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Super-Poincarè algebras, space-times and supergravities (I)

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Abstract

A new formulation of theories of supergravity as theories satisfying a generalized Principle of General Covariance is given. It is a generalization of the superspace formulation of simple 4D-supergravity of Wess and Zumino and it is designed to obtain geometric descriptions for the supergravities that correspond to the super Poincarè algebras of Alekseevsky and Cortés' classification.

1 Introduction

Up to now various theories of supergravity, in diverse dimensions and based on many super-extensions of Poincarè algebras, have been constructed. Although super-extensions of Poincarè algebras and algebras of Lorentzian symmetric spaces have been already classified under various natural hypothesis (see, e.g., [1, 12, 23, 30]), to the best of our knowledge, there does not

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exist a methodical presentation of supergravity theories that parallels those lists of super-extensions.

We also recall that for gauge theories of classical Poincarè algebras, like General Relativity, the requirement of invariance under localizations of translations is just a re-formulation of the classical Principle of General Covariance, i.e., the principle of invariance under local changes of coordinates (or, equivalently, local diffeomorphisms) of the space-time (see, e.g., [2, 19, 28, 29]). By analogy, it is natural to expect that, also for supergravity theories, the supersymmetries (analogues of localizations of translations) can be identified with Lie derivatives along vector fields of an appropriate super-manifold and that the requirement of supersymmetric invariance can be stated as a suitably generalized Principle of General Covariance.

On this regard, we would like to point out that when a supergravity can be presented in a *manifestly covariant* way, i.e., in terms of tensorial equations, the Principle of General Covariance is automatically satisfied and the off-shell invariance of the theory is assured, with no need of explicit computations in coordinates or components.

The expectation that the invariance conditions of supergravity can be stated in terms of Lie derivatives is supported by the very first superspace formulation of simple four-dimensional (4D)-supergravity ([33]). However, an explicit and clear formulation in such terms seems to us still missing. So, here and in [24], we offer a presentation of supergravities based on a generalized Principle of General Covariance and involving a very small number of tensorial objects.

It can be considered as a generalization of the superspace formulation of Wess and Zumino: as in [33], the physical fields are presented as restrictions to space-time M_{o} (not necessarily 4D) of fields defined over a superspace M, which has M_{o} as a body, and the usual supersymmetries are presented as appropriate (infinitesimal) local diffeomorphisms of M.

Our definitions are designed so as to depend in a canonical way on an initial choice of a super-extension $\mathfrak{g}$ of a Poincarè algebra. We consider only the super-Poincarè algebras classified by Alekseevsky and Cortés ([1]) corresponding to $N = 1$ supergravities, but the whole scheme can be easily repeated for other super-algebras and $N = p$ supergravities with $p \geq 2$. Note also that our main goal was to reach a simple and economical description of existing supergravity theories in terms of objects that can be studied with standard techniques of Differential Geometry. We *did not* address questions on the construction of Lagrangians, but we do expect interesting consequences on this topic too.

Here is a more detailed description of our results.

In Section 2, after recalling some facts on $\mathbb{Z}_2$-graded and super-extensions of Poincarè algebras $\mathfrak{g} = \mathfrak{so}(V) + V + S$ of a pseudo-Riemannian space $V = \mathbb{R}^{p,q}$, we introduce the notion of *space-time of type* $\mathfrak{g}$, which is a (super) manifold M with a distinguished submanifold $M_o \subset M$ and a non-integrable distribution $\mathcal{D}$, whose Levi form $\mathcal{L}$ is modeled on the Lie brackets of elements in $S \subset \mathfrak{g}$. Then we define as *gravity field* any pair (g, ∇) formed by a tensor field g on M of type $(0, 2)$, inducing a pseudo-Riemannian metric on the g-orthogonal distribution $\mathcal{D}^{\perp}$ and by a covariant derivation ∇ preserving $\mathcal{D}$, g and $\mathcal{L}$. Properties of these connections are also given.

In Section 3, we define as *supergravity of type* $\mathfrak{g}$ any pair formed by a space-time $(M, M_o, \mathcal{D})$ of type $\mathfrak{g}$ and a gravity field (g, ∇). Any supergravity induces on M_o (which represents the space-time of Physics) the following *physical fields*: two covariant derivations, called *metric* and *spinor connections*, and three tensor fields, corresponding to the *graviton*, the *gravitino* and the *auxiliary field(s)*. Then we state our generalized *Principle of Infinitesimal General Covariance* and the notion of *manifestly covariance* for constraints and equations.

In Section 4, we consider the class of *(strict) Levi–Civita supergravities of type* $\mathfrak{g}$, characterized by the vanishing of certain parts of the torsion T of ∇. The connections satisfying these conditions are generalizations of the Levi–Civita connections of pseudo-Riemannian manifolds and *we prove for them an existence and uniqueness theorem*. From this result it follows that the physical fields of strict Levi–Civita supergravities are completely determined by the graviton, the gravitino and the auxiliary field(s), as in supergravities formulated in the component approach. Finally, we determine the transformation rules for the graviton, gravitino and the auxiliary field of a Levi–Civita supergravity. The expressions nicely match the well-known rules of simple 4D-supergravity and other supergravities.

In Section 5, we give examples on how known theories of supergravity can be presented as theories of Levi–Civita supergravities of type $\mathfrak{g}$.

Our results is re-formulated and formalized in the language of supermanifolds in [24]. We chose to postpone such formalization in a second paper for the following reasons. It is very common to deal with supermanifolds in a naive way and consider them just as smooth manifolds with points labeled by two kinds of coordinates, the bosonic and the fermionic ones. Following this habit, we give here definitions and results on gauge theories of super and non-super extensions of Poincarè algebras, with proofs that can be considered rigorous only for what concerns the latter and essentially correct for the former only if one consider supermanifolds "as if" they were smooth manifolds. In [24], we convert everything into rigorous statements on supermanifolds and on the gauge theories of super Poincarè algebras.

Before concluding, we need to recall that a presentation of supergravity, which is based on a Principle of General Covariance, appears also in the so-called "rheonomic approach" of Regge, Ne'eman, Castellani, D'Adda, D'Auria, Fré and van Nieuwenhuizen (see, e.g., [2–4, 13, 14]), where supergravities are described as theories of fields on a soft-group manifold P, a sort of principle bundle over the superspace M. We also remark that our approach is crucially based on the notion of the non-integrable distribution $\mathcal{D}$ modeled on $\mathfrak{g}$: To the best of our knowledge, similar non-integrable distributions have only been considered in the geometrical approach of Ogievetsky, Sokatchev, Roslyĭ, Schwarz et al. (see, e.g., [9, 10, 15–18, 20–22, 25]) and in the superspace formulation of supergravity of Deligne [5]. Analogies and differences will be carefully discussed elsewhere.

Notation. Throughout the paper, we consider Clifford algebras as defined, e.g., in [8]. According to this, the Clifford product of vectors of the standard basis of $\mathbb{R}^{p,q}$ is $e_i \cdot e_j = -2\eta_{ij}$ and not "$+2\eta_{ij}$" as it is often assumed in Physics literature.

2 Space-times and gravity fields of type $\mathfrak{g}$

2.1 Extended Poincarè algebras and associated space-times

Let $V = \mathbb{R}^{p,q}$ and $\mathfrak{p}(V) = Lie(Iso(\mathbb{R}^{p,q})) = \mathfrak{so}(V) + V$ its *Poincarè algebra.*

Definition 2.1. A $\mathbb{Z}_2$-graded Lie algebra (resp. a super-algebra) $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ is called *extended* (resp. *super*) *Poincarè algebra* if

(a) $\mathfrak{g}_0 = \mathfrak{p}(V) = \mathfrak{so}(V) + V$;
(b) $\mathfrak{g}_1 = S$ is an irreducible spinor module (i.e., an irreducible real representation of the Clifford algebra $\mathcal{C}\ell(V)$ of V) and the adjoint action $\mathrm{ad}_{\mathfrak{so}(V)}|_S : S \longrightarrow S$ coincides with the standard action of $\mathfrak{so}(V)$ on S (i.e., $[A, s] = A \cdot s$ for any $A \in \mathfrak{so}(V)$, $s \in S$);
(c) $[V, S] = 0$;
(d) $[S, S] \subseteq V$.

If $\mathfrak{g}$ is an extended (resp. super) Poincarè algebra, any connected homogeneous (super) space $M = G/H$, with $Lie(G) = \mathfrak{g}$ and $Lie(H) = \mathfrak{so}(V)$, will be called *flat space-time of type* $\mathfrak{g}$. The submanifold $M_{\mathrm{o}} = G_{\mathrm{o}}/H \subset M$, with $G_o \subset G$ connected and $Lie(G_o) = \mathfrak{so}(V) + V$, is called *body* of the space-time.

As we have done in this definition, all statements and arguments of this paper have a "super" and a "non-super" version. But, hoping to be clear and at the same time rigorous, from now on *we give exact and precise definitions and statements only for the "non-super" case.* Corresponding accurate definitions and statements for the "super" case will be given in [24]. Nonetheless, it should not be hard to understand their contents on the base of analogies.

We now want to introduce a generalization of the notion of flat space-time, which is fundamental in our presentation of supergravity theories. For this, we need to recall some notion, commonly used in studying CR structures and non-integrable distributions. Let M be a manifold of dimension m and $\mathcal{D} \subset TM$ a distribution of rank $p \leq m$ on M. At any point $x \in M$, we may consider the map

$$\mathcal{L}_x : \Lambda^2 \mathcal{D}_x \longrightarrow T_x M/\mathcal{D}_x, \quad \mathcal{L}_x(v, w) = [X^{(v)}, X^{(w)}]_x \mod \mathcal{D}_x \tag{1}$$

where $X^{(v)}$, $X^{(w)}$ are vector fields in $\mathcal{D}$ with $X^{(v)}_x = v$ and $X^{(w)}_x = w$. A simple check shows that $\mathcal{L}_x(v, w)$ depends only on v and w and that (1) is a well-defined bilinear map. It is called *Levi form of $\mathcal{D}$ at x*.

We say that $\mathcal{D}$ is *of uniform type* if its Levi form $\mathcal{L}_x$ is independent on x up to linear isomorphisms (i.e., if for any $x, y \in M$ there exists an isomorphism $\imath : T_x M \xrightarrow{\simeq} T_y M$, so that $\imath(\mathcal{D}_x) = \mathcal{D}_y$ and $\imath^*(\mathcal{L}_y) = \mathcal{L}_x$).

Example 2.2. Any flat space-time $M = G/H$ is naturally endowed with a G-invariant distribution, i.e., the unique invariant distribution $\mathcal{D}^{\mathfrak{g}}$ such that

$$\mathcal{D}^{\mathfrak{g}}|_o = S, \quad o = eH$$

(we use the standard identification $T_o G/H \simeq V + S$). This distribution is of uniform type, transversal to the body $M_o = G_o/H$ and with Levi form at o

$$\mathcal{L}^{\mathfrak{g}}_o(s, s') = [s, s'], \quad s, s' \in S.$$

If G/H is simply connected, $\mathcal{D}^{\mathfrak{g}}$ is described in coordinates as follows. Let (e_i, e_α) be a basis for $V + S$ with $e_i \in V, e_\alpha \in S$. The exponential map exp :

$\mathfrak{g} \longrightarrow G$ induces a diffeomorphism

$$\exp : V + S \xrightarrow{\simeq} G/H \tag{2}$$

and we may consider the global system of coordinates $\xi : G/H \longrightarrow \mathbb{R}^{\widehat{n}}, \widehat{n} = \dim V + \dim S$, that associates to any $x = \exp(x^i e_i + \theta^\alpha e_\alpha)$ the coordinates $\xi(x) = (x^1, \ldots, x^n, \theta^1, \ldots, \theta^{\widehat{n}-n})$.

A vector $v = v^i e_i + v^\alpha e_\alpha \in V + S \simeq T_o G/H$ is represented in the coordinate basis as $v = v^i \left.\frac{\partial}{\partial x^i}\right|_o + v^\alpha \left.\frac{\partial}{\partial \theta^\alpha}\right|_o$ and it is the tangent vector at $t = 0$ of the curve $\gamma_t = \exp(t(v^i e_i + v^\alpha e_\alpha)) \in G/H$. By Baker-Campbell-Hausdorff (BCH)-formula, an element $g = \exp(x^j e_j + \theta^\beta e_\beta) \in \exp(V + S) \subset G$ maps γ_t into the curve

$$g \cdot \gamma_t = \exp\left(x^j e_j + \theta^\beta e_\beta + t(v^i e_i + v^\alpha e_\alpha) + \frac{1}{2} t v^\alpha \theta^\beta \mathcal{L}^k_{\beta\alpha} e_k\right),$$

where $\mathcal{L}^i_{\alpha\beta}$ are the components of the Levi form $\mathcal{L}^{\mathfrak{g}}_o$ in the basis (e_i, e_α). From this it follows that

$$g_*(v) = v^i \left.\frac{\partial}{\partial x^i}\right|_{(x^j,\theta^\beta)} + v^\alpha \left(\left.\frac{\partial}{\partial \theta^\alpha}\right|_{(x^j,\theta^b)} + \frac{1}{2}\theta^\beta \mathcal{L}^i_{\beta\alpha} \left.\frac{\partial}{\partial x^i}\right|_{(x^j,\theta^\beta)} \right)$$

and hence that any linear combination of the vector fields

$$E_i \stackrel{\text{def}}{=} \frac{\partial}{\partial x^i}, \quad E_\alpha \stackrel{\text{def}}{=} \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2}\theta^\beta \mathcal{L}^i_{\beta\alpha} \frac{\partial}{\partial x^i} \tag{3}$$

is $\exp(V + S)$-invariant (see, e.g., [34], Ch. 14). Finally, the G-invariant distribution $\mathcal{D}^{\mathfrak{g}}$ of G/H is generated by the fields E_α, i.e., $\mathcal{D}^{\mathfrak{g}}_x = \mathrm{Span}_{\mathbb{R}} \{E_\alpha|_x\}$.

The properties of $\mathcal{D}^{\mathfrak{g}}$ of previous example motivate the following notion.

Definition 2.3. A *space-time of type* $\mathfrak{g}$ is any triple $(M, M_o, \mathcal{D})$ given by:

(i) a connected manifold M of dimension $\widehat{n} = \dim V + \dim S$;
(ii) a connected submanifold $M_o \subset M$ of dimension $n = \dim V$;
(iii) a distribution $\mathcal{D} \subset TM$ of rank $n^S = \dim S$ and transversal to M_o (i.e., with $T_x M_o \cap \mathcal{D}_x = \{0\}$ at any $x \in M_o$) satisfying the following "uniformity assumption":
for any $x \in M$ there exists a neighborhood $\mathcal{U} \subset M$ of x and a smooth family of vector space isomorphisms $\imath^{(y)} : V + S \longrightarrow T_y M$, $y \in \mathcal{U}$,

so that

$$\imath^{(y)}(S) = \mathcal{D}_x \quad \text{and} \quad \imath^{(y)*}(\mathcal{L}_y) = \mathcal{L}^{\mathfrak{g}}_{\mathrm{o}};$$

if $S = S^+ + S^-$ is sum of irreducible $\mathfrak{so}(V)$-moduli, we also assume $\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$ for distributions $\mathcal{D}^\pm$ and $\imath^{(y)}(S^\pm) = \mathcal{D}^\pm_y$.

The submanifold M_{o} is called *body* of the space-time.

Notice that, if $(M, M_{\mathrm{o}}, \mathcal{D})$ is a (non-flat) space-time and (E_α) is a set of local generators for $\mathcal{D}$ around a point $x_{\mathrm{o}} \in M_{\mathrm{o}}$, then it is always possible to determine a system of coordinates (x^i, θ^α) on a neighborhood $\mathcal{U}$ of x_{o}, so that

$$M_{\mathrm{o}} \cap \mathcal{U} = \{\ \theta^\alpha = 0\ \}, \quad E_\alpha|_{M_{\mathrm{o}}} = \left.\frac{\partial}{\partial\theta^\alpha}\right|_{M_{\mathrm{o}}},$$

as it occurs on the flat space-time considered above. On the other hand, it goes without saying that the expressions for the E_α's outside the body M_{o} are in general quite different from the (3).

2.2 Admissible extended (or super) Poincarè algebras and associated gravity fields

2.2.1 Admissible extended and admissible super Poincarè algebras

In [1] it was observed that, given an irreducible spinor module S, any extension $\mathfrak{g} = \mathfrak{so}(V) + V + S$ of $\mathfrak{p}(V)$ is completely determined by the tensor

$$L \in \Lambda^2 S^* \otimes V \quad (\text{resp. } \vee^2 S^* \otimes V)$$

that defines the Lie brackets $[\cdot,\cdot]$ between elements in S. This tensor is $\mathfrak{so}(V)$-invariant and any $\mathfrak{so}(V)$-invariant tensor of this kind corresponds to a unique structure of extended (or super) Poincarè algebra on $\mathfrak{so}(V) + V + S$.

A tensor $L \in \Lambda^2 S^* \otimes V$ or $\vee^2 S^* \otimes V$ is called *admissible* if the associated tensor

$$L^* \in S^* \otimes S^* \otimes V^*, \quad L^*(s, s', v) \overset{\text{def}}{=} \langle L(s, s'), v\rangle$$

is of the form

$$L^*(s, s', v) = \beta(v \cdot s, s'), \tag{4}$$

for some non-degenerate $\mathfrak{so}(V)$-invariant bilinear form β on S such that:

(1) it is either symmetric or skew-symmetric;
(2) the Clifford multiplications $v \cdot (\cdot) : S \longrightarrow S$, $v \in V$, are either all β-symmetric or all β-skew symmetric;
(3) if S is sum of irreducible $\mathfrak{so}(V)$-moduli $S = S^+ + S^-$, then $S^\pm$ are either mutually β-orthogonal or both β-isotropic.

Any admissible tensor is $\mathfrak{so}(V)$-invariant, it corresponds to an extended (or super) Poincarè algebra and *the spaces* $(\Lambda^2 S^* \otimes V)^{\mathfrak{so}(V)}$ *and* $(\vee^2 S^* \otimes V)^{\mathfrak{so}(V)}$ *have bases of admissible elements* ([1]).

Definition 2.4. An extended (or super) Poincarè algebra $\mathfrak{g} = \mathfrak{so}(V) + V + S$ is called *admissible* if it is determined by an admissible tensor L. In this case, if β is the bilinear form (4), we call *extended inner product of* $V + S$ the non-degenerate bilinear form $(\cdot,\cdot)$, defined by

$$(\cdot,\cdot)|_{V\times S} = 0, \quad (\cdot,\cdot)|_{V\times V} = \langle\cdot,\cdot\rangle, \quad (\cdot,\cdot)|_{S\times S} = \beta. \tag{5}$$

From now on, *any extended (super) Poincarè algebra will be assumed to be admissible* and $(\cdot,\cdot)$ will always indicate the bilinear form (5). ([1])

Example 2.5. Let $V = \mathbb{R}^{3,1}$ and denote by $(e_0, \dots, e_3)$ its standard basis with $\langle e_i, e_j\rangle = \varepsilon_i \delta_{ij}$ with $\varepsilon_0 = -1, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +1$. Let also $S = \mathbb{C}^4$ and denote by $\rho : \mathcal{C}\ell_{3,1} \longrightarrow \mathrm{End}\,(S)$ the Dirac representation of $\mathcal{C}\ell_{3,1}$, determined by the Γ-matrices

$$\Gamma_0 = \rho(e_0) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Gamma_i = \rho(e_i) = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

where the σ_i are the usual Pauli matrices $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Let also

$$\Gamma_5 \overset{\text{def}}{=} i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{6}$$

and $S = S^+ + S^- = \mathbb{C}^2 + \mathbb{C}^2$ be the corresponding decomposition of S in Γ_5-eigenspaces, i.e., into irreducible $\mathfrak{so}(V)$-moduli of Weyl spinors, on which

[1]Actually, for many of our results, it is sufficient to consider a non-degenerate $\mathfrak{so}(V)$-invariant bilinear form (5), with β not necessarily equal to the one in (4).

$\mathfrak{so}(V)$ acts by conjugate representations. Finally, let ε be the standard volume form of $\mathbb{C}^2 = S^+ = S^-$ and $\omega \in \Lambda^2 S \simeq \Lambda^2\mathbb{C}^4$ the 2-form

$$\omega(s, s') = \varepsilon(s^+, s'^+) - \varepsilon(s^-, s^{'-}) = s^T C s', \tag{7}$$

where we considered the decompositions $s = s^+ + s^-, s' = s^{'+} + s^{'-}$ into $S^{\pm}$-components and $C = -i\Gamma_0\Gamma_2$ is the charge conjugation matrix.

The admissible bilinear forms

$$\begin{gathered}\beta_1(s, s') = \operatorname{Re}\omega(s, s) = -\operatorname{Re}(\mathrm{i}s^T\Gamma_0\Gamma_2 s'),\\ \beta_2(s, s') = \operatorname{Im}\omega(s, s') = -\operatorname{Im}(\mathrm{i}s^T\Gamma_0\Gamma_2 s'),\\ \beta_3(s, s') = \operatorname{Re}(\overline{s}^T\Gamma_0 s'), \quad \beta_4(s, s') = \operatorname{Re}(\overline{s}^T\Gamma_5\Gamma_0 s').\end{gathered}$$

give a basis for the space of tensors associated with *super* extensions of $\mathfrak{p}(\mathbb{R}^{3,1})$, while the admissible bilinear forms

$$\begin{gathered}\widetilde{\beta}_1(s, s') = \operatorname{Im}(s^T\Gamma_1\Gamma_3 s'), \quad \widetilde{\beta}_2(s, s') = \operatorname{Re}(s^T\Gamma_1\Gamma_3 s'),\\ \widetilde{\beta}_3(s, s') = \operatorname{Im}(\overline{s}^T\Gamma_0 s'), \quad \widetilde{\beta}_4(s, s') = \operatorname{Im}(\overline{s}^T\Gamma_5\Gamma_0 s'),\end{gathered}$$

give a basis for the space of tensors associated with *non-super* extensions.

Example 2.6. Let $V = \mathbb{R}^{10,1}$ and again denote by $(e_0, \dots, e_{10})$ its standard basis with $\langle e_i, e_j\rangle = \varepsilon_i\delta_{ij}$ with $\varepsilon_0 = -1$, $\varepsilon_i = +1$ for $1 \le i \le 10$. Let $S = \mathbb{C}^{32}$ and $\rho : C\ell_{10,1} \longrightarrow \mathrm{End}\ (S)$ the Dirac representation of $C\ell_{10,1}$, determined by purely imaginary Γ-matrices $\Gamma_i = \rho(e_i)$ (see, e.g., [11]). The admissible bilinear forms

$$\beta_1(s, s') = \operatorname{Re}(is^T\Gamma_0 s'), \quad \beta_2(s, s') = \operatorname{Im}(is^T\Gamma_0 s'), \quad \beta_3(s, s') = \operatorname{Re}(\overline{s}^T\Gamma_0 s'),$$

give a basis for the space of tensors associated with *super* extensions of $\mathfrak{p}(\mathbb{R}^{10,1})$, while the admissible bilinear form

$$\beta(s, s') = \operatorname{Im}(\overline{s}^T\Gamma_0 s')$$

is a basis for the space of tensors associated with *non-super* extensions.

2.2.2 Gravity fields of type $\mathfrak{g}$

In the following definition, we denote by $\mathfrak{g}$ an admissible extended Poincarè algebra with extended inner product $(\cdot,\cdot)$ and by $(M, M_o, \mathcal{D})$ a space-time of type $\mathfrak{g}$ with Levi form $\mathcal{L}$.

Definition 2.7. A *gravity field on* $(M, M_o, \mathcal{D})$ is a pair (g, ∇) formed by a tensor field g of type $(0, 2)$ and a connection ∇ on M so that:

(i) the tensor g is so that, for any $x \in M$, there exists a neighborhood $\mathcal{U} \subset M$ of x and a smooth family of vector space isomorphisms $\imath^{(y)} : V + S \longrightarrow T_y M$, $y \in \mathcal{U}$, so that:
  (a) $\imath^{(y)}(S) = \mathcal{D}_y, \imath^{(y)}(V) = \mathcal{D}^\perp_y$ and, if $S = S^+ + S^-$, $\imath^{(y)}(S^\pm) = \mathcal{D}^\pm_y$;
  (b) $\imath^{(y)}_*(\cdot, \cdot) = g_y$;
  (c) $\imath^{(y)}_*(\mathcal{L}^{\mathfrak{g}}_o) = \mathcal{L}^g_y$, where $\mathcal{L}^g_y \in \mathrm{Hom}(\mathcal{D}_y \times \mathcal{D}_y, \mathcal{D}^\perp_y)$ is

$$\mathcal{L}^g_y \stackrel{\text{def}}{=} (\pi|_{\mathcal{D}^\perp})^{-1} \circ \mathcal{L}_y$$

  and $\pi|_{\mathcal{D}^\perp} : \mathcal{D}^\perp \longrightarrow TM/\mathcal{D}$ is the natural isomorphism between the g-orthogonal distribution $\mathcal{D}^\perp$ to $\mathcal{D}$ and the bundle $TM/\mathcal{D}$;

(ii) the distribution $\mathcal{D}$ is ∇-invariant and, if $S = S^+ + S^-$, the distributions $\mathcal{D}^\pm$ are ∇-invariant;

(iii) $\nabla g = 0$ and $\nabla \mathcal{L}^g = 0$.

In this case, we say that g is the *extended metric* and ∇ the *extended metric connection.*

The name "extended metric" stems from the notion of "extended inner product" (see Definition 2.4) and one should keep in mind that g is not always a symmetric tensor field. Notice also that, from (ii) and (iii), any extended metric connection ∇ preserves also the complementary distribution $\mathcal{D}^\perp$.

Let (g, ∇) be a gravity field on a space time $(M, M_o, \mathcal{D})$ of type $\mathfrak{g}$. We call *bundle of orthonormal frames* of g the collection $\mathrm{O}_g(M, \mathcal{D})$ of all vector spaces isomorphism $\imath : V + S \longrightarrow T_x M$ satisfying (a)–(c) of previous definition. Using (i), one can check that $\mathrm{O}_g(M, \mathcal{D})$ is indeed a principal bundle over M with a structure group G, whose identity component G^0 is the subgroup of $\mathrm{GL}(V + S)$

$$G^0 = \left\{ \begin{pmatrix} k & 0 \\ 0 & k \circ h \end{pmatrix},\ k \in \mathrm{Spin}^0(V), h \in H^0 \right\} = \mathrm{Spin}^0(V) \cdot H^0,$$

where H^0 is the identity component of $H = \mathrm{O}(S, \beta) \cap C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V))$ or of the subgroup of H, which preserves S^+ and S^-, when $S = S^+ + S^-$.

By definitions, the extended metric connection ∇ preserves $\mathrm{O}_g(M, \mathcal{D})$ and it can be considered as the covariant derivation on M determined by a connection form ω on $\mathrm{O}_g(M, \mathcal{D})$.

Consider now the connections ∇^o and $\nabla^{o'}$ induced by ∇ on the vector bundles

$$\pi : \mathcal{D}^\perp \longrightarrow M, \quad \pi' : \mathcal{D} \longrightarrow M.$$

They may be considered as the covariant derivations determined by connection forms $\omega^o, \omega^{o'}$ on the bundles $\mathrm{O}_g(\mathcal{D}^\perp)$ and $\mathrm{O}_g(\mathcal{D})$ of the g-orthonormal frames of the spaces $\mathcal{D}^\perp_x \subset T_xM$ and $\mathcal{D}_x \subset T_xM$, respectively. On the other hand, using a (local) field of frames $\imath^{(y)} : V + S \longrightarrow T_yM$, one can identify any space $\mathcal{D}_y$ with the spinor module S and identify (at least locally) the bundle $\pi' : \mathcal{D} \longrightarrow M$ with the spinor bundle associated with $\mathrm{O}_g(\mathcal{D}^\perp)$, i.e.,

$$\mathcal{D} \simeq \mathrm{Spin}_g(\mathcal{D}^\perp) \times_{\mathrm{Spin}^0(V)} S. \tag{8}$$

For a fixed (local) identification (8), we may consider on $\mathcal{D}$ the covariant derivation induced by the covariant derivation ∇^o of $\mathcal{D}^\perp$ (be aware that the induced derivation depends on the identification (8)). If we denote also this covariant derivation by ∇^o, we have that

$$\nabla^{o'}_X s = \nabla^o_X s + C_X(s), \quad X \in \mathfrak{X}(M), \quad s \in \Gamma(\mathcal{D}), \tag{9}$$

for some field C in $T^*_xM \otimes \mathcal{D}^*_x \otimes \mathcal{D}_x \simeq (V + S)^* \otimes S^* \otimes S$ at any $x \in M$.

In particular, ∇ can be locally written as a sum of the form $\nabla = \nabla^o + C$, where C is defined in (9) and ∇^o is sum of the connection on $\mathcal{D}^\perp$ and the induced connection on $\mathcal{D}$. Note that ∇^o satisfies (ii), (iii) of Definition 2.7.

As we pointed out above, such decomposition (and the field C) depends in principle on the chosen identification (8). However, the next proposition shows that in many cases C is trivial, no matter what is the used identification.

Proposition 2.8. *Let (g, ∇) be a gravity field on $(M, M_{\mathrm{o}}, \mathcal{D})$ and $\nabla = \nabla^o + C$ a decomposition determined by an identification (8). For any $x \in M$, the tensor C_x belongs to $(V + S)^* \otimes \mathfrak{h}$, where $\mathfrak{h} = Lie(H)$ is contained in one of the subspaces of $C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V))$ described in Table 1:*

In particular, $\nabla = \nabla^o$ when $p - q = 0, 1, 6, 7 \mod 8$.

Proof. By the properties of ∇ and ∇^o, for any vector fields $X, v, s, s' \in \mathfrak{X}(M)$, with $v_x \in \mathcal{D}^\perp_x (\simeq V)$ and $s_x, s'_x \in \mathcal{D}_x (\simeq S)$ at all points, we have that

$$g(v, \mathcal{L}^g(C_X(s), s')) + g(v, \mathcal{L}^g(s, C_X(s'))) = 0, \quad g(C_X(s), s') + g(s, C_X(s')) = 0.$$

Table 1

$p-q \mod 8$	0	1	2	3
$C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V))$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$
irr. $\mathfrak{so}(V)$*moduli in* S	$S^+ \not\simeq S^-$	$S^+ \simeq S^-$	$S^+ \simeq S^-$	S
$\mathfrak{h}$ *is contained in*	*0*	*0*	$\mathrm{Span}_{\mathbb{R}}\{i\}$	$\mathrm{Span}_{\mathbb{R}}\{i,j,k\}$
$p-q \mod 8$	4	5	6	7
$C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V))$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$
irr. $\mathfrak{so}(V)$*moduli in* S	$S^+ \not\simeq S^-$	S	S	S
$\mathfrak{h}$ *is contained in*	$\mathrm{Span}_{\mathbb{R}}\{i,j,k\}$	$\mathrm{Span}_{\mathbb{R}}\{i\}$	0	0

Hence, using the identifications $T_xM \simeq V + S$ and the admissibility of $\mathfrak{g}$, we get that for any $X \in V + S$, $v \in V$ and $s, s' \in S$

$$\beta(v \cdot C_X(s), s') + \beta(v \cdot s, C_X(s')) = 0, \quad \beta(C_X(v \cdot s), s') + \beta(v \cdot s, C_X(s')) = 0.$$

By non degeneracy of β, these conditions are equivalent to

$$v \cdot C_X(\cdot) = C_X(v \cdot (\cdot)), \quad \beta(C_X(\cdot), \cdot) + \beta(\cdot, C_X(\cdot)) = 0,$$

i.e., $C_X \in \mathfrak{o}(S, \beta) \cap C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V))$ (in addition, if $S = S^+ + S^-$, the conditions on ∇ and ∇^o imply that C_X preserves S^+ and S^-).

For any given signature $s = p - q$, the centralizer $C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V))$ is immediately determined recalling that $\mathcal{C}\ell(V) \simeq \mathbb{K}(N)$ or $\mathbb{K}(N) \oplus \mathbb{K}(N)$ for some suitable N, with $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In all cases, one can determine the $\mathfrak{so}(V)$-moduli in S and the elements in $C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V))$ that preserve these moduli (see [1], Prop. 1.5 and [6], Tables 1 and 2). Excluding the elements which are real multiples of the identity (which cannot be in $\mathfrak{o}(S, \beta)$), one gets the spaces listed in the last row of Table 1. □

Remark 2.9. It should be stressed that Table 1 gives just an upper bound for $\dim \mathfrak{h}$. When β is explicitly given, one gets a finer result by direct computations.

3 Theories of supergravity

3.1 Gravities and supergravities

Definition 3.1. Let M_o be a manifold of dimension $n = \dim V$. We call *(super) gravity of type* $\mathfrak{g}$ *on* M_o any pair $\mathcal{G} = ((M, M_o, \mathcal{D}), (g, \nabla))$ formed by

(a) a space time $(M, M_o, \mathcal{D})$ of type $\mathfrak{g}$ with body M_o;

(b) a gravity field (g, ∇) on $(M, M_o, \mathcal{D})$.

Given a (super) gravity $\mathcal{G} = ((M, M_o, \mathcal{D}), (g, \nabla))$, we call *spinor bundle of* $\mathcal{G}$ the pullback bundle

$$\pi : \mathcal{S} = \mathcal{D}|_{M_o} \longrightarrow M_o.$$

We also call *physical fields of* $\mathcal{G}$ the following objects:

(a) the tensor field in $T^*M_o \otimes_{M_o} \mathcal{S}$, called *gravitino*, defined by

$$\vartheta(X) \stackrel{\text{def}}{=} \pi^{\mathcal{D}}(X), \tag{10}$$

where, for any $x \in M$, we denote by $\pi^{\mathcal{D}}_x : T_xM \longrightarrow \mathcal{D}_x$ the g-orthogonal projection onto $\mathcal{D}_x$;

(b) the tensor field in $\vee^2 T^*M_o$, called *graviton*, defined by

$$\widehat{g}(X, Y) \stackrel{\text{def}}{=} g(X, Y) - g(\vartheta(X), \vartheta(Y)) = g(\pi^{\mathcal{D}^\perp}(X), \pi^{\mathcal{D}^\perp}(Y)), \tag{11}$$

where, for any $x \in M$, we denote by $\pi^{\mathcal{D}^\perp}_x : T_xM \longrightarrow \mathcal{D}^\perp_x$ the g-orthogonal projection onto $\mathcal{D}^\perp_x$;

(c) the tensor field in $T^*M_o \otimes_{M_o} \mathcal{S}^* \otimes_{M_o} \mathcal{S}$, called *A-field*, defined by

$$\mathbb{A}_{Xs} \stackrel{\text{def}}{=} -\pi^{\mathcal{D}}(T_{Xs}), \tag{12}$$

where we denoted by T the torsion of the connection ∇;

(d) the connection $D : \mathfrak{X}(M_o) \times \mathfrak{X}(M_o) \longrightarrow \mathfrak{X}(M_o)$, called *metric connection*, defined by ([2])

$$D_XY \stackrel{\text{def}}{=} (\pi^{\mathcal{D}^\perp}\Big|_{TM_o})^{-1}\left(\nabla_X\, \pi^{\mathcal{D}^\perp}(Y)\right); \tag{13}$$

(e) the connection $\mathbb{D} : \mathfrak{X}(M_o) \times \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S})$ on the space $\Gamma(\mathcal{S})$ of the sections of $\pi : \mathcal{S} \longrightarrow M_o$, called *spinor connection*, defined by

$$\mathbb{D}_X s \stackrel{\text{def}}{=} \nabla_X s - \pi^{\mathcal{D}}(T_{Xs}) = \nabla_X s + \mathbb{A}_X s. \tag{14}$$

Finally, we call *non-physical fields of* $\mathcal{G}$ the tensor fields in $\mathcal{S}^* \otimes_{M_o} \mathcal{S}^* \otimes_{M_o} \mathcal{S}$ and $T^*M_o \otimes_{M_o} \mathcal{S}^* \otimes_{M_o} \mathcal{S}^* \otimes_{M_o} \mathcal{S}$, called *B-field* and *C-field*, defined by

$$\mathbb{B}_{ss'} \stackrel{\text{def}}{=} -\pi^{\mathcal{D}}\left(T_{ss'}\right), \quad \mathbb{C}_{Xss'} \stackrel{\text{def}}{=} -\pi^{\mathcal{D}}\left((\nabla_{s'}T)_{Xs}\right). \tag{15}$$

[2]Notice that D_XY is equal to the projection of ∇_XY onto TM_o w.r.t. to the decomposition $TM|_{M_o} = TM_o + \mathcal{D}|_{M_o}$. In other words, D is the connection on the submanifold $M_o \subset M$, induced by ∇, by identifying the normal bundle $TM|_{M_o}/TM_o$ with $\mathcal{D}|_{M_o}$.

Remark 3.2. Our presentation of supergravity theories is essentially based on this definition and the contents of next subsection. In Section 5, we will indicate how *various $N = 1$ supergravities can be presented as theories on physical fields of supergravities of type $\mathfrak{g}$.*

As it is suggested by our choice of names, the above defined “graviton” and “gravitino” are precisely the objects, which we want to use to formalize the common notions of graviton and gravitino in standard supergravity theories.

In fact, from Definition 2.7, one can check that *the graviton $\widehat{g}$ is a pseudo-Riemannian metric of signature (p, q) and that $D\widehat{g} = 0$.* On the other hand, given a fixed orthonormal basis (e_i, e_α) for $(V + S, (\cdot, \cdot))$ and a corresponding local frame field

$$\Big(E_i(x) = \imath^{(x)}(e_i), \quad E_\alpha(x) = \imath^{(x)}(e_\alpha)\Big), \quad \imath^{(x)} \in \mathrm{O}_g(M, \mathcal{D})|_{M_\mathrm{o}},$$

the field ϑ is of the form $\vartheta = \psi^\alpha_i E_\alpha \otimes E^i|_{TM_\mathrm{o}}$, as the usual gravitino (see [32]).

3.2 The principle of general covariance and manifestly covariant equations

As we mentioned in the Introduction, we want to present the transformation rules of a supergravity theory as actions of infinitesimal diffeomorphisms (= Lie derivatives) and generalize the Principle of General Covariance.

We first remark that for any (super) gravity $\mathcal{G} = ((M, M_\mathrm{o}, \mathcal{D}), (g, \nabla))$ of type $\mathfrak{g}$, any (local) diffeomorphism $\varphi : M \longrightarrow M$, sufficiently close to Id_M, determines a new a pair

$$\mathcal{G}' = \varphi_*(\mathcal{G}) \overset{\text{def}}{=} ((M, M_\mathrm{o}, \varphi_*(\mathcal{D})), \quad ((\varphi^{-1})^* g, (\varphi^{-1})^* \nabla)) \tag{16}$$

which still is a (super) gravity of type $\mathfrak{g}$. This suggests the following two notions:

> *A collection $\mathcal{E}_\mathrm{o}$ of constraints and equations on the physical fields of (super) gravities of type $\mathfrak{g}$ satisfies the* Generalized Principle of Infinitesimal General Covariance *if:*
>
> (i) *there exists a system $\mathcal{E}$ of constraints and equations on $(\mathcal{D}, g, \nabla)$, so that any (local) solution of $\mathcal{E}$ determines physical fields, which solve $\mathcal{E}_\mathrm{o}$ and every (local) solution of $\mathcal{E}_\mathrm{o}$ is of this form;*

(ii) *the class of (local) solutions of* $\mathcal{E}_{\mathrm{o}}$ *is invariant under all actions* (16), *where* $\mathcal{G}$ *is given by a solution of* $\mathcal{E}$ *and* φ *is of the form* $\varphi = \Phi_t^X$ *for some* $X \in \mathfrak{X}_{\mathrm{loc}}(M)$.

The system $\mathcal{E}_{\mathrm{o}}$ *is said* manifestly covariant *if there exists a system* $\mathcal{E}$ *as in (i), which is of tensorial type.*

Any manifestly covariant system $\mathcal{E}_{\mathrm{o}}$ *automatically satisfies (ii)* and hence *also the Generalized Principle of General Covariance.*

Now, for a given (super) gravity $\mathcal{G} = ((M, M_{\mathrm{o}}, \mathcal{D}), (g, \nabla))$ of type $\mathfrak{g}$, let us consider the following class of (local) vector fields on M:

$$\mathfrak{X}_{\mathrm{loc}}(M; M_{\mathrm{o}}) = \{X \in \mathfrak{X}_{\mathrm{loc}}(M) : X_x \in T_x M_{\mathrm{o}},\ x \in M_{\mathrm{o}}\},$$
$$\mathfrak{X}_{\mathrm{loc}}(M; \mathcal{S}) = \{X \in \mathfrak{X}_{\mathrm{loc}}(M) : X \in \Gamma_{\mathrm{loc}}(\mathcal{D}),\ (\nabla_s X)_x = 0,\ s \in \mathcal{D}_x,\ x \in M_{\mathrm{o}}\}.$$

Clearly, any $X \in \mathfrak{X}_{\mathrm{loc}}(M_{\mathrm{o}})$ admits an extension $\widehat{X} \in \mathfrak{X}_{\mathrm{loc}}(M; M_{\mathrm{o}})$ and one can check that any local section s of $\mathcal{S}$ admits an extension $\widehat{s} \in \mathfrak{X}_{\mathrm{loc}}(M; \mathcal{S})$.

The actions of the fields in $\mathfrak{X}_{\mathrm{loc}}(M; M_{\mathrm{o}})$ can be considered as generalizations of the actions of the vector fields of M_{o}. In fact, for any $X \in \mathfrak{X}_{\mathrm{loc}}(M_{\mathrm{o}})$ with extension $\widehat{X} \in \mathfrak{X}_{\mathrm{loc}}(M_{\mathrm{o}})$, the family of metrics $\Phi^X_{t\,*}(\widehat{g})$ coincides with the family of gravitons $\widehat{g}^t$ of the (super) gravities $\mathcal{G}_t = \Phi^{\widehat{X}}_{t\,*}(\mathcal{G})$.

As we will see later (Section 5), the actions of fields in $\mathfrak{X}_{\mathrm{loc}}(M; \mathcal{S})$ coincide with the supersymmetries of simple $4D$-supergravity and other supergravities.

In other words, those supergravity theories are invariant under the class of vector fields

$$\mathfrak{X}_{\mathrm{loc}}(M; M_{\mathrm{o}}) + \mathfrak{X}_{\mathrm{loc}}(M; \mathcal{S}), \tag{17}$$

which is properly included in $\mathfrak{X}_{\mathrm{loc}}(M)$.

We recall that a class $\mathcal{A} \subset \mathfrak{X}_{\mathrm{loc}}(M)$ of vector fields is called *Lie pseudo-algebra* if for any $\lambda, \mu \in \mathbb{R}$ and any pair $X, X' \in \mathcal{A}$, defined on two open subsets $\mathcal{U}, \mathcal{U}' \subset M$, the fields $\lambda X + \mu X'$ and $[X, X']$ on $\mathcal{U} \cap \mathcal{U}'$ are both elements of $\mathcal{A}$. Lie pseudo-algebras share many basic properties with usual Lie algebras of vector fields (see, e.g., [27]).

It is hardly to be expected that brackets between elements in (17) are still in (17), i.e., that (17) is a Lie pseudo-algebra. Hence, if one is looking for a Lie pseudo-algebra of symmetries, it is more natural to consider the whole

$\mathfrak{X}_{\mathrm{loc}}(M)$. In fact, as we will shortly see, equations of simple $4D$-supergravity are manifestly covariant and hence invariant under all elements in $\mathfrak{X}_{\mathrm{loc}}(M)$.

4 Levi–Civita supergravities and the transformations laws for their physical fields

4.1 Levi–Civita supergravities

Let $\mathcal{G} = ((M, M_{\mathrm{o}}, \mathcal{D}), (g, \nabla))$ be a (super) gravity of type $\mathfrak{g}$. At any $x \in M$, the torsion T_x of ∇ decomposes into a sum of the form

$$T_x = T_x^{\mathcal{D}^\perp} + T_x^{\mathcal{D}} + \mathcal{C}_x^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}} + \mathcal{C}_x^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp} + \mathcal{H}_x^{\Lambda^2\mathcal{D}^\perp;\mathcal{D}} + \mathcal{H}_x^{\Lambda^2\mathcal{D};\mathcal{D}^\perp}, \qquad (18)$$

with summands belonging to the following $\mathfrak{so}(\mathcal{D}_x^\perp)$-modules:

$$\begin{gathered} T_x^{\mathcal{D}^\perp} \in \mathrm{Hom}(\mathcal{D}_x^\perp \wedge \mathcal{D}_x^\perp, \mathcal{D}_x^\perp), \quad T_x^{\mathcal{D}} \in \mathrm{Hom}(\mathcal{D}_x \wedge \mathcal{D}_x, \mathcal{D}_x), \\ \mathcal{C}_x^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}} \in \mathrm{Hom}(\mathcal{D}_x \times \mathcal{D}_x^\perp, \mathcal{D}_x), \quad \mathcal{C}_x^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp} \in \mathrm{Hom}(\mathcal{D}_x \times \mathcal{D}_x^\perp, \mathcal{D}_x^\perp), \\ \mathcal{H}_x^{\Lambda^2\mathcal{D}^\perp;\mathcal{D}} \in \mathrm{Hom}(\mathcal{D}_x^\perp \wedge \mathcal{D}_x^\perp, \mathcal{D}_x), \quad \mathcal{H}_x^{\Lambda^2\mathcal{D};\mathcal{D}^\perp} \in \mathrm{Hom}(\mathcal{D}_x \wedge \mathcal{D}_x, \mathcal{D}_x^\perp). \end{gathered}$$

Note that *the decomposition* (18) *is preserved by any action* (16).

Since ∇ preserves $\mathcal{D}$ and $\mathcal{D}^\perp$, it follows that

$$\mathcal{H}_x^{\Lambda^2\mathcal{D};\mathcal{D}^\perp} = -\mathcal{L}_x^g \simeq -\mathcal{L}_{\mathrm{o}}^{\mathfrak{g}}. \qquad (19)$$

From this, at any $x_{\mathrm{o}} \in M_{\mathrm{o}}$ and for any $X, Y \in \mathfrak{X}(M_{\mathrm{o}})$ with $[X,Y]|_{x_{\mathrm{o}}} = 0$, the value $T^D_{XY}\big|_{x_{\mathrm{o}}}$ of the torsion of the metric connection D is equal to

$$\begin{aligned} T^D_{XY}\big|_{x_{\mathrm{o}}} &= (\pi^{\mathcal{D}^\perp}\big|_{TM_{\mathrm{o}}})^{-1}\left(\nabla_X(\pi^{\mathcal{D}^\perp}(Y)) - \nabla_Y(\pi^{\mathcal{D}^\perp}(X))\right)\Big|_{x_{\mathrm{o}}} \\ &= (\pi^{\mathcal{D}^\perp}\big|_{TM_{\mathrm{o}}})^{-1}\left(\pi^{\mathcal{D}^\perp}(T_{XY})\right)\Big|_{x_{\mathrm{o}}} = (\pi^{\mathcal{D}^\perp}\big|_{TM_{\mathrm{o}}})^{-1} \circ \\ \circ \Big(T^{\mathcal{D}^\perp}_{X^\perp Y^\perp} &+ \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}(\vartheta(X), Y^\perp) - \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}(\vartheta(Y), X^\perp) \\ &- \mathcal{L}^g(\vartheta(X), \vartheta(Y))\Big)\big|_{x_{\mathrm{o}}} \end{aligned} \qquad (20)$$

(here $X^\perp$, $Y^\perp$ denote the components of X, Y along $\mathcal{D}^\perp$). Hence, for any admissible extended (or super) Poincarè algebra $\mathfrak{g} = \mathfrak{so}(V) + V + S$, the torsion T^D has a non-trivial term, depending quadratically on ϑ. Due to this, *there is no way to require the vanishing of T^D for all values of ϑ*, in contrast with the well-known property of Levi Civita connections.

However, the following theorem holds (in the statement, $\mathrm{Sym}^g(\mathcal{D}_x^\perp)$ denotes the space of endomorphisms of $\mathcal{D}_x^\perp$ that are symmetric w.r.t. g_x).

Theorem 4.1. *For any $((M, M_o, \mathcal{D}), g)$ that satisfies Definition 2.7 (i), there exists a connection ∇ satisfying also (ii), (iii) and the constraints*

$$T_x^{\mathcal{D}^\perp} = 0 \quad \text{and} \quad \mathcal{C}_x^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp} \in \mathcal{D}_x^* \otimes \mathrm{Sym}^g(\mathcal{D}_x^\perp) \quad \text{at any } x \in M. \tag{21}$$

This connection is uniquely determined up to the field C, defined in (9). In particular, ∇ is unique whenever $\mathfrak{o}(S, \beta) \cap C_{\mathfrak{gl}(S)}(\mathcal{C}\ell(V)) = 0$.

Proof. Assume that g is an extended metric on $(M, M_o, \mathcal{D})$ satisfying Definition 2.7. For a fixed choice of local frames in $\mathrm{O}_g(M, \mathcal{D})$, let ∇ be the unique, locally defined, $\mathcal{D}$ and $\mathcal{D}^\perp$ preserving connection, for which the field C in (9) is 0 and for any $v, w, z \in \mathcal{D}^\perp$, $s \in \mathcal{D}$

$$\begin{aligned} g(\nabla_v w, z) = \frac{1}{2}\Big(&v \cdot g(w, z) + w \cdot g(z, v) - z \cdot g(v, w) \\ &+ g(\pi^{\mathcal{D}^\perp}([v, w]), z) - g(\pi^{\mathcal{D}^\perp}([w, z]), v) - g(\pi^{\mathcal{D}^\perp}([v, z]), w)\Big), \end{aligned} \tag{22}$$

$$g(\nabla_s w, z) = \frac{1}{2}\Big(s \cdot g(w, z) + g(\pi^{\mathcal{D}^\perp}([s, w]), z) - g(\pi^{\mathcal{D}^\perp}([s, z]), w)\Big). \tag{23}$$

One can check that it satisfies (21) and (ii), (iii). Using a partition of unity, one gets the existence part of the theorem.

About the uniqueness part, assume that $\mathcal{G} = ((M, M_o, \mathcal{D}), (g, \nabla))$ and $\widetilde{\mathcal{G}} = ((M, M_o, \mathcal{D}), (g, \widetilde{\nabla}))$ are two (super) gravities of the same type $\mathfrak{g}$, both satisfying (21). Fix a local identification (8) so that we may consider the decompositions $\nabla = \nabla^o + C$ and $\widetilde{\nabla} = \widetilde{\nabla}^o + \widetilde{C}$ described in (9). By definitions, for any $X \in \mathfrak{X}(M)$, the operators ∇^o_X and $\widetilde{\nabla}^o_X$ act on the vector fields in $\mathcal{D}^\perp$ just as the covariant derivations ∇_X and $\widetilde{\nabla}_X$, while they act on the fields in $\mathcal{D}$ by means of the corresponding spinorial connections. In particular, ∇^o and $\widetilde{\nabla}^o$ are uniquely determined by their restrictions $\nabla^o|_{\mathfrak{X}(M)\times\mathcal{D}^\perp}$ and $\widetilde{\nabla}^o|_{\mathfrak{X}(M)\times\mathcal{D}^\perp}$.

On the other hand, by definitions,

$$\widetilde{\nabla}^o|_{\mathfrak{X}(M)\times\mathcal{D}^\perp} = \nabla^o|_{\mathfrak{X}(M)\times\mathcal{D}^\perp} + F, \tag{24}$$

for some suitable tensor field F taking values in $T^*M \otimes \mathfrak{so}(\mathcal{D}^\perp)$, i.e., so that for any $X \in \mathfrak{X}(M)$, $v, v' \in \Gamma(\mathcal{D}^\perp)$

$$g(F_X(v), v') + g(v, F_X(v')) = 0. \tag{25}$$

Now, for a given F in $T^*M \otimes \mathfrak{so}(\mathcal{D}^\perp)$, let us denote by $\partial_F T^o = T^o - \widetilde{T}^o$ the difference between the torsions T^o and $\widetilde{T}^o$ of the connections ∇^o and $\widetilde{\nabla}^o$, respectively. Simple arguments based just on definitions imply that

$$\partial_F T^o_{XY} = F_X(Y) - F_Y(X), \quad \text{for any } X, Y \in \mathfrak{X}(M)$$

and that, for any $x \in M$, the map

$$\begin{aligned} &\varphi_1 : \mathcal{D}^*_x \otimes \mathfrak{so}(\mathcal{D}^\perp_x) \longrightarrow \mathcal{D}^*_x \otimes \mathrm{End}\,(\mathcal{D}^\perp_x), \\ &\varphi_1\left(F_x|_{\mathcal{D}_x\times\mathcal{D}^\perp_x}\right) \stackrel{\text{def}}{=} \pi^{\mathcal{D}^\perp} \circ (\partial_F T^o)|_{\mathcal{D}_x\times\mathcal{D}^\perp_x} \end{aligned} \tag{26}$$

coincides with the trivial embedding of $\mathcal{D}^*_x \otimes \mathfrak{so}(\mathcal{D}^\perp_x)$ into $\mathcal{D}^*_x \otimes \mathrm{End}\,(\mathcal{D}^\perp_x)$. Due to this, by the fact that the antisymmetric parts of the tensors $\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}_x$ of ∇ and $\widetilde{\nabla}$ are both 0, one gets $F_x|_{\mathcal{D}_x\times\mathcal{D}^\perp_x} = 0$ at any $x \in M$.

Consider now the map

$$\begin{aligned} &\varphi_2 : \mathcal{D}^{\perp *}_x \otimes \mathfrak{so}(\mathcal{D}^\perp_x) \longrightarrow \Lambda^2\mathcal{D}^{\perp *}_x \otimes \mathcal{D}^\perp_x, \\ &\varphi_2\left(F_x|_{\mathcal{D}^\perp_x\times\mathcal{D}^\perp_x}\right) = \pi^{\mathcal{D}^\perp} \circ (\partial_F T^o)|_{\mathcal{D}^\perp_x\times\mathcal{D}^\perp_x}. \end{aligned} \tag{27}$$

We claim that this is a vector space isomorphism between $\mathcal{D}^{\perp *}_x \otimes \mathfrak{so}(\mathcal{D}^\perp_x)$ and $\Lambda^2\mathcal{D}^{\perp *}_x \otimes \mathcal{D}^\perp_x$. In fact, if we identify T_xM with $V + S$ by means of a frame in $\mathrm{O}_g(M, \mathcal{D})$, the map $F_x|_{\mathcal{D}^\perp_x\times\mathcal{D}^\perp_x}$ is identifiable with an element of

$$V^* \otimes \mathfrak{so}(V, <,>) \simeq V^* \otimes \Lambda^2 V^*,$$

while $\pi^{\mathcal{D}^\perp} \circ (\partial_F T^o)|_{\mathcal{D}^\perp_x\times\mathcal{D}^\perp_x}$ is identifiable with an element of $\Lambda^2V^* \otimes V \simeq \Lambda^2V^* \otimes V^*$. The map (27) is equal to the so-called "Spencer operator"

$$\begin{aligned} &\partial : V^* \otimes \Lambda^2V^* \longrightarrow \Lambda^2V^* \otimes V^*, \\ &(\partial\alpha)(v_1, v_2, w) = \alpha(v_1, v_2, w) - \alpha(v_2, v_1, w), \end{aligned} \tag{28}$$

which is well known to be an isomorphism. Due to this, since ∇ and $\widetilde{\nabla}$ has $T^{\mathcal{D}^\perp} \equiv 0$, then $F_x|_{\mathcal{D}_x^\perp \times \mathcal{D}_x^\perp} = 0$ at any $x \in M$.

Hence, $F \equiv 0$ and $\nabla^o = \widetilde{\nabla}^o$. The claim is then a consequence of the fact that variations of C do not affect $T^{\mathcal{D}^\perp}$ and $\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}$. □

This result motivates the following definition.

Definition 4.2. A (super) gravity $((M, M_{\mathrm{o}}, \mathcal{D}), (g, \nabla))$ is called *Levi-Civita* if ∇ satisfy (21). In this case, ∇ is called a *Levi–Civita connection of* g.

Let $\mathcal{G} = ((M, M_{\mathrm{o}}, \mathcal{D}), (g, \nabla))$ be a Levi–Civita (super) gravity. By (20), the value of the torsion of the metric connection D on commuting fields is

$$T^D_{XY} = \left(\pi^{\mathcal{D}^\perp}\Big|_{TM_{\mathrm{o}}} \right)^{-1} \circ$$
$$\circ \left(\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}(\vartheta(X), Y^\perp) - \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}(\vartheta(Y), X^\perp) - \mathcal{L}^g(\vartheta(X), \vartheta(Y)) \right)\Big|_{M_{\mathrm{o}}}.$$

This shows that T^D (and hence D, being any metric connection recoverable from its torsion, through the associated contorsion) is completely determined by the graviton, the gravitino and the tensor field $\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}|_{M_{\mathrm{o}}}$.

A common assumption in supergravity is $\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}|_{M_{\mathrm{o}}} = 0$ (see Section 5). It is therefore convenient to introduce the following definition.

Definition 4.3. A (super) gravity $\mathcal{G} = ((M, M_{\mathrm{o}}, \mathcal{D}), (g, \nabla))$ is called *strict Levi-Civita* if the torsion of ∇ satisfies the conditions $T^{\mathcal{D}^\perp} \equiv 0 \equiv \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}$.

Since the difference between spinor and metric connections is given by the A-field (and the field C in (9), in special signatures), it follows that *all physical fields of strict Levi-Civita (super) gravities are completely determined by the graviton, gravitino and A-field* (and, sometimes, by C).

4.2 Transformations rules for gravitons, gravitinos and A-fields

In this section we give explicit formulae for the actions of vector fields in $\mathfrak{X}_{\mathrm{loc}}(M; \mathcal{S})$ on the graviton, gravitino and A-field of a strict Levi-Civita (super) gravity. We perform computations in local coordinates and components to show that the obtained expressions nicely match the well-known rules of simple 4D-supergravity and other supergravities.

Let $\mathcal{G} = ((M, M_o, \mathcal{D}), (g, \nabla))$ be a *strict Levi-Civita supergravity* and (E_a, E_α) a (local) field of g-orthonormal frames with $E_a \in \mathcal{D}^\perp$, $E_\alpha \in \mathcal{D}$ and w.r.t. which the Levi form $\mathcal{L}^g$ has constant components $\mathcal{L}^a_{\alpha\beta}$. Let also (E^a, E^α) be the dual coframe field.

Now, if we consider the $\widehat{g}$-orthonormal coframes $(e^a = E^a|_{TM_o})$ on M_o, we have that the graviton, the gravitino and the A-field are of the form

$$\widehat{g} = \eta_{ab}\, e^a \otimes e^b, \quad \vartheta = \psi^\alpha_b\, E_\alpha|_{M_o} \otimes e^b, \quad \mathbb{A} = \mathbb{A}_{a\beta}^{\ \alpha}\, E_\alpha|_{M_o} \otimes e^a \otimes E^\beta|_{\mathcal{S}}, \tag{29}$$

where $\eta_{ab} = \epsilon_a \delta_{ab}$, and ψ^α_b, $\mathbb{A}_{a\beta}^{\ \alpha}$ are suitable smooth functions. Indeed, $\widehat{g}$, ϑ, $\mathbb{A}$ are the restriction to TM_o and $\mathcal{S}$ of the tensor fields of M

$$g(\pi^{\mathcal{D}^\perp}(\cdot), \pi^{\mathcal{D}^\perp}(\cdot)) = \eta_{ab} E^a \otimes E^b, \quad \pi^{\mathcal{D}} = E_\alpha \otimes E^\alpha,$$
$$-\pi^{\mathcal{D}} \circ T = -\, T^\alpha_{ab}\, E_\alpha \otimes E^a \otimes E^b - T^\alpha_{a\beta}\, E_\alpha \otimes E^a \otimes E^\beta - T^\alpha_{\gamma\beta}\, E_\alpha \otimes E^\gamma \otimes E^\beta.$$

In particular, the ψ^α_b are the components of the 1-forms $E^\alpha|_{TM_o} = \psi^\alpha_b e^b$, while the $\mathbb{A}_{a\beta}^{\ \alpha}$ are the functions $\mathbb{A}_{a\beta}^{\ \alpha} = -T^\alpha_{a\beta}|_{M_o} - \psi^\gamma_a \cdot T^\alpha_{\gamma\beta}|_{M_o}$.

Motivated by the above remark, we call *variations of the graviton, the gravitino and the A-field along* $X \in \mathfrak{X}_{\mathrm{loc}}(M)$ the fields on M_o defined by

$$\delta_X e^a := (\mathcal{L}_X E^a)|_{TM_o}, \quad \delta_X \vartheta := (\mathcal{L}_X \pi^{\mathcal{D}})|_{TM_o},$$
$$\delta_X \mathbb{A} := -(\mathcal{L}_X(\pi^{\mathcal{D}} \circ T))|_{TM_o \times \mathcal{S}}$$

where "$\mathcal{L}_X$" denotes the usual Lie derivative along the vector field X. As we will see in Section 5, the variations along the vector fields in $\mathfrak{X}_{\mathrm{loc}}(M; \mathcal{S})$ correspond to the so-called "supersymmetry transformations" in simple 4D-supergravity and other supergravity theories. We thus consider the following definition.

Definition 4.4. Let $\varepsilon = \varepsilon^\alpha E_\alpha|_{M_o}$ be a (locally defined) spinor field in $\mathcal{S}$. We call (*super*) *variations along* ε the infinitesimal variations

$$\delta_\varepsilon e^a \overset{\mathrm{def}}{=} \delta_{X^{(\varepsilon)}} e^a, \quad \delta_\varepsilon \vartheta \overset{\mathrm{def}}{=} \delta_{X^{(\varepsilon)}} \vartheta, \quad \delta_\varepsilon \mathbb{A} \overset{\mathrm{def}}{=} \delta_{X^{(\varepsilon)}} \mathbb{A},$$

determined by an arbitrary vector field $X^{(\varepsilon)} = \mathfrak{X}_{\mathrm{loc}}(M; \mathcal{S})$ with $X^{(\varepsilon)}|_{M_o} = \varepsilon$.

The (super) variations along ε are clearly determined by the functions $\delta_\varepsilon e^a_b$, $\delta_\varepsilon \psi^\alpha_a$, $\delta_\varepsilon \psi^b_a$, $\delta_\varepsilon \mathbb{A}_{a\beta}^{\ \alpha}$ and $\delta_\varepsilon \mathbb{A}_{a\beta}^{\ b}$ defined by the relations

$$\delta_\varepsilon e^a := (\delta_\varepsilon e^a_b)\ e^b, \quad \delta_\varepsilon \vartheta := (\delta_\varepsilon \psi^\alpha_a)\ E_\alpha|_{M_o} \otimes e^a + (\delta_\varepsilon \psi^b_a)\ E_b|_{M_o} \otimes e^a, \quad (30)$$

$$\delta_\varepsilon \mathbb{A} := (\delta_\varepsilon \mathbb{A}_{a\beta}^{\ \alpha})\ E_\alpha|_{M_o} \otimes e^a \otimes E^\beta|_{\mathcal{S}} + (\delta_\varepsilon \mathbb{A}_{a\beta}^{\ b})\ E_b|_{M_o} \otimes e^a \otimes E^\beta|_{\mathcal{S}}. \quad (31)$$

We now compute explicitly those functions, proving also that they are independent of the choice of the extension $X^{(\varepsilon)}$. In the following, (e_a) is the $\widehat{g}$-orthonormal frames field on M_o defined by $\pi^{\mathcal{D}^\perp}(e_a) = E_a|_{M_o}$.

Proposition 4.5. *Given a spinor field $\varepsilon = \varepsilon^\alpha E_\alpha|_{M_o}$, the components of the corresponding (super) variations of graviton and gravitino are of the form*

$$\delta_\varepsilon e^b_a = -\varepsilon^\alpha \psi^\beta_a \mathcal{L}^b_{\alpha\beta} + L^b_a \quad \text{for some } L = (L^b_a) \in \mathfrak{so}(V), \quad (32)$$

$$\delta_\varepsilon \psi^\alpha_a = e_a(\varepsilon^\alpha) + \varepsilon^\beta(\mathcal{H}_{a\beta}^{\ \alpha} + \mathbb{A}_{a\beta}^{\ \alpha} + \psi^\gamma_a \mathbb{B}^\alpha_{\beta\gamma}), \quad (33)$$

$$\delta_\varepsilon \psi^b_a = \varepsilon^\alpha \mathcal{L}^b_{\alpha\beta} \psi^\beta_a, \quad (34)$$

where $\mathcal{H}_{a\beta}^{\ \alpha}$, $\mathbb{B}^\alpha_{\beta\gamma}$ are the Christoffel symbols $\mathcal{H}_{a\beta}^{\ \alpha} \overset{\text{def}}{=} E^\alpha(\nabla_{e_a} E_\beta)$ and the components of the $\mathbb{B}$-field $\mathbb{B}^\alpha_{\beta\gamma} \overset{\text{def}}{=} E^\alpha(\mathbb{B}_{E_\beta E_\gamma})$, respectively.

Proof. For simplicity of notation, let us denote by the symbol "ε" also the field $X^{(\varepsilon)} \in \mathfrak{X}_{\text{loc}}(M; \mathcal{S})$. We first need one simple observation. In order to compute the functions which determine the variations (30), one has to evaluate the tensor fields $\mathcal{L}_\varepsilon E^a$, $\mathcal{L}_\varepsilon \pi^{\mathcal{D}}$ on elements of TM_o. One can also check that it is always possible to extend the field of $\widehat{g}$-orthonormal frames (e_a) on M_o to vector fields (e_a) on an open subset $\mathcal{U} \subset M$ so that $\pi^{\mathcal{D}^\perp}(e_a) = E_a$ and $\nabla_{E_\alpha} \pi^{\mathcal{D}}(e_a)|_{M_o} = 0$. By tensoriality, one is allowed to evaluate $\mathcal{L}_\varepsilon E^a$ and $\mathcal{L}_\varepsilon \pi^{\mathcal{D}}$ on these special extensions and then restrict the result to M_o.

With these remarks in mind, one has that

$$\begin{aligned}\delta_\varepsilon e^b_a = (\mathcal{L}_\varepsilon E^b)(e_a) = -E^b([\varepsilon, e_a]) = -E^b(\nabla_\varepsilon e_a) + E^b(\nabla_{e_a} \varepsilon) \\ + E^c(e_a) E^b(\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}(\varepsilon, E_c)) + E^\alpha(e_a) E^b(\mathcal{H}^{\Lambda^2\mathcal{D};\mathcal{D}^\perp}(\varepsilon, E_\alpha))\end{aligned}$$

and that the matrix $L^b_a = -E^b(\nabla_\varepsilon e_a)|_x$ belongs to $\mathfrak{so}(V)$ for any $x \in M_o$. From $E^b(\nabla_{e_a} \varepsilon) = 0$, $\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp} = 0$ and (19), equality (32) follows.

Similarly, we have that

$$\delta_\varepsilon \psi_a^\alpha = E^\alpha((\mathcal{L}_\varepsilon \pi^{\mathcal{D}})(e_a))\big|_{M_o} = E^\alpha\left([\varepsilon, \pi^{\mathcal{D}}(e_a)]\right)\big|_{M_o} - E^\alpha\left(\pi^{\mathcal{D}}([\varepsilon, e_a])\right)\big|_{M_o}.$$

Since

$$\begin{aligned} E^\alpha([\varepsilon, \pi^{\mathcal{D}}(e_a)]) &= E^\alpha(\nabla_\varepsilon \pi^{\mathcal{D}}(e_a)) - E^\alpha(\nabla_{\pi^{\mathcal{D}}(e_a)}\varepsilon) - E^\alpha(T^{\mathcal{D}}_{\varepsilon\ \pi^{\mathcal{D}}(e_a)}), \\ E^\alpha\left(\pi^{\mathcal{D}}([\varepsilon, e_a]\right) &= E^\alpha\left([\varepsilon, e_a]\right) = E^\alpha(\nabla_\varepsilon e_a) - E^\alpha(\nabla_{e_a}\varepsilon) - E^\alpha(T_{\varepsilon e_a}) \\ &= E^\alpha(\nabla_\varepsilon \pi^{\mathcal{D}}(e_a)) - E^\alpha(\nabla_{e_a}\varepsilon) - E^\alpha(\pi^{\mathcal{D}}(T_{\varepsilon e_a})), \end{aligned}$$

we have that

$$\begin{aligned} \delta_\varepsilon \psi_a^\alpha &= E^\alpha(\nabla_{e_a}\varepsilon + \mathbb{A}_{e_a\varepsilon} + \mathbb{B}_{\varepsilon\vartheta(e_a)}) = E^\alpha(\mathbb{D}_{e_a}\varepsilon + \mathbb{B}_{\varepsilon\vartheta(e_a)}) \\ &= e_a(\varepsilon^\alpha) + \varepsilon^\beta E^\alpha(\mathbb{D}_{e_a}E_\beta) + E^\alpha(\mathbb{B}_{\varepsilon\vartheta(e_a)}), \end{aligned}$$

and (33) follows. Finally, (34) follows immediately from

$$\delta_\varepsilon \psi_a^b = E^b\left([\varepsilon, \pi^{\mathcal{D}}(e_a)]\right)\Big|_{M_o} - E^b\left(\pi^{\mathcal{D}}([\varepsilon, e_a])\right)\Big|_{M_o} = E^b(\mathcal{L}^g_{\varepsilon\ \pi^{\mathcal{D}}(e_a)})\Big|_{M_o}.$$

□

Proposition 4.6. *Given a spinor field $\varepsilon = \varepsilon^\alpha E_\alpha|_{M_o}$, the components of the corresponding (super) variation of the A-field are of the form*

$$\begin{aligned} \delta_\varepsilon \mathbb{A}_{a\beta}^{\ \alpha} &= E^\delta(\mathbb{D}_{e_a}\varepsilon)\mathbb{B}_{\delta\beta}^{\ \alpha} + \varepsilon^\gamma\left(\mathbb{A}_{a\beta}^{\ \delta}\mathbb{B}_{\gamma\delta}^{\ \alpha} + \mathbb{A}_{a\gamma}^{\ \delta}\mathbb{B}_{\beta\delta}^{\ \alpha} - R_{\gamma a\beta}^{\ \ \ \alpha} - R_{a\beta\gamma}^{\ \ \ \alpha} - \psi_a^\delta R_{\beta\gamma\delta}^{\ \ \ \alpha}\right) \\ &\quad + \varepsilon^\gamma\left(\mathbb{A}_{a\beta}^{\ \zeta}\mathbb{B}_{\zeta\gamma}^{\ \alpha} - \psi_a^\zeta\psi_c^\xi\mathcal{L}^c_{\zeta\beta}\mathbb{B}_{\xi\gamma}^{\ \alpha} + \psi_a^\zeta\mathcal{L}^c_{\zeta\beta}\mathbb{A}_{c\gamma}^{\ \alpha} + \mathbb{C}_{a\gamma\beta}^{\ \ \ \alpha} - \mathbb{B}_{\beta\gamma|a}^{\ \alpha}\right), \quad (35) \\ \delta_\varepsilon \mathbb{A}_{a\beta}^{\ b} &= \varepsilon^\gamma\mathcal{L}^b_{\gamma\delta}\mathbb{A}_{a\beta}^{\ \delta}, \quad (36) \end{aligned}$$

where:

- *where $\mathbb{B}_{\gamma\beta}^{\ \alpha}$, $\mathbb{C}_{a\gamma\beta}^{\ \ \ \alpha}$, $R_{ABC}^{\ \ \ \ D}$ are the components of B- and C-fields and of the Riemann tensor R of ∇ w.r.t. $(E_a|_{M_o}, E_\alpha|_{M_o}, e^a, E^\alpha|_{\mathcal{S}})$;*
- *$\mathbb{B}_{\beta\gamma|a}^{\ \alpha} \stackrel{\text{def}}{=} E^\alpha\left((\mathbb{D}_{e_a}\mathbb{B})_{E_\beta E_\gamma} - (\mathbb{A}_{e_a}\cdot\mathbb{B})_{E_\beta E_\gamma}\right)$, where $\mathbb{A}_{e_a}\cdot$ denotes the natural action of $\mathbb{A}_{e_a}|_x \in \mathrm{Hom}(\mathcal{D}_x, \mathcal{D}_x)$ on $\mathbb{B}_x \in \mathrm{Hom}(\mathcal{D}_x \times \mathcal{D}_x, \mathcal{D}_x)$.*

Proof. As in the previous proof, for simplicity of notation, we denote by "ε" also the extension $X^{(\varepsilon)} \in \mathfrak{X}_{\mathrm{loc}}(M, \mathcal{S})$. By definition of Lie derivative and first Bianchi identity, we have that

$$\begin{aligned}\big(\mathcal{L}_\varepsilon(\pi^{\mathcal{D}}\circ T)\big)_{YZ} &= \big[\varepsilon,(\pi^{\mathcal{D}}\circ T)_{YZ}\big]-(\pi^{\mathcal{D}}\circ T)_{[\varepsilon,Y]Z}-(\pi^{\mathcal{D}}\circ T)_{Y[\varepsilon,Z]}\\ &= (\mathcal{L}_\varepsilon-\nabla_\varepsilon)(\pi^{\mathcal{D}}(T_{YZ}))+\pi^{\mathcal{D}}\left((\nabla_\varepsilon T)_{YZ}+T_{\nabla_\varepsilon YZ}+T_{Y\nabla_\varepsilon Z}\right.\\ &\quad\left.-T_{[\varepsilon,Y]Z}-T_{Y[\varepsilon,Z]}\right)\\ &= (\mathcal{L}_\varepsilon-\nabla_\varepsilon)(\pi^{\mathcal{D}}(T_{YZ}))+\pi^{\mathcal{D}}\left((\nabla_\varepsilon T)_{YZ}+T_{\nabla_Y\varepsilon Z}+T_{Y\nabla_Z\varepsilon}\right.\\ &\quad\left.+T_{T_{\varepsilon Y}Z}+T_{T_{Z\varepsilon}Y}\right)\\ &\overset{\text{Bianchi id.}}{=} (\mathcal{L}_\varepsilon-\nabla_\varepsilon)(\pi^{\mathcal{D}}(T_{YZ}))+\pi^{\mathcal{D}}\left(T_{\nabla_Y\varepsilon Z}+T_{Y\nabla_Z\varepsilon}\right)\\ &\quad+\pi^{\mathcal{D}}\left(R_{\varepsilon Y}Z+R_{YZ}\varepsilon+R_{Z\varepsilon}Y-T_{T_{YZ}\varepsilon}\right.\\ &\quad\left.-(\nabla_Y T)_{Z\varepsilon}-(\nabla_Z T)_{\varepsilon Y}\right). \end{aligned}\tag{37}$$

On the other hand,

$$\delta_\varepsilon\mathbb{A}_{a\beta}^{\ \alpha}=E^\alpha((\delta_\varepsilon\mathbb{A})_{e_aE_\beta})=-\,E^\alpha\big(\big(\mathcal{L}_\varepsilon(\pi^{\mathcal{D}}\circ T)\big)_{e_aE_\beta}\big)\Big|_{M_o}.$$

Hence, from (37), we get that

$$\begin{aligned}\delta_\varepsilon\mathbb{A}_{a\beta}^{\ \alpha} &= -E^\delta(T_{e_aE_\beta})E^\alpha([\varepsilon,E_\delta]-\nabla_\varepsilon E_\delta)-E^\alpha\left(T_{\nabla_{e_a}\varepsilon E_\beta}\right)-E^\alpha\left(R_{\varepsilon e_a}E_\beta\right.\\ &\quad\left.+R_{e_aE_\beta}\varepsilon+R_{E_\beta\varepsilon}e_a-T_{T_{e_aE_\beta}\varepsilon}-(\nabla_{e_a}T)_{E_\beta\varepsilon}-(\nabla_{E_\beta}T)_{\varepsilon e_a}\right)\\ &= \varepsilon^\gamma\mathbb{A}_{a\beta}^{\ \delta}\mathbb{B}_{\gamma\delta}^{\ \alpha}+E^\delta(\nabla_{e_a}\varepsilon)\mathbb{B}_{\delta\beta}^{\ \alpha}-\varepsilon^\gamma\left(R_{\gamma a\beta}^{\ \ \ \alpha}+R_{a\beta\gamma}^{\ \ \ \alpha}+\psi_a^\delta R_{\beta\gamma\delta}^{\ \ \ \alpha}\right)\\ &\quad+\varepsilon^\gamma E^\alpha\left(T_{T_{e_aE_\beta}E_\gamma}+(\nabla_{e_a}T)_{E_\beta E_\gamma}+(\nabla_{E_\beta}T)_{E_\gamma e_a}\right). \end{aligned}\tag{38}$$

Now, we remark that at the points of M_o,

(1) $E^\delta(\nabla_{e_a}\varepsilon)=E^\delta(\mathbb{D}_{e_a}\varepsilon)-\varepsilon^\gamma\mathbb{A}_{a\gamma}^{\ \delta}$;
(2) $E^\alpha(T_{T_{e_aE_\beta}E_\gamma})=\mathbb{A}_{a\beta}^{\ \zeta}\mathbb{B}_{\zeta\gamma}^{\ \alpha}+\psi_a^\zeta\mathcal{L}_{\zeta\beta}^c\mathbb{A}_{c\gamma}^{\ \alpha}-\psi_c^\xi\psi_a^\zeta\mathcal{L}_{\zeta\beta}^c\mathbb{B}_{\xi\gamma}^{\ \alpha}$.

Replacing (1) and (2) in (38), we get (35). Similarly, from (37),

$$\delta_\varepsilon\mathbb{A}_{a\beta}^{\ b}=-\,E^b\left(\big[\varepsilon,\pi^{\mathcal{D}}(T_{e_aE_\beta})\big]\right)\Big|_{M_o},$$

from which (36) follows immediately. □

Corollary 4.7. *If $\mathcal{G}$ is strict Levi-Civita and satisfies the constraint*

$$T^{\mathcal{D}}\equiv 0,\tag{39}$$

then (33) *simplifies into*

$$\delta_\varepsilon\psi_a^\alpha=\mathrm{e}_a(\varepsilon^\alpha)+\varepsilon^\beta(\mathcal{H}_{a\beta}^{\ \alpha}+\mathbb{A}_{a\beta}^{\ \alpha}),\tag{40}$$

while (35) *simplifies into*

$$\delta_\varepsilon \mathbb{A}_{a\beta}^{\ \ \alpha} = \varepsilon^\gamma \left(-R_{\gamma a \beta}^{\ \ \ \ \alpha} - R_{a\beta\gamma}^{\ \ \ \ \alpha} - \psi_a^\delta R_{\beta\gamma\delta}^{\ \ \ \ \alpha} + \psi_a^\zeta \mathcal{L}_{\zeta\beta}^c \mathbb{A}_{c\gamma}^{\ \ \alpha} + \mathbb{C}_{a\gamma\beta}^{\ \ \ \ \alpha} \right). \tag{41}$$

5 Classical supergravities as supergravities of type $\mathfrak{g}$

In this section, we want to indicate how simple supergravity in four dimension might be encoded in the language of supergravities of type $\mathfrak{g}$. We also give short remarks on other supergravities in four and higher dimensions, supporting the expectation that they can be presented as supergravities of type $\mathfrak{g}$ too. In the following, the discussion is forced to be informal. Indeed, a rigorous presentation of supergravity should be based on various notions of supergeometry, which will be introduced in [24].

5.1 Notations

In all the following, $\mathcal{G} = ((M, M_{\mathrm{o}}, \mathcal{D}), (g, \nabla))$ is a fixed (super) gravity of type $\mathfrak{g}$.

5.1.1 Clifford product between elements of TM and $\mathcal{D}$

For any $x \in M$, $w \in T_xM$ and $\imath^{(x)} \in O_g(M, \mathcal{D})$, we denote by $w = w^V + w^S$ the g-orthogonal decomposition of w into $\mathcal{D}^\perp$- and $\mathcal{D}$-components and we set

$$\widehat{w} = \imath^{(x)-1}(w) \in V + S, \quad \widehat{w}^V = \imath^{(x)-1}(w^V) \in V, \quad \widehat{w}^S = \imath^{(x)-1}(w^S) \in S.$$

For any $s \in \mathcal{D}_x$, we call *Clifford product between* w *and* s the element in $\mathcal{D}_x$

$$w \cdot s \stackrel{\text{def}}{=} \imath^{(x)}(\widehat{w}^V \cdot \widehat{s}), \tag{42}$$

where "$\widehat{w}^V \cdot \widehat{s}$" is the usual Clifford product. One can check that (42) does not depend on the choice of $\imath^{(x)} \in O_g(M, \mathcal{D})$. We extend canonically (42) to a product $\alpha \cdot s \in \mathcal{D}_x$ between any $\alpha \in \Lambda T_xM$ and $s \in \mathcal{D}_x$ and, by g-duality, also to a product $\omega \cdot s$ between any $\omega \in \Lambda T_x^*M$ and $s \in \mathcal{D}_x$.

We remark that *any such Clifford product is preserved by the action* (16).

5.1.2 $\mathcal{D}^\perp$-curvatures and Rarita–Schwinger form

We denote by $\mathrm{Ric}^{\mathcal{D}^\perp}$ and $s^{\mathcal{D}^\perp}$ the tensor field and the scalar function, defined at any $x \in M$ by

$$\mathrm{Ric}^{\mathcal{D}^\perp}_x(v_1, v_2) = \sum_{i=1}^n \epsilon_i g((\pi^{\mathcal{D}^\perp} \circ R)_{v_1 E_i} v_2, E_i), \quad s^{\mathcal{D}^\perp} = \sum_{j=1}^n \epsilon_j \,\mathrm{Ric}^{\mathcal{D}^\perp}(E_j, E_j),$$

where (E_i) is any g-orthonormal basis of $\mathcal{D}^\perp_x$ and $\epsilon_i = g(E_i, E_i) = \pm 1$. These objects are related with Ricci and scalar curvature of the metric connection D on M_{o} as follows. Since the curvature R^D of D is at any $x \in M_{\mathrm{o}}$ given by

$$R^D_x = (\pi^{\mathcal{D}^\perp}|_{TM_{\mathrm{o}}})^{-1} \left(\pi^{\mathcal{D}^\perp} \circ R|_{TM_{\mathrm{o}} \times TM_{\mathrm{o}} \times TM_{\mathrm{o}}} \right),$$

we get that Ricci curvature Ric^D and scalar curvature s^D of D are given by

$$\mathrm{Ric}^D_x(v_1, v_2) = \mathrm{Ric}^{\mathcal{D}^\perp}(v_1, v_2) + \sum_{i=1}^n \epsilon_i g((\pi^{\mathcal{D}^\perp} \circ R)_{v_1 \pi^{\mathcal{D}}(e_i)} v_2, e_i), \tag{43}$$

$$s^D_x = s^{\mathcal{D}^\perp}_x + \sum_{i,j=1}^n \epsilon_i \epsilon_j g_x((\pi^{\mathcal{D}^\perp} \circ R)(\pi^{\mathcal{D}}(e_j), \pi^{\mathcal{D}}(e_i)) e_j, e_i), \tag{44}$$

for any $v_1, v_2 \in T_x M_{\mathrm{o}}$, where (e_i) is a $\widehat{g}$-orthonormal basis for $T_x M_{\mathrm{o}}$.

We call *Rarita–Schwinger 3-form* the tensor field $\mathcal{R} \in \Lambda^3 T^*M \otimes_M \mathcal{D}$ defined at any $x \in M$, $v_1, v_2, v_3 \in T_x M$, by

$$\mathcal{R}_x(v_1, v_2, v_3) \stackrel{\mathrm{def}}{=} \sum_{\sigma \in P_3} (-1)^{\epsilon(\sigma)} v_{\sigma(1)} \cdot \left((\pi^{\mathcal{D}} \circ T)_x \left(v_{\sigma(2)}, v_{\sigma(3)} \right) \right).$$

Using coordinates on M_{o}, the 3-form $\mathcal{R}|_{\Lambda^3 TM_{\mathrm{o}}}$ is related with ϑ by

$$\left(\mathcal{R}|_{\Lambda^3 TM_{\mathrm{o}}}\right)_{\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{i_2}}, \frac{\partial}{\partial x^{i_3}}} = 2 \sum_{\sigma \in P_3} (-1)^{\epsilon(\sigma)} \left(\frac{\partial}{\partial x^{i_{\sigma(1)}}} \cdot \nabla_{\frac{\partial}{\partial x^{i_{\sigma(2)}}}} \left(\vartheta(\frac{\partial}{\partial x^{i_{\sigma(3)}}}) \right) \right). \tag{45}$$

5.2 Simple 4D-supergravity

Let $V = \mathbb{R}^{3,1}$ and $\mathfrak{g} = \mathfrak{so}(V) + V + S$ the super-Poincarè algebra determined by the admissible bilinear form $\beta(s, s') = \mathrm{Re}\, \omega(s, s') = -\mathrm{Re}(is^T \Gamma_0 \Gamma_2 s')$ on the irreducible spinor module $S = S^+ + S^-$ of $\mathcal{C}\ell_{3,1}$ (see Example 2.5). Simple $4D$-supergravity can be interpreted as a supergravity

$$\mathcal{G} = ((M, M_{\rm o}, \mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-), (g, \nabla))$$

of type $\mathfrak{g}$ ([3]), subjected to the following constraints and equations, which are equivalent to Wess and Zumino's constraints and the usual Euler–Lagrange equations [31–33].

Constraints

(1) ∇ is strict Levi-Civita (i.e., $T^{\mathcal{D}^\perp} = 0 = \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}$);
(2) $T^{\mathcal{D}} \equiv 0$.

Equations

(i) $\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}}\Big|_{\mathcal{S}\otimes TM_{\rm o}} = 0$ (*vanishing of auxiliary fields*);

(ii) $\mathcal{R}|_{\Lambda^3 TM_{\rm o}} = 0$ (*Rarita–Schwinger eq.*);

(iii) $\mathrm{Ric}^{\mathcal{D}^\perp}\Big|_{TM_{\rm o}\times TM_{\rm o}} - \frac{1}{2}\, s^{\mathcal{D}^\perp} g(\pi^{\mathcal{D}^\perp}(\cdot), \pi^{\mathcal{D}^\perp}(\cdot))\Big|_{TM_{\rm o}\times TM_{\rm o}} = 0$ (*Einstein eq.*).

The first equation corresponds to the vanishing of the "auxiliary fields", the second one to the so-called Rarita–Schwinger equation for gravitinos, whereas the last one corresponds to the Euler–Lagrange equation for gravitons.

Firstly, from constraints (1) and (2) and Bianchi identities, one gets that the A-field $\mathbb{A}$ is of the following very special form (see [32], Ch. XV)

$$\begin{aligned}\mathbb{A}_X s &= \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}}\Big|_{\mathcal{S}\otimes TM_{\rm o}}(s, X)\\ &= -\operatorname{Re}(a) X\cdot\Gamma_5\cdot s + i\operatorname{Im}(a) X\cdot s + iA(X)\Gamma_5\cdot s + \frac{i}{3} X\cdot A\cdot\Gamma_5\cdot s,\end{aligned} \tag{46}$$

for a complex function $a : M_{\rm o} \to \mathbb{C}$ and a 1-form $A \in T^*M_{\rm o}$, usually called *auxiliary fields*, and hence that (i) is equivalent to equations $a = 0$, $A = 0$.

Equation (ii) is equivalent to the Rarita–Schwinger equation by simply comparing the coordinate expression (45) with [31], formula (5) at p. 222.

[3]For this super-algebra, the space $\mathfrak{h}$ is trivial (see Remark 2.9) and $\nabla = \nabla^o$ for $\mathcal{G}$.

Now, assume constraints (1), (2) and equations (i), (ii) hold. By equations (43), (44) and Bianchi identities, one can prove $s^{\mathcal{D}^\perp} = s^D$ so that (iii) reads

$$\left(\mathrm{Ric}^D(X,Y) - \frac{1}{2}s^D\widehat{g}(X,Y)\right) - \sum_{i=1}^n \epsilon_i g((\pi^{\mathcal{D}^\perp} \circ R)_{X\pi^{\mathcal{D}}(e_i)}Y, e_i) = 0, \quad (47)$$

for any $X, Y \in \mathfrak{X}(M_o)$. Using again Bianchi identities and (ii), the equation (47) becomes equivalent to

$$\left(\mathrm{Ric}^D(X,Y) - \frac{1}{2}s^D\widehat{g}(X,Y)\right) - \sum_{i=1}^n \epsilon_i g(\pi^{\mathcal{D}}(e_i), X \cdot \left(\pi^{\mathcal{D}} \circ T)_{Ye_i}\right) = 0,$$

for any $X, Y \in \mathfrak{X}(M_o)$. From the expression in coordinates

$$\pi^{\mathcal{D}}\left(T_{\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}}\right) = \nabla_{\frac{\partial}{\partial x^i}}\left(\vartheta(\frac{\partial}{\partial x^j})\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\vartheta(\frac{\partial}{\partial x^i})\right)$$

and [31], formula (10) at p. 222, one gets that (iii) is equivalent to the usual Euler–Lagrange equations for gravitons (see [31], formula (6) at p.222).

Finally, we remark that, under the constraints (1) and (2), the usual transformation rules for graviton and gravitino (see [32], Ch. XVIII) coincide with those in Proposition 4.5 and Corollary 4.7 and it is reasonable to expect that, via (46), the usual transformation rules of auxiliary fields imply the variations for the A-field, determined in Corollary 4.7. We plan to check carefully this point in the near future.

In any case, we claim that the above constraints and equations are *manifestly covariant* and hence *invariant under all super-variations* of Definition 4.4, by the following reasons.

Consider the system $\mathcal{E}$ on $(\mathcal{D}, g, \nabla)$ given by the tensorial equations

$$T^{\mathcal{D}^\perp} = 0, \quad \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp} = 0, \quad T^{\mathcal{D}} = 0,$$
$$\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}} = 0, \quad \mathcal{R} = 0, \quad \mathrm{Ric}^{\mathcal{D}^\perp} - \frac{1}{2}s^{\mathcal{D}^\perp} g(\pi^{\mathcal{D}^\perp}(\cdot), \pi^{\mathcal{D}^\perp}(\cdot)) = 0.$$

Any (local) solution of $\mathcal{E}$ gives physical fields satisfying the system $\mathcal{E}_o$ of (1), (2), (i), (ii), (iii). So, being $\mathcal{E}$ of tensorial type, in order to check the manifest covariance, it remains to show that any (local) solution of $\mathcal{E}_o$ is given by the physical fields of some (local) solution of $\mathcal{E}$.

Indeed, following the same arguments used in [24] to check the manifest covariance of the 11D supergravity equations and constraints, one can see that the conditions $\mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp} = T^{\mathcal{D}} = \mathcal{C}^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}} = 0$, together with the relations $R|_{\mathcal{D}\otimes\mathcal{D}} = 0$ and (15.21) of [32] (which come from the first Bianchi identities of ∇), coincide with the *rheonomic constraints* considered by Castellani, D'Auria and Frè in [2], Ch.III.3.5. By the results of [2], one gets that all equations of the system $\mathcal{E}$ are consequences of such rheonomic constraints and Bianchi identities and that the required one-to-one correspondence between solutions of $\mathcal{E}_{\mathrm{o}}$ and solutions of $\mathcal{E}$ is a corollary of the properties of the rheonomic constraints (we refer to [24] for more details on this line of arguments).

5.3 Other supergravities

5.3.1 Gates and Siegel's supergravities

Simple 4D-supergravity is one of the supergravities, parameterized by $\zeta \in \mathbb{R} \cup \{\infty\}$, introduced by Gates, Siegel in [7, 26] (see also [21, 22]). All of them can be interpreted as supergravities

$$\mathcal{G} = ((M, M_{\mathrm{o}}, \mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-), (g, \nabla))$$

of the same type $\mathfrak{g}$ of simple supergravity and they are subjected to the following constraints for $\zeta \neq -\frac{1}{3}$ (the case $\zeta = -\frac{1}{3}$ is simple supergravity).

Constraints

(1) ∇ is (non-strict) Levi-Civita with $\mathcal{C}_x^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp}$ of the form

$$\mathcal{C}_x^{\mathcal{D},\mathcal{D}^\perp;\mathcal{D}^\perp} = \frac{\zeta}{3\zeta+1}(\mathrm{Re}(\mathcal{T}) \circ \pi^{\mathcal{D}} - \mathrm{i}\,\mathrm{Im}(\mathcal{T}) \circ \pi^{\mathcal{D}} \circ \Gamma_5)_x \otimes \pi_x^{\mathcal{D}^\perp},$$

for some complex-valued 1-form $\mathcal{T} \in T^*M$;

(2) $T^{\mathcal{D}}$ is of the form

$$\begin{aligned} T_x^{\mathcal{D}}(v_1, v_2) = \frac{1}{2}\frac{\zeta+1}{3\zeta+1}\Big(&\pi_x^{\mathcal{D}^\pm}(v_1)(\mathrm{Re}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} - \mathrm{i}\,\mathrm{Im}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} \circ \Gamma_5)_x(v_2) \\ &+ \pi_x^{\mathcal{D}^\pm}(v_2)(\mathrm{Re}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} - i\,\mathrm{Im}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} \circ \Gamma_5)_x(v_1)\Big) \\ + \frac{1}{2}\frac{\zeta-1}{3\zeta+1}\Big(&\pi_x^{\mathcal{D}^\mp}(v_1)(\mathrm{Re}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} - i\,\mathrm{Im}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} \circ \Gamma_5)_x(v_2) \\ &+ \pi_x^{\mathcal{D}^\mp}(v_2)(\mathrm{Re}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} - i\,\mathrm{Im}(\mathcal{T}) \circ \pi^{\mathcal{D}^\pm} \circ \Gamma_5)_x(v_1)\Big), \end{aligned}$$

for any v_1, $v_2 \in T_xM$;

(3) the torsion components $\mathcal{C}^{\mathcal{D}^+,\mathcal{D}^\perp;\mathcal{D}^-}$ and $\mathcal{C}^{\mathcal{D}^-,\mathcal{D}^\perp;\mathcal{D}^+}$ vanish.

These constraints are manifestly covariant. We expect that also the Euler–Lagrangian equations of these supergravities are manifestly covariant, as it occurs for simple 4D supergravity.

5.3.2 Supergravities in dimensions $n \geq 5$

We recall that the Poincarè superalgebra $\mathfrak{g} = \mathfrak{so}_{3,1} + \mathbb{R}^{3,1} + S$ of simple 4D supergravity is the algebra of rigid supersymmetries of maximally supersymmetric vacua solutions and that the theory is actually determined by "gauging" such symmetries.

Supergravities in dimensions $n \geq 5$ are similarly obtained from algebras $\mathfrak{g}$ of rigid supersymmetries of homogenous manifolds playing the role of vacua.

The superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ is usually taken from Nahm's classification ([12]), i.e., it is a simple Lie superalgebra with $\mathfrak{g}_0 = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is reductive and $\mathfrak{p}$ is a conformal or de Sitter algebra, and with $\mathfrak{g}_1 = S$ a spinor module.

The associated simply connected, homogeneous supermanifold is of the form G/H, with $\mathfrak{h} = \mathfrak{so}_{p,1} \oplus \mathfrak{k} \subset \mathfrak{p} \oplus \mathfrak{k}$ and it is endowed with the G-invariant distribution $\mathcal{D}$ with $\mathcal{D}|_{eH} = S$. Its Levi form at eH is the $\mathfrak{so}_{p,1}$-invariant tensor

$$L \in S^2 S^* \otimes \mathbb{R}^{p,1}, \quad L(s, s') = [s, s'] \mod \mathfrak{h}.$$

This means that $(G/H, G_0/H, \mathcal{D})$ *is a space-time of type* $\mathfrak{g}'$, where $\mathfrak{g}'$ is the super Poincarè algebra $\mathfrak{g}' = (\mathfrak{so}_{p,1} + \mathbb{R}^{p,1}) + S$, with brackets $[\cdot,\cdot]|_{S\times S} = L$.

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References

[1] D.V. Alekseevsky and V. Cortés, *Classification of N-(super)-extended Poincaré algebras and bilinear invariants of the Spinor representation of* Spin(p, q), Comm. Math. Phys. **183** (1997), 477–510.

[2] L. Castellani, R. D'Auria and P. Fré, *Supergravity and superstrings: a geometric perspective* — Vol. II, *World Scientific Publ. Co.*, New Jersey, 1991.

[3] A. D'Adda, R. D'Auria, P. Fré and T. Regge, *Geometrical formulation of supergravity theories on orthosymplectic supergroup manifolds*, Riv. Nuovo Cimento **3** (1980), 81.

[4] R. D'Auria, P. Fré and T. Regge, *Group manifold approach to gravity and supergravity theories*, in "*Supergravity '81, Proceedings of the First School held in Trieste*, April 22–May 6 (1981)" eds. S. Ferrara and J. G. Taylor, *Cambridge University Press*, Cambridge–New York, 1982.

[5] P. Deligne, Super space descriptions of super gravity, in "*Quantum fields and strings: a course for mathematicians — Vol. II*", ed P. Deligne et al., American Mathematical Society, Providence, 1999.

[6] J. Figueroa-O'Farrill, *Majorana spinors*, unpublished notes. www.maths.ed.ac.uk/∼jmf/Teaching/Lectures/Majorana.pdf

[7] S.J. Gates, Jr. and W. Siegel, *Understanding constraints in superspace formulations of supergravity*, Nucl. Phys. B **163** (1980), 519–545.

[8] H. Blaine Lawson, Jr. and M.-L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, 1989.

[9] J. Lott, *Torsion constraints in supergeometry*, Commun. Math. Phys. **133** (1990), 563–615.

[10] Yu. Manin, *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Berlin, 1988.

[11] A. Miemiec and I. Schnakenburg, *Basics of M-theory*, Fortschr. Phys. **54** (2006), 5–72.

[12] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. B **135** (1978), 149–166.

[13] Y. Ne'eman and T. Regge, *Gravity and supergravity as gauge theories on a group manifold*, Phys. Lett. B **74** (1978), 54–56.

[14] Y. Ne'eman and T. Regge, *Gauge theory of gravity and supergravity on a group manifold*, Riv. Nuovo Cimento **1** (1978), 1–43.

[15] V.I. Ogievetsky and E.S. Sokatchev, *Structure of the supergravity group*, Phys. Lett. B **79** (1978), 222–224.

[16] V.I. Ogievetsky and E.S. Sokatchev, *The axial superfield and the supergravity group*, Yad. Fiz. **28** (1978), 1631–1639.

[17] V.I. Ogievetsky and E.S. Sokatchev, *The simplest group of Einstein supergravity*, Yad. Fiz. **31** (1980), 264–279.

[18] V.I. Ogievetsky and E.S. Sokatchev, *The axial gravitational superfield and the formalism of differential geometry*, Yad. Fiz. **31** (1980), 821–840.

[19] K.A. Pilch, *Geometrical meaning of the Poincarè group gauge theory*, Lett. Math. Phys. **4** (1980), 49–51.

[20] A.A. Roslyĭ, O.M. Khudaverdyan and A.S. Schwarz, *Supersymmetry and Complex Geometry* in "Encyclopaedia of Mathematical Sciences vol. 9 — Several Complex Variables III: Geometric Function Theory", ed. G. M. Khenkin, Springer, Berlin, 1989.

[21] A.A. Roslyĭ and A.S. Schwarz, *Geometry of $N = 1$ supergravity*, Comm. Math. Phys. **95** (1984), 161–184.

[22] A.A. Roslyĭ and A.S. Schwarz, *Geometry of $N = 1$ supergravity (II)*, Comm. Math. Phys. **96** (1984), 285–309.

[23] A. Santi, *Superizations of Cahen–Wallach symmetric spaces and spin representations of the Heisenberg algebra*, J. Geom. Phys. **60** (2010), 295–325.

[24] A. Santi and A. Spiro, *Super Poincarè algebras, space-times and supergravities (II)*, J. Math. Phys. **53** (2012), 032505.

[25] A.S. Schwarz, *Supergravity, Complex Geometry and G-structures*, Commun. Math. Phys. **87** (1982), 37–63.

[26] W. Siegel and S.J. Gates, Jr., *Superfield supergravity*, Nucl. Phys. B **147** (1979), 77–104.

[27] I.M. Singer and S. Sternberg, *The infinite groups of Lie and Cartan — Part I (the Transitive Groups)*, J. Analyse Math. **15** (1965), 1–114.

[28] A. Spiro and S. Tantucci, *The two ways of gauging the Poincarè group*, Int. J. Geom. Methods Mod. Phys. **6** (2009), 1115–1134.

[29] S. Sternberg, *The interaction of spin and torsion. II. The principle of general covariance*, Ann. Phys. **162** (1985), 85–99.

[30] J. Strathdee, *Extended Poincarè supersymmetry*, Int. J. Modern Physic A **2** (1987), 273–300.

[31] P. van Nieuwenhuizen, *Supergravity*, Phys. Rep. **68** (1981), 189–398.

[32] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, New Jersey, 1992.

[33] J. Wess and B. Zumino, *Superspace formulation of supergravity*, Phys. Lett. B **66** (1977), 361–364.

[34] P. West, *Introduction to Supersymmetry and Supergravity*, World Scientific, Singapore, 1986.