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Equivariant modular categories via Dijkgraaf–Witten theory

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Abstract

Based on a weak action of a finite group J on a finite group G, we present a geometric construction of J-equivariant Dijkgraaf–Witten theory as an extended topological field theory. The construction yields an explicitly accessible class of equivariant modular tensor categories. For the action of a group J on a group G, the category is described as the representation category of a J-ribbon algebra that generalizes the Drinfel'd double of the finite group G.

Contents

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1 Introduction

This paper has two seemingly different motivations and, correspondingly, can be read from two different points of view, a more algebraic and a more geometric one. Both in the introduction and the main body of the paper, we try to separate these two points of view as much as possible, in the hope to keep the paper accessible for readers with specific interests.

1.1 Algebraic motivation: equivariant modular categories

Among tensor categories, modular tensor categories are of particular interest for representation theory and mathematical physics. The representation categories of several algebraic structures give examples of semisimple modular tensor categories:

(1) Left modules over connected factorizable ribbon weak Hopf algebras with Haar integral over an algebraically closed field [38].
(2) Local sectors of a finite μ-index net of von Neumann algebras on $\mathbb{R}$, if the net is strongly additive and split [27].
(3) Representations of self-dual C_2-cofinite vertex algebras with an additional finiteness condition on the homogeneous components and which have semisimple representation categories [22].

Despite this list and the rather different fields in which modular tensor categories arise, it is fair to say that modular tensor categories are rare mathematical objects. Arguably, the simplest incarnation of the first algebraic structure in the list is the Drinfel'd double $\mathcal{D}(G)$ of a finite group G. Bantay [2] has suggested a more general source for modular tensor categories: a pair, consisting of a finite group H and a normal subgroup $G \triangleleft H$. (In fact, Bantay has suggested general finite crossed modules, but for this paper, only the case of a normal subgroup is relevant.) In this situation, Bantay constructs a ribbon category which is, in a natural way, a representation category of a ribbon Hopf algebra $\mathcal{B}(G \triangleleft H)$. Unfortunately, it turns out that, for a proper subgroup inclusion, the category $\mathcal{B}(G \triangleleft H)$-mod is only premodular and not modular.

Still, the category $\mathcal{B}(G \triangleleft H)$-mod is modularizable in the sense of Bruguières [8], and the next candidate for new modular tensor categories is the modularization of $\mathcal{B}(G \triangleleft H)$-mod. However, it has been shown [37] that this modularization is equivalent to the representation category of the Drinfel'd double $\mathcal{D}(G)$.

The modularization procedure of Bruguières is based on the observation that the violation of modularity of a modularizable tensor category $\mathcal{C}$ is captured in terms of a canonical Tannakian subcategory of $\mathcal{C}$. For the category $\mathcal{B}(G \triangleleft H)$-mod, this subcategory can be realized as the representation category of the the quotient group $J := H/G$ [37]. The modularization functor

$$\mathcal{B}(G \triangleleft H)\text{-mod} \to \mathcal{D}(G)\text{-mod}$$

is induction along the commutative Frobenius algebra given by the regular representation of J. This has the important consequence that the modularized category $\mathcal{D}(G)$ is endowed with a J-action.

Experience with orbifold constructions, see [26, 43] for a categorical formulation, raises the question of whether the category $\mathcal{D}(G)$-mod with this J-action can be seen in a natural way as the neutral sector of a J-modular tensor category.

We thus want to complete the following square of tensor categories:

$$\begin{tikzcd}
\mathcal{D}(G)\text{-mod} \arrow[loop above, "J"] \arrow[r, hook] \arrow[d, shift left, "\text{orbifold}"] & ??? \arrow[loop above, "J"] \arrow[d, shift left, "\text{orbifold}"] \\
\mathcal{B}(G \triangleleft H)\text{-mod} \arrow[r, hook] \arrow[u, shift left, "\text{modularization}"] & ??? \arrow[u, shift left]
\end{tikzcd} \tag{1.1}$$

Here vertical arrows pointing upwards stand for induction functors along the commutative algebra given by the regular representation of J, while downwards pointing arrows indicate orbifoldization. In the upper right corner, we wish to place a J-modular category, and in the lower right corner its J-orbifold which, on general grounds [26], has to be a modular tensor category. Horizontal arrows indicate the inclusion of neutral sectors.

In general, such a completion need not exist. Even if it exists, there might be inequivalent choices of J-modular tensor categories of which a given modular tensor category with J-action is the neutral sector [15].

1.2 Geometric motivation: equivariant extended topological field theory (TFT)

TFT is a mathematical structure that has been inspired by physical theories [46] and which has developed into an important tool in low-dimensional topology. Recently, these theories have received increased attention due to the advent of *extended* topological field theories [31, 42]. The present paper focuses on three-dimensional (3D) TFT.

Dijkgraaf–Witten theories provide a class of extended topological field theories. They can be seen as discrete variants of Chern–Simons theories, which provide invariants of three-manifolds and play an important role in knot theory [46]. Dijkgraaf–Witten theories have the advantage of being particularly tractable and admitting a very conceptual geometric construction.

A Dijkgraaf–Witten theory is based on a finite group G; in this case the 'field configurations' on a manifold M are given by G-bundles over M, denoted by $\mathcal{A}_G(M)$. Furthermore, one has to choose a suitable action functional $S : \mathcal{A}_G(M) \to \mathbb{C}$ (which we choose here in fact to be trivial) on field configurations; this allows us to make the structure suggested by formal path integration rigorous and to obtain a topological field theory. A conceptually very clear way to carry this construction out rigorously is described in [17, 36]; see Section 2 of this paper for a review.

Let us now assume that as a further input datum we have another finite group J which acts on G. In this situation, we get an action of J on the Dijkgraaf–Witten theory based on G. But it turns out that this topological field theory together with the J-action does not fully reflect the equivariance of the situation: it has been an important insight that the right notion is the one of equivariant topological field theories, which have been another point of recent interest [26, 43]. Roughly speaking, equivariant topological field theories require that all geometric objects (i.e., manifolds of different dimensions) have to be decorated by a J-cover (see Definitions 3.11 and 3.13 for details). Equivariant field theories also provide a conceptual setting for the orbifold construction, one of the standard tools for model building in conformal field theory and string theory.

Given the action of a finite group J on a finite group G, these considerations lead to the question of whether Dijkgraaf–Witten theory based on G

can be enlarged to a J-equivariant topological field theory. Let us pose this question more in detail:

- What exactly is the right notion of an action of J on G that leads to interesting theories? To keep equivariant Dijkgraaf–Witten theory as explicit as the non-equivariant theory, one needs notions to keep control of this action as explicitly as possible.
- Ordinary Dijkgraaf–Witten theory is mainly determined by the choice of field configurations $\mathcal{A}_G(M)$ to be G-bundles. As mentioned before, for J-equivariant theories, we should replace manifolds by manifolds with J-covers. We thus need a geometric notion of a G-bundle that is "twisted" by this J-cover in order to develop the theory parallel to the non-equivariant one.

Based on an answer to these two points, we wish to construct equivariant Dijkgraaf–Witten theory as explicitly as possible.

1.3 Summary of the results

This paper solves both the algebraic and the geometric problem we have just described. In fact, the two problems turn out to be closely related. We first solve the problem of explicitly constructing equivariant Dijkgraaf–Witten and then use our solution to construct the relevant modular categories that complete the square (1.1).

Despite this strong mathematical interrelation, we have taken some effort to write the paper in such a way that it is accessible to readers sharing only a geometric or algebraic interest. The geometrically minded reader might wish to restrict his attention to Sections 2 and 3, and only take notice of the result about J-modularity stated in Theorem 4.34. An algebraically oriented reader, on the other hand, might simply accept the categories described in Proposition 3.22 together with the structure described in Propositions 3.23, 3.24 and 3.26 and then directly delve into Section 4.

For the benefit of all readers, we present here an outline of all our findings. In Section 2, we review the pertinent aspects of Dijkgraaf–Witten theory and in particular the specific construction given in [36]. Section 3 is devoted to the equivariant case: we observe that the correct notion of J-action on G is what we call a weak action of the group J on the group G; this notion is introduced in Definition 3.1. Based on this notion, we can very explicitly construct for every J-cover $P \to M$ a category $\mathcal{A}_G(P \to M)$ of P-twisted G-bundles. For the definition and elementary properties of twisted bundles, we refer to Section 3.2 and for a local description to Appendix A.1. We are then ready to construct equivariant Dijkgraaf Witten theory along the lines of the construction described in [36]. This is carried out in Sections 3.3 and 3.4. We obtain a construction of equivariant Dijkgraaf–Witten theory that is so explicit that we can read off the category $\mathcal{C}^J(G)$ it assigns to the circle $\mathbb{S}^1$. The equivariant topological field theory induces additional structure on this category, which can also be computed by geometric methods due to the explicit control of the theory, and part of which we compute in Section 3.5. This finishes the geometric part of our work. It remains to show that the category $\mathcal{C}^J(G)$ is indeed J-modular.

To establish the J-modularity of the category $\mathcal{C}^J(G)$, we have to resort to algebraic tools. Our discussion is based on the Appendix 6 of [43] by Virélizier. At the same time, we explain the solution of the algebraic problems described in Section 1.1. The Hopf algebraic notions we encounter in Section 4, in particular Hopf algebras with a weak group action and their orbifold Hopf algebras might be of independent algebraic interest.

In Section 4, we introduce the notion of a J-equivariant ribbon Hopf algebra. It turns out that it is natural to relax some strictness requirements on the J-action on such a Hopf algebra. Given a weak action of a finite group J on a finite group G, we describe in Proposition 4.23 a specific ribbon Hopf algebra which we call the equivariant Drinfel'd double $\mathcal{D}^J(G)$. This ribbon Hopf algebra is designed in such a way that its representation category is equivalent to the geometric category $\mathcal{C}^J(G)$ constructed in Section 3, compare Proposition 4.24.

The J-modularity of $\mathcal{C}^J(G)$ is established via the modularity of its orbifold category. The corresponding notion of an orbifold algebra is introduced in Section 4.4. In the case of the equivariant Drinfel'd double $\mathcal{D}^J(G)$, this orbifold algebra is shown to be isomorphic, as a ribbon Hopf algebra, to a Drinfel'd double. This implies modularity of the orbifold theory and, by a result of [26], J-modularity of the category $\mathcal{C}^J(G)$; cf. Theorem 4.34.

In the course of our construction, we develop several notions of independent interest. In fact, our paper might be seen as a study of the geometry of chiral backgrounds. It allows for various generalizations, some of which are briefly sketched in the conclusions. These generalizations include in particular twists by 3-cocycles in group cohomology and, possibly, even the case of non-semi simple chiral backgrounds.

2 Dijkgraaf–Witten theory and Drinfel'd double

This section contains a short review of Dijkgraaf–Witten theory as an extended 3D topological field theory, covering the contributions of many authors, including in particular the work of Dijkgraaf–Witten [14], of Freed–Quinn [17] and of Morton [36]. We explain how these extended 3D TFTs give rise to modular tensor categories. These specific modular tensor categories are the representation categories of a well-known class of quantum groups, the Drinfel'd doubles of finite groups.

While this section does not contain original material, we present the ideas in such a way that equivariant generalizations of the theories can be conveniently discussed. In this section, we also introduce some categories and functors that we need for later sections.

2.1 Motivation for Dijkgraaf–Witten theory

We start with a brief motivation for Dijkgraaf–Witten theory from physical principles. A reader already familiar with Dijkgraaf–Witten theory might wish to take at least notice of the Definition 2.2 and of Proposition 2.3.

It is an old, yet successful idea to extract invariants of manifolds from quantum field theories, in particular from quantum field theories for which the fields are G-bundles with connection, where G is some group. In this paper we mostly consider the case of a finite group and only occasionally make reference to the case of a compact Lie group.

Let M be a compact oriented manifold of dimension 1, 2 or 3, possibly with boundary. As the *'space' of field configurations*, we choose G bundles with connection,

$$\mathcal{A}_G(M) := \mathcal{B}un_G^{\nabla}(M).$$

In this way, we really assign to a manifold a groupoid, rather than an actual space. The morphisms of the category take gauge transformations into account. We will nevertheless keep on calling it 'space' since the correct framework to handle $\mathcal{A}_G(M)$ is as a stack on the category of smooth manifolds.

Moreover, another piece of data specifying the model is a function defined on manifolds of a specific dimension,

$$S : \mathcal{A}_G(M) \to \mathbb{C}$$

called the *action*. In the simplest case, when G is a finite group, a field configuration is given by a G-bundle, since all bundles are canonically flat and no connection data are involved. Then, the simplest action is given by

$S[P] := 0$ for all P. In the case of a compact, simple, simply connected Lie group G, consider a 3-manifold M. In this situation, each G-bundle P over M is globally of the form $P \cong G \times M$, because $\pi_1(G) = \pi_2(G) = 0$. Hence, a field configuration is given by a connection on the trivial bundle, which is a 1-form $A \in \Omega^1(M, \mathfrak{g})$ with values in the Lie algebra of G. An example of an action yielding a topological field theory that can be defined in this situation is the Chern–Simons action

$$S[A] := \int_M \langle A \wedge dA \rangle - \frac{1}{6} \langle A \wedge A \wedge A \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the basic invariant inner product on the Lie algebra $\mathfrak{g}$.

The heuristic idea is then to introduce an invariant $Z(M)$ for a 3-manifold M by integration over all field configurations:

$$Z(M) := \text{“} \int_{\mathcal{A}_G(M)} d\phi \, \mathrm{e}^{\mathrm{i}S[\phi]} \text{ ”}.$$

Warning 2.1. *In general, this path integral has only a heuristic meaning. In the case of a finite group, however, one can choose a counting measure $d\phi$ and thereby reduce the integral to a well-defined finite sum. The definition of Dijkgraaf–Witten theory [14] is based on this idea.*

Instead of giving a well-defined meaning to the invariant $Z(M)$ as a path-integral, we exhibit some formal properties these invariants are expected to satisfy. To this end, it is crucial to allow for manifolds that are not closed, as well. This allows us to cut a three-manifold into several simpler three-manifolds with boundaries so that the computation of the invariant can be reduced to the computation of the invariants of simpler pieces.

Hence, we consider a three-manifold M with a 2D boundary ∂M. We fix boundary values $\phi_1 \in \mathcal{A}_G(\partial M)$ and consider the space $\mathcal{A}_G(M, \phi_1)$ of all fields ϕ on M that restrict to the given boundary values ϕ_1. We then introduce, again at a heuristic level, the quantity

$$Z(M)_{\phi_1} := \text{“} \int_{\mathcal{A}_G(M, \phi_1)} d\phi \, \mathrm{e}^{\mathrm{i}S[\phi]} \text{ ”}. \tag{2.1}$$

The assignment $\phi_1 \mapsto Z(M)_{\phi_1}$ could be called a ‘wave function’ on the space $\mathcal{A}_G(\partial M)$ of boundary values of fields. These ‘wave functions’ form a vector space $\mathcal{H}_{\partial M}$, the *state space*

$$\mathcal{H}_{\partial M} := \text{“}L^2\big(\mathcal{A}_G(\partial M), \mathbb{C}\big)\text{”}$$

that we assign to the boundary ∂M. The transition to wave functions amounts to a linearization. The notation L^2 should be taken with a grain of salt and should indicate the choice of an appropriate vector space for the category $\mathcal{A}_G(\partial M)$; it should not suggest the existence of any distinguished measure on the category.

In the case of Dijkgraaf–Witten theory based on a finite group G, the space of states has a basis consisting of δ-functions on the set of isomorphism classes of field configurations on the boundary ∂M:

$$\mathcal{H}_{\partial M} = \mathbb{C}\big\langle \delta_{\phi_1} \mid \phi_1 \in Iso\mathcal{A}_G(\partial M)\big\rangle.$$

In this way, we associate finite-dimensional vector spaces $\mathcal{H}_\Sigma$ to compact oriented 2-manifolds Σ. The heuristic path integral in (2.1) suggests to associate to a 3-manifold M with boundary ∂M an element

$$Z(M) \in \mathcal{H}_{\partial M},$$

or, equivalently, a linear map $\mathbb{C} \to \mathcal{H}_{\partial M}$.

A natural generalization of this situation are cobordisms $M : \Sigma \to \Sigma'$, where Σ and Σ' are compact oriented two-manifolds. A cobordism is a compact oriented three-manifold M with boundary $\partial M \cong \bar{\Sigma} \sqcup \Sigma'$ where $\bar{\Sigma}$ denotes Σ, with the opposite orientation. To a cobordism, we wish to associate a linear map

$$Z(M) : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$$

by giving its matrix elements in terms of the path integral

$$Z(M)_{\phi_0,\phi_1} := \text{“}\int_{\mathcal{A}_G(M,\phi_0,\phi_1)} d\phi \; \mathrm{e}^{\mathrm{i}S[\phi]}\text{”}$$

with fixed boundary values $\phi_0 \in \mathcal{A}_G(\Sigma)$ and $\phi_1 \in \mathcal{A}_G(\Sigma')$. Here $\mathcal{A}_G(M, \phi_0, \phi_1)$ is the space of field configurations on M that restrict to the field configuration ϕ_0 on the ingoing boundary Σ and to the field configuration ϕ_1 on the outgoing boundary Σ'. One can now show that the linear maps $Z(M)$ are compatible with gluing of cobordisms along boundaries. (If the group G is not finite, additional subtleties arise; e.g., $Z(M)_{\phi_0,\phi_1}$ has to be interpreted as an integral kernel.)

Atiyah [1] has given a definition of a topological field theory that formalizes these properties: it describes a topological field theory as a symmetric monoidal functor from a geometric tensor category to an algebraic category. To make this definition explicit, let $\mathfrak{C}\mathrm{ob}(2,3)$ be the category which has 2D compact oriented smooth manifolds as objects. Its morphisms $M : \Sigma \to \Sigma'$ are given by (orientation preserving) diffeomorphism classes of 3D, compact oriented cobordism from Σ to Σ' which we write as

$$\Sigma \hookrightarrow M \hookleftarrow \Sigma'.$$

Composition of morphisms is given by gluing cobordisms together along the boundary. The disjoint union of 2D manifolds and cobordisms equips this category with the structure of a symmetric monoidal category. For the algebraic category, we choose the symmetric tensor category $\mathrm{Vect}_{\mathbb{K}}$ of finite-dimensional vector spaces over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Definition 2.2 (Atiyah)**.** A *3D TFT* is a symmetric monoidal functor

$$Z : \mathfrak{C}\mathrm{ob}(2,3) \to \mathrm{Vect}_{\mathbb{K}}.$$

Let us set up such a functor for Dijkgraaf–Witten theory, i.e., fix a finite group G and choose the trivial action $S : A_G(M) \to \mathbb{C}$, i.e., $S[P] = 0$ for all G-bundles P on M. Then the path integrals reduce to finite sums over 1 hence simply count the number of elements in the category $\mathcal{A}_G$. Since we are counting objects in a category, the stabilizers have to be taken appropriately into account, for details see e.g., [34, Section 4]. This is achieved by the *groupoid cardinality* (which is sometimes also called the Euler-characteristic of the groupoid Γ)

$$|\Gamma| := \sum_{[g]\in Iso(\Gamma)} \frac{1}{|\mathrm{Aut}(g)|}.$$

A detailed discussion of groupoid cardinality can be found in [5,30].

We summarize the discussion:

Proposition 2.3 ([14,17])**.** *Given a finite group G, the following assignment Z_G defines a 3D TFT: to a closed, oriented 2-manifold Σ, we assign the vector space freely generated by the isomorphism classes of G-bundles on Σ,*

$$\Sigma \quad \longmapsto \quad \mathcal{H}_\Sigma := \mathbb{K}\big\langle \delta_P \mid P \in Iso\mathcal{A}_G(\Sigma)\big\rangle.$$

To a 3D cobordism M, we associate the linear map

$$Z_G\Big(\Sigma \hookrightarrow M \hookleftarrow \Sigma'\Big): \quad \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$$

with matrix elements given by the groupoid cardinality of the categories $\mathcal{A}_G(M, P_0, P_1)$:

$$Z_G(M)_{P_0,P_1} := \big|\mathcal{A}_G(M, P_0, P_1)\big|.$$

Remark 2.4. 1) In the original paper [14], a generalization of the trivial action $S[P] = 0$, induced by an element η in the group cohomology $H^3_{Gp}\big(G, \mathrm{U}(1)\big)$ with values in U(1), has been studied. We postpone the treatment of this generalization to a separate paper: in the present paper, the term *Dijkgraaf–Witten theory* refers to the 3D TFT of Proposition 2.3 or its extended version.

2) In the case of a compact, simple, simply-connected Lie group G, a definition of a 3D TFT by a path integral is not available. Instead, the combinatorial definition of Reshetikin–Turaev [39] can be used to set up a 3D TFT, which has the properties expected for Chern–Simons theory.

3) The vector spaces $\mathcal{H}_\Sigma$ can be described rather explicitly. Since every compact, closed, oriented 2-manifold is given by a disjoint union of surfaces Σ_g of genus g, it suffices to compute the dimension of $\mathcal{H}_{\Sigma_g}$. This can be done using the well-known description of moduli spaces of flat G-bundles in terms of homomorphisms from the fundamental group $\pi_1(\Sigma_g)$ to the group G, modulo conjugation,

$$Iso\mathcal{A}_G(\Sigma_g) \cong \mathrm{Hom}(\pi_1(\Sigma_g), G)/G,$$

which can be combined with the usual description of the fundamental group $\pi_1(\Sigma_g)$ in terms of generators and relations. In this way, one finds that the space is 1D for surfaces of genus 0. In the case of surfaces of genus 1, it is generated by pairs of commuting group elements, modulo simultaneous conjugation.

4) Following the same line of argument, one can show that for a closed 3-manifold M, one has

$$\big|\mathcal{A}_G(M)\big| = \big|\mathrm{Hom}(\pi_1(M), G)\big| \ / \ |G|.$$

This expresses the three-manifold invariants in terms of the fundamental group of M.

2.2 Dijkgraaf–Witten theory as an extended TFT

Up to this point, we have considered a version of Dijkgraaf–Witten theory, which assigns invariants to closed three-manifolds $Z(M)$ and vector spaces to 2D manifolds Σ. Iterating the argument that has lead us to consider three-manifolds with boundaries, we might wish to cut the two-manifolds into smaller pieces as well, and thereby introduce two-manifolds with boundaries into the picture.

Hence, we drop the requirement on the two-manifold Σ to be closed and allow Σ to be a compact, oriented two-manifold with 1D boundary $\partial\Sigma$. Given a field configuration $\phi_1 \in \mathcal{A}_G(\partial\Sigma)$ on the boundary of the surface Σ, we consider the space of all field configurations $\mathcal{A}_G(\Sigma, \phi_1)$ on Σ that restrict to the given field configuration ϕ_1 on the boundary $\partial\Sigma$. Again, we linearize the situation and consider for each field configuration ϕ_1 on the 1D boundary $\partial\Sigma$ the vector space freely generated by the isomorphism classes of field configurations on Σ,

$$\mathcal{H}_{\Sigma,\phi_1} := \text{“}L^2\big(\mathcal{A}_G(\Sigma, \phi_1)\big)\text{”} = \mathbb{C}\big\langle \delta_\phi \mid \phi \in Iso\mathcal{A}_G(\Sigma, \phi_1)\big\rangle.$$

The object we associate to the 1D boundary $\partial\Sigma$ of a two-manifold Σ is thus a map $\phi_1 \mapsto \mathcal{H}_{\Sigma,\phi_1}$ of field configurations to vector spaces, i.e., a complex vector bundle over the space of all fields on the boundary. In the case of a finite group G, we prefer to see these vector bundles as objects of the functor category from the essentially small category $\mathcal{A}_G(\partial\Sigma)$ to the category $\mathrm{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces, i.e., as an element of

$$\mathrm{Vect}(\mathcal{A}_G(\partial\Sigma)) = \big[\mathcal{A}_G(\partial\Sigma), \mathrm{Vect}_{\mathbb{C}}\big].$$

Thus the extended version of the theory assigns the category $Z(S) = [\mathcal{A}_G(S), \mathrm{Vect}_{\mathbb{C}}]$ to a 1D, compact oriented manifold S. These categories possess certain additional properties, which can be summarized by saying that they are 2-vector spaces in the sense of [28]:

Definition 2.5.
1) A *2-vector space* (over a field $\mathbb{K}$) is a $\mathbb{K}$-linear, abelian, finitely semi-simple category. Here finitely semi-simple means that the category has finitely many isomorphism classes of simple objects and each object is a finite direct sum of simple objects.
2) Morphisms between 2-vector spaces are $\mathbb{K}$-linear functors and 2-morphisms are natural transformations. We denote the 2-category of 2-vector spaces by $2\mathrm{Vect}_{\mathbb{K}}$
3) The *Deligne tensor product* $\boxtimes$ endows $2\mathrm{Vect}_{\mathbb{K}}$ with the structure of a symmetric monoidal 2-category.

For the Deligne tensor product, we refer to [11, Section 5] or [6, Definition 1.1.15]. The definition and the properties of symmetric monoidal bicategories (resp. 2-categories) can be found in [42, Chapter 3].

In the spirit of Definition 2.2, we formalize the properties of the extended theory Z by describing it as a functor from a cobordism 2-category to the algebraic category $2\mathrm{Vect}_{\mathbb{K}}$. It remains to state the formal definition of the relevant geometric category. Here, we ought to be a little bit more careful, since we expect a 2-category and hence can not identify diffeomorphic two-manifolds. For precise statements on how to address the difficulties in gluing smooth manifolds with corners, we refer to [35, 4.3]; here, we restrict ourselves to the following short definition:

Definition 2.6. $\mathfrak{Cob}(1,2,3)$ is the following symmetric monoidal bicategory:

- Objects are compact, closed, oriented one-manifolds S.
- 1-Morphisms are 2D, compact, oriented collared cobordisms $S \times I \hookrightarrow \Sigma \hookleftarrow S' \times I$.
- 2-Morphisms are generated by diffeomorphisms of cobordisms fixing the collar and 3D collared, oriented cobordisms with corners M, up to diffeomorphisms preserving the orientation and boundary.
- Composition is by gluing along collars.
- The monoidal structure is given by disjoint union with the empty set $\emptyset$ as the monoidal unit.

Remark 2.7. The 1-morphisms are defined as collared surfaces, since in the case of extended cobordism categories, we consider surfaces rather than diffeomorphism classes of surfaces. A choice of collar is always possible, but not unique. The choice of collars ensures that the glued surface has a well-defined smooth structure. Different choices for the collars yield equivalent 1-morphisms in $\mathfrak{Cob}(1,2,3)$.

Obviously, extended cobordism categories can be defined in dimensions different from three as well. We are now ready to give the definition of an extended TFT, which goes essentially back to Lawrence [29]:

Definition 2.8. An *extended 3D TFT* is a weak symmetric monoidal 2-functor

$$Z : \mathfrak{Cob}(1,2,3) \to 2\mathrm{Vect}_{\mathbb{K}}.$$

We pause to explain in which sense extended TFTs extend the TFTs defined in Definition 2.2. To this end, we note that the monoidal 2-functor Z has to send the monoidal unit in $\mathfrak{Cob}(1,2,3)$ to the monoidal unit in $2\mathrm{Vect}_{\mathbb{K}}$. The monoidal unit in $\mathfrak{Cob}(1,2,3)$ is the empty set $\emptyset$, and the unit in $2\mathrm{Vect}_{\mathbb{K}}$ is the category $\mathrm{Vect}_{\mathbb{K}}$. The functor Z restricts to a functor $Z|_{\emptyset}$ from the endomorphisms of $\emptyset$ in $\mathfrak{Cob}(1,2,3)$ to the endomorphisms of $\mathrm{Vect}_{\mathbb{K}}$ in $2\mathrm{Vect}_{\mathbb{K}}$. It follows directly from the definition that $\mathrm{End}_{\mathfrak{Cob}(1,2,3)}(\emptyset) \cong \mathfrak{Cob}(2,3)$. Using the fact that the morphisms in $2\mathrm{Vect}_{\mathbb{K}}$ are additive (which follows from $\mathbb{C}$-linearity of functors in the definition of 2-vector spaces), it is also easy to see that the equivalence of categories $\mathrm{End}_{2\mathrm{Vect}_{\mathbb{K}}}(\mathrm{Vect}_{\mathbb{K}}) \cong \mathrm{Vect}_{\mathbb{K}}$ holds. Hence we have deduced:

Lemma 2.9. *Let Z be an extended 3D TFT. Then $Z|_{\emptyset}$ is a 3D TFT in the sense of Definition* 2.2.

At this point, the question arises whether a given (non-extended) 3d TFT can be extended. In general, there is no reason for this to be true. For Dijkgraaf–Witten theory, however, such an extension can be constructed based on ideas which we described at the beginning of this section. A very conceptual presentation of this this construction based on important ideas of [17, 19] can be found in [36]. Before we describe this construction in more detail in Section 2.3, we first state the result:

Proposition 2.10 ([36])**.** *Given a finite group G, there exists an extended 3D TFT Z_G which assigns the categories*

$$\left[\mathcal{A}_G(S), \mathrm{Vect}_{\mathbb{K}}\right]$$

to 1D, closed oriented manifolds S and whose restriction $Z_G|_{\emptyset}$ is (isomorphic to) the Dijkgraaf–Witten TFT described in Proposition 2.3.

Remark 2.11. One can iterate the procedure of extension and introduce the notion of a fully extended TFT, which also assigns quantities to points rather than just 1-manifolds. It can be shown that Dijkgraaf–Witten theory can be turned into a fully extended TFT, see [16]. The full extension will not be needed in the present article.

2.3 Construction via 2-linearization

In this subsection, we describe in detail the construction of the extended 3d TFT of Proposition 2.10. An impatient reader may skip this subsection and should still be able to understand most of the paper. He might, however, wish to take notice of the technique of 2-linearization in Proposition 2.14, which is also an essential ingredient in our construction of equivariant Dijkgraaf–Witten theory in sequel of this paper.

As emphasized in particular by Morton [36], the construction of the extended TFT is naturally split into two steps, which have already been implicitly present in preceding sections. The first step is to assign to manifolds and cobordisms the configuration spaces $\mathcal{A}_G$ of G bundles. We now restrict ourselves to the case when G is a finite group. The following fact is standard:

- The assignment $M \mapsto \mathcal{A}_G(M) := \mathcal{B}un_G$ is a contravariant 2-functor from the category of manifolds to the 2-category of groupoids. Smooth maps between manifolds are mapped to the corresponding pullback functors on categories of bundles.

A few comments are in order: for a connected manifold M, the category $\mathcal{A}_G(M)$ can be replaced by the equivalent category given by the action groupoid $\mathrm{Hom}\big(\pi_1(M), G\big)//G$ where G acts by conjugation. In particular, the category $\mathcal{A}_G(M)$ is essentially finite, if M is compact. It should be appreciated that at this stage no restriction is imposed on the dimension of the manifold M.

The functor $\mathcal{A}_G(-)$ can be evaluated on a 2D cobordism $S \hookrightarrow \Sigma \hookleftarrow S'$ or a 3D cobordism $\Sigma \hookrightarrow M \hookleftarrow \Sigma'$. It then yields diagrams of the form

$$\mathcal{A}_G(S) \longleftarrow \mathcal{A}_G(\Sigma) \longrightarrow \mathcal{A}_G(S'),$$
$$\mathcal{A}_G(\Sigma) \longleftarrow \mathcal{A}_G(M) \longrightarrow \mathcal{A}_G(\Sigma').$$

Such diagrams are called spans. They are the morphisms of a symmetric monoidal bicategory $\mathfrak{Span}$ of spans of groupoids as follows (see e.g., [12,35]):

- Objects are (essentially finite) groupoids.
- Morphisms are spans of essentially finite groupoids.
- 2-Morphisms are isomorphism classes of spans of span-maps.
- Composition is given by forming weak fibre products.
- The monoidal structure is given by the cartesian product $\times$ of groupoids.

Proposition 2.12 ([36]). *$\mathcal{A}_G$ induces a symmetric monoidal 2-functor*

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}(1,2,3) \to \mathfrak{Span}.$$

This functor assigns to a 1D manifold S the groupoid $\mathcal{A}_G(S)$, to a 2D cobordism $S \hookrightarrow \Sigma \hookleftarrow S'$ the span $\mathcal{A}_G(S) \longleftarrow \mathcal{A}_G(\Sigma) \longrightarrow \mathcal{A}_G(S')$ and to a 3-cobordism with corners a span of span-maps.

Proof. It only remains to be shown that composition of morphisms and the monoidal structure is respected. The first assertion is shown in [36, Theorem 2] and the second assertion follows immediately from the fact that bundles over disjoint unions are given by pairs of bundles over the components, i.e., $\mathcal{A}_G(M \sqcup M') = \mathcal{A}_G(M) \times \mathcal{A}_G(M')$. □

The second step in the construction of extended Dijkgraaf–Witten theory is the 2-linearization of [34]. As we have explained in Section 2.1, the idea is to associate to a groupoid Γ its category of vector bundles $\mathrm{Vect}_{\mathbb{K}}(\Gamma)$. If Γ is essentially finite, the category of vector bundles is conveniently defined as the functor category $\big[\Gamma, \mathrm{Vect}_{\mathbb{K}}\big]$. If $\mathbb{K}$ is algebraically closed of characteristic zero, this category is a 2-vector space, see [34, Lemma 4.1.1].

- The assignment $\Gamma \mapsto \mathrm{Vect}_{\mathbb{K}}\big(\Gamma\big) := \big[\Gamma, \mathrm{Vect}_{\mathbb{K}}\big]$ is a contravariant 2-functor from the bicategory of (essentially finite) groupoids to the 2-category of 2-vector spaces. Functors between groupoids are sent to pullback functors.

We next need to explain what 2-linearization assigns to spans of groupoids. To this end, we use the following lemma due to [34, 4.2.1]:

Lemma 2.13. *Let $f : \Gamma \to \Gamma'$ be a functor between essentially finite groupoids. Then the pullback functor $f^* : \mathrm{Vect}\big(\Gamma'\big) \to \mathrm{Vect}\big(\Gamma\big)$ admits a 2-sided adjoint $f_* : \mathrm{Vect}\big(\Gamma\big) \to \mathrm{Vect}\big(\Gamma'\big)$, called the pushforward.*

Two-sided adjoints are also called 'ambidextrous' adjoint, see [3, ch. 5] for a discussion. We use this pushforward to associate to a span

$$\Gamma \xleftarrow{p_0} \Lambda \xrightarrow{p_1} \Gamma'$$

of (essentially finite) groupoids the *'pull-push'-functor*

$$(p_1)_* \circ (p_0)^* : \quad \mathrm{Vect}_{\mathbb{K}}\big(\Gamma\big) \longrightarrow \mathrm{Vect}_{\mathbb{K}}\big(\Gamma'\big).$$

A similar construction [34] associates to spans of span-morphisms a natural transformation. Altogether we have:

Proposition 2.14 ([34]). *The functor $\Gamma \mapsto \mathrm{Vect}_{\mathbb{K}}(\Gamma)$ can be extended to a symmetric monoidal 2-functor on the category of spans of groupoids*

$$\widetilde{\mathcal{V}_{\mathbb{K}}} : \mathfrak{S}\mathrm{pan} \to 2\mathrm{Vect}_{\mathbb{K}}.$$

This 2-functor is called 2-linearization*.*

Proof. The proof that $\widetilde{\mathcal{V}_{\mathbb{K}}}$ is a 2-functor is in [34]. The fact that $\widetilde{\mathcal{V}_{\mathbb{K}}}$ is monoidal follows from the fact that $\mathrm{Vect}_{\mathbb{K}}(\Gamma \times \Gamma') \cong \mathrm{Vect}_{\mathbb{K}}(\Gamma) \boxtimes \mathrm{Vect}_{\mathbb{K}}(\Gamma')$ for a product $\Gamma \times \Gamma'$ of essentially finite groupoids. □

Arguments similar to the ones in [12, Proposition 1.10] which are based on the universal property of the span category can be used to show that such an extension is essentially unique.

We are now in a position to give the functor Z_G described in Proposition 2.10 which is Dijkgraaf–Witten theory as an extended 3d TFT as the composition of functors

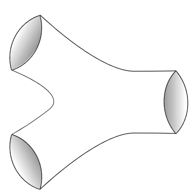
$$Z_G := \widetilde{\mathcal{V}_{\mathbb{K}}} \circ \widetilde{\mathcal{A}_G} : \mathfrak{Cob}(1,2,3) \longrightarrow 2\mathrm{Vect}_{\mathbb{K}}.$$

It follows from Propositions 2.12 and 2.14 that Z_G is an extended 3d TFT in the sense of Definition 2.8. For the proof of Proposition 2.10, it remains to be shown that $Z_G|_{\emptyset}$ is the Dijkgraaf–Witten 3D TFT from Proposition 2.3; this follows from a calculation which can be found in [36, Section 5.2].

2.4 Evaluation on the circle

The goal of this subsection is a more detailed discussion of extended Dijkgraaf–Witten theory Z_G as described in Proposition 2.10. Our focus is on the object assigned to the 1-manifold $\mathbb{S}^1$ given by the circle with its standard orientation. We start our discussion by evaluating an arbitrary extended 3D TFT Z as in Definition 2.8 on certain manifolds of different dimensions:

1) To the circle $\mathbb{S}^1$, the extended TFT assigns a $\mathbb{K}$-linear, abelian finitely semisimple category $\mathcal{C}_Z := Z(\mathbb{S}^1)$.
2) To the 2D sphere with three boundary components, two incoming and one outgoing, also known as the *pair of pants*,

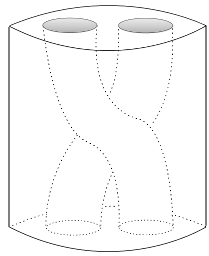


the TFT associates a functor

$$\otimes : \ \mathcal{C}_Z \boxtimes \mathcal{C}_Z \to \mathcal{C}_Z,$$

which turns out to provide a tensor product on the category $\mathcal{C}_Z$.

3) The figure



shows a 2-morphism between two three-punctured spheres, drawn as the upper and lower lid. Both lids are 1-morphisms from two copies of the circle to the circle. The outgoing circle is drawn as the boundary of the big disc. To this cobordism, the TFT associates a natural transformation

$$\otimes \Rightarrow \otimes^{\mathrm{opp}}$$

which turns out to be a braiding.

Moreover, the TFT provides coherence cells, in particular associators and relations between the given structures. This endows the category $\mathcal{C}_Z$ with much additional structure. This structure can be summarized as follows:

Proposition 2.15. *For Z an extended 3D TFT, the category $\mathcal{C}_Z := Z(\mathbb{S}^1)$ is naturally endowed with the structure of a braided tensor category.*

For details, we refer to [9, 18–20]. This is not yet the complete structure that can be extracted: from the braiding-picture above it is intuitively clear that the braiding is not symmetric; in fact, the braiding is 'maximally non-symmetric' in a precise sense that is explained in Definition 2.20. We discuss this in the next section for the category obtained from the Dijkgraaf–Witten extended TFT.

We now specialize to the case of extended Dijkgraaf–Witten TFT Z_G. We first determine the category $\mathcal{C}(G) := \mathcal{C}_Z$; it is by definition

$$\mathcal{C}(G) = \big[\mathcal{A}_G(\mathbb{S}^1), \mathrm{Vect}_{\mathbb{K}}\big].$$

It is a standard result in the theory of coverings that G-covers on $\mathbb{S}^1$ are described by group homomorphisms $\pi_1(\mathbb{S}^1) \to G$ and their morphisms by group elements acting by conjugation. Thus the category $\mathcal{A}_G(\mathbb{S}^1)$ is equivalent to the action groupoid $G//G$ for the conjugation action. As a consequence, we obtain the abelian category $\mathcal{C}(G) \cong [G//G, \mathrm{Vect}_{\mathbb{K}}]$. We spell out this functor category explicitly:

Proposition 2.16. *For the extended Dijkgraaf–Witten 3d TFT, the category $\mathcal{C}(G)$ associated to the circle $\mathbb{S}^1$ is given by the category of G-graded* *vector spaces* $V = \bigoplus_{g\in G} V_g$ *together with a* G*-action on* V *such that for all* $x, y \in G$

$$x.V_g \subset V_{xgx^{-1}}.$$

As a next step we determine the tensor product on $\mathcal{C}(G)$. Since the fundamental group of the pair of pants is the free group on two generators, the relevant category of G-bundles is equivalent to the action groupoid $(G \times G)//G$ where G acts by simultaneous conjugation on the two copies of G. The 2-linearization $\widetilde{\mathcal{V}_{\mathbb{K}}}$ on the span

$$(G//G) \times (G//G) \leftarrow (G \times G)//G \rightarrow G//G.$$

is treated in detail in [36, Remark 5]; the result of this calculation yields the following tensor product:

Proposition 2.17. *The tensor product of* V *and* W *is given by the* G*-graded vector space*

$$(V \otimes W)_g = \bigoplus_{st=g} V_s \otimes W_t$$

together with the G*-action* $g.(v, w) = (gv, gw)$*. The associators are the obvious ones induced by the tensor product in* $\mathrm{Vect}_{\mathbb{K}}$*.*

In the same vein, the braiding can be calculated:

Proposition 2.18. *The braiding* $V \otimes W \rightarrow W \otimes V$ *is for* $v \in V_g$ *and* $w \in W$ *given by*

$$v \otimes w \mapsto gw \otimes v.$$

2.5 Drinfel'd double and modularity

The braided tensor category $\mathcal{C}(G)$ we just computed from the last section has a well-known description as the category of modules over a braided Hopf-algebra $\mathcal{D}(G)$, the Drinfel'd double $\mathcal{D}(G) := \mathcal{D}(\mathbb{K}[G])$ of the group algebra $\mathbb{K}[G]$ of G, see e.g., [24, Chapter 9.4]. The Hopf-algebra $\mathcal{D}(G)$ is defined as follows:

As a vector space, $\mathcal{D}(G)$ is the tensor product $\mathbb{K}(G) \otimes \mathbb{K}[G]$ of the algebra of functions on G and the group algebra of G, i.e., we have the canonical basis $(\delta_g \otimes h)_{g,h\in G}$. The algebra structure can be described as a smash product [33], an analogue of the semi-direct product for groups: in the canonical basis, we have

$$(\delta_g \otimes h)(\delta_{g'} \otimes h') = \begin{cases} \delta_g \otimes hh' \text{ for } g = hg'h^{-1}, \\ 0 \text{ else.} \end{cases}$$

where the unit is given by the tensor product of the two units: $\sum_{g\in G} \delta_g \otimes 1$. The coalgebra structure of $\mathcal{D}(G)$ is given by the tensor product of the coalgebras $\mathbb{K}(G)$ and $\mathbb{K}[G]$, i.e., the coproduct reads

$$\Delta(\delta_g \otimes h) = \sum_{g'g''=g} (\delta_{g'} \otimes h) \otimes (\delta_{g''} \otimes h)$$

and the counit is given by $\epsilon(\delta_1 \otimes h) = 1$ and $\epsilon(\delta_g \otimes h) = 0$ for $g \neq 1$ for all $h \in G$. It can easily be checked that this defines a bialgebra structure on $\mathbb{K}(G) \otimes \mathbb{K}[G]$ and that furthermore the linear map

$$S : (\delta_g \otimes h) \mapsto (\delta_{h^{-1}g^{-1}h} \otimes h^{-1})$$

is an antipode for this bialgebra so that $\mathcal{D}(G)$ is a Hopf algebra. Furthermore, the element

$$R := \sum_{g,h\in G} (\delta_g \otimes 1) \otimes (\delta_h \otimes g) \in \mathcal{D}(G) \otimes \mathcal{D}(G)$$

is a universal R-matrix, which fulfils the defining identities of a braided bialgebra and corresponds to the braiding in Proposition 2.18. At last, the element

$$\theta := \sum_{g\in G} (\delta_g \otimes g^{-1}) \in \mathcal{D}(G)$$

is a ribbon-element in $\mathcal{D}(G)$, which gives $\mathcal{D}(G)$ the structure of a ribbon Hopf-algebra (as defined in [24, Definition 14.6.1]). Comparison with Propositions 2.17 and 2.18 shows

Proposition 2.19. *The category $\mathcal{C}(G)$ is isomorphic, as a braided tensor category, to the category $\mathcal{D}(G)$-*mod.

The category $\mathcal{D}(G)$-mod is actually endowed with more structure than the one of a braided monoidal category. Since $\mathcal{D}(G)$ is a ribbon Hopf-algebra, the category of representations $\mathcal{D}(G)$-mod has also dualities and a compatible twist, i.e., has the structure of a ribbon category (see [24, Proposition 16.6.2] or [6, Definition 2.2.1] for the notion of a ribbon category). Moreover, the category $\mathcal{D}(G)$-mod is a 2-vector space over $\mathbb{K}$ and thus, in particular, finitely semi-simple. We finally make explicit the non-degeneracy condition on the braiding that was mentioned in the last subsection.

Definition 2.20. 1) Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A *premodular tensor category* over $\mathbb{K}$ is a $\mathbb{K}$-linear, abelian, finitely semisimple category $\mathcal{C}$ which has the structure of a ribbon category such that the tensor product is linear in each variable and the tensor unit is absolutely simple, i.e., $\mathrm{End}(\mathbf{1}) = \mathbb{K}$.

2) Denote by $\Lambda_{\mathcal{C}}$ a set of representatives for the isomorphism classes of simple objects. The braiding on $\mathcal{C}$ allows us to define the *S-matrix* with entries in the field $\mathbb{K}$

$$s_{XY} := tr(R_{YX} \circ R_{XY}),$$

where $X, Y \in \Lambda_{\mathcal{C}}$. A premodular category is called *modular*, if the S-matrix is invertible.

In the case of the Drinfel'd double, the S-matrix can be expressed explicitly in terms of characters of finite groups [6, Section 3.2]. Using orthogonality relations, one shows:

Proposition 2.21. *The category* $\mathcal{C}(G) \cong \mathcal{D}(G)$-mod *is modular.*

The notion of a modular tensor category first arose as a formalization of the Moore–Seiberg data of a 2D rational conformal field theory. They are the input for the Turaev–Reshetikhin construction of 3D topological field theories.

3 Equivariant Dijkgraaf–Witten theory

We are now ready to turn to the construction of equivariant generalization of the results of Section 2. We denote again by G a finite group. Equivariance will be with respect to another finite group J that acts on G in a way we will have to explain. As usual, 'twisted sectors' [44] have to be taken into account for a consistent equivariant theory. A description of these twisted sectors in terms of bundles twisted by J-covers is one important result of this section.

3.1 Weak actions and extensions

Our first task is to identify the appropriate definition of a J-action. The first idea that comes to mind — a genuine action of the group J acting on G by group automorphisms — turns out to need a modification. For reasons that will become apparent in a moment, we only require an action up to inner automorphism.

Definition 3.1. 1) A *weak action* of a group J on a group G consists of a collection of group automorphisms $\rho_j : G \to G$, one automorphism for each $j \in J$, and a collection of group elements $c_{i,j} \in G$, one group element for each pair of elements $i, j \in J$. These data are required to obey the relations:

$$\rho_i \circ \rho_j = \mathrm{Inn}_{c_{i,j}} \circ \rho_{ij} \quad \rho_i(c_{j,k}) \cdot c_{i,jk} = c_{i,j} \cdot c_{ij,k} \quad \text{and} \quad c_{1,1} = 1$$

for all $i, j, k \in J$. Here Inn_g denotes the inner automorphism $G \to G$ associated to an element $g \in G$. We will also use the short hand notation ${}^j g := \rho_j(g)$.

2) Two weak actions $(\rho_j, c_{i,j})$ and $(\rho'_j, c'_{i,j})$ of a group J on a group G are called isomorphic, if there is a collection of group elements $h_j \in G$, one group element for each $j \in J$, such that

$$\rho'_j = \mathrm{Inn}_{h_j} \circ \rho_j \quad \text{and } c'_{ij} \cdot h_{ij} = h_i \cdot \rho_i(h_j) \cdot c_{ij}.$$

Remark 3.2. 1) If all group elements $c_{i,j}$ equal the neutral element, $c_{i,j} = 1$, the weak action reduces to a strict action of J on G by group automorphisms.

2) A weak action induces a strict action of J on the group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ of outer automorphisms.

3) In more abstract terms, a weak action amounts to a (weak) 2-group homomorphism $J \to \mathrm{AUT}(G)$. Here $\mathrm{AUT}(G)$ denotes the automorphism 2-group of G. This automorphism 2-group can be described as the monoidal category of endofunctors of the one-object-category with morphisms G. The group J is considered as a discrete 2-group with only identities as morphisms. For more details on 2-groups, we refer to [7].

Weak actions are also known under the name *Dedecker cocycles*, due to the work [10]. The correspondence between weak actions and extensions of groups is also termed *Schreier theory*, with reference to [40]. Let us briefly sketch this correspondence:

- Let $(\rho_j, c_{i,j})$ be a weak action of J on G. On the set $H := G \times J$, we define a multiplication by

$$(g, i) \cdot (g', j) := \big(g \cdot {}^{i}(g') \cdot c_{i,j}\ ,\ ij\big). \tag{3.1}$$

  One can check that this turns H into a group in such a way that the sequence $G \to H \to J$ consisting of the inclusion $g \mapsto (g, 1)$ and the projection $(g, j) \mapsto j$ is exact.
- Conversely, let $G \longrightarrow H \xrightarrow{\pi} J$ be an extension of groups. Choose a set theoretic section $s : J \to H$ of π with $s(1) = 1$. Conjugation with the group element $s(j) \in H$ leaves the normal subgroup G invariant. We thus obtain for $j \in J$ the automorphism $\rho_j(g) := s(j)\ g\ s(j)^{-1}$ of G. Furthermore, the element $c_{i,j} := s(i)s(j)s(ij)^{-1}$ is in the kernel of π and thus actually contained in the normal subgroup G. It is then straightforward to check that $(\rho_j, c_{i,j})$ defines a weak action of J on G.
- Two different set-theoretic sections s and s' of the extension $G \to H \to J$ differ by a map $J \to G$. This map defines an isomorphism of the induced weak actions in the sense of Definition 3.1.2.

We have thus arrived at the

Proposition 3.3 (Dedecker, Schreier)**.** *There is a 1-1 correspondence between isomorphism classes of weak actions of J on G and isomorphism classes of group extensions $G \to H \to J$.*

Remark 3.4. 1) One can easily turn this statement into an equivalence of categories. Since we do not need such a statement in this paper, we leave a precise formulation to the reader.
2) Under this correspondence, *strict actions* of J on G correspond to *split extensions*. This can be easily seen as follows: given a *split* extension $G \to H \to J$, one can choose the section $J \to H$ as a group homomorphism and thus obtains a *strict* action of J on G. Conversely for a strict action of J on G it is easy to see that the group constructed in (3.1) is a semidirect product and thus the sequence of groups splits. To cover all extensions, we thus really need to consider weak actions.

3.2 Twisted bundles

It is a common lesson from field theory that in an equivariant situation, one has to include "twisted sectors" to obtain a complete theory. Our next task is to construct the parameters labelling twisted sectors for a given weak action of a finite group J on G, with corresponding extension $G \to H \to J$

of groups and chosen set-theoretic section $J \to H$. We will adhere to a two-step procedure as outlined after Proposition 2.14. To this end, we will first construct for any smooth manifold a category of twisted bundles. Then, the linearization functor can be applied to spans of such categories.

We start our discussion of twisted G-bundles with the most familiar case of the circle, $M = \mathbb{S}^1$.

The isomorphism classes of G-bundles on $\mathbb{S}^1$ are in bijection to connected components of the free loop space $\mathcal{L}BG$ of the classifying space BG:

$$Iso\big(\mathcal{A}_G(\mathbb{S}^1)\big) = \mathrm{Hom}_{\mathrm{Ho(Top)}}(\mathbb{S}^1, BG) = \pi_0(\mathcal{L}BG).$$

Given a (weak) action of J on G, one can introduce twisted loop spaces. For any element $j \in J$, we have a group automorphism $j : G \to G$ and thus a homeomorphism $j : BG \to BG$. The j-twisted loop space is then defined to be

$$\mathcal{L}^j BG := \big\{ f : [0,1] \to BG \mid f(0) = j \cdot f(1) \big\}.$$

Our goal is to introduce for every group element $j \in J$ a category $\mathcal{A}_G(\mathbb{S}^1, j)$ of j-twisted G-bundles on $\mathbb{S}^1$ such that

$$Iso\big(\mathcal{A}_G(\mathbb{S}^1, j)\big) = \pi_0(\mathcal{L}^j BG).$$

In the case of the circle $\mathbb{S}^1$, the twist parameter was a group element $j \in J$. A more geometric description uses a family of J-covers P_j over $\mathbb{S}^1$, with $j \in J$. The cover P_j is uniquely determined by its monodromy j for the base point $1 \in \mathbb{S}^1$ and a fixed point in the fibre over 1. A concrete construction of the cover P_j is given by the quotient $P_j := [0,1] \times J/\sim$ where $(0,i) \sim (1, ji)$ for all $i \in J$. In terms of these J-covers, we can write

$$\mathcal{L}^j BG = \big\{ f : P_j \to BG \mid f \text{ is } J\text{-equivariant} \big\}.$$

This description generalizes to an arbitrary smooth manifold M. The natural twist parameter in the general case is a J-cover $P \xrightarrow{J} M$.

Suppose, we have a weak J-action on G and construct the corresponding extension $G \to H \xrightarrow{\pi} J$. The category of bundles we need are H-lifts of the given J-cover:

Definition 3.5. Let J act weakly on G. Let $P \xrightarrow{J} M$ be a J-cover over M.

- A P-twisted G-bundle over M is a pair (Q,φ), consisting of an H-bundle Q over M and a smooth map $\varphi : Q \to P$ over M that is required to obey
$$\varphi(q \cdot h) = \varphi(q) \cdot \pi(h)$$
for all $q \in Q$ and $h \in H$. Put differently, a $P \xrightarrow{J} M$-twisted G-bundle is a lift of the J-cover P reduction along the group homomorphism $\pi : \ H \to J$.
- A morphism of P-twisted bundles (Q,φ) and (Q',φ') is a morphism $f : Q \to Q'$ of H-bundles such that $\varphi' \circ f = \varphi$.
- We denote the category of P-twisted G-bundles by $\mathcal{A}_G\big(P \to M\big)$. For $M = \mathbb{S}^1$, we introduce the abbreviation $\mathcal{A}_G\big(\mathbb{S}^1, j\big) := \mathcal{A}_G\big(P_j \to \mathbb{S}^1\big)$ for the standard covers of the circle.

Remark 3.6. There is an alternative point of view on a P-twisted bundle (Q,φ): the subgroup $G \subset H$ acts on the total space Q in such a way that the map $\varphi : Q \to P$ endows Q with the structure of a G-bundle on P. Both the structure group H of the bundle Q and the bundle P itself carry an action of G; for twisted bundles, an equivariance condition on this action has to be imposed. Unfortunately this equivariance property is relatively involved; therefore, we have opted for the definition in the form given above.

A morphism $f : P \to P'$ of J-covers over the same manifold induces a functor $f_* : \mathcal{A}_G\big(P \to M\big) \to \mathcal{A}_G\big(P' \to M\big)$ by $f_*(Q,\varphi) := (Q, f \circ \varphi)$. Furthermore, for a smooth map $f : M \to N$, we can pull back the twist data $P \to M$ and get a *pullback functor* of twisted G-bundles:
$$f^* : \mathcal{A}_G\big(P \to N\big) \to \mathcal{A}_G\big(f^*P \to M\big)$$
by $f^*(Q,\varphi) = (f^*Q, f^*\varphi)$. Before we discuss more sophisticated properties of twisted bundles, we have to make sure that our definition is consistent with 'untwisted' bundles:

Lemma 3.7. *Let the group J act weakly on the group G. For G-bundles twisted by the trivial J-cover $M \times J \to M$, we have a canonical equivalence of categories*
$$\mathcal{A}_G\big(M \times J \to M\big) \cong \mathcal{A}_G(M).$$

Proof. We have to show that for an element $(Q,\varphi) \in \mathcal{A}_G\big(M \times J \to M\big)$ the H-bundle Q can be reduced to a G-bundle. Such a reduction is the same as a section of the associated fibre bundle $\pi_*(Q) \in \mathcal{B}un_J(M)$ see e.g., [4, Satz 2.14]). Now $\varphi : Q \to M \times J$ induces an isomorphism of J-covers $Q \times_H J \cong (M \times J) \times_H J \cong M \times J$ so that the bundle $Q \times_H J$ is trivial as a J-cover and in particular admits global sections.

Since morphisms of twisted bundles have to commute with these sections, we obtain in that way a functor $\mathcal{A}_G\big(M \times J \to M\big) \to \mathcal{A}_G(M)$. Its inverse is given by extension of G-bundles on M to H-bundles on M. □

We also give a description of twisted bundles using standard covering theory; for an alternative description using Čech-cohomology, we refer to Appendix A.1. We start by recalling the following standard fact from covering theory, see e.g., [21, 1.3] that has already been used to prove Proposition 2.16: for a finite group J, the category of J-covers is equivalent to the action groupoid $\mathrm{Hom}(\pi_1(M), J)//J$. (Note that this equivalence involves choices and is not canonical.)

To give a similar description of twisted bundles, fix a J-cover P. Next, we choose a basepoint $m \in M$ and a point p in the fibre P_m over m. These data determine a unique group morphism $\omega : \pi_1(M, m) \to J$ representing P.

Proposition 3.8. *Let J act weakly on G. Let M be a connected manifold and P be a J-cover over M represented after the choices just indicated by the group homomorphism $\omega : \pi_1(M) \to J$. Then there is a (non-canonical) equivalence of categories*

$$\mathcal{A}_G\big(P \to M\big) \cong \mathrm{Hom}^{\omega}\big(\pi_1(M), H\big)//G$$

where we consider group homomorphisms

$$\mathrm{Hom}^{\omega}\big(\pi_1(M), H\big) := \big\{\mu : \pi_1(M) \to H \mid \pi \circ \mu = \omega\big\}$$

whose composition restricts to the group homomorphism ω describing the J-cover P. The group G acts on $\mathrm{Hom}^{\omega}\big(\pi_1(M), H\big)$ via pointwise conjugation using the inclusion $G \to H$.

Proof. Let $m \in M$ and $p \in P$ over m be the choices of base point in the J-cover $P \to M$ that lead to the homomorphism ω. Consider a $(P \to M)$ twisted bundle $Q \to M$. Since $\varphi :\ Q \to P$ is surjective, we can choose a base point q in the fibre of Q over m such that $\varphi(q) = p$. The group homomorphism $\pi_1(M) \to H$ describing the H-bundle Q is obtained by lifting closed paths in M starting in m to paths in Q starting in q. They are mapped under φ to lifts of the same path to P starting at p, and these lifts are just described by the group homomorphism $\omega : \pi_1(M) \to J$ describing the cover P. If the end point of the path in Q is qh for some $h \in H$, then by the defining property of φ, the lifted path in P has endpoint $\varphi(qh) = \varphi(q)\pi(h) = p\pi(h)$. Thus $\pi \circ \mu = \omega$. □

Remark 3.9. For non-connected manifolds, a description as in Proposition 3.8 can be obtained for every component. Again the equivalence involves choices of base points on M and in the fibres over the base points. This could be fixed by working with pointed manifolds, but pointed manifolds cause problems when we consider cobordisms. Alternatively, we could use the fundamental groupoid instead of the fundamental group, see e.g., [32].

Example 3.10. We now calculate the categories of twisted bundles over certain manifolds using Proposition 3.8.

1) For the circle $\mathbb{S}^1$, $\omega \in \mathrm{Hom}(\pi_1(\mathbb{S}^1), J) = \mathrm{Hom}(\mathbb{Z}, J)$ is determined by an element $j \in J$ and the condition $\pi \circ \mu = \omega$ requires $\mu(1) \in H$ to be in the preimage $H_j := \pi^{-1}(j)$ of j. Thus, we have $\mathcal{A}_G(\mathbb{S}^1, j) \cong H_j//G$.
2) For the 3-Sphere $\mathbb{S}^3$, all twists P and all G-bundles are trivial. Thus, we have $\mathcal{A}_G(P \to \mathbb{S}^3) \cong \mathcal{A}_G(\mathbb{S}^3) \cong pt//G$.

3.3 Equivariant Dijkgraaf–Witten theory

The key idea in the construction of equivariant Dijkgraaf–Witten theory is to take twisted bundles $\mathcal{A}_G(P \to M)$ as the field configurations, taking the place of G-bundles in Section 2. We cannot expect to get then invariants of closed three-manifolds M, but rather invariants of three-manifolds M together with a twist datum, i.e., a J-cover P over M. Analogous statements apply to manifolds with boundary and cobordisms. Therefore we need to introduce extended cobordism-categories as $\mathfrak{Cob}(1,2,3)$ in Definition 2.6, but endowed with the extra datum of a J-cover over each manifold.

Definition 3.11. $\mathfrak{Cob}^J(1,2,3)$ is the following symmetric monoidal bicategory:

- Objects are compact, closed, oriented 1-manifolds S, together with a J-cover $P_S \xrightarrow{J} S$.

- 1-Morphisms are collared cobordisms

$$S \times I \hookrightarrow \Sigma \hookleftarrow S' \times I$$

where Σ is a 2D, compact, oriented cobordism, together with a J-cover $P_\Sigma \to \Sigma$ and isomorphisms

$$P_\Sigma|_{(S\times I)} \xrightarrow{\sim} P_S \times I \quad \text{and} \quad P_\Sigma|_{(S'\times I)} \xrightarrow{\sim} P_{S'} \times I.$$

over the collars.
- 2-Morphisms are generated by
  - orientation preserving diffeomorphisms $\varphi : \Sigma \to \Sigma'$ of cobordisms fixing the collar together with an isomorphism $\widetilde{\varphi} : P_\Sigma \to P_{\Sigma'}$ covering φ.
  - 3D collared, oriented cobordisms with corners M with cover $P_M \to M$ together with covering isomorphisms over the collars (as before) up to diffeomorphisms preserving the orientation and boundary.
- Composition is by gluing cobordisms and covers along collars.
- The monoidal structure is given by disjoint union.

Remark 3.12. In analogy to Remark 2.7, we point out that the isomorphisms of covers are defined over the collars, rather than only over the the boundaries. This endows the glued cover with a well-defined smooth structure.

Definition 3.13. An extended 3d J-TFT is a symmetric monoidal 2-functor

$$Z : \mathfrak{Cob}^J(1,2,3) \to 2\mathrm{Vect}_{\mathbb{K}}.$$

Just for the sake of completeness, we will also give a definition of non-extended J-TFT. Therefore define the symmetric monoidal category $\mathfrak{Cob}^J(2,3)$ to be the endomorphism category of the monoidal unit $\emptyset$ in $\mathfrak{Cob}(1,2,3)$. More concretely, this category has as objects closed, oriented 2-manifolds with J-cover and as morphisms J-cobordisms between them.

Definition 3.14. A (non-extended) 3D J-TFT is a symmetric monoidal 2-functor

$$\mathfrak{Cob}^J(2,3) \to \mathrm{Vect}_{\mathbb{K}}.$$

Similarly as in the non-equivariant case (lemma 2.9), we get

Lemma 3.15. *Let Z be an extended 3d J-TFT. Then $Z|_\emptyset$ is a (non-extended) 3D J-TFT.*

Now we can state the main result of this section:

Theorem 3.16. *For a finite group G and a weak J-action on G, there is an extended 3D J-TFT called Z_G^J which assigns the categories*

$$\mathrm{Vect}_{\mathbb{K}}\big(\mathcal{A}_G(P \to S)\big) = \big[\mathcal{A}_G(P \to S), \mathrm{Vect}_{\mathbb{K}}\big]$$

to 1D, closed oriented manifolds S with J-cover $P \to S$.

We will give a proof of this theorem in the next sections. Having twisted bundles at our disposal, the main ingredient will again be the 2-linearization described in Section 2.3.

3.4 Construction via spans

As in the case of ordinary Dijkgraaf–Witten theory, cf. Section 2.3, equivariant Dijkgraaf–Witten Z_G^J theory is constructed as the composition of the symmetric monoidal 2-functors

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \to \mathfrak{Span} \quad \text{and} \quad \widetilde{\mathcal{V}_{\mathbb{K}}} : \mathfrak{Span} \to 2\mathrm{Vect}_{\mathbb{K}}.$$

The second functor will be exactly the 2-linearization functor of Proposition 2.14. Hence we can limit our discussion to the construction of the first functor $\widetilde{\mathcal{A}_G}$. As it will turn out, our definition of twisted bundles is set up precisely in such a way that the construction of the corresponding functor in Proposition 2.12 can be generalized.

Our starting point is the following observation:

- The assignment $(P_M \xrightarrow{J} M) \longmapsto \mathcal{A}_G(P_M \xrightarrow{J} M)$ of twisted bundles to a twist datum $P_M \to M$ constitutes a contravariant 2-functor from the category of manifolds with J-cover to the 2-category of groupoids. Maps between manifolds with cover are mapped to the corresponding pullback functors of bundles.

From this functor that is defined on manifolds of any dimension, we construct a functor $\widetilde{\mathcal{A}_G}$ on J-cobordisms with values in the 2-category $\mathfrak{Span}$ of spans of groupoids, where the category $\mathfrak{Span}$ is defined in Section 2.3. To an object in $\mathfrak{Cob}^J(1,2,3)$, i.e., to a J-cover $P_S \to M$, we assign the category

$\mathcal{A}_G(P_S \to S)$ of J-covers. To a 1-morphism $P_S \hookrightarrow P_\Sigma \hookleftarrow P'_S$ in $\mathfrak{Cob}^J(1,2,3)$, we associate the span

$$\mathcal{A}_G(P_S \to S) \leftarrow \mathcal{A}_G(P_\Sigma \to \Sigma) \to \mathcal{A}_G(P_{S'} \to S') \tag{3.2}$$

and to a 2-morphism of the type $P_\Sigma \hookrightarrow P_M \hookleftarrow P_{\Sigma'}$ the span

$$\mathcal{A}_G(P_\Sigma \to \Sigma) \leftarrow \mathcal{A}_G(P_M \to M) \to \mathcal{A}_G(P_{\Sigma'} \to \Sigma'). \tag{3.3}$$

We have to show that this defines a symmetric monoidal functor $\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \to \mathfrak{Span}$.

In particular, we have to show that the composition of morphisms is respected.

Lemma 3.17. *Let $P_\Sigma \to \Sigma$ and $P_{\Sigma'} \to \Sigma'$ be two 1-morphisms in $\mathfrak{Cob}^J$ $(1,2,3)$ which can be composed at the object $P_S \to S$ to get the 1-morphism*

$$P_\Sigma \circ P_{\Sigma'} := \big(P_\Sigma \sqcup_{P_S \times I} P_{\Sigma'} \to \Sigma \sqcup_{S\times I} \Sigma'\big),$$

where $I = [0,1]$ is the standard interval. (Recall that we are gluing over collars.) Then the category $\mathcal{A}_G\big(P_\Sigma \circ P_{\Sigma'}\big)$ is the weak pullback of $\mathcal{A}_G(P_\Sigma \to \Sigma)$ and $\mathcal{A}_G(P_{\Sigma'} \to \Sigma')$ over $\mathcal{A}_G(P_S \to S)$.

Proof. By definition the category

$$\mathcal{A}_G\big(P_\Sigma \circ P_{\Sigma'}\big)$$

has as objects twisted G-bundles over the two-manifold $\Sigma \sqcup_{S\times I} \Sigma' =: N$. The manifold N admits an open covering $N = U_0 \cup U_1$ with $U_0 = \Sigma \setminus S$ and $U_1 = \Sigma' \setminus S$ where the intersection is the cylinder $U_0 \cap U_1 = S \times (0,1)$. By construction, the restrictions of the glued bundle $P_N \to N$ to U_0 and U_1 are given by $P_\Sigma \setminus P_S$ and $P_{\Sigma'} \setminus P_S$.

The natural inclusions $U_0 \to \Sigma$ and $U_1 \to \Sigma'$ induce equivalences

$$\mathcal{A}_G(P_\Sigma \to \Sigma) \xrightarrow{\sim} \mathcal{A}_G(P_N|_{U_0} \to U_0)$$
$$\mathcal{A}_G(P_{\Sigma'} \to \Sigma') \xrightarrow{\sim} \mathcal{A}_G(P_N|_{U_1} \to U_1)$$

Analogously, we have an equivalence

$$\mathcal{A}_G\big(P_N|_{U_0\cap U_1} \to U_0 \cap U_1\big) \xrightarrow{\sim} \mathcal{A}_G(P_S \to S).$$

At this point, we have reduced the claim to an assertion about descent of twisted bundles which we will prove in Corollary 3.20. This corollary implies that $\mathcal{A}_G(P_N \to N)$ is the weak pullback of $\mathcal{A}_G(P_N|_{U_0} \to U_0)$ and $\mathcal{A}_G(P_N|_{U_1} \to U_1)$ over $\mathcal{A}_G\big(P_N|_{U_0 \cap U_1}\big)$. Since weak pullbacks are invariant under equivalence of groupoids, this shows the claim. $\square$

We now turn to the promised results about descent of twisted bundles. Let $P \to M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M, where for the sake of generality we allow for arbitrary open coverings. We want to show that twisted bundles can be glued together like ordinary bundles; while the precise meaning of this statement is straightforward, we briefly summarize the relevant definitions for the sake of completeness:

Definition 3.18. Let $P \to M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M. The descent category $\mathcal{D}esc(U_\alpha, P)$ has

- Objects: families of $P|_{U_\alpha}$-twisted bundles Q_α over U_α, together with isomorphisms of twisted bundles $\varphi_{\alpha\beta} : Q_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} Q_\beta|_{U_\alpha \cap U_\beta}$ satisfying the cocycle condition $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$.
- Morphisms: families of morphisms $f_\alpha : Q_\alpha \to Q'_\alpha$ of twisted bundles such that over $U_{\alpha\beta}$ we have $\varphi'_{\alpha\beta} \circ (f_\alpha)|_{U_{\alpha\beta}} = (f_\beta)|_{U_{\alpha\beta}} \circ \varphi_{\alpha\beta}$.

Proposition 3.19 (Descent for twisted bundles)**.** *Let $P \to M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M. Then the groupoid $\mathcal{A}_G(P \to M)$ is equivalent to the descent category $\mathcal{D}esc(U_\alpha, P)$.*

Proof. Note that the corresponding statements are true for H-bundles and for J-covers. Then the description in Definition 3.5 of a twisted bundle as an H-bundle together with a morphism of the associated J-cover immediately implies the claim. $\square$

Corollary 3.20. *For an open covering of M by two open sets U_0 and U_1 the category $\mathcal{A}_G(P \to M)$ is the weak pullback of $\mathcal{A}_G(P|_{U_0} \to U_0)$ and $\mathcal{A}_G(P|_{U_1} \to U_1)$ over $\mathcal{A}_G(P|_{U_0 \cap U_1} \to U_0 \cap U_1)$.*

In order to prove that the assignment (3.2) and (3.3) really promotes $\mathcal{A}_G$ to a symmetric monoidal functor $\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \to \mathfrak{Span}$, it remains to show that $\mathcal{A}_G$ preserves the monoidal structure.

Now a bundle over a disjoint union is given by a pair of bundles over each component. Thus, for a disjoint union of J-manifolds $P \to M = (P_1 \sqcup P_2) \to (M_1 \sqcup M_2)$, we have $\mathcal{A}_G(P \to M) \cong \mathcal{A}_G(P_1 \to M_1) \times \mathcal{A}_G(P_2 \to M_2)$. Note that the manifolds M, M_1 and M_2 can also be cobordisms. The isomorphism of categories is clearly associative and preserves the symmetric structure. Together with Lemma 3.17, this proves the next proposition.

Proposition 3.21. *$\mathcal{A}_G$ induces a symmetric monoidal functor*

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \to \mathfrak{Span}$$

which assigns the spans (3.2) *and* (3.3) *to 2D and 3D cobordisms with J-cover.*

3.5 Twisted sectors and fusion

We next proceed to evaluate the J-equivariant TFT Z_G^J constructed in the last section on the circle, as we did in Section 2.4 for the non-equivariant TFT. We recall from Section 3.2 the fact that over the circle $\mathbb{S}^1$ we have for each $j \in J$ a standard cover P_j. The associated category

$$\mathcal{C}(G)_j := Z_G^J\big(P_j \to \mathbb{S}^1\big)$$

is called the j-twisted sector of the theory; the sector $\mathcal{C}(G)_1$ is called the neutral sector. By Lemma 3.7, we have an equivalence $\mathcal{A}_G(P_1 \to \mathbb{S}^1) \cong \mathcal{A}_G(\mathbb{S}^1)$; hence we get an equivalence of categories $\mathcal{C}(G)_1 \cong \mathcal{C}(G)$, where $\mathcal{C}(G)$ is the category arising in the non-equivariant Dijkgraaf–Witten model, we discussed in Section 2.4. We have already computed the twisted sectors as abelian categories in Example 3.10 and note the result for future reference:

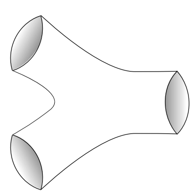
Proposition 3.22. *For the j-twisted sector of equivariant Dijkgraaf–Witten theory, we have an equivalence of abelian categories*

$$\mathcal{C}(G)_j \cong [H_j//G, \mathrm{Vect}_{\mathbb{K}}],$$

where $H_j//G$ is the action groupoid given by the conjugation action of G on $H_j := \pi^{-1}(j)$. More concretely, the category $\mathcal{C}(G)_j$ is equivalent to the category of H_j-graded vector spaces $V = \bigoplus_{h \in H_j} V_h$ together with a G-action on V such that

$$g.V_h \subset V_{ghg^{-1}}.$$

As a next step, we want to make explicit additional structure on the categories $\mathcal{C}(G)_j$ coming from certain cobordisms. Therefore, consider the pair of pants $\Sigma(2,1)$:



The fundamental group of $\Sigma(2,1)$ is the free group on two generators. Thus, given a pair of group elements $j,k \in J$, there is a J-cover $P^{\Sigma(2,1)}_{j,k} \to \Sigma(2,1)$ which restricts to the standard covers P_j and P_k on the two ingoing boundaries and to the standard cover P_{jk} on the outgoing boundary circle. (To find a concrete construction, one should fix a parametrization of the pair of pants $\Sigma(2,1)$.) The cobordism $P^{\Sigma(2,1)}_{j,k}$ is a morphism

$$P^{\Sigma(2,1)}_{j,k} : \big(P_j \to \mathbb{S}^1\big) \sqcup \big(P_k \to \mathbb{S}^1\big) \longrightarrow \big(P_{jk} \to \mathbb{S}^1\big) \tag{3.4}$$

in the category $\mathfrak{Cob}^J(1,2,3)$. Applying the equivariant TFT-functor Z^J_G yields a functor

$$\otimes_{jk} : \quad \mathcal{C}(G)_j \boxtimes \mathcal{C}(G)_k \longrightarrow \mathcal{C}(G)_{jk}.$$

We describe this functor in terms of the equivalent categories of graded vector spaces as a functor

$$H_j//G\text{-mod} \times H_k//G\text{-mod} \to H_{jk}//G\text{-mod}.$$

Proposition 3.23. *For* $V = \bigoplus_{h\in H_j} V_h \in H_j//G\text{-mod}$ *and* $W = \bigoplus W_h \in H_k//G\text{-mod}$ *the product* $V \otimes_{jk} W \in H_{jk}//G\text{-mod}$ *is given by*

$$(V \otimes_{jk} W)_h = \bigoplus_{st=h} V_s \otimes W_t$$

together with the action $g.(v \otimes w) = g.v \otimes g.w$.

Proof. As a first step we have to compute the span $\widetilde{\mathcal{A}_G}(P^{\Sigma(2,1)}_{j,k})$ associated to the cobordism $P^H_{j,k}$. From the description of twisted bundles in Proposition 3.8 and the fact that the fundamental group of $\Sigma(2,1)$ is the free group on two generators, we derive the following equivalence of categories:

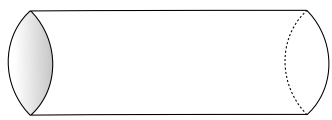
$$\mathcal{A}_G\big(P^{\Sigma(2,1)}_{jk} \to \Sigma(2,1)\big) \cong (H_j \times H_k)//G.$$

Here we have $H_j \times H_k = \{(h,h') \in H \times H \mid \pi(h) = j,\ \pi(h') = k\}$, on which G acts by simultaneous conjugation. This leads to the span of action groupoids

$$H_j//G \times H_k//G \longleftarrow (H_j \times H_k)//G \longrightarrow H_{jk}//G$$

where the left map is given be projection to the factors and the right hand map by multiplication. Applying the 2-linearization functor $\widetilde{\mathcal{V}_{\mathbb{K}}}$ from Proposition 2.14 amounts to computing the corresponding pull–push functor. This yields the result. □

Next, we consider the two-manifold $\Sigma(1,1)$ given by the cylinder over $\mathbb{S}^1$, i.e., $\Sigma(1,1) = \mathbb{S}^1 \times I$:



There exists a cover $P_{j,x}^{\Sigma(1,1)} \to \Sigma(1,1)$ for $j, x \in J$ that restricts to P_j on the ingoing circle and to $P_{xjx^{-1}}$ on the outgoing circle. The simplest way to construct such a cover is to consider the cylinder $P_{xjx^{-1}} \times I \to \mathbb{S}^1 \times I$ and to use the identification of $P_{j,x}^{\Sigma(1,1)}$ over (a collaring neighbourhood of) the outgoing circle by the identity and over the ingoing circle the identification by the morphism $P_{\Sigma(1,1)}|_{\mathbb{S}^1\times 1} = P_j \to P_{x^{-1}jx}$ given by conjugation with x. In this way, we obtain a cobordism that is a 1-morphism

$$P_{j,x}^{\Sigma(1,1)} : (P_j \to \mathbb{S}^1) \longrightarrow (P_{xjx^{-1}} \to \mathbb{S}^1) \tag{3.5}$$

in the category $\mathfrak{Cob}^J(1,2,3)$ and hence induces a functor

$$\phi_x : \mathcal{C}(G)_j \to \mathcal{C}(G)_{xjx^{-1}}.$$

We compute the functor on the equivalent action groupoids explicitly:

Proposition 3.24. *The image under ϕ_x of an object $V = \bigoplus V_h \in H_j//G$-mod is the graded vector space with homogeneous component*

$$\phi_x(V)_h = V_{s(x^{-1})hs(x^{-1})^{-1}}$$

for $h \in H_{xjx^{-1}}$ and with G-action on $v \in V_h$ given by $s(x^{-1})gs\left(x^{-1}\right)^{-1} \cdot v$.

Proof. As before we compute the span $\widetilde{\mathcal{A}_G}(P_{j,x}^{\Sigma(1,1)})$. Using explicitly the equivalence given in the proof of Proposition 3.8, we obtain the span of action groupoids

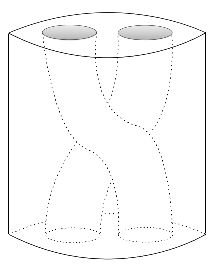
$$H_j//G \leftarrow H_{xjx^{-1}}//G \rightarrow H_{xjx^{-1}}//G$$

where the right-hand map is the identity and the left-hand map is given by

$$(h,g) \mapsto \big(s(x^{-1})hs(x^{-1})^{-1}, s(x^{-1})gs(x^{-1})^{-1}\big).$$

Computing the corresponding pull–push functor, which here in fact only consists of a pullback, shows the claim. $\square$

Finally we come to the structure corresponding to the braiding of Section 2.4. Note that the cobordism that interchanges the two ingoing circles of the pair of pants $\Sigma(2,1)$, as in the following picture,



can also be realized as the diffeomorphism $F : \Sigma(2,1) \to \Sigma(2,1)$ of the pair of pants that rotates the ingoing circles counterclockwise around each other and leaves the outgoing circle fixed. In this picture, we think of the cobordism as the cylinder $\Sigma(2,1) \times I$ where the identification with $\Sigma(2,1)$ on the top is the identity and on the bottom is given by the diffeomorphism F. More explicitly, denote by $\tau : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ the map that interchanges the two copies. We then consider the following diagram in the 2-category $\mathfrak{Cob}(1,2,3)$:

$$\begin{tikzcd}
 & & \Sigma(2,1) \arrow[dd, phantom, "\Downarrow F"] & & \\
\mathbb{S}^1 \times \mathbb{S}^1 \arrow[urr, "\iota"] \arrow[dr, "\tau"'] & & & & \mathbb{S}^1 \arrow[ull] \arrow[dl] \\
 & \mathbb{S}^1 \times \mathbb{S}^1 \arrow[rr] & & \Sigma(2,1) &
\end{tikzcd}$$

where $\iota : \mathbb{S}^1 \times \mathbb{S}^1 \to \Sigma(2,1)$ is the standard inclusion of the two ingoing boundary circles into the trinion $\Sigma(2,1)$.

Our next task is to lift this situation to manifolds with J-covers. On the ingoing trinion, we take the J cover $P_{jk}^{\Sigma(2,1)}$. We denote the symmetry isomorphism in $\mathfrak{Cob}^J(1,2,3)$ by τ as well. Applying the diffeomorphism of the trinion explicitly, one sees that the outgoing trinion will have monodromies jkj^{-1} and j on the ingoing circles. Hence we have to apply a J-cover $P_{j,k}^{\Sigma(1,1)}$ of the cylinder $\Sigma(1,1)$ first to one insertion. The next lemma asserts that then the 2-morphism in $\mathfrak{Cob}^J(1,2,3)$ is fixed:

Lemma 3.25. *In the 2-category* $\mathfrak{Cob}^J(1,2,3)$*, there is a unique 2-morphism*

$$\hat{F} : \quad P_{j,k}^{\Sigma(2,1)} \Longrightarrow \big(P_{jkj^{-1},j}^{\Sigma(2,1)}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P_{j,k}^{\Sigma(1,1)}\big)$$

that covers the 2-morphism F in $\mathfrak{Cob}(1,2,3)$.

Proof. First we show that a morphism $\tilde{F} : P_{jkj^{-1},j}^{\Sigma(2,1)} \to P_{j,k}^{\Sigma(2,1)}$ can be found that covers the diffeomorphism $F : \Sigma(2,1) \to \Sigma(2,1)$. This morphism is most easily described using the action of F on the fundamental group $\pi_1(\Sigma(2,1))$ of the pair of pants. The latter is a free group with two generators which can be chosen as the paths a, b around the two ingoing circles, $\pi_1(\Sigma(2,1)) = \mathbb{Z} * \mathbb{Z} = \langle a, b\rangle$. Then the induced action of F on the generators is $\pi_1(F)(a) = aba^{-1}$ and $\pi_1(F)(b) = a$. Hence, we find on the covers $F^* P_{j,k} \cong P_{jkj^{-1},j}$. This implies that we have a diffeomorphism $\tilde{F} : P_{jkj^{-1},j} \to P_{j,k}$ covering F.

To extend $\tilde{F}$ to a 2-morphism in $\mathfrak{Cob}^J(1,2,3)$, we have to be a bit careful about how we consider the cover $P_{jkj^{-1},j}^{\Sigma(2,1)} \to \Sigma(2,1)$ of the trinion as a 1-morphism. In fact, it has to be considered as a morphism $(P_j \to \mathbb{S}^1) \sqcup (P_k \to \mathbb{S}^1) \longrightarrow (P_{jk} \to \mathbb{S}^1)$ where the ingoing components are first exchanged and then the identification of $P_k \to \mathbb{S}^1$ and $P_{jkj^{-1}} \to \mathbb{S}^1$ via the conjugation isomorphisms $P_{j,k}^{\Sigma(1,1)}$ induced by covers of the cylinders is used first, compare the lower arrows in the preceding commuting diagram. This yields the composition $\big(P_{jkj^{-1},j}^{\Sigma(2,1)}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P_{j,k}^{\Sigma(1,1)}\big)$ on the right hand side of the diagram. $\square$

The next step is to apply the TFT functor Z_G^J to the 2-morphism $\hat{F}$. The target 1-morphism of $\hat{F}$ can be computed using the fact that Z_G^J is a symmetric monoidal 2-functor; we find the following functor $\mathcal{C}(G)_j \otimes \mathcal{C}(G)_k \to \mathcal{C}(G)_{jk}$:

$$Z_G^J\Big(\big(P^{\Sigma(2,1)}_{jkj^{-1},j}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P^{\Sigma(1,1)}_{j,k}\big)\Big) = (-)^j \otimes^{op}_{jkj^{-1},j} (-)$$

We thus have the functor that acts on objects as $(V, W) \mapsto \phi_j(W) \otimes V$ for $V \in \mathcal{C}(G)_j$ and $W \in \mathcal{C}(G)_k$.

Then $c := Z_G^J(\hat{F})$ is a natural transformation $(-) \otimes_{j,k} (-) \Longrightarrow (-)^j \otimes^{op}_{jkj^{-1},j} (-)$ i.e., a family of isomorphisms

$$c_{V,W} : V \otimes_{j,k} W \xrightarrow{\sim} \quad \phi_j(W) \otimes_{jkj^{-1},j} V \tag{3.6}$$

in $\mathcal{C}(G)_{jk}$ for $V \in \mathcal{C}(G)_j$ and $W \in \mathcal{C}(G)_k$.

We next show how this natural transformation is expressed when we use the equivalent description of the categories $\mathcal{C}(G)_j$ as vector bundles on action groupoids:

Proposition 3.26. *For* $V = \bigoplus V_h \in H_j//G$-mod *and* $W = \bigoplus W_h \in H_k//G$-mod *the natural isomorphism* $c_{V,W} : V \otimes W \to \phi_j(W) \otimes V$ *is given by*

$$v \otimes w \mapsto (s(j^{-1})h).w \otimes v$$

for $v \in V_h$ *with* $h \in H_j$ *and* $w \in W$.

Proof. We first compute the 1-morphism in the category $\mathfrak{S}$pan of spans of finite groupoids that corresponds to the target 1-morphism $\big(P^{\Sigma(2,1)}_{jkj^{-1},j}\big) \circ \tau \circ \big(id \sqcup P^{\Sigma(1,1)}_{j,k}\big)$. From the previous proposition, we obtain the following zigzag diagram:

$$H_j//G \times H_k//G \leftarrow H_{jkj^{-1}}//G \times H_j//G \leftarrow (H_{jkj^{-1}} \times H_k)//G \to H_{jk}//G.$$

The first morphism is given by the morphisms implementing the J-action that has been computed in the proof of Proposition 3.24, composed with the exchange of factors. The second 1-morphism is obtained from the two projections and the last 1-morphism is the product in the group H.

Thus, the 2-morphism $\hat{F}$ from Lemma 3.25 yields a 2-morphism $\hat{F}_G$ in the diagram

$$\begin{tikzcd}
 & & H_j \times H_k//G \arrow[dll] \arrow[dr] \arrow[dd, Rightarrow, "\hat{F}_G"] & \\
H_j//G \times H_k//G & & & H_{jk}//G \\
 & H_{jkj^{-1}}//G \times H_j//G \arrow[ul] & (H_{jkj^{-1}} \times H_j)//G \arrow[l] \arrow[ur] &
\end{tikzcd}$$

where $\hat{F}_G$ is induced by the equivariant map $(h, h') \mapsto (hh'h^{-1}, h)$. Once the situation is presented in this way, one can carry out explicitly the calculation along the lines described in [36, Section 4.3] and obtain the result. $\square$

A similar discussion can in principle be carried out to compute the associators. More generally, structural morphisms on $H//G$-mod can be derived from suitable 3-cobordisms. The relevant computations become rather involved. On the other hand, the category $H//G$-mod also inherits structural morphisms from the underlying category of vector spaces. We will use in the sequel the latter type of structural morphism.

4 Equivariant Drinfel'd double

The goal of this section is to show that the category $\mathcal{C}^J(G) := \bigoplus_{j\in J} \mathcal{C}(G)_j$ comprising the categories we have constructed in Proposition 3.22 has a natural structure of a J-modular category.

Very much like ordinary modularity, J-modularity is a completeness requirement for the relevant tensor category that is suggested by principles of field theory. Indeed, it ensures that one can construct a J-equivariant topological field theory, see [43]. For the definition of J-modularity, we refer to [26, Definition 10.1].

To establish the structure of a modular tensor category on the category found in the previous sections, we realize this category as the representation category of a finite-dimensional algebra, more precisely of a J-Hopf algebra. This section is organized as follows: we first recall the notions of equivariant ribbon categories and of equivariant ribbon algebras, taking into account a suitable form of weak actions. In Section 4.3, we then present the appropriate generalization of the Drinfel'd double that describes the category $\mathcal{C}^J(G)$. We then describe its orbifold category as the category of representations of a braided Hopf algebra, which allows us to establish the modularity of the orbifold category. We then apply a result of [26] to deduce that the structure with which we have endowed $\mathcal{C}^J(G)$ is the one of a J-modular tensor category.

The Hopf algebraic structures endowed with weak actions we introduce in this section might be of independent interest.

4.1 Equivariant braided categories

Let $1 \to G \to H \xrightarrow{\pi} J \to 1$ be an exact sequence of finite groups. The normal subgroup G acts on H by conjugation; denote by $H//G$ the corresponding action groupoid. We consider the functor category $H//G$-mod $:= [H//G, \mathrm{Vect}_{\mathbb{K}}]$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. The category $H//G$-mod is the category of H-graded vector spaces, endowed with an action of the subgroup G such that $g.V_h \subset V_{ghg^{-1}}$ for all $g \in G, h \in H$.

An immediate corollary of Proposition 3.22 is the following description of the category $\mathcal{C}^J(G) := \bigoplus_{j\in J} \mathcal{C}(G)_j$ as an abelian category:

Proposition 4.1. *The category* $\mathcal{C}^J(G)$ *is equivalent, as an abelian category, to the category* $H//G$-mod*. In particular, the category* $\mathcal{C}^J(G)$ *is a 2-vector space in the sense of Definition* 2.5.

Proof. With $H_j := \pi^{-1}(j)$, Proposition 3.22 gives the equivalence $\mathcal{C}(G)_j \cong H_j//G$-mod of abelian categories. The equivalence of categories $\mathcal{C}^J(G) \cong H//G$-mod now follows from the decomposition $H = \bigsqcup_{j\in J} H_j$. By [34, Lemma 4.1.1], the representation category of a finite groupoid is a 2-vector space. □

Representation categories of finite groupoids are very close in structure to representation categories of finite groups. In particular, there is a complete character theory that describes the simple objects, see Appendix A.2.

We next introduce equivariant categories.

Definition 4.2. Let J be a finite group and $\mathcal{C}$ a category.

1) A *categorical action* of the group J on the category $\mathcal{C}$ consists of the following data:
   – A functor $\phi_j : \mathcal{C} \to \mathcal{C}$ for every group element $j \in J$.

– A functorial isomorphism $\alpha_{i,j} : \phi_i \circ \phi_j \xrightarrow{\sim} \phi_{ij}$, called *compositors*, for every pair of group elements $i, j \in J$

such that the coherence conditions

$$\alpha_{ij,k} \circ \alpha_{i,j} = \alpha_{i,jk} \circ \phi_i(\alpha_{j,k}) \quad \text{and} \quad \phi_1 = \mathrm{id} \tag{4.1}$$

hold.

2) If $\mathcal{C}$ is a monoidal category, we only consider actions by monoidal functors ϕ_j and require the natural transformations to be monoidal natural transformations. In particular, for each group element $j \in J$, we have the additional datum of a natural isomorphism

$$\gamma_j(U, V) : \phi_j(U) \otimes \phi_j(V) \xrightarrow{\sim} \phi_j(U \otimes V)$$

for each pair of objects U, V of $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd}
{}^{jk}X \otimes {}^{jk}Y \arrow[r, "\gamma_{jk}(X,Y)"] \arrow[d, "\alpha_{jk}(X)\otimes\alpha_{jk}(Y)"'] & {}^{jk}(X \otimes Y) \arrow[d, "\alpha_{jk}(X\otimes Y)"] \\
{}^{j}({}^{k}(X)) \otimes {}^{j}({}^{k}(Y)) \arrow[r, "{}^{j}\gamma_k(X,Y)\circ\gamma_j({}^{k}X,{}^{k}Y)"'] & {}^{j}({}^{k}(X \otimes Y))
\end{tikzcd}$$

(The data of a monoidal functor includes an isomorphism $\phi_j(1) \to 1$ in principal, but in this paper, the isomorphism will be the identity and therefore we will suppress it in our discussion.) We use the notation ${}^{j}U := \phi_j(U)$ for the image of an object $U \in \mathcal{C}$ under the functor ϕ_j.

3) A J-*equivariant* category $\mathcal{C}$ is a category with a decomposition $\mathcal{C} = \bigoplus_{j\in J} \mathcal{C}_j$ and a categorical action of J, subject to the compatibility requirement

$$\phi_i \mathcal{C}_j \subset \mathcal{C}_{iji^{-1}}$$

with the grading.

4) A J-*equivariant tensor category* is a J-equivariant monoidal category $\mathcal{C}$, subject to the compatibility requirement that the tensor product of two homogeneous elements $U \in \mathcal{C}_i, V \in \mathcal{C}_j$ is again homogeneous, $U \otimes V \in \mathcal{C}_{ij}$.

Remark 4.3. We remark that the condition $\phi_1 = \mathrm{id}$ 4.1 should in general be replaced by an extra datum, an isomorphism $\eta : \mathrm{id} \xrightarrow{\sim} \phi_1$ and two coherence conditions which involve the compositors $\alpha_{i,j}$. The diagrams can be found as follows: For any category $\mathcal{C}$, consider the category $\mathrm{AUT}(\mathcal{C})$ whose objects are automorphisms of $\mathcal{C}$ and whose morphisms are natural isomorphisms. The composition of functors and natural transformations endow $\mathrm{AUT}(\mathcal{C})$ with the natural structure of a strict tensor category. A categorical action of a finite group J on a category $\mathcal{C}$ then amounts to a tensor functor $\phi : J \to \mathrm{AUT}(\mathcal{C})$, where J is seen as a tensor category with only identity morphisms, compare also Remark 3.2.3. The condition $\phi_1 = \mathrm{id}$ holds in the categories we are interested in, hence we impose it.

Similarly, we consider for a monoidal category $\mathcal{C}$ the category $\mathrm{AUT}\,\mathrm{mon}(\mathcal{C})$ whose objects are monoidal automorphisms of $\mathcal{C}$ and whose morphisms are monoidal natural automorphisms. The categorical actions we consider for monoidal categories are then tensor functors $\phi : J \to \mathrm{AUT}\,\mathrm{mon}(\mathcal{C})$ For more details, we refer to [43] Appendix 5.

The category $H//G$-mod has a natural structure of a monoidal category: the tensor product of two objects $V = \oplus_{h\in H} V_h$ and $W = \oplus_{h\in H} W_h$ is the vector space $V \otimes W$ with H grading given by $(V \otimes W)_h := \oplus_{h_1 h_2 = h} V_{h_1} \otimes W_{h_2}$ and G action given by $g.(v \otimes w) = g.v \otimes g.w$. The associators are inherited from the underlying category of vector spaces.

Proposition 4.4. *Consider an exact sequence of groups $1 \to G \to H \to J \to 1$. Any choice of a a set-theoretic section $s : J \to H$ allows us to endow the abelian category $H//G$-*mod *with the structure of a J-equivariant tensor category as follows: the functor ϕ_j is given by shifting the grading from h to $s(j)hs(j)^{-1}$ and replacing the action by g by the action of $s(j)gs(j)^{-1}$. The isomorphism $\alpha_{i,j} : \phi_i \circ \phi_k \to \phi_{ij}$ is given by the left action action of the element*

$$\alpha_{i,j} = s(i)s(j)s(ij)^{-1}.$$

The fact that the action is only a weak action thus accounts for the failure of s to be a section in the category of groups.

Proof. Only the coherence conditions $\alpha_{ij,k} \circ \alpha_{i,j} = \alpha_{i,jk} \circ \phi_i(\alpha_{j,k})$ remain to be checked. By the results of Dedecker and Schreier, cf. Proposition 3.3, the group elements $s(i)s(j)s(ij)^{-1} \in G$ are the coherence cells of a weak group action of J on H. By Definition 3.1, this implies the coherence identities, once one takes into account that that composition of functors is written in different order than group multiplication. □

We have derived in Section 3.5 from the geometry of extended cobordism categories more structure on the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$.

In particular, we collect the functors $\otimes_{jk} : \mathcal{C}(G)_j \boxtimes \mathcal{C}(G)_k \to \mathcal{C}(G)_{jk}$ from Proposition 3.23 into a functor

$$\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}. \tag{4.2}$$

Another structure are the isomorphisms $V \otimes W \to \phi_j(W) \otimes V$ for $V \in \mathcal{C}(G)_j$, described in Proposition 3.26. Together with the associators, this suggests to endow the category $\mathcal{C}^J(G)$ with a structure of a braided J-equivariant tensor category:

Definition 4.5. A *braiding* on a J-equivariant tensor category is a family

$$c_{U,V} : U \otimes V \to {}^jV \otimes U$$

of isomorphisms, one for every pair of objects $U \in \mathcal{C}_i, V \in \mathcal{C}_j$, which are natural in U and V. Moreover, a braiding is required to satisfy an analogue of the hexagon axioms (see [43, Appendix A.5]) and to be preserved under the action of J, i.e., the following diagram commutes for all objects U, V with $U \in \mathcal{C}_j$ and $i \in J$

$$\begin{tikzcd}
{}^i(U \otimes V) \arrow[r, "{}^i(c_{U,V})"] \arrow[d, "\gamma_i"'] & {}^i({}^jV \otimes U) \arrow[r, "\gamma_i"] & {}^i({}^jV) \otimes {}^iU \arrow[d, "\alpha_{ij}(V)\otimes \mathrm{id}"] \\
{}^iU \otimes {}^iV \arrow[r, "c_{{}^iU,{}^iV}"'] & {}^{ij^{-1}}({}^iV) \otimes {}^iU \arrow[r, "\alpha_{iji^{-1},i(V)}\otimes \mathrm{id}"'] & {}^{ij}V \otimes {}^iU
\end{tikzcd} \tag{4.3}$$

Remark 4.6. 1) It should be appreciated that a braided J-equivariant category is not, in general, a braided category. Its *neutral component* $\mathcal{C}_1$ with $1 \in J$ the neutral element, is a braided tensor category.

2) By replacing the underlying category by an equivalent category, one can replace a weak action by a strict action, compare [43, Appendix A5]. In our case, weak actions actually lead to simpler algebraic structures.
3) The J-equivariant monoidal category $H//G$-mod has a natural braiding isomorphism that has been described in Proposition 3.26

We use the equivalence of abelian categories between $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ and $H//G$-mod to endow the category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ with associators. The category has now enough structure that we can state our next result:

Proposition 4.7. *The category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$, with the tensor product functor from (4.2), can be endowed with the structure of a braided*

J-equivariant tensor category such that the isomorphism $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j \cong H//G\text{-mod}$ becomes an isomorphism of braided J-equivariant tensor categories.

Proof. The compatibility with the grading is implemented by definition via the graded components $\otimes_{jk}$ of $\otimes$ and the graded components of $c_{V,W}$. It remains to check that the action is by tensor functors and that the braiding satisfies the hexagon axiom. The second boils down to a simple calculation and the first is seen by noting that the action is essentially an index shift which is preserved by tensoring together the respective components. □

4.2 Equivariant ribbon algebras

In the following, let J again be a finite group. To identify the structure of a J-modular tensor category on the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$, we need dualities. This will lead us to the discussion of (equivariant) ribbon algebras. Apart from strictness issues, our discussion closely follows [43]. We start our discussion with the relevant category-theoretic structures, which generalize premodular categories (cf. Definition 2.20) to the equivariant setting.

Definition 4.8. 1) A *J-equivariant ribbon category* is a J-braided category with dualities and a family of isomorphisms $\theta_V : V \to {}^jV$ for all $j \in J, V \in \mathcal{C}_j$, such that θ is compatible with duality and the action of J (see [43, VI.2.3] for the identities). In contrast to [43], we allow weak J-actions and thus require the diagram

$$\begin{tikzcd}
U \otimes V \arrow[rrr, "\theta_{U\otimes V}"] \arrow[d, "\theta_U\otimes\theta_V"'] & & & {}^{ij}(U\otimes V) \\
{}^iU \otimes {}^jV \arrow[dr, "c_{{}^iU,{}^jV}"'] & & & {}^{iji^{-1}}({}^iU)\otimes {}^{ij}V \arrow[u, "\gamma_{ij}\circ(\alpha_{iji^{-1},i}\otimes\mathrm{id})"'] \\
& {}^i({}^jV)\otimes {}^iU \arrow[r, "\alpha_{i,j}\otimes\mathrm{id}"'] & {}^{ij}V\otimes {}^iU \arrow[ur, "c_{{}^{ij}V,{}^iU}"'] &
\end{tikzcd} \tag{4.4}$$

involving compositors, to commute for $U \in \mathcal{C}_i$ and $V \in \mathcal{C}_j$.

2) A *J-premodular category* is a $\mathbb{K}$-linear, abelian, finitely semi-simple J-equivariant ribbon category such that the tensor product is a $\mathbb{K}$-bilinear functor and the tensor unit is absolutely simple.

Remark 4.9. The following facts directly follow from the definition of J-equivariant ribbon category: the neutral component $\mathcal{C}_1$ is itself a braided tensor category. In particular, it contains the tensor unit of the J-equivariant tensor category. The dual object of an object $V \in \mathcal{C}_j$ is in the category $\mathcal{C}_{j^{-1}}$.

We will not be able to directly endow the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ with the structure of a J-equivariant ribbon category. Rather, we will realize an equivalent category as the category of modules over a suitable algebra. To this end, we introduce in several steps the notions of a J-ribbon algebra and analyze the extra structure induced on its representation category.

Definition 4.10. Let A be an (associative, unital) algebra over a field $\mathbb{K}$. A *weak J-action* on A consists of algebra automorphisms $\varphi_j \in \mathrm{Aut}(A)$, one for every element $j \in J$, and invertible elements $c_{ij} \in A$, one for every pair of elements $i, j \in J$, such that for all $i, j, k \in J$ the following conditions hold:

$$\varphi_i \circ \varphi_j = \mathrm{Inn}_{c_{i,j}} \circ \varphi_{ij} \quad \varphi_i(c_{j,k}) \cdot c_{i,jk} = c_{i,j} \cdot c_{ij,k} \quad \text{and} \quad c_{1,1} = 1 \tag{4.5}$$

Here Inn_x with x an invertible element of A denotes the algebra automorphism $a \mapsto xax^{-1}$. A weak action of a group J is called strict, if $c_{i,j} = 1$ for all pairs $i, j \in J$.

Remark 4.11. As discussed for weak actions on groups in Remark 3.2, a weak action on a $\mathbb{K}$-algebra A can be seen as a categorical action on the category which has one object and the elements of A as endomorphisms.

We now want to relate a weak action $(\varphi_j, c_{i,j})$ of a group J on an algebra A to a categorical action on the representation category A-mod. To this end, we define for each element $j \in J$ a functor on objects by

$$^j(M, \rho) := (M, \rho \circ (\varphi_{j^{-1}} \otimes \mathrm{id}_M))$$

and on morphisms by $^jf = f$. For the functorial isomorphisms, we take

$$\alpha_{i,j}(M, \rho) := \rho\big((c_{j^{-1},i^{-1}})^{-1} \otimes \mathrm{id}_M\big).$$

The inversions in the above formulas make sure that the action on the level of categories really becomes a left action.

Lemma 4.12. *Given a weak action of J on a $\mathbb{K}$-algebra A, these data define a categorical action on the category A-mod.*

Proof. Let V be an A-Modul. We first show that $\alpha_{i,j}$ is a morphism ${}^{i}({}^{j}V) \rightarrow^{ij} V$ in A-mod: Let $V \in A$-mod, then for every $v \in V, a \in A$, we have:

$$\begin{aligned}\alpha_{i,j}(V) \circ \rho_{{}^{i}({}^{j}V)}(a \otimes v) &= (c_{i^{-1},j^{-1}})^{-1} \varphi_{j^{-1}} \circ \varphi_{i^{-1}}(a).v \\ &= \varphi_{(ij)^{-1}}(a)(c_{i^{-1},j^{-1}})^{-1}.v \\ &= \rho_{({}^{ij}V)} \circ (\mathrm{id}_A \otimes \alpha_{i,j}(V))(a \otimes v),\end{aligned}$$

where we used the abbreviation $a.v := \rho(a \otimes v)$. The validity of the coherence condition (4.1) for the $\alpha_{i,j}$ follows from the second equation in (4.5), since the right side of (4.1) evaluated on an element $v \in V$ reads

$$\alpha_{i,jk} \circ \phi_i(\alpha_{j,k})(v) = (c_{(jk)^{-1},i^{-1}})^{-1}(c_{k^{-1},j^{-1}})^{-1}.v$$

and the left one is:

$$\alpha_{ij,k} \circ \alpha_{i,j}(v) = (c_{k^{-1},(ij)^{-1}})^{-1} \varphi_{k^{-1}}(c_{j^{-1},i^{-1}})^{-1}.v$$

□

We next turn to an algebraic structure that yields J-equivariant tensor categories.

Definition 4.13. A *J-Hopf algebra* over $\mathbb{K}$ is a Hopf algebra A with a J-grading $A = \bigoplus_{j \in J} A_j$ and a weak J-action such that:

- The algebra structure of A restricts to the structure of an associative algebra on each homogeneous component so that A is the direct sum of the components A_j as an algebra.
- J acts by homomorphisms of Hopf algebras.
- The action of J is compatible with the grading, i.e., $\varphi_i(A_j) \subset A_{iji^{-1}}$
- The coproduct $\Delta : \ A \rightarrow A \otimes A$ respects the grading, i.e.,

$$\Delta(A_j) \subset \bigoplus_{p,q \in J, pq=j} A_p \otimes A_q.$$

- The elements $(c_{i,j})_{i,j \in J}$ are group-like, i.e $\Delta(c_{i,j}) = c_{i,j} \otimes c_{i,j}$.

Remark 4.14. 1) For the counit ϵ and the antipode S of a J-Hopf algebra, the compatibility relations with the grading $\epsilon(A_j) = 0$ for $j \neq 1$ and $S(A_j) \subset A_{j^{-1}}$ are immediate consequences of the definitions.
2) The restrictions of the structure maps endow the homogeneous component A_1 of A with the structure of a Hopf algebra with a weak J-action.

3) J-Hopf algebras with strict J-action have been considered under the name "J-crossed Hopf coalgebra" in [43, Chapter VII.1.2].

The category A-mod of finite-dimensional modules over a J-Hopf algebra inherits a natural duality from the duality of the underlying category of $\mathbb{K}$-vector spaces. The weak action described in Lemma 4.12 is even a monoidal action, since J acts by Hopf algebra morphisms. A grading on A-mod can be given by taking $(A\text{-mod})_j = A_j$-mod as the j-homogeneous component. From the properties of a J-Hopf algebra one can finally deduce that the tensor product, duality and grading are compatible with the J-action. We have thus arrived at the following statement:

Lemma 4.15. *The category of representations of a J-Hopf algebra has a natural structure of a $\mathbb{K}$-linear, abelian J-equivariant tensor category with compatible duality as introduced in Definition* 4.5.

Proof. We show, that the grading and the action on A-mod are compatible. Let $V \in A_j$-mod, then the j-component 1_j of the unit in A acts as the identity on V. We have to check, that ${}^iV \in A_{iji^{-1}}$-mod, i.e that $1_{iji^{-1}}$ acts as the identity on ${}^iV$. For $v \in^i V$, we have:

$$\rho_i(1_{iji^{-1}} \otimes v) = \varphi_{i^{-1}}(1_{iji^{-1}}).v = 1_j.v = v$$

This shows ${}^iV \in A_{iji^{-1}}$-mod. $\square$

The representation category of a braided Hopf algebra is a braided tensor category. If the Hopf algebra has, moreover, a twist element, its representation category is even a ribbon category. We now present J-equivariant generalizations of these structures.

Definition 4.16. Let A be a J-Hopf algebra.

1) A J-equivariant R-matrix is an element $R = R_{(1)} \otimes R_{(2)} \in A \otimes A$ such that for $V \in (A\text{-mod})_i$, $W \in A$-mod, the map

$$\begin{aligned} c_{VW} :& V \otimes W \to^i W \otimes V \\ & v \otimes w \mapsto R_{(2)}.w \otimes R_{(1)}.v \end{aligned}$$

is a J-braiding on the category A-mod according to Definition 4.5.

2) A J-twist is an invertible element $\theta \in A$ such that for every object $V \in (A\text{-mod})_i$ the induced map

$$\begin{aligned}\theta_V :& V \to^i V \\ & v \mapsto \theta^{-1}.v\end{aligned}$$

is a J-twist on A-mod as defined in 4.8.

If A has an R-matrix and a twist, we call it a *J-ribbon-algebra.*

Remark 4.17.
- A J-ribbon algebra is not, in general, a ribbon Hopf algebra.
- The component A_1 with the obvious restrictions of R and θ is a ribbon algebra.
- The conditions that the category A-mod is braided resp. ribbon can be translated into algebraic conditions on the elements R and θ. Since we are mainly interested in the categorical structure we refrain from doing that here.

Taking all the introduced algebraic structure into account we get the following proposition.

Proposition 4.18. *The representation category of a J-ribbon algebra is a J-ribbon category.*

Remark 4.19. In [43], Hopf algebras and ribbon Hopf algebras with strict J-action have been considered. The next subsection will give an illustrative example where the natural action is not strict.

4.3 Equivariant Drinfel'd double

The goal of this subsection is to construct a J-ribbon algebra, given a finite group G with a weak J-action. As explained in Section 3.1, such a weak J-action amounts to a group extension

$$1 \to G \longrightarrow H \xrightarrow{\pi} J \to 1 \tag{4.6}$$

with a set-theoretical splitting $s : J \to H$.

We start from the well-known fact reviewed in Section 2.5 that the Drinfel'd double $\mathcal{D}(H)$ of the finite group H is a ribbon Hopf algebra. The double

$\mathcal{D}(H)$ has a canonical basis $\delta_{h_1} \otimes h_2$ indexed by pairs h_1, h_2 of elements of H. Let $G \subset H$ be a subgroup. We are interested in the vector subspace $\mathcal{D}^J(G)$ spanned by the basis vectors $\delta_h \otimes g$ with $h \in H$ and $g \in G$.

Lemma 4.20. *The structure maps of the Hopf algebra $\mathcal{D}(H)$ restrict to the vector subspace $\mathcal{D}^J(G)$ in such a way that the latter is endowed with the structure of a Hopf subalgebra.*

Remark 4.21. The induced algebra structure on $\mathcal{D}^J(G)$ is the one of the groupoid algebra of the action groupoid $H//G$.

The Drinfel'd double $\mathcal{D}(H)$ of a group H has also the structure of a ribbon algebra. However, neither the R-matrix nor the the ribbon element yield an R-matrix or a ribbon element of $\mathcal{D}^J(G) \subset \mathcal{D}(H)$. Rather, this Hopf subalgebra can be endowed with the structure of a J-ribbon Hopf algebra as in Definition 4.16.

To this end, consider the partition of the group H into the subsets $H_j := \pi^{-1}(j) \subset H$. It gives a J-grading of the algebra A as a direct sum of subalgebras:

$$A_j := \langle \delta_h \otimes g \rangle_{h \in H_j, g \in G}.$$

The set-theoretical section s gives a weak action of J on A that can be described by its action on the canonical basis of A_j:

$$\varphi_j(\delta_h \otimes g) := (\delta_{s(j)hs(j)^{-1}} \otimes s(j)gs(j)^{-1});$$

the coherence elements are

$$c_{ij} := \sum_{h \in H} \delta_h \otimes s(i)s(j)s(ij)^{-1}.$$

Proposition 4.22. *The Hopf algebra $\mathcal{D}^J(G)$, together with the grading and weak J-action derived from the weak J-action on the group G, has the structure of a J-Hopf algebra.*

Proof. It only remains to check the compatibility relations of grading and weak J-action with the Hopf algebra structure that have been formulated in Definition 4.13. The fact that $\varphi_i(A_j) \subset A_{iji^{-1}}$ is immediate, since $s(i)H_j s(i^{-1}) \subset H_{iji^{-1}}$. The axioms follow are essentially equivalent to the property that conjugation commutes with the product and coproduct of the Drinfel'd double. □

We now turn to the last piece of structure, an R-matrix and twist element in $\mathcal{D}^J(G)$. Consider the element $R = \sum_{ij} R_{i,j} \in \mathcal{D}^J(G) \otimes \mathcal{D}^J(G)$ with homogeneous elements $R_{i,j}$ defined as

$$R_{i,j} := \sum_{h_1 \in H_i, h_2 \in H_j} (\delta_{h_1} \otimes 1) \otimes (\delta_{h_2} \otimes s(i^{-1})h_1). \tag{4.7}$$

(Note that $\pi(s(i^{-1})h_1) = 1$ for $h_1 \in H_i$ and thus $s(i^{-1})h_1 \in G$.)

The element $R_{i,j}$ is invertible with inverse

$$R_{i,j}^{-1} = \sum_{h_1 \in H_i, h_2 \in H_j} (\delta_{h_1} \otimes 1) \otimes (\delta_{h_2} \otimes h_1^{-1} s(i^{-1})^{-1}).$$

We also introduce a twist element $\theta = \sum_{j \in J} \theta_j \in \mathcal{D}^J(G)$ with

$$\theta_j^{-1} := \sum_{h \in H_j} \delta_h \otimes s\big(j^{-1}\big)h \in A_j \tag{4.8}$$

for every element $j \in J$.

Proposition 4.23. *The elements R and θ endow the J-Hopf algebra $\mathcal{D}^J(G)$ with the structure of a J-ribbon algebra that we call the J-Drinfel'd double of G.*

Proof. By Definition 4.16 we have to check that the induced transformations on the level of categories satisfy the axioms for a braiding and a twist. We first compute the induced braiding by using the R-matrix given in (4.7): For $V \in (\mathcal{D}^J(G)\text{-mod})_j = \mathcal{D}^J(G)_j\text{-mod}$ and $W \in \mathcal{D}^J(G)\text{-mod}$, we get the linear map

$$\begin{aligned} c_{V,W} : V \otimes W &\to {}^j W \otimes V \\ v \otimes w &\mapsto s(j^{-1})h.w \otimes v \quad \text{for} \quad v \in V_h \end{aligned}$$

First of all, we show that this is a morphism in $\mathcal{D}^J(G)$-mod. Let $v \in V_h$ and $w \in W_{h'}$. We have $v \otimes w \in (V \otimes W)_{hh'}$ and $c_{V,W}(v \otimes w) = s(j^{-1})h.w \otimes v \in ({}^j W \otimes V)_{hh'}$, since the element $s(j^{-1})h.w$ is in the component $\big(s(j^{-1})hh'h^{-1}s(j^{-1})^{-1}\big)$ of W which is the component $hh'h^{-1}$ of ${}^jW$. So $c_{V,W}$ respects the grading. As for the action of G, we observe that for

$g \in G$, the element $g.v$ lies in the component $V_{ghg^{-1}}$, and so

$$\begin{aligned} c_{V,W}(g.(v \otimes w)) &= \left(s(j^{-1})ghg^{-1}\right) g.w \otimes g.v = s(j^{-1})gh.w \otimes g.v \\ g.c_{V,W}(v \otimes w) &= \left(s(j^{-1})gs(j^{-1})^{-1}\right) s(j^{-1})h.w \otimes g.v = s(j^{-1})gh.w \otimes g.v \end{aligned}$$

which shows that $c_{V,W}$ commutes with the action of G.

Furthermore, it needs to be checked, that $c_{V,W}$ satisfies the hexagon axioms, which in our case reduce to the two diagrams

$$\begin{tikzcd} U \otimes V \otimes W \arrow[rr, "c_{U,V\otimes W}"] \arrow[d, "c_{U,V}\otimes \mathrm{id}_W"'] & & {}^iV \otimes {}^i W \otimes U \\ {}^iV \otimes U \otimes W \arrow[urr, "\mathrm{id}_{{}^iV}\otimes c_{U,W}"'] & & \end{tikzcd}$$

and

$$\begin{tikzcd} U \otimes V \otimes W \arrow[r, "c_{U\otimes V,W}"] \arrow[d, "\mathrm{id}_U \otimes c_{V,W}"'] & {}^{ij}W \otimes U \otimes V \\ U \otimes {}^j W \otimes V \arrow[r, "c_{U,{}^jW}\otimes \mathrm{id}_V"'] & {}^i({}^jW) \otimes U \otimes V \arrow[u, "\alpha_{i,j}\otimes \mathrm{id}_U"'] \end{tikzcd}$$

where $U \in \mathcal{C}_i, V \in \mathcal{C}_j$ and we suppressed the associators in A-mod.

And at last we need to check the compatibility of the braiding and the J-action, i.e., diagram (4.3). All of that can be proven by straightforward calculations, which show, that the morphism induced by the element R is really a braiding in the module category.

We now compute the morphism given by the action with the inverse twist element (4.8). For $V \in (\mathcal{D}^J(G)\text{-mod})_j = \mathcal{D}^J(G)_j\text{-mod}$ we get the linear map:

$$\begin{aligned} \theta_V : V &\to {}^j V \\ v &\mapsto s(j^{-1})h.v \end{aligned}$$

for $v \in V_h$. This map is compatible with the H-grading since for $v \in V_h$, we have $s(j^{-1})h.v \in V_{s(j^{-1})hs(j^{-1})^{-1}} = ({}^jV)_h$ and it is also compatible with the

G-action, since

$$\theta_V(g.v) = s(j^{-1})(ghg^{-1})g.v = s(j^{-1})gh.v$$
$$g.\theta_V(v) = \left(s(j^{-1})gs(j^{-1})^{-1}\right) s(j^{-1})h.v = s(j^{-1})gh.v$$

So the induced map is a morphism in the category $\mathcal{D}^J(G)$. In order to proof that it is a twist, we have to check the commutativity of diagram (4.4) as well as for every $i, j \in J$ and object V in the component j, the commutativity of the diagrams

$$\begin{tikzcd}
{}^{j}V^{\vee} \arrow[r, "(\theta_V)^{\vee}"] \arrow[d, "\theta_{{}^{j}V^{\vee}}"'] & V^{\vee} \\
{}^{j^{-1}}({}^{j}V^{\vee}) \arrow[ur, "\alpha_{j^{-1},j}"'] &
\end{tikzcd}$$

(which displays the compatibility of the twist with the duality), and

$$\begin{tikzcd}
{}^{i}V \arrow[r, "{}^{i}\theta_V"] \arrow[d, "\theta_{{}^{i}V}"'] & {}^{i}({}^{j}V) \arrow[d, "\alpha_{i,j}"] \\
{}^{iji^{-1}}({}^{i}V) \arrow[r, "\alpha_{iji^{-1},i}"'] & {}^{ij}V
\end{tikzcd}$$

(which is the compatibility of the twist with the J-action). This again can be checked by straightforward calculations. □

We are now ready to come back to the J-equivariant tensor category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ described in Proposition 4.7. From this proposition, we know that the category $\mathcal{C}^J(G)$ is equivalent to $H//G$-mod $\cong \mathcal{D}^J(G)$-mod as a J-equivariant tensor category. Also J-action and tensor product coincide with the ones on $\mathcal{D}^J(G)$-mod. Moreover, the equivariant braiding of $\mathcal{C}^J(G)$ computed in Proposition 3.26 coincides with the braiding on $\mathcal{D}^J(G)$ computed in the proof of the last Proposition 4.23.

This allows us to transfer also the other structure on the representation category of the J-Drinfel'd double $\mathcal{D}^J(G)$ described in Proposition 4.23 to the category $\mathcal{C}^J(G)$:

Proposition 4.24. *The J-equivariant tensor category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ described in Proposition* 4.7 *can be endowed with the structure of a J-premodular category such that it is equivalent, as a J-premodular category, to the category $\mathcal{D}^J(G)$*-mod.

Remark 4.25. At this point, we have constructed in paticular a J-equivariant braided fusion category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ (see [15]) with neutral component $\mathcal{C}(G)_1 \cong \mathcal{D}(G)$-mod from a weak action of the group J on the group G, or in different words, from a 2-group homomorphisms $J \to \mathrm{AUT}(G)$ with $\mathrm{AUT}(G)$ the automorphism 2-group of G.

In this remark, we very briefly sketch the relation to the description of J-equivariant braided fusion categories with given neutral sector $\mathcal{B}$ in terms of 3-group homomorphisms $J \to \underline{\mathrm{Pic}}(\mathcal{B})$ given in [15]. Here $\underline{\mathrm{Pic}}(\mathcal{B})$ denotes the so called Picard 3-group whose objects are invertible module-categories of the category $\mathcal{B}$. The group structure comes from the tensor product of module categories, which can be defined since the braiding on $\mathcal{B}$ allows it to turn module categories into bimodule categories.

Using this setting, we give a description of our J-equivariant braided fusion category $\mathcal{D}^J(G)$-mod in terms of a functor $\Xi : J \to \underline{\mathrm{Pic}}(\mathcal{D}(G))$. To this end, we construct a 3-group homomorphism $\mathrm{AUT}(G) \to \underline{\mathrm{Pic}}(\mathcal{D}(G))$ and write Ξ as the composition of this functor and the functor $J \to \mathrm{AUT}(G)$ defining the weak J-action.

The 3-group homomorphism $\mathrm{AUT}(G) \to \underline{\mathrm{Pic}}(\mathcal{D}(G))$ is given as follows: to an object $\varphi \in \mathrm{AUT}(G)$ we associate the twisted conjugation groupoid $G//^{\varphi}G$, where G acts on itself by twisted conjugation, $g.x := gx\varphi(g)^{-1}$. This yields the category $G//^{\varphi}G$-mod $:= [G//^{\varphi}G, \mathrm{Vect}_{\mathbb{K}}]$ which is naturally a module category over $\mathcal{D}(G)$-mod. Morphisms $\varphi \to \psi$ in $\mathrm{AUT}(G)$ are given by group elements $g \in G$ with $g\varphi g^{-1} = \psi$; to such a morphism we associate the functor $L_g : G//^{\varphi}G \to G//^{\psi}G$ given by conjugating with $g \in G$ on objects and morphisms. This induces functors of module categories $G//^{\varphi}G$-mod $\to G//^{\psi}G$-mod. Natural coherence data exist; one then shows that this really establishes the desired 3-group homomorphism.

4.4 Orbifold category and orbifold algebra

It remains to show that the J-equivariant ribbon category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ described in 4.23 is J-modular. To this end, we will use the orbifold category of the J-equivariant category:

Definition 4.26. Let $\mathcal{C}$ be a J-equivariant category. The *orbifold category* $\mathcal{C}^J$ of $\mathcal{C}$ has:

- as objects pairs $(V, (\psi_j)_{j\in J})$ consisting of an object $V \in \mathcal{C}$ and a family of isomorphisms $\psi_j : {}^jV \to V$ with $j \in J$ such that $\psi_i \circ {}^i\psi_j = \psi_{ij} \circ \alpha_{i,j}$.

- as morphisms $f : (V, \psi_j^V) \to (W, \psi_j^W)$ those morphisms $f : V \to W$ in $\mathcal{C}$ for which $\psi_j \circ {}^j(f) = f \circ \psi_j$ holds for all $j \in J$.

In [26], it has been shown that the orbifold category of a J-ribbon category is an ordinary, non-equivariant ribbon category:

Proposition 4.27. 1) *Let $\mathcal{C}$ be a J-ribbon category. Then the orbifold category $\mathcal{C}^J$ is naturally endowed with the structure of a ribbon category by the following data:*

- *The tensor product of the objects $(V, (\psi_j^V))$ and $(W, (\psi_j^W))$ is defined as the object $(V \otimes W, (\psi_j^V \otimes \psi_j^W))$.*
- *The tensor unit for this tensor product is $\mathbf{1} = (\mathbf{1}, (\mathrm{id}))$*
- *The dual object of $(V, (\psi_j))$ is the object $(V^*, (\psi_j^*)^{-1})$, where V^* denotes the dual object in $\mathcal{C}$.*
- *The braiding of the two objects $(V, (\psi_j^V))$ and $(W, (\psi_j^W))$ with $V \in \mathcal{C}_j$ is given by the isomorphism $(\psi_j \otimes \mathrm{id}_V) \circ c_{V,W}$, where $c_{V,W} : V \otimes W \to^j W \otimes V$ is the J-braiding in $\mathcal{C}$.*
- *The twist on an object $(V, (\psi_j))$ is $\psi_j \circ \theta$, where $\theta : V \to {}^jV$ is the twist in $\mathcal{C}$.*

2) *If $\mathcal{C}$ is a J-premodular category, then the orbifold category $\mathcal{C}^J$ is even a premodular category.*

It has been shown in [26] that the J-modularity of a J-premodular category is equivalent to the modularity as in Definition 2.20 of its orbifold category. Our problem is thus reduced to showing modularity of the orbifold category of $\mathcal{D}^J(G)$-mod.

To this end, we describe orbifoldization on the level of (Hopf-)algebras: given a J-equivariant algebra A, we introduce an orbifold algebra $\widehat{A}^J$ such that its representation category $\widehat{A}^J$-mod is isomorphic to the orbifold category of A-mod.

Definition 4.28. Let A be an algebra with a weak J-action (φ_j, c_{ij}). We endow the vector space $\widehat{A}^J := A \otimes \mathbb{K}[J]$ with a unital associative multiplication which is defined on an element of the form $(a \otimes j)$ with $a \in A$ and $j \in J$ by

$$(a \otimes i)(b \otimes j) := a\varphi_i(b)c_{ij} \otimes ij.$$

This algebra is called the *orbifold algebra* $\widehat{A}^J$ of the J-equivariant algebra A with respect to the weak J-action.

If A is even a J-Hopf algebra, it is possible to endow the orbifold algebra with even more structure. To define the coalgebra structure on the orbifold algebra, we use the standard coalgebra structure on the group algebra $\mathbb{K}[J]$ with coproduct $\Delta_J(j) = j \otimes j$ and counit $\epsilon_J(j) = 1$ on the canonical basis $(j)_{j \in J}$. The tensor product coalgebra on $A \otimes \mathbb{K}[J]$ has the coproduct and counit

$$\Delta(a \otimes j) = (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_{\mathbb{K}[J]})(\Delta_A(a) \otimes j \otimes j), \quad \text{and} \quad \epsilon(a \otimes j) = \epsilon_A(a) \tag{4.9}$$

which is clearly coassociative and counital.

To show that this endows the orbifold algebra with the structure of a bialgebra, we have first to show that the coproduct Δ is a unital algebra morphism. This follows from the fact, that Δ_A is already an algebra morphism and that the action of J is by coalgebra morphisms. Next, we have to show that the counit ϵ is a unital algebra morphism as well. This follows from the fact that the action of J commutes with the counit and from the fact that the elements $c_{i,j}$ are group-like. The compatibility of ϵ with the unit is obvious.

In a final step, one verifies that the endomorphism

$$S(a \otimes j) = (c_{j^{-1},j})^{-1} \varphi_{j^{-1}}(S_A(a)) \otimes j^{-1}$$

is an antipode. Altogether, one arrives at

Proposition 4.29. *If A is a J-Hopf-algebra, then the orbifold algebra $\widehat{A}^J$ has a natural structure of a Hopf algebra.*

Remark 4.30.
1) The algebra $\widehat{A}^J$ is not the fixed point subalgebra A^J of A; in general, the categories A^J-mod and $\widehat{A}^J$-mod are inequivalent.
2) Given any Hopf algebra A with weak J-action, we have an exact sequence of Hopf algebras

$$A \to \widehat{A}^J \to \mathbb{K}[J]. \tag{4.10}$$

   In particular, A is a sub-Hopf algebra of $\widehat{A}^J$. In general, there is no inclusion of $\mathbb{K}[J]$ into $\widehat{A}^J$ as a Hopf algebra.
3) If the action of J on the algebra A is strict, then the algebra A is a module algebra over the Hopf algebra $\mathbb{K}[J]$ (i.e., an algebra in the tensor category $\mathbb{K}[J]$-mod). Then the orbifold algebra is the smash product $A\#\mathbb{K}[J]$ (see [33, Section 4] for the definitions). The situation described occurs, if and only if the exact sequence (4.10) splits.

The next proposition justifies the name “orbifold algebra” for $\widehat{A}^J$:

Proposition 4.31. *Let A be a J-Hopf algebra. Then there is an equivalence of tensor categories*

$$\widehat{A}^J\text{-mod} \cong (A\text{-mod})^J.$$

Proof.

- An object of $(A\text{-mod})^J$ consists of a $\mathbb{K}$-vector space M, an A-action $\rho : A \to \mathrm{End}(M)$ and a family of A-module morphisms $(\psi_j)_{j\in J}$. We define on the same $\mathbb{K}$-vector space M the structure of an $\widehat{A}^J$ module by $\tilde{\rho} : \widehat{A}^J \to \mathrm{End}(M)$ with $\tilde{\rho}(a \otimes j) := \rho(a) \circ (\psi_{j^{-1}})^{-1}$.

  One next checks that, given two objects (M, ρ, ψ) and (M', ρ', ψ') in $(A\text{-mod})^J$, a $\mathbb{K}$-linear map $f \in \mathrm{Hom}_{\mathbb{K}}(M, M')$ is in the subspace $\mathrm{Hom}_{(A\text{-mod})^J}(M, M')$ if and only if it is in the subspace $\mathrm{Hom}_{\widehat{A}^J\text{-mod}}$ $(M, \tilde{\rho}), (M', \tilde{\rho}'))$.

  We can thus consider a $\mathbb{K}$-linear functor

  $$F : (A\text{-mod})^J \to \widehat{A}^J\text{-mod}, \tag{4.11}$$

  which maps on objects by $(M, \rho, \psi) \mapsto (M, \tilde{\rho})$ and on morphisms as the identity. This functor is clearly fully faithful.

  To show that the functor is also essentially surjective, we note that for any object $(M, \tilde{\rho})$ in $\widehat{A}^J$-mod, an object in $(A\text{-mod})^J$ can be obtained as follows: on the underlying vector space, we have the structure of an A-module by restriction, $\rho(a) := \tilde{\rho}(a \otimes 1_J)$. A family of equivariant morphisms is given by $\psi_j := (\tilde{\rho}(\mathbf{1} \otimes j^{-1}))^{-1}$. Clearly its image under F is isomorphic to $(M, \tilde{\rho})$. This shows that the functor F is an equivalence of categories, indeed even an isomorphism of categories.
- The functor F is also a strict tensor functor: consider two objects $F(M, \rho, \psi)$ and $F(M', \rho', \psi')$ in $(A\text{-mod})^J$. The functor F yields the following action of the orbifold Hopf algebra $\widehat{A}^J$ on the $\mathbb{K}$-vector space $M \otimes_{\mathbb{K}} M'$:

  $$\tilde{\rho}_{M\otimes M'}(a \otimes j) = \rho \otimes \rho'(\Delta(a)) \circ ((\psi_{j^{-1}})^{-1} \otimes (\psi'_{j^{-1}})^{-1}).$$

  Since the coproduct on $\widehat{A}^J$ was just given by the tensor product of coproducts on A and $\mathbb{K}[J]$, this coincides with the tensor product of $F(M, \rho, \psi)$ and $F(M', \rho', \psi')$ in $\widehat{A}^J$-mod. $\square$

In a final step, we assume that the J-equivariant algebra A has the additional structure of a J-ribbon algebra. Then, by Proposition 4.18, the category A-mod is a J-ribbon category and by Proposition 4.27 the orbifold category $(A\text{-mod})^J$ is a ribbon category. The strict isomorphism (4.11) of tensor categories allows us to transport both the braiding and the ribbon structure to the representation category of the orbifold Hopf algebra $\widehat{A}^J$. General results [24, Proposition 16.6.2] assert that this amounts to a natural structure of a ribbon algebra on $\widehat{A}^J$. In fact, we directly read off the R-matrix and the ribbon element. For example, the R-matrix $\hat{R}$ of $\hat{A}^J$ equals

$$\hat{R} = \hat{\tau}\; c_{\widehat{A}^J,\widehat{A}^J}(1_{\widehat{A}^J} \otimes 1_{\widehat{A}^J}) \in \widehat{A}^J \otimes \widehat{A}^J,$$

where the linear map $\hat{\tau}$ flips the two components of the tensor product $\widehat{A}^J \otimes \widehat{A}^J$. This expression can be explicitly evaluated, using the fact that $A \otimes \mathbb{K}[J]$ is an object in $(A\text{-mod})^J$ with A-module structure given by left action on the first component and that the morphisms ψ_j are given by left multiplication on the second component. We find for the R-matrix of $\widehat{A}^J$

$$\begin{aligned}\hat{R} &= \sum_{i,j\in J} (\mathrm{id} \otimes \psi_j)(\rho \otimes \rho)(R_{ij})((1_A \otimes 1_J) \otimes 1_A \otimes 1_J) \\ &= \sum_{i,j\in J} ((R_{i,j})_1 \otimes 1_J) \otimes (1_A \otimes j^{-1})^{-1}((R_{i,j})_2 \otimes 1_J)),\end{aligned}$$

where R is the R-matrix of A. The twist element of $\widehat{A}^J$ can be computed similarly; one finds

$$\theta^{-1} = \sum_{j\in J} \psi_j \circ \rho(\theta_j)(1_{A_j} \otimes 1_J) = \sum_{j\in J} (1_A \otimes j^{-1})^{-1}(\theta_j \otimes 1_J)$$

We summarize our findings:

Corollary 4.32. *If A is a J-ribbon algebra, then the orbifold algebra $\widehat{A}^J$ inherits a natural structure of a ribbon algebra such that the equivalence of tensor categories in Proposition* 4.31 *is an equivalence of ribbon categories.*

4.5 Equivariant modular categories

In this subsection, we show that the orbifold category of the J-equivariant ribbon category $\mathcal{C}^J(G)$-mod is J-modular. A theorem of Kirillov [26, Theorem 10.5] then immediately implies that the category $\mathcal{C}^J(G)$-mod is J-modular.

Since we have already seen in Corollary 4.32 that the orbifold category is equivalent, as a ribbon category, to the representation category of the orbifold Hopf algebra, it suffices the compute this Hopf algebra explicitly. Our final result asserts that this Hopf algebra is an ordinary Drinfel'd double:

Proposition 4.33. *The* $\mathbb{K}$*-linear map*

$$\begin{aligned}\Psi : \widehat{\mathcal{D}^J(G)}^J &\to \mathcal{D}(H)\\ (\delta_h \otimes g \otimes j) &\mapsto (\delta_h \otimes gs(j))\end{aligned} \tag{4.12}$$

is an isomorphism of ribbon algebras, where the Drinfel'd double $\mathcal{D}(H)$ *is taken with the standard ribbon structure introduced in Section 2.5.*

This result immediately implies the equivalence

$$(\widehat{\mathcal{D}^J(G)}\text{-mod})^J \cong \mathcal{D}(H)\text{-mod}$$

of ribbon categories and thus, by Proposition 2.21, the modularity of the orbifold category, so that we have finally proven:

Theorem 4.34. *The category* $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ *has a natural structure of a* J*-modular tensor category.*

Proof of Proposition 4.33. We show by direct computations that the linear map Ψ preserves product, coproduct, R-matrix and twist element:

- Compatibility with the product:

$$\begin{aligned}
&\Psi((\delta_h \otimes g \otimes j)(\delta'_h \otimes g' \otimes j'))\\
&\quad = \Psi((\delta_h \otimes g) \cdot {}^j(\delta'_h \otimes g')c_{jj'} \otimes jj')\\
&\quad = \Psi\Bigg((\delta_h \otimes g) \cdot (\delta_{s(j)h's(j)^{-1}} \otimes s(j)g's(j)^{-1})\\
&\qquad \cdot \sum_{h''\in H} (\delta_{h''} \otimes s(j)s(j')s(jj')^{-1}) \otimes jj'\Bigg)\\
&\quad = \Psi\left(\delta(h, gs(j)hs(j)^{-1}g^{-1})(\delta_h \otimes gs(j)g's(j')s(jj')^{-1}) \otimes jj'\right)\\
&\quad = \delta(h, gs(j)hs(j)^{-1}g^{-1})(\delta_h \otimes gs(j)g's(j'))\\
&\quad = (\delta_h \otimes gs(j)) \cdot (\delta_{h'} \otimes g's(j'))\\
&\quad = \Psi(\delta_h \otimes g \otimes j)\Psi(\delta_{h'} \otimes g' \otimes j'.
\end{aligned}$$

- Compatibility with the coproduct:

$$\begin{aligned}(\Psi \otimes \Psi)\Delta(\delta_h \otimes g \otimes j) &= \sum_{h'h''=h} \Psi(\delta_{h'} \otimes g \otimes j) \otimes \Psi(\delta_{h''} \otimes g \otimes j) \\ &= \sum_{h'h''=h} (\delta_{h'} \otimes gs(j)) \otimes (\delta_{h''} \otimes gs(j)) \\ &= \Delta\left(\Psi(\delta_h \otimes g \otimes j)\right).\end{aligned}$$

- The R-matrix of the orbifold algebra $\widehat{\mathcal{D}^J(G)}^J$ can be determined using the lines preceding C.orollary 4.32 and the definition of the R-Matrix of $\mathcal{D}^J(G)$ given in (4.7):

$$R = \sum_{j,j'\in J} \sum_{h\in H_j, h'\in H_{j'}} (\delta_h \otimes 1_G \otimes 1_J) \otimes (1 \otimes 1_G \otimes j^{-1})^{-1}(\delta_{h'} \otimes s(j^{-1})h \otimes 1_J).$$

This implies

$$\begin{aligned}(\Psi \otimes \Psi)(R) &= \sum_{j,j'\in J} \sum_{h\in H_j, h'\in H_{j'}} \Psi(\delta_h \otimes 1_G \otimes 1_J) \otimes \Psi(1 \otimes 1_G \otimes j^{-1})^{-1} \\ &\quad \cdot \Psi(\delta_{h'} \otimes s(j^{-1})h \otimes 1_J) \\ &= \sum_{j,j'\in J} \sum_{h\in H_j, h'\in H_{j'}} (\delta_h \otimes 1_H) \otimes (1 \otimes s(j^{-1})^{-1}) \cdot (\delta_{h'} \otimes s(j^{-1})h) \\ &= \sum_{h\ h'\in H} (\delta_h \otimes 1) \otimes (\delta_{h'} \otimes h),\end{aligned}$$

which is the standard R-matrix of the Drinfel'd double $\mathcal{D}(H)$.

- The twist in $\widehat{\mathcal{D}^J(G)}^J$ is by Corollary 4.32 equal to

$$\theta^{-1} = \sum_{j\in J} \sum_{h\in H_j} (\delta_h \otimes s(j^{-1})h \otimes 1_J)$$

and thus it gets mapped to the element

$$\begin{aligned}\Psi(\theta^{-1}) &= \sum_{j\in J} \sum_{h\in H_j} \Psi(1 \otimes 1_G \otimes j^{-1})^{-1}\Psi(\delta_h \otimes s(j^{-1})h \otimes 1_J) \\ &= \sum_{j\in J} \sum_{h\in H_j} (1 \otimes s(j^{-1})^{-1}) \cdot (\delta_h \otimes s(j^{-1})h) \\ &= \sum_{h\in H} (\delta_h \otimes h)\end{aligned}$$

which is the inverse of the twist element in $\mathcal{D}(H)$. □

4.6 Summary of all tensor categories involved

We summarize our findings by discussing again the four tensor categories mentioned in the introduction, in the square of (1.1), thereby presenting the explicit solution of the algebraic problem described in Section 1.1. Given a finite group G with a weak action of a finite group J, we get an extension $1 \to G \to H \to J \to 1$ of finite groups, together with a set-theoretic section $s : J \to H$.

Proposition 4.35. *We have the following natural realizations of the categories in question in terms of categories of finite-dimensional representations over finite-dimensional ribbon algebras:*

1) *The premodular category introduced in [2] is* $\mathcal{B}(G \triangleleft H)$-mod*. As an abelian category, it is equivalent to the representation category* $G//H$-mod *of the action groupoid* $G//H$*, i.e., to the category of* G*-graded* $\mathbb{K}$*-vector spaces with compatible action of* H*.*
2) *The modular category obtained by modularization is* $\mathcal{D}(G)$-mod*. As an abelian category, it is equivalent to* $G//G$-mod*.*
3) *The* J*-modular category constructed in this paper is* $\mathcal{D}^J(G)$-mod*. As an abelian category, it is equivalent to* $H//G$-mod*.*
4) *The modular category obtained by orbifoldization from the* J*-modular category* $\mathcal{D}(G)$-mod *is equivalent to* $\mathcal{D}(H)$-mod*. As an abelian category, it is equivalent to* $H//H$-mod*.*

*Equivalently, the diagram in (*1.1*), has the explicit realization:*

$$\begin{tikzcd} \mathcal{D}(G)\text{-mod} \arrow[loop above, "J"] \arrow[r, hook] \arrow[d, shift left, "orbifold"] & \mathcal{D}^J(G)\text{-mod} \arrow[loop above, "J"] \arrow[d, shift left, "orbifold"] \\ \mathcal{B}(G \triangleleft H)\text{-mod} \arrow[u, shift left, "modularization"] \arrow[r, hook] & \mathcal{D}(H)\text{-mod} \arrow[u, shift left] \end{tikzcd} \tag{4.13}$$

We could have chosen the inclusion in the lower line as an alternative starting point for the solution of the algebraic problem presented in Introduction 1.1. Recall from the Introduction that the category $\mathcal{B}(G \triangleleft H)$-mod contains a Tannakian subcategory that can be identified with the category of representations of the quotient group $J = H/G$. The Tannakian subcategory and thus the category $\mathcal{B}(G \triangleleft H)$-mod contain a commutative Frobenius algebra given by the algebra of functions on J; recall that the modularization function was just induction along this algebra. The image of this algebra under the inclusion in the lower line yields a commutative Frobenius algebra in the category $\mathcal{D}(H)$-mod. In a next step, one can consider induction along this algebra to obtain another tensor category which, by general results [26, Theorem 4.2] is a J-modular category.

In this approach, it remains to show that this J-modular tensor category is equivalent, as a J-modular tensor category, to $\mathcal{D}^J(G)$-mod and, in a next step that the modularization $\mathcal{D}(G)$-mod can be naturally identified with the neutral sector of the J-modular category. This line of thought has been discussed in [25, Lemma 2.2] including the square (4.13) of Hopf algebras. Our results directly lead to a natural Hopf algebra $\mathcal{D}^J(G)$ and additionally show how the various categories arise from extended topological field theories which are built on clear geometric principles and through which all additional structure of the algebraic categories become explicitly computable.

5 Outlook

Our results very explicitly provide an interesting class J-modular tensor categories. All data of these theories, including the representations of the modular group $SL(2,\mathbb{Z})$ on the vector spaces assigned to the torus, are directly accessible in terms of representations of finite groups. Also series of examples exist in which closed formulae for all quantities can be derived, e.g., for the inclusion of the alternating group in the symmetric group.

Our results admit generalizations in various directions. In fact, in this paper, we have only studied a subclass of Dijkgraaf–Witten theories. The general case requires, apart from the choice of a finite group G, the choice of an element of

$$H^3_{Gp}(G, U(1)) = H^4(\mathcal{A}_G, \mathbb{Z}).$$

This element can be interpreted [45] geometrically as a 2-gerbe on $\mathcal{A}_G$. It is known that in this case a quasi-triangular Hopf algebra can be extracted that is exactly the one discussed in [13]. Indeed, our results can also be generalized by including the additional choice of a non-trivial element

$$\omega \in H^4_J(\mathcal{A}_G, \mathbb{Z}) \equiv H^4(\mathcal{A}_G//J, \mathbb{Z}).$$

Only all these data together allow us to investigate in a similar manner the categories constructed by Bantay [2] for crossed modules with a boundary map that is not necessarily injective any longer. We plan to explain this general case in a subsequent publication.

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Appendix A

A.1 Cohomological description of twisted bundles

In this appendix, we give a description of P-twisted bundles as introduced in Definition 3.5 in terms of local data. This local description will also serve as a motivation for the term ‘twisted’ in twisted bundles. Recall the relevant situation: $1 \to G \to H \xrightarrow{\pi} J \to 1$ is an exact sequence of groups. Let $P \xrightarrow{J} M$ be a J-cover. A P-twisted bundle on a smooth manifold M is an H-bundle $Q \to M$, together with a smooth map $\varphi : Q \to P$ such that $\varphi(qh) = \varphi(q)\pi(h)$ for all $q \in Q$ and $h \in H$.

We start with the choice of a contractible open covering $\{U_\alpha\}$ of M, i.e., a covering for which all open sets U_α are contractible. Then the J-cover P admits local sections over U_α. By choosing local sections s_α, we obtain the cocycle

$$j_{\alpha\beta} := s_\alpha^{-1} \cdot s_\beta : U_\alpha \cap U_\beta \to J$$

describing P.

Let (Q, φ) be a P-twisted G-bundle over M. We claim that we can find local sections

$$t_\alpha : U_\alpha \to Q$$

of the H-bundle Q which are compatible with the local section of the J-cover P in the sense that $\varphi \circ t_\alpha = s_\alpha$ holds for all α.

To see this, consider the map $\varphi : Q \to P$; restricting the H-action on Q along the inclusion $G \to H$, we get a G-action on Q that covers the identity on P. Hence Q has the structure of a G-bundle over P. Note that the image of s_α is contractible, since U_α is contractible. Thus the G-bundle $Q \to P$ admits a section s'_α over the image of s_α. Then $t_\alpha := s'_\alpha \circ s_\alpha$ is a section of the H-bundle $Q \to M$ that does the job.

With these sections $t_\alpha : U_\alpha \to Q$, we obtain the cocycle description

$$h_{\alpha\beta} := t_\alpha^{-1} \cdot t_\beta : U_\alpha \cap U_\beta \to H$$

of Q.

The set underlying the group H is isomorphic to the set $G \times J$. The relevant multiplication on this set depends on the choice of a section $J \to H$; it has been described in (3.1):

$$(g,i) \cdot (g',j) := \big(g \cdot {}^{i}(g') \cdot c_{i,j} \;,\; ij\big).$$

This allows us to express the H-valued cocycles $h_{\alpha\beta}$ in terms of J-valued and G-valued functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G.$$

By the condition $\varphi \circ t_\alpha = s_\alpha$, the J-valued functions are determined to be the J-valued cocycles $j_{\alpha\beta}$. Using the multiplication on the set $G \times J$, the cocycle condition $h_{\alpha\beta} \cdot h_{\beta\gamma} = h_{\alpha\gamma}$ can be translated into the following condition for $g_{\alpha\beta}$

$$g_{\alpha\beta} \cdot {}^{j_{\beta\gamma}}\big(g_{\beta\gamma}\big) \cdot c_{j_{\alpha\beta}, j_{\beta\gamma}} = g_{\alpha\gamma} \tag{A.1}$$

over $U_\alpha \cap U_\beta \cap U_\gamma$. This local expression can serve as a justification of the term P-twisted G-bundle.

We next turn to morphisms. A morphism f between P-twisted bundles (Q,φ) and (Q',ψ) which are represented by twisted cocycles $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is represented by a coboundary

$$l_\alpha := (t'_\alpha)^{-1} \cdot f(t_\alpha) : U_\alpha \to H$$

between the H-valued cocycles $h_{\alpha\beta}$ and $h'_{\alpha\beta}$. Since f satisfies $\psi \circ f = \varphi$, the J-component $\pi \circ l_\alpha : U_\alpha \to H \to J$ is given by the constant function to $e \in J$. Hence the local data describing the morphism f reduce to a family of functions

$$k_\alpha : U_\alpha \to G.$$

Under the multiplication (3.1), the coboundary relation $l_\alpha \cdot h_{\alpha\beta} = h'_{\alpha\beta} \cdot l_\beta$ translates into

$$k_\alpha \cdot \ {}^{e}\big(g_{\alpha\beta}\big) \cdot c_{e,j_{\alpha\beta}} = g'_{\alpha\beta} \cdot \ {}^{j_{\alpha\beta}}\big(k_\beta\big) \cdot c_{j_{\alpha\beta},e}.$$

One can easily conclude from the Definition 3.1 of a weak action that ${}^{e}g = g$ and $c_{e,g} = c_{g,e} = e$ for all $g \in G$. Hence this condition reduces to the condition

$$k_\alpha \cdot g_{\alpha\beta} = g'_{\alpha\beta} \cdot \ {}^{j_{\alpha\beta}}\big(k_\beta\big). \tag{A.2}$$

We are now ready to present a classification of P-twisted bundles in terms of Čech-cohomology.

Therefore we define the relevant cohomology set:

Definition A.1.1. Let $\{U_\alpha\}$ be a contractible cover of M and $(j_{\alpha\beta})$ be a Čech-cocycle with values in J.

- A $(j_{\alpha\beta})$-twisted Čech-cocycle is given by a family

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

  satisfying relation (A.1).
- Two such cocycles $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are cobordant if there exists a coboundary, that is a family of functions $k_a : U_\alpha \to G$ satisfying relation (A.2).
- The twisted Čech-cohomology set $\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M, G)$ is defined as the quotient of twisted cocycles modulo coboundaries.

Warning A.1.2. *It might be natural to guess that twisted Čech-cohomology* $\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M, G)$ *agrees with the preimage of the class* $[j_{\alpha\beta}]$ *under the map* $\pi_* : \check{\mathrm{H}}^1(M, H) \to \check{\mathrm{H}}^1(M, J)$*. This turns out to be wrong: The natural map*

$$\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M, G) \to \check{\mathrm{H}}^1(M, H), \tag{A.3}$$

$$[g_{\alpha\beta}] \mapsto [(g_{\alpha,\beta}, j_{\alpha\beta})], \tag{A.4}$$

is, in general, not injective. The image of this map is always the fibre ${\pi_*}^{-1}[j_{\alpha\beta}]$.

We summarize our findings:

Proposition A.1.3. *Let P be a J-cover of M, described by the cocycle $j_{\alpha\beta}$ over the contractible open cover $\{U_\alpha\}$. Then there is a canonical bijection*

$$\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G) \cong \left\{\begin{matrix}\text{Isomorphism classes of P-twisted} \\ \text{G-bundles over M}\end{matrix}\right\}.$$

A.2 Character theory for action groupoids

In this subsection, we explicitly work out a character theory for finite action groupoids $M//G$; in the case of $M = pt$, this theory specializes to the character theory of a finite group (cf. [23,41]). In the special case of a finite action groupoid coming from a finite crossed module, a character theory including orthogonality relation has been presented in [2]. In the sequel, let $\mathbb{K}$ be a field and denote by $\mathrm{Vect}_{\mathbb{K}}(M//G)$ the category of $\mathbb{K}$-linear representations of $M//G$.

Definition A.2.4. Let $((V_m)_{m\in M}, (\rho(g))_{g\in G})$ be a $\mathbb{K}$- linear representation of the action groupoid $M//G$ and denote by $P(m)$ the projection of $V = \bigoplus_{n\in M} V_n$ to the homogeneous component V_m. We call the function

$$\begin{aligned}\chi : M \times G \to \mathbb{K},\\ \chi(m,g) := \mathrm{Tr}_V(\rho(g)P(m))\end{aligned}$$

the *character of the representation.*

Example A.2.5. On the $\mathbb{K}$-vector space $H := \mathbb{K}(M) \otimes \mathbb{K}[G]$ with canonical basis $(\delta_m \otimes g)_{m\in M, g\in G}$, we define a grading by $H_m = \bigoplus_g \mathbb{K}(\delta_{g.m} \otimes g)$ and a group action by $\rho(g)(\delta_m \otimes h) = \delta_m \otimes gh$. This defines an object in $\mathrm{Vect}_{\mathbb{K}}(M//G)$, called the regular representation. The character is easily calculated in the canonical basis and found to be

$$\chi_H(m,g) = \sum_{(n,h)\in M\times G} \delta(g,1)\delta(h.m,n) = \delta(g,1)|G|.$$

Definition A.2.6. We call a function

$$f : M \times G \to \mathbb{K}$$

an *action groupoid class function* on $M//G$, if it satisfies

$$f(m,g) = 0 \text{ if } g.m \neq m \quad \text{and} \quad f(h.m, hgh^{-1}) = f(m,g).$$

The character of any finite-dimensional representation is a class function.

¿ From now on, we assume that the characteristic of $\mathbb{K}$ does not divide the order $|G|$ of the group G. This assumption allows us to consider the following normalized non-degenerate symmetric bilinear form

$$\langle f, f' \rangle := \frac{1}{|G|} \sum_{g \in G, m \in M} f(m, g^{-1}) f'(m,g). \tag{A.5}$$

In the case of *complex* representations, one can show, precisely as in the case of groups, the equality $\chi(m, g^{-1}) = \overline{\chi(m,g)}$ which allows introduce the Hermitian scalar product

$$(\chi, \chi') := \frac{1}{|G|} \sum_{g \in G, m \in M} \overline{\chi(m,g)} \chi'(m,g). \tag{A.6}$$

Lemma A.2.7. *Let $\mathbb{K}$ be algebraically closed. The characters of irreducible $M//G$-representations are orthogonal and of unit length with respect to the bilinear form (*A.5*).*

Proof. The proof proceeds as in the case of finite groups: for a linear map $f : V \to W$ on the vector spaces underlying two irreducible representations, one considers the intertwiner

$$f^0 = \frac{1}{|G|} \sum_{g \in G, m \in M} \rho_W(g^{-1}) P_W(m) f P_V(m) \rho_V(g), \tag{A.7}$$

and applies Schur's lemma. □

A second orthogonality relation

$$\sum_{i \in I} \chi_i(m,g) \chi_i(n, h^{-1}) = \sum_{z \in G} \delta(n, z.m) \delta(h, zgz^{-1})$$

can be derived as in the case of finite groups, as well.

Combining the orthogonality relations with the explicit form for the character of the regular representation, we derive in the case of an algebraically closed field whose characteristic does not divide the order $|G|$ use a standard reasoning:

Lemma A.2.8. *Every irreducible representation V_i is contained in the regular representation with multiplicity $d_i := \dim_{\mathbb{K}} V_i$.*

As a consequence, the following generalization of Burnside's Theorem holds:

Proposition A.2.9. *Denote by $(V_i)_{i\in I}$ a set of representatives for the isomorphism classes of simple representations of the action groupoid and by $d_i := \dim_{\mathbb{K}} V_i$ the dimension of the simple object. Then*

$$\sum_{i\in I} |d_i|^2 = |M||G|.$$

Proof. One combines the relation $\dim H = \sum_{i\in I} d_i \dim V_i$ from Lemma A.2.8 with the relation $\dim H = |M||G|$. □

In complete analogy to the case of finite groups, one then shows:

Proposition A.2.10. *The irreducible characters of $M//G$ form an orthogonal basis of the space of class functions with respect to the scalar product (*A.5*).*

The above proposition allows us to count the number of irreducible representations. On the set

$$A := \{(m, g)|g.m = m\} \subset M \times G$$

the group G naturally acts by $h.(m, g) := (h.m, hgh^{-1})$. A class function of $M//G$ is constant on G-orbits of A; it vanishes on the complement of A in $M \times G$. We conclude that the number of irreducible characters equals the number of G-orbits of A.

This can be rephrased as follows: the set A is equal to the set of objects of the inertia groupoid $\Lambda(M//G) := [\bullet//\mathbb{Z}, M//G]$. Thus the number of G-orbits of A equals the number of isomorphism classes of objects in $\Lambda(M//G)$, thus $|I| = |Iso(\Lambda(M//G))|$.

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