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Direct integration for general Ω backgrounds

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Abstract

We extend the direct integration method of the holomorphic anomaly equations to general Ω backgrounds $\epsilon_1 \neq -\epsilon_2$ for pure SU(2) $N = 2$ Super-Yang–Mills theory and topological string theory on non-compact Calabi–Yau threefolds. We find that an extension of the holomorphic anomaly equation, modularity and boundary conditions provided by the perturbative terms as well as by the gap condition at the conifold are sufficient to solve the generalized theory in the above cases. In particular, we use the method to solve the topological string for the general Ω backgrounds on non-compact toric Calabi–Yau spaces. The conifold boundary condition follows from that the $N = 2$ Schwinger-loop calculation with Bogomol'nyi-Prasad-Sommerfield (BPS) states coupled to a self-dual and an anti-self-dual field strength. We calculate such BPS states also for the large base decompactification limit of Calabi–Yau spaces with regular K3 fibrations and half $K3$s embedded in Calabi–Yau backgrounds.

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Contents

1 Introduction

Nekrasov's instanton calculations for the $N = 2$ supersymmetric gauge theory [1] completes the program of [2] and confirms the Seiberg–Witten prepotential as the leading contribution in the asymptotically free region from the microscopic field theory perspective. These instanton calculations have been made mathematically more rigorous in [3, 4]. Higher-order contributions in Nekrasov's partition function correspond to gravitational couplings of the gauge theory, and are organized by a topological genus expansion. The genus one formula is also mathematically proven in [5]. In previous works, we computed the higher genus terms in $SU(2)$ Seiberg–Witten theory (with fundamental matter) [6, 7] in terms of generators of modular forms w.r.t. the monodromy group, which is a subgroup of $SL(2, Z)$, using holomorphic anomaly equations [8] and novel boundary conditions at the special points of the moduli space. Our formulae constitute well-defined mathematical conjectures that sum up all instanton contribution of Nekrasov's partition function at fixed genus in a closed from, which defines it explicitly at every point on the Coulomb branch.

There are two deformation parameters ϵ_1, ϵ_2 in Nekrasov's partition function. Our higher genus formulae in [6, 7] correspond to the case $\epsilon_1 = -\epsilon_2$ or $\beta := -\frac{\epsilon_1}{\epsilon_2} = 1$, where the technique of holomorphic anomaly equations from the topological string theory is applicable. Recently, it has become an interesting question to study the general case of arbitrary β-backgrounds due to several developments.

Firstly, the AGT (Alday–Gaiotto–Tachikawa) conjecture [9] relates the Nekrasov function at general deformation parameter (at a fixed instanton number) to correlation functions in the Liouville theory. A matrix model with a modified measure, the called β-ensemble [43], was related to the general β-deformations of gauge theories.

Secondly, in the BPS interpretation of the topological string partition function, there is a natural meaning of the ϵ_1, ϵ_2 expansion. It gives refined information about the cohomology of the moduli space of the BPS states, while the $\epsilon_1 = -\epsilon_2$ slice computes complex structure invariant indices. A refined topological vertex was proposed in [10] that generalizes the topological string partition function for non-compact toric Calabi–Yau manifolds, which have no complex structure deformations. It was shown to reduce to the Nekrasov partition function for general deformation parameters ϵ_1, ϵ_2 in the field theory limit.

In this paper, we describe first the B-model approach of direct integration for the simplest deformed $N = 2$ gauge theory in Section 2. Similar results have been also obtained recently, in fact in more generality in [11]. It is clear already from the perturbative test of [44] that the β-ensemble for the matrix models associated to Seiberg–Witten theories, suggested in [43], leads to the possibility to remodel the B-model along the line of [48] from the spectral curve using the formalism of [49].

A direct implementation of the deformed β-ensemble to the matrix models associated to the topological string on local Calabi–Yau spaces seems not straightforward. We found that even the β-ensemble for the Chern–Simons matrix model, describing the resolved conifold in the canonical parameterization, fails[1] to reproduce the known results [10] for this geometry if $\epsilon_1 \neq -\epsilon_2$. We therefore move on to calculate general deformations in topological string theory in Section 3.

In Section 3.1, we explain first the interpretation of the refined topological string expansion in terms of the cohomology of the moduli space of BPS states. The BPS picture yields the generalized gap condition at the conifold locus in Section 3.2 and the large radius conifold expansion in Section 3.3.

We then make predictions for the generalized BPS invariants in the decompactification limit of $K3$ fibered Calabi–Yau spaces for large base space using heterotic type II duality in Section 3.5.2 and analyze in Section 4.1 a similar setting for the half $K3$, which by a T-duality predicts a sector of the partition function for compactifications of $N = 4$ Super-Yang-Mills (SYM) on manifolds with $b_+ = 1$.

Using the generalized holomorphic anomaly equation and boundary condition we extend the methods of [6, 7, 40]. I.e. we perform the direct integration for general β-backgrounds on local toric Calabi–Yau spaces using the generalized gap condition discussed in 3.2. We then consider non-compact manifolds, such as $\mathcal{O}(K_B) \to B$, where the base B is a toric manifold. As examples, we present the cases $B = \mathbb{P}^2$ and $B = \mathbb{P}^1 \times \mathbb{P}^1$ in Sections 4.2 and 4.3, respectively. We hope that our analytic expressions for the amplitudes will help to find a matrix model description.

2 Seiberg–Witten gauge theory

The Nekrasov partition function consists of the perturbative contributions and the instanton contributions

$$Z(a, \epsilon_1, \epsilon_2) = Z_{\text{pert}}(a, \epsilon_1, \epsilon_2) Z_{\text{inst}}(a, \epsilon_1, \epsilon_2). \tag{2.1}$$

[1] We thank Marcos Mariño for sharing insights into similar attempts.

In this paper, we only consider the pure $SU(2)$ case, so there is only one Seiberg–Witten period a and we choose the cut-off parameter in [3] to be $\Lambda = 1/16$ which can be recovered by dimensional analysis in the formula. So the function essentially depends on three parameters $a, \epsilon_1, \epsilon_2$. The logarithm of the Nekrasov function can be expanded as

$$\log Z(a, \epsilon_1, \epsilon_2) = \sum_{i,j=0}^{\infty} (\epsilon_1 + \epsilon_2)^i (\epsilon_1 \epsilon_2)^{j-1} F^{(\frac{i}{2}, j)}(a). \tag{2.2}$$

Then the genus zero $F^{(0,0)}$ is the prepotential well known from the work of Seiberg and Witten [12], and the formula F^g with $g > 1$ for the case of $\epsilon_1 + \epsilon_2 = 0$ in [6, 7] correspond to $F^g = F^{(0,g)}$ in our current notation. It turns out that for the models that we will study, when i is an odd integer, $F^{(\frac{i}{2},j)}(a)$ vanishes except a trivial term from the perturbative contributions. So we will only need to consider $F^{(i,j)}(a)$ with i, j non-negative integers. This is not always true for all models. In particular, the $F^{(\frac{1}{2},0)}$ is non-vanishing for $SU(2)$ Seiberg–Witten theory with $N_\mathrm{f} = 1$ massless flavor. For this interesting case, as well as the massless $N_\mathrm{f} = 2, 3$ theories; see the recent paper [11].

2.1 Generalized holomorphic anomaly equations

It turns out that the topological amplitudes $F^{(g_1,g_2)}$ satisfy for $g_1 + g_2 \geq 2$ a generalized holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}} F^{(g_1,g_2)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \big(D_j D_k F^{(g_1,g_2-1)} + {\sum_{r_1,r_2}}' D_j F^{(r_1,r_2)} D_k F^{(g_1-r_1,g_2-r_2)} \big), \tag{2.3}$$

where the prime denotes that the sum over r_1, r_2 does not include $(r_1, r_2) = 0$ and $(r_1, r_2) = (g_1, g_2)$, and the first term on the right-hand side is understood to be zero, if $g_2 = 0$. This equation reduces to the ordinary Bershadsky-Cecotti-Ooguri-Vafa (BCOV) holomorphic anomaly equation when $g_1 = 0$, and is a simplification of the extended holomorphic anomaly equation in [11] without the so-called Griffiths infinitesimal invariant, which turns out to be vanishing for the models we study.

To integrate the holomorphic anomaly equation and write out the compact expressions for the higher genus amplitudes, we first express the Seiberg–Witten period a and Coulomb modulus u in terms of modular functions of the coupling $\tau \sim \frac{1}{2\pi \mathrm{i}} \frac{\partial^2 F^{(0,0)}}{\partial a^2}$ as

$$a = \frac{E_2(\tau) + \theta_3^4(\tau) + \theta_4^4(\tau)}{3\theta_2^2(\tau)}, \quad u = \frac{\theta_3^4(\tau) + \theta_4^4(\tau)}{\theta_2^4(\tau)}. \tag{2.4}$$

In the cusp limit $\tau \to \mathrm{i}\infty$, we find $q = \mathrm{e}^{2\pi \mathrm{i}\tau} \to 0$ and $a \sim q^{-\frac{1}{4}}$, $u \sim q^{-\frac{1}{2}}$. We can express a, u, q in terms of series expansion of each other, from the above relations and the well known series expansion formulae of the Theta functions and Eisenstein series. It is proven in [5] that the genus one formulae for Nekrasov function are

$$F^{(0,1)} = -\log(\eta(\tau)), \quad F^{(1,0)} = -\frac{1}{6}\log\left(\frac{\theta_2^2}{\theta_3\theta_4}\right). \tag{2.5}$$

The anholomorphic generator in the topological amplitudes is the shifted Eisenstein series $\hat{E}_2 = E_2(\tau) + \frac{6i}{\pi(\bar{\tau}-\tau)}$. Using some well-known results about the three-point coupling and the relations between parameters a, u, τ in (2.4), we find that (2.3) becomes

$$48\frac{\partial F^{(g_1,g_2)}}{\partial E_2} = \frac{d^2}{da^2}F^{(g_1,g_2-1)} + \left(\sum_{r_1=0}^{g_1}\sum_{r_2=0}^{g_2}\right)' \left(\frac{dF^{(r_1,r_2)}}{da}\right)\left(\frac{dF^{(g_1-r_1,g_2-r_2)}}{da}\right). \tag{2.6}$$

If the above generalized holomorphic anomaly equation is true, then it will determine $F^{(g_1,g_2)}$ recursively up to a rational function of modulus u with a pole at the discriminant of Seiberg–Witten curve $u^2 - 1$ of degree $2(g_1 + g_2) - 2$.

The equation (2.6) applies to the case of $g_1 + g_2 \geq 2$. At genus one, we note that $F^{(0,1)}$ satisfies the ordinary BCOV holomorphic anomaly equation after we pass to the usual modular but an-holomorphic completion of $\eta \to \sqrt{\mathrm{Im}(\tau)}|\eta(\tau)|^2$.

As for $F^{(1,0)}$, we write

$$F^{(1,0)} = -\frac{1}{6}\log\left(\frac{\theta_2^2}{\theta_3\theta_4}\right) = \frac{1}{24}\log(u^2 - 1). \tag{2.7}$$

We see that $F^{(1,0)}$ has only a logarithmic cut at the discriminant $u^2 - 1$. It is already modular, needs no an-holomorphic modular completion and has therefore no holomorphic anomaly.

2.2 Higher genus formulae and the dual expansion

Problems associated to Riemann surfaces $\mathcal{C}_1$ of genus one such as $SU(2)$ $N=2$ SYM theories, topological string related to local del Pezzo surfaces or cubic matrix models have only one an-holomorphic generator $\hat{E}_2$ in the ring of modular objects generating all $F^{(g_1,g_2)}$. It is convenient to define an an-holomorphic generator of weight zero, e.g., $X = \hat{E}_2/\theta_2^4$ in the case above.

In cases with one an-holomorphic generator X the direct integration of the generalized holomorphic anomaly equation of the type (2.6) leads to the following general form of the $F^{(g_1,g_2)}$:

$$F^{(g_1,g_2)} = \frac{1}{\Delta^{2(g_1+g_2-1)}(u)} \sum_{k=0}^{3g_2-3-g_1} X^k c_k^{(g_1,g_2)}(u), \tag{2.8}$$

where $\Delta(u)$ is the conifold discriminant of $\mathcal{C}_1$ and u are holomorphic monodromy invariant parameters. All $c_i^{(g_1,g_2)}(u)$ are polynomial in these parameters. The extension of the generalized anomaly equations and the general form (2.8) to cases with more an-holomorphic generators X_{ij} for theories related to Riemann surfaces $\mathcal{C}_{g>1}$ works along the lines discussed in [33, 38, 42].

In (2.8), all $c_{i>0}^{(g_1,g_2)}(u)$ are determined by the generalized holomorphic anomaly equation, while the holomorphic ambiguity $c_0^{(g_1,g_2)}(u)$ must be determined from the boundary conditions. We find that the expansion at the conifold divisor in the moduli space and in particular the gap condition in this expansion together with regularity at other limits in the moduli space and the knowledge of the classical terms are sufficient to completely fix $c_0^{(g_1,g_2)}(u)$. Note that regularity of $F^{(g_1,g_2)}$ the $u \to \infty$ limit implies that the $c_i^{(g_1,g_2)}(u)$ are finite degree polynomials. We will explain the gap condition in more details in the context of topological string theory on Calabi–Yau manifolds in Section 3.2.

To determine now the holomorphic ambiguity for the pure $SU(2)$ theory, we expand the topological amplitudes around the monopole point $u=1$. This can be achieved by an S-duality transformation. Under an S-duality transformation $\tau \to -\frac{1}{\tau}$, the shifted E_2 transforms with weight 2, and the Theta functions transform as $\theta_2^4 \to -\theta_4^4$, $\theta_3^4 \to -\theta_3^4$, $\theta_4^4 \to -\theta_2^4$. The parameter u and a become[2]

$$a_D = \frac{2}{3\theta_4^2(\tau)}(E_2(\tau) - \theta_3^4(\tau) - \theta_2^4(\tau)), \quad u_D = \frac{\theta_3^4(\tau)+\theta_2^4(\tau)}{\theta_4^4(\tau)}. \tag{2.9}$$

[2]Here, we normalize a_D by a factor of $2i$ for the consistence of conventions.

We find that in the cusp limit $\tau \to i\infty$, the parameters go to $a_D \sim q^{\frac{1}{2}} \to 0$, $u_D \to 1$. This is similar to the conifold point in the moduli space Calabi–Yau manifolds. We find the gap condition [6] around this point completely fixes the holomorphic ambiguity.

We obtain compact formulae for higher genus $F^{(g_1,g_2)}$ similar to those in [6] for $F^{(0,g)}$. The genus two formulae are

$$\begin{aligned} F^{(0,2)} &= \frac{200X^3 - 360uX^2 + (60u^2+180)X - 19u^3 - 45u}{12960(u^2-1)^2}, \\ F^{(1,1)} &= \frac{20uX^2 - (40u^2+60)X + 3u^3 + 45u}{2160(u^2-1)^2}, \\ F^{(2,0)} &= \frac{10u^2X + u^3 - 75u}{4320(u^2-1)^2}. \end{aligned} \tag{2.10}$$

We note that X, u are modular invariant under the monodromy group $\Gamma(2) \subset SL(2,Z)$ if we shift the second Eisenstein series by an anholomorphic piece $E_2 \to \hat{E}_2 = E_2 + \frac{6i}{\pi(\bar{\tau}-\tau)}$.

We expand the genus one and genus two formulae (2.5) and (2.10) around the conifold point.

$$\begin{aligned} F_D^{(0,1)} &= -\frac{1}{12}\log(a_D) + c_{0,1} - \frac{a_D}{2^5} + \mathcal{O}(a_D^2), \\ F_D^{(1,0)} &= \frac{1}{24}\log(a_D) + c_{1,0} - \frac{3a_D}{2^6} + \mathcal{O}(a_D^2), \\ F_D^{(0,2)} &= -\frac{1}{240a_D^2} - \frac{a_D}{2^{13}} + \mathcal{O}(a_D^2), \\ F_D^{(1,1)} &= \frac{7}{1440a_D^2} + \frac{3}{2^{11}} + \frac{25a_D}{2^{14}} + \mathcal{O}(a_D^2), \\ F_D^{(2,0)} &= -\frac{7}{5760a_D^2} + \frac{9}{2^{13}} + \frac{135a_D}{2^{16}} + \mathcal{O}(a_D^2). \end{aligned} \tag{2.11}$$

Here $c_{0,1}$ and $c_{1,0}$ are two irrelevant constants. We see that the genus two functions satisfy the gap condition with the absence of $\frac{1}{a_D}$ term. We present gap structure and results for $g_1 + g_2 = 3$ in Appendix A.1. Our exact formulae (2.10) sum up the genus two parts of all instanton contributions of the Nekrasov's function. We can check the agreements with Nekrasov's function up to some instanton number, by expanding the expressions around the large complex structure parameter point $u \sim \infty$.

3 The refined topological string theory

In this section, we discuss general aspects of refined topological string theory, such as the description of the expansions in terms of refined BPS states and their invariance under complex structure deformations. As examples we treat the conifold and K3 fibrations.

First we interpret the general $-\frac{\epsilon_1}{\epsilon_2} = \beta \neq 1$ deformation for the BPS states related to topological string theory from the generalized Schwinger-loop amplitude.

3.1 The Schwinger-loop amplitude

It will be convenient to define

$$\epsilon_{\mathrm{R/L}} = \epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2). \tag{3.1}$$

In [10, 13, 14], it was suggested to integrate out BPS states in the Schwinger-loop amplitude leading to an F-term in $N = 2$ supergravity

$$R_-^2 T_-^{2m-2} F_+^{2n-2}. \tag{3.2}$$

Here R_- and T_- are the anti-self-dual curvature and anti-self-dual graviphoton field strength and F_+ is a self-dual field strength. More precisely [10] considers for F_+ the self-dual part of the graviphoton. In this case the amplitude cannot lead to an F-term. In [14] for F_+ the self-dual part of the field strength associated to the heterotic dilaton and claim that this gives rise to an F-term.

The term can be calculated in a 5d M-theory compactification on $S^1 \times M$ or on an Type II compactification on the Calabi–Yau M. Following the former picture and denoting the general field strength $G = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4$ then integrating out a massive particle of mass m in the representation $\mathcal{R}$ of the little group of the 5D Lorentz $SO(4) \sim SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ gives the following contribution to the Schwinger-loop amplitude:

$$F(\epsilon_1, \epsilon_2) = -\int_\epsilon^\infty \frac{ds}{s} \frac{\mathrm{Tr}_{\mathcal{R}}(-1)^{\sigma_{\mathrm{L}}+\sigma_{\mathrm{R}}}\, \mathrm{e}^{-sm}\, \mathrm{e}^{-2\mathrm{i}s(\sigma_{\mathrm{L}}\epsilon_{\mathrm{L}}+\sigma_{\mathrm{R}}\epsilon_{\mathrm{R}})}}{4\left(\sin^2\left(\frac{s\epsilon_{\mathrm{L}}}{2}\right) - \sin^2\left(\frac{s\epsilon_{\mathrm{R}}}{2}\right)\right)}, \tag{3.3}$$

where we denoted by $\epsilon_{\mathrm{R/L}} = \epsilon_\pm = \mathrm{i}eG_\pm$ the self-dual or anti-self-dual part of field strengths coupling to the BPS state respectively.

At large complex structure we expect to be able to count BPS numbers for the D-brane charges $\vec{Q} = (Q^6, Q^4, Q^2, Q^0) = (1, 0, \beta, n)$ with $\beta \in H_2(M, \mathbb{Z})$ and $n \in \mathbb{Z}$. More precisely in M-theory compactifications on Calabi–Yau threefolds M the BPS invariants related to topological string theory have been interpreted as an index in the cohomology of the moduli space $H^*(\mathcal{M}_\beta)$ of an $M2$ brane wrapping a curve in the class $\beta \in H_2(M, \mathbb{Z})$ [15]. After compactification on the M-theory S^1 the moduli space $\mathcal{M}_\beta$ can be described equivalently as the one of a $D2/D0$ bound state in the type IIA compactification on M, where the D2 wraps now the curve and n is the degeneracy of the D0 branes.

The $SU(2)_{\mathrm{L}/R}$ factors of the little group $SO(4) \sim SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ of the 5D Lorentz group of the M-theory compactification on M act as two Lefshetz actions on the cohomology of the moduli space of the brane system $H^*(\mathcal{M}_\beta)$ and these factors of the spacetime group are the same that were used in the localization procedure in [1]. I.e. ϵ_{L} and ϵ_{R} are identified with the eigenvalues of the $j^3_{\mathrm{L/R}}$ in the corresponding $SU(2)_{\mathrm{L/R}}$, which label the integer BPS numbers[3] $n^\beta_{j_{\mathrm{L}},j_{\mathrm{R}}}$.

One important point here is that $n^\beta_{j_{\mathrm{L}},j_{\mathrm{R}}}$ is not invariant under complex structure deformations but only the index

$$n^\beta_{j_{\mathrm{L}}} = \sum_{j_{\mathrm{R}}} (-1)^{2j_{\mathrm{R}}}(2j_{\mathrm{R}} + 1) n^\beta_{j_{\mathrm{L}},j_{\mathrm{R}}}. \tag{3.4}$$

This relies on the fact only the F-term $R_-^2 T_-^{2g-2}$, which defines the invariant topological string amplitude, depends exclusively on the Kähler moduli. In the corresponding one loop amplitude the anti-self-dual graviphoton field strength T_- as well as the anti-self-dual curvature 2-form R_- couple only to the left spin, while the right spin content enters the calculation of $R_-^2 T_-^{2g-2}$ merely with its multiplicity weighted with -1 for fermions and 1 for bosons leading to 3.4).

In order to compare with the genus expansion of the topological string the the left representations have to be organized into

$$I^n_{\mathrm{L}} = \left[\left(\frac{1}{2}\right) + 2(0)\right]^{\otimes n}, \tag{3.5}$$

i.e., one defines

$$\sum n^\beta_{j_{\mathrm{L}},j_{\mathrm{R}}} (-1)^{2j_{\mathrm{R}}}(2j_{\mathrm{R}} + 1)[j_{\mathrm{L}}] = \sum_g n^\beta_g I^g_{\mathrm{L}}. \tag{3.6}$$

[3]Below we drop the index 3 on $j^3_{\mathrm{L/R}}$.

There is an amusing fact about the expansion $I^n = \sum_j c_j^{2n}[j/2]$: the coefficients $c_i^n \in \mathbb{N}$ are the distributions of random walk in the half plane with reflective boundary conditions after n steps, i.e., $c_i^0 = \delta_{i,0}$, $c_i^k = 0$ for all k and $i < 0$

$$c_i^k = \begin{cases} c_i^{k-1} + c_{i+1}^{k-1}, & k \text{ even}, \\ c_{i-1}^{k-1} + c_i^{k-1}, & k \text{ odd}. \end{cases}$$

Since $c_n^{2n} = 1$ the $[j/2]$ basis can be expressed in terms of the I^r with integer coefficient.

Using (3.4) in (3.3),

$$\mathrm{Tr}_{I_{\mathrm{L}}^n}(-1)^{\sigma_{\mathrm{L}}} \mathrm{e}^{-2\pi \mathrm{i}\sigma_{\mathrm{L}} s} = (2\sin(s/2))^{2n}, \tag{3.7}$$

the formula for mass of the $D2/D0$ brane system $m^2 = t + 2\pi \mathrm{i} n$ as well as sum over the $D0$ brane momenta n on the M-theory S^1 yields the formal expression [15]

$$\mathcal{F}^{\mathrm{hol}}(\lambda = \epsilon_-, t) = \sum_{g=0}^{\infty} \sum_{\beta \in H_2(M,\mathbb{Z})} \sum_{m=1}^{\infty} n_g^{\beta} \frac{1}{m} \left(2\sin\frac{m\lambda}{2}\right)^{2g-2} \mathrm{e}^{m(\beta,t)}. \tag{3.8}$$

Similarly if one calculates along the same lines the $R_-^2 T_-^{2m-2} F_+^{2n-2}$ amplitude one obtains

$$\mathcal{F}^{\mathrm{hol}}(\epsilon_{\mathrm{R/L}}, t) = \sum_{\substack{j_{\mathrm{L}}, j_{\mathrm{R}}=0 \\ m=1}}^{\infty} \sum_{\beta \in H_2(M,\mathbb{Z})} \frac{n^{\beta}_{j_{\mathrm{L}}, j_{\mathrm{R}}}}{m} \times \frac{(-1)^{2j_{\mathrm{L}}+2j_{\mathrm{R}}} \left(\sum_{n=-j_{\mathrm{L}}}^{j_{\mathrm{L}}} y_{\mathrm{L}}^{mn}\right)\left(\sum_{n=-j_{\mathrm{R}}}^{j_{\mathrm{R}}} y_{\mathrm{R}}^{mn}\right) \mathrm{e}^{m(\beta,t)}}{4\left(\sin^2\left(\frac{m\epsilon_{\mathrm{L}}}{2}\right) - \sin^2\left(\frac{m\epsilon_{\mathrm{R}}}{2}\right)\right)} \tag{3.9}$$

with $y_{L/R} = \mathrm{e}^{\mathrm{i}\epsilon_{L/R}}$. It is convenient to rewrite (3.9) in terms of the $I_{\mathrm{L}}^{g_{\mathrm{L}}}$ and $I_{\mathrm{R}}^{g_{\mathrm{R}}}$ basis using $\sum n^{\beta}{}_{j_{\mathrm{L}}, j_{\mathrm{R}}}[J_{\mathrm{L}}, J_{\mathrm{R}}] = \sum \tilde{n}^{\beta}_{g_{\mathrm{L}}, g_{\mathrm{R}}} I_{\mathrm{L}}^{g_{\mathrm{L}}} \otimes I_{\mathrm{R}}^{g_{\mathrm{R}}}$ and (3.7) for the left and the right spins

$$\begin{aligned} \mathcal{F}^{\mathrm{hol}}&(\epsilon_{1/2}, t) \\ &= \sum_{\substack{g_{\mathrm{L}}, g_{\mathrm{R}}=0 \\ m=1}}^{\infty} \sum_{\beta \in H_2(M,\mathbb{Z})} \frac{\tilde{n}^{\beta}_{g_{\mathrm{L}}, g_{\mathrm{R}}}}{m} \frac{\sin\left(\frac{m(\epsilon_1-\epsilon_2)}{4}\right)^{2g_{\mathrm{L}}} \sin\left(\frac{m(\epsilon_1+\epsilon_2)}{4}\right)^{2g_{\mathrm{R}}} \mathrm{e}^{m(\beta,t)}}{4\left(\sin\left(\frac{m\epsilon_1}{2}\right)\left(\sin\left(\frac{m\epsilon_2}{2}\right)\right)\right.} \\ &= \sum_{\substack{g_1, g_1=0 \\ g_1+g_1=0 \bmod 2}}^{\infty} \epsilon_1^{g_1-1} \epsilon_2^{g_2-1} \tilde{F}_{g_1, g_2}(t). \end{aligned} \tag{3.10}$$

Here, the $\tilde F_{g_1,g_2}$ are easily extracted since at every power of $\epsilon_{1/2}$ they involve only finitely many $\tilde n^\beta_{g_1,g_2}$. We list the first few

$$
\begin{aligned}
\tilde F_{0,0} &= \sum_\beta \tilde n^\beta_{0,0}\mathrm{Li}_3\left(\mathrm{e}^{(t,\beta)}\right),\\
\tilde F_{0,2} &= \sum_\beta \left(\frac{\tilde n^\beta_{0,1}}{24}+\frac{1}{4}(\tilde n^\beta_{01}+\tilde n^\beta_{1,0})\right)\mathrm{Li}_1\left(\mathrm{e}^{(t,\beta)}\right),\\
\tilde F_{1,3} &= \sum_\beta \left(\frac{1}{4}(\tilde n^\beta_{02}-\tilde n^\beta_{2,0})\right)\mathrm{Li}_{-1}\left(\mathrm{e}^{(t,\beta)}\right),\\
\tilde F_{2,2} &= \sum_\beta \left(\frac{\tilde n^\beta_{0,1}}{576}-\frac{1}{96}(\tilde n^\beta_{02}-\tilde n^\beta_{2,0})+\frac{3}{8}(\tilde n^\beta_{02}-\tilde n^\beta_{2,0})+\frac{1}{8}\tilde n^\beta_{1,1}\right)\\
&\quad\times \mathrm{Li}_{-1}\left(\mathrm{e}^{(t,\beta)}\right),\\
&\quad \text{etc}
\end{aligned}
\tag{3.11}
$$

and note that generally the Polylogarithm $\mathrm{Li}_{3-g_1-g_2}(x) := \sum_{k=1}^\infty \frac{x^k}{(3-g_1-g_2)^k}$ describes the multi covering of the curve in the class β contributing to $\tilde F_{g_1,g_2}(t)$.

Moreover note, that if (3.3) is the correct starting point for the general generalized topological string instanton expansion then there will be no odd powers in ϵ_1, ϵ_2. By comparison with expansion of the type (2.2) we see that there are no contributions from the instantons to $F^{(n/2,m)}(t)$ for n odd. Since $F^{(1/2,0)}(t)$ is related to the Griffith infinitesimal invariant in the holomorphic anomaly of [11], it seems that for the topological string the version of the generalized holomorphic anomaly equation (2.6) is generally applicable for the topological string.

By geometrical engineering the $SU(2)$ SYM theory with $N_\mathrm{f}=1$, for which a non-trivial modification of (2.6) seems necessary [11], is related to the field theory limit of topological string on the blow up of $\mathbb{F}_1$ [20]. One would expect to see the non-trivial contribution by a subtle effect in this field theory limit.

3.2 The gap condition at the conifold point

Here, we provide a general derivation of the singular terms in the dual expansion near the generic conifold, such as (2.11, A.2). We will be able to explain the gap condition as well as computing the leading coefficients.

Our argument is a generalization of that of [16], and has been also presented recently in [11]. Basically, the singular terms in dual expansion come from integrating out nearly massless particles near the conifold point. Generically the massless BPS state has the charge $\vec{Q} = (1, 0, 0, 0)$ and has identified as a massless extremal black hole [17]. In Type II string theory, on a Calabi–Yau space it comes from a D3-brane wrapping the S^3, which shrinks at the conifold and its mass squared is $t_c = \int_{S^3} \Omega/t_0$. Here Ω is the holomorphic $(3, 0)$ form and t_0 is a period which starts with the constant one at the conifold. In the non-compact limit leading to the Seiberg–Witten gauge theory the local reduction of that period becomes $a_D = \int_{S^1} \lambda$, where λ is the meromorphic Seiberg–Witten differential. In the gauge theory, the vanishing mass squared is that of a magnetic monopole. Following the arguments of Gopakumar and Vafa [15] and integrating out the nearly massless particle generates the singular terms in the dual expansion of

$$F(\epsilon_1, \epsilon_2, a_D) = -\int_0^\infty \frac{ds}{s} \frac{\exp(-s a_D)}{4\sin(s\epsilon_1/2)\sin(s\epsilon_2/2)} + \mathcal{O}(a_D^0). \tag{3.12}$$

Since the calculation is local we present it only for the gauge theory case. For the string case, a_D is simply to be replaced with the flat coordinate t_c.

It is straightforward to expand the integrand in small ϵ_1, ϵ_2 and perform the integral. We compute the first few orders[4]

$$\begin{aligned}
&F(\epsilon_1, \epsilon_2, a_D) \\
&\quad = \left[-\frac{1}{12} + \frac{1}{24}(\epsilon_1+\epsilon_2)^2(\epsilon_1\epsilon_2)^{-1}\right]\log(a_D) \\
&\qquad + \left[-\frac{1}{240}(\epsilon_1\epsilon_2) + \frac{7}{1440}(\epsilon_1+\epsilon_2)^2 - \frac{7}{5760}(\epsilon_1+\epsilon_2)^4(\epsilon_1\epsilon_2)^{-1}\right]\frac{1}{a_D^2} \\
&\qquad + \left[\frac{1}{1008}(\epsilon_1\epsilon_2)^2 - \frac{41}{20160}(\epsilon_1+\epsilon_2)^2(\epsilon_1\epsilon_2) + \frac{31}{26880}(\epsilon_1+\epsilon_2)^4\right. \\
&\qquad \left. - \frac{31}{161280}(\epsilon_1+\epsilon_2)^6(\epsilon_1\epsilon_2)^{-1}\right]\frac{1}{a_D^4} + \mathcal{O}\left(\frac{1}{a_D^6}\right) + \mathcal{O}(a_D^0).
\end{aligned} \tag{3.13}$$

We see the gap structure in the dual expansion around the conifold point, and the leading coefficients exactly match those in (2.11, A.2). This universal behavior will enable us to fixed the holomorphic ambiguity in the refined topological string theory.

[4]The logarithmic term $\log(a_D)$ comes from the regularization near $s = 0$ of the integral $\int_0^\infty \frac{ds}{s}\, e^{-s a_D} = -\log(a_D) + \mathcal{O}(a_D^0)$.

3.3 *B*-model for the resolved conifold

The resolved conifold can be represented as line bundle over a sphere $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. This is one of simplest local Calabi–Yau models where the Gopakumar–Vafa correspondence between topological strings and Chern–Simons gauge theory was first discovered [18]. The topological A-model on resolved conifold is particularly simple and the Chern–Simons gauge theory become a matrix model in the small Kähler parameter limit. From the B-model perspective, a Picard–Fuchs differential equation for the model was provided in [19], where the complex structure parameter of the mirror curve is simply related to the exponential of the Kähler parameter T in the A-model by $Q = \mathrm{e}^{-T}$. One can see the Christoffel symbol of the moduli space metric and the propagator defined in [8] are rational functions of Q. Therefore in this case, we do not need to integrate the holomorphic anomaly equation in B-model because the higher genus amplitudes are simply rational functions Q. We can determine this rational function by the gap condition near the small Kähler parameter limit $T \sim 0$.

It turns out these ideas are also valid in the refined case. In this case, the geometry supports only the rigid $\mathbb{P}^1$ as smooth curve. As it is rigid $[j_{\mathrm{R}}] = [0]$ and as it is genus zero $[j_{\mathrm{L}}] = [0]$. Hence $n^{\mathbb{P}^1}_{0,0} = 1$ and all other $n^{\mathbb{P}^1}_{j_{\mathrm{L}},j_{\mathrm{R}}}$ vanish. The specialization of (3.9) yields

$$F = -\sum_{n=1}^{\infty} \frac{Q^n}{n(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}. \tag{3.14}$$

Here $Q = \mathrm{e}^{-T}$ and $q = \mathrm{e}^{\epsilon_1}, t = \mathrm{e}^{-\epsilon_2}$. We can easily extract the refined topological string amplitudes as

$$\log(Z) = \sum_{i,j=0}^{\infty} (\epsilon_1 + \epsilon_2)^i (\epsilon_1 \epsilon_2)^{j-1} F^{(\frac{i}{2},j)}(Q). \tag{3.15}$$

One can find

$$F^{(g_1,g_2)} \sim \sum_{n>0} n^{2g_1+2g_2-3} Q^n = \mathrm{Li}_{3-2g_1-2g_2}(Q). \tag{3.16}$$

Here for convenience we only consider the instanton part of the amplitudes, and the classical contribution and constant map contributions at $g > 2$ can be easily accounted for. It is possible to sum the infinite series and we found at genus one

$$F^{(0,1)} = -\frac{1}{12}\log(1-Q), \quad F^{(1,0)} = \frac{1}{24}\log(1-Q). \tag{3.17}$$

From the B-model perspective, we can compute the higher genus $F^{(g_1,g_2)}$ by requiring it to be a rational function of the form

$$F^{(g_1,g_2)}(Q) = \frac{\sum_{n=1}^{2g_1+2g_2-3} c_n Q^n}{(1-Q)^{2g_1+2g_2-2}}, \tag{3.18}$$

where we have used the boundary condition $F^{(g_1,g_2)}(Q) \sim \frac{1}{T^{2g_1+2g_2-2}}$ when $T \sim 0$ and $Q = \mathrm{e}^{-T} \sim 1$, and $F^{(g_1,g_2)}(Q)$ vanishes in both limits $Q \sim 0$ and $Q \sim +\infty$. Furthermore, the following gap condition can completely fix the polynomial in the numerator of $F^{(g_1,g_2)}(Q)$:

$$F^{(g_1,g_2)}(Q) \sim \frac{1}{T^{2g_1+2g_2-2}} + \mathcal{O}(T^0), \tag{3.19}$$

where the leading coefficients can be found either from the expansion in (3.14) or the analysis from integrating massless charged particles in Section 3.2. We find for example

$$\begin{aligned} F^{(0,2)} &= -\frac{Q}{240(1-Q)^2}, \quad F^{(1,1)} = \frac{7Q}{1440(1-Q)^2}, \\ F^{(2,0)} &= -\frac{7Q}{5760(1-Q)^2}, \quad F^{(0,3)} = \frac{Q(1+4Q+Q^2)}{6048(1-Q)^4}, \\ F^{(1,2)} &= -\frac{41Q(1+4Q+Q^2)}{120960(1-Q)^4}, \quad F^{(2,1)} = \frac{31Q(1+4Q+Q^2)}{161280(1-Q)^4}, \\ F^{(3,0)} &= -\frac{31Q(1+4Q+Q^2)}{967680(1-Q)^4}. \end{aligned} \tag{3.20}$$

In this case, the gap condition is understood as coming from the matrix model description at the small Kähler parameter limit. One can also see from (3.16) that the higher genus amplitudes can be directly obtained from genus one amplitude by operating with the operator Θ^{2g-2}, where $\Theta = Q\frac{d}{dQ} = -\frac{d}{dT}$. The operator Θ^{2g-2} transform the logarithmic singularity $\log(T)$ at genus one to $\frac{1}{T^{2g-2}}$ at genus g, and also determine the rational function form of the higher genus amplitudes.

3.4 The index and complex structure deformations

Let us point out how complex structure specialization can lead to different models for $\mathcal{M}_\beta$ for which the individual $n^\beta_{j_L,j_R}$ change, but not the index. Particular simple examples occur for rational curves embedded with degree one [20, 21]: Calabi–Yau hypersurfaces M in weighted projective spaces with a Z_2 singularity over a smooth genus g curve $\mathcal{C}_g$, e.g., the octic in $\mathbb{P}(1,1,2,2,2)$ where $\mathcal{C}_3$ is the degree 4 hypersurface depending on the last three coordinates, contain after resolving the $\mathbb{Z}_2$ singularity by an $\mathbb{P}^1$ a rational fibration over $\mathcal{C}_g$. We want to discuss the moduli space and the associated BPS numbers of the smooth rational curve in the fibration. It represents the basis $[B]$ of the hypersurface M viewed as an $K3$ fibration. Because of the rational fibration the moduli space of the fiber $\mathbb{P}^1$ is $\mathcal{M}_{[B]} = \mathcal{C}_g$, with Dolbeault homology dimensions $\begin{matrix} & 1 & \\ g & & g \\ & 1 & \end{matrix}$. The $\mathbb{P}^1$ represents the highest (and lowest) left spin for a curve in the class $[B]$. It is $[0]_\mathrm{L}$ in this case. Since the right Lefshetz action $SU(2)_\mathrm{R}$ for the highest left spin is just the usual Lefshetz action on the deformation space $\mathcal{C}_g$ [15, 20], one can read off immediately the right representation as

$$\left[\frac{1}{2}\right]_\mathrm{R} + 2g[0]_\mathrm{R}. \tag{3.21}$$

One can then show that $\mathcal{C}_g$ exists only for the toric embedding of the hypersurface, which freezes $g-1$ complex structure moduli to fixed values. If one considers general complex structure deformations, the so called non-toric deformations, a superpotential of degree $2g-2$ develops [23], which restricts the $\mathbb{P}^1$ to sit at $2g-2$ points hence the right spin content is now the one for $\mathcal{M}_\beta = (2g-2)$ points, i.e.,

$$(2g-2)[0]_\mathrm{R}. \tag{3.22}$$

The weighted sum yields the Euler number of the deformation space with a sign, i.e., $n^\beta_{j^{\max}_\mathrm{L}}$ equals $n^\beta_{g^{\max}} = (-1)^{\dim(\mathcal{M}_\beta)} e(\mathcal{M}_\beta)$ and yields in the cases discussed above for which $g^{\max} = 0$ invariantly $n^{[B]}_0 = 2g-2$. Related considerations for rational curves on the quintic [21] show generally that at complex codimension one loci in the complex moduli space the moduli space of the rational curves embedded with degree one can jump from isolated points to higher genus curves in a way, which preserves the index. Generally, the virtual dimension of the moduli of holomorphic curves is zero; however, even for the most general complex structure deformation the actual dimension of the moduli space can be positive.

The calculation of $n^{\beta}_{J_{\mathrm{L}},J_{\mathrm{R}}}$ is a well defined but difficult problem on compact Calabi–Yau spaces, which sheds light, e.g., on the deformation space of holomorphic curves. However, for compact Calabi–Yau spaces there can be in general no generating function depending just on the Kähler moduli. To avoid this problem one can try fix the complex moduli in a canonical way. The most obvious possibility is to consider local Calabi–Yau, which have no complex structure moduli. Another canonical choice can arise for decompactification limit of regular $K3$ fibrations.

3.5 Refined BPS state counting on $K3$ fibrations

Here we discuss a refined Göttsche formula, which incorporates the left and the right spin degeneracies and relate them to an one loop amplitude in topological string theory.

3.5.1 The refined Göttsche formula

Geometrical Lefshetz decompositions yielding only the index have been defined in models for $\mathcal{M}_\beta$ and checked in [20] using the Abel–Jacobi map for a variety of geometric settings. Specially simple situation arise for curves in surfaces in a CY threefold. The easiest example is $K3 \times T_2$ where one specializes to classes in the $K3$. Strictly speaking this case is degenerate, because as far as Gromov–Witten — and Gopakumar–Vafa invariants are concerned, there is a multiplicative zero coming from the T_2. However the description of the moduli space of BPS states below finds application for CY, which are regular $K3$ fibrations. And in this case a relation to Gromov–Witten invariants exists. The moduli space for the BPS states of $(D2, D0)$ brane system with charge (β, g) is the canonical resolution $S^{[g]}$ of the Hilbert scheme of g points, i.e., $S^{\otimes g}$ divided by the permutation group Sym^g. The dependence of the BPS invariant on the class β is only via $\beta\cdot\beta = 2g-2$ and the g points correspond to the nodes of the general genus g curve and can be interpreted as positions of the D0 branes.

For general $S^{[g]}$ Göttsche derived a generating function $P(X,z) = \sum_i b_i(X)z^i$ capturing the Betti numbers of all $S^{[g]}$

$$\sum_{g=0}^{\infty} P(S^{[g]},z)q^g = \prod_{m=1}^{\infty} \frac{(1+z^{2m-1}q^m)^{b_1(S)}(1+z^{2m+1}q^m)^{b_1(S)}}{(1-z^{2m-2}q^m)^{b_0(S)}(1-z^{2m}q^m)^{b_2(S)}(1-z^{2m+2}q^m)^{b_0(S)}}. \tag{3.23}$$

This can be interpreted as partition function $b_1(S)$ chiral fermions and $b_0(S)+b_2(S)$ chiral bosons, whose oscillators are in addition distinguished by the ordinary $SU(2)$ Lefshetz charge j_3. For $z=-1$ that specializes to

$$\sum_{g=0}^{\infty} e(S^{[g]})t^g = \prod_{m=1}^{\infty}(1-q^m)^{-e(S)} = \frac{q^{\frac{e(S)}{24}}}{\eta(q)^{e(S)}} \tag{3.24}$$

and for $K3$, where there is no odd cohomology and $\chi(K3)=24$, the formula can be explained within heterotic type II/duality in six dimensions, as counting literally the energy degeneracy of the 24 left (l) moving bosonic oscillators α_{-k} in the index $\mathrm{Tr}(-1)^{\bar F_r} q^{L_0-\frac{c}{24}} q^{\bar L_0-\frac{\bar c}{24}}$ of the heterotic string theory [24].

Let us assume that $b_1(S)=0$ and $b_0(S)=1$ which is true for the relevant cases. Then the picture can be refined to implement the left and right $SU(2)_{\rm L}\times SU(2)_{\rm R}$ quantum numbers on surfaces S by assigning to all bosonic oscillators α_{-k} instead of the representation $(b_2(S)+1)[0]+[1]$ in the diagonal $SU(2)_{L+R}$, which lead to (3.23), the $SU(2)_{\rm L}\times SU(2)_{\rm R}$ representation [25, 26]

$$\alpha_{-k}: \quad b_2(S)[0,0]+\left[\frac{1}{2},\frac{1}{2}\right]. \tag{3.25}$$

Let us define

$$G^S(q,z_{\rm L},z_{\rm R}) := \sum_{g=0}^{\infty} P(S^{[g]},z_{\rm L},z_{\rm R})q^g \tag{3.26}$$

with $P(X,z_{\rm L},z_{\rm R})=\sum_{J_{\rm L},J_{\rm R}} b_{J_{\rm L},J_{\rm R}}(X) z_{\rm L}^{J_{\rm L}} z_{\rm R}^{J_{\rm R}}$. The generalization of (3.23) to the representation (3.25) for surfaces with $b_1(S)=0$ reads [25]

$$\begin{aligned}&\sum_{g=0}^{\infty} P(S^{[g]},z_{\rm L},z_{\rm R})q^g \\ &\quad = \prod_{m=1}^{\infty}\frac{1}{(1-(z_{\rm L}z_{\rm R})^{m-1}q^m)(1-(z_{\rm L}z_{\rm R})^{m+1}q^m)(1-(z_{\rm L}z_{\rm R})^{m}q^m)^{b_2(S)-2}} \\ &\qquad \times \frac{1}{(1-z_{\rm R}^2(z_{\rm L}z_{\rm R})^{m-1}q^m)(1-z_{\rm L}^2(z_{\rm L}z_{\rm R})^{m-1}q^m)}.\end{aligned} \tag{3.27}$$

From the description of the Lefshetz decomposition of the cohomology of the moduli space (3.27) one can get the genus expansion of the topological string in terms of the n_g^β for $K3$ can using (3.5) and (3.7) [25, 26]

$$\begin{aligned} G^{K3}(q/y, y, 1) &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2} \\ &= \sum_{g=0,\beta}^{\infty} (-1)^g n_g^\beta (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2g} q^\beta. \end{aligned} \tag{3.28}$$

While (3.28) counts quantities, which are invariant under complex structure deformations, (3.27) contains more complete information of the $H_{(\mathcal{M}_\beta)}$ cohomology for a fixed complex structure. The $n^\beta_{j_L, j_R}$ encode the information of the deformed Ω-background $\epsilon_1 \neq -\epsilon_2$, i.e., $\epsilon_+ \neq 0$.

3.5.2 Heterotic/type II duality

Next we compare the result (3.27) with the modified heterotic string one loop contribution suggested by [13, 14]. Let us point out the difference of the latter to the heterotic one-loop integral, which leads to the successful evaluation of BPS invariants in K3-fibered Calabi–Yau spaces [27–30]. Here the integral is over the fundamental region of the WS-torus parameterized by $\tau = \tau_1 + i\tau_2$

$$F_g(t) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \tau^{2g-2} \sum_J \mathcal{I}_J^g, \tag{3.29}$$

where

$$\mathcal{I}_J^g = \frac{\hat{\mathcal{P}}_g}{Y^{g-1}} \bar{\Theta}_J^g(q) f_J(q). \tag{3.30}$$

The sum over J labels orbifold sectors for which $\bar{\Theta}_J^g(q)$ is orbifold projection of the Siegel–Narain Θ-function with p_r^{2g-2} insertions and f_J capture the oscillators contributions in the orbifold sectors. This sum will combine to a modular form of appropriate weight, see (3.39). The amplitude depends only on the vector moduli via their occurrence in $\Theta_J^g(q)$ and $Y = \mathrm{e}^{-K}$, where K is the Kählerpotential of the vector moduli metric. It is possible to write down all terms using[5]

$$\mathrm{e}^{-\frac{\pi\lambda^2}{\tau_2}} \left(\frac{2\pi\eta^3\lambda}{\theta_1(\lambda|\tau)} \right)^2 = \sum_{g=0} (2\pi\lambda)^{2g} \hat{\mathcal{P}}_g = -\exp\left(2\sum_{k=1}^{\infty} \frac{\xi(2k)}{k} \hat{E}_{2k}(\tau) \lambda^{2k} \right). \tag{3.31}$$

[5]One could also use (3.31) to write the total amplitude $F(\lambda, t) = \sum_{\lambda=0^\infty} \lambda^{2g-2} F_g(t)$ directly as integral, but the notation get more clumsy.

Here $\hat{E}_{2k}$ are the holomorphic Eisenstein series E_{2k} for $k > 1$ and $\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2}$ the almost holomorphic second Eisenstein series, i.e., all $\hat{E}_{2k}$ transform as modular forms of weight $2k$.

In [13, 14] it was suggested to couple the BPS states in the Schwinger-loop amplitude to an additional self-dual matter vector field strength F_+, i.e., they consider the one loop amplitude $R_-^2 T_-^{2g-2} F_+^{2n-2}$. The effect is merely to split (3.31) as

$$\begin{aligned} &\mathrm{e}^{-\frac{\pi(\epsilon_-^2+\epsilon_+^2)}{\tau_2}} \left(\frac{2\pi(\epsilon_- + \epsilon_+)\eta^3}{\theta_1((\epsilon_- + \epsilon_+)|\tau)}\right)\left(\frac{2\pi(\epsilon_- - \epsilon_+)\eta^3}{\theta_1((\epsilon_- - \epsilon_+)|\tau)}\right) \\ &\quad = \sum_{m,n} (2\pi(\epsilon_- + \epsilon_+))^{2m} (2\pi(\epsilon_- - \epsilon_+))^{2n} \hat{\mathcal{P}}_{m,n}, \end{aligned} \tag{3.32}$$

where

$$\hat{\mathcal{P}}_{m,n} = \mathcal{S}_m(x_1, \ldots, x_m)\mathcal{S}_n(x_1, \ldots, x_n) \tag{3.33}$$

are almost modular forms of weight $2m + 2n$. Concretely, $x_k = \frac{|B_{2k}|}{2k(2k)!}\hat{E}_{2k}$ and $\mathcal{S}_m(\underline{x})$ is defined by $\exp(\sum_{n=1} x_i z^i) = \sum_{n=0}^{\infty} \mathcal{S}_m(\underline{x}) z^m$. This allows us to define

$$F_{m,n}(t) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \tau^{2(m+n)-2} \sum_J \mathcal{I}_J^{m,n} \tag{3.34}$$

with

$$\mathcal{I}_J^{m,n} = \frac{\hat{\mathcal{P}}_{m,n}}{Y^{m+n-1}} \bar{\Theta}_J^{m+n}(q) f_J(q). \tag{3.35}$$

Now we can point out the difference in the calculation of the heterotic one-loop amplitude (3.29) and (3.34). First for $\epsilon_+ = 0$ (3.34) specializes to (3.29) and this has been calculated for many examples of heterotic/Type II pairs starting with the *STU* model in [28] in more general situations [29–31]. If the Calabi–Yau space is a regular $K3$-fibration the result for the one-loop amplitude in the holomorphic limit is expressed using the expansion [29]

$$\begin{aligned} \mathcal{G}_{K3}^{\mathrm{hol}}(\lambda, t) &= \frac{M(q)}{q} \left(\frac{\lambda}{2\sin(\frac{\lambda}{2})}\right)^2 G^{K3}(q/y, y, 1) \\ &= \sum_{g=0,d=-1}^{\infty} c_g(d) \lambda^{2g-2} q^d, \end{aligned} \tag{3.36}$$

where $y = \mathrm{e}^{\mathrm{i}\lambda}$ and the Kähler of $K3$ enters via $q = \mathrm{e}^{2\pi \mathrm{i} t}$, by

$$\mathcal{F}^{\mathrm{hol}}_{K3}(\lambda, t) = \sum_{g=0}^{\infty} \sum_{\alpha \in H_2^{prim}(K3,\mathbb{Z})} \lambda^{2g-2} c_g(\alpha^2/(2r)) \frac{Li_{3-2g}(\mathrm{e}^{(\alpha,t)})}{(2\pi i)^{3-2g}} \,. \tag{3.37}$$

The function $\frac{M(q)}{q}$ has been determined in many cases. r depends on the Picard–Lattice of the generic $K3$ fiber. First one notes that there will be always a factor $\frac{1}{\eta(q)^{24}} = \frac{1}{q\prod_{n=1}^{\infty}(1-q^n)^{24}}$ in $\mathcal{G}^{\mathrm{hol}}_{K3}(\lambda, t)$, which comes from the left moving bosonic oscillator modes of the heterotic string.[6] It is therefore convenient to define

$$\frac{M(q)}{q} = \frac{\Theta(q)}{q\prod_{n=1}^{\infty}(1-q^n)^4}. \tag{3.38}$$

$\Theta(q)$ is a form under, in general, a subgroup $SL(2,\mathbb{Z})$ of weight $11 - \frac{r}{2}$ where r is the rank of the Picard Lattice of the $K3$. E.g., for the ST ($r = 2$) and STU ($r = 1$) model one has

$$\Theta^{ST}(q) = \theta_3(\tau/2) E_4 F_6, \quad \Theta^{\mathrm{STU}}(q) = E_4 E_6 \tag{3.39}$$

and $F_6 = E_6 - 2F_2(\theta_3^4(\tau/2) - 2F_2)(\theta_3^4(\tau/2) - 16F_2)$, where $F_2(q) = \sum_{n\in\mathbb{Z}_+,odd} \sigma_1(n) q^{\frac{n^2}{4}}$. Much more general examples have been discussed in [29, 32].

It is worthwhile to stress that the $c_g(\alpha^2/2) \in \mathbb{Z}$ are not the BPS invariants n^α_g. To get the latter, we have to compare (3.37) with (3.8) for classes $\beta \in H_2(K3,\mathbb{Z})$.

Going over the calculation leading to (3.29) one recognized the difference for the evaluation of (3.34) does not affect the $\frac{\Theta(q)}{q}$ part, which is clear as it is completely determined by the genus zero contributions, which are deformation invariant. Incorporating the ϵ_+ deformation we can use as before the Jacobi triple function identity.

$$\theta_1(z,\tau) = -2q^{\frac{1}{8}} \sin(\pi z) \prod_{m=1} (1 - q^m)(1 - 2\cos(2\pi z) q^m + q^{2m}) \tag{3.40}$$

[6]In fact applied to the six-dimensional heterotic on T^4 versus type IIA on $K3$ duality it reproduces the famous (3.24) as counting function of nodal curves on $K3$ as observed by Yau and Zaslow.

to obtain

$$\begin{aligned}
\mathcal{G}^{\rm hol}_{K3}(\epsilon_{\rm R/L},t) &= \frac{M(q)}{q}\frac{1}{4\left(\sin^2\left(\frac{\epsilon_{\rm L}}{2}\right)-\sin^2\left(\frac{\epsilon_{\rm R}}{2}\right)\right)}\\
&\quad\times\prod_{n>0}\frac{1}{(1-y_{\rm L}y_{\rm R}q^n)(1-y_{\rm L}^{-1}y_{\rm R}^{-1}q^n)}\\
&\quad\times\frac{1}{(1-y_{\rm L}y_{\rm R}^{-1}q^n)(1-y_{\rm L}^{-1}y_{\rm R}q^n)(1-q^n)^{20}} \qquad (3.41)\\
&= \frac{M(q)}{q}\frac{1}{4\left(\sin^2\left(\frac{\epsilon_{\rm L}}{2}\right)-\sin^2\left(\frac{\epsilon_{\rm R}}{2}\right)\right)}G^{K3}(q/(y_{\rm L}y_{\rm R}),y_{\rm L},y_{\rm R})\\
&=: \sum_{n,m=0,d=-1}^{\infty} c_{n,m}(d)\epsilon_1^{m-1}\epsilon_1^{n-1}q^d.
\end{aligned}$$

Here we rescaled $\epsilon_{\rm R/L}$ by $2\pi i$ and set $y_{\rm R/L}={\rm e}^{{\rm i}\epsilon_{\rm R/L}}$. The formula gives the desired interpretation of the BPS contributions to the heterotic one-loop integral [13, 14] in terms of the Lefshetz decomposition the moduli spaces of curves on the $K3$ fiber. The free energy is then given by

$$\mathcal{F}^{\rm hol}_{K3}(\epsilon_{L/R},t) = \sum_{m,n=0}^{\infty}\sum_{\alpha\in H_2^{prim}(K3,\mathbb{Z})}\epsilon_1^{m-1}\epsilon_2^{n-1}c_{m,n}(\alpha^2/(2r))\frac{Li_{3-m-n}({\rm e}^{(\alpha,t)})}{(2\pi i)^{3-m-n}}. \qquad (3.42)$$

To read of the $n^\alpha_{j_{\rm L},j_{\rm R}}$ one compares (3.10) for classes in the $K3$ fiber with (3.42) and then re-express the result in terms of the $(J_{\rm L},J_{\rm R})$ basis. Let us label the classes in the $K3$ fibre of the STU model by $(1,n)$, we get then

$$\begin{aligned}
n=1:&\quad 488\,(\mathbf{0},\mathbf{0})-2\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)\\
n=2:&\quad 280962\,(\mathbf{0},\mathbf{0})+486\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)-2\,(\mathbf{1},\mathbf{1}),\\
n=3:&\quad 15298438\,(\mathbf{0},\mathbf{0})+281448\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)+486\,(\mathbf{1},\mathbf{1})-2\left(\frac{\mathbf{3}}{\mathbf{2}},\frac{\mathbf{3}}{\mathbf{2}}\right)\\
&\qquad -2((\mathbf{1},\mathbf{0})+(\mathbf{0},\mathbf{1})),\\
n=4:&\quad 410133612\,(\mathbf{0},\mathbf{0})+16209886\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)+281446\,(\mathbf{1},\mathbf{1})
\end{aligned}$$

$$+ 486 \left(\mathbf{\frac{3}{2}}, \mathbf{\frac{3}{2}}\right) - 2\,(\mathbf{2{,}2}) + 486((\mathbf{1{,}0}) + (\mathbf{0{,}1}))$$
$$- 2\left(\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{3}{2}}\right) + \left(\mathbf{\frac{1}{2}}, \mathbf{\frac{3}{2}}\right)\right). \tag{3.43}$$

From (3.29) and (3.31), one can work out the full an-holomorphic dependence of (3.37), as was done for the STU model in Appendix C.1 of [33], and check that it is compatible with the holomorphic anomaly equation of [8]. It is not hard to trace the anholomorphic dependence under the factorization of (3.31) into (3.32) and show that it leads to the sum structure in the second term of the right hand side in the generalized holomorphic anomaly equation (2.6). Results for other regular $K3$ fibrations are obtained similarly.

As it is clear from the explanations in the introduction to this chapter, it cannot be true in general that the description of right-handed BPS states does not depend on the complex structure, i.e., the hyper multiplets in the heterotic string. One possible interpretation is that in the strict weak coupling limit, which corresponds to infinite volume of the base $\mathbb{P}^1$ the description and the right Lefshetz decomposition of the moduli space of curves in the $K3$ fiber is invariant. This would explain, why the authors [13,14] do not find hyper multiplet dependence in their perturbative calculation. From the examples of the K3 fibered hypersurfaces in $\mathbb{P}^4(1, 1, w_1, w_2, w_3)$, discussed in the introduction one can conclude that the hyper multiplet must couple in the non-perturbative sector of the heterotic string. In Section 4, we will apply the above results to a situation where the decoupling of the complex moduli is obvious in the geometrical context.

4 Local Calabi–Yau manifolds

One obvious possibility to decouple the complex moduli is to look at local models. In particular, for all del Pezzo surfaces B embedded in a Calabi–Yau three manifold one can take a local limit in which the local non-compact Calabi–Yau is described by the total space of the canonical line bundle $\mathcal{O}(K_B) \to B$. We start with the del Pezzo's, which are elliptic fibrations over $\mathbb{P}^1$

4.1 A rational elliptic surface: half $K3$

In [34] simple expressions for the BPS number generating function in terms of $SL(2, \mathbb{Z})$ modular forms for the rational elliptic surface B_n embedded in a Calabi–Yau threefold M were found. In the simplest example, B_9 was embedded in an elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$ and two of the ten classes in $H_2(B_9, \mathbb{Z})$ were independent in M, namely the base P and the fiber F of the elliptically fibered B_9.

The expression found in [34] described genus zero Gromov–Witten invariants r_0^{P+nF}, which are wound one times around the base $P = \mathbb{P}^1$ and n times around the elliptic fiber

$$H_1^{(0)}(q) = \sum_{n=0}^{\infty} r_0^{P+nF} q^n = \frac{q^{\frac{1}{2}} E_4(q)}{\eta^{12}(q)} \tag{4.1}$$

and has given an explanation in terms of tensionless strings [34]; see also [35]. Note that $r_0^{P+nF} = n_0^{p+nF}$ by virtue of (3.8).

Higher genus curves of genus g and degree one in the base have a simple geometry. They are copies of the elliptic fiber over g points in the basis $P = \mathbb{P}^1$. The moduli space $\mathcal{M}_{P+nF}$ consist of the possible position of these fibers on the base $\mathbb{P}^1$ and the $U(1)$ connection on each of the g elliptic fibers. By T-duality on the fiber fixing a $U(1)$ connection equivalently corresponds to picking point on the dual T^2. Therefore one expects that the $SU(1)_{\mathrm{L}} \times SU(1)_{\mathrm{R}}$ decomposition of the cohomology of the moduli space of the $P + nF$ curves in this is described again by the generating function [25] $G^{B_9}(q/(y_{\mathrm{L}} y_{\mathrm{R}}), y_{\mathrm{L}}, y_{\mathrm{R}})$, with $\chi(B_9) = 12$ and $b_2(B_9) = 10$. Similar as in (3.36) it should be supplemented by the $SL(2, \mathbb{Z})$ modular part from genus zero. Therefore the result for the refined topological invariants is obtained from

$$\mathcal{G}_{hol}^{B_9} = \frac{q^{\frac{1}{2}} E_4(q)}{\eta^4(q)} \frac{1}{4\left(\sin^2\left(\frac{\epsilon_{\mathrm{L}}}{2}\right) - \sin^2\left(\frac{\epsilon_{\mathrm{R}}}{2}\right)\right)} G^{B_9}(q/(y_{\mathrm{L}} y_{\mathrm{R}}), y_{\mathrm{L}}, y_{\mathrm{R}}) \tag{4.2}$$

by the same steps that lead to (3.43)

$$\begin{aligned}
g = 1: &\quad 248\,(\mathbf{0}, \mathbf{0}) + \left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right), \\
g = 2: &\quad 4125\,(\mathbf{0}, \mathbf{0}) + 249\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right) + 1\,(\mathbf{1{,}1}), \\
n = 3: &\quad 35001\,(\mathbf{0}, \mathbf{0}) + 4374\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right) + 249\,(\mathbf{1{,}1}) + \left(\mathbf{\frac{3}{2}}, \mathbf{\frac{3}{2}}\right) \\
&\qquad + ((\mathbf{1{,}0}) + (\mathbf{0{,}1})),
\end{aligned}$$

$$n=4: \quad 217501\,(\mathbf{0},\mathbf{0})+39375\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)+4375\,(\mathbf{1,1})+249\left(\frac{\mathbf{3}}{\mathbf{2}},\frac{\mathbf{3}}{\mathbf{2}}\right)+(\mathbf{2,2})$$
$$+\,249((\mathbf{1,0})+(\mathbf{0,1}))+\left(\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{3}}{\mathbf{2}}\right)+\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{3}}{\mathbf{2}}\right)\right). \tag{4.3}$$

The new point is that these results can be also interpreted as refined gauge theory invariants for $N=4$ $SU(2)$ theory on the half K3 [36]. Since b_+ $(B_9)=1$, we expect the refined individual cohomology numbers of the moduli space of gauge theory instantons to be invariant [37]. Moreover, one expects by direct integration of a holomorphic anomaly in the base degree as in [25] to be able to extend this result to higher rank gauge groups and to further classes of the B_n surfaces [36].

4.2 The refined topological string on local $\mathbb{P}^1\times\mathbb{P}^1$

The local $\mathbb{P}^1\times\mathbb{P}^1$ is a well studied non-compact Calabi–Yau manifold. This Calabi-Yau geometrically engineers $SU(2)$ Seiberg-Witten theory, and the topological string partition function reduces to the Nekrasov partition function at $\epsilon_1+\epsilon_2=0$ in the field theory limit [20, 22]. One can also use the refined topological vertex to construct the *refined* topological string amplitude that reduces to Nekrasov partition function in general Ω background. This was studied in [10, 39]. It is found that the world-sheet instanton contribution to the refined topological partition function for this model is

$$Z(Q_1,Q_2,t,q)=\sum_{\nu_1,\nu_2} Q_1^{|\nu_1|+|\nu_2|} q^{||\nu_2^t||^2} t^{||\nu_1^t||^2} \tilde{Z}_{\nu_1}(t,q)\tilde{Z}_{\nu_2^t}(t,q)\tilde{Z}_{\nu_2}(q,t)\tilde{Z}_{\nu_1^t}(q,t)$$
$$\times\prod_{i,j=1}^{\infty}(1-Q_2t^{i-1-\nu_{2,j}}q^{j-\nu_{1,i}})^{-1}(1-Q_2q^{i-1-\nu_{1,j}}t^{j-\nu_{2,i}})^{-1}. \tag{4.4}$$

Some explanations of the notations are the followings. The $Q_1=\mathrm{e}^{2\pi i T_1}$, $Q_2=\mathrm{e}^{2\pi \mathrm{i} T_2}$ are exponentials of the Kähler parameters T_1,T_2 of the model. The string coupling constant is refined into two parameters ϵ_1,ϵ_2 and are related to q,t as $q=\mathrm{e}^{\epsilon_1}, t=\mathrm{e}^{-\epsilon_2}$. The sum in the above equation (4.4) is over all two-dimensional (2D) Young tableaux ν_1,ν_2. A 2D Young tableau is a sequence of non-increasing non-negative integers $\nu=\{\nu_{,1}\geq\nu_{,2}\geq\cdots\geq 0\}$, and ν^t denotes the transpose of the Young tableau ν. Some definitions related to the Young tableaux are the followings:

$$
\begin{aligned}
|\nu| &= \sum_{i=1}^{\infty} \nu_{,i}, \\
||\nu||^2 &= \sum_{i=1}^{\infty} (\nu_{,i})^2, \\
\tilde{Z}_\nu(t,q) &= \prod_{(i,j)\in\nu} (1 - t^{(\nu^t)_{,j}-i+1} q^{\nu_{,i}-j})^{-1}.
\end{aligned} \tag{4.5}
$$

We perform the sum over Young tableaux in (4.4) to a finite order in Q_1, Q_2. In order to perform the infinite product exactly, we use the formula $(1-x)^{-1} = \exp(\sum_{n\geq 0} \frac{x^n}{n})$ to convert the infinite product to infinite sums of geometric series of t and q. To compare with the B-model calculations from Picard–Fuchs equation and holomorphic anomaly, we should expand the logarithm of the partition function as in (2.2), where $F^{(\frac{i}{2},j)}(Q_1,Q_2)$ depends now on the flat Kähler coordinates $T_{1,2}$ of the $\mathbb{P}^1 \times \mathbb{P}^1$ geometry via $Q_{1,2} = \mathrm{e}^{2\pi i T_{1,2}}$. As explained at the end of Section 3.1, only $F^{(i,j)}$ with integers i, j should appear in the expansion. The first non-trivial contribution from the refinement of the topological string is $F^{(1,0)}$. We perform the computation on the refined partition function (4.4) to first few orders and the results for $(g_1 + g_2) = 1$ are

$$
\begin{aligned}
F^{(0,1)} &= -\frac{1}{6}(Q_1+Q_2) - \frac{1}{12}(Q_1^2 + 4Q_1Q_2 + Q_2^2) \\
&\quad - \frac{1}{18}(Q_1^3 + 9Q_1^2Q_2 + 9Q_1Q_2^2 + Q_2^3) \\
&\quad - \frac{1}{24}(Q_1^4 + 16Q_1^3Q_2 - 148Q_1^2Q_2^2 + 16Q_1Q_2^3 + Q_2^4) + \mathcal{O}(Q^5),
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
F^{(1,0)} &= -\frac{1}{6}(Q_1+Q_2) - \frac{1}{12}(Q_1^2 + 28Q_1Q_2 + Q_2^2) - \frac{1}{18}(Q_1^3 + 153Q_1^2Q_2 \\
&\quad + 153Q_1Q_2^2 + Q_2^3) - \frac{1}{24}(Q_1^4 + 496Q_1^3Q_2 + 2204Q_1^2Q_2^2 + 496Q_1Q_2^3 \\
&\quad + Q_2^4) + \mathcal{O}(Q^5).
\end{aligned} \tag{4.7}
$$

For $(g_1 + g_2) = 2$ we get

$$
\begin{aligned}
F^{(0,2)} &= -\frac{1}{120}\left(Q_1+Q_2\right) - \frac{1}{60}\left(Q_1^2 + Q_1Q_2 + Q_2^2\right) - \frac{1}{40}\left(Q_1^3 + Q_1^2Q_2\right. \\
&\quad \left. + Q_1Q_2^2 + Q_2^3\right) - \frac{1}{30}\left(Q_1^4 + Q_1^3Q_2 + 5Q_1^2Q_2^2 + Q_1Q_2^3 + Q_2^4\right) \\
&\quad + \mathcal{O}(Q^5),
\end{aligned}
$$

$$
\begin{aligned}
F^{(1,1)} &= -\frac{1}{90}\left(Q_1+Q_2\right)-\frac{1}{90}\left(2Q_1^2+17Q_1Q_2+2Q_2^2\right)\\
&\quad -\frac{1}{30}\left(Q_1^3+21Q_1^2Q_2+21Q_1Q_2^2+Q_2^3\right)\\
&\quad -\frac{1}{45}\left(2Q_1^4+77Q_1^3Q_2-995Q_1^2Q_2^2+77Q_1Q_2^3+2Q_2^4\right)+\mathcal{O}(Q^5),\\
F^{(2,0)} &= \frac{1}{360}\left(Q_1+Q_2\right)+\frac{1}{180}\left(Q_1^2-59Q_1Q_2+Q_2^2\right)\\
&\quad +\frac{1}{120}\left(Q_1^3-399Q_1^2Q_2-399Q_1Q_2^2+Q_2^3\right)\\
&\quad +\frac{1}{90}\left(Q_1^4-1379Q_1^3Q_2-7495Q_1^2Q_2^2-1379Q_1Q_2^3+Q_2^4\right)+\mathcal{O}(Q^5).
\end{aligned}
\tag{4.8}
$$

The $(g_1+g_2)=3$ results are relegated to Appendix A.2.

The $F^{(0,g)}$ is the ordinary topological string amplitude and has been well studied before from B-model using mirror symmetry; see e.g., [40]. Here we include it for the purpose of checking the calculations. The Kähler parameters $Q_1=\mathrm{e}^{2\pi\mathrm{i}\,T_1}, Q_2=\mathrm{e}^{2\pi\mathrm{i}\,T_2}$ are related to the complex structure parameters z_1, z_2 of the mirror manifold through Picard–Fuchs equations. The two Picard–Fuchs operators are

$$
\begin{aligned}
L_1 &= \Theta_1^2-2z_1(\Theta_1+\Theta_2)(1+2\Theta_1+\Theta_2),\\
L_2 &= \Theta_2^2-2z_2(\Theta_1+\Theta_2)(1+2\Theta_1+\Theta_2),
\end{aligned}
\tag{4.9}
$$

where $\Theta_i=z_i\frac{\partial}{\partial z_i}, i=1,2$. The discriminant is $z_1z_2\Delta=0$ where the conifold discriminant is given by

$$
\Delta = 1-8(z_1+z_2)+16(z_1-z_2)^2. \tag{4.10}
$$

In figure 1, we depict the moduli space in figure 1 following [40]. The conifold loci C is parameterized by $\Delta=0$, and intersect tangentially with the other singular loci $z_1=0$, $z_2=0$, and $\frac{1}{z_1+z_2}=0$. To study the model around the tangent intersection points, we need to blow up the points by adding the extra lines F, F_1, F_2 in the Figure 1, so that the intersections become normal. In this paper we will study the model around the large volume point $z_1=z_2=0$, and around a generic point on the conifold loci not intersecting with other singular point.

The Picard–Fuchs equations of local models have a constant solution and the Kähler moduli T_1, T_2 in the mirror are the two logarithmic solutions of

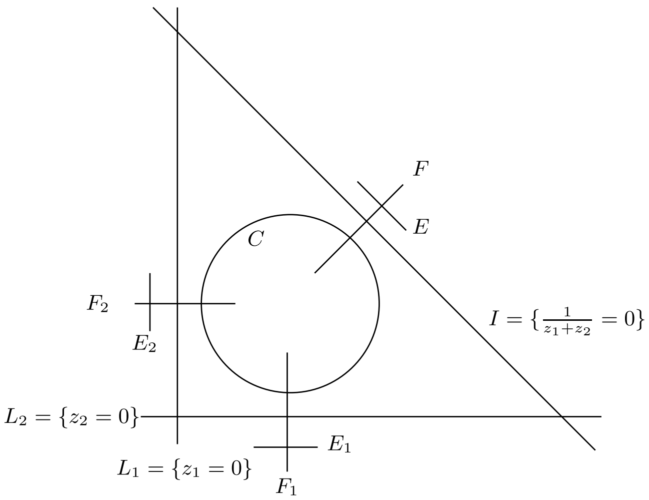

Figure 1: Resolved moduli space of $\mathbb{F}_0$.

the Picard–Fuchs equations around the large volume point $z_1 = z_2 = 0$,

$$\begin{aligned} 2\pi i T_1 &= \log(z_1) + 2(z_1 + z_2) + 3(z_1 + 4z_1 z_2 + z_2^2) + \mathcal{O}(z^3), \\ 2\pi i T_2 &= \log(z_2) + 2(z_1 + z_2) + 3(z_1 + 4z_1 z_2 + z_2^2) + \mathcal{O}(z^3). \end{aligned} \tag{4.11}$$

Exponentiate and invert the above series expansion one finds

$$\begin{aligned} z_1(Q_1, Q_2) &= Q_1 - 2Q_1(Q_1 + Q_2) + 3Q_1(Q_1^2 + Q_2^2) + \mathcal{O}(Q^4), \\ z_2(Q_1, Q_2) &= Q_2 - 2Q_2(Q_1 + Q_2) + 3Q_2(Q_1^2 + Q_2^2) + \mathcal{O}(Q^4). \end{aligned} \tag{4.12}$$

We expect the first non-trivial amplitude $F^{(1,0)}$ to be holomorphic and proportional the logarithm of the discriminant of the Calabi-Yau geometry. Indeed the simplest ansatz that matches (3.13), is symmetric in (z_1, z_2) and regular for large z_1, z_2 is

$$\begin{aligned} \frac{1}{24}\log\left(\frac{\Delta}{z_1 z_2}\right) &= -\frac{1}{24}\log(Q_1 Q_2) - \frac{1}{6}(Q_1 + Q_2) - \frac{1}{12}(Q_1^2 + 28Q_1Q_2 + Q_2^2) \\ &\quad - \frac{1}{18}(Q_1^3 + 153Q_1^2Q_2 + 153Q_1Q_2^2 + Q_2^3) + \mathcal{O}(Q^4)\ . \end{aligned} \tag{4.13}$$

The perturbative terms agree with the results in (4.7) from the refined topological vertex calculations.

We then compute the higher genus amplitudes in the refined topological string using the extended holomorphic anomaly equations and the boundary conditions at singular points of the moduli space in the mirror. We follow the techniques developed in [40] for dealing with multi-parameter Calabi–Yau models. For the local model we consider, the Kähler potential is a constant in the holomorphic limit, so the covariant derivative with respect to the vacuum line bundle $\mathcal{L}$ vanishes, and we only need to use the Christoffel connection Γ^i_{jk} with respect to the metric in the covariant derivative. In the holomorphic limit, the metric and Christoffel connection can be calculated from the mirror maps (4.11) (4.12)

$$G_{i\bar{j}} \sim \frac{\partial T_j}{\partial z_i}, \quad \Gamma^i_{jk} = \frac{\partial z_i}{\partial T_l}\frac{\partial^2 T_l}{\partial z_j \partial z_k}. \tag{4.14}$$

To integrate the holomorphic anomaly equation, we should write the topological string as polynomials of some generators following the approach in [16,41]. It turns out that for multi-parameter models, it is more convenient to use the propagators S^{jk}, S^k, S as anholomorphic generators of the polynomials [42]. The propagators were originally introduced by BCOV [8] to integrate the holomorphic anomaly equations, and for local models we only need the two-index propagators, which are defined in terms of the three point coupling as $\bar{\partial}_{\bar{i}} S^{jk} = \bar{C}^{jk}_{\bar{i}}$. The main difference with one-parameter model is that the topological string amplitudes will be polynomials of the anholomorphic generators S^{jk} with the coefficients as rational functions of the moduli z_i, where in one-parameter models one can also include a holomorphic generator, which is a rational function of the modulus, and the topological string amplitudes would be truly polynomials with constant coefficients. Assuming the anti-holomorphic derivative of the propagators $\bar{\partial}_{\bar{i}} S^{jk} = \bar{C}^{jk}_{\bar{i}}$ are linearly independent, the generalized holomorphic anomaly equation (2.3) can be written as

$$\frac{\partial F^{(g_1,g_2)}}{\partial S^{jk}} = \frac{1}{2}\left(D_j D_k F^{(g_1,g_2-1)} + {\sum_{r_1,r_2}}' D_j F^{(r_1,r_2)} D_k F^{(g_1-r_1,g_2-r_2)} \right). \tag{4.15}$$

This equation can be integrated with respect to S^{jk} and we solve for $F^{(g_1,g_2)}$ recursively as a polynomial of S^{jk} with rational function coefficients, up to a rational function ambiguity. We note S^{jk} is a symmetric tensor, so for the two-parameter model such as the local $\mathbb{P}^1 \times \mathbb{P}^1$ model we consider here, we have three independent generators S^{11}, S^{12}, S^{22}. To carry out the polynomial formalism, we need the formula for the derivative of the propagators and the Christoffel symbol. This can be derived from the special geometry relation; see e.g., [40, 42].

$$\begin{aligned} D_i S^{jk} &= -C_{imn} S^{jm} S^{kn} + f_i^{jk}, && (4.16) \\ \Gamma_{ij}^k &= -C_{ijl} S^{kl} + \tilde{f}_{ij}^k, && (4.17) \end{aligned}$$

where the three point coupling C_{ijk} are

$$\begin{aligned} &C_{111} = \frac{(1-4z_2)^2 - 16z_1(1+z_1)}{4z_1^3\Delta}, \quad C_{112}\frac{16z_1^2 - (1-4z_2)^2}{4z_1^2 z_2\Delta}, \\ &C_{122}\frac{16z_2^2 - (1-4z_1)^2}{4z_1 z_2^2\Delta}, \quad C_{222} = \frac{(1-4z_1)^2 - 16z_2(1+z_2)}{4z_2^3\Delta}, \qquad (4.18) \end{aligned}$$

and the other combinations follow by symmetry. The rational functions $f, \tilde{f}$ are

$$\begin{aligned} &\tilde{f}_{11}^1 = -\frac{1}{z_1}, \quad \tilde{f}_{12}^1 = \tilde{f}_{21}^1 = -\frac{1}{4z_2}, \quad \tilde{f}_{22}^1 = 0, \\ &\tilde{f}_{11}^2 = 0, \quad \tilde{f}_{12}^2 = \tilde{f}_{21}^2 = -\frac{1}{4z_1}, \quad \tilde{f}_{22}^2 = -\frac{1}{z_2}, \qquad (4.19) \\ &f_1^{11} = -\frac{1}{8}z_1(1+4z_1-4z_2), \quad f_1^{12} = f_1^{21} = -\frac{1}{8}z_2(1+4z_1-4z_2), \\ &f_1^{22} = -\frac{z_2^2}{8z_1}(1+4z_1-4z_2), \\ &f_2^{11} = -\frac{z_1^2}{8z_2}(1+4z_2-4z_1), \quad f_2^{12} = f_2^{21} = -\frac{1}{8}z_1(1+4z_2-4z_1), \\ &f_2^{22} = -\frac{1}{8}z_2(1+4z_2-4z_1). \qquad (4.20) \end{aligned}$$

Here we note that the $\tilde{f}$ are chosen so that the overdetermined equations for the propagators (4.17) are solvable. Under this choice of $\tilde{f}$ in (4.19), the propagators are related and have only one independent component

$$S^{ij} = \begin{pmatrix} S(z_1,z_2) & \frac{z_2}{z_1} S(z_1,z_2) \\ \frac{z_2}{z_1} S(z_1,z_2) & \frac{z_2^2}{z_1^2} S(z_1,z_2) \end{pmatrix}. \qquad (4.21)$$

Like the Christoffel symbol, the propagators are in general not rational functions of z_i. We calculate the series expansion of the propagators around the

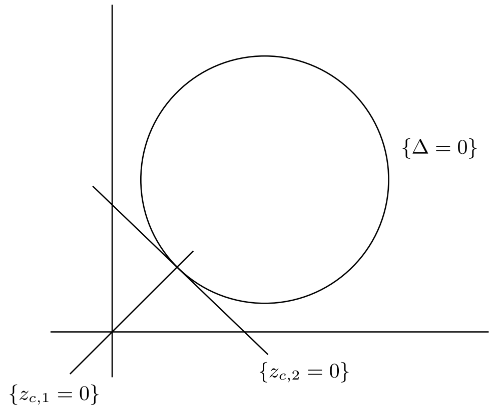

Figure 2: Conifold coordinates.

large volume point $z_1 = z_2 = 0$ in terms of the mirror maps Q_1, Q_2

$$\begin{aligned} S = {} & \frac{Q_1^2}{2} - 4Q_1^2 (Q_1 + Q_2) + Q_1^2 \left(17Q_1^2 + 20Q_1Q_2 + 17Q_2^2\right) \\ & - 4Q_1^2 \left(13Q_1^3 + 17Q_1^2Q_2 + 17Q_1Q_2^2 + 13Q_2^3\right) + \mathcal{O}(Q^5). \end{aligned} \tag{4.22}$$

The fix the rational function of z_i appearing as the constant term in the integration of the holomorphic anomaly equations, we need to expand the topological strings around the conifold point of the moduli space. This is depicted in figure 2. Here, we choose to expand around a symmetric point $z_{\mathrm{c}1} = z_{\mathrm{c}2} = 0$, where the coordinates are

$$z_{\mathrm{c}1} = 1 - \frac{z_1}{z_2}, \quad z_{\mathrm{c}2} = 1 - \frac{z_2}{\frac{1}{8} - z_1}. \tag{4.23}$$

We can solve the Picard–Fuchs system of differential equations around this point, find the mirror maps

$$\begin{aligned} t_{\mathrm{c}1} = {} & -\log(1 - z_{\mathrm{c}1}), \qquad (4.24) \\ t_{\mathrm{c}2} = {} & z_2 + \frac{1}{16}\left(2z_1^2 + 8z_1z_2 + 13z_2^2\right) + \frac{1}{768}\left(96z_1^3 + 228z_1^2z_2 + 240z_1z_2^2\right. \\ & \left. + 521z_2^3\right) + \frac{1}{8192}(904z_1^4 + 1600z_1^3z_2 + 1172z_1^2z_2^2 + 1680z_1z_2^3 \\ & + 4749z_2^4) + \mathcal{O}(z^5), \end{aligned}$$

and the inverse series

$$
\begin{aligned}
z_{\mathrm{c}1} &= 1 - \mathrm{e}^{-t_{\mathrm{c}1}}, \qquad (4.25)\\
z_{\mathrm{c}2} &= t_2 - \frac{1}{16}\left(2t_1^2 + 8t_1t_2 + 13t_2^2\right) + \frac{1}{768}\left(48t_1^3 + 312t_1^2t_2 + 696t_1t_2^2 + 493t_2^3\right)\\
&\quad - \frac{1}{24576}(832t_1^4 + 7040t_1^3t_2 + 21216t_1^2t_2^2 + 27856t_1t_2^3 + 12427t_2^4)\\
&\quad + \mathcal{O}(t^5).
\end{aligned}
$$

The coordinate $t_{\mathrm{c}2}$ is normal to the conifold loci, so the expansion of the topological string amplitude around this point should be singular as $t_{\mathrm{c}2} \to 0$ and exhibit the gap condition. To expand the topological strings around the conifold point $z_{\mathrm{c}1} = z_{\mathrm{c}2} = 0$ in terms of the mirror maps $t_{\mathrm{c}1}, t_{\mathrm{c}2}$, we transform the coordinates and the propagators to the conifold coordinate

$$S^{z_iz_j} = \frac{\partial z_i}{\partial z_{\mathrm{c},k}}\frac{\partial z_j}{\partial z_{\mathrm{c},l}} S^{z_{\mathrm{c},k}z_{\mathrm{c},l}}. \qquad (4.26)$$

In order to compute the series expansion of the propagator at the conifold point using (4.17), we also need to transform the three point Yukawa couplings, the holomorphic ambiguity $\tilde{f}$ to the conifold coordinates. The Yukawa couplings transform as a tensor, and the holomorphic ambiguity $\tilde{f}$ transform according the rules

$$\tilde{f}^{z_{\mathrm{c},i}}_{z_{\mathrm{c},j}z_{\mathrm{c},k}} = \frac{\partial z_{\mathrm{c},i}}{\partial z_l}\frac{\partial^2 z_l}{\partial z_{\mathrm{c},j}\partial z_{\mathrm{c},k}} + \frac{\partial z_{\mathrm{c},i}}{\partial z_l}\frac{\partial z_m}{\partial z_{\mathrm{c},j}}\frac{\partial z_n}{\partial z_{\mathrm{c},k}}\tilde{f}^{z_l}_{z_mz_n}. \qquad (4.27)$$

We also need to calculate the Christoffel connection around the conifold point using the mirror maps (4.24, 4.25), and the relation (4.14). It turns out that the propagators have only one non-vanishing component around the conifold point. The three components vanish $S^{z_{\mathrm{c}1}z_{\mathrm{c}1}} = S^{z_{\mathrm{c}1}z_{\mathrm{c}2}} = S^{z_{\mathrm{c}2}z_{\mathrm{c}1}} = 0$, and the last component is computed as

$$S^{z_{\mathrm{c}2}z_{\mathrm{c}2}} = \frac{t_2}{2} - \frac{1}{8}\left(4t_1 + 13t_2\right)t_2 + \frac{t_2\left(840t_1^2 + 4032t_1t_2 + 4987t_2^2\right)}{1536} + \mathcal{O}(t^4).$$

The gap conditions at the conifold point plus one more condition from the constant map contribution at large volume point are sufficient to fix the topological string amplitudes. Here for convenience, we fix the large volume behavior of the topological string amplitudes to be vanishing, i.e., $F^{(g_1,g_2)} \sim \mathcal{O}(Q)$. The constant map contribution can be simply recovered by adding the appropriate constant to the topological strings without effects on the gap condition around the conifold point. We found the genus two results as the followings:

$$
\begin{aligned}
F^{(0,2)} &= \frac{5S^3}{24z_1^6\Delta^2} + \frac{S^2}{48z_1^4\Delta^2}(48z_1^2 - 96z_1z_2 + 40z_1 + 48z_2^2 + 40z_2 - 13) \\
&\quad + \frac{S}{144z_1^2\Delta^2}(384z_1^3 - 384z_1^2z_2 + 80z_1^2 - 384z_1z_2^2 + 736z_1z_2 - 112z_1 \\
&\quad + 384z_2^3 + 80z_2^2 - 112z_2 + 17) + \frac{1}{1440\Delta^2}(2688z_1^4 + 1536z_1^3z_2 \\
&\quad - 416z_1^3 - 8448z_2^2z_1^2 + 6560z_2z_1^2 - 696z_1^2 + 1536z_1z_2^3 + 6560z_1z_2^2 \\
&\quad - 2768z_1z_2 + 258z_1 + 2688z_2^4 - 416z_2^3 - 696z_2^2 + 258z_2 - 25).
\end{aligned} \tag{4.28}
$$

$$
\begin{aligned}
F^{(1,1)} &= \frac{S^2\,(1 - 4z_1 - 4z_2)}{24z_1^4\Delta^2} + \frac{S}{144z_1^2\Delta^2}(-192z_1^3 + 192z_1^2z_2 + 16z_1^2 \\
&\quad + 192z_1z_2^2 - 544z_1z_2 + 28z_1 - 192z_2^3 + 16z_2^2 + 28z_2 - 5) \\
&\quad + \frac{1}{720\Delta^2}(-1408z_1^4 - 1536z_1^3z_2 + 736z_1^3 + 5888z_1^2z_2^2 - 4320z_1^2z_2 \\
&\quad - 24z_1^2 - 1536z_1z_2^3 - 4320z_1z_2^2 + 1328z_1z_2 - 38z_1 - 1408z_2^4 + 736z_2^3 \\
&\quad - 24z_2^2 - 38z_2 + 5),
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
F^{(2,0)} &= \frac{S}{288z_1^2\Delta^2}\left(16z_1^2 + 32z_1z_2 - 8z_1 + 16z_2^2 - 8z_2 + 1\right) \\
&\quad + \frac{1}{2880\Delta^2}(-512z_1^4 + 9216z_1^3z_2 + 704z_1^3 - 17408z_1^2z_2^2 + 2880z_1^2z_2 \\
&\quad - 336z_1^2 + 9216z_1z_2^3 + 2880z_1z_2^2 - 1568z_1z_2 + 68z_1 - 512z_2^4 \\
&\quad + 704z_2^3 - 336z_2^2 + 68z_2 - 5),
\end{aligned} \tag{4.30}
$$

where we have used the large volume coordinate and $S = S^{z_1z_1}$, but the expressions are exact can be expanded around any point in the moduli space. We also solve the topological string amplitudes at genus three. We check that the expansion around large volume point $z_1 = z_2 = 0$ agree with the calculations (4.8), (A.3) from the refined topological vertex. Using (2.8), the form of the conifold discriminant (4.10) and the finiteness of the $F^{(g_1,g_2)}$ in the large $z_{1,2}$ limit one can easily see as in [40] that the deformed model is completely integrable, i.e., all $c_0^{(g_1,g_2)}(z_{1,2})$ are fixed by the boundary conditions.

4.3 The refined topological string on local $\mathbb{P}^2$

The refined topological vertex formalism [10] is not applicable to the local $\mathcal{O}(-3) \to \mathbb{P}^2$ model, because it does not give rise to a gauge theory.

It remains an interesting problem to find a refined topological vertex formalism that applies directly to this model. Nevertheless, the homology of local $\mathbb{P}^2$ can be embedded into the one of another toric geometry, the local $\mathbb{F}_1$ model, which is the simply the blow up of $\mathbb{P}^2$ and geometrically engineers the $SU(2)$ Seiberg–Witten theory with $N_\mathrm{f} = 0$ fundamental flavor in four dimensions. One can calculate the refined topological string amplitudes of the local $\mathbb{F}_1$ model with the refined topological vertex formalism, and extract the BPS numbers $N_{j_\mathrm{L},j_\mathrm{R}}$ with spins in both left and right $SU(2)$ subgroups of the Lorentz group. These BPS numbers determine also the refined BPS numbers $N_{j_\mathrm{L},j_\mathrm{R}}$ of the $\mathbb{P}^2$ geometry [10].

Using these refined BPS numbers provided in [10] we find the instanton part of the refined topological string amplitudes to the first few orders. The genus one and two results are

$$\begin{aligned}
F^{(0,1)} &= \frac{Q}{4} - \frac{3Q^2}{8} - \frac{23Q^3}{3} + \frac{3437Q^4}{16} + \mathcal{O}(Q^5),\\
F^{(1,0)} &= \frac{7Q}{8} - \frac{129Q^2}{16} + \frac{589Q^3}{6} - \frac{43009Q^4}{32} + \mathcal{O}(Q^5),
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
F^{(0,2)} &= \frac{Q}{80} + \frac{3Q^3}{20} - \frac{514Q^4}{5} + \mathcal{O}(Q^5),\\
F^{(1,1)} &= \frac{11Q}{160} - \frac{9Q^2}{16} - \frac{1317Q^3}{40} + \frac{72019Q^4}{40} + \mathcal{O}(Q^5),\\
F^{(2,0)} &= \frac{29Q}{640} - \frac{207Q^2}{64} + \frac{18447Q^3}{160} - \frac{526859Q^4}{160} + \mathcal{O}(Q^5).
\end{aligned} \tag{4.32}$$

The genus three results

$$\begin{aligned}
F^{(0,3)} &= \frac{Q}{2016} + \frac{Q^2}{336} + \frac{Q^3}{56} + \frac{1480Q^4}{63} + O\left(Q^5\right),\\
F^{(1,2)} &= \frac{127Q}{40320} - \frac{31Q^2}{3360} + \frac{547Q^3}{1120} - \frac{293777Q^4}{315} + \mathcal{O}(Q^5),\\
F^{(2,1)} &= \frac{143Q}{53760} - \frac{547Q^2}{2240} - \frac{182901Q^3}{4480} + \frac{4107139Q^4}{840} + \mathcal{O}(Q^5),\\
F^{(3,0)} &= \frac{137Q}{322560} - \frac{7573Q^2}{13440} + \frac{608717Q^3}{8960} - \frac{21873839Q^4}{5040} + \mathcal{O}(Q^5).
\end{aligned} \tag{4.33}$$

Now we turn to the B-model calculations. We extended the approach in [40] to solve the extended holomorphic anomaly equations. The Picard–Fuchs differential equation is well known $\mathcal{L}f = 0$ where the Picard–Fuchs operator is

$$\mathcal{L} = \theta^3 + 3z(3\Theta + 2)(3\Theta + 1)\Theta. \tag{4.34}$$

Here $\Theta = z\frac{\partial}{\partial z}$. The discriminant of the Picard–Fuchs operator is $\Delta = 1 + 27z$. The large volume point is $z \sim 0$ and the conifold point is $z \sim -\frac{1}{27}$. The three linearly independent solutions at large volume point are the constant, a logarithmic solution and a double logarithmic solution. The Kähler parameter in the A-model is the logarithmic solution and its exponential $Q = \mathrm{e}^{-T}$ is expanded as the following:

$$Q = z - 6z^2 + 63z^3 - 866z^4 + 13899z^5 + \mathcal{O}(z^6). \tag{4.35}$$

From the double logarithmic solution we can find the three-point Yukawa coupling

$$C_{zzz} = -\frac{1}{3}\frac{1}{z^3(1+27z)}. \tag{4.36}$$

The genus one topological amplitude $F^{(0,1)}$ is well known

$$F^{(0,1)} = -\frac{1}{2}\log\left(\frac{\partial T}{\partial z}\right) - \frac{1}{12}\log(z^7\Delta). \tag{4.37}$$

From our general analysis, we expect the refined topological string amplitude $F^{(1,0)}$ to be 1/24 the logarithm of the discriminant, plus a piece proportional to $\log(z)$. We use the results (4.31) from A-model to fix this constant, and we find

$$F^{(1,0)} = \frac{1}{24}\log\left(\frac{\Delta}{z}\right). \tag{4.38}$$

Once the constant is fixed, the large volume expansion of the above equation agrees with (4.31) using the mirror map (4.35).

To compute the higher genus refined amplitudes, we need the propagator S^{zz} and its relation with the Christoffel connection. This is also fixed in [40]

$$\begin{aligned}\Gamma^z_{zz} &= -C_{zzz}S^{zz} - \frac{7+216z}{6z\Delta},\\ D_z S^{zz} &= -C_{zzz}(S^{zz})^2 - \frac{z}{12\Delta}.\end{aligned} \tag{4.39}$$

The Christoffel symbol and the propagator can be expanded around any point in the moduli space using the corresponding mirror map around the relevant point. For example, at the large volume point $z \sim 0$, the propagator $S \equiv S^{zz}$ is

$$S = \frac{Q^2}{2} + 15Q^3 + 135Q^4 + 785Q^5 + \frac{4473Q^6}{2} + \mathcal{O}(Q^7). \tag{4.40}$$

The extended holomorphic anomaly equation of one-parameter model without the Griffiths infinitesimal invariant for genus greater than one is

$$\frac{\partial F^{(g_1,g_2)}}{\partial S} = \frac{1}{2}\left[D_z^2 F^{(g_1,g_2-1)} + \left(\sum_{r_1=0}^{g_1}\sum_{r_2=0}^{g_2}\right)' D_z F^{(r_1,r_2)} D_z F^{(g_1-r_1,g_2-r_2)}\right], \tag{4.41}$$

where $(\sum)'$ denotes the sum excludes $r_1 = r_2 = 0$ and $r_1 = g_1, r_2 = g_2$. Integrating the holomorphic anomaly equation can determine the refined topological amplitude $F^{(g_1,g_2)}$ as a polynomial of S whose coefficients are rational functions of z, up to a rational of function of the form $\frac{f(z)}{\Delta(z)^{2g_1+2g_2-2}}$, where $f(z)$ is a degree $2g_1+2g_2-2$ polynomial. Expanding again the $F^{(g_1,g_2)}$ around the conifold point, we can fix the ambiguous rational function up to a constant, which can be further fixed by the constant map contribution at the large volume point. Since the constant map contribution can be simply recovered by adding the appropriate constant to the topological strings without effects on the gap condition around the conifold point, here for convenience we simply require $f(z)$ to a polynomial of one degree less, thus completely fix the ambiguity. The results at the first few orders are the followings:

$$\begin{aligned}F^{(0,2)} &= \frac{100S^3 - 90S^2z^2 + 30Sz^4 + 3(9z-1)z^6}{4320z^6\Delta^2},\\ F^{(1,1)} &= \frac{10S^2 + 5S(108z-1)z^2 + 2(1-54z)z^4}{1440z^4\Delta^2},\\ F^{(2,0)} &= \frac{10S + (1296z+11)z^2}{11520z^2\Delta^2}.\end{aligned} \tag{4.42}$$

The genus three results are

$$
\begin{aligned}
F^{(0,3)} = & \frac{1}{8709120z^{12}\Delta^4}[33600S^6 - 84000S^5z^2 - 6720S^4(189z-13)z^4 \\
& - 280S^3\left(17496z^2 - 8964z + 173\right)z^6 + 3024S^2\left(4941z^2\right. \\
& \left. - 561z + 5\right)z^8 + 504S\left(26244z^3 - 20169z^2 + 981z - 5\right)z^{10} \\
& + 3\left(-3254256z^3 + 649296z^2 - 18288z + 53\right)z^{12}], \\
F^{(1,2)} = & \frac{1}{8709120z^{10}\Delta^4}[8400S^5 + 2520S^4(180z-7)z^2 + 420S^3\left(19440z^2\right. \\
& \left. - 4572z + 35\right)z^4 - 42S^2\left(1382184z^2 - 64584z + 145\right)z^6 \\
& - 252S\left(629856z^3 - 221859z^2 + 4977z - 5\right)z^8 \\
& + \left(116680824z^3 - 13878702z^2 + 189810z - 89\right)z^{10}], \\
F^{(2,1)} = & \frac{1}{11612160z^8\Delta^4}[1120S^4 + 280S^3(432z-7)z^2 + 252S^2\left(36288z^2\right. \\
& \left. - 1788z + 5\right)z^4 + 14S\left(22674816z^3 - 3936600z^2 + 38988z - 25\right)z^6 \\
& - \left(365316480z^3 - 22651488z^2 + 175608z + 253\right)z^8], \\
F^{(3,0)} = & \frac{1}{69672960z^6\Delta^4}[280S^3 + 420S^2(108z-1)z^2 + 42S\left(209952z^2\right. \\
& \left. - 4212z + 5\right)z^4 + \left(1167753024z^3 - 29387448z^2 + 355536z\right. \\
& \left. + 2269\right)z^6].
\end{aligned} \tag{4.43}
$$

The expansions of these exact results around the large volume point agree with the A-model results (4.32), (4.33). Again a counting of the parameters in the $c^{(g_1,g_2)})_0$ polynomials shows similar as in [40], that all ambiguities are fixed for the deformed topological string on $\mathbb{P}^2$.

5 Conclusion and directions for further work

We have extended the direct integration method developed in [6, 7, 40] to solve pure Seiberg–Witten theory and topological string theory on local Calabi–Yau spaces. We found a generalized holomorphic anomaly equation, which as we argued by comparing with the general expansion of $F^{(g_1,g_2)}$ in terms of BPS invariants should hold in full generality for the topological string on non-compact Calabi–Yau manifolds. We have demonstrated that the gap condition of the $F^{(g_1,g_2)}$ at the conifold provides together with regularity of the $F^{(g_1,g_2)}$ at other boundary divisors enough boundary conditions to solve these models. Our formalism implies that the $F^{(g_1,g_2)}$ are expressible in terms of generators of a ring of an-holomorphic modular forms and that $F^{(g_1=0,g_2)} = F^{(g_1)}$ is the most an-holomorphic object. Our expressions, e.g., (4.42) and (4.43) can be readily expanded near the $\mathbb{C}/\mathbb{Z}_3$ orbifold point in the local $\mathbb{P}^2$ moduli space using the flat coordinates provided in [38] to yield refined orbifold Gromov–Witten invariants.

The holomorphic anomaly equation in this paper also applies to the pure $N = 2$ supersymmetric gauge theories. However it it seems not to apply to the cases with general matter. It was found, e.g., for the $SU(2)$ gauge theory with one fundamental flavor, that it has to be modified in an interesting way by a Griffith infinitesimal invariant [11]. From the point of view of direct integration, which is based on the fact that the $F^{(g_1,g_2)}$ are finitely generated by independent generators, it would be interesting to clarify the modularity property of this function.

So far the versions of the holomorphic anomaly in this paper and in [11] are not derived from first principles. In the case of topological string case on local Calabi–Yau manifolds, it can be argued that the $F^{(g_1,g_2)}$ should obey T-duality of the spacetime geometry, which in our cases is an elliptic curve. Because of the special properties of the quasi modular generator at weight 2 E_2, the failure of holomorphicity is then closely related to a failure of T-duality, which can maybe be explained from the space-time point of view.

Since the β-ensemble of the matrix model was already shown to describe the gauge theory perturbatively [44], it seems clear that a proof of the generalized holomorphic anomaly equations along the lines of [46] should be possible for the $N = 2$ supersymmetric gauge theories, once the program of the paper [48] is established for the deformed case. One expects that in this program only the recursion relation changes as a consequence of the β dependent measure, that affects the loop equations, while the spectral curve stays the same.

As was mentioned a greater challenge is to extend in general the Chern-Simons matrix model [47] to refined topological string backgrounds as already the perturbative calculation for the blown up conifold fails to be related to the Chern-Simons model in the β-ensemble. It could be that a more general coordinate transformation, which involve the $\epsilon_{1/2}$ in the mirror map, is necessary to relate the topological string on compact Calabi–Yau manifolds to the β-ensemble. This would be very useful to extend the analysis to the open topological string invariants and to proof the extended holomorphic anomaly equation as in [46]. Moreover it is known that the deformed β-ensemble calculates for $\beta = \frac{1}{2}$ $SP(N)$ orientifold graphs and for $\beta = 2$ $SO(N)$ orientifold graphs [45], which should have the corresponding interpretation in the topological string theory.

We generalized the heterotic amplitude and made predictions for the refined topological invariants related to left and right $SU(2)_{\mathrm{R/L}}$ Lefshetz action in the $K3$-fiber of a regular $K3$ fibered Calabi–Yau space and made a connection to a Göttsche formula for Hilbert schemes on $K3$ and checked that the refined holomorphic anomaly equation holds in this case.

The $K3$ results could bear implications for the micro- and macroscopic description of small black holes in $K3$ fibered Calabi–Yau spaces.

The duality on the elliptic fiber of a half $K3$ relates the refined topological string invariants to refined topological invariants on the moduli space of $N = 4$ Yang–Mills instantons on del Pezzo surfaces. Some predictions along these lines can be found in Section 4.1

Let us finally mentioned that all our results are symmetric in $\epsilon_{1/2}$. That is not necessarily the case for general refined BPS invariants in compact Calabi–Yau manifolds. It is also clear that in this case the individual refined BPS numbers are not symplectic invariants, but depend rather on the complex structure. It is nevertheless remarkable that a straightforward generalization of the formalism to the quintic yields an integer structure. If we extend the Ansätze for $F^{(1,0)}$ and $F^{(0,1)}$ in a natural way to the case of the quintic in $\mathbb{P}^4$,[7]

$$F^{(0,1)} = \log\left(G^{-1/2}_{\psi,\bar\psi} \exp\left[\frac{31}{3}K\right] |\psi^{\frac{31}{3}}(1-\psi^5)^{-\frac{1}{12}}|^2\right) \tag{5.1}$$

and

$$F^{(1,0)} = \log\left(\exp\left[\frac{31}{3}K\right] \psi^{\frac{31}{3}}(1-\psi^5)^{\frac{1}{24}}\right) , \tag{5.2}$$

which is compatible with regularity of $F^{(1,0)}$ at the orbifold point, the universal conifold behavior and assumes that $F^{(1,0)}$ is the section of the same Kähler line bundle then $F^{(0,1)}$ one gets the following integers:

$$n^{(1,0)}_d = -1492, -171409, 123200314, 381613562015, \ldots. \tag{5.3}$$

The integrality, which is nontrivial from the multi covering formula and the subtraction of the genus zero contribution, has been checked up to high degree $d = 50$. It would be interesting to understand, whether these integers

[7]Here we use the conventions of [50] with an additional 1/2 in front of $F^{(1)} = F^{(0,1)}$, which was corrected in [8] as it is essential for the higher genus calculations.

count refined cohomologies of D-branes for some canonical choice of the complex structure of the quintic.

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Appendix A

A.1 $g_1 + g_2 = 3$ results for pure $N = 2$ SU(2) SYM theory

The genus 3 formulae are

$$
\begin{aligned}
F^{(0,3)} &= \frac{1}{2916(u^2-1)^4}\bigg[5X^6 - 25uX^5 + (50u^2+30)X^4 \\
&\quad - \frac{u}{12}(559u^2+1557)X^3 + \frac{1}{80}(1223u^4+13794u^2+3735)X^2 \\
&\quad - \frac{u}{40}(155u^4+3060u^2+3537)X \\
&\quad + \frac{1}{3360}(236u^6+43299u^4+111078u^2+16875)\bigg], \\
F^{(1,2)} &= \frac{1}{5832(u^2-1)^4}\bigg[5uX^5 - (26u^2+15)X^4 + \frac{3u}{4}(75u^2+233)X^3 \\
&\quad - \frac{1}{20}(1163u^4+13419u^2+3150)X^2 + \frac{u}{20}(287u^4+11751u^2 \\
&\quad + 13194)X - \frac{3}{1120}(516u^6+52231u^4+152238u^2+23175)\bigg], \\
F^{(2,1)} &= \frac{1}{5832(u^2-1)^4}\bigg[u^2X^4 - u\left(\frac{11}{2}u^2+6\right)X^3 \\
&\quad + \frac{9}{20}(23u^4+184u^2+35)X^2 - \frac{u}{40}(53u^4+15849u^2+16182)X \\
&\quad + \frac{1}{560}(1216u^6+93615u^4+307008u^2+45225)\bigg],
\end{aligned}
$$

$$F^{(3,0)} = \frac{1}{69984(u^2-1)^4}\Big[u^3X^3 - 3u^2(2u^2+3)X^2 - \frac{3u}{20}(u^4 - 1347u^2 - 450)X - \frac{1}{140}(769u^6 + 87012u^4 + 310500u^2 + 43875)\Big]. \quad \text{(A.1)}$$

The dual expansions are

$$\begin{aligned}
F_D^{(0,3)} &= \frac{1}{1008a_D^4} - \frac{9a_D}{2^{20}} + \mathcal{O}(a_D^2),\\
F_D^{(1,2)} &= -\frac{41}{20160a_D^4} + \frac{15}{2^{18}} + \frac{1239a_D}{5\cdot 2^{20}} + \mathcal{O}(a_D^2),\\
F_D^{(2,1)} &= \frac{31}{26880a_D^4} - \frac{117}{2^{20}} - \frac{5799a_D}{5\cdot 2^{23}} + \mathcal{O}(a_D^2),\\
F_D^{(3,0)} &= -\frac{31}{161280a_D^4} - \frac{243}{2^{21}} - \frac{41607a_D}{5\cdot 2^{24}} + \mathcal{O}(a_D^2).
\end{aligned} \quad \text{(A.2)}$$

Similar to the genus two case, we observe the gap structure in the dual expansion around the conifold point with the absence of $\frac{1}{a_D}, \frac{1}{a_D^2}, \frac{1}{a_D^3}$ terms.

A.2 $g_1 + g_2 = 3$ results for local $\mathbb{P}^1 \times \mathbb{P}^1$

The genus three results are

$$\begin{aligned}
F^{(0,3)} &= -\frac{Q_1+Q_2}{3024} - \frac{4Q_1^2 + Q_1Q_2 + 4Q_2^2}{1512} - \frac{9Q_1^3 + Q_1^2Q_2 + Q_1Q_2^2 + 9Q_2^3}{1008}\\
&\quad - \frac{1}{756}\left(16Q_1^4 + Q_1^3Q_2 + 8Q_1^2Q_2^2 + Q_1Q_2^3 + 16Q_2^4\right) + \mathcal{O}(Q^5),\\
F^{(1,2)} &= -\frac{11}{30240}(Q_1+Q_2) - \frac{1}{15120}(44Q_1^2 + 137Q_1Q_2 + 44Q_2^2)\\
&\quad - \frac{1}{10080}(99Q_1^3 + 347Q_1^2Q_2 + 347Q_1Q_2^2 + 99Q_2^3)\\
&\quad - \frac{1}{7560}(176Q_1^4 + 641Q_1^3Q_2 + 3364Q_1^2Q_2^2 + 641Q_1Q_2^3 + 176Q_2^4),\\
F^{(2,1)} &= \frac{1}{2520}(Q_1+Q_2) + \frac{1}{2520}(8Q_1^2 - 61Q_1Q_2 + 8Q_2^2)\\
&\quad + \frac{1}{840}(9Q_1^3 - 223Q_1^2Q_2 - 223Q_1Q_2^2 + 9Q_2^3)\\
&\quad + \frac{1}{1260}(32Q_1^4 - 1573Q_1^3Q_2 + 28408Q_1^2Q_2^2 - 1573Q_1Q_2^3 + 32Q_2^4),
\end{aligned} \quad \text{(A.3)}$$

$$
\begin{aligned}
F^{(3,0)} = &-\frac{1}{15120}(Q_1+Q_2) - \frac{1}{7560}(4Q_1^2+127Q_1Q_2+4Q_2^2)\\
&-\frac{1}{5040}(9Q_1^3+2857Q_1^2Q_2+2857Q_1Q_2^2+9Q_2^3)\\
&-\frac{1}{3780}(16Q_1^4+19531Q_1^3Q_2+143144Q_1^2Q_2^2\\
&+19531Q_1Q_2^3+16Q_2^4).
\end{aligned}
$$

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