On the mechanics of crystalline solids with a continuous distribution of dislocations

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Abstract

We formulate the laws governing the dynamics of a crystalline solid in which a continuous distribution of dislocations is present. Our formulation is based on new differential geometric concepts, which in particular relate to Lie groups. We then consider the static case, which describes crystalline bodies in equilibrium in free space. The mathematical problem in this case is the free minimization of an energy integral, and the associated Euler–Lagrange equations constitute a nonlinear elliptic system of partial differential equations. We solve the problem in the simplest cases of interest.

Introduction and summary

Classical elasticity theory rests on the hypothesis that a solid may exist in a stress-free state in Euclidean space. The dynamics of the solid is then

described by a time-dependent mapping taking the position of each material particle in this relaxed reference state to its actual position at a given time. Now the hypothesis of the existence of a relaxed state is violated when dislocations are present in the solid. These cause internal stresses even in the absence of external forces.

Solids in which a continuous distribution of dislocations is present have been treated in the linear approximation [10,11,13]. However, the conceptual framework of this linear theory is inadequate for the formulation of an exact theory. The appropriate conceptual framework was introduced in [5] and an exact nonlinear theory was proposed. The basic concept introduced in [5] is that of the material manifold, which captures those properties of a crystalline solid which are intrinsic to it, being independent of the way it is embedded in space. The dynamics of the elastic body is then described by a mapping from space-time into this material manifold.

While an exact theory of crystalline solids with a continuous distribution of dislocations was formulated in [5], the theory was left undeveloped up to the present time. The aim of the present work is to develop the theory and derive results which may be brought into contact with experimental data. Our focus in the present paper is on the static case, where we have a crystalline solid with a uniform distribution of elementary dislocations in equilibrium in free space. As a basic example we study, in the continuum limit, a two-dimensional (2d) crystalline solid with a uniform distribution of edge dislocations, one of the two elementary types of dislocations. In this case, the material manifold is the affine group of the real line, the hyperbolic plane. We then study, in the continuum limit, a three-dimensional (3d) crystalline solid with a uniform distribution of screw dislocations, the other elementary type of dislocation. In this case, the material manifold is the Heisenberg group.

The purpose of the present work is to introduce to the mathematics and theoretical physics community a field where beautiful differential geometric structures, in particular Lie groups, form the basis of a classical physical theory. Moreover, the laws of the theory form a nonlinear system of variational partial differential equations, which in the static case is elliptic and in the dynamical case is of hyperbolic type. Both cases constitute a worthy challenge for the geometric analyst.

This paper is organized as follows. In Part I, we give a general introduction to the theory of crystalline solids containing an arbitrary distribution of dislocations, based on the work of Christodoulou [4], [5]. We introduce the basic concepts of material manifold and crystalline structure. In the present work, the fundamental notion is the canonical form, which is used

to define the dislocation density. We illustrate the theory by giving two basic examples of solids with a uniform distribution of dislocations. In order to state the laws governing the dynamics, the thermodynamic space is introduced and the energy function defined. The dynamics is described by a mapping from space-time into the material manifold. The equivalence relations for crystalline structures and for the mechanical properties of a solid are discussed to give the proper physical interpretation of the theory. Finally, the Eulerian picture is given including the non-relativistic limit. In the Eulerian picture, the material manifold is eliminated.

In Part II, we focus our attention on the static case. We derive the boundary value problem from an action principle and give the Legendre–Hadamard conditions for the energy function. We then consider the two examples of uniform distributions of elementary dislocations and motivate the choice of our model energy function. We separately discuss the special cases of uniform distributions of edge and screw dislocations in two and three dimensions, respectively. In concluding the second part, we derive the scaling properties of the theory.

Part III is devoted to the analysis of equilibrium configurations of a crystalline solid with a uniform distribution of elementary dislocations in two and three dimensions. We solve the problem in two dimensions and give the method for the solution of the anisotropic problem in the case of a uniform distribution of screw dislocations in three dimensions.

Part I The general setting

1 The material manifold

Let $\mathcal{N}$ be an oriented n-dimensional differentiable manifold, called the **material manifold**. It describes a material together with those of its properties which are intrinsic to it being independent of the way in which it is to be embedded in physical space. A point $y \in \mathcal{N}$ represents a particle. We denote by $\mathcal{X}(\mathcal{N})$ the C^∞-vectorfields on the material manifold $\mathcal{N}$. Let

$$\begin{aligned}
\epsilon_y : \mathcal{X}(\mathcal{N}) &\to T_y\mathcal{N} \\
X &\mapsto \epsilon_y(X) = X(y)
\end{aligned} \tag{1.1}$$

be the evaluation map at a point $y \in \mathcal{N}$.

Definition I.1. *A* **crystalline structure** *on $\mathcal{N}$ is a distinguished linear subspace $\mathcal{V}$ of $\mathcal{N}(\mathcal{N})$ such that the evaluation map ϵ_y restricted to $\mathcal{V}$ is an isomorphism for each $y \in \mathcal{N}$.*

Remark I.1. *The orientation of $\mathcal{N}$ induces an orientation in $\mathcal{V}$ such that ϵ_y is orientation preserving. $\mathcal{N}$ admits a crystalline structure if and only if $\mathcal{N}$ is parallelizable, see [5].*

We introduce a 1-form ν on $\mathcal{N}$ with values in $\mathcal{V}$ defined by

$$\nu_y(Y_y) = \epsilon_y^{-1}(Y_y) \quad : \forall Y_y \in T_y\mathcal{N}. \tag{1.2}$$

In the case that $(\mathcal{N}, \mathcal{V})$ is a Lie group with corresponding Lie algebra, ν is the *Maurer–Cartan* form.

Remark I.2. *The most fundamental object is the 1-form ν. We may in fact replace $\mathcal{V}$ by an abstract real vectorspace W of the same dimension, n, as the manifold $\mathcal{N}$. Then we define a canonical form ν to be a W-valued 1-form on $\mathcal{N}$ such that*

$$\nu_y := \nu|_{T_y\mathcal{N}} : T_y\mathcal{N} \to W$$

is an isomorphism for each $y \in \mathcal{N}$. Given an element $v \in W$, we may then define a vectorfield Y_v on $\mathcal{N}$ by

$$Y_v(y) = \nu_y^{-1}(v) : \quad \forall y \in \mathcal{N}.$$

We then define the crystalline structure $\mathcal{V}$ by

$$\mathcal{V} = \{X_v : v \in W\}.$$

Then the canonical form corresponds to the 1-form ν defined in (1.2).

We say that the crystalline structure $\mathcal{V}$ on $\mathcal{N}$ is **complete** if each $X \in \mathcal{V}$ is a complete vectorfield on $\mathcal{N}$. Then each element of $\mathcal{V}$ generates a 1-parameter group of diffeomorphisms of $\mathcal{N}$, which represents (in the continuum limit) a group of translations of the crystal lattice with parameter proportional to the number of atoms traversed.

A complete crystalline structure $\mathcal{V}$ on the material manifold $\mathcal{N}$ defines an **exponential map**

$$\exp : \mathcal{N} \times \mathcal{V} \to \mathcal{N}$$

as follows. Let $\exp(y, X)$ be the point in $\mathcal{N}$ that is at parameter value 1 from y along the integral curve of X initiating at y. For each $y \in \mathcal{N}$, let

$$\exp_y : \mathcal{V} \to \mathcal{N}$$
$$X \mapsto \exp_y(X) = \exp(y, X).$$

We have

$$\exp_y(0) = y, \quad d\exp_y(0) = \epsilon_y.$$

Thus, $d\exp_y(0)$ is an isomorphism for each $y \in \mathcal{N}$. By the implicit function theorem it follows that, for each $y \in \mathcal{N}$, there is a neighborhood $\mathcal{U}_y$ of the zero vector in $\mathcal{V}$ such that $\exp_y$ restricted to $\mathcal{U}_y$ is a diffeomorphism onto its image in $\mathcal{N}$.

Now choose a totally antisymmetric n-linear form ω on $\mathcal{V}$ which is positive when evaluated on a positive basis. The n-form ω defines a **volume form** $d\mu_\omega$, called **mass form** on $\mathcal{N}$ by:

$$d\mu_\omega\,(Y_{1,y}, \ldots, Y_{n,y}) = \omega\left(\epsilon_y^{-1}(Y_{1,y}), \ldots, \epsilon_y^{-1}(Y_{n,y})\right)$$
$$: \forall y \in \mathcal{N}, \forall Y_{1,y}, \ldots, Y_{n,y} \in T_y\mathcal{N}. \tag{1.3}$$

The volume assigned by $d\mu_\omega$ to a domain $\mathcal{D} \subset \mathcal{N}$,

$$\int_{\mathcal{D}} d\mu_\omega,$$

is the rest mass of $\mathcal{D}$.

Definition I.2. *Given a crystalline structure $\mathcal{V}$ on $\mathcal{N}$ we can define a mapping*

$$\Lambda : \mathcal{N} \to \mathcal{L}(\mathcal{V} \wedge \mathcal{V}, \mathcal{V})$$

by:

$$\Lambda(y)(X, Y) = \epsilon_y^{-1}\left([X, Y](y)\right) \in \mathcal{V}, \forall y \in \mathcal{N}, X, Y \in \mathcal{V}. \tag{1.4}$$

*We call Λ **dislocation density**.*

Suppose $X, Y \in \mathcal{V}$ are complete and generate 1-parameter groups of diffeomorphisms Φ_t, Ψ_t, $t \in \mathbb{R}$ from $\mathcal{N}$ to $\mathcal{N}$. For $y \in \mathcal{N}$

$$(t, s) \mapsto \Psi_{-s}\left(\Phi_{-t}\left(\Psi_s\left(\Phi_t(y)\right)\right)\right)$$

coincides to order ts with $(t, s) \mapsto \Xi_{ts}$, where Ξ_t is the 1-parameter group of diffeomorphisms of $\mathcal{N}$ generated by $\Lambda(y)(X, Y)$.

The dislocation density is a concept that arises in the continuum limit of a distribution of elementary dislocations in a crystal lattice. An elementary dislocation has the property that, if we start at an atom in the crystal lattice

and move according to one group of lattice transformations k atoms in one direction, then according to a different group l atoms in a second direction, according to the first $-k$ atoms and finally $-l$ atoms in the second directions, then we arrive at a different atom than we started from, but which, provided the circuit encloses a single elementary dislocation, is reached at in a single step corresponding to a lattice translation. This step is called **Burger's vector**.

If Λ is constant on $\mathcal{N}$ then, for all $X, Y \in \mathcal{V}$, there is a $Z \in \mathcal{V}$ such that

$$[X, Y] = Z,$$

corresponding to a uniform distribution of elementary dislocations of the same kind, see below. Thus $\mathcal{V}$ constitutes in this case a **Lie algebra**, i.e., a vectorspace $\mathcal{V}$ over $\mathbb{R}$ with a bracket

$$[., .] : \mathcal{V} \wedge \mathcal{V} \to \mathcal{V}$$

satisfying the Jacobi identity.

By the fundamental theorems of Lie group theory, upon choosing an identity element $e \in \mathcal{N}$, the material manifold $\mathcal{N}$ can then be given the structure of a **Lie group** such that $\mathcal{V}$ is the space of vectorfields on $\mathcal{N}$ which generate the right action of the group on itself. $\mathcal{V}$ is then the space of vectorfields on $\mathcal{N}$ which are left invariant, i.e., invariant under left group multiplication. The dual space $\mathcal{V}^*$ is then the space of left invariant 1-forms on $\mathcal{N}$.

Let us consider $d\nu$, a 2-form on $\mathcal{N}$: for any pair of vectorfields X, Y on $\mathcal{N}$ we have

$$d\nu(X, Y) = X(\nu(Y)) - Y(\nu(X)) - \nu([X, Y]) \qquad (1.5)$$

by the formula for the exterior derivative of a 1-form. In particular, this holds for $X, Y \in \mathcal{V}$. Now for $X \in \mathcal{V}$ we have

$$\nu(X)(y) = \nu_y(X_y) = \epsilon_y^{-1}(X_y) = X, \quad \forall y \in \mathcal{N}, X \in \mathcal{V},$$

a constant $\mathcal{V}$-valued function on $\mathcal{N}$. Similarly with X replaced by Y. Therefore, from (1.5) we have

$$d\nu(X, Y) = -\nu([X, Y]), \forall X, Y \in \mathcal{V}. \qquad (1.6)$$

On the other hand,

$$\Lambda(y)(X, Y) = \epsilon_y^{-1}([X, Y](y)) = \nu_y([X, Y](y)), \quad \forall y \in \mathcal{N}, X, Y \in \mathcal{V}. \quad (1.7)$$

Thus, comparing (1.6) and (1.7), we obtain

$$(d\nu(X,Y))(y) = -\nu_y([X,Y](y)) = -\Lambda(y)(X,Y). \qquad (1.8)$$

Let us then define the $\mathcal{V}$-valued 2-form λ on $\mathcal{N}$ by:

$$\lambda_y(Y_{1,y}, Y_{2,y}) = \Lambda(y)\left(\epsilon_y^{-1}(Y_{1,y}), \epsilon_y^{-1}(Y_{2,y})\right): \quad \forall Y_{1,y}, Y_{2,y} \in T_y\mathcal{N},$$

at any point $y \in \mathcal{N}$. We conclude from (1.8) that

$$d\nu = -\lambda.$$

Let Γ be a closed curve in $\mathcal{N}$ and let Σ be any surface spanning Γ, i.e., $\partial\Sigma = \Gamma$. We finally conclude

$$-\int_\Gamma \nu = \int_\Sigma \lambda. \qquad (1.9)$$

The right-hand side of (1.9) is the sum of all Burger's vectors enclosed by the curve Γ (or threading Σ).

1.1 Uniform dislocation distributions and lie groups

At the atomic level, two kinds of elementary dislocations are found. They are called **edge** and **screw** dislocations. In the following, we consider these two types of dislocations taking them as our model cases. For a uniform distribution of these two types of elementary dislocations, we determine the corresponding Lie groups, the affine group and the Heisenberg group, respectively. For a detailed description see [8].

1.1.1 Edge dislocations and the affine group

The most basic type of a dislocation in a 2d crystal lattice is an edge dislocation. It appears in a 2d lattice in which an extra half-line of atoms has been inserted along the positive 1st axis. A circuit of translations in the directions of the 1st and 2nd axis, alternately, which encloses the origin, ends at an atom, which is reached in a single step by a translation in the direction of the 2nd axis. On the other hand, circuits not enclosing the origin close. Mathematically, this phenomenon is represented by the commutation relation $[E_1, E_2] = E_2$, where E_1, E_2 are the vectorfields along the coordinate axis.

We want to show that a uniform distribution of edge dislocations in a 2d lattice gives rise in the continuum limit to the **affine group**. This group is characterized by transformations of the real line of the form

$$\mathbb{R} \to \mathbb{R}$$
$$x \mapsto e^{y^1} x + y^2,$$

where $(y^1, y^2) \in \mathbb{R}^2$ are two parameters. The subgroups of the affine group are $t \mapsto e^{y^1} t$ (multiplication) and $s \mapsto s + y^2$ (translation).

We have as the group manifold $\mathbb{R}^2$ equipped with the following multiplication

$$(y^1, y^2)(\tilde{y}^1, \tilde{y}^2) = (y^1 + \tilde{y}^1, y^2 + e^{y^1} \tilde{y}^2).$$
$$X = \frac{\partial}{\partial y^1} \quad \text{and} \quad Y = e^{y^1} \frac{\partial}{\partial y^2}$$

generate the right action with

$$[X, Y] = \frac{\partial}{\partial y^1} e^{y^1} \frac{\partial}{\partial y^2} - e^{y^1} \frac{\partial}{\partial y^2} \frac{\partial}{\partial y^1} = e^{y^1} \frac{\partial}{\partial y^2} = Y.$$

The Lie algebra of the affine group is thus generated by the vectorfields X, Y, which satisfy the commutation relation

$$[X, Y] = Y.$$

Therefore, the affine group is the material manifold $\mathcal{N}$ endowed with the crystalline structure.

If we take $\{E_1 = X, E_2 = Y\}$ as a basis of $\mathcal{V}$, we have the following dual basis $\{\omega^1, \omega^2\}$ for $\mathcal{V}^*$

$$\omega^1 = dy^1, \quad \omega^2 = e^{-y^1} dy^2.$$

The corresponding left invariant metric

$$\overset{\circ}{n} = (\omega^1)^2 + (\omega^2)^2 = (dy^1)^2 + e^{-2y^1}(dy^2)^2 \tag{1.10}$$

on $\mathcal{N}$ makes $\mathcal{N}$ the **hyperbolic plane**.

The dislocation density λ is

$$\lambda(E_1, E_2)(y) = \Lambda(y)(E_1, E_2) = \epsilon_y^{-1}\left([E_1, E_2](y)\right) = \epsilon_y^{-1}\left(\epsilon_y(E_2)\right) = E_2.$$

Since $\omega_1 \wedge \omega_2 = e^{-y_1}\left(dy^1 \wedge dy^2\right)$, it follows

$$\lambda = e^{-y_1}\left(dy^1 \wedge dy^2\right) E_2,$$

and therefore

$$\int_\Sigma \lambda = \int_\Sigma e^{-y_1}\left(dy^1 \wedge dy^2\right) E_2 = A(\Sigma)E_2,$$

where $A(\Sigma)$ is the area of the surface Σ, a domain in $\mathcal{N}$. This makes sense since the sum of the Burger vectors associated to a domain in a uniform distribution of edge dislocations should be proportional to the area of the domain.

1.1.2   Screw dislocations and the Heisenberg group

The second kind of elementary dislocation is called a **screw dislocation**. It appears in a 3d lattice in the following way. A circuit of translations along the direction of the 1st and 2nd axis, alternately, which encloses the 3rd, ends at an atom which is reached at a single step by a translation in the direction of the 3rd axis, while circuits not enclosing the 3rd axis close. Mathematically, this phenomenon is represented by the commutation relations $[E_1, E_2] = E_3$, $[E_1, E_3] = [E_2, E_3] = 0$, where E_1, E_2, E_3 are the vectorfields along the coordinate axis.

We want to show that a uniform distribution of dislocations of the screw type give rise in the continuum limit to the **Heisenberg group**. This group is represented as a group of unitary transformations on the space of square integrable complex valued functions Ψ on $\mathbb{R}$ as follows:

$$L^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$$
$$\Psi(x) \mapsto \Psi'(x) = e^{i(y^2 x + y^3)}\Psi(x + y^1),$$

where $(y^1, y^2, y^3) \in \mathbb{R}^3$ are three parameters. The subgroups of the Heisenberg group are $t \mapsto \Psi(x + t)$ (translation in position), $s \mapsto e^{isx}\Psi(x)$ (translation in momentum), and $u \mapsto e^{iu}\Psi(x)$ (multiplication by a phase).

We have as the group manifold $\mathbb{R}^3$ equipped with the following multiplication:

$$(y^1, y^2, y^3)(\tilde{y}^1, \tilde{y}^2, \tilde{y}^3) = (y^1 + \tilde{y}^1, y^2 + \tilde{y}^2, y^3 + \tilde{y}^3 + y^1 \tilde{y}^2).$$

$$X = \frac{\partial}{\partial y^1}, \quad Y = \frac{\partial}{\partial y^2} + y^1 \frac{\partial}{\partial y^3} \quad \text{and} \quad Z = \frac{\partial}{\partial y^3}$$

generate the right action with

$$[X, Y] = \left[\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} + y^1 \frac{\partial}{\partial y^3} \right] = \frac{\partial}{\partial y^3} = Z, \quad [X, Z] = [Y, Z] = 0.$$

The Heisenberg group is thus generated by the vectorfields $E_1 = X, E_2 = Y, E_3 = Z$, which fulfill the commutation relations

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0,$$

and generate the right multiplication. The linear span of (E_1, E_2, E_3) forms a Lie algebra (crystalline structure) corresponding to a uniform distribution of screw dislocations in a 3d crystal lattice. Therefore, we can associate the Heisenberg group, which is the group corresponding to the crystalline structure, with the material manifold $\mathcal{N}$.

If we take $\{E_1, E_2, E_3\}$ as a basis of $\mathcal{V}$, we have the following dual basis $\{\omega^1, \omega^2, \omega^3\}$ for $\mathcal{V}^*$

$$\omega^1 = dy^1, \quad \omega^2 = dy^2, \quad \omega^3 = dy^3 - y^1 dy^2,$$

and the corresponding metric

$$\overset{\circ}{n} = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = (dy^1)^2 + (dy^2)^2 + (dy^3 - y^1 dy^2)^2, \quad (1.11)$$

which is a **Bianchi type VII** metric. The manifold $\mathcal{N}$ endowed with this metric is a **homogeneous space**.

Here, the dislocation density λ turns out to be

$$\lambda(E_1, E_2)(y) = E_3, \ \lambda(E_1, E_3)(y) = \lambda(E_2, E_3)(y) = 0,$$

and therefore

$$\lambda = (\omega_1 \wedge \omega_2) E_3 = \left(dy^1 \wedge dy^2 \right) E_3.$$

The integral of λ over a surface Σ in $\mathcal{N}$ is

$$\int_\Sigma \lambda = \int_\Sigma \left(dy^1 \wedge dy^2\right) E_3 = A(\Pi\Sigma)E_3,$$

where Π is the projection map of the line bundle of the homogeneous space (1.11) over $\mathbb{R}^2$ with the standard metric on the base (the curvature of the bundle being $-dy^1 \wedge dy^2$).

2 The thermodynamic state space

Consider the space $S_2^+(\mathcal{V})$ of inner products on the crystalline structure $\mathcal{V}$. The **thermodynamic state space** is defined as the product

$$S_2^+(\mathcal{V}) \times \mathbb{R}^+$$

and its elements are (γ, σ), where $\gamma \in S_2^+(\mathcal{V})$ is the **configuration** and $\sigma \in \mathbb{R}^+$ is the **entropy per unit mass**.

Each $\gamma \in S_2^+(\mathcal{V})$ defines a volume form ω_γ on the crystalline structure $\mathcal{V}$ by the condition that if $(E_1, \ldots, E_n)$ is a positive basis for $\mathcal{V}$, orthonormal with respect to γ, i.e., $\gamma_{AB} := \gamma(E_A, E_B) = \delta_{AB}$, then

$$\omega_\gamma(E_1, \ldots, E_n) = 1.$$

It follows that there is a positive function V on $S_2^+(\mathcal{V})$ such that

$$\omega_\gamma = V(\gamma)\omega.$$

The positive real number $V(\gamma)$ is the **volume per unit mass** corresponding to the configuration γ.

The **thermodynamic state function** κ is a real-valued function on the thermodynamic state space $S_2^+(\mathcal{V}) \times \mathbb{R}^+$. The Lagrangian which determines the dynamics will be defined through this function.

The **thermodynamic stress** corresponding to a thermodynamic state (γ, σ) is the element $\pi(\gamma, \sigma)$ of $(S_2(\mathcal{V}))^*$ defined by

$$\frac{\partial \left(\kappa(\gamma, \sigma) V(\gamma) \right)}{\partial \gamma} = -\frac{1}{2} \pi(\gamma, \sigma) V(\gamma).$$

The **temperature** corresponding to a thermodynamic state (γ, σ) is the real number $\vartheta(\gamma, \sigma)$ given by

$$\vartheta(\gamma, \sigma) = \frac{\partial \left(\kappa(\gamma, \sigma) V(\gamma) \right)}{\partial \sigma},$$

with the requirement that $\vartheta(\gamma, \sigma)$ is positive and tends to zero as σ tends to zero.

3 The dynamics

In the **general theory of relativity**, the space-time manifold is an oriented $(n + 1)$-dimensional differentiable manifold $\mathcal{M}$ endowed with a **Lorentzian metric** g, that is a continuous assignment of a symmetric bilinear form g_x of index 1 in $T_x\mathcal{M}$, $\forall x \in \mathcal{M}$. The Lorentzian metric divides $T_x\mathcal{M}$ into three different subsets I_x, N_x, S_x, the set of **timelike**, **null** and **spacelike** vectors at x, respectively, according to whether g_x restricted to the corresponding subset is negative, zero or positive. The subset N_x is a double cone called the null cone at x. The subset I_x is the interior of this cone, an open set of two components, the future and past component. S_x is the subset outside the null cone, a connected open set for $n > 1$. A curve γ is called **causal** if its tangent vector belongs to $I_x \cup N_x$, $\forall x \in \gamma$, and it is called timelike if its tangent vector belongs to I_x, $\forall x \in \gamma$. We assume that $(\mathcal{M}, g)$ is time oriented, that is a continuous choice of future component I_x^+ of I_x can and has been made $\forall x \in \mathcal{M}$. This choice determines the future component N_x^+ of N_x at each $x \in \mathcal{M}$. A timelike or causal curve is then future or past directed, according to which component its tangent vector belongs to. A hypersurface $\mathcal{H}$ is called spacelike if at each $x \in \mathcal{H}$ the restriction of g_x to $T_x\mathcal{H}$ is positive definite. A spacelike hypersurface in $\mathcal{M}$ is called a **Cauchy hypersurface** if each causal curve in $\mathcal{M}$ intersects $\mathcal{H}$ exactly once. We assume that $(\mathcal{M}, g)$ possesses such a Cauchy hypersurface [3].

The motion of the material continuum is described by a mapping f from the space-time manifold $\mathcal{M}$ into the material manifold $\mathcal{N}$,

$$f : \mathcal{M} \rightarrow \mathcal{N}. \tag{3.1}$$

This mapping specifies which material particle is at a given event in space-time. It is subject to the following conditions:

(i) The mapping f, restricted to a Cauchy hypersurface $\mathcal{H}$, $f|_{\mathcal{H}}$, is one to one.

(ii) The differential of the mapping f, $df(x)$, has a 1d kernel contained in I_x, $\forall x \in \mathcal{M}$.

Then, for each $y \in f(\mathcal{M}) \subset \mathcal{N}$, $f^{-1}(y)$ is a timelike curve in $\mathcal{M}$. The **material velocity** u is the future directed unit tangent vectorfield of the timelike curve f^{-1}:

$$\mathrm{span}(u_x) = \ker\left(df(x)\right) = T_x f^{-1}(y),\ f(x) = y,\ g(u_x, u_x) = -1, \forall x \in \mathcal{M}.$$

The **simultaneous space** at x is the orthogonal complement of the linear span of u_x:

$$\Sigma_x = (\mathrm{span}(u_x))^{\perp}.$$

Note that $g_{\Sigma_x} = g_x|_{\Sigma_x}$ is positive definite. The restriction of the differential $df(x)$ to Σ_x is an isomorphism of Σ_x onto $T_y \mathcal{N}$, where $y = f(x)$.

(iii) The isomorphism $df(x)|_{\Sigma_x}$ is orientation preserving.

The **equations of motion**, a second order system of partial differential equations for the mapping f, are derived from a **Lagrangian** L, a function on $\mathcal{M}$ which is constructed from f. The action $\mathcal{A}$ in a domain $\mathcal{D} \subset \mathcal{M}$ is the integral

$$\int_{\mathcal{D}} L \, d\mu_g,$$

where $d\mu_g$ is the volume form of $(\mathcal{M}, g)$.

Any mapping f fulfilling the three requirements stated above, defines an orientation preserving isomorphism $j_{f,x}$ of $\mathcal{V}$ onto Σ_x by

$$j_{f,x} = \left(df(x)|_{\Sigma_x}\right)^{-1} \circ \epsilon_{f(x)}. \tag{3.2}$$

Define

$$j_{f,x}^* g_x = \gamma, \tag{3.3}$$

$\gamma \in S_2^+(\mathcal{V})$ and depends only on f and x. Note that

$$\mu(x) = \frac{1}{V\left(j_{f,x}^* g_x\right)}$$

is the rest mass density at x.

To take into account thermal effects in the Lagrangian picture we invoke the adiabatic condition which states that the entropy per unit mass of any element of the material remains unchanged. This allows us to consider the entropy per unit mass σ as a given function on the material manifold $\mathcal{N}$.

The **Lagrangian** function L on $\mathcal{M}$ is defined by:

$$L(x) = \kappa\left(j_{f,x}^* g_x, \sigma(f(x))\right), \ \forall x \in \mathcal{M}, \tag{3.4}$$

where κ is the thermodynamic state function on $S_2^+(\mathcal{V}) \times \mathbb{R}^+$. Note that $L(x)$ depends on g only through g_x and not on derivatives of g.

The **energy–momentum–stress tensor** T_x at x is an element of the dual space $(S_2(T_x\mathcal{M}))^*$ defined by

$$\frac{\partial\left(L(x)d\mu_g(x)\right)}{\partial g_x} = -\frac{1}{2}T_x d\mu_g(x). \tag{3.5}$$

From [5] we have the following:

Proposition I.1. *We can write the energy–momentum–stress tensor at x as follows:*

$$T_x = \rho(x)u_x \otimes u_x + S_x \tag{3.6}$$

with the mass-energy density ρ (since we are in the relativistic framework this includes the rest mass energy) given by

$$\rho(x) = \kappa(j_{f,x}^* g_x, \sigma(f(x))). \tag{3.7}$$

The stress tensor is given by

$$S_x(\dot{g}_x) = \pi(j_{f,x}^* g_x, \sigma(f(x)))(j_{f,x}^* \dot{g}_x).$$

Since $j_{f,x}^* \dot{g}_x = j_{f,x}^* \dot{g}_{\Sigma_x}$, S_x can be viewed as an element of $(S_2(\Sigma_x))^*$.

Remark I.3. *Hence, we see from Proposition* I.1 *that in the case of pure continuum mechanics (in the absence of electromagnetic fields) the energy per unit mass is $e = \kappa V$.*

Define the **principal pressures** p_i, $i = 1, \ldots, n$ as the eigenvalues of $S_{\flat\flat}$ relative to $g|_\Sigma$, where the subscript $\flat\flat$ means lowering of the indices with respect to g. Note that the principal pressures $p_i : i = 1, \ldots, n$ can equivalently be described as the eigenvalues of $\pi_{\flat\flat}$ relative to γ.

The positivity condition on T_x requires that

$$H_x^+ \to T_x\mathcal{M}$$
$$v^\mu \mapsto -T^\mu_\nu v^\nu \tag{3.8}$$

maps into $\overline{I_x^+}$. This is equivalent to the condition that $|p_i| \leq \rho$: $i = 1, \ldots, n$. Here, in the case of crystalline solids, we assume the stronger condition that the range of (3.8) lies in I_x^+ which is in turn equivalent to $|p_i| < \rho$: $i = 1, \ldots, n$.

3.1 Variation of the Lagrangian

Let $v = df(x) \in U_{(x,y)} \subset \mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$, $y = f(x)$, where $U_{(x,y)}$ is the open subset of the linear space $\mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$ consisting of those $v \in \mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$ which verify the two conditions

(1) $\ker v$ is a time like line in $T_x\mathcal{M}$,
(2) with Σ_x the g_x-orthogonal complement of $\ker v$ in $T_x\mathcal{M}$, the isomorphism

$$v|_{\Sigma_x} : \Sigma_x \to T_y\mathcal{N}$$

is orientation-preserving.

These above conditions (1) and (2) correspond to the conditions **(ii)** and **(iii)**, respectively. Note that **(i)** is not a local condition so it does not reduce to a condition on v.

Since the isomorphism

$$j_{f,x} = \left(v|_{\Sigma_x}\right)^{-1} \circ \epsilon_y,$$

as defined in (3.2), depends only on v, we may write

$$i(v) = j_{f,x},$$

so the configuration $\gamma_{f,x}$ corresponding to the isomorphism $v \in U_{(x,y)}$ is given by the positive quadratic form on $\mathcal{V}$

$$\gamma(v) = i^*(v) \, g|_{\Sigma_x}. \tag{3.9}$$

Now $v \mapsto \gamma(v)$ is a mapping of

$$\mathcal{B} := \bigcup_{(x,y)\in\mathcal{M}\times\mathcal{N}} U_{(x,y)}, \tag{3.10}$$

into $S_2^+(\mathcal{V})$. The bundle $\mathcal{B}$, a bundle over $\mathcal{M} \times \mathcal{N}$, is the bundle over which the Lagrangian is defined.

The mapping $v \mapsto \gamma(v)$ is described by the functions $\gamma_{AB}(v)$ on $\mathcal{B}$, where

$$\gamma_{AB}(v) = \gamma(v)\,(E_A, E_B), \tag{3.11}$$

where $E_A : A = 1, \ldots, n$ is a basis of $\mathcal{V}$ such that $\omega(E_1, \ldots, E_n) = 1$.

By the properties of v we have a positive basis $(X_A(x) : A = 1, \ldots, n)$ of Σ_x defined by:

$$v \cdot X_A(x) = E_A(y) : A = 1, \ldots, n,$$

where $E_A(y) = \epsilon_y(E_A) \in T_y\mathcal{N}(A = 1, \ldots, n)$, and thus from (3.9) and (3.11)

$$\gamma_{AB} = g|_{\Sigma_x}\,(X_A, X_B).$$

Note that $(u, X_1, \ldots, X_n)$ is a frame field for $\mathcal{M}$. Let $\dot{v} \in \mathcal{L}(T_x\mathcal{M}, T_y\mathcal{N})$ be a variation of $v \in U_{(x,y)}$. To describe $\dot{v}$ we must give $\dot{v} \cdot u \in T_y\mathcal{N}$ and $\dot{v} \cdot X_A \in T_y\mathcal{N}$,

$$\dot{v} \cdot u(x) = \dot{v}_0^A E_A(y),$$
$$\dot{v} \cdot X_A(x) = \dot{v}_A^B E_B(y).$$

This is because $(E_A(y) : A = 1, \ldots, n)$ is a basis for $T_y\mathcal{N}$. So the $(\dot{v}_0^A : A = 1, \ldots, n; \dot{v}_A^B : A, B = 1, \ldots, n)$ can be thought of as the the components of $\dot{v}$.

In view of (3.4) and (3.7) the Lagrangian function L is $L(v) = \rho(\gamma(v), \sigma(y))$, where ρ is the relativistic energy-density which includes the rest mass

contribution, a function of a thermodynamic state $(\gamma(v), \sigma(y)) \in S_2^+(\mathcal{V}) \times \mathbb{R}^+$. L is a function on the bundle $\mathcal{B}$ (3.10). The Lagrangian form, a top degree form on $\mathcal{M}$, is $L d\mu_g$.

The Lagrangian L depends on v through the configuration γ, $L(\gamma_{AB})$, and the first variation reads

$$\dot{L} = \frac{\partial \rho}{\partial \gamma_{AB}} \dot{\gamma}_{AB},$$

where the first variation of γ_{AB} is given by

$$\dot{\gamma}_{AB} = -\dot{v}_A^C \gamma_{BC} - \dot{v}_B^C \gamma_{AC}$$

(see [4]). To formulate the hyperbolicity condition (see below), we need to consider the second variation of L with respect to v,

$$\ddot{L} = \frac{\partial^2 L}{\partial v^2} \cdot (\dot{v}, \dot{v}) = h(\dot{v}, \dot{v}), \tag{3.12}$$

where, in general,

$$h(\dot{v}, \dot{v}) = h_{AB}^{00} \dot{v}_0^A \dot{v}_0^B + 2 h_{AB}^{C0} \dot{v}_C^A \dot{v}_0^B + h_{AB}^{CD} \dot{v}_C^A \dot{v}_D^B. \tag{3.13}$$

For $L = L(\gamma_{ab})$, using the formula for the second variation of γ_{AB},

$$\ddot{\gamma}_{AB} = 2 \left(\gamma_{AC} \gamma_{BD} \dot{v}_0^C \dot{v}_0^D + \gamma_{CD} \dot{v}_A^C \dot{v}_B^D + \gamma_{AC} \dot{v}_B^D \dot{v}_D^C + \gamma_{BC} \dot{v}_A^D \dot{v}_D^C \right)$$

(see [4]), we obtain

$$\begin{aligned} \ddot{L} &= \frac{\partial \rho}{\partial \gamma_{AB}} \ddot{\gamma}_{AB} + \frac{\partial^2 \rho}{\partial \gamma_{AB} \partial \gamma_{CD}} \dot{\gamma}_{AB} \dot{\gamma}_{CD} \\ &= 2 \frac{\partial \rho}{\partial \gamma_{AB}} \left(\gamma_{AC} \gamma_{BD} \dot{v}_0^C \dot{v}_0^D + \gamma_{AC} \dot{v}_B^D \dot{v}_D^C + \gamma_{BC} \dot{v}_A^D \dot{v}_D^C + \gamma_{CD} \dot{v}_A^C \dot{v}_B^D \right) \\ &\quad + \frac{\partial^2 \rho}{\partial \gamma_{AB} \partial \gamma_{CD}} \left(\dot{v}_A^E \gamma_{EB} + \dot{v}_B^E \gamma_{EA} \right) \left(\dot{v}_C^F \gamma_{FD} + \dot{v}_D^F \gamma_{FC} \right) \\ &= 2 \frac{\partial \rho}{\partial \gamma_{CD}} \gamma_{AC} \gamma_{BD} \dot{v}_0^A \dot{v}_0^B + 2 \frac{\partial \rho}{\partial \gamma_{CD}} \gamma_{AB} \dot{v}_C^A \dot{v}_D^B \\ &\quad + 2 \left(\frac{\partial \rho}{\partial \gamma_{CE}} \gamma_{BE} \delta_A^D + \frac{\partial \rho}{\partial \gamma_{DE}} \gamma_{AE} \delta_B^C \right) \dot{v}_C^A \dot{v}_D^B \\ &\quad + 4 \frac{\partial^2 \rho}{\partial \gamma_{CE} \partial \gamma_{DF}} \gamma_{AE} \gamma_{BF} \dot{v}_C^A \dot{v}_D^B. \end{aligned}$$

Comparing coefficients with (3.13) using (3.12), we see that:

$$h^{00}_{AB} = 2 \frac{\partial \rho}{\partial \gamma_{CD}} \gamma_{AC} \gamma_{BD}, \quad h^{C0}_{AB} = 0,$$

$$h^{CD}_{AB} = 4 \frac{\partial^2 \rho}{\partial \gamma_{CE} \partial \gamma_{DF}} \gamma_{AE} \gamma_{BF}$$

$$+ 2 \left(\frac{\partial \rho}{\partial \gamma_{CD}} \gamma_{AB} + \frac{\partial \rho}{\partial \gamma_{CE}} \gamma_{BE} \delta^D_A + \frac{\partial \rho}{\partial \gamma_{DE}} \gamma_{AE} \delta^C_B \right),$$

where the last expression will be made use of in the derivation of the Legendre–Hadamard conditions in Part II.

3.2 Hyperbolicity and characteristic speeds

Set

$$G_{AB} = -h^{00}_{AB}, \quad M^{CD}_{AB} = h^{CD}_{AB}.$$

Then

$$h(\dot{v}, \dot{v}) = -G_{AB} \dot{v}^A_0 \dot{v}^B_0 + M^{CD}_{AB} \dot{v}^A_C \dot{v}^B_D.$$

The hyperbolicity condition in general is that there is a pair $(T, \theta) \in T_x \mathcal{M} \times T^*_x \mathcal{M}$ with $\theta \cdot T > 0$ such that h is *negative-definite* on

$$\{ \dot{v} : \dot{v} = \theta \otimes Y, Y \in T_y \mathcal{N} \} \tag{3.14}$$

and *positive definite* on

$$\{ \dot{v} : \dot{v} = \kappa \otimes Y, \kappa \cdot T = 0, Y \in T_y \mathcal{N} \} \tag{3.15}$$

(see [4]). We stipulate here that the above conditions hold with (T, θ) defined by the rest frame of the material at x, i.e., $T = u_x$, $\Sigma_x = \ker \theta$, $\theta \cdot T = 1$.

For $\dot{v}$ in (3.14) we have:

$$\dot{v} \cdot u = Y, \quad \dot{v} \cdot X_A = 0,$$

i.e.,

$$\dot{v}_0^A = Y^A, \quad \dot{v}_A^B = 0,$$

and the condition that h is negative definite on (3.14) is the condition that

$$G_{AB}Y^A Y^B > 0: \quad \forall Y \neq 0, \tag{3.16}$$

i.e., that G is positive definite.

Since

$$G_{AB} = \pi_{AB} + \rho\gamma_{AB}$$

(we are raising and lowering indices with respect to γ_{AB}) and the principal pressures $p_1, \ldots, p_n$ are the eigenvalues of π_{AB} with respect to γ_{AB}, this condition is:

$$\min_i p_i > -\rho,$$

which of course follows from $\max_i |p_i| < \rho$. For $\dot{v}$ in (3.15) we have:

$$\dot{v} \cdot u = 0, \quad \dot{v} \cdot X_A = \kappa_A Y,$$

where $\kappa_A = \kappa \cdot X_A$, i.e.,

$$\dot{v}_0^A = 0, \quad \dot{v}_A^B = \kappa_A Y^B,$$

and the condition that h is positive definite on (3.15) is the condition that

$$M_{AB}^{CD}\kappa_C\kappa_D Y^A Y^B > 0 \quad : \forall \kappa \neq 0, \forall Y \neq 0. \tag{3.17}$$

Set

$$H_{AB}(\kappa) = M_{AB}^{CD}\kappa_C\kappa_D.$$

Then condition (3.17) is that $H_{AB}(\kappa)$ is positive definite for all $\kappa \neq 0$.

If the first condition (3.16) is satisfied, the second condition (3.17) becomes the condition that for $\kappa \neq 0$ the eigenvalues $\lambda_1(\kappa), \ldots, \lambda_n(\kappa)$ of $H_{AB}(\kappa)$ with respect to G_{AB} are all positive. Note that $H_{AB}(\kappa)$ is homogeneous of degree 2 in κ, hence so are the $\lambda_1(\kappa), \ldots, \lambda_n(\kappa)$.

Now the characteristic matrix is

$$\chi_{AB}(\xi) = h_{AB}^{\mu\nu}\xi_\mu\xi_\nu.$$

We use the basis $(u, X_1, \ldots, X_n)$ for $T_x\mathcal{M}$. We denote by ω the frequency and by κ_A, $A = 1, \ldots, n$, the wave number components, i.e.,

$$\omega := \xi_0 := \xi \cdot u, \quad \kappa_A := \xi_A := \xi \cdot X_A.$$

Then

$$\chi_{AB} = -G_{AB}\omega^2 + H_{AB}(\kappa),$$

so

$$\det\chi = \det G \cdot \prod_{i=1}^{n}\left(\lambda_i(\kappa) - \omega^2\right).$$

We see that the second condition is equivalent to the $2n$ roots $\pm\sqrt{\lambda_i(\kappa)}$, $i : 1, \ldots, n$, of the characteristic polynomial $\det\chi$ as a polynomial in ω being real. The characteristic speeds are

$$\eta_i = \frac{\sqrt{\lambda_i(\kappa)}}{|\kappa|}, \quad \text{where } |\kappa| = \sqrt{(\gamma^{-1})^{AB}\kappa_A\kappa_B}. \tag{3.18}$$

Note that the η_i are homogeneous of degree 0 in κ. The causality condition is that the inner characteristic core in $T_x^*\mathcal{M}$ contains the null cone of g in $T_x^*\mathcal{M}$. This reads (in units $c = 1$)

$$\eta_i < 1, \quad \forall i = 1, \ldots, n.$$

4   Properties of the energy per unit mass

Postulate I.1. *We stipulate that the energy per unit mass $e = \rho V$ has a strict minimum at a certain inner product $\mathring{\gamma}$.*

Remark I.4. *The above postulate has an intuitive physical interpretation. Note that the set of inner products γ is an open positive cone in the linear space of quadratic forms. For large γ (large expansion) and also for γ near the boundary (large compression) the energy e is physically expected to blow up, so we can restrict ourselves to a compact set of inner products, where e necessarily attains a minimum.*

We choose $(E_1, \ldots, E_n)$ to be an orthonormal basis relative to $\overset{\circ}{\gamma}$, so $\overset{\circ}{\gamma}_{AB} = \delta_{AB}$. This is compatible with the previous condition $\omega(E_1, \ldots, E_n) = 1$ for a suitable volume form ω on $\mathcal{V}$. This choice of ω corresponds to a choice of unit of mass so that the mass density associated to the configuration $\overset{\circ}{\gamma}$ is equal to 1. $\overset{\circ}{\gamma}$ defines a metric $\overset{\circ}{n}$ on $\mathcal{N}$ by

$$\overset{\circ}{n} = \sum_{A,B=1}^{n} \overset{\circ}{\gamma}_{AB}\, \omega^A \otimes \omega^B = \sum_{A=1}^{n} \omega^A \otimes \omega^A,$$

where $(\omega^1, \ldots, \omega^n)$ is the dual basis to $(E_1, \ldots, E_n)$. If $\mathcal{V}$ is a Lie algebra, so that $\mathcal{N}$ is a Lie group, then $\overset{\circ}{n}$ is a left-invariant metric, i.e., invariant under the actions of $\mathcal{N}$ on itself by left multiplications by elements of $\mathcal{N}$. Thus $(\mathcal{N}, \overset{\circ}{n})$ is a homogeneous Riemannian manifold (which in general is not isotropic). The Riemannian manifold $(\mathcal{N}, \overset{\circ}{n})$ has curvature except when $\mathcal{V}$ is Abelian, so there are no dislocations.

Note that $d\mu_{\overset{\circ}{n}} = \sqrt{\det \overset{\circ}{n}}\, d^n y = d\mu_\omega$; therefore we have for any domain Ω in $\mathcal{N}$

$$M(\Omega) = \int_\Omega d\mu_\omega = \int_\Omega d\mu_{\overset{\circ}{n}},$$

that is, in our choice of units the mass of Ω is equal to its volume with respect to $\overset{\circ}{n}$.

4.1 The isotropic case

Let us define the orthogonal group corresponding to $\overset{\circ}{\gamma}$,

$$O_{\overset{\circ}{\gamma}} = \left\{ O \in \mathcal{L}(\mathcal{V}, \mathcal{V}) : \overset{\circ}{\gamma}(OX, OY) = \overset{\circ}{\gamma}(X, Y), \forall X, Y \in \mathcal{V} \right\}.$$

$O_{\overset{\circ}{\gamma}}$ acts on $S_2^+(\mathcal{V})$ in the following way: $\gamma \mapsto O\gamma$, where

$$(O\gamma)(X, Y) = \gamma(OX, OY).$$

Since $\overset{\circ}{\gamma} \mapsto O\overset{\circ}{\gamma} = \overset{\circ}{\gamma}$, the inner product $\overset{\circ}{\gamma}$ is a fixed point of the action of $O_{\overset{\circ}{\gamma}}$ on $S_2^+(\mathcal{V})$.

If the energy per unit mass $e(\gamma)$ is invariant under $O_{\overset{\circ}{\gamma}}$ we are in the case of **isotropic elasticity**. Then $e(\gamma)$ depends only on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of γ relative to $\overset{\circ}{\gamma}$,

$$e(\gamma) = e(\lambda_1, \ldots, \lambda_n),$$

where $e(\lambda_1, \ldots, \lambda_n)$ is totally symmetric in its arguments. This is the simplest case of an energy density.

Consider

$$q = f^* \overset{\circ}{n}.$$

This is a 2-covariant symmetric tensorfield on $\mathcal{M}$, i.e., at each $x \in \mathcal{M}$ q_x is a quadratic form in $T_x \mathcal{M}$. The vector u_x belongs to the null space of q_x and the restriction $q_x|_{\Sigma_x}$ is positive definite.

Let $\Lambda_1, \ldots, \Lambda_n$ be the eigenvalues of $q_x|_{\Sigma_x}$ relative to $g_x|_{\Sigma_x}$. We then have

Proposition I.2. *The eigenvalues $\Lambda_1, \ldots, \Lambda_n$ are the inverses of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\gamma = j_{f,x}^* g_x$ relative to $\overset{\circ}{\gamma}$. In particular,*

$$\sqrt{\Lambda_1 \cdot \ldots \cdot \Lambda_n} = \frac{1}{\sqrt{\lambda_1 \cdot \ldots \cdot \lambda_n}} = \frac{1}{v} = N.$$

Therefore, in the variational principle, the crystalline structure $\mathcal{V}$ on $\mathcal{N}$ is eliminated in favor of the Riemannian metric $\overset{\circ}{n}$.

5   Equivalences

5.1   Equivalence of crystalline structures

Definition I.3. *Two crystalline structures $\mathcal{V}$ and $\mathcal{V}'$ on $\mathcal{N}$ are said to be equivalent if there is a diffeomorphism ψ of $\mathcal{N}$ onto itself such that ψ_*, the pushforward of ψ, induces an isomorphism of $\mathcal{V}$ onto $\mathcal{V}'$.*

Let ν be the canonical 1-form (1.2) associated to $\mathcal{V}$ and ν' the one associated to $\mathcal{V}'$. We have

$$\begin{aligned}
\nu(Y_y) &= Y \in \mathcal{V} &&: Y_y \in T_y \mathcal{N}, \\
\nu'(d\psi \cdot Y_y) &= \psi_* Y \in \mathcal{V}' &&: d\psi \cdot Y_y \in T_{\psi(y)} \mathcal{N}.
\end{aligned} \tag{5.1}$$

We may take the above as the definition of pullback for $\mathcal{V}$-valued 1-forms:

$$\nu = \psi^* \nu'.$$

It then follows that:

$$\lambda = \psi^* \lambda', \tag{5.2}$$

with the natural extension of the above definition of pullback to $\mathcal{V}$-valued 2-forms, i.e.,

$$\lambda' \left(d\psi \cdot X_y, d\psi \cdot Y_y \right) = \psi_* (\lambda(X_y, Y_y)) : \quad \forall X, Y \in \mathcal{V}, \forall y \in \mathcal{N}.$$

In fact, we have for all $X, Y \in \mathcal{V}$ and $y \in \mathcal{N}$:

$$\begin{aligned}
\Lambda'(\psi_* X, \psi_* Y) \left(\psi(y) \right) &= \nu' \left([\psi_* X, \psi_* Y](\psi(y)) \right) \\
&= \nu' \left(\psi_*[X, Y](\psi(y)) \right) \\
&= \nu' \left(d\psi \cdot [X, Y](y) \right) \\
&= \psi_* \left(\nu([X, Y](y)) \right) \\
&= \psi_* \left(\Lambda(X, Y)(y) \right),
\end{aligned}$$

where we have made use of the fact that $[\psi_* X, \psi_* Y] = \psi_*[X, Y]$.

5.2   Equivalence of mechanical properties

We investigate the question of the equivalence of the mechanical properties of a solid. A certain solid phase of a certain substance is described by an equation of state.

Let $A : \mathcal{V} \to \mathcal{V}'$ be a linear isomorphism. The group of all such A is homomorphic to $GL_n(\mathbb{R})$. Then $A^* : S_2^+(\mathcal{V}') \to S_2^+(\mathcal{V})$ is the induced isomorphism defined by $\gamma = A^* \gamma'$, where

$$\gamma(X, Y) = \gamma'(AX, AY), \quad \forall X, Y \in \mathcal{V}.$$

The corresponding energy functions on $S_2^+(\mathcal{V})$ and $S_2^+(\mathcal{V}')$ are denoted by e and e', respectively.

Definition I.4. *Two energy functions e on $\mathcal{V}$ and e' on $\mathcal{V}$ are equivalent if there exists a linear isomorphism $A : \mathcal{V} \to \mathcal{V}'$ such that $e'(\gamma') = e(\gamma)$, where $\gamma = A^* \gamma'$, for all $\gamma' \in S_2^+(\mathcal{V}')$.*

We illustrate the above definition of mechanical equivalence by giving an example in the isotropic case. Let $\gamma \in S_2^+(\mathcal{V})$ be given, and define $M \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ by

$$\overset{\circ}{\gamma}(MX, Y) = \gamma(X, Y), \quad \forall X, Y \in \mathcal{V}.$$

Similarly, given an isomorphism A as above (i.e., $\gamma = A^*\gamma'$), we define $M' \in \mathcal{L}(\mathcal{V}', \mathcal{V}')$ from the corresponding $\gamma' \in S_2^+(\mathcal{V}')$ as

$$\overset{\circ}{\gamma'}(M'X', Y') = \gamma'(X', Y'), \quad \forall X', Y' \in \mathcal{V}'.$$

In the above, $\overset{\circ}{\gamma} = A^* \overset{\circ}{\gamma'}$. Then $M' = AMA^{-1}$. For, given any $X', Y' \in \mathcal{V}'$ let $X, Y \in \mathcal{V}$ be $X = A^{-1}X'$, $Y = A^{-1}Y'$. Then

$$\overset{\circ}{\gamma'}(M'X', Y') = \gamma'(X', Y') = \gamma(X, Y) = \overset{\circ}{\gamma}(MX, Y)$$

$$= \overset{\circ}{\gamma'}(AMX, AY) = \overset{\circ}{\gamma'}(AMA^{-1}X', Y').$$

Therefore, the eigenvalues $\lambda_1', \ldots, \lambda_n'$ of M' coincide with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of M. In the isotropic case, $e'(\gamma') = e(\gamma)$ means

$$e'(\lambda_1', \ldots, \lambda_n') = e(\lambda_1, \ldots, \lambda_n),$$

where both sides of the above equation are symmetric functions, so the energy functions coincide, i.e., $e' = e$.

The equivalence of the energy functions, however, does not fully capture the equivalence of having two solids of the same substance in the same phase. What is required in addition is to have the same equilibrium mass density of infinitesimal portions. In fact, it is the triplet $(\mathcal{V}, \omega, e)$ that defines a solid with its mechanical properties. Two solids of the same substance in the same phase, but with possibly different dislocation structures, to the extent that they can be described by the same manifold $\mathcal{N}$ (which is true if they are diffeomorphic) are defined by the triplets $(\mathcal{V}, \omega, e)$ and $(\mathcal{V}', \omega', e')$, respectively. Additionally, there is an isomorphism $A : \mathcal{V} \to \mathcal{V}'$, such that

$$\omega = A^*\omega',$$

i.e., $\omega(X_1, \ldots, X_n) = \omega'(AX_1, \ldots, AX_n) : \forall X_1, \ldots, X_n \in \mathcal{V}$ as well as

$$e'(\gamma') = e(\gamma), \quad \gamma = A^*\gamma' \quad : \forall \gamma' \in S_2^+(\mathcal{V}')$$

in accordance with Definition I.4. Thus if $(E_1, \ldots, E_n)$ is a positive basis for $\mathcal{V}$ which is orthonormal relative to $\overset{\circ}{\gamma}$, then

$$\omega_{\overset{\circ}{\gamma}}(E_1, \ldots, E_n) = 1,$$

while $\omega(E_1, \ldots, E_n) = \mu_0$. On the other hand, with $E_i' = AE_i : i = 1, \ldots, n$, $(E_1', \ldots, E_n')$ is orthonormal relative to $\overset{\circ}{\gamma}'$, thus

$$\omega_{\overset{\circ}{\gamma}'}(E_1', \ldots, E_n') = 1,$$

while $\omega'(E_1', \ldots, E_n') = \omega(E_1, \ldots, E_n) = \mu_0$. Consequently, the mass density corresponding to $\overset{\circ}{\gamma}'$ relative to the triplet $(\mathcal{V}', \omega', e')$ is the same as the mass density corresponding to $\overset{\circ}{\gamma}$ relative to the triplet $(\mathcal{V}, \omega, e)$.

Remark I.5. *If we have two triplets $(\mathcal{V}, \omega, e)$ and $(\mathcal{V}', \omega', e')$ which correspond to the same substance in the same phase and, additionally, the isomorphism $A : \mathcal{V} \to \mathcal{V}'$ is of the form $A = \psi_*$, where ψ is a diffeomorphism of $\mathcal{N}$ onto itself, then the dislocation structures are also equivalent. In this case, if $f : \mathcal{M} \to \mathcal{N}$ is a dynamical solution of the problem corresponding to $(\mathcal{V}, \omega, e)$, then $f' := \psi \circ f : \mathcal{M} \to \mathcal{N}$ is a dynamical solution of the problem corresponding to $(\mathcal{V}', \omega', e')$.*

5.3 Similarity

Let now $\mathcal{V}$ be fixed, so that $S_2^+(\mathcal{V})$ is also fixed. Two different energy functions e and e' may be related as follows. There is a constant $a > 0$ and an isomorphism $\mathcal{V} \to \mathcal{V}$, defined by

$$X \mapsto aX : \quad \forall X \in \mathcal{V}.$$

This defines an isomorphism $S_2^+(\mathcal{V}) \to S_2^+(\mathcal{V})$ by $\gamma \mapsto \gamma'$, where

$$\gamma'(X, Y) = \gamma(aX, aY) = a^2 \gamma(X, Y) : \quad \forall X, Y \in \mathcal{V},$$

i.e., $\gamma \mapsto \gamma' = a^2 \gamma$.

According to the above discussion of equivalence of mechanical properties, we must have $e(\gamma) = e'(\gamma')$, $\forall \gamma \in S_2^+(\mathcal{V})$, *and*, additionally $\omega' = a^n \omega$ for the

same material in the same phase, for

$$\omega'(X_1, \ldots, X_n) = \omega(aX_1, \ldots, X_n) = a^n \omega(X_1, \ldots, X_n).$$

Then the two theories (primed and unprimed) represent the same material in the same phase.

Moreover, since $\mathcal{V}$ is identical in the two theories, the dislocation structures may seem identical. However, for a given domain $\Omega \subset\subset \mathcal{N}$, which has the same crystalline structure $\mathcal{V}|_\Omega$ (the restriction to Ω of the vectorfields in $\mathcal{V}$), the primed theory actually assigns a *physical dislocation density* a^{-2} times that of the unprimed theory. This is due to the fact that $\omega' = a^n \omega$ and the observation that in any dimension, the elementary dislocations are co-dimension two objects. Thus, their density refers to the 2d measure of a cross-section.

6 The Eulerian picture

As a prelude to formulating the Eulerian picture, we view the crystalline structure $\mathcal{V}$ as an abstract n-dimensional real vector space, as in Remark I.2. We then view the canonical form ν as a 1-form on $\mathcal{N}$ with values in $\mathcal{V}$ such that $\nu|_{T_y \mathcal{N}}$ is an isomorphism from $T_y \mathcal{N}$ onto $\mathcal{V}$ for all $y \in \mathcal{N}$. Thus, given any $v \in \mathcal{V}$, we have a tangent vector

$$\left(\nu|_{T_y \mathcal{N}} \right)^{-1} \cdot v = Y_y(v) \in T_y \mathcal{N} : \quad \forall y \in \mathcal{N},$$

that is a vectorfield $Y(v)$ on $\mathcal{N}$. Then the crystalline structure in the original sense is the space

$$\{ Y(v) : v \in \mathcal{V} \}$$

of vectorfields on $\mathcal{N}$.

To obtain the Eulerian description we must eliminate the material manifold $\mathcal{N}$. A fundamental variable is the material velocity u, a future-directed timelike unit vectorfield on $\mathcal{M}$. This defines the distribution of local simultaneous spaces

$$\Sigma = \{ \Sigma_x : x \in \mathcal{M} \}, \tag{6.1}$$

where Σ_x is the orthogonal complement of u_x in $T_x \mathcal{M}$.

We then need another entity defined on $\mathcal{M}$ to play the role of the canonical form ν. Consider

$$\xi = f^*\nu.$$

This is a 1-form on $\mathcal{M}$ with values in $\mathcal{V}$, viewed as an abstract n-dimensional vector space.

The 1-form ξ has the following properties:

(1) $\xi \cdot u = 0$,
(2) for each $x \in \mathcal{M}$, $\xi|_{\Sigma_x}$ is an isomorphism from Σ_x onto $\mathcal{V}$, *and*
(3) $\mathcal{L}_u \xi = 0$.

We thus introduce *ab initio* a 1-form ξ on $\mathcal{M}$ with values in $\mathcal{V}$ possessing the above three properties, as another fundamental Eulerian variable besides u.

By **(2)**, given any $v \in \mathcal{V}$, we have a tangent vector

$$\left(\xi|_{\Sigma_x}\right)^{-1} \cdot v = X_x(v) \in \Sigma_x$$

at each $x \in \mathcal{M}$, that is we obtain a vectorfield $X(v)$ whose value at each point belongs to the distribution (6.1), i.e., it is orthogonal to u.

A mapping $\mathcal{M} \to S_2^+(\mathcal{V})$, $x \mapsto \gamma_x$, is then defined by:

$$\gamma_x(v_1, v_2) = g_x(X_x(v_1), X_x(v_2)) = g|_{\Sigma_x}(X_x(v_1), X_x(v_2)) \quad : \forall v_1, v_2 \in \mathcal{V}. \tag{6.2}$$

The last fundamental variable is the entropy s, a positive function on $\mathcal{M}$.

The volume form ω, the space of thermodynamic configurations $S_2^+(\mathcal{V})$, and the volume per unit mass V are defined as in Section 2. The thermodynamic state space is $S_2^+(\mathcal{V}) \times \mathbb{R}^+$ and the energy per unit mass e is a function on this space, as before. The thermodynamic stress π at each $(\gamma, s) \in S_2^+(\mathcal{V}) \times \mathbb{R}^+$, $\pi(\gamma, s) \in (S_2(\mathcal{V}))^*$, is defined by

$$\frac{\partial e(\gamma, s)}{\partial \gamma} = -\frac{1}{2} V(\gamma) \pi(\gamma, s),$$

as in Section 2. Also, the temperature θ is defined, as before, by

$$\theta(\gamma, s) = \frac{\partial e(\gamma, s)}{\partial s}.$$

Thus

$$de = -\frac{1}{2}V\pi \cdot d\gamma + \theta ds$$

expresses the first law of thermodynamics in the present framework.

The stress S is a 2-contravariant symmetric tensorfield on $\mathcal{M}$, an assignment of an element $S_x \in (S_2(\Sigma_x))^*$ at each $x \in \mathcal{M}$. This is defined as follows. Given any $\dot{g}_x \in S_2(T_x\mathcal{M})$, we define $\dot{\gamma}_x \in S_2(\mathcal{V})$ as in (6.2) by:

$$\dot{\gamma}_x(v_1, v_2) = \dot{g}_x(X_x(v_1), X_x(v_2)) = \dot{g}|_{\Sigma_x}(X_x(v_1), X_x(v_2)) \quad : \forall v_1, v_2 \in \mathcal{V}.$$

Then

$$S_x(\dot{g}_x) = \pi(\gamma_x, s(x))(\dot{\gamma}_x).$$

The mass-energy density ρ is then the positive function on $\mathcal{M}$ given by

$$\rho(x) = \frac{e(\gamma_x, s(x))}{V(\gamma_x)},$$

and the energy–momentum–stress tensor is defined according to (3.6).

The Eulerian equations of motion are a first-order system of partial differential equations consisting of

  (a) $\mathcal{L}_u\xi = 0$, i.e., property (3) of ξ, and
  (b) $\nabla \cdot T = 0$.

Remark I.6. *In n-space dimensions, $\dim\mathcal{M} = n + 1$, there are $n^2 + n + 1$-dependent variables, the n^2 algebraically independent components of ξ, the n algebraically independent components of u, and also s. The Eulerian equations are also $n^2 + n + 1$ in number, n^2 independent equations in (a) and $n + 1$ independent equations in (b).*

For solutions of the Eulerian equations such that u, ξ, and s are continuous, (b) implies the adiabatic condition on s:

$$u(s) = 0.$$

For the proof of this fact see [5].

Remark I.7. *The case of absence of dislocations is the case $d\xi = 0$.*

Let $(E_A : A = 1, \ldots, n)$ be a basis for $\mathcal{V}$. Given any $\gamma \in S_2^+(\mathcal{V})$, we set

$$\gamma_{AB} = \gamma(E_A, E_B).$$

According to the above, at each $x \in \mathcal{M}$ there is a unique $X_A \in \Sigma_x$ such that

$$\xi \cdot X_A = E_A : \quad A = 1, \ldots, n.$$

Thus, X_A is a vectorfield on $\mathcal{M}$ belonging to Σ. Then

$$S = \pi^{AB} X_A \otimes X_B,$$

where $\pi^{AB} \in S_2^+(\mathcal{V})^*$ is the thermodynamic stress defined above.

Fix an element $v \in \mathcal{V}$ and consider the vectorfield $X \subset \Sigma$, such that

$$\xi \cdot X = v.$$

Then

$$0 = u(\xi \cdot X) = (\mathcal{L}_u \xi) \cdot X + \xi \cdot [u, X],$$

i.e., (according to **(3)**) $\xi \cdot [u, X] = 0$. It follows that

$$\Pi\,[u, X] = 0, \tag{6.3}$$

where Π is the *g*-orthogonal projection to Σ.

Proposition I.3. *Define $\mu := \frac{1}{V}$. Then the mass current $I = \mu u$ satisfies the equation of continuity:*

$$\nabla \cdot I = 0. \tag{6.4}$$

Proof. $\mathcal{V}$ is endowed with a volume form ω and $\omega_\gamma = V\omega$. Choose a basis $(E_1, \ldots, E_n)$ for $\mathcal{V}$ such that $\omega(E_1, \ldots, E_n) = 1$. Then $V = \sqrt{\det \gamma}$, and, by (6.2),

$$\gamma_{AB} = \gamma(E_A, E_B) = g(X_A, X_B), \tag{6.5}$$

hence

$$u(V) = \frac{1}{2} V \left(\gamma^{-1}\right)^{AB} u(\gamma_{AB}), \tag{6.6}$$

and, using (6.5),

$$\begin{aligned}
u(\gamma_{AB}) &= u(g(X_A, X_B)) \\
&= g(\nabla_u X_A, X_B) + g(X_A, \nabla_u X_B) \\
&= g(\nabla_{X_A} u, X_B) + g(X_A, \nabla_{X_B} u) + g([u, X_A], X_B) + g(X_A, [u, X_B]).
\end{aligned}$$

By (6.3), the last two terms in the above equation vanish. Let us define κ by

$$\nabla_{X_A} u = \kappa_A^B X_B. \tag{6.7}$$

$[(X_A : A = 1, \ldots, n)$ is a basis for Σ_x at each point.] Then

$$g(\nabla_{X_A} u, X_B) = \kappa_A^C g(E_C, E_B) = \gamma_{CB} \kappa_A^C.$$

Hence, we obtain

$$u(\gamma_{AB}) = \gamma_{CB} \kappa_A^C + \gamma_{AC} \kappa_B^C.$$

Substituting into (6.6) yields

$$u(V) = \frac{1}{2} V \left(\gamma^{-1}\right)^{AB} u(\gamma_{AB}) = V \operatorname{tr} \kappa.$$

Then $\mu = V^{-1}$ satisfies

$$u(\mu) + \mu \operatorname{tr} \kappa = 0,$$

and since, from (6.7), $\operatorname{tr} \kappa = \nabla \cdot u$, we obtain

$$\nabla \cdot I = \nabla(\mu u) = u(\mu) + \mu \nabla \cdot u = 0, \tag{6.8}$$

which establishes (6.4). $\qquad \square$

6.1   The non-relativistic limit

First of all, we restrict ourselves to the case where $(\mathcal{M}, g)$ is the Minkowski space-time. Then we consider the non-relativistic limit, where Minkowski space-time is replaced by Galilean space-time. There, we have the hyperplanes Σ_t of absolute simultaneity, which are isometric to n-dimensional Euclidean space. Any family of parallel lines transversal to the Σ_t represents a Galilean frame, that is, a family of observers in uniform motion and at rest relative to each other. Any such family of parallel lines defines an isometry of the Σ_t onto each other.

In terms of a Galilean frame and a rectangular coordinate system $(x^1, \ldots, x^n)$ in Euclidean space, and with $x^0 = ct$ (c: the speed of light in vacuum), the space-time velocity $u = u^\mu \frac{\partial}{\partial x^\mu}$ is represented in terms of the space velocity $v = v^i \frac{\partial}{\partial x^i}$ by

$$u^0 = \frac{1}{\sqrt{1 - |v|^2/c^2}}, \quad u^i = \frac{\frac{v^i}{c}}{\sqrt{1 - |v|^2/c^2}}.$$

Therefore, in the non-relativistic limit $c \to \infty$

$$cu = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}.$$

Also, the condition

$$\xi_\mu u^\mu = 0$$

is simply $\xi_0 = 0$ and thus ξ becomes a $\mathcal{V}$-valued 1-form on each Σ_t,

$$\xi = \xi_i dx^i.$$

Equations **(a)** become

$$\frac{\partial \xi}{\partial t} + \mathcal{L}_v \xi = 0 \quad \left(\text{or} \quad \frac{\partial \xi_i}{\partial t} + v^j \frac{\partial \xi_i}{\partial x^j} + \xi_j \frac{\partial v^j}{\partial x^i} = 0 \right), \tag{6.9}$$

and

$$S^{\mu\nu} u_\nu = 0, \quad u_\nu = g_{\mu\nu} u^\mu,$$

reads (note that $u_\nu = g_{\kappa\nu} u^\kappa$, i.e., $u_0 = -u^0$, while $u_i = u^i$)

$$S^{i0} - S^{ij} \frac{v^j}{c} = 0, \tag{6.10}$$

$$S^{00} - S^{0i} \frac{v^i}{c} = 0. \tag{6.11}$$

So, in the limit $c \to \infty$, we have $S^{i0} = S^{00} = 0$ and, therefore,

$$S = S^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

Let

$$e = c^2 + e',$$

$$\rho = \mu e = \mu c^2 + \varepsilon, \quad \varepsilon = \mu e'.$$

We have for the mass current $I^\nu = \mu u^\nu$:

$$I^0 = \frac{\mu}{\sqrt{1 - |v|^2/c^2}} = \mu + \frac{1}{2} \frac{\mu |v|^2}{c^2} + O(c^{-4}),$$

$$I^i = \frac{\mu \frac{v^i}{c}}{\sqrt{1 - |v|^2/c}} = \frac{\mu v^i}{c} + \frac{1}{2} \frac{\mu |v|^2 v^i}{c^3} + O(c^{-5}).$$

In the non-relativistic limit, the equation of continuity $\nabla_\nu I^\nu = 0$ becomes the classical continuity equation:

$$\frac{\partial \mu}{\partial t} + \frac{\partial \left(\mu v^i \right)}{\partial x^i} = 0. \tag{6.12}$$

Similarly, for the energy–momentum–stress tensor $T_{\kappa\lambda}$, we have

$$T^{00} = \rho(u^0)^2 + S^{00} = \mu c^2 + (\varepsilon + \mu v^2) + O(c^{-2}),$$

$$T^{0i} = \rho u^0 u^i + S^{0i} = \mu v^i c + (\varepsilon + \mu v^2) \frac{v^i}{c^2} - S^{ij} \frac{v^j}{c} + O(c^{-2}),$$

$$T^{ij} = \frac{\rho v^i v^j / c^2}{1 - |v|^2/c^2} + S^{ij} = \mu v^i v^j + S^{ij} + O(c^{-2}).$$

The equations of motion $\nabla_\lambda T^{\kappa\lambda} = 0$ read

$$\kappa = 0: \quad \frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{0i}}{\partial x^i} = 0 \xrightarrow{c \to \infty} \frac{\partial \mu}{\partial t} + \frac{\partial \left(\mu v^i \right)}{\partial x^i} = 0,$$

$$\kappa = i: \quad \frac{\partial T^{i0}}{\partial x^0} + \frac{\partial T^{ij}}{\partial x^j} = 0 \xrightarrow{c \to \infty} \frac{\partial \left(\mu v^i \right)}{\partial t} + \frac{\partial \left(\mu v^i v^j + S^{ij} \right)}{\partial x^j} = 0.$$

In order to obtain the energy equation, let us consider

$$F^\nu = T^{0\nu} - c^2 I^\nu, \quad \text{where} \quad \nabla_\nu F^\nu = 0.$$

For the components of F^ν we have, using (6.11), i.e., $S^{i0} = S^{ij} v^j / c$ and $S^{00} = S^{ij} v^i v^j / c^2$,

$$F^0 = \frac{\mu |v|^2 + \varepsilon}{1 - |v|^2/c^2} + \frac{S^{ij} v^i v^j}{c^2} - \frac{c^2 \mu}{\sqrt{1 - |v|^2/c^2}} = \frac{1}{2}\mu |v|^2 + \varepsilon + O(c^{-2}),$$

$$F^i = \frac{\left(\mu |v|^2 + \varepsilon\right) v^i / c}{1 - |v|^2/c^2} + S^{0i} - \frac{c\mu v^i}{\sqrt{1 - |v|^2/c^2}}$$

$$= \left(\mu |v|^2 + \varepsilon\right) \frac{v^i}{c} + S^{ij} \frac{v^j}{c} - \frac{1}{2}\mu |v|^2 \frac{v^i}{c} + O(c^{-2}).$$

Hence,

$$c\nabla_\nu F^\nu = \frac{\partial F^0}{\partial t} + c\frac{\partial F^i}{\partial x^i} = 0$$

becomes the energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mu |v|^2 + \varepsilon\right) + \frac{\partial}{\partial x^i}\left[\left(\frac{1}{2}\mu |v|^2 + \varepsilon\right) v^i + S^{ij} v^j\right] = 0. \qquad (6.13)$$

In conclusion non-relativistic Eulerian equations are

(a) $\quad \dfrac{\partial \xi}{\partial t} + \mathcal{L}_v \xi = 0,$

(b0) $\quad \dfrac{\partial \mu}{\partial t} + \dfrac{\partial}{\partial x^i}\left(\mu v^i\right) = 0,$

(b1) $\quad \dfrac{\partial \left(\mu v^i\right)}{\partial t} + \dfrac{\partial}{\partial x^j}\left(\mu v^i v^j + S^{ij}\right) = 0,$

(b2) $\quad \dfrac{\partial}{\partial t}\left(\dfrac{1}{2}\mu |v|^2 + \varepsilon\right) + \dfrac{\partial}{\partial x^i}\left[\left(\dfrac{1}{2}\mu |v|^2 + \varepsilon\right) v^i + S^{ij} v^j\right] = 0.$

Note:

- **(b0)** is the differential mass conservation law, a consequence of **(a)**,
- **(b1)** is the differential momentum conservation law, *and*
- **(b2)** is the differential energy conservation law.

Remark I.8. (a) *may be thought of as expressing a law of conservation of dislocations. In fact, let C_0 be a closed curve on Σ_0. Let ϕ_t be the flow in Galilean space-time generated by the vectorfield*

$$u = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}.$$

Then $\phi_t|_{\Sigma_0}$ is a diffeomorphism from Σ_0 onto Σ_t. Let $C_t = \phi_t(C_0)$. Then, according to **(a)**,

$$\int_{C_t} \xi = \int_{C_0} \xi.$$

The left-hand side corresponds, in the continuum limit, to minus the sum of the Burger's vectors of all the dislocation lines enclosed by C_t.

Remark I.9. *For continuous solutions, i.e., (ξ, v, s) continuous,* **(b2)** *is equivalent to the adiabatic condition*

(c) $\frac{\partial s}{\partial t} + v^i \frac{\partial s}{\partial x^i} = 0,$

modulo the other equations. When discontinuities such as shocks develop, this equivalence no longer holds. However, the Eulerian equations **(a)**–**(b)** *still hold, but in a weak or integral sense.*

6.2   From the Eulerian to the Lagrangian picture

In order to go from the Eulerian picture to the Lagrangian formulation we have to extract the canonical form ν from ξ. First note that

$$\mathcal{L}_u \xi = 0 \quad \Leftrightarrow \quad \phi_t^* \xi = \xi, \tag{6.14}$$

where ϕ_t is the flow generated by u.

Let $\mathcal{H}$ be a Cauchy hypersurface in $\mathcal{M}$. We identify $\mathcal{H}$ with $\mathcal{N}$ and define $f : \mathcal{M} \to \mathcal{N}$ as follows:

$$f(x) = y \in \mathcal{H}$$

is the point at which the integral curve of u through x intersects $\mathcal{H}$.

For $X_x \in T_x \mathcal{M}$, we have from (6.14)

$$(\phi^* \xi) = \xi \left(d\phi_t \cdot X_x \right) = \xi(X_x).$$

Define ν to be the $\mathcal{V}$-valued 1-form induced by ξ on $\mathcal{H}$:

$$\nu(X_x) = \xi(X_x) \quad : \forall X_x \in T_x \mathcal{H}, x \in \mathcal{H}.$$

We must show

Proposition I.4.

$$f^* \nu = \xi \,.$$

Proof. Let $V_p \in T_p \mathcal{M}$, $p \in \phi_t(\mathcal{H})$ for some $t \in \mathbb{R}$. V_p uniquely decomposes into

$$V_p = \lambda u + Y_p, \text{ where } Y_p \in T_p \phi_t(\mathcal{H}).$$

Therefore, it suffices to show that $(f^* \nu)(u_p) = \xi(u_p) = 0$, which is obvious, and

$$(f^* \nu)(Y_p) = \xi(Y_p).$$

Since $\phi_t|_{\mathcal{H}}$ is a diffeomorphism from $\mathcal{H}$ onto $\phi_t(\mathcal{H})$, $d\phi_t(q)$ is an isomorphism from $T_q \mathcal{H}$ onto $T_p \phi_t(\mathcal{H})$, $\phi_t(q) = p$. Therefore, there exists a unique $X_q \in T_q \mathcal{H}$, such that

$$d\phi_t \cdot X_q = Y_p.$$

Now,

$$\xi(Y_p) = \xi(d\phi_t \cdot X_q) = (\phi_t^* \xi)(X_q) = \xi(X_q) = \nu(X_q)$$

and

$$(f^* \nu) = (f^* \nu)(d\phi_t \cdot X_q) = (\phi_t^* f^* \nu)(X_q) = (f^* \nu)(X_q) = \nu(df \cdot X_q).$$

But $f|_{\mathcal{H}} = id$, hence $df \cdot X_q = X_q$ because

$$df|_{T_q \mathcal{H}} = id. \qquad \square$$

Remark I.10. *The Eulerian picture is the one more related to direct phys-ical experience and also the one which may serve as a basis for extending the theory beyond the domain of elasticity theory, where the dislocations are no longer anchored in the solid. We shall see in the next part, however, that the Lagrangian picture provides the suitable framework for the study of static problems.*

Part II   The static case

1   Formulation of the static problem

Let $\mathcal{N} = \Omega \subset \mathbb{R}^n$ be the material manifold and $\mathcal{M} = E^n$ Euclidean space ($n = 2, 3$ being the cases of interest). In the static case, we adopt the material picture, that is, we consider one to one mappings ϕ from the material manifold into space,

$$\begin{aligned} \phi : \mathcal{N} &\to \mathcal{M} \\ y &\mapsto \phi(y) = x. \end{aligned} \tag{1.1}$$

They correspond to mappings $f : \mathcal{M} \times \mathbb{R} \to \mathcal{N}$ such that

$$f(x, t) = \phi^{-1}(x) \quad : \forall t.$$

In the material picture, the interchange of the roles of the domain and target space transforms a free boundary problem into a fixed boundary problem.

We recall that the *configuration* γ is an element of the space of inner products on the crystalline structure $\mathcal{V}$, $\gamma \in S_2^+(\mathcal{V})$. In terms of the mapping ϕ it is defined as the pullback of the metric g on $\mathcal{M}$ by the isomorphism $i_{\phi,y}$

$$\gamma = i_{\phi,y}^* g, \tag{1.2}$$

where $i_{\phi,y} = d\phi(y) \circ \epsilon_y$ and ϵ_y is the evaluation map (1.1) from $\mathcal{V}$ to $T_y \mathcal{N}$. The energy per unit mass $e(\gamma)$ defines the *thermodynamic stress* π, an element in $(S_2(\mathcal{V}))^*$, by

$$\frac{\partial e}{\partial \gamma} = -\frac{1}{2}\pi V, \tag{1.3}$$

where $V \equiv V(\gamma)$ is the volume per unit mass, related to the mass density μ by

$$V(\gamma) = \frac{1}{\mu(x)}, \quad x = \phi(y).$$

In the following, we assume the entropy σ to be constant, what is called isentropic.

Since we are in the relativistic framework, e includes the contribution c^2 of the rest mass–energy, so

$$e = c^2 + e'$$

(in conventional units), where e' is what we would call energy per unit mass in the non-relativistic framework. Note that the additive constant c^2 does *not* affect the definition (1.3) of the thermodynamic stress π, which may equally well be written in the form

$$\frac{\partial e'}{\partial \gamma} = -\frac{1}{2}\pi V.$$

Also, in view of (2.1) below, we have

$$E = Mc^2 + E',$$

where

$$M = \int_\Omega d\mu_\omega, \quad E' = \int_\Omega e'(\gamma) d\mu_\omega.$$

The additive constant Mc^2 does not affect variations of E, which are the same as those of E'. For this reason, we shall not distinguish in the static theory e', E' from e, E, and we shall denote the former by the latter.

Remark II.1. *The static theory would be formally identical if formulated wholly within the non-relativistic framework. What distinguishes the two theories is that the non-relativistic theory is applicable only when the characteristic speeds η_i (3.18) are negligible in comparison to c, while the relativistic theory holds for any values of the η_i less than c.*

2 Boundary value problem and Legendre–Hadamard condition

2.1 Euler–Lagrange equations

We choose a basis $E_1, \ldots, E_n$ of $\mathcal{V}$ such that $\omega(E_1, \ldots, E_n) = 1$, where ω is the volume form on $\mathcal{V}$. We denote by $\gamma_{AB} = \gamma(E_A, E_B)$ the matrix representing the configuration in this frame. We then have

$$\omega_\gamma = V(\gamma)\omega = \sqrt{\det \gamma} \cdot \omega,$$

and for the total energy of a domain Ω in the material manifold

$$E = \int_\Omega e(\gamma) \, d\mu_\omega, \tag{2.1}$$

where $d\mu_\omega$ is the mass element on $\mathcal{N}$ induced by ω:

$$d\mu_\omega \left(\frac{\partial}{\partial y_1}\bigg|_y, \ldots, \frac{\partial}{\partial y_n}\bigg|_y \right) = \omega \left(\epsilon_y^{-1}\left(\frac{\partial}{\partial y_1}\bigg|_y\right), \ldots, \epsilon_y^{-1}\left(\frac{\partial}{\partial y_n}\bigg|_y\right) \right). \tag{2.2}$$

Let $(\omega^A : A = 1, \ldots, n)$ be the basis for $\mathcal{V}^*$ which is dual to the basis $(E_A : A = 1, \ldots, n)$ for $\mathcal{V}$. The ω^A are 1-forms on $\mathcal{N}$ such that $\omega^A(E_B) = \delta_B^A$. Since $(E_A(y) : A = 1, \ldots, n)$ is a basis for $T_y\mathcal{N}$, given any $Y \in T_y\mathcal{N}$, there are coefficients $Y^A : A = 1, \ldots, n$ such that $Y = Y^A E_A(y)$. Then:

$$\omega^A(Y) = \omega^A \left(Y^B E_B(y) \right) = Y^B(\omega^A(E_B))(y) = Y^B \delta_B^A = Y^A.$$

Setting $Y = \frac{\partial}{\partial y^a}\big|_y$ we obtain

$$Y^A = \omega^A \left(\frac{\partial}{\partial y^a}\bigg|_y \right) = \omega_a^A(y),$$

and we have $\epsilon_y^{-1}(Y) = Y^A E_A \in \mathcal{V}$. Thus,

$$\epsilon_y^{-1}\left(\frac{\partial}{\partial y^a}\bigg|_y \right) = \omega_a^A(y)E_A. \tag{2.3}$$

Let us denote by $\det \omega(y)$ the determinant of the n-dimensional matrix with entries $\omega_a^A(y)$. We conclude from (2.2), (2.3) that

$$d\mu_\omega = \det \omega(y)d^n y. \tag{2.4}$$

The first variation of the energy (2.1) is:

$$\dot{E} = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} E(\gamma + \lambda\dot{\gamma}) = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \int_\Omega e(\gamma + \lambda\dot{\gamma})d\mu_\omega = \int_\Omega \frac{\partial e(\gamma)}{\partial\gamma} \cdot \dot{\gamma}\, d\mu_\omega,$$

(2.5)

where $\dot{\gamma} \in S_2^+(\mathcal{V})$ is the variation of the configuration γ. By definition of the thermodynamic stress (1.3), we conclude, in view of (2.4),

$$\dot{E} = -\int_\Omega \frac{1}{2}\pi^{AB}\dot{\gamma}_{AB}\sqrt{\det\gamma}\,\det\omega(y)\; d^n y.$$

Denoting by m_{ab} the pullback by ϕ to $\mathcal{N}$ of the Euclidean metric g on $\mathcal{M}$, we have

$$m = \phi^* g, \tag{2.6}$$

$$m_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}g_{ij} \qquad [x^i = \phi^i(y)], \tag{2.7}$$

$$\gamma_{AB} = E_A^a E_B^b m_{ab}, \tag{2.8}$$

where the n-dimensional matrix with entries $E_A^a(y)$ is the reciprocal of the n-dimensional matrix with entries $\omega_a^A(y)$. Thus, $\det E(y) = (\det\omega(y))^{-1}$, and by (2.8) we have

$$\det\gamma = \det m\,(\det E)^2\,,$$
$$\sqrt{\det\gamma}\,\det\omega = \sqrt{\det m}.$$

For the stresses S on $\mathcal{N}$ and $T = \phi_* S$ on $\mathcal{M}$ we have

$$S^{ab} = \pi^{AB} E_A^a E_B^b, \tag{2.9}$$

$$T^{ij} = S^{ab}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}. \tag{2.10}$$

We will now formulate the Euler–Lagrange equations on both the material manifold and on space. We calculate for the variation of γ_{AB} from (2.8)

using linear coordinates in E^n (i.e., $\partial_k g_{ij} = 0$)

$$\dot{\gamma}_{AB} = E_A^a E_B^b \dot{m}_{ab}, \quad \dot{m}_{ab} = g_{ij}\left(\frac{\partial \dot{x}^i}{\partial y^a}\frac{\partial x^j}{\partial y^b} + \frac{\partial x^i}{\partial y^a}\frac{\partial \dot{x}^j}{\partial y^b}\right).$$

The first variation of the energy thus reads

$$\dot{E} = -\int_\Omega \frac{1}{2}S^{ab}\dot{m}_{ab}\sqrt{\det m}\, d^n y = -\int_\Omega S^{ab}g_{ij}\frac{\partial \dot{x}^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\sqrt{\det m}\, d^n y, \quad (2.11)$$

where $\dot{x}^i$ is the variation of x^i. Hence, (2.11) is

$$-\int_\Omega S^{ab}g_{ij}\frac{\partial \dot{x}^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}d\mu_m, \quad (2.12)$$

where $d\mu_m$ is the volume form on Ω associated to the metric m. By the divergence theorem, (2.12) becomes

$$-\int_{\partial\Omega} S^{ab}g_{ij}\frac{\partial x^j}{\partial y^b}M_a\dot{x}^i\, dA_m + \int_\Omega \dot{x}^i\overset{m}{\nabla}_a Q_i^a d\mu_m, \quad (2.13)$$

where Q_i are the vectorfields with components

$$Q_i^a = S^{ab}g_{ij}\frac{\partial x^j}{\partial y^b} \quad (2.14)$$

and $\overset{m}{\nabla}$ is the covariant derivative operator on $\mathcal{N}$ associated to m. Thus, in local coordinates on $\mathcal{N}$,

$$\overset{m}{\nabla}_a Q_i^a = \frac{1}{\sqrt{\det m}}\frac{\partial}{\partial y^a}\left(\sqrt{\det m}Q_i^a\right). \quad (2.15)$$

In (2.13), M^a are the components of the outward unit normal to $\partial\Omega$ with respect to m, $M_a = m_{ab}M^b$, and dA_m is the area element of $\partial\Omega$ associated to the metric induced by m. So, M_a are the components of the covectorfield along $\partial\Omega$ whose null space is the tangent plane to $\partial\Omega$ at each point.

The Euler–Lagrange equations are obtained by requiring $\dot{E} = 0$ for variations $\dot{x}^i$, which are supported in Ω and vanish on $\partial\Omega$. In view

of (2.13), (2.14), (2.15), the Euler–Lagrange equations are:

$$\frac{\partial}{\partial y^a}\left(Q_i^a \sqrt{\det m}\right) = 0 \quad \text{on } \Omega. \tag{2.16}$$

Note that with Q_i^a defined in (2.14) we have using (2.7)

$$Q_i^a \frac{\partial x^i}{\partial y^b} = g_{ij}S^{ac}\frac{\partial x^j}{\partial y^c}\frac{\partial x^i}{\partial y^b} = m_{bc}S^{ac} = S_b^a. \tag{2.17}$$

Consider

$$\overset{m}{\nabla}_a S_b^a = \frac{1}{\sqrt{\det m}}\frac{\partial}{\partial y^a}\left(\sqrt{\det m}\,S_b^a\right) - \overset{m}{\Gamma^c_{ab}}S_c^a. \tag{2.18}$$

The first term on the right-hand side of (2.18) is, by (2.16), (2.17),

$$\frac{1}{\sqrt{\det m}}\frac{\partial}{\partial y^a}\left(\sqrt{\det m}\,Q_i^a\frac{\partial x^i}{\partial y^b}\right) = Q_i^a\frac{\partial^2 x^i}{\partial y^a \partial y^b} = S_c^a\frac{\partial y^c}{\partial x^i}\frac{\partial^2 x^i}{\partial y^a \partial y^b}.$$

Thus (2.16) is equivalent to:

$$\overset{m}{\nabla}_a S_b^a = \left(\frac{\partial y^c}{\partial x^i}\frac{\partial^2 x^i}{\partial y^a \partial y^b} - \overset{m}{\Gamma^c_{ab}}\right)S_c^a. \tag{2.19}$$

We shall show now that

$$\overset{m}{\Gamma^c_{ab}} = \frac{\partial y^c}{\partial x^i}\frac{\partial^2 x^i}{\partial y^a \partial y^b}, \tag{2.20}$$

so that the factor in parenthesis in (2.19) vanishes. In fact, (2.20) is equivalent to:

$$\overset{m}{\Gamma}_{ab,c} = m_{cd}\frac{\partial y^d}{\partial x^i}\frac{\partial^2 x^i}{\partial y^a \partial y^b},$$

which, since $m_{cd}\frac{\partial y^d}{\partial x^i} = g_{ij}\frac{\partial x^j}{\partial y^c}$, is in turn equivalent to

$$\overset{m}{\Gamma}_{ab,c} = g_{ij}\frac{\partial x^j}{\partial y^c}\frac{\partial^2 x^i}{\partial y^a \partial y^b}. \tag{2.21}$$

Now we have by the definition of the Christoffel symbols $\overset{m}{\Gamma}_{ab,c}$ and m from (2.7):

$$\overset{m}{\Gamma}_{ab,c} = \frac{1}{2}\left(\frac{\partial m_{ac}}{\partial y^b} + \frac{\partial m_{bc}}{\partial y^a} - \frac{\partial m_{ab}}{\partial y^c}\right) = g_{ij}\frac{\partial x^j}{\partial y^c}\frac{\partial^2 x^i}{\partial y^a \partial y^b}.$$

Thus (2.21) is indeed verified. We conclude that the Euler–Lagrange equations (2.16) are equivalent to:

$$\overset{m}{\nabla}_a S_b^a = 0, \text{ or } \quad \overset{m}{\nabla}_a S^{ab} = 0 \quad \text{on } \Omega. \tag{2.22}$$

Remark II.2. T^{ij} *being the pushforward of* S^{ab} *by* ϕ *and* m_{ab} *being the pullback of* g_{ij} *by* ϕ, *the last equations (2.22) are in turn equivalent to*

$$\overset{g}{\nabla}_i T^{ij} = 0, \text{ or } \quad \partial_i T^{ij} = 0 \quad \text{on } \phi(\Omega) \tag{2.23}$$

in linear coordinates.

2.2 Boundary conditions

Let now equations (2.16) hold. Requiring again $\dot{E} = 0$, this time for arbitrary variations $\dot{x}$, we obtain from the first term of (2.13)

$$0 = \int\limits_{\partial\Omega} S^{ab} g_{ij} \frac{\partial x^j}{\partial y^b} M_a \dot{x}^i \, dA_m. \tag{2.24}$$

Here the covectorfield along $\partial\Omega$ with components M_a is defined by

$$M_a Y^a = 0 \quad : \forall Y \in T_y \partial\Omega,$$

together with the condition $M_a Y^a > 0$ for all $Y \in T_y\Omega$, $y \in \partial\Omega$, such that Y points to the exterior of Ω, and the normalization condition $(m^{-1})^{ab} M_a M_b = 1$. As (2.24) is to hold for arbitrary variations $\dot{x}^i$, we obtain the boundary conditions

$$S^{ab} g_{ij} \frac{\partial x^j}{\partial y^b} M_a = 0, \text{ or } \quad S^{ab} M_b = 0 \quad \text{on } \partial\Omega. \tag{2.25}$$

Let $X^i = \frac{\partial x^i}{\partial y^a} Y^a$. Setting $M_a = \frac{\partial x^i}{\partial y^a} N_i$ we have:

$$N_i X^i = 0 \quad : \forall X \in T_x \phi(\partial\Omega). \tag{2.26}$$

In fact, the covectorfield along $\phi(\partial\Omega)$ with components N_i is defined by (2.26), together with the condition that

$$N_i X^i > 0 : \quad \forall X \in T_x \phi(\Omega),$$

$x \in \phi(\partial\Omega) = \partial(\phi(\Omega))$, such that X points to the exterior of $\phi(\Omega)$, and the normalization condition $(g^{-1})^{ij} N_i N_j = 1$. In other words, $N^i := (g^{-1})^{ij} N_j$

are the components of the outward unit normal to $\partial(\phi(\Omega))$ in E^n. In view of the fact that $S^{ab}\frac{\partial x^j}{\partial y^b} = T^{ij}\frac{\partial y^a}{\partial x^i}$, the boundary conditions (2.25) are equivalent to

$$T^{ij}N_i = 0 \quad \text{on } \phi(\partial\Omega) = \partial\phi(\Omega), \qquad (2.27)$$

which means that no forces are acting on the boundary $\partial\phi(\Omega)$.

2.3   Legendre–Hadamard condition

The *Legendre–Hadamard* condition in the static case is just the second part of the hyperbolicity condition stated in (3.15), see [4] for details. Here, it is formulated in terms of $\frac{\partial\phi^i}{\partial y^a}(y) = v_a^i$. It requires for $\xi_a, \xi_b \in T_y^*\mathcal{N}, \eta^i, \eta^j \in T_{\phi(y)}\mathcal{M}$ that

$$\frac{1}{4}\frac{\partial^2 e}{\partial v_a^i \partial v_b^j}\xi_a\xi_b\eta^i\eta^j \quad : \quad \text{positive for } \xi, \eta \neq 0. \qquad (2.28)$$

We differentiate e with respect to v_b^j:

$$\frac{\partial e}{\partial v_b^j} = \frac{\partial e}{\partial\gamma_{AB}}\frac{\partial\gamma_{AB}}{\partial v_b^j},$$

where $\gamma_{AB} = E_A^a E_B^b g_{ij} v_a^i v_b^j$, hence

$$\frac{\partial\gamma_{AB}}{\partial v_b^j} = 2E_{(A}^a E_{B)}^b g_{ij} v_a^i,$$

$(E_{(A}^a E_{B)}^b$ denotes symmetrization in A and B) and thus

$$\frac{1}{2}\frac{\partial e}{\partial v_b^j} = \frac{\partial e}{\partial\gamma_{AB}}E_A^a E_B^b g_{ij} v_a^i. \qquad (2.29)$$

We differentiate (2.29) once more with respect to v_a^i to get

$$\frac{1}{4}\frac{\partial^2 e}{\partial v_a^i \partial v_b^j} = \frac{\partial^2 e}{\partial\gamma_{AB}\partial\gamma_{CD}}g_{li}g_{kj}v_c^l v_d^k E_A^d E_B^b E_C^c E_D^a + \frac{1}{2}\frac{\partial e}{\partial\gamma_{AB}}E_A^a E_B^b g_{ij}.$$

Therefore,

$$\frac{1}{4}\frac{\partial^2 e}{\partial v_a^i \partial v_b^j}\xi_a\xi_b\eta^i\eta^j = \frac{1}{2}\frac{\partial e}{\partial\gamma_{AB}}|\eta|^2\xi_A\xi_B + \frac{\partial^2 e}{\partial\gamma_{AB}\partial\gamma_{CD}}\eta_C\eta_A\xi_B\xi_D, \qquad (2.30)$$

where $|\eta|^2 = g_{ij}\eta^i\eta^j$, $\xi_A = E_A^a\xi_a$ and $\eta_C = E_C^c v_c^l g_{li}\eta^i$. Thus, the Legendre–Hadamard condition reads:

$$\frac{\partial^2 e}{\partial\gamma_{AB}\partial\gamma_{CD}}\eta_C\eta_A\xi_B\xi_D + \frac{1}{2}\frac{\partial e}{\partial\gamma_{AB}}|\eta|^2\xi_A\xi_B > 0 \quad (\eta,\xi \neq 0).$$

At $\overset{\circ}{\gamma}_{AB}$ the Legendre–Hadamard condition reduces to

$$\frac{\partial^2 e}{\partial\gamma_{AB}\partial\gamma_{CD}}\eta_C\eta_A\xi_B\xi_D > 0 \qquad (\eta,\,\xi \neq 0),$$

because the first term on the right-hand side of (2.30) (containing $\frac{\partial e}{\partial\gamma_{AB}}$) vanishes. Note that this condition is satisfied by virtue of the Postulate I.1 that e has a strict minimum at $\overset{\circ}{\gamma}$, since this implies

$$\frac{\partial^2 e}{\partial\gamma_{AB}\partial\gamma_{CD}}(\overset{\circ}{\gamma})\sigma_{AB}\sigma_{CD} > 0$$

for any non-zero symmetric σ_{AB} and, in particular, for

$$\sigma_{AB} = \xi_A\eta_B + \eta_A\xi_B.$$

In the general case, we define η^A by $\eta^A E_A^a v_a^i = \eta^i$ and we have

$$|\eta|^2 = g_{ij}\eta^i\eta^j = g_{ij}v_a^i v_b^j E_A^a E_B^b\eta^A\eta^B = \gamma_{AB}\eta^A\eta^B.$$

Additionally,

$$\gamma_{AB}\eta^B = g_{ij}v_a^i v_b^j E_A^a E_B^b\eta^B = g_{ij}v_a^i E_A^a\eta^j = \eta_A,$$

and hence $\eta^A = (\gamma^{-1})^{AB}\eta_B$. We finally have for the inverse $(\gamma^{-1})^{AB}$ of γ_{AB} with $|\eta|^2 = \eta_A\eta^A = \eta_A(\gamma^{-1})^{AB}\eta_B$

$$|\eta|^2 = (\gamma^{-1})^{AB}\eta_A\eta_B.$$

3   Uniform distribution of edge dislocations in 2d

The case of a uniform distribution of 2d *edge dislocations* is realized by the material manifold $\mathcal{N}$ being the affine group, see Section 1.1.1. The corresponding metric $\overset{\circ}{n}$ from (1.10) is isometric to the metric of the hyperbolic plane.

In general, we consider the total energy E (2.1) of a domain Ω in the material manifold $\mathcal{N}$. In agreement with Postulate I.1, we assume that the energy per unit mass e has a strict minimum at $\overset{\circ}{\gamma}$. The symmetry of the problem motivates the choice of an isotropic energy function e that is a symmetric function of the eigenvalues of $m = \phi^* g$ relative to $\overset{\circ}{n}$. Therefore, in this model case, the crystalline structure is eliminated in favor of the Riemannian manifold $(\mathcal{N}, \overset{\circ}{n})$.

3.1 Isotropic energy function

Concerning our toy energy function, we make the following basic choice:

$$e(\gamma) = e(\lambda_1, \dots, \lambda_n) = \frac{1}{2} \sum_{k=1}^n (\lambda_k - 1)^2 \geq 0 \qquad (3.1)$$

that satisfies $e(\lambda_1 = 1, \dots, \lambda_n = 1) = 0$. Thus, we have a strict minimum of the energy density at $\gamma = \overset{\circ}{\gamma}$. Note that, in (3.1), we are subtracting the rest mass energy, something which does not affect the variation of E, see also the discussion at the end of Section 1.

Therefore,

$$e(\gamma) = \frac{1}{2} \sum_{k=1}^n \left(\lambda_k^2 - 2\lambda_k + 1 \right) = \frac{1}{2} \operatorname{tr} \gamma^2 - \operatorname{tr} \gamma + \frac{n}{2}$$

$$= \frac{1}{2} \sum_{A,B=1}^n \gamma_{AB}^2 - \sum_{A=1}^n \gamma_{AA} + \frac{n}{2}. \qquad (3.2)$$

Here and in the following the basis $(E_A : A = 1, \dots, n)$ for $\mathcal{V}$ is chosen to be orthonormal relative to $\overset{\circ}{\gamma}$, hence $\overset{\circ}{\gamma}_{AB} = \delta_{AB}$. For such a choice to be compatible with the condition $\omega(E_1, \dots, E_n) = 1$, we must choose the physical unit of mass so that the mass density corresponding to $\overset{\circ}{\gamma}$ is equal to 1.

In case that the metric of the target space is Euclidean, $g_{ij} = \delta_{ij}$, we find from (2.8)

$$\gamma_{AB} = E_A^a E_B^b \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \delta_{ij} = E_A^a E_B^b \frac{\partial \phi}{\partial y^a} \cdot \frac{\partial \phi}{\partial y^b}, \qquad (3.3)$$

where the dot denotes the Euclidean inner product. For the total energy E (2.1) of a domain $\Omega \subset\subset \mathcal{N}$, where ω coincides with the volume form on $\mathcal{V}$

induced by $\overset{\circ}{\gamma}$, we have $d\mu_\omega = d\mu_{\overset{\circ}{n}}$ (the volume form on $\mathcal{N}$ corresponding to the Riemannian metric $\overset{\circ}{n}$), and we obtain for the total energy E expressed in terms of the components of γ,

$$E = \int_\Omega \left(\frac{1}{2} \sum_{A,B=1}^n \gamma_{AB}^2 - \sum_{A=1}^n \gamma_{AA} + \frac{n}{2} \right) d\mu_{\overset{\circ}{n}}. \tag{3.4}$$

In the case $n = 2$, the invariants of a linear mapping are, respectively, the trace and the determinant. Expressed in terms of the eigenvalues λ_1, λ_2, they read:

$$\tau = \lambda_1 + \lambda_2 = \operatorname{tr} m,$$
$$\delta = \lambda_1 \lambda_2 = \det m.$$

We have:

$$\tau^2 = \lambda_1^2 + \lambda_2^2 + 2\delta.$$

The toy energy (3.1) thus reads

$$e(\lambda_1, \lambda_2) = \frac{1}{2} \left((\lambda_1 - 1)^2 + (\lambda_1 - 1)^2 \right) = \frac{1}{2} \left(\tau^2 - 2\tau - 2\delta + 2 \right).$$

3.2   Expansion of the hyperbolic metric

We shall perform a dilation of the hyperbolic plane in order to be able to establish an existence result for a fixed domain, from which we can deduce an analogous result for energy minimizing mappings from a suitably small domain in the original hyperbolic plane to the Euclidean plane.

We start with the usual metric, of constant curvature -1 for the hyperbolic plane H, expressed in polar normal coordinates (R, θ):

$$ds_H^2 = dR^2 + \sinh^2 R\, d\theta^2.$$

Given a parameter $\varepsilon \in \mathbb{R}^+$, we dilate H by a factor ε^{-1} obtaining H_ε. The corresponding metric, of curvature $-\varepsilon^2$, is:

$$ds_{H_\varepsilon}^2 = \varepsilon^{-2} \left(dR^2 + \sinh^2 R\, d\theta^2 \right) = dr^2 + F_\varepsilon^2(r)\, d\theta^2,$$

where $r = \varepsilon^{-1} R$ and

$$F_\varepsilon(r) = \varepsilon^{-1} \sinh(\varepsilon r) = \varepsilon^{-1} \left(\varepsilon r + \frac{1}{3!} (\varepsilon r)^3 + \frac{1}{5!} (\varepsilon r)^5 + \cdots \right).$$

(r, θ) are polar normal coordinates in H_ε. The expression for $F_\varepsilon^2(r)$ thus reads:

$$F_\varepsilon^2(r) = r^2 \left(1 + \frac{1}{3} (\varepsilon r)^2 + \frac{2}{45} (\varepsilon r)^4 + \cdots \right). \tag{3.5}$$

The main point in the expansion (3.5) is that $F_\varepsilon^2(r)$ is a real analytic function converging to r^2 for $\varepsilon \to 0$, corresponding to the transition of the negatively curved hyperbolic plane to the flat Euclidean plane.

In order to simplify the calculations, we go back to rectangular coordinates. We denote rectangular normal coordinates in H_ε by $(y^i : i = 1, 2)$

$$y^1 = r \cos \theta,$$
$$y^2 = r \sin \theta.$$

Then

$$ds_{H_\varepsilon}^2 = \left(dy^1 \right)^2 + \left(dy^2 \right)^2 + \varepsilon^2 \left(\frac{1}{3} + \frac{2}{45} (\varepsilon r)^2 + \ldots \right) r^4 d\theta^2.$$

Since

$$d\theta = \frac{1}{r^2} \left(-y^2 dy^1 + y^1 dy^2 \right),$$

denoting by ds_E^2 the Euclidean metric

$$ds_E^2 = \left(dy^1 \right)^2 + \left(dy^2 \right)^2,$$

the metric of H_ε takes the form:

$$ds_{H_\varepsilon}^2 = ds_E^2 + \varepsilon^2 \left(\frac{1}{3} + \frac{2}{45} (\varepsilon r)^2 + \ldots \right) \left(-y^2 dy^1 + y^1 dy^2 \right)^2. \tag{3.6}$$

The crucial point in the above expression for the metric h of H_ε is that it takes the form

$$h_{ab} = \delta_{ab} + \varepsilon^2 f_{ab}, \tag{3.7}$$

where f_{ab} are analytic functions in (y^1, y^2) and ε^2. In fact,

$$\left(-y^2 dy^1 + y^1 dy^2\right)^2 = \left(r^2 \delta_{ab} - y^a y^b\right) dy^a dy^b, \tag{3.8}$$

hence, by (3.6), (3.7) and (3.8),

$$f_{ab} = \left(\frac{1}{3} + \frac{2}{45}(\varepsilon r)^2 + \cdots\right)\left(r^2 \delta_{ab} - y^a y^b\right). \tag{3.9}$$

That is,

$$f_{ab} = f(\varepsilon^2 r^2)\left(r^2 \delta_{ab} - y^a y^b\right), \quad \text{where} \quad f(x) = \frac{1}{3} + \frac{2}{45}x + \cdots.$$

4   Uniform distribution of screw dislocations in 3d

The case of a uniform distribution of (3d) *screw dislocations* is realized when the material manifold $\mathcal{N}$ is the Heisenberg group, see 1.1.2. The corresponding metric $\overset{\circ}{n}$ from (1.11) is isometric to the metric of a homogeneous but anisotropic space.

4.1   Anisotropic energy function

The symmetry of the problem in this case motivates the choice of an anisotropic energy function e, since the dislocation lines have a preferred direction. Therefore, in this model case, the crystalline structure cannot be eliminated in favor of the Riemannian manifold $(\mathcal{N}, \overset{\circ}{n})$, in contrast to the 2d case previously discussed.

The question of the right choice of energy per unit mass for the Heisenberg group is investigated in the following. Consider the orthogonal transformation O of $\mathcal{V}$, given by $(X', Y', Z') = O(X, Y, Z)$ with

$$\begin{aligned} X' &= \cos\theta \cdot X + \sin\theta \cdot Y, \\ Y' &= -\sin\theta \cdot X + \cos\theta \cdot Y, \\ Z' &= Z. \end{aligned} \tag{4.1}$$

The commutation relations $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$ are preserved by this transformation since

$$\left[X', Y' \right] = \cos^2 \theta [X, Y] - \sin^2 \theta [Y, X] = \left(\cos^2 \theta + \sin^2 \theta \right) [X, Y] = Z = Z',$$
$$\left[X', Z' \right] = [Y', Z'] = 0.$$

Consider the metric $m = \phi^* g$. Since m is symmetric, we have generally six independent components in three dimensions. This also holds for γ, the corresponding inner product on the crystalline structure:

$$\gamma = \begin{pmatrix} \bar{\gamma}_{AB} & \theta_A \\ \theta_A^T & \rho \end{pmatrix},$$

where the six components are

$$\bar{\gamma}_{AB} = \gamma_{AB}, \qquad A, B = 1, 2,$$
$$\theta_A = \gamma_{A3}, \qquad A = 1, 2,$$
$$\rho = \gamma_{33}.$$

The energy density e depends on the variables $(\bar{\gamma}, \theta, \rho)$. e must be invariant under orthogonal transformations (4.1), that is $e(\bar{\gamma}, \theta, \rho) = e(O\bar{\gamma}\tilde{O}, O\theta, \rho)$, for any O given by (4.1). $|\theta|^2$ is such an invariant; together with the invariants $\operatorname{tr} \bar{\gamma}$ and $\operatorname{tr} \bar{\gamma}^2$, which are the same as in the 2d case, we have the following four invariants:

$$\operatorname{tr} \bar{\gamma}, \ \operatorname{tr} \bar{\gamma}^2, \ |\theta|^2, \ \rho. \tag{4.2}$$

Thus, starting with six variables, we have eliminated two and are left with an energy per unit mass of the form

$$e = e(\mu_1, \mu_2, |\theta|^2, \rho),$$

where $\mu_{1,2}$ are the eigenvalues of $\bar{\gamma}_{AB}$ with respect to $\overset{\circ}{\bar{\gamma}}_{AB}$ in the XY-plane $(A, B = 1, 2)$.

We may choose in particular:

$$e = \frac{1}{2} \left((\mu_1 - 1)^2 + (\mu_2 - 1)^2 \right) + \frac{\alpha}{2} |\theta|^2 + \frac{\beta}{2} (\rho - 1)^2, \tag{4.3}$$

which α, β are positive constants, which has a minimum at the identity $\gamma_{AB} = \overset{\circ}{\gamma}_{AB} = \delta_{AB}$. It is not surprising that the screw dislocations, which have a distinguished direction, break the isotropy and we are lead to an anisotropic energy of the form (4.3), which is invariant under the transformation (4.1).

4.2   The Heisenberg group as a homogeneous space

Let

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z} \tag{4.4}$$

be, as in Section 4, the basis for the Lie algebra $\mathcal{V}$ of the Heisenberg group. We have the commutation relations

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0. \tag{4.5}$$

The basis of 1-forms (ξ, ζ, η) dual to X, Y, Z is

$$\xi = dx, \quad \zeta = dy, \quad \eta = dz - xdy.$$

A left-invariant metric on the Heisenberg group is, up to isometries and an overall scale factor, given by:

$$\overset{\circ}{n} = \xi \otimes \xi + \zeta \otimes \zeta + e^{2\beta} \eta \otimes \eta, \tag{4.6}$$

that is:

$$ds^2 = dx^2 + dy^2 + e^{2\beta}(dz - xdy)^2, \tag{4.7}$$

where β is a real constant. The metric (4.7) is called a Bianchi type VII metric (for the classification into Bianchi types, see [12]). It represents a homogeneous space which is not isotropic.

The inner product $\overset{\circ}{\gamma}$ on $\mathcal{V}$ giving rise to the metric $\overset{\circ}{n}$ is given in the basis (X, Y, Z) by:

$$\overset{\circ}{\gamma}(X, X) = \overset{\circ}{\gamma}(Y, Y), \quad \overset{\circ}{\gamma}(Z, Z) = e^{2\beta},$$
$$\overset{\circ}{\gamma}(X, Y) = \overset{\circ}{\gamma}(X, Z) = \overset{\circ}{\gamma}(Y, Z) = 0.$$

Thus defining

$$E_1 = X, \quad E_2 = Y, \quad E_3 = e^{-\beta} Z, \tag{4.8}$$

$(E_A : A = 1, 2, 3)$ is an orthonormal basis for $\overset{\circ}{\gamma}$:

$$\overset{\circ}{\gamma}_{AB} := \overset{\circ}{\gamma}(E_A, E_B) = \delta_{AB}. \tag{4.9}$$

The dual basis is:

$$\omega^1 = \xi, \quad \omega^2 = \zeta, \quad \omega^3 = e^\beta \eta. \tag{4.10}$$

In the following, we shall denote the metric $\overset{\circ}{n}$ by h. Thus, in terms of the basis $(\omega^A : A = 1, 2, 3)$ we have:

$$h = \sum_A \omega^A \otimes \omega^A.$$

Now $(\mathcal{N}, h)$ being a homogeneous space, any point in $\mathcal{N}$ may be taken as the origin. Consider a local coordinate system $(y^a : a = 1, 2, 3)$ on $\mathcal{N}$ with origin a given point. Since in the Euclidean space E^3 we may set up a rectangular coordinate system $(x^i : i = 1, 2, 3)$, a mapping id is then defined in the domain of the coordinate system on $\mathcal{N}$ by:

$$id(y^1, y^2, y^3) = (y^1, y^2, y^3) \in E^3,$$

that is, id is expressed in the respective coordinate systems as the identity mapping. Thus m, the pullback to $\mathcal{N}$ by id of the Euclidean metric $\delta_{ij} dx^i \otimes dx^j$ is

$$m = \delta_{ab} dy^a \otimes dy^b$$

in the local coordinates $(y^a : a = 1, 2, 3)$ on $\mathcal{N}$. The following problem then arises in connection with the analytic method to be presented below: given any point in $\mathcal{N}$, find a local coordinate system $(y^a : a = 1, 2, 3)$ on $\mathcal{N}$ with origin the given point such that γ_{AB}, the components at y of the corresponding inner product on $\mathcal{V}$ in the basis $(E_A : A = 1, 2, 3)$, that is:

$$\gamma_{AB}(y) = m_{ab}(y) E_A^a(y) E_B^b(y) = \delta_{ab} E_A^a(y) E_B^b(y)$$

satisfy:

$$\gamma_{AB}(y) - \overset{\circ}{\gamma}_{AB} = O(|y|^2).$$

(Recall that $\overset{\circ}{\gamma}_{AB} = \delta_{AB}$.) The solution is simply to set up Riemannian normal coordinates on $(\mathcal{N}, h)$ with origin the given point. For, the components of h in such coordinates satisfy:

$$h_{ab}(y) = \delta_{ab} + O(|y|^2).$$

Since

$$h_{ab}(y)E_A^a(y)E_B^b(y) = \delta_{AB}$$

identically, then

$$\gamma_{AB}(y) - \delta_{AB} = (\delta_{ab} - h_{ab}(y)) E_A^a(y)E_B^b(y) = O(|y|^2),$$

as required.

5   Scaling properties

5.1   The general isotropic case

Let $\mathcal{N} = \Omega \subset \mathbb{R}^n$ and $\phi(\Omega) \subset E^n$ be the domain and the target space, respectively, of the mapping ϕ with the same properties as in (1.1). We fix coordinates in $\mathcal{N}$ and also fix an origin in E^n. We may assume that the coordinate origin in $\mathcal{N}$ is mapped by ϕ to the origin in E^n. Consider

$$\tilde{\phi}(y) = l\phi\left(\frac{y}{l}\right),$$

where $l > 0$ is a given positive constant. If U is a domain in E^n, we denote by lU the domain $\{lx : x \in U\}$. Similarly for a domain in $\mathbb{R}^n$. The domain of $\tilde{\phi}$ is $\tilde{\Omega} = l\Omega$, and

$$\tilde{\phi} : \tilde{\Omega} = l\Omega \to \tilde{\phi}(\tilde{\Omega}) = l\phi(\Omega).$$

The change from ϕ to $\tilde{\phi}$ induces a change from m to $\tilde{m}$ (and similarly for the inverses) as follows:

$$\tilde{m}_{ab}(y) = \sum_{i=1}^n \frac{\partial\tilde{\phi}^i(y)}{\partial y^a}\frac{\partial\tilde{\phi}^i(y)}{\partial y^b} = m_{ab}\left(\frac{y}{l}\right), \quad \left(\tilde{m}^{-1}\right)^{ab}(y) = \left(m^{-1}\right)^{ab}\left(\frac{y}{l}\right).$$
(5.1)

For, since $\tilde{\phi}^i(y) = l\phi^i\left(\frac{y}{l}\right)$, we have

$$\frac{\partial\tilde{\phi}^i(y)}{\partial y^a} = l\frac{\partial\phi^i(y/l)}{\partial y^a}\frac{1}{l} = \frac{\partial\phi^i(y/l)}{\partial y^a}.$$
(5.2)

We consider here the case of an isotropic energy function. We have seen that in this case the crystalline structure $\mathcal{V}$ can be eliminated in favor of a Riemannian metric h.

Let us define a new metric $\tilde{h}$ on $\mathcal{N}$ by $\tilde{h}_{ab}(y) = h_{ab}(\frac{y}{l})$. Note that $\tilde{h}$ is *not* isometric to h. In the case of the toy energy function (3.2), we have (see (1.1) below),

$$\sqrt{\frac{\det \tilde{m}}{\det \tilde{h}}} \tilde{S}^{ab} = 2 \left(\tilde{h}^{-1} \right)^{ac} \left(\tilde{h}^{-1} \right)^{bd} \left(\tilde{h}_{cd} - \tilde{m}_{cd} \right),$$

and similarly with $\tilde{h}$, $\tilde{m}$, $\tilde{S}$ replaced by h, m, S. Using (5.1) we then obtain:

$$\tilde{S}^{ab}(y) = S^{ab}\left(\frac{y}{l}\right). \tag{5.3}$$

Now equations (2.22) transform as follows. From the definition of the covariant derivative, we have

$$\overset{m}{\nabla}_b S^{ab} = \frac{\partial S^{ab}}{\partial y^b} + \overset{m}{\Gamma^a_{bc}} S^{bc} + \overset{m}{\Gamma^b_{ac}} S^{ac}, \tag{5.4}$$

where $\overset{m}{\Gamma^a_{bc}}$ are the Christoffel symbols with respect to m defined by

$$\overset{m}{\Gamma^a_{bc}} = \frac{1}{2}(m^{-1})^{ad} \left(\frac{\partial m_{bd}}{\partial y^c} + \frac{\partial m_{cd}}{\partial y^b} - \frac{\partial m_{bc}}{\partial y^d} \right), \tag{5.5}$$

as well as analogous expressions for $\overset{\tilde{m}}{\nabla}_b \tilde{S}^{ab}$ and $\overset{\tilde{m}}{\Gamma^a_{bc}}$. From (5.1) and (5.3) we then obtain

$$\frac{\partial \tilde{m}_{bc}}{\partial y^d}(y) = \frac{1}{l} \frac{\partial m_{bc}}{\partial y^d}\left(\frac{y}{l}\right) \quad \text{and} \quad \frac{\partial \tilde{S}^{ab}}{\partial y^b}(y) = \frac{1}{l} \frac{\partial S^{ab}}{\partial y^b}\left(\frac{y}{l}\right), \tag{5.6}$$

and, consequently, from (5.5),

$$\overset{\tilde{m}}{\Gamma^a_{bc}}(y) = \frac{1}{l} \overset{m}{\Gamma^a_{bc}}\left(\frac{y}{l}\right). \tag{5.7}$$

We conclude, using (5.4) with (5.6) and (5.7),

$$\overset{\tilde{m}}{\nabla}_b \tilde{S}^{ab}(y) = \frac{1}{l} \left(\overset{m}{\nabla}_b S^{ab} \right)\left(\frac{y}{l}\right).$$

So, if ϕ is a solution relative to h and Ω, $\tilde{\phi}$ is a solution relative to $\tilde{h}$ and $\tilde{\Omega}$, the tangent plane to $\tilde{\Omega} = l\Omega$ at the point $ly \in \partial\tilde{\Omega}$ being parallel to the tangent plane to $\partial\Omega$ at y relative to the linear structure of $\mathbb{R}^n \supset \Omega$.

If $(E_A : A = 1, \ldots, n)$ is an orthonormal frame field for h then $(\tilde{E}_A : A = 1, \ldots, n)$, with $\tilde{E}_A$ defined by

$$\tilde{E}_A^a(y) = E_A^a\left(\frac{y}{l}\right),$$

is an orthonormal frame field for $\tilde{h}$. Also, using a formula similar to (5.7) with h in the role of m, we deduce

$$\tilde{R}^a_{bcd}(y) = l^{-2} R^a_{bcd}\left(\frac{y}{l}\right).$$

It follows that, if K_Π is the sectional curvature of h corresponding to the plane Π at y and $\tilde{K}_{\tilde{\Pi}}$ is the sectional curvature of $\tilde{h}$ corresponding to the plane $\tilde{\Pi}$ at $\tilde{y} = S_l y = ly$, where $\tilde{\Pi} = dS_l(\Pi)$ (S_l the scaling map), then

$$\tilde{K}_{\tilde{\Pi}} = l^{-2} K_\Pi.$$

5.2   The 2d case

Consider the 2d case. The metric of the hyperbolic plane of curvature $-\varepsilon^2$ is given in Riemannian normal coordinates by:

$$h_{ab} = \delta_{ab} + \varepsilon^2 f_{ab}(y),$$
$$f_{ab}(y) = f(\varepsilon^2 r^2) l_{ab}(y),$$
$$l_{ab}(y) = \delta_{ab} r^2 - y^a y^b,$$

and $f(z)$ is an entire function with $f(0) = \frac{1}{3}$.

We now consider the metric $\tilde{h}$, the components of which in the above coordinates are: $\tilde{h}_{ab}(y) = h_{ab}(y/l)$. We have:

$$\tilde{h}_{ab}(y) = \delta_{ab} + \varepsilon^2 f_{ab}\left(\frac{y}{l}\right),$$
$$f_{ab}\left(\frac{y}{l}\right) = f\left(\varepsilon^2 \frac{r^2}{l^2}\right) \frac{1}{l^2} l_{ab}(y),$$

therefore,

$$\tilde{h}_{ab}(y) = \delta_{ab} + \tilde{\varepsilon}^2 f\left(\tilde{\varepsilon}^2 r^2\right) l_{ab}(y) = \delta_{ab} + \tilde{\varepsilon}^2 \tilde{f}_{ab}(y),$$

where $\tilde{\varepsilon} = \varepsilon/l$ and $\tilde{f}_{ab}(y)$ is $f_{ab}(y)$ with ε replaced by $\tilde{\varepsilon}$. While the curvature K of h is $-\varepsilon^2$, the curvature $\tilde{K}$ of $\tilde{h}$ is

$$-\tilde{\varepsilon}^2 = -\frac{\varepsilon^2}{l^2}.$$

Given a domain Ω_1, we shall show in Part III below that there is a $\varepsilon_1 > 0$ such that we can solve equations (2.22) for all $0 < \varepsilon < \varepsilon_1$. Let $\phi_{1,\varepsilon}$ be the solution corresponding to Ω_1 and $\varepsilon \in (0, \varepsilon_1)$. Ω_1 is a domain in the hyperbolic plane of curvature $-\varepsilon^2$. We define $\phi_{l,\varepsilon/l}$ by

$$\phi_{l,\varepsilon/l}(y) = l\phi_{1,\varepsilon}\left(\frac{y}{l}\right). \tag{5.8}$$

This is the solution corresponding to the domain $\Omega_l = l\Omega_1$ in the hyperbolic plane of curvature $-\varepsilon^2/l^2$.

Choosing then $l = \varepsilon$ we have from (5.8)

$$\phi_{l,1}(y) = l\phi_{1,l}\left(\frac{y}{l}\right) = l\phi_{1,\varepsilon}\left(\frac{y}{l}\right)$$

a solution of the problem for the domain $\Omega_l := l\Omega_1 = \varepsilon\Omega_1$ in the hyperbolic plane of curvature $-\varepsilon^2/l^2 = -1$, that is, the standard hyperbolic plane. In conclusion, once we have a solution for the domain Ω_1 and curvature $-\varepsilon^2$, we automatically have a solution for the smaller (rescaled) domain $\Omega_l = \varepsilon\Omega_1$ and curvature -1.

Consider the stress T^{ij} and its rescaled version $\tilde{T}^{ij}$ at the respective points in Euclidean space. From (5.3) and (5.2) we conclude:

$$T^{ij}\left(\phi(y)\right) = S^{ab}(y)\frac{\partial \phi^i(y)}{\partial y^a}\frac{\partial \phi^j(y)}{\partial y^b},$$

$$\tilde{T}^{ij}\left(\tilde{\phi}(y)\right) = \tilde{S}^{ab}(y)\frac{\partial \tilde{\phi}^i(y)}{\partial y^a}\frac{\partial \tilde{\phi}^j(y)}{\partial y^b} = S^{ab}\left(\frac{y}{l}\right)\frac{\partial \phi^i(y/l)}{\partial y^a}\frac{\partial \phi^j(y/l)}{\partial y^b}$$

$$= T^{ij}\left(\phi\left(\frac{y}{l}\right)\right).$$

The rescaled stress $\tilde{T}$ at the rescaled point $\tilde{\phi}(y)$ is thus the same as the original stress T at the point $\phi(y/l)$.

5.3 The 3d case

Let $(E_A : A = 1, \ldots, 3)$ be the vectorfields (4.8) that satisfy the commutation relations $[E_1, E_2] = e^\beta E_3$, $[E_1, E_3] = [E_2, E_3] = 0$ and $(\omega^A : A = 1, \ldots, 3)$ the dual 1-forms (4.10), so that

$$h = \sum_{A=1}^{3} \omega^A \otimes \omega^A, \quad h_{AB} = h(E_A, E_B) = \delta_{AB}. \tag{5.9}$$

We introduce Riemannian normal coordinates $(y^a : a = 1, 2, 3)$ for h at a given point in $\mathcal{N}$, which we take as the origin. The components of the metric h in these coordinates are of the form:

$$h_{ab}(y) = \delta_{ab} + \epsilon_{ab}(y), \quad \epsilon_{ab}(y) = O(|y|^2). \tag{5.10}$$

Let id be the identity map defined in Section 4.2 and m the pullback by id of the Euclidean metric $\delta_{ij} dx^i \otimes dx^j$. Then the components m_{ab} of m in the coordinates $(y^a : a = 1, 2, 3)$ are simply $m_{ab} = \delta_{ab}$ and the corresponding inner product on $\mathcal{V}$, which depends on y, is given by:

$$\begin{aligned} \gamma_{AB}(y) &:= \gamma(y)(E_A, E_B) \\ &= m_{ab}(y) E_A^a(y) E_B^b(y) \\ &= \delta_{ab} E_A^a(y) E_B^b(y) \\ &= \left(h_{ab}(y) - \epsilon_{ab}(y) \right) E_A^a(y) E_B^b(y) \\ &= \delta_{AB} - \epsilon_{AB}(y), \end{aligned}$$

where $\epsilon_{AB}(y) = \epsilon_{ab}(y) E_A^a(y) E_B^b(y) = O(|y|^2)$.

We now dilate the Heisenberg group metric h by the factor $l > 1$, i.e., we set:

$$\tilde{h} = l^2 h, \quad \tilde{h}(E_A, E_B) = l^2 h(E_A, E_B) = l^2 \delta_{AB}.$$

Define now $\tilde{E}_A = l^{-1} E_A$, so

$$\tilde{h}(\tilde{E}_A, \tilde{E}_B) = \delta_{AB},$$

that is $(\tilde{E}_A : A = 1, 2, 3)$ is an orthonormal frame field relative to $\tilde{h}$. We denote by $\overset{\circ}{\tilde{\gamma}}$ the corresponding inner product on $\mathcal{V}$, i.e., $\overset{\circ}{\tilde{\gamma}}(\tilde{E}_A, \tilde{E}_B) = \delta_{AB}$.

Remark that the commutation relations of the frame field $(\tilde{E}_A : A = 1, 2, 3)$ read

$$[\tilde{E}_1, \tilde{E}_2] = l^{-1} e^\beta \tilde{E}_3, \quad [\tilde{E}_1, \tilde{E}_3] = [\tilde{E}_2, \tilde{E}_3] = 0.$$

Now:

$$\tilde{h} = l^2 h = l^2 h_{ab}(y) dy^a \otimes dy^b.$$

Set $\tilde{y}^a = ly^a$, then $\tilde{h} = \tilde{h}_{ab}(\tilde{y}) d\tilde{y}^a \otimes \tilde{y}^b$ and hence

$$\tilde{h}_{ab}(\tilde{y}) = h_{ab}\left(\frac{\tilde{y}}{l}\right). \tag{5.11}$$

Consider the geodesic ray through the origin of the coordinate system y^a:

$$y^a = \lambda^a t, \quad \tilde{y}^a = \tilde{\lambda}^a t, \quad \text{where} \quad \tilde{\lambda}^a = l\lambda^a.$$

For the Christoffel symbols of the metric h with respect to the coordinates y^a, we have:

$$\Gamma^a_{bc}(\lambda t) \lambda^b \lambda^c = 0.$$

But from (5.7),

$$\tilde{\Gamma}^a_{bc}(\tilde{y}) = \frac{1}{l} \Gamma^a_{bc}\left(\frac{\tilde{y}}{l}\right),$$

hence:

$$\tilde{\Gamma}^a_{bc}(\tilde{\lambda} t) \tilde{\lambda}^b \tilde{\lambda}^c = \frac{1}{l} \Gamma^a_{bc}(\lambda t) l^2 \lambda^b \lambda^c = 0. \tag{5.12}$$

Thus $(\tilde{y}^a : a = 1, 2, 3)$ are Riemannian normal coordinates for $\tilde{h}$.

We remark that the mapping id defined in Section 4.2 depends on the choice of local coordinates in $\mathcal{N}$. Let us denote by $\tilde{id}$ the mapping associated to the coordinates $(\tilde{y}^a : a = 1, 2, 3)$, reserving the notation id for the mapping associated to the original coordinates $(y^a : a = 1, 2, 3)$. Let $\tilde{m}$ be the pullback of the Euclidean metric $\delta_{ij} dx^i \otimes dx^j$ by $\tilde{id}$. Then the components $\tilde{m}_{ab}$ of $\tilde{m}$ in the coordinates $(\tilde{y}^a : a = 1, 2, 3)$ are again simply $\tilde{m}_{ab} = \delta_{ab}$.

Now, by (5.10) and (5.11),

$$\tilde{h}_{ab}(\tilde{y}) = \delta_{ab} + \tilde{\epsilon}_{ab}(\tilde{y}),$$

where

$$\tilde{\epsilon}_{ab}(\tilde{y}) = \epsilon_{ab}\left(\frac{\tilde{y}}{l}\right).$$

Therefore,

$$\tilde{\epsilon}_{ab}(\tilde{y}) = O\left(l^{-2}|\tilde{y}|^2\right),$$

and we obtain, in analogy with the above:

$$\begin{aligned}
\tilde{\gamma}_{AB}(\tilde{y}) &:= \tilde{\gamma}(\tilde{y})(\tilde{E}_A, \tilde{E}_B) \\
&= \tilde{m}_{ab}(\tilde{y})\tilde{E}_A^a(\tilde{y})\tilde{E}_B^b(\tilde{y}) \\
&= \delta_{ab}\tilde{E}_A^a(\tilde{y})\tilde{E}_B^b(\tilde{y}) \\
&= \left(\tilde{h}_{ab}(\tilde{y}) - \tilde{\epsilon}_{ab}(\tilde{y})\right)\tilde{E}_A^a(\tilde{y})\tilde{E}_B^b(\tilde{y}) \\
&= \delta_{AB} - \tilde{\epsilon}_{AB}(\tilde{y}),
\end{aligned}$$

where $\tilde{\epsilon}_{AB}(\tilde{y}) = \tilde{\epsilon}_{ab}(\tilde{y})\tilde{E}_A^a(\tilde{y})\tilde{E}_B^b(\tilde{y}) = O(l^{-2}|\tilde{y}|^2)$. Here $\tilde{E}_A^a$ are the components of the vectorfield $\tilde{E}_A$ in the coordinate system $(\tilde{y}^a : a = 1, 2, 3)$.

Let now $\tilde{\Omega}$ be a fixed domain in the $\tilde{y}$ coordinates containing the origin. Let $\tilde{\phi}$ be a solution of the boundary value problem

$$\begin{cases} \overset{\tilde{m}}{\nabla}_b \tilde{S}^{ab} = 0 & : \quad \text{in} \quad \tilde{\Omega}, \\ \tilde{S}^{ab}\tilde{M}_b = 0 & : \quad \text{on} \quad \partial\tilde{\Omega}, \end{cases} \tag{5.13}$$

such that $\tilde{\phi}$ takes the $\tilde{y}$ coordinate origin in $\mathcal{N}$ to the x-coordinate origin in E^3. In (5.13) $\tilde{S}^{ab} = \tilde{\pi}^{AB}\tilde{E}_A^a\tilde{E}_B^b$ and, by definition,

$$\sqrt{\frac{\det\tilde{\gamma}}{\det\overset{\circ}{\tilde{\gamma}}}}\tilde{\pi}^{AB} = -2\frac{\partial\tilde{e}}{\partial\tilde{\gamma}_{AB}}.$$

Since $\tilde{h} = \tilde{h}_{ab}(\tilde{y})d\tilde{y}^a \otimes d\tilde{y}^b$ and $y^a = l^{-1}\tilde{y}^a$ we have

$$h = l^{-2}\tilde{h} = l^{-2}\tilde{h}_{ab}(\tilde{y})d\tilde{y}^a \otimes d\tilde{y}^b = \tilde{h}_{ab}(ly)dy^a \otimes dy^b,$$

hence $h_{ab}(y) = \tilde{h}_{ab}(ly)$. We now consider the rescaled (smaller) domain $\Omega = \{y = l^{-1}\tilde{y}, \tilde{y} \in \tilde{\Omega}\}$ and define the mapping

$$\phi(y) = l^{-1}\tilde{\phi}(ly): \quad \forall y \in \Omega.$$

The pullback by ϕ of the Euclidean metric is m, the components of which in the y^a coordinates are

$$m_{ab}(y) = \delta_{ij}\frac{\partial \phi^i}{\partial y^a}(y)\frac{\partial \phi^j}{\partial y^b}(y).$$

Since

$$\frac{\partial \phi^i}{\partial y^a}(y) = \frac{\partial \tilde{\phi}^i}{\partial \tilde{y}^a}(\tilde{y}),$$

we have $m_{ab}(y) = \tilde{m}_{ab}(\tilde{y})$.

Hence, the corresponding inner product on $\mathcal{V}$, $\gamma_{AB}(y) = \gamma(y)(E_A, E_B)$ in terms of the original basis $(E_A : A = 1, 2, 3)$, is:

$$\gamma_{AB}(y) = m_{ab}(y)E_A^a(y)E_B^b(y) = \tilde{m}_{ab}(\tilde{y})\tilde{E}_A^a(\tilde{y})\tilde{E}_B^b(\tilde{y}) = \tilde{\gamma}_{AB}(\tilde{y}).$$

For,

$$\tilde{E}_A^a(\tilde{y})\frac{\partial}{\partial \tilde{y}^a} = l^{-1}E_A^a(y)\frac{\partial}{\partial y^a},$$

hence $\tilde{E}_A^a(\tilde{y}) = E_A^a(y)$. For the stress tensors in both coordinate systems we have $\pi^{AB}(\gamma(y)) = \tilde{\pi}^{AB}(\tilde{\gamma}(\tilde{y}))$ because intrinsically $\tilde{e}(\tilde{\gamma}) = e(\gamma)$. Therefore:

$$\tilde{S}^{ab}(\tilde{y}) = \tilde{\pi}^{AB}(\tilde{\gamma}(\tilde{y}))\tilde{E}_A^a(\tilde{y})\tilde{E}_B^b(\tilde{y}) = \pi^{AB}(\gamma(y))E_A^a(y)E_B^b(y) = S^{ab}(y),$$

which shows that the stress is scaling invariant, as expected.

Remark II.3. *The energy per unit mass e is a function of the configuration γ, an inner product on $\mathcal{V}$. However, when representing e as a function of the components γ_{AB} of γ in a basis $(E_A : A = 1, \ldots, n)$, which is orthonormal relative to $\overset{\circ}{\gamma}$, e in this representation depends indirectly on $\overset{\circ}{\gamma}$.*

Consider the equations in (5.13). Starting from $\tilde{m}_{ab}(\tilde{y}) = m_{ab}(y)$,

$$\frac{\partial \tilde{m}_{ab}}{\partial \tilde{y}^c}(\tilde{y}) = l^{-1}\frac{\partial \tilde{m}_{ab}}{\partial y^c}(\tilde{y}) = l^{-1}\frac{\partial m_{ab}}{\partial y^c}(y),$$

the Christoffel symbols Γ^c_{ab} transform accordingly

$$\overset{\tilde{m}}{\Gamma}{}^c_{ab}(\tilde{y}) = l^{-1}\overset{m}{\Gamma}{}^c_{ab}(y). \qquad (5.14)$$

Since we have

$$\frac{\partial \tilde{S}^{ab}}{\partial \tilde{y}^c}(\tilde{y}) = l^{-1}\frac{\partial S^{ab}}{\partial y^c}(y), \qquad (5.15)$$

we finally obtain, using (5.14), (5.15),

$$\left(\overset{\tilde{m}}{\nabla}_b\tilde{S}^{ab}\right)(\tilde{y}) = \frac{\partial \tilde{S}^{ab}}{\partial \tilde{y}^b}(\tilde{y}) + \overset{\tilde{m}}{\Gamma}{}^a_{bc}(\tilde{y})\tilde{S}^{bc}(\tilde{y}) + \overset{\tilde{m}}{\Gamma}{}^b_{ac}(\tilde{y})\tilde{S}^{ac}(\tilde{y}) = l^{-1}\left(\overset{m}{\nabla}_bS^{ab}\right)(y).$$

Therefore, once we have a solution for the equation in the $\tilde{y}$ coordinates, we immediately obtain the solution for the original equation and the boundary condition is also satisfied since $\tilde{M}_a(\tilde{y}) = M_a(y)$.

Part III   The analysis of equilibrium configurations

1   Stress tensor

We will now show that the stress tensor S on the material manifold $\mathcal{N}$ is given by

$$\sqrt{\frac{\det m}{\det \overset{\circ}{n}}}S^{ab} = -2\frac{\partial e}{\partial m_{ab}}. \qquad (1.1)$$

Recall first the definition (1.3) of the thermodynamic stress π on the crystalline structure $\mathcal{V}$

$$\sqrt{\frac{\det \gamma}{\det \overset{\circ}{\gamma}}}\pi^{AB} = -2\frac{\partial e}{\partial \gamma_{AB}}, \qquad (1.2)$$

where we have made use of the definition of the volume $V(\gamma) = \sqrt{\frac{\det \gamma}{\det \mathring{\gamma}}}$ in the frame $(E_1, \ldots, E_n)$ satisfying $\omega(E_1, \ldots, E_n) = 1$, $\mathring{\gamma}_{AB} = \delta_{AB}$ (so in fact $\det \mathring{\gamma} = 1$).

Now, from (2.8) we have

$$\frac{\partial e}{\partial m_{ab}} = \frac{\partial e}{\partial \gamma_{AB}} \frac{\partial \gamma_{AB}}{\partial m_{ab}} = \frac{\partial e}{\partial \gamma_{AB}} E_A^a E_B^b,$$

hence, with $E(y) = \omega(y)^{-1}$,

$$\frac{\partial e}{\partial \gamma_{AB}} = \frac{\partial e}{\partial m_{ab}} \omega_a^A \omega_b^B, \tag{1.3}$$

and finally, using (2.9), (1.2), (1.3) and the equality

$$\frac{\det m}{\det \mathring{n}} = \frac{\det \gamma}{\det \mathring{\gamma}}, \tag{1.4}$$

we obtain:

$$\sqrt{\frac{\det m}{\det \mathring{n}}} S^{ab} \omega_a^A \omega_b^B = \sqrt{\frac{\det \gamma}{\det \mathring{\gamma}}} \pi^{AB} = -2 \frac{\partial e}{\partial \gamma_{AB}} = -2 \frac{\partial e}{\partial m_{ab}} \omega_a^A \omega_b^B,$$

which directly implies (1.1). The equality (1.4) is a special case of the following proposition, which applies to the isotropic case.

Proposition III.1. *The eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of* γ *with respect to* $\mathring{\gamma}$ *coincide with the eigenvalues of* m *with respect to* $\mathring{n}$.

Proof. We define $A(y) \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ by

$$\mathring{\gamma}(A(y)Y_1, Y_2) = \gamma(y)(Y_1, Y_2) : \quad \forall Y_1, Y_2 \in \mathcal{V}.$$

Hence, $A(y)E_B = A_B^A(y)E_A$, where

$$A_B^A(y) = (\mathring{\gamma}^{-1})^{AC} \gamma_{BC}(y).$$

Thus, $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

Similarly, we define $B(y) \in \mathcal{L}(T_y\mathcal{N}, T_y\mathcal{N})$ by

$$\overset{\circ}{n}(B(y)Y_{1,y}, Y_{2,y}) = m(y)(Y_{1,y}, Y_{2,y}) \quad : \forall Y_{1,y}, Y_{2,y} \in T_y\mathcal{N}.$$

and using $B(y)\frac{\partial}{\partial y^a}|_y = B_a^b(y)\frac{\partial}{\partial y^b}|_y$, we want to express $A(y)$ in terms of $B(y)$.

Since the evaluation map ϵ_y is an isomorphism from $\mathcal{V}$ to $T_y\mathcal{N}$ for each $y \in \mathcal{N}$, we have

$$A(y) = \epsilon_y^{-1} \circ B(y) \circ \epsilon_y.$$

Hence, using $E_B(y) = E_B^b(y)\frac{\partial}{\partial y^b}|_y$,

$$A_B^A(y)E_A = A(y)E_B = \epsilon_y^{-1}(B(y)E_B(y))$$

$$= \epsilon_y^{-1}\left(E_B^b(y)B_b^a(y)\left.\frac{\partial}{\partial y^a}\right|_y\right)$$

$$= E_B^b(y)B_b^a(y)\omega_a^A(y)E_A,$$

we conclude

$$A(y) = E(y)B(y)\omega(y) = \omega(y)^{-1}B(y)\omega(y), \quad E(y) = \omega(y)^{-1},$$

i.e., the linear mappings $A(y)$ and $B(y)$ are conjugate and therefore $\lambda_1, \ldots, \lambda_n$ are also the eigenvalues of $B(y)$. $\qquad\square$

Let us now calculate the stress tensor S^{ab} corresponding to the toy energy (3.1). Here, we denote $\overset{\circ}{n}$ by h, the metric of the hyperbolic plane that is the metric of the material manifold for a uniform distribution of elementary edge dislocations. Recall that

$$e = \frac{1}{2}\left((\lambda_1 - 1)^2 + (\lambda_2 - 1)^2\right)$$

$$= \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 - 2(\lambda_1 + \lambda_2) + 2\right)$$

$$= \frac{1}{2}\text{tr}_h(m^2) - \text{tr}_h m + 1$$

$$= \frac{1}{2}\left(h^{-1}\right)^{ac}\left(h^{-1}\right)^{bd}m_{ab}m_{cd} - \left(h^{-1}\right)^{ab}m_{ab} + 1.$$

Then,

$$\sqrt{\frac{\det m}{\det h}} S^{ab} = -2 \frac{\partial e}{\partial m_{ab}} = -2 \left(\left(h^{-1}\right)^{ac} \left(h^{-1}\right)^{bd} m_{cd} - \left(h^{-1}\right)^{ab} \right), \quad (1.5)$$

or,

$$S^{ab} = 2\sqrt{\frac{\det h}{\det m}} \left(h^{-1}\right)^{ac} \left(h^{-1}\right)^{bd} \left(h_{cd} - m_{cd}\right). \quad (1.6)$$

The physical interpretation of (1.6) is the following:

(i) If m is smaller than h then the stress is positive,
(ii) if m is larger than h then the stress is negative.

Recall that a quadratic form $q = h - m$ on $T_y \mathcal{N}$ is said to be positive (negative) if, for all $v \in T_y \mathcal{N}$,

$$q(v, v) > 0 \, (< 0): \quad \forall v \neq 0.$$

2   Setup and method in 2d

As was shown in Part I, Section 1.1.1, the material manifold $\mathcal{N}$ for the case of a uniform distribution of edge dislocations in two dimensions is given by the affine group, and a left-invariant metric on $\mathcal{N}$ gives $\mathcal{N}$ the structure of the hyperbolic plane H_ε of curvature $-\varepsilon^2$.

To solve the problem in this case, we fix an origin in H_ε and set up Riemannian normal coordinates $(y^a : a = 1, 2)$ as in Section 4.2 of Part II. Let Ω be any smooth bounded domain in these coordinates, containing the origin. Note that as $\varepsilon \to 0$, $H_\varepsilon|_\Omega$ tends to $H_0|_\Omega$, where $H_0 = E$ is the Euclidean plane. We also choose an origin and set up rectangular coordinates $(x^i : i = 1, 2)$ in E. An identity mapping

$$\begin{aligned} id : H_\varepsilon &\to E \\ (y^1, y^2) &\mapsto (x^1, x^2) = (y^1, y^2) \end{aligned} \quad (2.1)$$

is then defined as in Section 4.2 of Part II.

We now restrict the allowed mappings $\phi : \Omega \subset H_\varepsilon \to E$ by the following two requirements. First, ϕ should map the origin in H_ε into the origin in E. Second, $d\phi(0)$ should map the vector $\frac{\partial}{\partial y^1}|_0$ into a vector of the form $\lambda \frac{\partial}{\partial x^1}|_0$ for some $\lambda > 0$. By virtue of this restriction, the identity mapping

(2.1) (restricted to Ω) is for $\varepsilon = 0$ the unique minimizer of our toy energy e from (3.1).

Remark III.1. *The restriction is needed to ensure uniqueness. Otherwise, composition on the left with a rigid motion of E gives another minimizer. The appropriate restriction in the 3d case will be stated below in Section 3.1. Analogously, it can be formulated in any number of space dimensions. Thus uniqueness is ensured in general. The argument of Part I, Section 3.2 applies with the hyperbolic plane H_ε and the Euclidean plane replaced by the n-dimensional hyperbolic space H_ε^n and the n-dimensional Euclidean space. This is as long as the toy energy (3.1) is considered.*

From (2.22) and (2.25) the system of partial differential equations and the corresponding boundary conditions for the static problem of a uniform distribution of edge dislocations in $n = 2$ dimensions is of the form

$$F_\varepsilon[\phi] = 0,$$

where

$$F_\varepsilon[\phi] = \begin{pmatrix} \overset{m}{\nabla}_b S^{ab} \\ S^{ab} M_b \end{pmatrix}. \tag{2.2}$$

We linearize the equations at the identity mapping, which is a solution for $\varepsilon = 0$. First, we solve the linearized problem using the theorem of Lax–Milgram (as in [2]). An iteration will then show that there exists a solution to the nonlinear problem for sufficiently small ε, and, therefore, by the scaling argument of Section 5 of Part II, that there is a mapping from a rescaled domain $\tilde{\Omega}$ in the standard hyperbolic plane $H_1 = H$ of curvature -1 to the Euclidean plane, $\phi : \tilde{\Omega} \subset H \to E$, satisfying the conditions of the problem.

We set $\phi = id + \psi$, where ψ is a small deviation from the identity mapping. We have:

$$F_0[id] = 0.$$

In a neighborhood of the identity F_ε is of the form

$$F_\varepsilon[\phi] = F_\varepsilon[id] + D_{id}F_\varepsilon \cdot \psi + N_\varepsilon[\psi],$$

where $N_\varepsilon[\psi]$ is to leading order quadratic in ψ. We denote by L_ε the linearized operator $D_{id}F_\varepsilon$. Then $F_\varepsilon[\phi] = 0$ reads

$$L_\varepsilon \cdot \psi = -F_\varepsilon[id] - N_\varepsilon[\psi]. \tag{2.3}$$

We will solve this by an iteration starting at $\psi_0 = 0$. In the first step of the iteration, we have the linearized equations

$$L_\varepsilon \cdot \psi_1 = -F_\varepsilon\,[id]\,. \tag{2.4}$$

The term $-F_\varepsilon[id]$ in (2.3), (2.4) can be interpreted as a source term.

A possible approach to the problem is to study the iteration

$$L_\varepsilon \cdot \psi_{n+1} = -F_\varepsilon\,[id] - N_\varepsilon[\psi_n] \tag{2.5}$$

and show that ψ_n converges to a solution $\psi = \phi - id$ of (2.3), provided that we choose ε appropriately small. However, we follow a different approach.

Adding $L_0 \cdot \psi$ on both sides of (2.3) yields

$$L_0 \cdot \psi = -\,(L_\varepsilon - L_0) \cdot \psi - F_\varepsilon[id] - N_\varepsilon[\psi].$$

What we actually do is to consider the iteration

$$L_0 \cdot \psi_{n+1} = -\,(L_\varepsilon - L_0) \cdot \psi_n - F_\varepsilon[id] - N_\varepsilon[\psi_n]. \tag{2.6}$$

Note that while both (2.5), (2.6) are linear in the next iterate ψ_{n+1}, in (2.5) the operator L_ε which refers to H_ε acts on ψ_{n+1} whereas in (2.6) the operator which refers to $H_0 = E$ acts on ψ_{n+1}.

In (2.6), $L_\varepsilon - L_0$ is a pair of linear operators, a second-order operator in Ω and a first-order operator on $\partial\Omega$. The coefficients of these operators are of order ε^2, which we may write symbolically in the form:

$$L_\varepsilon - L_0 \sim \partial\,(h - \delta) = \varepsilon^2 \partial f, \tag{2.7}$$

since, from (3.7), $h - \delta = \varepsilon^2 f$.

The iteration (2.6) starts also with $\psi_0 = 0$. Then, setting $n = 0$ in (2.6), we have:

$$L_0 \cdot \psi_1 = -F_\varepsilon[id], \tag{2.8}$$

thus, taking into account the fact that $\det h = 1 + O(\varepsilon^2)$, (1.6) implies $S^{ab}(id) = -2\varepsilon^2 f_{ab}|_{\varepsilon=0} + O(\varepsilon^4)$. It follows that ψ_1 is of order ε^2. This is step one of the iteration, the linear level.

For $n \geq 1$, taking the difference between (2.6) and the same with n replaced by $n - 1$, we obtain

$$L_0 \cdot (\psi_{n+1} - \psi_n) = -(L_\varepsilon - L_0) \cdot (\psi_n - \psi_{n-1}) - (N_\varepsilon[\psi_n] - N_\varepsilon[\psi_{n-1}]).$$

In view of (2.7) and the fact that $N_\varepsilon[\psi]$ is to leading order quadratic in ψ, for sufficiently small ε, contraction will hold and ψ_n will converge to a solution ψ of (2.3). Thus, $\phi = id + \psi$ solves the problem (2.2).

Consider now the equations in (2.2). To analyze the problem at the linearized level, we consider the variation $\dot{m}_{ab}$ of the metric m_{ab} at the identity mapping id. Setting

$$\phi^i = y^i + s\psi^i, \tag{2.9}$$

and recalling that

$$m_{ab}(y) = \frac{\partial \phi^i(y)}{\partial y^a} \frac{\partial \phi^i(y)}{\partial y^b},$$

we obtain:

$$\begin{aligned}
\dot{m}_{ab} &= \frac{d}{ds}\bigg|_{s=0} \left[\left(\delta_a^i + s\frac{\partial \psi^i}{\partial y^a} \right) \cdot \left(\delta_b^i + s\frac{\partial \psi^i}{\partial y^b} \right) \right] \\
&= \frac{\partial \psi^i}{\partial y^a} \delta_b^i + \delta_a^i \frac{\partial \psi^i}{\partial y^b} = \frac{\partial \psi^b}{\partial y^a} + \frac{\partial \psi^a}{\partial y^b}.
\end{aligned} \tag{2.10}$$

Hence, $\dot{m}_{ab}$ is the Lie derivative of the metric δ_{ab} on H_ε (the pullback by id of the metric δ_{ij} on E) with respect to the vectorfield $\psi^a \frac{\partial}{\partial y^a}$ on H_ε (which the pushforward by id takes to the vectorfield $\psi^i \frac{\partial}{\partial x^i}$ on E).

3   Analogous geometric linear problem

In the work on the stability of the Minkowski space-time [6], a linear geometric problem was studied analogous to the linear problems in (2.5) and (2.6).

Let (M, g) be a compact Riemannian manifold with boundary ∂M and X a vectorfield on M. We set

$$\pi = \mathcal{L}_X g, \quad \pi_{ij} = \nabla_i X_j + \nabla_j X_i, \quad X_i = g_{ij} X^j, \tag{3.1}$$

the Lie derivative of the metric g along X, a symmetric 2-covariant tensorfield on M. The variation of π with respect to X reads:

$$\dot{\pi}_{ij} = \nabla_i \dot{X}_j + \nabla_j \dot{X}_i. \tag{3.2}$$

We consider the problem of free minimization of the action integral

$$A = \int_M \left(\frac{1}{4} |\pi|_g^2 + \rho^i X_i \right) d\mu_g - \int_{\partial M} \tau^i X_i \, d\mu_g|_{\partial M} . \tag{3.3}$$

Here ρ is a given vectorfield on M and τ is a given vectorfield along ∂M. In mechanical terms, ρ is the body force and τ the boundary force.

The first variation of (3.3), using (3.2), is

$$\dot{A} = \int_M \left(\pi^{ij} \nabla_j \dot{X}_i + \rho^i \dot{X}_i \right) d\mu_g - \int_{\partial M} \tau^i \dot{X}_i \, d\mu_g|_{\partial M} . \tag{3.4}$$

For variations of X_i which vanish near the boundary we have

$$\dot{A} = - \int_M \left(\nabla_j \pi^{ij} - \rho^i \right) \dot{X}_i d\mu_g,$$

and requiring $\dot{A} = 0$ for such variations yields the Euler–Lagrange equations:

$$\nabla_j \pi^{ij} = \rho^i \quad : \text{in } M. \tag{3.5}$$

Requiring then $\dot{A} = 0$ for arbitrary variations:

$$- \int_{\partial M} \tau^i \dot{X}_i \, d\mu_g|_{\partial M} = \int_{\partial M} \left(\pi^{ij} N_j - \tau^i \right) \dot{X}_i \, d\mu_g|_{\partial M} = 0,$$

yields the boundary conditions:

$$\pi^{ij} N_j = \tau^i \quad : \text{on } \partial M. \tag{3.6}$$

Equations (3.5), together with the boundary conditions (3.6), correspond to the linearized boundary value problem (2.7) corresponding to a crystalline solid with a uniform distribution of dislocations in equilibrium if we identify $(\Omega, \dot{m}, id^*\delta)$ with (M, π, g), δ being the Euclidean metric.

We proceed in showing self adjointness of the above operators in the following sense. Let Y be a vectorfield on M, $\sigma = \mathcal{L}_Y g$, i.e.,

$$\sigma_{ij} = \nabla_i Y_j + \nabla_j Y_i = \sigma_{ji}, \quad Y_i = g_{ij} Y^j. \tag{3.7}$$

Then we have by repeated partial integration using (3.1),

$$\langle Y, \nabla \cdot \pi \rangle_{L^2(M)} - \langle Y, \pi \cdot N \rangle_{L^2(\partial M)} = \langle X, \nabla \cdot \sigma \rangle_{L^2(M)} - \langle X, \sigma \cdot N \rangle_{L^2(\partial M)}.$$

Suppose now that Y is a Killing field, i.e., $\sigma = \mathcal{L}_Y g = 0$. Then we have by (3.5), (3.6) and (3.7)

$$\int_M Y_i \rho^i = \int_M Y_i \nabla_j \pi^{ij} = -\frac{1}{2} \int_M \sigma_{ij} \pi^{ij} + \int_{\partial M} Y_i \pi^{ij} N_j = \int_{\partial M} Y_i \tau^i,$$

since $\sigma_{ij} = 0$ on M. Thus, the integrability condition reads:

$$\int_M Y_i \rho^i = \int_{\partial M} Y_i \tau^i. \tag{3.8}$$

This condition guarantees the existence of a solution for the boundary value problem (3.5), (3.6) by the theorem of Lax–Milgram.

3.1   Uniqueness of the solution

In fact, the solution is unique up to an additive Killing field. For, if we take two solutions X_1 and X_2 of (3.5) with (3.6), their difference $X = X_1 - X_2$ satisfies the homogeneous equation of (3.5), i.e., $\rho = 0$ with zero boundary conditions, (3.6) for $\tau = 0$. Therefore, we have, setting $\pi = \mathcal{L}_X g$, $\nabla_j(\pi^{ij} X_i) = \pi^{ij} \nabla_j X_i$,

$$A = \frac{1}{4} \int_M |\pi|^2 = \frac{1}{2} \int_M \pi^{ij} \nabla_j X_i = \frac{1}{2} \int_M \nabla_j \left(\pi^{ij} X_i \right).$$

Using Gauss's theorem we obtain:

$$A = \frac{1}{2} \int_M \nabla_j \left(\pi^{ij} X_i \right) = \frac{1}{2} \int_{\partial M} \underbrace{\left(\pi^{ij} N_j \right)}_{=0} X_i = 0.$$

It follows that $\pi = 0$, and thus $X = X_1 - X_2$ is a Killing field, i.e., the solutions X_1, X_2 only differ by a Killing field.

For our problem, where (M, g) is isometric to a domain in the Euclidean plane, recalling the restriction

$$\phi(0) = 0,$$

$$d\phi(0) \cdot \left.\frac{\partial}{\partial y^1}\right|_0 = \lambda \left.\frac{\partial}{\partial x^1}\right|_0,$$

we obtain, setting

$$\phi^i(x) = y^i + sX^i(y),$$

the conditions

$$sX^i(0) = 0,$$

$$\delta^i_1 + s\frac{\partial X^i}{\partial y^1}(0) = \lambda(s)\delta^i_1.$$

Taking then the derivative with respect to s at $s = 0$ yields the linearized conditions:

$$X^i(0) = 0, \tag{3.9}$$

$$\frac{\partial X^i}{\partial y^1}(0) = \mu\delta^i_1, \tag{3.10}$$

where $\mu = \dot{\lambda}(0)$. The second of the above conditions is

$$\frac{\partial X^2}{\partial y^1}(0) = 0.$$

Substituting the general form of a Killing field,

$$X^i = \alpha^i_j y^j + \beta^i, \quad \alpha^i_j = -\alpha^j_i,$$

it follows from the first condition (3.9) that $\beta^i = 0$, and from the second condition (3.10) that $\alpha^2_1 = 0$, hence $\alpha^i_j = 0$, i.e., $X = 0$. So the solution of the linearized problem is in fact unique.

We remark that in three dimensions, one must add, for uniqueness, the condition that

$$d\phi(0) \cdot \left.\frac{\partial}{\partial y^2}\right|_0$$

is a vector at the origin contained in the plane spanned by $\left.\frac{\partial}{\partial x^1}\right|_0$ and $\left.\frac{\partial}{\partial x^2}\right|_0$. This can always be arranged by a suitable rotation in 3d Euclidean space.

Thus

$$d\phi(0) \cdot \left.\frac{\partial}{\partial y^2}\right|_0 = \lambda_1 \left.\frac{\partial}{\partial x^1}\right|_0 + \lambda_2 \left.\frac{\partial}{\partial x^2}\right|_0.$$

At the linearized level, this additional condition gives:

$$\frac{\partial X^i}{\partial y^2}(0) = \mu_1 \delta_1^i + \mu_2 \delta_2^i, \tag{3.11}$$

where $\mu_1 = \dot{\lambda}_1(0)$, $\mu_2 = \dot{\lambda}_2(0)$. Setting $i = 3$ in (3.11), we have:

$$\frac{\partial X^3}{\partial y^2}(0) = 0,$$

while the conditions from (3.10) read

$$\frac{\partial X^2}{\partial y^1}(0) = \frac{\partial X^3}{\partial y^1}(0) = 0.$$

Hence, if X is a Killing field,

$$\alpha_1^2 = \alpha_1^3 = 0 \quad \text{and} \quad \alpha_2^3 = 0,$$

or, taking into account that also $\beta^i = 0$ $(i = 1, \ldots, 3)$, we conclude that $X = 0$.

4   The linear problem

The unknown of the problem being the mapping $\phi : \Omega \to E$, the pullback metric $m = \phi^* \delta$ depends on ϕ according to:

$$m_{ab}[\phi](y) = \sum_i \frac{\partial \phi^i(y)}{\partial y^a} \frac{\partial \phi^i(y)}{\partial y^b}.$$

Since $m_{ab}(id) = \delta_{ab}$, and $\phi = id + \psi$, i.e., $\phi^i = y^i + \psi^i(y)$, we have:

$$m_{ab} = \delta_{ab} + \dot{m}_{ab} + \mu_{ab}[\psi], \quad \text{where} \quad \dot{m}_{ab} = \frac{\partial \psi^b}{\partial y^a} + \frac{\partial \psi^a}{\partial y^b}$$

and $\mu_{ab}[\psi]$ is quadratic in ψ. It follows from (2.6), together with (2.7), that $\psi = O(\varepsilon^2)$ at the linear level, whence $\psi^2 = O(\varepsilon^4)$. Furthermore, from the

expansion of the hyperbolic metric in rectangular coordinates we have

$$h_{ab} = \delta_{ab} + \varepsilon^2 l_{ab} + O(\varepsilon^4),$$

where $l_{ab} = f_{ab}|_{\varepsilon=0}$.

Linearizing the operator F_ε from (2.2) at the identity mapping, $L_0\psi = D_{id}F_0 \cdot \psi$, we find:

$$L_0\psi = \begin{cases} -2\partial_b \dot{m}_{ab} & : \text{ in } \Omega, \\ -2\dot{m}_{ab}M_b & : \text{ on } \partial\Omega, \end{cases}$$

with $\dot{m}_{ab}$ as above. Moreover, for $\phi = id$ we have, $m_{ab} = \delta_{ab}$, $\overset{m}{\nabla}_b = \frac{\partial}{\partial y^b}$. Hence:

$$\frac{1}{2}S^{ab}(id) = \sqrt{\det h}\,(h^{-1})^{ac}(h^{-1})^{bd}(h_{cd} - m_{cd}) = \varepsilon^2 l_{ab} + O(\varepsilon^4), \qquad (4.1)$$

where we have made use of

$$\sqrt{\det h} = \sqrt{\det(\delta + \varepsilon^2 l)} = 1 + O(\varepsilon^2).$$

Therefore,

$$F_\varepsilon(id) = \begin{cases} \partial_b S^{ab}(id) = 2\varepsilon^2 \partial_b l^{ab} + O(\varepsilon^4) & : \text{ in } \Omega, \\ S^{ab}(id)M_b = 2\varepsilon^2 l^{ab} M_b + O(\varepsilon^4) & : \text{ on } \partial\Omega. \end{cases}$$

Thus, dropping terms of $O(\varepsilon^4)$ in $F_\varepsilon(id)$ which come from the $O(\varepsilon^4)$ terms in (4.1), the linearized problem (2.8) reduces to the boundary value problem

$$\begin{cases} \frac{\partial}{\partial y^b}\left(\dot{m}_{ab} - \varepsilon^2 l_{ab}\right) = 0 & : \quad \text{in} \quad \Omega, \\ \left(\dot{m}_{ab} - \varepsilon^2 l_{ab}\right) M_b = 0 & : \quad \text{on} \quad \partial\Omega. \end{cases} \qquad (4.2)$$

We consider the following problem analogous to (4.2):

$$\begin{cases} \nabla_j\left(\pi^{ij} - \sigma^{ij}\right) = 0 & : \quad \text{in} \quad M, \\ \left(\pi^{ij} - \sigma^{ij}\right)N_j = 0 & : \quad \text{on} \quad \partial M, \end{cases} \qquad (4.3)$$

where $\pi_{ij} = (\mathcal{L}_X g)_{ij}$, σ_{ij} is a given symmetric 2-covariant tensorfield on M, N_j is a covector whose null space is the tangent plane $T_x M$ at $x \in \partial M$, and $N^i = (g^{-1})^{ij}N_j$ is the corresponding outer unit normal vector to ∂M.

To obtain the solution of (4.3), we find a vectorfield $\tilde{X}$ such that $\tilde{\pi} = \mathcal{L}_{\tilde{X}} g$ satisfies

$$\left(\tilde{\pi}^{ij} - \sigma^{ij}\right) N_j = 0 \qquad : \quad \text{on} \quad \partial M. \tag{4.4}$$

We define $X' = X - \tilde{X}$ and $\pi' = \mathcal{L}_{X'} g = \mathcal{L}_X g - \mathcal{L}_{\tilde{X}} g = \pi - \tilde{\pi}$, to obtain from (4.4) and the boundary conditions of (4.3):

$$(\pi')^{ij} N_j = \left(\pi^{ij} - \tilde{\pi}^{ij}\right) N_j = \left(\pi^{ij} - \sigma^{ij}\right) N_j = 0 \qquad : \quad \text{on} \quad \partial M.$$

Defining the vectorfield ρ in M by $\rho^i = \nabla_j \left(\sigma^{ij} - \tilde{\pi}^{ij}\right)$, the problem then reduces to:

$$\begin{cases} \nabla_j (\pi')^{ij} = \rho^i & : \quad \text{in} \quad M, \\ (\pi')^{ij} N_j = 0 & : \quad \text{on} \quad \partial M. \end{cases}$$

This is of the same form as (4.3), but with zero boundary conditions. To see whether this problem has a solution, we need to check the integrability condition (3.8) (i.e., orthogonality to the Killing fields in the L^2 sense). Here, the boundary terms vanish since we have zero boundary conditions and it remains for us to show that

$$\int_M \xi_i \rho^i d\mu_g = 0,$$

for all Killing fields ξ. We have:

$$\begin{aligned} \int_M \xi_i \rho^i d\mu_g &= \int_M \xi_i \nabla_j \left(\sigma^{ij} - \tilde{\pi}^{ij}\right) d\mu_g \\ &= \int_{\partial M} \xi_i \left(\sigma^{ij} - \tilde{\pi}^{ij}\right) N_j \, d\mu_g|_{\partial M} - \int_M \nabla_j \xi_i \left(\sigma^{ij} - \tilde{\pi}^{ij}\right) d\mu_g = 0, \end{aligned}$$

where the first term is zero due to (4.4) (that is, the choice of $\tilde{X}$), and the second one vanishes by virtue of the fact that ξ is a Killing field,

$$\nabla_j \xi_i \left(\sigma^{ij} - \tilde{\pi}^{ij}\right) = \frac{1}{2} \left(\nabla_i \xi_j + \nabla_j \xi_i\right) \left(\sigma^{ij} - \tilde{\pi}^{ij}\right) = 0.$$

By applying the Lax–Milgram theorem as in [2], we conclude that there is a solution X to the generalized linear problem, which is unique up to an additive Killing field. Thus, also the linear case of the original problem (4.2), viewed as a special case of the above, has a unique solution up to an additive Euclidean Killing field.

5   The nonlinear case

We will show that the nonlinear system (2.2) is solvable in Ω under a certain smallness assumption on the parameter ε. Let us first state the nonlinear case of the original problem (P), again.

$$(P) \begin{cases} \overset{m}{\nabla}_b S^{ab} = 0 & : \quad \text{in} \quad \Omega, \\ S^{ab} M_b = 0 & : \quad \text{on} \quad \partial\Omega, \end{cases} \qquad (5.1)$$

where S^{ab} is given by (1.6) and M_b is a covector whose null space is the tangent plane $T_y\Omega$ at $y \in \partial\Omega$. The orientation may be defined by $M_b X^a > 0$, whenever X^b is a vector pointing to the exterior of Ω. $M^a = (m^{-1})^{ab} M_b$ is the outer normal to $\partial\Omega$. The strategy for solving the problem (P) is the following. We set up an iteration scheme, where the first step is the linearized problem. By the result for problem (4.3), the linearized problem is solvable because the integrability condition is automatically satisfied. The integrability condition of the iteration can then be satisfied by applying a doping technique similar to the one in [9].

5.1   Iteration

We first study the iteration scheme. For the analogous (generalized) problem (AP) we have from (2.6), (5.1) (now written in terms of the coordinates y^a on Ω, and π denoting the linearized metric $\dot{m}$):

$$(AP) \begin{cases} \dfrac{\partial \pi_{n+1}^{ab}}{\partial y^b} = \rho_n^a & : \quad \text{in} \quad \Omega, \\ \left(\pi_{n+1}^{ab} - \sigma_n^{ab} \right) M_b = 0 & : \quad \text{on} \quad \partial\Omega, \end{cases} \qquad (5.2)$$

where

$$\pi_{n+1}^{ab} = \frac{\partial \psi_{n+1}^b}{\partial y^a} + \frac{\partial \psi_{n+1}^a}{\partial y^b}.$$

Now we set $\psi'_{n+1} = \psi_{n+1} - \tilde{\psi}_{n+1}$, where the $\tilde{\psi}$-part satisfies the boundary conditions, i.e.,

$$\left(\tilde{\pi}_{n+1}^{ab} - \sigma_n^{ab} \right) M_b = 0, \qquad (5.3)$$

and we have a modified problem (AP') with zero boundary conditions

$$(AP') \begin{cases} \frac{\partial \pi'^{ab}_{n+1}}{\partial y^b} = \rho^a_n - \frac{\partial \tilde{\pi}^{ab}_{n+1}}{\partial y^b} & : \quad \text{in} \quad \Omega, \\ \pi'^{ab}_{n+1} M_b = 0 & : \quad \text{on} \quad \partial\Omega. \end{cases}$$

The integrability condition, which yields existence of solutions of the problem by the Lax–Milgram theorem [7], now reads:

$$0 = \int_\Omega \xi^a \left(\rho^a_n - \frac{\partial \tilde{\pi}^{ab}_{n+1}}{\partial y^b} \right) = \int_\Omega \xi^a \rho^a_n - \int_{\partial\Omega} \xi^a \tilde{\pi}^{ab}_{n+1} M_b + \int_\Omega \tilde{\pi}^{ab}_{n+1} \frac{\partial \xi^a}{\partial y^b}.$$

That is:

$$\int_\Omega \xi^a \rho^a_n - \int_{\partial\Omega} \xi^a \sigma^{ab}_n M_b = 0, \tag{5.4}$$

for every Killing field ξ of a background Euclidean metric on Ω. This metric is $id^*\delta$, where δ is the Euclidean metric of E. The y^a are rectangular coordinates on Ω relative to this metric. In particular, M_b is a unit covector relative to this Euclidean metric and the integral $\int_{\partial\Omega}$ is taken with respect to the measure on $\partial\Omega$ corresponding to the metric (arc length if $\dim \Omega = 2$) induced on $\partial\Omega$ by this Euclidean metric on Ω. In the above, we have made use of (5.3) and the properties of ξ as a Killing field. In particular,

$$\xi^a = \alpha^a_b y^b + \beta^a, \quad \alpha^a_b = -\alpha^b_a \quad \Rightarrow \quad \frac{\partial \xi^a}{\partial y^b} = \alpha^a_b = \frac{1}{2} \left(\alpha^b_a - \alpha^a_b \right),$$

and thus the contraction of the symmetric tensors $\tilde{\pi}^{ab}_{n+1}$, respectively, σ^{ab}_n, with $\frac{\partial \xi^a}{\partial y^b}$ vanishes. In conclusion, the integrability condition at the $n+1$ step of the iteration is (5.4).

5.2   Killing fields and doping technique

In n-dimensional Euclidean space we have the following linearly independent Killing fields:

  (i) n translations,
  (ii) $\frac{n(n-1)}{2}$ rotations.

So, the space of Killing fields on E^n is N-dimensional, where

$$N = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Let $(\xi_A : A = 1, \ldots, N)$ be a basis for the space of Killing fields. In the spirit of [9], we have to modify ρ, that is the inhomogeneity on the right-hand side of (5.2). This technique is called *doping*. We replace ρ by

$$\rho' = \rho + \sum_A c_A \xi_A,$$

and require in accordance with (5.4) that

$$\int_\Omega \xi_A \cdot \rho' = \int_{\partial\Omega} \xi_A \cdot \sigma_M : \quad \forall A = 1, \ldots, N, \tag{5.5}$$

where $\sigma_M^a = \sigma^{ab} M_b$. Since

$$\int_\Omega \xi_A \cdot \xi_B = M_{AB}$$

is positive definite, we obtain a linear system of equations for the coefficients $c_A : A = 1, \ldots, N$ as follows:

$$\int_\Omega \xi_A \cdot \rho' = \int_\Omega \xi_A \cdot \rho + \int_\Omega \xi_A \left(\sum_B c_B \xi_B \right) = \int_{\partial\Omega} \xi_A \cdot \sigma_M : \quad \forall A = 1, \ldots, N,$$

and thus

$$\sum_B M_{AB} c_B = \int_{\partial\Omega} \xi_A \cdot \sigma_M - \int_\Omega \xi_A \cdot \rho =: \sigma_A : \quad \forall A = 1, \ldots, N.$$

The system $\sum_B M_{AB} c_B = \sigma_A$ can be solved, and there is a solution

$$c_A = \sum_B \left(M^{-1} \right)_{AB} \sigma_B, \tag{5.6}$$

M_{AB} being positive definite, hence non-singular.

We reformulate the problem (AP) as follows:

$$(\widetilde{AP}) \begin{cases} \frac{\partial \pi_{n+1}^{ab}}{\partial y^b} = \rho_n'^{\,a} & : \quad \text{in} \quad \Omega, \\ \left(\pi_{n+1}^{ab} - \sigma_n^{ab}\right) M_b = 0 & : \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.7}$$

Writing $\psi_{n+1}^a = X^a$, $\rho_n'^{\,a} = \rho'^{\,a}$, $\sigma_n^{ab} M_b = \tau^a$, (5.7) is of the form

$$\begin{cases} \frac{\partial}{\partial y^b} \left(\frac{\partial X^b}{\partial y^a} + \frac{\partial X^a}{\partial y^b} \right) = \rho'^{\,a} & : \quad \text{in} \quad \Omega, \\ \left(\frac{\partial X^b}{\partial y^a} + \frac{\partial X^a}{\partial y^b} \right) M_b = \tau^a & : \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.8}$$

Under the restriction discussed above, which if X is a Euclidean Killing field forces X to vanish identically, the linear system (5.8) has no kernel. The following estimate then holds (see [1])

$$||X||_{H_{s+2}(\Omega)} \leq C \left\{ ||\rho'||_{H_s(\Omega)} + ||\tau||_{H_{s+1/2}(\partial\Omega)} \right\}, \quad C = C(\Omega). \tag{5.9}$$

Applying this estimate to (5.7) and taking ε suitably small we can prove contraction of the sequence (ψ_n) in $H_{s+2}(\Omega)$ for $s > \frac{n}{2}$, $H_s(\Omega)$ and $H_{s-1/2}(\partial\Omega)$ being under this condition Hilbert algebras.

$$\left. \begin{array}{rrcl} & \sigma_{n,A} & \to & \sigma_A \\ & c_{n,A} & \to & c_A \\ \text{in } H_s(\Omega) & \rho_n^a & \to & \rho^a \\ \text{in } H_{s+1/2}(\partial\Omega) & \sigma_n^{ab} & \to & \sigma^{ab} \\ \text{in } H_{s+1}(\Omega) & \pi_{n+1}^{ab} & \to & \pi^{ab} \\ \text{in } H_s(\Omega) & \rho_n'^{\,a} & \to & \rho'^{\,a} \end{array} \right\} \quad \text{for} \quad n \to \infty,$$

and, in the limit $n \to \infty$ we deduce in terms of the modified inhomogeneity ρ'

$$\begin{cases} \frac{\partial \pi^{ab}}{\partial y^b} = \rho'^{\,a} & : \quad \text{in} \quad \Omega, \\ \left(\pi^{ab} - \sigma^{ab}\right) M_b = 0 & : \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.10}$$

We have:

$$\rho'^{\,a} = \rho^a + \sum_A c_A \xi_A^a, \tag{5.11}$$

and c_A is given in terms of σ_A by (5.6). From (5.1), (5.10) and (5.11), we find

$$\begin{cases} \overset{m}{\nabla}_b S^{ab} = \frac{\partial \pi^{ab}}{\partial y^b} - \rho^a = \rho'^a - \rho^a =: X^a & : \text{in} \quad \Omega, \\ S^{ab} M_b = 0 & : \text{on} \quad \partial\Omega. \end{cases} \tag{5.12}$$

We have to show that

Proposition III.2.

$$X^a = \sum_A c_A \xi_A^a = 0 \quad : a = 1, \ldots, n, \tag{5.13}$$

if ε is suitably small.

Proof. Let ξ^a be a Killing field of δ_{ab}. We have:

$$\int_\Omega \delta_{ab} \xi^a X^b d^2 y = \int_\Omega m_{ab} \xi^a X^b d\mu_m + O(\varepsilon^2 |X|_\infty), \tag{5.14}$$

where we say that a real-valued function $Q(\varepsilon, X)$ is $O(\varepsilon^2 |X|_\infty)$, if there is a constant C such that

$$|Q(\varepsilon, X)| \leq C \varepsilon^2 |X|_\infty .$$

(5.14) holds because $m_{ab} = \delta_{ab} + O(\varepsilon^2)$. Consider the vectorfield ζ on Ω, the pushforward of which by ϕ to the Euclidean plane E coincides with ξ

$$\zeta^a = \xi^i \frac{\partial y^a}{\partial x^i} \quad , \quad \left[\xi^i = \frac{\partial \phi^i}{\partial y^a} \zeta^a \right] .$$

Then ζ is a Killing field of the metric $m = \phi^* \delta$ on Ω. Hence:

$$\int_\Omega m_{ab} \zeta^a X^b d\mu_m = \int_\Omega \zeta_a \overset{m}{\nabla}_b S^{ab} d\mu_m$$

$$= - \int_\Omega \overset{m}{\nabla}_b \zeta_a S^{ab} d\mu_m + \int_{\partial\Omega} \zeta_a S^{ab} M_b \, d\mu_m|_{\partial\Omega} , \tag{5.15}$$

where $\zeta_a = m_{ab} \zeta^b$. The integral on Ω in (5.15) vanishes because

$$\overset{m}{\nabla}_b \zeta_a S^{ab} = \frac{1}{2} \left(\overset{m}{\nabla}_b \zeta_a + \overset{m}{\nabla}_a \zeta_b \right) S^{ab} = 0 .$$

The integral on $\partial\Omega$ vanishes by virtue of the boundary condition in (5.12). Then, since

$$\xi^a = \zeta^a + O(\varepsilon^2),$$

we deduce:

$$\int_\Omega \delta_{ab} \xi^a X^b d^2 y = \int_\Omega m_{ab} \zeta^a X^b d\mu_m + O(\varepsilon^2 |X|_\infty) = O(\varepsilon^2 |X|_\infty). \tag{5.16}$$

However,

$$X^a = \sum_A c_A \xi_A^a \quad : \quad a = 1, \ldots, n,$$

and since Killing fields are analytic functions, we have on a bounded domain Ω:

$$|X|_\infty \leq C_1 \max_A |c_A| . \tag{5.17}$$

From (5.16),

$$\left| \int_\Omega \xi_A \cdot X \, d^2y \right| \leq C\varepsilon^2 |X|_\infty . \tag{5.18}$$

But from the definition of X in (5.13),

$$\int_\Omega \xi_A \cdot X \, d^2y = \sum_B \int_\Omega \xi_A \xi_B c_B \, d^2y = \sum_B M_{AB} c_B = \sigma_A,$$

and hence, from (5.18),

$$|\sigma_A| \leq C\varepsilon^2 |X|_\infty . \tag{5.19}$$

On the other hand, we see from (5.6) that

$$\max_A |c_A| \leq \tilde{C} \max_A |\sigma_A| , \tag{5.20}$$

where $\tilde{C} = ||M^{-1}||$. Finally, using (5.17), (5.20) and (5.19), we obtain

$$|X|_\infty \leq C_1 \max_A |c_A| \leq C_2 \max_A |\sigma_A| \leq C_3\varepsilon^2 |X|_\infty ,$$

which implies $X \equiv 0$ for ε sufficiently small ($C_3\varepsilon^2 < 1$). This finishes the proof. $\qquad\square$

Remark III.2. *Concerning the regularity of the solution for C^∞ domain Ω (smooth boundary) the solution is also C^∞. For, $\psi \in H_{s+2}$, $s > n/2$ implies $\psi \in H_{s+3}(\Omega)$. Therefore, by induction, $\psi \in H_k(\Omega)$ for every k.*

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