Y^{11}) \times \mathbf{H}(Y^{11}, \mathbf{B}^3U(1)_{\text{conn}}) \rightarrow U(1) .$$

2. For the case of the M5-brane we consider the dual C_6 of the C -field on the 6-dimensional smooth oriented worldvolume Σ_6 . We again have sigma model maps $\phi : \Sigma_6 \rightarrow Y^{11}$ which form the space $\mathbf{H}(\Sigma_6, Y^{11})$. The holonomy of the dual of the C -field is then

$$(3.5.17) \quad \text{hol}_{\text{M5}} : \mathbf{H}(\Sigma_6, Y^{11}) \times \mathbf{H}(Y^{11}, \mathbf{B}^6U(1)_{\text{conn}}) \rightarrow U(1) .$$

In the special case that the C -field happens to be given by a globally defined 3-form C_3 on Y^{11} , we have the explicit formula

$$(3.5.18) \quad \text{hol}_{\text{M2}}(\phi, C_3) = \exp(2\pi i \int_{\Sigma_3} \phi^* C_3) .$$

This is the familiar higher gauge coupling or *Wess-Zumino-Witten* term in the M2-brane action. The above construction generalizes this to general C -field configurations (and with just slight adaption to branes with boundary). The description of holonomy for the M5-brane is analogous. The partition functions in this setting are described in [67].

3.6. Differential characteristic maps

We have seen abelian differential cohomology in Sections 3.3 and 3.5 and are about to consider higher nonabelian cohomology in the next Section 3.7. The link between the two is given by *characteristic classes*, or rather by their refinement to smooth and differential characteristic maps between smooth moduli stacks. These differentially refined characteristic maps are the structures from which we obtain higher Chern-Simons action functionals in Section 4. Here we briefly review some basic ideas. Details are in [30] and in Sections 2.3.18 and 3.3.14 of [79].

Let G be a Lie group and $[c] \in H^{n+1}(BG; \mathbb{Z})$ an integral characteristic class on its classifying space. If G is compact, then by Theorem 3.3.29 in [79]

there is an isomorphism

$$(3.6.1) \quad H^{n+1}(BG; \mathbb{Z}) \cong \pi_0 \mathbf{H}(BG; \mathbf{B}^n U(1)) ,$$

between the integral cohomology in degree $n + 1$ of the topological space BG , and the gauge equivalence classes of circle n -bundles over the moduli stack $\mathbf{B}G$. In other words, there is, up to equivalence, a unique morphism

$$(3.6.2) \quad \mathbf{c} : \mathbf{B}G \rightarrow \mathbf{B}^n U(1)$$

of smooth stacks, such that for $E : X \rightarrow \mathbf{B}G$ the map of stacks classifying a G -principal bundle on X , the integral class $c(E) \in H^{n+1}(X, \mathbb{Z})$ is that classifying the circle n -bundle given by the composite $\mathbf{c}(E) : X \rightarrow \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^n U(1)$.

A simple example of this that we have already seen was the smooth refinement \mathbf{c}_1 of the first Chern class in (3.2.1). For this example we obtained in (3.2.5) furthermore a *differential refinement*, namely a morphism $\hat{\mathbf{c}}$ between the corresponding moduli stacks for bundles with connection, which refines \mathbf{c} by fitting into a homotopy commuting diagram

$$(3.6.3) \quad \begin{array}{ccc} \mathbf{B}G_{\text{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^n U(1)_{\text{conn}} \xrightarrow{\text{curv}} \Omega_{\text{cl}}^{n+1} \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}^n U(1) \end{array} .$$

This diagram in particular says that for (E, ∇) a G -principal bundle E with connection ∇ , then the $(n + 1)$ -form curvature of the circle n -bundle $\hat{\mathbf{c}}(E, \nabla)$ is a de Rham representative of the integral class of the underlying circle n -bundle $\mathbf{c}(E)$.

When restricted to gauge equivalence classes and to cohomology, this is a construction that is provided by classical *Chern-Weil theory*, simply by evaluating the curvature form F_∇ in an *invariant polynomial* $\langle - \rangle$ on the Lie algebra of \mathfrak{g} . We need to refine this construction away from gauge equivalence classes to the full higher moduli stacks. A step in this direction has been provided by Brylinski and McLaughlin in [15], who showed how to construct for every single Čech-Deligne cocycle for a G -principal bundle representing a map $(E, \nabla) : X \rightarrow \mathbf{B}G_{\text{conn}}$, a corresponding Čech-Deligne cocycle for the the circle n -bundle with connection $\hat{\mathbf{c}}(E) : X \rightarrow \mathbf{B}^n U(1)_{\text{conn}}$. Based on this and on L_∞ -algebraic resolutions considered in [76], we gave in [30] a construction of genuine morphisms $\hat{\mathbf{c}}$ of higher moduli stacks by a procedure

that may be understood as a higher analog of Lie integration of Lie algebra homomorphism, generalized to Lie n -algebras. However, the morphisms of higher stacks obtained this way typically go out of *higher connected covers* of moduli stacks for Lie groups, and extra work is required in pushing them down again. This phenomenon is, in low degree, already familiar from classical Lie theory: there, morphisms of (finite dimensional) Lie algebras correspond to morphisms between the *connected* and *simply connected* Lie groups integrating these. In order to get morphisms between non-simply connected Lie groups from Lie integration, extra quotienting is required, which may or may not be respected by the morphism.

Example. The simplest instance of this phenomenon occurs with $G = O(2)$, the two-dimensional orthogonal group. Namely, the Lie algebra of $O(2)$ is \mathfrak{so}_2 and there is a (unique up to a scalar factor) nontrivial real valued Lie algebra 1-cocycle μ_1 on \mathfrak{so}_2 (i.e., a nontrivial Lie algebra morphism $\mu_1 : \mathfrak{so}_2 \rightarrow \mathbb{R}$). Yet, μ_1 cannot be integrated to a Lie group 1-cocycle (i.e., to a morphism of Lie groups) $\rho : O(2) \rightarrow U(1)$; indeed, the only nontrivial Lie group homomorphism from $O(2)$ to $U(1)$ is $\det : O(2) \rightarrow \{\pm 1\} \subseteq U(1)$. Here the topological obstruction to the integration of μ_1 is the nonconnectedness of $O(2)$. And indeed, μ_1 can be integrated if we pass to the “connected cover” $SO(2)$ of $O(2)$; its integration is nothing but the well known isomorphism $SO(2) \xrightarrow{\sim} U(1)$. In terms of integral characteristic classes, this discussion is summarized by the fact that $H^2(BO(2); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and so there are, up to equivalence, only two morphisms of stacks $\mathbf{BO}(2)_{\text{conn}} \rightarrow \mathbf{BU}(1)_{\text{conn}}$ corresponding to characteristic classes for $O(2)$: a principal $O(2)$ bundle with connection given by local data (g_{ij}, A_i) can be mapped either to the trivial $U(1)$ -bundle with connection $(1_{ij}, 0_i)$ or to the $U(1)$ -bundle with connection locally given by $(\det(g_{ij}), 0_i)$. Notice that in both cases the map at the level of local connection data is the zero morphism $\mathfrak{so}_2 \rightarrow \mathbb{R}$.

The general statement is the natural generalization of the example just presented: a real valued Lie algebra n -cocycle on \mathfrak{g} universally integrates to a morphism of moduli n -stacks

$$(3.6.4) \quad \mathbf{BG}\langle n \rangle \rightarrow \mathbf{B}^n \mathbf{U}(1) ,$$

where $G\langle n \rangle$ is an $(n - 1)$ -connected cover of the Lie group G . For instance, if O denotes the infinite orthogonal group and \mathfrak{so} its Lie algebra, then there is no nontrivial real valued Lie algebra 1-cocycle on \mathfrak{so} ; the canonical Lie algebra 3-cocycle on \mathfrak{so} integrates to the first fractional Pontrjagin class

$$(3.6.5) \quad \frac{1}{2} \mathbf{p}_1 : \mathbf{BSpin} \rightarrow \mathbf{B}^3 \mathbf{U}(1) ,$$

where $\text{Spin}(n)$ is the 1-connected cover of $\text{O}(n)$.

While this machine goes through for all n , a crucial subtlety but also a source for much of the structure that we discuss here, is that further higher connected covers of Lie groups *do not exist as Lie groups*, in general. Instead, they only exist as *higher Lie groups*. Notably, the canonical Lie algebra 7-cocycle on \mathfrak{so} integrates to the second fractional Pontrjagin class. But this is no longer defined on the stack \mathbf{BSpin} itself, but on a higher nonabelian stack (a 2-stack, in this case), which we denote $\mathbf{BString}$. The reader may think of this as a *twisted product* of the nonabelian 1-stack \mathbf{BSpin} with the abelian 2-stack $\mathbf{B}^2\text{U}(1)$ discussed in Section 3.3. We describe this in more detail in a moment, below in Section 3.7. So in terms of this String 2-stack, the degree-7 cocycle on \mathfrak{so} turns out to integrate the second fractional Pontrjagin class refined to a higher stack morphism of the form

$$(3.6.6) \quad \frac{1}{6}\mathbf{p}_2 : \mathbf{BString} \rightarrow \mathbf{B}^7\text{U}(1) .$$

Moreover, both these Pontrjagin classes are induced by Lie algebra n -cocycles, and have differential refinements

$$(3.6.7) \quad \frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{BSpin}_{\text{conn}} \rightarrow \mathbf{B}^3\text{U}(1)_{\text{conn}}$$

and

$$(3.6.8) \quad \frac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{BString}_{\text{conn}} \rightarrow \mathbf{B}^7\text{U}(1)_{\text{conn}} .$$

A proof of these results, together with a construction of $G\langle n \rangle$ as an higher smooth group via (a version of) the Sullivan construction, and a treatment of the integration of Lie algebra n -cocycles can be found in [30]. The stacks $\mathbf{BString}$ and $\mathbf{BString}_{\text{conn}}$ mentioned above will be defined in Section 3.7.

Remark 3.6.1. One may wonder whether the n -cocycles $\mathbf{B}G\langle n \rangle \rightarrow \mathbf{B}^n\text{U}(1)$ and $\mathbf{B}G\langle n \rangle_{\text{conn}} \rightarrow \mathbf{B}^n\text{U}(1)_{\text{conn}}$ mentioned above descend to n -cocycles $\mathbf{B}G \rightarrow \mathbf{B}^n\text{U}(1)$ and $\mathbf{B}G_{\text{conn}} \rightarrow \mathbf{B}^n\text{U}(1)_{\text{conn}}$. As we mentioned above, the obstruction to this descent is of a topological kind: one has to see whether the characteristic class in $H^{n+1}(\mathbf{B}G\langle n \rangle; \mathbb{Z})$ defined by the cocycle is the pullback of an integral characteristic class in $H^{n+1}(\mathbf{B}G; \mathbb{Z})$ or not. For instance, the first Pontrjagin class is an element in $H^4(\mathbf{B}SO; \mathbb{Z})$. It refines to a morphism of stacks $\mathbf{p}_1 : \mathbf{B}SO \rightarrow \mathbf{B}^3\text{U}(1)$, whose infinitesimal version is twice the standard Lie algebra 3-cocycle μ_3 on \mathfrak{so}_3 . As mentioned above, the 3-cocycle μ_3 integrates to $\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{BSpin}_{\text{conn}} \rightarrow \mathbf{B}^3\text{U}(1)_{\text{conn}}$, and so $2\mu_3$ integrates to $\hat{\mathbf{p}}_1 : \mathbf{BSpin}_{\text{conn}} \rightarrow \mathbf{B}^3\text{U}(1)_{\text{conn}}$. Since the underlying characteristic class p_1

actually lives on BSO , there are no obstructions to the descent of $\hat{\mathbf{p}}_1$ to $\mathbf{BSO}_{\text{conn}}$, and so one has a differential refinement of the first Pontrjagin class to a morphism of stacks

$$(3.6.9) \quad \hat{\mathbf{p}}_1 : \mathbf{BSO}_{\text{conn}} \rightarrow \mathbf{B}^3\mathbf{U}(1)_{\text{conn}} .$$

For $n = 3$ and G a compact simple Lie group, one can find a full detailed treatment of the descent problem for differential cocycles in [34].

As anticipated in the previous section, we will only be concerned with top degree local differential forms data for n -connections given by morphisms $X \rightarrow \mathbf{B}^n\mathbf{U}(1)_{\text{conn}}$. So the relevant question for us now is: what is the degree- n differential form $A_{n;i}$ associated with the \mathfrak{g} -valued local 1-form ω_i of a \mathfrak{g} -connection and with a real valued Lie algebra n -cocycle μ inducing the characteristic map $\hat{\mathbf{c}}$? And the answer is everything but unexpected:

$$(3.6.10) \quad A_{n;i} = \text{CS}_n(\omega_i) ,$$

where CS_n is a *Chern-Simons element* for the Lie algebra cocycle μ . Again, see [74],[30] or [79] for details and for the generalization of the notion of Chern-Simons elements to Lie n -algebras.

In what follows, we will need the full local data for the $\mathbf{U}(1)$ -principal 3-bundle with connection that is associated by $\frac{1}{2}\hat{\mathbf{p}}_1$ to a \mathbf{Spin} -principal bundle with connection given by local data (ω_i, g_{ij}) . These are obtained as follows:

- since the group \mathbf{Spin} is connected, over each double intersections one can lift the transition function g_{ij} to a smooth family of based paths in G , $\hat{g}_{ij} : U_{ij} \times \Delta^1 \rightarrow G$, with $\hat{g}_{ij}(0) = e$ and $\hat{g}_{ij}(1) = g_{ij}$;
- since \mathbf{Spin} is 1-connected, over each triple intersection one can find a smooth family of based 2-simplices in G , $\hat{g}_{ijk} : U_{ijk} \times \Delta^2 \rightarrow G$, with boundaries labeled by the based paths on double overlaps:

$$(3.6.11) \quad \begin{array}{ccc} & g_{ij} & \\ \hat{g}_{ij} \nearrow & & \searrow g_{ij} \cdot \hat{g}_{jk} \\ e & \hat{g}_{ijk} & g_{ik} \\ & \hat{g}_{ik} & \end{array} .$$

- since \mathbf{Spin} is 2-connected, on each quadruple intersection one can find a smooth family of based 3-simplices in G , $\hat{g}_{ijkl} : U_{ijkl} \times \Delta^3 \rightarrow G$, cobounding the union of the 2-simplices corresponding to the triple intersections;

- for $n = 1, 2, 3$, let $\hat{\omega}_{i_0, \dots, i_n}$ be the Δ^n -family of \mathfrak{so} -valued 1-form on U_{i_0, \dots, i_n} obtained via the gauge action of $\hat{g}_{i_0, \dots, i_n}$ on $\omega_{i_0}|_{U_{i_0, \dots, i_n}}$, i.e., $\hat{\omega}_{i_0, \dots, i_n} \in \Omega^1(U_{i_0, \dots, i_n} \times \Delta^n; \mathfrak{so})$ is defined as

$$(3.6.12) \quad \hat{g}_{i_0, \dots, i_n} \omega_{i_0} \hat{g}_{i_0, \dots, i_n}^{-1} + \hat{g}_{i_0, \dots, i_n} d_U \hat{g}_{i_0, \dots, i_n}^{-1},$$

where d_U denotes the de Rham differential in the U_{i_0, \dots, i_n} -direction on $U_{i_0, \dots, i_n} \times \Delta^n$.

These pieces of data are sent to the Čech-Deligne 3-cocycle

$$(3.6.13) \quad (C_i, B_{ij}, A_{ijk}, g_{ijkl}) := \left(\text{CS}_3(\omega_i), \int_{\Delta^1} \text{CS}_3(\hat{\omega}_{ij}), \int_{\Delta^2} \text{CS}_3(\hat{\omega}_{ijk}), \int_{\Delta^3} \mu_3(\hat{\omega}_{ijkl} \wedge \hat{\omega}_{ijkl} \wedge \hat{\omega}_{ijkl}) \bmod \mathbb{Z} \right),$$

where μ_3 and cs_3 are the standard 3-cocycle and Chern-Simons element for \mathfrak{so} . Notice that the integration over the 3-simplex Δ^3 in the above formula only reads the vertical part of the 3-form $\mu_3(\hat{\omega}_{ijkl} \wedge \hat{\omega}_{ijkl} \wedge \hat{\omega}_{ijkl})$ with respect to the projection $U_{ijkl} \times \Delta^k \rightarrow U_{ijkl}$, which, by construction, is $\hat{g}_{ijkl}^* \mu_3(\theta \wedge \theta \wedge \theta)$, where $\theta \in \Omega^1(\text{Spin}; \mathfrak{so})$ is the Maurer-Cartan form on the group Spin.

3.7. The 2-stacks **BString** and **BString_{conn}**

We now discuss the moduli 2-stacks **BString** and **BString_{conn}** of String-principal 2-bundles and of String-principal 2-bundles with 2-connection ([30] and [79], Section 4.1.3). The definitions and discussion here proceeds in direct analogy with the discussion in 3.2, simply by replacing the first Chern class with the first fractional Pontrjagin class.

Let **BSpin** be the smooth stack of smooth Spin-principal bundles. As remarked above around (3.6.5), the first fractional Pontrjagin class $\frac{1}{2}p_1$ refines to a morphism of smooth 3-stacks $\frac{1}{2}\mathbf{p}_1 : \mathbf{BSpin} \rightarrow \mathbf{B}^3\mathbf{U}(1)$ and so we can define **BString** as the homotopy fiber of this morphism, by analogy with (3.2.2), hence as the homotopy pullback

$$(3.7.1) \quad \begin{array}{ccc} \mathbf{BString} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{BSpin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3\mathbf{U}(1) \end{array}$$

of smooth higher stacks. One way to characterize what this means in more concrete terms (another one we get to in a moment) is to say that that a

String-principal 2-bundle over a smooth manifold X , hence a stack morphism $X \rightarrow \mathbf{BString}$, is

- a Čech cocycle datum (g_{ij}) of a Spin-principal bundle on X ;
- together with a *choice* of a trivialization of the induced Čech 3-cocycle $(\int_{\Delta^3} \hat{g}_{ijkl}^* \mu_3(\theta \wedge \theta \wedge \theta) \bmod \mathbb{Z})$ given in (3.6.13).

Homotopy pullbacks may be more familiar in the context of topological spaces. There, for $f : X \rightarrow Y$ any continuous map of pointed topological spaces, its homotopy fiber may be defined, up to weak homotopy equivalence, as the ordinary pullback of the *based path space* projection $PY \rightarrow Y$ of Y along f . But there are many other constructions that give the same result up to weak homotopy equivalence.

To relate the homotopy pullbacks of higher stacks with those of topological spaces, we invoke the geometric realization map from (3.1.2). By the discussion there, the image of $\frac{1}{2}\mathbf{p}_1$ under geometric realization is a continuous map of topological spaces

$$(3.7.2) \quad \frac{1}{2}p_1 : BSpin \rightarrow K(\mathbb{Z}, 4),$$

hence a class in degree-4 integral cohomology $H^4(BSpin, \mathbb{Z})$. This is the ordinary Pontryagin class of which $\frac{1}{2}\mathbf{p}_1$ is the smooth refinement.

While, in general, geometric realization of higher stacks does not preserve homotopy fibers, one can show that in some special cases, such as the one above, it does (Theorem 3.3.25 in [79]). This means that

$$(3.7.3) \quad |\mathbf{BString}| \simeq BString$$

is indeed the classifying space of the topological String group as traditionally defined, fitting into a homotopy pullback

$$(3.7.4) \quad \begin{array}{ccc} BString & \longrightarrow & * \\ \downarrow & & \downarrow \\ BSpin & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z}, 4) \end{array}$$

of topological spaces. In this form we can see more directly than otherwise in which way the String group is indeed related to something *stringy*. For if we form topological loop spaces twice on this last homotopy pullback diagram,

we get the homotopy pullbacks

$$(3.7.5) \quad \begin{array}{ccccc} \mathrm{U}(1) & \longrightarrow & \Omega\mathrm{String} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \Omega\mathrm{Spin} & \xrightarrow{\Omega^2 \frac{1}{2} p_1} & \mathrm{BU}(1) \end{array}$$

and hence the long fiber sequence

$$(3.7.6) \quad \mathrm{U}(1) \longrightarrow \Omega\mathrm{String} \longrightarrow \Omega\mathrm{Spin} \xrightarrow{\frac{1}{2} p_1} K(\mathbb{Z}, 4) .$$

This exhibits an equivalence between the loop space of the topological String group and the level-1 Kac-Moody central extension $\hat{\Omega}\mathrm{Spin}$ of the topological loop group of Spin

$$(3.7.7) \quad \Omega\mathrm{String} \simeq \hat{\Omega}\mathrm{Spin} .$$

It is in this reduced form that String bundles first appeared in the string theory literature [89]: as central extensions of the structure loop group on the loop space of spacetime (hence the configuration space of the closed string). But some torsion information is lost in this double looping ($\Omega \frac{1}{2} p_1(\Omega X)$ may vanish even if $\frac{1}{2} p_1(X)$ does not), and the precise structure needed in the heterotic string are genuine (twisted) String 2-bundles [76] as discussed here.

While the above simple argument, derivng the topological Kac-Moody loop group from String, does not go through quite in this form for the fully-fledged *smooth* (stacky) version of String, it turns out that a refinement of this state of affairs does hold true: In [7] it is shown that when the Kac-Moody loop group $\hat{\Omega}\mathrm{Spin}$ is regarded as a smooth group (a Fréchet group), then the evident smooth map

$$(3.7.8) \quad \hat{\Omega}\mathrm{Spin} \rightarrow P\mathrm{Spin}$$

to the smooth group $P\mathrm{Spin}$ of based paths in Spin is equipped with an action of the path group, such that it becomes what is called a *crossed module* of smooth groups. This may naturally be thought of as defining a smooth 2-stack (def. 1.3.5 in [79]), written $\mathbf{B}(\hat{\Omega}\mathrm{Spin} \rightarrow P\mathrm{Spin})$. By Theorem 4.1.29 in [79], this turns out to be an equivalent incarnation of the moduli 2-stack of String-2-bundles

$$(3.7.9) \quad \mathbf{B}\mathrm{String} \simeq \mathbf{B}(\hat{\Omega}\mathrm{Spin} \rightarrow P\mathrm{Spin}) .$$

Having two such seemingly different descriptions for the stack of principal String bundles should not be surprising: **BString** is defined as a homotopy fiber, and so it is well defined only up to weak equivalence.

Analogous considerations apply for the infinitesimal approximation to the smooth String-2-group: its Lie 2-algebra **string** (see [74]). This Lie 2-algebra is abstractly defined (def. 4.1.15 in [79]) as the L_∞ -algebra homotopy fiber of the canonical degree-3 Lie algebra cocycle μ_3 on \mathfrak{so} , which we may identify with a map of Lie 3-algebras

$$(3.7.10) \quad \mu_3 : \mathfrak{so} \rightarrow b^2\mathfrak{u}(1)$$

from \mathfrak{so} to the *line Lie 3-algebra*, the two-fold delooping of $\mathfrak{u}(1) \simeq \mathbb{R}$. Again, by the very definition of homotopy pullbacks, this homotopy fiber may be presented by a Lie 2-algebra, denoted \mathfrak{so}_{μ_3} , defined by the fact that a morphism of Lie 2-algebroids (see [74])

$$(3.7.11) \quad (\omega, B) : TX \rightarrow b(\mathfrak{so}_{\mu_3})$$

from the tangent Lie algebroid of a smooth manifold X is

- a flat \mathfrak{so} -valued 1-form ω ;
- equipped with a choice of 2-form B which trivializes the Chern-Simons 3-form of ω , in that

$$(3.7.12) \quad dB = \text{tr}(\omega \wedge \omega \wedge \omega).$$

But, as before for the finite String-2-group, there is, now by Theorem 30 in [7], a quite different looking but equivalent incarnation of the Lie 2-algebra **string**. This is given by a pair of ordinary non-abelian Lie algebras, the Kac-Moody loop Lie algebra $\hat{\Omega}\mathfrak{so}$, and the based path Lie algebra $P\mathfrak{so}$, equipped with the canonical Lie algebra homomorphism $(\hat{\Omega}\mathfrak{so} \xrightarrow{h} P_*\mathfrak{so})$, which defines what is called a “strict” Lie 2-algebra (all this is reviewed in the Appendix). This is an equivalent presentation (an equivalent higher gauge incarnation) of **string**:

$$(3.7.13) \quad \mathfrak{so}_{\mu_3} \simeq (\hat{\Omega}\mathfrak{so} \xrightarrow{h} P_*\mathfrak{so}).$$

Using this infinitesimal description of the String-2-group, we obtain now, by [30], a differential refinement of **BString**, namely the moduli 2-stack of String-2-bundles equipped with 2-connections. As remarked in Section 3.6 (see Equation (3.6.7)), using this, the morphism of 3-stacks $\frac{1}{2}\mathbf{p}_1$ can

be further refined to a morphism of differentially refined 3-stacks $\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{BSpin}_{\text{conn}} \rightarrow \mathbf{B}^3\mathbf{U}(1)_{\text{conn}}$. Therefore, we could define, analogous to (3.2.10), the moduli 2-stack of String bundles with connections as the homotopy fiber of $\frac{1}{2}\hat{\mathbf{p}}_1$ over the trivial $\mathbf{U}(1)$ -2-gerbe endowed with the trivial connection. However, this turns out to be too a strict notion, so one prefers to adopt a more flexible one, i.e. to define the stack $\mathbf{BString}_{\text{conn}}$ by analogy with the homotopy fiber of (3.2.11), to be the homotopy fiber of $\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{BSpin}_{\text{conn}} \rightarrow \mathbf{B}^3\mathbf{U}(1)$ over the the trivial $\mathbf{U}(1)$ -2-gerbe. In other words, one asks that the underlying Chern-Simons $\mathbf{U}(1)$ -2-gerbe $\frac{1}{2}\mathbf{p}_1(P)$ of a Spin-principal bundle P with connection trivializes, but without requiring that the induced connection on it be trivial. String structures of this kind have been considered in [87]; they are particular examples of the more general notion of *twisted String structure* considered in [76], to which we turn in Section 3.8.

In more detail, $\mathbf{BString}_{\text{conn}}$ is defined by the homotopy pullback

$$(3.7.14) \quad \begin{array}{ccc} \mathbf{BString}_{\text{conn}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{BSpin}_{\text{conn}} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3\mathbf{U}(1) , \end{array}$$

which is the higher analog of (3.2.11). Since we have a homotopy pullback

$$(3.7.15) \quad \begin{array}{ccc} \Omega^{1 \leq \bullet \leq 3} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}^3\mathbf{U}(1)_{\text{conn}} & \longrightarrow & \mathbf{B}^3\mathbf{U}(1) \end{array}$$

analogous to (3.2.12), the 2-stack $\mathbf{BString}_{\text{conn}}$ is equivalently given by the homotopy pullback

$$(3.7.16) \quad \begin{array}{ccc} \mathbf{BString}_{\text{conn}} & \longrightarrow & \Omega^{1 \leq \bullet \leq 3} \\ \downarrow & & \downarrow \\ \mathbf{BSpin}_{\text{conn}} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3\mathbf{U}(1)_{\text{conn}} , \end{array}$$

in analogy with (3.2.13). As we have seen in Section 3.6, a realization of $\frac{1}{2}\hat{\mathbf{p}}_1$ maps the local data (ω_i, g_{ij}) of a Spin-bundle with connection on a manifold X to the Čech-Deligne 3-cocycle

$$(3.7.17) \quad (C_i, B_{ij}, A_{ijk}, g_{ijkl}) := \left(\text{CS}_3(\omega_i), \int_{\Delta^1} \text{CS}_3(\hat{\omega}_{ij}), \int_{\Delta^2} \text{CS}_3(\hat{\omega}_{ijk}), \int_{\Delta^3} \mu_3(\hat{\omega}_{ijkl} \wedge \hat{\omega}_{ijkl} \wedge \hat{\omega}_{ijkl}) \bmod \mathbb{Z} \right) .$$

So one can derive local expressions for local differential form data of a String connection in perfect analogy with formulae at the end of Section 3.3 . Alternatively, one can derive all these local differential forms data and the equations they satisfy by simplicial integration of the string Lie 2-algebra \mathfrak{so}_{μ_3} (as in Section 6.3 of [30], based on the L_∞ -algebraic resolutions in [76]). Whichever of these equivalent presentations one adopts, one finds that on each U_i of the chosen open cover \mathcal{U} of X the datum of a String 2-connection is the datum of an \mathfrak{so} -valued 1-form ω_i and of a real valued 2-form B_i on U_i , with 3-form curvature

$$(3.7.18) \quad \mathcal{H}_i := dB_i + \text{CS}_3(\omega_i) ,$$

satisfying a system of compatibility conditions on double and triple overlaps of the patches in the cover. On the other hand, due to the equivalence of Lie 2-algebras (3.7.13), there is an equivalent but rather different looking higher gauge in which a String 2-connection is locally on any U_i given by a pair (A_i, \hat{B}_i) of nonabelian differential forms, with $A_i \in \Omega^1(U_i, P\mathfrak{g})$, and $\hat{B}_i \in \Omega^2(U_i, \Omega\mathfrak{g} \oplus \mathbb{R})$. Notice that this has correspondingly a pair $(\mathcal{F}_i, \mathcal{H}_i)$ of curvature forms, with

$$(3.7.19) \quad \mathcal{F}_i = dA_i + \frac{1}{2}[A_i \wedge A_i] + h(B_i) \in \Omega^2(U_i, P_*\mathfrak{so})$$

$$(3.7.20) \quad \mathcal{H}_i = dB_i + [A_i \wedge B_i] \in \Omega^3(U_i, \hat{\Omega}\mathfrak{so})$$

(see [80] for details). Here the bracket in the first line is the Lie bracket on the $P_*\mathfrak{so}$ -components of the 1-form A , and in the second line it is the action of the $P_*\mathfrak{so}$ -components of A on the $\hat{\Omega}\mathfrak{so}$ -components of \hat{B} (see [7] for details of this action). The map h in the first line sends the $\hat{\Omega}\mathfrak{so}$ -valued 2-form to the underlying $P_*\mathfrak{so}$ -valued 2-form obtained by forgetting the central extension and regarding a loop as a special based path.

A higher nonabelian curvature structure of this form was also proposed in [63] for a description of nonabelian 2-forms on 5-branes, as recalled above around (2.3.1). There, also a Chern-Simons term was added, as in (3.7.18). We will see further Chern-Simons terms appear and acts as twists of the above curvature relations when we pass to *twisted* String-2-connections in

Section 3.8 below, (see Equation (3.8.9) there), though it seems that the details that we derive differ from the proposal in [63]. Precisely in the special case that the 2-form curvature (3.7.19) vanishes, there is a natural notion of non-Abelian Wilson-*surface* observables for 2-connections. This and the full local data for such 2-connections is derived and spelled out in [80, 81].

3.8. The 2-stack $\mathbf{BString}_{\text{conn}}^{2a}$ of twisted String 2-connections

We discuss now a notion of *twisted* String-2-connections. These relate to the String-2-connections from Section 3.7 as *twisted vector bundles with connection* (as in twisted K-theory) relate to ordinary vector bundles.⁷ In [29] we find twisted String-2-connections in boundary field configurations of C -fields, reviewed below in Section 4.3, and in this form we will identify them as the field configurations for 7d Chern-Simons theories in Section 4.4.

The construction of the refinement of the first fractional Pontrjagin class to a morphism of stacks described in Section 3.7 rests only on the fact that Spin is a compact and simply connected simple Lie group, and so the same argument applies to the exceptional Lie group E_8 . By classical results [14] its first non-vanishing homotopy group is $\pi_3(E_8) \simeq \mathbb{Z}$ and so it follows by the Hurewicz theorem that $H^4(BE_8, \mathbb{Z}) \simeq \mathbb{Z}$. Therefore the generator of this group is, up to sign, a canonical characteristic class, which we write $[a] \in H^4(BE_8, \mathbb{Z})$, corresponding to a characteristic map

$$(3.8.1) \quad a : BE_8 \rightarrow K(\mathbb{Z}, 4) .$$

For any integer k , the characteristic class $k[a] \in H^4(BE_8, \mathbb{Z})$ has an essentially unique refinement

$$(3.8.2) \quad k\mathbf{a} : \mathbf{B}E_8 \rightarrow \mathbf{B}^3\mathbf{U}(1)$$

to a morphism of smooth stacks, a representative of which is provided by the Lie integration of $k\mu_3^{\mathfrak{e}_8}$ according to [30], where $\mu_3^{\mathfrak{e}_8}$ denotes the canonical Lie algebra 3-cocycle on \mathfrak{e}_8 . Therefore, we can consider the smooth 2-groups $\mathbf{String}^{k\mathbf{a}}$, defined to be the loop space objects of the homotopy pullback in

⁷For more context in string theory see [76], for general theory see Sections 2.3.5, 3.3.7 and 4.4 of [79].

the top left corner of

$$(3.8.3) \quad \begin{array}{ccc} \mathbf{BString}^{ka} & \longrightarrow & \mathbf{BE}_8 \\ \downarrow & & \downarrow ka \\ \mathbf{BSpin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3\mathbf{U}(1) \end{array} .$$

This is the construction alluded to in the introduction around (2.1.13). By using the (higher) abelian group structure on $\mathbf{B}^3\mathbf{U}(1)$, the stacks $\mathbf{BString}^{ka}$ can be equivalently seen as the homotopy fibers of the difference $\frac{1}{2}\mathbf{p}_1 - ka$, via a stacky generalization of the description in [76],

$$(3.8.4) \quad \begin{array}{ccc} \mathbf{BString}^{ka} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathbf{Spin} \times E_8) & \xrightarrow{\frac{1}{2}\mathbf{p}_1 - ka} & \mathbf{B}^3\mathbf{U}(1) \end{array} .$$

By the defining nature of homotopy pullback, this means, in generalization of the discussion below (3.7.1), that a \mathbf{String}^{ka} -principal 2-bundle, classified by a morphism of 2-stacks $X \rightarrow \mathbf{BString}^{ka}$, is equivalently the data of

- an ordinary Spin-principal bundle P and an ordinary E_8 -principal bundle E ;
- equipped with a choice of gauge transformation

$$(3.8.5) \quad h : \frac{1}{2}\mathbf{p}(P) \xrightarrow{\simeq} ka(E)$$

between their Chern-Simons circle 3-bundles.

The image of this last condition in integral cohomology is

$$(3.8.6) \quad \frac{1}{2}p_1 = ka \in H^4(X, \mathbb{Z}) .$$

For $k = 2$ this is the “quantization condition” for supergravity C -field configurations on a 5-brane boundary. We come back to this in Section 4.3.

The 2-group \mathbf{String}^{ka} is related to the 2-group String in higher analogy of how the ordinary group \mathbf{Spin}^c is related to Spin. This is explained in [76]. As also discussed there, the Freed-Witten anomaly cancellation mechanism for type II strings on D-branes implies twisted \mathbf{Spin}^c -structures on D-branes. Here, the (twisted) \mathbf{String}^{2a} -structures that we find on M5-branes can therefore be understood as a direct higher generalization of this to higher dimension. Notice that for $k = 0$ we recover the untwisted string 2-group from

Section 3.7 together with a factor of E_8 :

$$(3.8.7) \quad \text{String}^{0\mathbf{a}} \simeq \text{String} \times E_8.$$

This is the higher analog of the fact that the “untwisted version” of $\text{Spin}^c(N)$ is $SO(N) \times U(1)$.

Also notice that, by a classical fact [14] which explains much of the role of E_8 in 11-dimensional supergravity, E_8 is 14-connected, so that for X of dimension 11 (and more generally for X of dimension ≤ 15) E_8 -bundles on X have the same classification as circle 3-bundles / 2-gerbes on X in that there is precisely one equivalence class of them for each element of $H^4(X, \mathbb{Z})$. Accordingly, configurations on such X that satisfy (3.8.6) for *some* E_8 bundle with class a are precisely the Spin-structures for which $\frac{1}{2}p_1$ — hence λ from expression (2.3.6) — is further divisible by k . To amplify this, observe that the identity morphism

$$(3.8.8) \quad \mathbf{DD}_2 : \mathbf{B}^3U(1) \rightarrow \mathbf{B}^3U(1)$$

is the canonical smooth refinement of the canonical 4-class of circle 3-bundles / 2-gerbes (the higher *Dixmier-Douady class*), which induces for each $k \in \mathbb{Z}$ the smooth 2-group $\text{String}^{k\mathbf{DD}_2}$. This is such that a lift from a Spin-structure to a $\text{String}^{2\mathbf{DD}_2}$ -structure exists precisely if λ is further divisible by k , irrespective of the dimension of X .

As before for String itself, with the methods of [30] we obtain a differential refinement to a moduli 2-stack $\mathbf{BString}_{\text{conn}}^{k\mathbf{a}}$ of $\text{String}^{2\mathbf{a}}$ -connections. This has a presentation by differential Lie integration of a Lie 2-algebra that extends the direct sum $\mathfrak{so} \oplus \mathfrak{e}_8$ via its canonical 3-cocycle. From this one finds, in generalization of the discussion around (3.7.18), that there is a higher gauge in which $\text{String}^{k\mathbf{a}}$ -connections are locally given by

- an \mathfrak{so} -valued 1-form ω_i ;
- an \mathfrak{e}_8 -valued 1-form A_i ;
- a 2-form B_i ;

with local curvature 3-form the sum of the (opposite of the) de Rham differential of B with the difference of the Chern-Simons form of ω and k times the Chern-Simons form of A , respectively:

$$(3.8.9) \quad \mathcal{H}_i = dB_i + \text{CS}_3(\omega_i) - k\text{CS}_3(A_i).$$

Note that this implies the equation

$$(3.8.10) \quad d\mathcal{H}_i = \langle F_{\omega_i} \wedge F_{\omega_i} \rangle - k \langle F_{A_i} \wedge F_{A_i} \rangle,$$

which is the de Rham image of the characteristic relation (3.8.5). It is no coincidence that these are the formulae known from the heterotic Green-Schwarz mechanism. See [76] for more on that.

Again, beware that these local formulae are a little deceptive, in that on the one hand there are other higher gauges in which they look rather different, and also the formulae on single patches U_i do not reflect the complexity of the data and its conditions on double and triple overlaps. As before for bare String-2-connections in Section 3.7 we have: due to the higher gauge freedom, there are other — very different looking local formulae — that are however higher gauge equivalent via an equivalence explained in the Appendix. In that other gauge, the 2-form B above is instead non-abelian and valued in a Kac-Moody loop Lie algebra; accordingly, the 3-form curvature is nonabelian and is given by a twisted version of Equation (3.7.20).

We discuss the role of these String^{ka}-2-connections in 5-brane physics below in Section 4.

3.9. The differential second Pontrjagin class

We discuss now the smooth and differential refinement of the *second* Pontrjagin class p_2 from [30, 74], defined on the 2-stack of String-2-connections. Below, in Section 4.5, this will give the indecomposable part of the non-abelian 7-dimensional Chern-Simons theory.

In Section 3.6 we saw, following [30], how the topology of the Spin group induces a natural morphism of higher smooth stacks $\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\text{Spin} \rightarrow \mathbf{B}^3\text{U}(1)$ which differentially refines the first fractional Pontrjagin class. Recall that this was constructed in terms of systems of smooth functions

$$(3.9.1) \quad U_{ijkl} \times \Delta^3 \rightarrow \text{Spin}$$

on patches of space U times a 3-simplex, which serve as a big resolution of cocycle data $g_{ij} : U_{ij} \rightarrow \text{Spin}$ for Spin-principal bundles. In order to extend this kind of construction directly to one that supports a construction of p_2 , we would need to pass all the way up to 7-simplices. But the nontriviality of the third homotopy group of Spin says that there is a topological obstruction to further extending a smooth function

$$(3.9.2) \quad U_{ijklm} \times \partial\Delta^4 \rightarrow \text{Spin},$$

to a map

$$(3.9.3) \quad U_{ijklm} \times \Delta^4 \rightarrow \text{Spin}.$$

Specifically, we have $\pi_3(\text{Spin}) = \mathbb{Z}$, and, via the Hurewicz isomorphism,

$$(3.9.4) \quad \pi_3(\text{Spin}) \cong H^4(B\text{Spin}, \mathbb{Z}).$$

Therefore, the generator of $\pi_3(\text{Spin})$ is canonically identified with the class $\frac{1}{2}p_1$, and so the vanishing of the first fractional Pontrjagin class of a Spin bundle precisely means that the above obstruction to further lifting of the nonabelian cocycles vanishes.

We need to be more precise here, and recall a bit of obstruction theory: the vanishing of the cohomology class for an obstruction cocycle does not mean that the cocycle is unobstructed, but that in the same cohomology class we can find *an* unobstructed cocycle. Moreover, the datum of a trivialization of the obstruction cocycle precisely tells us how to modify the cocycle in order to get an unobstructed cocycle. So the data of a String bundle can be read as the data of a Spin bundle together with the “instructions” to overcome the first topological obstruction to extend the transition functions of the bundle to higher and higher dimensional simplices. Once the first obstruction is passed, the construction will go on until the second obstruction is met. Since $\pi_i(\text{Spin})$ vanishes for $4 \leq i \leq 6$, while $\pi_7(\text{Spin}) = \mathbb{Z}$, the second topological obstruction for a Spin bundle is represented by the generator Q_2 of $H^8(B\text{Spin}, \mathbb{Z})$, which involves the *second* Pontrjagin class p_2 as $Q_2 := \frac{1}{2}(p_2 - \lambda^2)$, which is a power of 2 multiple of the one-loop polynomial [64]. Moreover, when further refined from Spin to String, the generator is simply $\frac{1}{6}p_2$ (see [75]).

This can be elegantly expressed in terms of classifying spaces: let $B\text{String}$ be the *classifying space of String bundles* defined as the homotopy fiber of the characteristic map

$$(3.9.5) \quad \frac{1}{2}p_1: B\text{Spin} \rightarrow K(\mathbb{Z}, 4).$$

The long exact homotopy sequence and Hurewicz theorem then tell us that $H^i(B\text{String}, \mathbb{Z}) = 0$ for $0 \leq i \leq 7$ and $H^8(B\text{String}, \mathbb{Z}) = \mathbb{Z}$, so that there is a distinguished map (unique up to homotopy)

$$(3.9.6) \quad \frac{1}{6}p_2: B\text{String} \rightarrow K(\mathbb{Z}, 8)$$

representing the generator of the eighth singular cohomology group. Hence, for X a topological space, we have a second fractional Pontrjagin class

$$(3.9.7) \quad [X, B\text{String}] \xrightarrow{\frac{1}{6}p_2} [X, K(\mathbb{Z}, 8)] \cong H^8(X, \mathbb{Z}) .$$

Moreover, the obstruction theoretic argument presented above tells us that String bundles are precisely those Spin bundles for which we can suitably define the $\hat{g}_{i_0, \dots, i_n}$ lifts of the transition functions up to $n = 7$, showing that $\frac{1}{6}p_2$ refines to a morphism of stacks

$$(3.9.8) \quad \frac{1}{6}\mathbf{p}_2: \mathbf{B}\text{String} \rightarrow \mathbf{B}^7\mathbf{U}(1) .$$

Furthermore, by looking at the construction of the differential refinement $\frac{1}{2}\hat{\mathbf{p}}_1$ presented in Section 3.6, one immediately sees that what is crucial are the extensions $\hat{g}_{i_0, \dots, i_n}$, since the extensions $\hat{\omega}_{i_0, \dots, i_n}$ are defined in terms of those. This means that also $\frac{1}{6}\mathbf{p}_2$ has an analogous differential refinement [30]

$$(3.9.9) \quad \frac{1}{6}\hat{\mathbf{p}}_2: \mathbf{B}\text{String}_{\text{conn}} \rightarrow \mathbf{B}^7\mathbf{U}(1)_{\text{conn}}$$

from the stack of principal String bundles with connection to the stack of $\mathbf{U}(1)$ -7-bundles with connections.

Explicit models for both $\frac{1}{6}\mathbf{p}_2$ and $\frac{1}{6}\hat{\mathbf{p}}_2$ can be obtained via Lie integration and 7-coskeletization from the canonical 7-cocycle μ_7 and Chern-Simons element cs_7 on \mathfrak{so} , as in [30]. The “from the top” Lie integration approach has the remarkable advantage of producing canonical stack morphism out of n -cocycles and Chern-Simons elements for a Lie algebra \mathfrak{g} , directly at the level of the $(n - 1)$ -connected cover of a Lie group G of the Lie algebra \mathfrak{g} . However, here we preferred to present $\frac{1}{6}\mathbf{p}_2$ and $\frac{1}{6}\hat{\mathbf{p}}_2$ in terms of a more classical “from the bottom” construction which is probably more familiar to a wider range of readers. However, in Sections 4.5 and 4.6 the Lie integration of L_∞ -cocycles will allow us to get explicit local formulae for the Chern-Simons functionals induced by these differentially refined characteristic maps on String 2-connection fields.

4. The 7-dimensional nonabelian gerbe theory

We indicate in this section a precise definition and some properties of a certain nonabelian 7-dimensional Chern-Simons theory whose configuration space is the smooth moduli 2-stack of boundary C -field configurations, identified with that of twisted String-2-connections, described in Section 3.7 and

Section 3.8. We show that it has the properties that the arguments in Section 2 suggest the Chern-Simons-dual of the 5-brane worldvolume theory should have. We present this in stages:

- in Section 4.1 we present the general construction principle of higher Chern-Simons functionals on higher gauge fields;
- in Section 4.2 we consider theories induced from cup product classes, such as the abelian Chern-Simons theory in 7d as well as the non-abelian theory induced from $(\frac{1}{2}p_1)^2$;
- in Section 4.3 we very briefly review some aspects of the details of C -field configurations from [29];
- in Section 4.4 we put these ingredients together to form an action functional of “7d abelian CS with $\frac{1}{4}p_1$ -background charge”; and keeping also the “ p_1 -background charge” dynamical this is refined to a nonabelian higher Chern-Simons theory on *twisted String-2-connection fields*;
- in Section 4.5 we consider the “indecomposable” 7d theory induced from p_2 on String-2-connection fields;
- finally, in Section 4.6 we put all the pieces together and discuss the full nonabelian Chern-Simons term of 7d supergravity on String-2-connection fields.

4.1. Higher Chern-Simons functionals and their level quantization

The general mechanism behind all these *natural Chern-Simons functionals* is the following (see in [79] Sections 2.3.21 and 4.6). Let G be any higher smooth group (such as for instance an ordinary Lie group or the String-2-group) and write $\mathbf{B}G_{\text{conn}}$ for the higher moduli stack of G -connections. Then assume any differential characteristic map is given

$$(4.1.1) \quad \hat{\mathbf{c}} : \mathbf{B}G_{\text{conn}} \rightarrow \mathbf{B}^n \mathbf{U}(1)_{\text{conn}} .$$

(The examples that we will shortly turn to are those from Section 3.7 and Section 3.9.) Here we may also think of the stack on the left as the higher moduli stack stack of higher nonabelian gauge fields for the higher gauge group G , in that for Σ any (n -dimensional) smooth manifold, a higher

form gauge field data on Σ is characterized by a morphism of higher stacks $\phi : \Sigma \rightarrow \mathbf{B}G_{\text{conn}}$. Simply by composing this representing map with the above map \hat{c} we send it to a map $\hat{c}(\phi) : \Sigma \rightarrow \mathbf{B}^n\mathbf{U}(1)_{\text{conn}}$. By the discussion in Section 3.3, this now represents an n -form gauge field on Σ . Since Σ itself is n -dimensional, we may identify this with a differential n -form on Σ . An assignment of a top-degree form to a field configuration we may think of as a *Lagrangian* L for a field theory

$$(4.1.2) \quad L_c := \mathbf{H}(\Sigma, \hat{c}) : \text{Fields}(\Sigma) \simeq \mathbf{H}(\Sigma, \mathbf{B}G_{\text{conn}}) \rightarrow \mathbf{H}(\Sigma, \mathbf{B}^n\mathbf{U}(1)_{\text{conn}}).$$

Here $\text{Fields}(\Sigma)$ is the higher groupoid of field configurations: its objects are nonabelian higher form fields, its morphisms are gauge transformations between these, and its 2-morphisms are gauge-of-gauge transformations, and so on. This is the higher groupoid that the BRST-complex / Lie n -algebroid of the gauge theory is the infinitesimal approximation to.

In order to make this into an action functional, it remains only to integrate the Lagrangian over Σ . By the discussion in Section 3.5 this may be understood as forming the n -volume holonomy of the n -form gauge fields in $\mathbf{B}^n\mathbf{U}(1)_{\text{conn}}$. Thus the exponentiated action functional on the integrated BRST-complex (Lie n -algebroid) $\text{Fields}(\Sigma)$ of field configurations and gauge transformation on Σ induced from \hat{c} is

$$(4.1.3) \quad \begin{aligned} \exp(iS_c(-)) &:= \exp(2\pi i \int_{\Sigma} L_c(-)) : \\ \text{Fields}(\Sigma) &\simeq \mathbf{H}(\Sigma, \mathbf{B}G_{\text{conn}}) \xrightarrow{L_c} \mathbf{H}(\Sigma, \mathbf{B}^n\mathbf{U}(1)_{\text{conn}}) \xrightarrow{\exp(2\pi i \int_{\Sigma} (-))} \mathbf{U}(1) , \end{aligned}$$

where the second morphism is the higher holonomy morphism from (3.5.12).

The action functionals obtained this way are guaranteed to satisfy the requirements on a Chern-Simons functionals associated with a class c :

1. The Lagrangian L_c is locally given by a higher Chern-Simons form for the de Rham image of the integral class c ;
2. The action functional $\exp(iS_c(-))$ is (higher) gauge invariant.

The first property follows from the nature of differential characteristic maps (3.6.3). It is the statement of traditional *Chern-Weil theory* refined to higher gauge fields. The higher Chern-Simons forms for higher gauge fields that appear here have been introduced in [74]. Their appearance in the higher Lagrangians as above has been established in [30]. For more on the general

context of higher Chern-Weil theory see in [79] Sections 2.3.18 and 3.3.14. However, just as ordinary degree-3 Chern-Simons forms, also these higher Chern-Simons forms by themselves are not gauge invariant.

The gauge invariance of the integrated and exponentiated Lagrangian, hence of the action functional, follows in the above general construction by the very nature of what it means to give a morphism from a higher groupoid/stack $\mathbf{H}(\Sigma, \mathbf{B}G_{\text{conn}})$ to a set such as $U(1)$. This is what the intrinsic formulation in terms of higher stacks here accomplishes for us. It is in fact impossible to make a non-gauge-invariant construction in higher stack theory as long as one sticks to universal constructions as in the above (as opposed to direct component presentations by local differential form data).

Notice here that the subtle quantization condition on the *level* that is the familiar condition on the ordinary 3-dimensional Chern-Simons action to be gauge invariant (see [32] for a review) is all encoded in the initial choice of the characteristic map c , which is a *discrete* choice. In generalization of this, the gauge invariance and the *level quantization* of the above $\exp(iS_{\mathbf{c}}(-))$ is all in the fact that \mathbf{c} is indeed a morphism of higher stacks to $\mathbf{B}^n U(1)$, and as such indeed a smooth refinement of an integral cohomology class.

Example. As the archetypical example for this phenomenon, consider the the case $G := \text{Spin}$ and $\hat{\mathbf{c}} := \frac{1}{2}\hat{\mathbf{p}}_1$, from (3.6.5). Feeding this into the above machine spits out the action functional of ordinary 3d Spin-Chern-Simons theory (this has essentially been observed in [17]) at *level 1* (or at level -1, depending on an inessential convention):

$$(4.1.4) \quad \begin{aligned} \exp(iS_{\frac{1}{2}p_1}(-)) : \omega &\mapsto \exp(2\pi i \int_{\Sigma} \text{CS}_3(\omega)) \\ &:= \exp(2\pi i \int_{\Sigma} \langle \omega \wedge d\omega \rangle + \frac{2}{3} \langle \omega \wedge \omega \wedge \omega \rangle). \end{aligned}$$

This is the direct reflection of the fact that $\frac{1}{2}\hat{\mathbf{p}}_1$ is the differential refinement of the *integral* class $\frac{1}{2}p_1 \in H^4(B\text{Spin}, \mathbb{Z})$, and that this, in turn, is the *generator* of $H^4(B\text{Spin}, \mathbb{Z}) \simeq \mathbb{Z}$. This is the reason for caring about the *fractional* Pontrjagin class here: precisely for every integer $k \in \mathbb{Z}$ do we get another class $\frac{k}{2}p_1 \in H^4(B\text{Spin}, \mathbb{Z})$ and its differential refinement $\frac{k}{2}\hat{\mathbf{p}}_1$. Feeding that into the above machine produces the ordinary action functional of 3d Spin-Chern-Simons theory at level k .

$$(4.1.5) \quad \exp(iS_{\frac{k}{2}p_1}(-)) : \omega \mapsto \exp(k 2\pi i \int_{\Sigma} \langle \omega \wedge d\omega \rangle + \frac{2}{3} \langle \omega \wedge \omega \wedge \omega \rangle).$$

While this issue of gauge invariance and quantized levels is classical and well understood by explicit computation for ordinary Chern-Simons theory, the analogous problem in constructing higher dimensional Chern-Simons theories on higher gauge fields quickly becomes intractable in terms of local differential form data and its higher gauge transformations. But the above general construction serves as an algorithm that tells us how to obtain gauge invariant higher Chern-Simons functionals and how exactly their levels have to be quantized.

We go through some examples in the following sections. For instance, for the indecomposable 7d theory induced by p_2 on String 2-connection fields (see Section 4.5) the quantization condition on the level is controlled by the fact that $H^8(BString, \mathbb{Z}) \simeq \mathbb{Z}$ and that the generator of this integral group is [75] the second fractional Pontrjagin class $\frac{1}{6}p_2$. Therefore the canonically induced 7d action $\exp(iS_{\frac{1}{6}p_2}(-))$ is precisely the *level-1* theory in 7d on non-abelian 2-connection fields, and we get the theory at another level precisely for any choice $k \in \mathbb{Z}$ by forming $\exp(iS_{\frac{k}{6}p_2}(-))$.

In the following we discuss some 7-dimensional action functionals of the above form.

4.2. The cup product of a 3d CS theory with itself

The action functional of the 7d theory involves decomposable as well as nondecomposable terms. In this section we consider the former, which is essentially the product of two copies of 3d Chern-Simons theory. The latter case is discussed in Section 4.5 below.

Let G be a compact and simply connected simple Lie group and let c the characteristic class given by the canonical generator of $H^4(G; \mathbb{Z})$. Then we have the cup product of c with itself defining a degree 8 integral cohomology class $c \cup c$. In terms of characteristic maps, this corresponds to the composition

$$(4.2.1) \quad c \cup c : BG \xrightarrow{(c,c)} K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \xrightarrow{\cup} K(\mathbb{Z}, 8) .$$

Such a structure is utilized in [69] to define particular such twists to Five-brane structures in relation to the M5-brane. Since the characteristic map c is induced by the canonical Lie algebra 3-cocycle on G , it has, by [30], a differential refinement to a morphism of stacks

$$(4.2.2) \quad \hat{c} : \mathbf{B}G_{\text{conn}} \rightarrow \mathbf{B}^3\mathbf{U}(1)_{\text{conn}} .$$

By itself, this induces ordinary 3d Chern-Simons theory, as discussed around (4.1.4). But using a differential refinement of the cup product, its cup square induces a 7d theory. Indeed, under the Dold-Kan correspondence (see Section 3.1), the *Beilinson-Deligne cup product*, which refines the cup product on differential cohomology classes to Deligne complexes, naturally gives a corresponding morphism of moduli stacks

$$(4.2.3) \quad \hat{\cup} : \mathbf{B}^k \mathbf{U}(1)_{\text{conn}} \times \mathbf{B}^l \mathbf{U}(1)_{\text{conn}} \rightarrow \mathbf{B}^{k+l+1} \mathbf{U}(1)_{\text{conn}} .$$

This cup product is such that for $\hat{C} : \Sigma \rightarrow \mathbf{B}^k \mathbf{U}(1)_{\text{conn}}$ a k -connection with local connection k -forms $\{C_i\}$ and globally defined curvature $(k + 1)$ -forms G , and for $\hat{B} : \Sigma \rightarrow \mathbf{B}^l \mathbf{U}(1)_{\text{conn}}$ an l -connection with with local connection l -forms $\{B_i\}$ and globally defined curvature $(l + 1)$ -form H , the cup product

$$\hat{C} \hat{\cup} \hat{B} : \Sigma \xrightarrow{(\hat{C}, \hat{B})} \mathbf{B}^k \mathbf{U}(1)_{\text{conn}} \times \mathbf{B}^l \mathbf{U}(1)_{\text{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^{k+l+1} \mathbf{U}(1)_{\text{conn}}$$

is a $k + l + 1$ -connection whose local connection $(k + l + 1)$ -forms can be taken to be $\{C_i \wedge H\}$ or $\{G \wedge B_i\}$, and hence whose curvature $(k + l + 2)$ -form is $G \wedge H$. Moreover, the underlying integral $(k + l + 2)$ -class of $\hat{C} \hat{\cup} \hat{B}$ is the ordinary cup product of the integral classes underlying \hat{C} and \hat{B} . Therefore, this induces a differential refinement of the ordinary integral cup square $c \cup c$ to a morphism of stacks

$$(4.2.5) \quad \hat{\mathbf{c}} \hat{\cup} \hat{\mathbf{c}} : \mathbf{B}G_{\text{conn}} \xrightarrow{(\hat{\mathbf{c}}, \hat{\mathbf{c}})} \mathbf{B}^3 \mathbf{U}(1)_{\text{conn}} \times \mathbf{B}^3 \mathbf{U}(1)_{\text{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^7 \mathbf{U}(1)_{\text{conn}} .$$

So if Σ is a compact oriented smooth manifold of dimension 7, we have a *cup product Chern-Simons theory* induced by \mathbf{c} : its Chern-Simons functional is

$$(4.2.6) \quad \exp(iS_{\mathbf{c} \cup \mathbf{c}}) : \mathbf{H}(\Sigma, \mathbf{B}G_{\text{conn}}) \xrightarrow{\hat{\mathbf{c}} \hat{\cup} \hat{\mathbf{c}}} \mathbf{H}(\Sigma, \mathbf{B}^7 \mathbf{U}(1)_{\text{conn}}) \xrightarrow{\int_{\Sigma}} \mathbf{U}(1) .$$

For ordinary Chern-Simons theory, the assumption that G is simply connected implies that BG is 3-connected, hence that every G -principal bundle on a 3-dimensional Σ is trivializable, so that G -principal connections on Σ can be identified with \mathfrak{g} -valued differential forms on Σ . This is no longer in general the case over a 7-dimensional Σ . Therefore, no simple explicit expression of the action $\exp(iS_{\mathbf{c} \cup \mathbf{c}}(\nabla))$ can be given in general (one can always describe in in terms of nonabelian Cech cocycles). However, if the underlying G -bundle of a field configuration happens to be trivial, then we do have such a simple expression. Namely, once a trivialization is chosen,

the G -connection ∇ is given by a globally defined \mathfrak{g} -valued 1-form A on Σ and the explicit expression of the Beilinson-Deligne cup product mentioned above implies that the cup product Chern-Simons action is

$$(4.2.7) \quad \begin{aligned} \exp(iS_{\text{cup}}(\nabla)) &= \exp\left(2\pi i \int_{\Sigma} \text{CS}_3(A) \wedge d\text{CS}_3(A)\right) \\ &= \exp\left(2\pi i \int_{\Sigma} \text{CS}_3(A) \wedge \langle F_A \wedge F_A \rangle\right), \end{aligned}$$

where $\text{CS}(A)$ is the usual Chern-Simons 3-form, and $\langle F_A \wedge F_A \rangle$ is the canonical de Rham representative for the cohomology class c .

Let, for instance, $c = \frac{1}{2}p_1$ be the first fractional Pontrjagin class on Spin-connections. Its smooth differential refinement

$$(4.2.8) \quad \frac{1}{2}\hat{\mathbf{P}}_1 : \mathbf{B}\text{Spin}_{\text{conn}} \rightarrow \mathbf{B}^3\mathbf{U}(1)_{\text{conn}}$$

was constructed in [30]. The corresponding cup product action $\exp(iS_{(\frac{1}{2}p_1^2)}(-))$ is the refinement to moduli stacks of the Chern-Simons term induced by the λ^2 -summand (2.3.5) in the quantum corrected supergravity action.

In direct analogy we obtain standard abelian 7d-Chern-Simons theory refined to its moduli 3-stack of field configurations $\mathbf{B}^3\mathbf{U}(1)_{\text{conn}}$. The *identity morphism*

$$(4.2.9) \quad \mathbf{DD}_2 : \mathbf{B}^3\mathbf{U}(1)_{\text{conn}} \rightarrow \mathbf{B}^3\mathbf{U}(1)_{\text{conn}}$$

we may think of as the differential refinement of the smooth refinement (3.8.8) of the higher Dixmier-Douady class on circle 3-bundles / bundle 2-gerbes. The corresponding cup product 7d Chern-Simons action functional is the composite

$$(4.2.10) \quad \begin{aligned} &\exp(iS_{\mathbf{DD}_2 \cup \mathbf{DD}_2}(-)) : \\ &\mathbf{H}(\Sigma, \mathbf{B}^3\mathbf{U}(1)_{\text{conn}}) \xrightarrow{\mathbf{DD}_2 \hat{\cup} \mathbf{DD}_2} \mathbf{H}(\Sigma, \mathbf{B}^7\mathbf{U}(1)_{\text{conn}}) \xrightarrow{\int_{\Sigma}} \mathbf{U}(1). \end{aligned}$$

Again, this has in general a complicated expression in terms of local data. But when restricted to fields C_3 in the inclusion $\Omega^3(-) \rightarrow \mathbf{B}^3\mathbf{U}(1)_{\text{conn}}$, the action has the simple expression

$$(4.2.11) \quad \exp(iS_{\mathbf{DD}_2 \cup \mathbf{DD}_2})(C) = \exp(2\pi i \int_{\Sigma} C_3 \wedge dC_3).$$

4.3. The moduli stack of supergravity C -field configurations

The 7d Chern-Simons functionals that we consider will be defined on the fields of 11-dimensional supergravity, their reduction to 7-dimensions and their restriction to 5-brane boundaries. The collection of these fields consists locally of a 3-form (the C -field), an \mathfrak{so} -valued 1-form (the field of gravity), and on the boundary also of an \mathfrak{e}_8 -valued 1-form, the gauge field, and a B -field that witnesses the gauge identifications on the boundary. Globally, however, these fields are interrelated and arrange to certain nonabelian twisted differential cocycles.

In [29] we give a detailed discussion of a smooth moduli 3-stack \mathbf{CField} of C -field configurations, as well as of a morphism $\mathbf{CField}^{\text{bdr}} \rightarrow \mathbf{CField}$ that exhibits the relative cohomology that classifies boundary C -field configurations. For convenient reference in the following discussion, we briefly list some basic statements from [29] here.

There is a close analogy with the discussion of B -field configurations on D-branes in Section 3.4. Recall that there the moduli of abelian bulk and nonabelian boundary field configurations were those of differential cohomology *relative* to the brane inclusion $Q \hookrightarrow X$ and relative to the characteristic map $\mathbf{dd} : \mathbf{BPU}(\mathcal{H}) \rightarrow \mathbf{B}^2U(1)$. The analog of this characteristic map for the C -field is the canonical (second Chern) map

$$(4.3.1) \quad \mathbf{a} : \mathbf{BE}_8 \rightarrow \mathbf{B}^3U(1)$$

from the moduli stack of E_8 -bundles to that of circle 3-bundles / bundle 2-gerbes, which is constructed as a morphism of smooth 3-stacks as in [30]. Under geometric realization (3.1.2) this becomes morphism $a : BE_8 \rightarrow K(\mathbb{Z}, 4)$ of topological spaces representing a generating degree-4 integral cohomology class in $H^4(BE_8) \simeq \mathbb{Z}$. By the higher connectedness of E_8 [14] this a is an equivalence on 15-coskeleta, which means that, while nonabelian E_8 -gauge fields have a very different differential geometry than abelian 3-form connections, the instanton sectors on both sides may be identified. All this is analogous to \mathbf{dd} and the situation for the B -field on D-branes around (3.4.8).

So let $Q \rightarrow X$ be a 5-brane worldvolume embedded into 11-dimensional spacetime X . A corresponding cocycle in \mathbf{a} -twisted relative differential cohomology is a homotopy commuting diagram of higher stacks

$$(4.3.2) \quad \begin{array}{ccc} Q & \xrightarrow{\hat{B}} & \mathbf{B}(E_8)_{\text{conn}} \\ \downarrow & \swarrow \simeq & \downarrow \hat{\mathbf{a}} \\ X & \xrightarrow{\hat{C}} & \mathbf{B}^3U(1)_{\text{conn}} \end{array}$$

analogous to (3.4.7). For fixed bulk field \hat{C} , this is equivalently an element in the homotopy pullback

$$(4.3.3) \quad \begin{array}{ccc} \mathbf{aStruc}_{\hat{C}|_Q}(Q) & \longrightarrow & * \\ \downarrow & & \downarrow \hat{C}|_Q \\ \mathbf{H}(Q, \mathbf{B}(E_8)_{\text{conn}}) & \xrightarrow{\mathbf{H}(Q, \hat{\mathbf{a}})} & \mathbf{H}(Q, \mathbf{B}^3U(1)_{\text{conn}}) \end{array}$$

of $\hat{C}|_Q$ -twisted differential $\text{String}(E_8)$ -structures [76], which are twisted $\text{String}(E_8)$ -2-connections on Q . Therefore, where the restriction of the abelian B -field on a D-brane gives rise to a nonabelian 1-form gauge field, the restriction of the C -field relative the \mathbf{a} -class gives rise to a nonabelian 2-form gauge field. This sets the main mechanism for the supergravity C -field boundary moduli. But there the situation is a little bit more complex, due to the additional presence of the dynamical Spin-connection.

We consider throughout the case of Spin-structures for which the class $\frac{1}{2}p_1$ is further divisible by 2. As discussed in [76], this may naturally be understood as given by $\text{String}^{2\mathbf{DD}_2}$ -structures, where, for \mathbf{c} some universal 4-class, $\text{String}^{\mathbf{c}}$ is a higher analog of Spin^c [68], and where \mathbf{DD}_2 is the universal Dixmier-Douady class for 2-gerbes / circle 3-bundles. Over the spacetimes of dimension ≤ 14 that we care about here, this are equivalently $\text{String}^{2\mathbf{a}}$ -structures, where \mathbf{a} is the universal 4-class of E_8 -principal bundles.

The moduli 3-stack of C -field configurations for $\frac{1}{2}p_1$ divisible by 2 is then the homotopy pullback

$$(4.3.4) \quad \begin{array}{ccc} \mathbf{CField} & \longrightarrow & \mathbf{B}^3U(1)_{\text{conn}} \\ \downarrow & & \downarrow \cdot 2 \\ \mathbf{BSpin}_{\text{conn}} \times \mathbf{BE}_8 & \xrightarrow{\frac{1}{2}p_1 + 2\mathbf{a}} & \mathbf{B}^3U(1) \end{array} ,$$

where $\frac{1}{2}p_1$ is the smooth refinement of the first fractional Pontrjagin class from [30]. This is a further refinement of the situation discussed in the

introduction around (2.1.13). Along the lines of the discussion there, one finds that a field configuration $\phi : \Sigma \rightarrow \mathbf{CField}$ has an underlying circle 3-connection \hat{C} , an underlying Spin-connection \hat{F}_ω , and an underlying E_8 -principal bundle with class a , as well as a choice of gauge transformation

$$(4.3.5) \quad H : G \xrightarrow{\simeq} a - \frac{1}{4}p_1$$

between the underlying circle 3-bundle of \hat{G} and the difference between the Chern-Simons circle 3-bundles of the Spin- and the E_8 -bundle.

There are two stages of boundary conditions for this data we consider, exhibited by a sequence of maps of moduli stacks

$$(4.3.6) \quad \mathbf{CField}^{\text{bdro}} \rightarrow \mathbf{CField}^{\text{bdr}} \rightarrow \mathbf{CField}.$$

For boundary field configurations $\phi : \Sigma \rightarrow \mathbf{CField}^{\text{bdr}}$ the integral cohomology class of \hat{G}_4 is required to vanish and a differential 3-form part may remain (this is the condition for restriction to an M5-brane), while for $\mathbf{CField}^{\text{bdro}}$ the full differential cohomology class of \hat{G}_4 is required to vanish (this is the condition for restriction to the orbifold fixed point of a Hořava-Witten boundary). In both cases the E_8 -bundle picks up a connection over the boundary.

In more detail, the boundary moduli $\mathbf{CField}^{\text{bdr}}$ are given by the homotopy pullback of smooth 2-stacks

$$(4.3.7) \quad \begin{array}{ccc} \mathbf{CField}^{\text{bdr}} & \xrightarrow{\quad} & \Omega^{1 \leq \bullet \leq 3}(-) \\ \downarrow & \swarrow \hat{B} & \downarrow \mathcal{H} \\ \mathbf{B}(\text{Spin} \times E_8)_{\text{conn}} & \xrightarrow{\frac{1}{2}\hat{p}_1 + 2\hat{a}} & \mathbf{B}^3U(1)_{\text{conn}} \end{array},$$

where the right morphism includes the moduli 3-stack for globally defined 3-forms and their gauge transformation canonically into the moduli 3-stack for 3-form field configurations. This is the analog for the C -field of the situation in (3.4.1) for the B -field. Over a patch U_i of a brane $Q \hookrightarrow X$ it encodes a 2-form B_i with differential $H_i = dB_i$ such that

$$(4.3.8) \quad \mathcal{H}_i = H_i + (\text{CS}_3(\omega) - 2\text{CS}_3(A)).$$

This corresponds to Equation (2.1.2) in the introduction.

The 2-stack $\mathbf{CField}^{\text{bdr}_0}$ is defined analogously, but with the right morphism being the inclusion $*$ $\rightarrow \mathbf{B}^3U(1)_{\text{conn}}$ of the entirely trivial 3-form connection. Comparison with (3.8.4) shows that therefore these C -field boundary moduli are equivalent to those of $2\mathbf{a}$ -twisted String 2-connections discuss in Section 3.8

$$(4.3.9) \quad \mathbf{CField}^{\text{bdr}} \simeq \mathbf{BString}_{\text{conn}}^{2\mathbf{a}} .$$

Then for $Q \rightarrow X$ a brane in an 11-dimensional spacetime X , the 3-stack of bulk and boundary field configurations is that of homotopy commuting squares of 3-stacks

$$(4.3.10) \quad \begin{array}{ccc} Q & \xrightarrow{\hat{B}} & \mathbf{CField}^{\text{bdr}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\hat{C}} & \mathbf{CField} . \end{array}$$

This is the C -field analog of what for the B -field is (3.4.7). More details are discussed in [29].

4.4. 7d CS theory with charges on the supergravity C -field

We can directly combine the above two kinds of Chern-Simons theories to their *direct product theory*, given by the action functional

$$(4.4.1) \quad \begin{array}{c} \exp(iS_{\frac{1}{2}p_1 \cup \frac{1}{2}p_1}) \exp(iS_{\text{DD}_2 \cup \text{DD}_2}) : \\ \mathbf{H}(\Sigma, \mathbf{BSpin}_{\text{conn}} \times \mathbf{B}^3U(1)_{\text{conn}}) \xrightarrow{(\frac{1}{2}\hat{p}_1 \cup \frac{1}{2}\hat{p}_1 + \hat{\mathbf{D}}\hat{\mathbf{D}}\hat{\mathbf{D}})} \mathbf{H}(\Sigma, \mathbf{B}^7U(1)_{\text{conn}}) \\ \exp(2\pi i \int_{\Sigma} (-)) \Big\downarrow \\ \mathbf{U}(1) , \end{array}$$

defined on pairs consisting of a Spin-connection and a 3-form field. To incorporate the interrelation between these fields in supergravity, we precompose this with the canonical projection map from the moduli of supergravity C -field configurations, as discussed in Section 4.3. Precomposing with the paired projections out of the defining homotopy pullback (4.3.4) gives the

action functional

$$\begin{array}{ccc}
 \mathbf{H}(\Sigma, \mathbf{CField}) & & \mathbf{U}(1). \\
 \downarrow & & \uparrow f_\Sigma \\
 \mathbf{H}(\Sigma, \mathbf{BSpin}_{\text{conn}} \times \mathbf{B}^3\mathbf{U}(1)_{\text{conn}}) & \xrightarrow{(\frac{1}{2}\hat{p}_1 \cup \frac{1}{2}\hat{p}_1 + \hat{D}\hat{D}\hat{U}\hat{D}\hat{D})} & \mathbf{H}(\Sigma, \mathbf{B}^7\mathbf{U}(1)_{\text{conn}})
 \end{array}$$

This is now a functional defined on 11-dimensional supergravity fields whose local form data is $\{C_i\}$ (the C -field) and $\{\omega_i\}$ (the spin connection), and whose integral classes satisfy the quantization condition

$$(4.4.2) \quad G_4 = a - \frac{1}{4}p_1 \in H^4(\Sigma, \mathbb{Z}).$$

The action functional is locally given by

$$(4.4.3) \quad (\omega_i, C_i) \mapsto \int (C_i \wedge dC_i - \text{CS}(\omega_i) \wedge \text{tr}(F_\omega \wedge F_\omega)).$$

This may be further restricted to the C -field *boundary* configuration along the morphism of moduli stacks $\mathbf{CField}^{\text{bdr}} \rightarrow \mathbf{CField}$ to an action functional

$$\begin{array}{ccc}
 \mathbf{H}(\Sigma, \mathbf{CField}^{\text{bdr}}) & & \mathbf{U}(1). \\
 \downarrow & & \uparrow f_\Sigma \\
 \mathbf{H}(\Sigma, \mathbf{BSpin}_{\text{conn}} \times \mathbf{B}^3\mathbf{U}(1)_{\text{conn}}) & \xrightarrow{(\frac{1}{2}\hat{p}_1 \cup \frac{1}{2}\hat{p}_1 + \hat{D}\hat{D}\hat{U}\hat{D}\hat{D})} & \mathbf{H}(\Sigma, \mathbf{B}^7\mathbf{U}(1)_{\text{conn}})
 \end{array}$$

By the equivalence (4.3.9) the fields in $\mathbf{H}(\Sigma, \mathbf{CField}^{\text{bdr}})$ are *twisted* String-2-connections. As discussed in Section 3.8, there is a higher gauge in which these are locally given by form data $(\omega_i, A_i, B_i, \mathcal{H}_i)$ subject to some constraints. In terms of this local data the local value of the action functional still reads as in the functional (4.4.3). However, the local data is insufficient to accurately judge the nature of the field content. For one, the relations (3.8.9) satisfied by the local form data means that equivalently the local action is given by expression such as

$$(4.4.4) \quad (\omega_i, A_i, B_i, \mathcal{H}_i) \mapsto \int (C_i \wedge dC_i - \mathcal{H}_i \wedge d\mathcal{H}_i - 2\mathcal{H}_i \wedge \langle F_A \wedge F_A \rangle - \text{CS}(A_i) \wedge \text{tr}(F_A \wedge F_A)).$$

Moreover, as discussed in Section 3.7, there are other gauges in which equivalently a nonabelian 2-form B_i appears. Finally, in either case the global

value of the action functional involves in general contributions from gauge transformations between local patches, and gauge-of-gauge transformations between these, which are not immediately evident from the above local formulae. A general account of the complete formulae in terms of nonabelian Čech cocycles with coefficients in L_∞ -algebra valued forms [74] is in [30].

Whichever way one uses to derive (or guess, should that indeed be possible) these correct explicit formulae for the higher nonabelian gauge field actions, the above simple constructions in terms of canonical Chern-Simons action functionals on mapping stacks guarantees that these formulae exist, are indeed gauge invariant, and are controlled by the defining characteristic classes in the way they should. It is then a straightforward matter of applying the general machinery to obtain any level of explicit detail as desired.

4.5. 7d indecomposable CS theory on String 2-connection fields

We now turn to the 7-dimensional Chern-Simons theory that is induced by the second Pontrjagin class p_2 on $B\text{Spin}$ and its fractional refinement $\frac{1}{6}p_2$ on $B\text{String}$. As discussed in Section 3.6 (Equation (3.6.8)) we have a canonical differential characteristic map

$$(4.5.1) \quad \frac{1}{6}\hat{p}_2 : \mathbf{BString}_{\text{conn}} \rightarrow \mathbf{B}^7\mathbf{U}(1)_{\text{conn}}$$

from the moduli 2-stack of String-2-connections to the moduli 7-stack of $\mathbf{U}(1)$ -6-gerbes with connection. By the general mechanism described at the beginning of Section 4, this induces a 7-dimensional Chern-Simons theory: for Σ a compact 7-dimensional oriented smooth manifold, define $\exp(iS_{\frac{1}{6}p_2}(-))$ to be the Chern-Simons action functional

$$(4.5.2) \quad \exp(iS_{\frac{1}{6}p_2}(-)) : \mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}}) \xrightarrow{\frac{1}{6}\hat{p}_2} \mathbf{H}(\Sigma, \mathbf{B}^7\mathbf{U}(1)_{\text{conn}}) \xrightarrow{\int_\Sigma} \mathbf{U}(1) .$$

This can be explicitly described as follows. To begin with notice that since the classifying space $B\text{String}$ of principal String bundles is 8-connected, the underlying String bundle to an object in $\mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}})$ is trivial, for any 7-dimensional Σ . Therefore the local differential forms data defining a String connection can actually be chosen to be globally defined. (But this is true only as long as we ignore here for the moment the twist that arises when passing to String^{2a} -connections, as in the previous section.)

Then, recall from Section 3.7 the different incarnations of the local differential form data for string 2-connections. With this in mind we have:

Proposition 4.5.1. (i) *in terms of the strict string Lie 2-algebra $(\hat{\Omega}\mathfrak{so} \rightarrow P_*\mathfrak{so})$, an object in $\mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}})$ is the datum of a pair of nonabelian differential forms $A \in \Omega^1(\Sigma, P_*\mathfrak{so})$, $B \in \Omega^2(\Sigma, \hat{\Omega}_*\mathfrak{so})$ and $\exp(iS_{\frac{1}{6}p_2}(-))$ takes this to*

$$(4.5.3) \quad \exp(iS_{\frac{1}{6}p_2}(A, B)) = \exp\left(2\pi i \int_{\Sigma} \text{CS}_7(A(1))\right) \quad ,$$

where $A(1) \in \Omega^1(\Sigma, \mathfrak{so})$ is the 1-form obtained by evaluating on the endpoint 1 the path Lie algebra-valued 1-form A , and CS_7 is the standard degree-7 Chern-Simons element on \mathfrak{so} from (2.3.14).

(ii) *in terms of the skeletal string Lie 2-algebra \mathfrak{so}_{μ_3} , an object in $\mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}})$ is the datum of a pair of differential forms $A \in \Omega^1(\Sigma, \mathfrak{so})$, $B \in \Omega^2(\Sigma, \mathbb{R})$, and $\exp(iS_{\frac{1}{6}p_2}(-))$ takes this to*

$$(4.5.4) \quad \exp(iS_{\frac{1}{6}p_2}(A, B)) = \exp\left(2\pi i \int_{\Sigma} \text{CS}_7(A)\right) \quad .$$

Notice that, while the 2-form B does not appear explicitly in the integrands on the right, it does nevertheless affect the kinematics of the theory. Its presence forces the connection A to be such that the first Pontrjagin term $\langle F_A \wedge F_A \rangle$ is exact (see [74] and [30] for details).

Note also that the universal differential map $\frac{1}{6}\hat{\mathbf{p}}_2$ plays a role already on the 10-dimensional boundary of spacetime, as the differential twist that induces the Green-Schwarz mechanism in magnetic heterotic String theory [75, 76]. In dimension 10 a String 2-connection field configuration $\phi : X \rightarrow \mathbf{BString}_{\text{conn}}$ is in general far from being given by globally defined differential form data, and the 2-form B appears more prominently in the relevant formulae, see [30, 76].

4.6. 7d CS theory in 11d supergravity on String-2-connection fields

In this section we can put the ingredients together and construct a 7-dimensional Chern-Simons theory induced by the quantum corrected Chern-Simons term in 11-dimensional supergravity (2.3.7). This is a twisted combination of the two 7-dimensional Chern-Simons action functionals from 4.2 and 4.5 which naturally lives on (a higher connected cover of) the moduli 2-stack $C\text{Field}(-)^{\text{bdr}}$ of boundary C -field configurations from Section 4.3.

Recall from the introduction that our task is to find a 7d Chern-Simons functional that is induced from a characteristic class of the form (2.3.5)

$I_8 = \frac{1}{48}(p_2 + (\frac{1}{2}p_1)^2)$ on fields that satisfy a quantization condition (2.3.10) given by $\frac{1}{2}p_1 = 2a$. For the *universal* characteristic class $p_2 + (\frac{1}{2}p_1)^2$ on the universal Spin-bundle over $B\text{Spin}$, the divisibility is (see [64])

$$(4.6.1) \quad 24I_8 = \frac{1}{2}(p_2 - (\frac{1}{2}p_1)^2) \in H^8(B\text{Spin}, \mathbb{Z}).$$

If, however, we fix an 8-dimensional oriented manifold $TX : X \rightarrow BSO(8)$ which admits a Spin-structure

$$(4.6.2) \quad \begin{array}{ccc} & & B\text{Spin}(8) \\ & \nearrow \text{---} & \downarrow \\ X & \xrightarrow{TX} & BSO(8) \end{array},$$

then index theory shows (Equation (3.2) in [92]) that the pullback of the universal class to X has now divisibility by 6

$$(4.6.3) \quad 8I_8(X) = \frac{1}{6}(p_2(X) - (\frac{1}{2}p_1(X))^2) \in H^8(X, \mathbb{Z}).$$

In order to eventually formulate this in the form of canonical Chern-Simpson theories defined on moduli stacks as in Section 4.1, observe that this means that we have the following diagram of differential cohomology sets/groups

$$(4.6.4) \quad \begin{array}{ccccc} & & H(\Sigma, \frac{1}{6}(\hat{p}_2 - (\frac{1}{2}\hat{p}_1)^2)) & \xrightarrow{\quad} & H(\Sigma, \mathbf{B}^7U(1)_{\text{conn}}) \\ & \nearrow & & & \downarrow \cdot 3 \\ \text{Fields}(\Sigma) & \xrightarrow{\quad} & H(\Sigma, \mathbf{B}\text{Spin}_{\text{conn}}) & \xrightarrow{H(\Sigma, \frac{1}{2}\hat{p}_2 - (\frac{1}{2}\hat{p}_1)^2)} & H(\Sigma, \mathbf{B}^7U(1)_{\text{conn}}) \\ & \downarrow & \downarrow & & \\ & & H(\Sigma, B\text{Spin}(8)) & & \\ & \downarrow & \downarrow & & \\ * & \xrightarrow{T\Sigma} & H(\Sigma, \mathbf{B}SO(8)) & & \end{array}$$

Here $\text{Field}(\Sigma)$ denotes the connected components of the homotopy pullback of $\mathbf{H}(\Sigma, \mathbf{B}\text{Spin}_{\text{conn}}) \rightarrow \mathbf{H}(\Sigma, \mathbf{B}SO)$ along $T\Sigma$, which encodes the gauge equivalence classes of Spin-connections on all possible Spin-structures on the tangent bundle TX . The existence of the top arrow expresses the fact that restricted to such fields, the universal differential class $\frac{1}{2}(\hat{p}_2 - (\frac{1}{2}p_1)^2)$ is

further divisible by 3, where up to here the differential refinements (denoted by the hats) can still be given by classical Chern-Weil theory.

However, this is still not the refined Chern-Simons functional (so far on gauge equivalence classes) that we need. While this cannot be further divided, also $\mathbf{Fields}(\Sigma)$ here is not the correct field configuration space yet. The correct configuration space for the boundary values of C -field configurations has to satisfy the quantization condition (2.3.9) saying that $\frac{1}{2}p_1$ is further divisible by 2. By (3.7.4) the classifying space for such configurations is $BString^{2DD}$, whose smooth and differential refinement is the moduli 2-stack $\mathbf{BString}_{\text{conn}}^{2DD}$, which is such that a map $X \rightarrow \mathbf{BString}_{\text{conn}}^{2DD}$ is a Spin-connection ω and a choice of divisibility morphism of the corresponding Pontrjagin class. So the quantization condition may be implemented in a gauge equivariant way by replacing in the above the moduli stack $\mathbf{BSpin}_{\text{conn}}$ with the moduli 2-stack $\mathbf{BString}_{\text{conn}}^{2DD}$. Then by the discussion on p. 9 of [92] we then have

(4.6.5)

$$\begin{array}{ccccc}
 & & H(\Sigma, \frac{1}{48}(\hat{p}_2 - (\frac{1}{2}\hat{p}_1)^2)) & \xrightarrow{\quad} & H(\Sigma, \mathbf{B}^7U(1)_{\text{conn}}) \\
 & \nearrow & & & \downarrow \cdot 24 \\
 \mathbf{Fields}_{\frac{1}{2}\lambda}(\Sigma) & \xrightarrow{\quad} & H(\Sigma, \mathbf{BString}_{\text{conn}}^{2DD}) & \xrightarrow{H(\Sigma, \frac{1}{2}\hat{p}_2 - (\frac{1}{2}\hat{p}_1)^2)} & H(\Sigma, \mathbf{B}^7U(1)_{\text{conn}}) \ . \\
 \downarrow & & \downarrow & & \\
 * & \xrightarrow{T\Sigma} & H(\Sigma, \mathbf{BSO}(8)) & &
 \end{array}$$

Here $\mathbf{Fields}_{\frac{1}{2}\lambda}$ is the connected components of the homotopy pullback of $\mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}}^{2DD}) \rightarrow \mathbf{H}(\Sigma, \mathbf{BSO}(8))$ along $T\Sigma$, which encodes the gauge equivalence classes of $String^{2DD}$ -2-connections on the tangent bundle of X . The curved morphism indicates that on this configuration space now the 7d Chern-Simons action has the full divisibility with prefactor $\frac{1}{48}$. In order to add dynamical E_8 -gauge fields on the boundary we invoke the canonical morphism $\mathbf{BString}^{2a} \rightarrow \mathbf{BString}^{2DD}$ which implements the condition that $\frac{1}{2}\hat{p}_1$ is not just divisible further by 2, but that half of it is the class of a given E_8 -bundle with given E_8 -connection.

In conclusion, the canonical boundary 7d Chern-Simons functional induced by the quantum correction term (2.3.7) and consistent at any given level, hence consistent for any number of 5-branes $N \in \mathbb{N}$ is the composite functional

$$\begin{array}{ccc}
 \mathbf{H}(\Sigma, \mathbf{CField}^{\text{bdr}}) & \xrightarrow{L_{I_8}} & \mathbf{H}(\Sigma, \mathbf{B}^7\mathbf{U}(1)_{\text{conn}}) \\
 \downarrow \simeq & & \downarrow \cdot 24 \\
 \mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}}^{2\mathbf{a}}) & \longrightarrow \mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}}^{2\mathbf{DD}2}) \xrightarrow{L_{24I_8}} & \mathbf{H}(\Sigma, \mathbf{B}^7\mathbf{U}(1)_{\text{conn}}) \\
 & & \downarrow \exp(2\pi i \int_{\Sigma} (-)) \\
 & & \mathbf{U}(1) .
 \end{array}$$

Notice here that, by the high connectedness of E_8 , we have over a 7-dimensional Σ that

$$(4.6.6) \quad \mathbf{H}(\Sigma, \mathbf{BE}_8) \rightarrow \mathbf{H}(\Sigma, \mathbf{B}^3\mathbf{U}(1))$$

is an isomorphism on gauge equivalence classes, identifying E_8 -instantons sectors with higher magnetic charge sectors of 3-bundles / 2-gerbes. But since as *smooth stacks* \mathbf{BE}_8 is different from $\mathbf{B}^3\mathbf{U}(1)$ (the gauge transformations are very different!) this is not an equivalence of the 2-groupoids of gauge transformations and gauge-of-gauge transformations. Moreover, the differential refinement

$$(4.6.7) \quad \mathbf{H}(\Sigma, (\mathbf{BE}_8)_{\text{conn}}) \rightarrow \mathbf{H}(\Sigma, \mathbf{B}^3\mathbf{U}(1)_{\text{conn}})$$

is not even an isomorphism on gauge equivalence classes: E_8 -gauge fields are much more refined than the 3-form Chern-Simons gauge fields induced from them. For these reasons, also the morphism

$$(4.6.8) \quad \mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}}^{2\mathbf{a}}) \rightarrow \mathbf{H}(\Sigma, \mathbf{BString}_{\text{conn}}^{2\mathbf{DD}})$$

appearing in the above composition is far from being an equivalence of of gauge field configuration data, even though the gauge equivalence classes of the underlying instanton/charge sectors are canonically identified.

By the discussion in Section 3.6 we have the Lagrangian $L_{I_8} := \mathbf{H}(\Sigma, \hat{\mathbf{I}}_8)$ universally defined, by higher Lie integration, on the universal 7-connected cover moduli 2-stack $\widehat{\mathbf{BString}}_{\text{conn}}^{2\mathbf{a}}$ of the moduli 2-stack $\mathbf{BString}_{\text{conn}}^{2\mathbf{a}}$, and the Lagrangian down on the latter is obtained by consistent quotienting from that. By the discussion in Section 4.1 the action $\exp(iS_{I_8})$ is guaranteed to locally be of the required form (2.3.12), while at the same time having the correct global properties and be gauge invariant under higher gauge

transformations of nonabelian String^{2a} 2-form connections fields, by level quantization.

An explicit presentation of the full Lagrangian on $\widehat{\mathbf{BString}}^{2a}_{\text{conn}}$ can be constructed directly with the tools in [30]. Instead of going here through the full details, which are spelled out there in general, we close by pointing out how the full functional $\exp(iS_{\mathbf{I}_8})$ restricts to the special cases which we discussed before. Namely, by the universal property of the homotopy pull-back, and by the “pasting law” for homotopy pullbacks we have a canonical morphism of moduli 2-stacks

$$(4.6.9) \quad \mathbf{BString} \rightarrow \mathbf{BString}^{2a}$$

given as the universal dashed arrow in the diagram

$$(4.6.10) \quad \begin{array}{ccc} \mathbf{BString} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{BString}^{2a} & \longrightarrow & \mathbf{BE}_8 \\ \downarrow & & \downarrow^{2a} \\ \mathbf{BSpin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3\mathbf{U}(1) \end{array} .$$

The action functional $\exp(iS_{\mathbf{I}_8})$ may be restricted along this morphism to the configuration 2-stack of untwisted String-2-connections. This restriction then coincides with the “indecomposable” 7-dimensional Chern-Simons functional $\exp(iS_{\frac{1}{6}\mathbf{p}_2})$ discussed in Section 4.5.

Acknowledgments

The research of H.S. is supported by NSF Grant PHY-1102218. U.S. gratefully acknowledges an invitation to the Department of Mathematics of Pittsburgh University in September 2011, which led to the work presented here.

Appendix: Two models for **string**, six models for **String**

In ordinary gauge theory, any two different but equivalent incarnations of, say, a Lie algebra are related by an *isomorphism*. In practice this typically appears simply as a linear transformation between two choices of basis of the underlying vector space, and is typically easily recognized and known as such. In contrast, there is a somewhat subtle new phenomenon that appears

when passing beyond ordinary nonabelian gauge theory to higher nonabelian gauge theory. Due to the higher gauge freedom, we have in general a plethora of possibly very different looking but nevertheless equivalent incarnations of a given higher gauge group, its Lie n -algebra and hence of the local differential form data of higher gauge fields. This accounts for a good bit of the subtlety of higher nonabelian gauge theory, the effects of which have not always been dealt with appropriately in existing proposals, in particular concerning the characterization and identification of what it means to have a “nonabelian 2-form theory”.

Discussing this requires a slightly higher level of mathematical detail than we wanted to use in the main text here, since these can be found discussed extensively in our previous publications. But because awareness of this phenomenon helps to put the higher nonabelian gauge theories discussed here in the proper perspective, we here briefly review some relevant facts for the case of the String-2-group and its Lie 2-algebra \mathbf{string} (see here Section 3.7 for more discussion). Analogous comments would apply to their twisted versions String^{2a} and \mathbf{string}^{es} from Section 3.8.

First we consider two different incarnations of the Lie 2-algebra \mathbf{string} , from one of main theorems in [7]. Let \mathfrak{g} be a semisimple Lie algebra. Write $\langle -, - \rangle : \mathfrak{g}^{\otimes 2} \rightarrow \mathbb{R}$ for its Killing form and

$$(A.1) \quad \mu = \langle -, [-, -] \rangle : \mathfrak{g}^{\otimes 3} \rightarrow \mathbb{R}$$

for the canonical 3-cocycle Lie algebra cocycle.

Definition A.1 (skeletal version of \mathbf{string}). Write \mathfrak{g}_μ for the Lie 2-algebra whose underlying graded vector space is

$$\mathfrak{g}_\mu = \mathfrak{g} \oplus \mathbb{R}[-1],$$

and whose nonvanishing brackets are defined as follows.

- The binary bracket is that of \mathfrak{g} when both arguments are from \mathfrak{g} and 0 otherwise.
- The ternary bracket is the 3-cocycle

$$[-, -, -]_{\mathfrak{g}_\mu} := \langle -, [-, -] \rangle : \mathfrak{g}^{\otimes 3} \rightarrow \mathbb{R}.$$

Definition A.2 (strict version of \mathbf{string}). Write $(\hat{\Omega}\mathfrak{g} \xrightarrow{h} P_*\mathfrak{g})$ for the Fréchet Lie 2-algebra whose underlying vector space is

$$(\hat{\Omega}\mathfrak{g} \rightarrow P_*\mathfrak{g}) = P_*\mathfrak{g} \oplus (\Omega\mathfrak{g} \oplus \mathbb{R})[-1],$$

where $P_*\mathfrak{g}$ is the vector space of smooth maps $\gamma : [0, 1] \rightarrow \mathfrak{g}$ such that $\gamma(0) = 0$, and where $\Omega\mathfrak{g}$ is the subspace for which also $\gamma(1) = 0$, and whose non-vanishing brackets are defined as follows

- $[-]_1 = \partial := \Omega\mathfrak{g} \oplus \mathbb{R} \rightarrow \Omega\mathfrak{g} \hookrightarrow P_*\mathfrak{g}$;
- $[-, -] : P_*\mathfrak{g} \otimes P_*\mathfrak{g} \rightarrow P_*\mathfrak{g}$ is given by the pointwise Lie bracket on \mathfrak{g} as

$$[\gamma_1, \gamma_2] = (\sigma \mapsto [\gamma_1(\sigma), \gamma_2(\sigma)]);$$

- $[-, -] : P_*\mathfrak{g} \otimes (\Omega\mathfrak{g} \oplus \mathbb{R}) \rightarrow \Omega\mathfrak{g} \oplus \mathbb{R}$ is given by pairs

$$(A.2) \quad [\gamma, (\ell, c)] := \left([\gamma, \ell], 2 \int_0^1 \langle \gamma(\sigma), \frac{d\ell}{d\sigma}(\sigma) \rangle d\sigma \right),$$

where the first term is again pointwise the Lie bracket in \mathfrak{g} .

Proposition A.1. *The linear map*

$$P_*\mathfrak{g} \oplus (\Omega\mathfrak{g} \oplus \mathbb{R})[-1] \rightarrow \mathfrak{g} \oplus \mathbb{R}[-1],$$

which in degree 0 is evaluation at the endpoint

$$\gamma \mapsto \gamma(1)$$

and which in degree 1 is projection onto the \mathbb{R} -summand, induces an equivalence of Lie 2-algebras

$$(\hat{\Omega}\mathfrak{g} \rightarrow P_*\mathfrak{g}) \simeq \mathfrak{g}_\mu .$$

This is Theorem 30 in [7].

By Section 3.4.1 in [79], Lie n -algebras \mathfrak{g} for all n can naturally be understood as *infinitesimal* smooth n -stacks $b\mathfrak{g} \hookrightarrow \mathbf{B}G$, forming the infinitesimal neighbourhood of the canonical base point in the moduli n -stacks $\mathbf{B}G$ integrating them. In this context, any Lie algebra n -cocycle μ is a morphism of infinitesimal n -stacks $\mu : b\mathfrak{g} \rightarrow b^n\mathbb{R}$, explicitly models for which are discussed in [74][30]. Therefore we may ask for the *homotopy fiber* of a Lie algebra cocycle and make the following definition.

Definition A.3. For \mathfrak{g} a semisimple Lie algebra as in (A.1), the Lie 2-algebra $\mathbf{string}(\mathfrak{g})$ is, up to weak equivalence, the loop space object of the

homotopy fiber $b\mathbf{string}(\mathfrak{g})$ in

$$\begin{array}{ccc} b\mathbf{string}(\mathfrak{g}) & \longrightarrow & * \\ \downarrow & & \downarrow \\ b\mathfrak{g} & \xrightarrow{\mu} & b^3\mathbb{R} \end{array} .$$

For the case $\mathfrak{g} = \mathfrak{so}_N$ we write for short

$$\mathbf{string} := \mathbf{string}(\mathfrak{so}) .$$

Notice the analogy to (3.7.1). With this we have

Proposition A.2. *With \mathfrak{g} as above, both \mathfrak{g}_μ as well as $(\hat{\Omega}\mathfrak{g} \rightarrow P_*\mathfrak{g})$ are equivalent incarnations of $\mathbf{string}(\mathfrak{g})$.*

Remark A.1. The two models \mathfrak{g}_μ and $(\hat{\Omega}\mathfrak{g} \rightarrow P_*\mathfrak{g})$ are at two opposite extremes of all possible models: while \mathfrak{g}_μ is singled out by having trivial unary bracket, $(\hat{\Omega}\mathfrak{g} \rightarrow P_*\mathfrak{g})$ is singled out by having trivial trinary bracket. Analogous statements apply to the models of the String 2-group, to which we now turn.

In direct analogy to how Lie algebras integrated to Lie groups in classical Lie theory, so higher Lie algebras integrate to higher smooth groups in higher Lie theory.

Proposition A.3. *The degreewise ordinary Lie integration of the differential crossed module $(\hat{\Omega}\mathfrak{so} \rightarrow P_*\mathfrak{so})$ yields the Fréchet Lie crossed module $(\hat{\Omega}\mathrm{Spin} \rightarrow P_*\mathrm{Spin})$, where $\hat{\Omega}\mathrm{Spin}$ is the level-1 Kac-Moody central extension of the smooth loop group of Spin . This is naturally a strict Fréchet Lie 2-group.*

The nontrivial part to check is that the action of $P_*\mathfrak{so}$ on $\hat{\Omega}\mathfrak{so}$ lifts to a compatible action of $P_*\mathrm{Spin}$ on $\hat{\Omega}\mathrm{Spin}$ which lifts the infinitesimal action and such as to satisfy the axioms of a crossed module. This is prop. 24 in [7]. In the above, the group operation in P_*G and in ΩG is the *pointwise* multiplication of parameterized paths in G , which in $\hat{\Omega}G$ is twisted by the action of a 2-cocycle on loops. There are two evident variants of this.

- One may consider forming *thin homotopy* equivalence classes of paths in G , which form a group not under pointwise multiplication, but under

composition. The corresponding strict 2-group

$$(A.3) \quad (\hat{\Omega}_{\text{th}}G \rightarrow P_{\text{th}}G)$$

is constructed in def. 4.1.20 of [79].

- One may consider a different cocycle on the loop group, known as *Mickelsson's cocycle*. This yields a strict 2-group $\text{String}_{\text{Mick}}$ given in prop. 4.1.26 of [79].

One may also form a universal higher Lie integration of \mathfrak{g}_μ as in [43], which in [30] was put in the context of higher smooth stacks as used here. This yields a weak smooth 2-group $\Omega\tau_2 \exp(\mathfrak{g}_\mu)$.

Theorem A.1. *All these 2-groups are equivalent models for the smooth 2-group String as defined in (3.7.1), as are their smooth moduli 2-stacks.*

This is Theorem 4.1.29 in [79].

There are further, very different looking models. In [55] a strict model is given whose degree-0 group is the actual topological string 2-group, but equipped with a smooth structure, and whose degree-1 group is is contractible group. All these models so far are degreewise presented by infinite-dimensional smooth spaces. In [77] an algorithm is given for constructing models of String by degreewise finite dimensional manifolds. In [88] this is explicitly related to the construction of *multiplicative bundle gerbes* on the group. This construction has at times been motivated as a plausible prerequisite for a tractable discussion of the differential geometry of String -geometry. But notice two things

1. Constructions as in [30] show that, contrary to that expectation, it is the model $\Omega\tau_2 \exp(\mathfrak{g}_\mu)$ which is most directly accessible by ordinary differential geometric methods and differential form computations as in [74]. Most of the local formulae for String 2-connections and their twisted version that we used here are constructed and controled by this model. One may understand this from the fact that, while when thought of as a graded (simplicial) space this model is degreewise infinite-dimensional, when thought of as a presentation of a higher stack — which is what we are actually interested in — then its *probes* by maps $X \rightarrow \Omega\tau_2 \exp(\mathfrak{g}_\mu)$ from a smooth manifold X are characterized by classical and tractable differential form data on X , which has immediate and useful interpretation in physics.

2. If one simply drops the requirement that a model has Kan fillers (which is not a necessary requirement) and allows spaces with arbitrary many connected components, then *every* higher smooth stack has a model that is degreewise a finite-dimensional smooth manifold. This is prop. 2.1.49 in [79]. The construction given there also shows that, while they are guaranteed to exist, these degreewise finite-dimensional models are typically not useful for practical computations with local differential form data. Instead, as shown there, their existence is most useful for abstract considerations in the homotopy theory of higher stacks, because they serve as *cofibrant models*.

On the other hand, of course every concrete model for a higher group has its advantages and disadvantages. The more models one has, the more aspects one sees of the abstractly defined higher group for which all these models are, after all, models. From the point of view of the corresponding higher gauge theory, the equivalences between the different models play the role of higher gauge transformations between higher gauges. Already from ordinary gauge theory it is a familiar fact that different gauges have their different uses, and the more of them are under control, the better for understanding the intrinsic, gauge invariant, nature of the theory.

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