Adv. Theor. Math. Phys. **15** (2011) 1817–1907

Algebraic deformations of toric varieties II: noncommutative instantons

Lucio Cirio[1,2], Giovanni Landi[3] and Richard J. Szabo[4]

[1]Grupo de Fisica Matemática da Universidade de Lisboa (GFM-UL), Instituto para a Investigação Interdisciplinar, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

[2]Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany.
lucio.cirio@gmail.com

[3]Dipartimento di Matematica e Informatica, Università di Trieste, Via A. Valerio 12/1, I-34127 Trieste, Italy and INFN, Sezione di Trieste, Trieste, Italy
landi@univ.trieste.it

[4]Department of Mathematics, Heriot-Watt University, Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K. and Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.
R.J.Szabo@ma.hw.ac.uk

e-print archive: http://lanl.arXiv.org/abs/1106.5708

Abstract

We continue our study of the noncommutative algebraic and differential geometry of a particular class of deformations of toric varieties, focusing on aspects pertinent to the construction and enumeration of noncommutative instantons on these varieties. We develop a noncommutative version of twistor theory, which introduces a new example of a noncommutative four-sphere. We develop a braided version of the ADHM construction and show that it parameterizes a certain moduli space of framed torsion free sheaves on a noncommutative projective plane. We use these constructions to explicitly build instanton gauge bundles with canonical connections on the noncommutative four-sphere that satisfy appropriate anti-selfduality equations. We construct projective moduli spaces for the torsion free sheaves and demonstrate that they are smooth. We define equivariant partition functions of these moduli spaces, finding that they coincide with the usual instanton partition functions for supersymmetric gauge theories on $\mathbb{C}^2$.

Contents

1 Introduction

This paper is the second part of a series of articles devoted to the construction and study of new noncommutative deformations of toric varieties. In the first part [13], the general theory was developed. In the present work, we elaborate on and extend some of these developments, and in particular derive a theory of instantons on the noncommutative projective planes $\mathbb{CP}^2_\theta$ constructed in [13] in several different contexts.

In the commutative situation, moduli spaces of framed sheaves on the complex projective plane have been studied intensively due to their connection with moduli spaces of framed instantons on the four-sphere; they are the basis for *instanton counting*. Generally, the Hitchin–Kobayashi correspondence establishes an identification between the moduli space of anti-selfdual irreducible connections on a Hermitean vector bundle E over a Kähler surface X and the set of equivalence classes of stable holomorphic bundles over X which are topologically equivalent to E. Instanton counting consists in computing BPS invariants of supersymmetric gauge theories in terms of "integrals" over equivariant cohomology classes of the moduli spaces. Equivariant cohomology groups of moduli spaces of framed sheaves on generic toric surfaces are much less well-understood in general than those of $\mathbb{CP}^2$.

One of the main results of this paper is a detailed description of the moduli space of torsion free sheaves on the noncommutative projective planes $\mathbb{CP}^2_\theta$ with a trivialization on a noncommutative line "at infinity". We will

identify points in this moduli space with classes of instantons on $\mathbb{CP}^2_\theta$. We shall establish a bijective correspondence between torsion free sheaves on $\mathbb{CP}^2_\theta$ and certain sets of braided ADHM data, which correspond to stable representations of the ADHM quiver with a certain q-deformation of the usual relation. We will study some details of the corresponding moduli spaces, which have good properties analogous to those of the commutative case making them tractable for uses in instanton counting problems, provided that one considers them as members of flat families in an appropriate sense [36]. This is in the same spirit as our general non-commutative deformations which occur as members of flat families of quantizations of toric varieties, and it agrees with general expectations [7, 8, 28] that instanton moduli spaces in the θ-deformed case produce "families" of instantons. Our results contain, in particular, an explicit realization of the construction of instantons on a q-deformed euclidean space $\mathbb{R}^4_q$ similar to the one sketched in [25, Section 9], our q-deformation being somewhat different.

Along the way we encounter some new constructions. A new noncommutative twistor theory is developed. In particular, we construct a new non-commutative four-sphere S^4_θ and describe some of its properties. Our construction of non-commutative instantons is partly inspired by the gluing construction of Frenkel and Jardim [18], although our non-commutative spaces and instanton gauge fields are rather different. Our analysis thus extends the existing examples of non-commutative instantons and spaces. Although our instanton moduli spaces are generically (commutative) deformations of those in the classical case, we will find that the equivariant counting problems, which define instanton partition functions of supersymmetric gauge theories, coincide with those of the classical limit in this case.

It is hoped that the constructions of non-commutative instantons presented here can be extended to more complicated toric varieties in four dimensions, such as ALE spaces or Hirzebruch surfaces, and ultimately to generalized instantons of six-dimensional non-commutative toric geometries pertinent to a non-commutative version of Donaldson–Thomas theory. The development of such generalizations would further extend the uses of instanton counting in enumerative geometry, and could possibly lead to new classes of enumerative invariants. They may also lead to new examples of BPS states in supersymmetric gauge theories and string theory. See [13, 37] for further motivation and background behind our constructions.

1.1 Outline of paper

In Section 2, we review and extend the quantization of toric varieties introduced in [13]. Our treatment follows the formalism of cocycle twist

quantization and non-commutative geometry in braided monoidal categories, see e.g., [8].

In Section 3 we introduce non-commutative projective varieties following [13], focusing on the non-commutative projective spaces $\mathbb{CP}^n_\theta$. We study coherent sheaves on these varieties and their invariants, and develop a theory of non-commutative monad complexes.

In Section 4, we develop a new non-commutative version of twistor theory in four dimensions following [13]. We construct a new example of a non-commutative sphere S^4_θ and corresponding twistor transforms, correspondences and fibrations. Although not developed here, it would be extremely interesting to investigate further the geometry of this non-commutative sphere, such as its cyclic cohomology and how it fits as the base of a non-commutative Hopf fibration whose total space is a seven-sphere.

In Section 5 we establish a bijection between framed torsion free sheaves on $\mathbb{CP}^2_\theta$ and a certain deformation of the usual ADHM construction. Our construction mimicks that of the commutative case and relies on many results of [6, 25] which were obtained in an analogous but rather different setting.

In Section 6 we explicitly construct instanton gauge bundles and canonical connections on S^4_θ using our non-commutative twistor correspondence and ADHM constructions. We verify that these connections satisfy appropriate anti-selfduality equations with respect to a natural metric, thus justifying them as bonafide non-commutative instantons. We do not develop a suitable (Yang–Mills) gauge theory for these instantons in this paper.

In Section 7 we use results of Nevins and Stafford [36] to construct corresponding moduli spaces of non-commutative instantons and demonstrate that they are smooth. We work out the corresponding deformation theory and explicitly compute the tangent spaces in terms of our braided ADHM construction. We work through many explicit examples which illustrate in what sense these moduli spaces are deformations of their classical counterparts, including a (commutative) deformation of the Hilbert scheme of points.

Finally, in Section 8 we use our moduli space constructions to compute standard equivariant counting functions for our non-commutative instantons with respect to a natural torus action on the instanton moduli space. We find a combinatorial classification of the torus fixed points in moduli space and show that it coincides with that of the classical limit. We comment on how it may be possible to construct instanton partition functions which capture more deeply the deformation of the moduli spaces.

1.2 Conventions and notation

Unless otherwise indicated, all tensor products are taken over the base field of complex numbers $\mathbb{C}$. All varieties considered are reduced separated schemes of finite type over $\mathbb{C}$. All algebras are associative over $\mathbb{C}$.

2 Braided symmetries of non-commutative toric varieties

2.1 Cocycle deformations of the algebraic torus

Let L be a lattice of rank n, and let $T = L \otimes_{\mathbb{Z}} \mathbb{C}^\times$ be the associated algebraic torus of dimension n over $\mathbb{C}$. The Pontrjagin dual group $\widehat{T} = \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C}^\times)$ is the group of characters $\{\chi_p\}_{p\in L^*}$ parameterized by elements of the dual lattice $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The dual pairing $L^* \times L \to \mathbb{Z}$ between lattices is denoted $(p, v) \mapsto p \cdot v$. Upon fixing a $\mathbb{Z}$-basis $e_1, \dots, e_n$ for L, with corresponding dual basis $e_1^*, \dots, e_n^*$ for L^*, one has $T \cong (\mathbb{C}^\times)^n$ and $\widehat{T} \cong \mathbb{Z}^n$. For $p = \sum_i p_i\, e_i^* \in L^*$ and $t = \sum_i e_i \otimes t_i \in T$, the characters are given by

$$\chi_p(t) = t^p := t_1^{p_1} \cdots t_n^{p_n}. \tag{2.1}$$

The unital algebra $\mathcal{H} = \mathcal{A}(T)$ of coordinate functions on the torus T is the Laurent polynomial algebra

$$\mathcal{H} := \mathbb{C}(t_1, \dots, t_n)$$

generated by elements t_i, $i = 1, \dots, n$. It is equipped with the Hopf algebra structure

$$\Delta(t^p) = t^p \otimes t^p, \quad \epsilon(t^p) = 1, \quad S(t^p) = t^{-p}$$

for $p \in L^*$, with the coproduct and the counit respectively extended as algebra morphisms $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and $\epsilon : \mathcal{H} \to \mathbb{C}$, and the antipode as an anti-algebra morphism $S : \mathcal{H} \to \mathcal{H}$. The canonical right action of T on itself by group multiplication dualizes to give a left $\mathcal{H}$-coaction

$$\begin{aligned} &\Delta_L : \mathcal{A}(T) \;\longrightarrow\; \mathcal{H} \otimes \mathcal{A}(T), \quad \Delta_L(u_i) = t_i \otimes u_i, \\ &\Delta_L\big(u_i^{-1}\big) = t_i^{-1} \otimes u_i^{-1}, \end{aligned} \tag{2.2}$$

where we write u_i, u_i^{-1}, $i = 1, \dots, n$ for the generators of $\mathcal{A}(T)$ viewed as a left comodule algebra over itself, when distinguishing the coordinate algebra $\mathcal{A}(T)$ from the Hopf algebra $\mathcal{H}$. This coaction is equivalent to a grading of

the algebra $\mathcal{A}(T)$ by the dual lattice L^*, for which the homogeneous elements are the characters (2.1).

A two-cocycle $F_\theta : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ on $\mathcal{H}$ is defined by choosing a complex skew-symmetric $n \times n$ matrix $\theta = (\theta^{ij})$, regarded as a homomorphism $\theta : L^* \to L \otimes_{\mathbb{Z}} \mathbb{C}$, setting

$$F_\theta(t^p, t^q) = \exp\left(\frac{\mathrm{i}}{2}\, p \cdot \theta q\right)$$

on characters $t^p, t^q \in \mathcal{H}$, and extending by linearity. The map F_θ is a convolution-invertible Hopf bicharacter and obeys

$$F_\theta \circ (S \otimes \mathrm{id}) = F_\theta^{-1} = F_\theta \circ (\mathrm{id} \otimes S), \quad F_\theta \circ (S \otimes S) = F_\theta .$$

It follows that F_θ is completely determined by its values on generators

$$F_\theta(t_i, t_j) = \exp\left(\frac{\mathrm{i}}{2}\, \theta^{ij}\right) =: q_{ij}$$

for $i, j = 1, \dots, n$.

Given such a map, one constructs the cotwisted Hopf algebra $\mathcal{H}_\theta$ which as a coalgebra is the same as $\mathcal{H}$ but which generally has a modified product and antipode. In the present case one easily finds that the product and antipode are in fact undeformed by F_θ, so $\mathcal{H} = \mathcal{H}_\theta$ as a Hopf algebra. On the other hand, $\mathcal{H}$ and $\mathcal{H}_\theta$ differ as coquasitriangular Hopf algebras. While the Hopf algebra $\mathcal{H}$ naturally carries the trivial coquasitriangular structure $\mathcal{R} = \epsilon \otimes \epsilon : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$, the cotwisted Hopf algebra $\mathcal{H}_\theta$ has twisted coquasitriangular structure given by the convolution-invertible Hopf bicharacter $\mathcal{R}_\theta : \mathcal{H}_\theta \otimes \mathcal{H}_\theta \to \mathbb{C}$ defined as $\mathcal{R}_\theta = F_\theta^{-2}$; on generators one has explicitly

$$\mathcal{R}_\theta(t_i, t_j) = F_\theta(t_j, t_i)\, F_\theta^{-1}(t_i, t_j) = F_\theta^{-2}(t_i, t_j) = q_{ij}^{-2} .$$

On the other hand, the algebra structure of $\mathcal{A}(T)$ gets deformed to a noncommutative product; this is a particular instance of the quantization functor we describe in Section 2.2.

2.2 Toric symmetries in braided monoidal categories

A left $\mathcal{H}$-comodule structure on a vector space V will be denoted as in (2.2) by $\Delta_L : V \to \mathcal{H} \otimes V$, together with a Sweedler notation $\Delta_L(v) = v^{(-1)} \otimes v^{(0)}$ for $v \in V$ and with implicit summation. Let ${}^{\mathcal{H}}\mathscr{M}$

denote the additive category of left $\mathcal{H}$-comodules. A map $V \xrightarrow{\sigma} W$ is a morphism of this category if and only if it is $\mathcal{H}$-coequivariant, i.e., it sits in the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\sigma} & W \\ {\scriptstyle \Delta_L}\downarrow & & \downarrow{\scriptstyle \Delta_L} \\ \mathcal{H}\otimes V & \xrightarrow[\mathrm{id}\otimes\sigma]{} & \mathcal{H}\otimes W \end{array}$$

or more explicitly $v^{(-1)} \otimes \sigma(v^{(0)}) = \sigma(v)^{(-1)} \otimes \sigma(v)^{(0)}$ for all $v \in V$. The category ${}^{\mathcal{H}}\mathscr{M}$ has a natural monoidal structure given by the tensor product coaction

$$\Delta_{V\otimes W}(v \otimes w) = v^{(-1)}\, w^{(-1)} \otimes \big(v^{(0)} \otimes w^{(0)}\big)$$

for $v \in V$, $w \in W$. With the trivial coquasitriangular structure $\mathcal{R} = \epsilon \otimes \epsilon$ on the Hopf algebra $\mathcal{H}$, the category ${}^{\mathcal{H}}\mathscr{M}$ of left $\mathcal{H}$-comodules is (trivially) braided by the collection $\Psi = \{\Psi_{V,W} : V \otimes W \to W \otimes V\}$ of functorial flip isomorphisms $\Psi_{V,W}(v \otimes w) = w \otimes v$ for each pair of objects V, W of ${}^{\mathcal{H}}\mathscr{M}$, and for all $v \in V$ and $w \in W$.

With the deformed coquasitriangular structure $\mathcal{R}_\theta = F_\theta^{-2}$ on the cotwisted Hopf algebra $\mathcal{H}_\theta$, the monoidal category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules is braided by the collection of functorial isomorphisms $\Psi^\theta = \{\Psi^\theta_{V,W} : V \otimes W \to W \otimes V\}$, where

$$\Psi^\theta_{V,W}(v \otimes w) = F_\theta^{-2}\big(w^{(-1)}, v^{(-1)}\big)\, w^{(0)} \otimes v^{(0)} \tag{2.3}$$

for each pair V, W of left $\mathcal{H}_\theta$-comodules with $v \in V$ and $w \in W$. Since the functor Ψ^θ is an involution, i.e., $\Psi^\theta \circ \Psi^\theta$ is isomorphic to the identity functor of the category ${}^{\mathcal{H}_\theta}\mathscr{M}$, the braiding is symmetric and makes ${}^{\mathcal{H}_\theta}\mathscr{M}$ into a tensor category. In particular, if A and B are left $\mathcal{H}_\theta$-comodule algebras with product maps $\mu_A : A \otimes A \to A$ and $\mu_B : B \otimes B \to B$, then one can define the braided tensor product algebra $A \otimes_\theta B$ to be the vector space $A \otimes B$ with the product map $\mu_{A\otimes_\theta B} : (A \otimes_\theta B) \otimes (A \otimes_\theta B) \to A \otimes_\theta B$ given by

$$\mu_{A\otimes_\theta B} = \big(\mu_A \otimes \mu_B\big) \circ \big(\mathrm{id}_A \otimes \Psi^\theta_{B,A} \otimes \mathrm{id}_B\big).$$

The resulting algebra is an object of the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ by the tensor product coaction.

The deformation in passing from $\mathcal{H}$ to $\mathcal{H}_\theta$ takes the form of a functorial isomorphism $\mathscr{F}_\theta : {}^{\mathcal{H}}\mathscr{M} \to {}^{\mathcal{H}_\theta}\mathscr{M}$ of braided monoidal categories. The functor $\mathscr{F}_\theta$ acts as the identity on objects and morphisms of ${}^{\mathcal{H}}\mathscr{M}$, but defines a new monoidal structure on the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ by

$$\begin{aligned}&\lambda_\theta : \mathscr{F}_\theta(V) \otimes \mathscr{F}_\theta(W) \ \longrightarrow \ \mathscr{F}_\theta(V \otimes W),\\ &\lambda_\theta(v \otimes w) = F_\theta\big(v^{(-1)}, w^{(-1)}\big)\, v^{(0)} \otimes w^{(0)}.\end{aligned}$$

This makes $\mathscr{F}_\theta$ into a monoidal functor which intertwines the braidings in ${}^{\mathcal{H}}\mathscr{M}$ and ${}^{\mathcal{H}_\theta}\mathscr{M}$, given respectively by the flip functor Ψ and the functor Ψ^θ defined in (2.3).

The quantization functor $\mathscr{F}_\theta$ simultaneously deforms all $\mathcal{H}$-coequivariant constructions to corresponding versions which are coequivariant under $\mathcal{H}_\theta$. For example, if A is an algebra in the category ${}^{\mathcal{H}}\mathscr{M}$, then the functor $\mathscr{F}_\theta$ takes its product map $\mu_A : A \otimes A \to A$ to a map $\mathscr{F}_\theta(\mu_A) : \mathscr{F}_\theta(A \otimes A) \to A_\theta := \mathscr{F}_\theta(A)$. Composing this morphism with λ_θ gives rise to a new product map

$$\mu_{A_\theta} := \mathscr{F}_\theta(\mu_A) \circ \lambda_\theta : A_\theta \otimes A_\theta \ \longrightarrow \ A_\theta$$

with

$$a \star_\theta b := \mu_{A_\theta}(a \otimes b) = F_\theta\big(a^{(-1)}, b^{(-1)}\big)\ \mu_A\big(a^{(0)} \otimes b^{(0)}\big), \tag{2.4}$$

which automatically makes A_θ into an $\mathcal{H}_\theta$-comodule algebra. If A is a commutative algebra, then the algebra A_θ is no longer commutative in general but only *braided commutative*, i.e., $\mu_{A_\theta} = \mu_{A_\theta} \circ \Psi^\theta_{A_\theta, A_\theta}$.

Another standard construction in braided monoidal categories yields deformations of exterior algebras. If V is a finite-dimensional object of the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules, then the exterior algebra of V in degree d is given by [13]

$$\textstyle\bigwedge^d_\theta V := V^{\otimes d} \,/\, \big\langle v_1 \otimes v_2 + \Psi^\theta_{V,V}(v_1 \otimes v_2) \big\rangle_{v_1, v_2 \in V}. \tag{2.5}$$

For $\theta = 0$ we recover the usual exterior algebra $\bigwedge^d V$ of the vector space V, while for $\theta \neq 0$ we obtain a braided skew-commutative algebra $\bigwedge^d_\theta V$. We also write

$$\textstyle\bigwedge^\bullet_\theta V := T(V) \,/\, \big\langle v_1 \otimes v_2 + \Psi^\theta_{V,V}(v_1 \otimes v_2) \big\rangle_{v_1, v_2 \in V} \tag{2.6}$$

where $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ is the free tensor algebra of the complex vector space V with $V^0 := \mathbb{C}$.

2.3 Quantization of toric varieties

As already mentioned at the end of Section 2.1, for $\mathcal{A} = \mathcal{A}(T)$ regarded as a left comodule algebra over itself, the functorial quantization constructed in Section 2.2 twists the algebra multiplication into a new product given by combining (2.2) and (2.4) to get

$$u_i \star_\theta u_j = F_\theta(t_i, t_j)\ u_i\, u_j = q_{ij}\ u_i\, u_j.$$

Let $\mathcal{A}(T_\theta)$ be the Laurent polynomial algebra generated by u_i with this product. It has relations

$$u_i \star_\theta u_j = q_{ij}^2\ u_j \star_\theta u_i, \quad u_i^{-1} \star_\theta u_j = q_{ij}^{-2}\ u_j \star_\theta u_i^{-1}$$

for each $i, j = 1, \dots, n$. This quantizes the torus T into the non-commutative algebraic torus $T_\theta = (\mathbb{C}^\times_\theta)^n$ dual to the algebra $\mathcal{A}(T_\theta)$. In the sequel we drop the star-product symbols $\star_\theta$ from the notation for simplicity.

A toric variety X of dimension n is a complex algebraic variety with an algebraic action of the torus $T = (\mathbb{C}^\times)^n$ and a T-equivariant injection $T \hookrightarrow X$ with dense image in the Zariski topology, where T acts on itself by group multiplication. In [13] we constructed a natural flat family of quantizations $X \to X_\theta$, with dual algebras $\mathcal{A}(X_\theta)$, over the coordinate algebra $\mathcal{A}(\bigwedge^2 T) = \mathbb{C}(q_{ij}, 1 \leq i < j \leq n)$ of the algebraic torus $\bigwedge^2 T := \mathrm{Hom}_{\mathbb{Z}}(\bigwedge^2 L^*, \mathbb{C}^\times)$ of dimension $\frac{1}{2}\, n\, (n-1)$. The original commutative toric variety $X = X_{\theta=0}$ is the fibre over the identity element of $\mathcal{A}(\bigwedge^2 T)$. Since the Zariski tangent space to the family at the identity is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\bigwedge^2 L^*, \mathbb{C})$, the family is universal for torus coinvariant non-commutative deformations in the following sense. Let Alg be the category of commutative unital noetherian $\mathbb{C}$-algebras and Set the category of sets. Let $\mathscr{F}_X : \mathsf{Alg} \to \mathsf{Set}$ be the covariant functor which sends an algebra A to the set $\mathscr{F}_X(A) = \mathcal{A}(X_A)$ of algebras dual to a flat family of T-coinvariant deformations $X \to X_A$ parameterized by A. Then there is a unique algebra homomorphism $\alpha : \mathcal{A}(\bigwedge^2 T) \to A$ such that $\mathscr{F}_X(A) \cong \mathscr{F}_X(\alpha)(\mathcal{A}(X_\theta))$. Whence the pair $\big(\mathcal{A}(\bigwedge^2 T), \mathcal{A}(X_\theta)\big)$ is a universal object representing the functor $\mathscr{F}_X$, and the toric variety X thus has a fine moduli space $\bigwedge^2 T$

of toric non-commutative deformations of dimension $\frac{1}{2}\,n\,(n-1)$ (see [24, Section 2.2]).

Noncommutative affine toric varieties correspond to finitely-generated $\mathcal{H}_\theta$-comodule subalgebras of the algebra $\mathcal{A}(T_\theta)$ of the non-commutative torus. Using the combinatorics of fans, one can glue these subalgebras together via algebra automorphisms in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ to form global non-commutative toric varieties; see [13, Section 3] for explicit constructions. The cones of the fan define the subcategory $\mathsf{Open}(X_\theta)$ of toric open sets of the non-commutative variety X_θ. In particular, the association of cones with $\mathcal{H}_\theta$-comodule subalgebras of $\mathcal{A}(T_\theta)$ defines the structure sheaf $\mathcal{O}_{X_\theta}$ of non-commutative $\mathbb{C}$-algebras on $\mathsf{Open}(X_\theta)$. The corresponding category of coherent sheaves of right $\mathcal{O}_{X_\theta}$-modules on $\mathsf{Open}(X_\theta)$ is denoted $\mathsf{coh}(X_\theta)$. While $\mathsf{coh}(X_\theta) \ncong \mathsf{coh}(X)$ in general, the coequivariant homological algebra of the non-commutative toric variety X_θ always coincides with that of the commutative fibre X (see [24, Section 7]).

The data defining a non-commutative deformation of a toric variety also defines a circle bundle over the (maximally) compact dual torus $T^*_{\mathbb{R}} = (L^* \otimes_{\mathbb{Z}} \mathbb{R})/L^* \cong \mathrm{U}(1)^n$, as both are given by pairings on the character lattice L^*. A basic operation in the theory of constructible sheaves yields "twisted" sheaves on $T^*_{\mathbb{R}}$ associated to this circle bundle, i.e., sheaves on the total space of the bundle whose monodromy around each fibre is given by a parameter $\lambda \in \mathbb{C}^\times$. A version of the coherent-constructible correspondence [16] then takes a coherent sheaf on the non-commutative toric variety X_θ to a complex of twisted constructible sheaves on the dual torus $T^*_{\mathbb{R}}$. These twisted complexes are equivalent to data in an analogue of the Fukaya category of lagrangian submanifolds of the cotangent bundle $\mathcal{T}^*$ of $T^*_{\mathbb{R}}$ [32]; in this sense the correspondence is a version of Kontsevich's homological mirror symmetry equivalence. In particular, this mirror correspondence relates the enumerative geometry of ideal sheaves (or instantons) on X_θ to that of Lagrangians in the mirror manifold $\mathcal{T}^*$. In topological string theory, it provides an equivalence between the category of non-commutative B-branes on X_θ and a certain category of twisted lagrangian A-branes on the mirror $\mathcal{T}^*$.

In this paper we work exclusively with non-commutative projective varieties, as defined in Section 3.1 below. The ensuing simplification is that we can work for the most part directly at the level of the homogeneous coordinate algebras, without resorting to the local picture provided by the combinatorial fan data of the toric variety. An alternative version of the homological mirror symmetry correspondence for $\mathbb{CP}^2_\theta$, and more generally for toric non-commutative weighted projective planes, is described in [5]; in this case the deformation parameter $\theta := \theta^{12} \in \mathbb{C}$ parameterizes

a non-exact deformation of the complexified Kähler class of the mirror Landau–Ginzburg model.

3 Coherent sheaves on non-commutative projective varieties

3.1 Projective spaces $\mathbb{CP}^n_\theta$

The homogeneous coordinate algebra $\mathcal{A}(\mathbb{CP}^n_\theta)$ of the non-commutative toric variety $\mathbb{CP}^n_\theta$ is the graded polynomial algebra in $n+1$ generators w_i, $i = 1, \ldots, n+1$ of degree one with the quadratic relations

$$\begin{aligned} w_{n+1}\, w_i &= w_i\, w_{n+1}, \quad i = 1, \ldots, n, \\ w_i\, w_j &= q_{ij}^2\; w_j\, w_i, \quad i, j = 1, \ldots, n. \end{aligned} \tag{3.1}$$

This algebra is naturally an $\mathcal{H}_\theta$-comodule algebra with left coaction

$$\Delta_L : \mathcal{A}(\mathbb{CP}^n_\theta) \;\longrightarrow\; \mathcal{H}_\theta \otimes \mathcal{A}(\mathbb{CP}^n_\theta)$$

given on generators by

$$\begin{aligned} \Delta_L(w_i) &= t_i \otimes w_i, \quad i = 1, \ldots, n, \\ \Delta_L(w_{n+1}) &= 1 \otimes w_{n+1}, \end{aligned} \tag{3.2}$$

and extended as an algebra morphism. As before, we denote the coaction on an arbitrary element $f \in \mathcal{A}(\mathbb{CP}^n_\theta)$ by $\Delta_L(f) = f^{(-1)} \otimes f^{(0)}$. The algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^n_\theta)$ is quadratic and graded by the usual polynomial degree as

$$\mathcal{A}(\mathbb{CP}^n_\theta) = \bigoplus_{k=0}^{\infty} \mathcal{A}_k,$$

with $\mathcal{A}_0 = \mathbb{C}$ and $\mathcal{A}_k = \bigoplus_{i_1+\cdots+i_{n+1}=k} \mathbb{C} w_1^{i_1} \cdots w_{n+1}^{i_{n+1}}$ for $k > 0$.

Each monomial w_i generates a left denominator set in $\mathcal{A}(\mathbb{CP}^n_\theta)$, and the degree zero subalgebra of the left Ore localization of $\mathcal{A}(\mathbb{CP}^n_\theta)$ with respect to w_i is naturally isomorphic to the non-commutative coordinate algebra of the ith maximal cone in the fan of $\mathbb{CP}^n_\theta$, for each $i = 1, \ldots, n+1$ [13, Thm. 5.4].

If $\mathcal{I} \subset \mathcal{A}(\mathbb{CP}^n_\theta)$ is a graded two-sided ideal generated by a set of homogeneous polynomials $f_1, \ldots, f_m$, then the quotient algebra $\mathcal{A}(\mathbb{CP}^n_\theta)/\mathcal{I}$ is identified as the graded homogeneous coordinate algebra of a non-commutative

projective variety $X_\theta(\mathcal{I})$. It has generators $w_1,\dots,w_{n+1}$ subject to the relations (3.1) and $f_1=\cdots=f_m=0$. We work only with varieties $X_\theta(\mathcal{I})$ whose coordinate algebras $\mathcal{A}(X_\theta(\mathcal{I}))=\mathcal{A}(\mathbb{CP}^n_\theta)/\mathcal{I}$ are regular. Noncommutative projective varieties inherit many of the properties of $\mathbb{CP}^n_\theta$ that we describe in the following.

The Koszul dual $\mathcal{A}^!=\bigoplus_{k\geq 0}\mathcal{A}^!_k$ of the quadratic algebra $\mathcal{A}=\mathcal{A}(\mathbb{CP}^n_\theta)$ was worked out in [13, Prop. 6.1] and it is a deformation of the exterior algebra of $\mathcal{A}^*$, graded again by polynomial degree, where throughout $(-)^*$ denotes the $\mathbb{C}$-dual $\mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$. In the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules, it is given by

$$\mathcal{A}^!_k=\bigwedge^k_\theta\mathcal{A}^*_1=\big(\mathcal{A}^*_1\big)^{\otimes k}\Big/\big\langle a_1\otimes a_2+\Psi^\theta_{\mathcal{A}^*_1,\mathcal{A}^*_1}(a_1\otimes a_2)\big\rangle_{a_1,a_2\in\mathcal{A}^*_1},$$

and hence the dual algebra $\mathcal{A}^!$ is generated by degree one elements $\check{w}_i\in\mathcal{A}^!_1$, $i=1,\dots,n+1$, with the relations

$$\begin{aligned}\check{w}_i^2&=0,\quad i=1,\dots,n+1,\\ \check{w}_i\,\check{w}_{n+1}+\check{w}_{n+1}\,\check{w}_i&=0,\quad i=1,\dots,n,\\ \check{w}_i\,\check{w}_j+q^2_{ij}\;\check{w}_j\,\check{w}_i&=0,\quad i,j=1,\dots,n.\end{aligned}\tag{3.3}$$

The coaction $\Delta_L:\mathcal{A}^!\to\mathcal{H}_\theta\otimes\mathcal{A}^!$ is dual to the coaction (3.2) and is given by

$$\begin{aligned}\Delta_L(\check{w}_i)&=t_i^{-1}\otimes\check{w}_i,\quad i=1,\dots,n,\\ \Delta_L(\check{w}_{n+1})&=1\otimes\check{w}_{n+1}.\end{aligned}\tag{3.4}$$

By using the associated Koszul resolution of the trivial right $\mathcal{A}$-module $\mathcal{A}_0=\mathbb{C}$ [13, Section 6.1], together with the fact that $\mathcal{A}=\mathcal{A}(\mathbb{CP}^n_\theta)$ is an Ore extension of a commutative polynomial algebra [5, Section 2.3], one shows that the algebra $\mathcal{A}$ is a noetherian regular algebra of homological dimension $n+1$ (see [5, Prop. 2.6] and [13, Cor. 6.5]).

3.2 Coherent sheaves

With $\mathcal{A}=\mathcal{A}(\mathbb{CP}^n_\theta)$, let $\mathsf{gr}(\mathcal{A})$ be the abelian category of finitely-generated graded right $\mathcal{A}$-modules $M=\bigoplus_{k\geq 0}M_k$ with morphisms given by module homomorphisms of degree zero. Let $\mathsf{tor}(\mathcal{A})$ be the full subcategory of $\mathsf{gr}(\mathcal{A})$ consisting of graded torsion $\mathcal{A}$-modules M which have finite dimension over $\mathbb{C}$, i.e., $M_k=0$ for $k\gg 0$. In [13] it was shown that one can identify the category $\mathsf{coh}(\mathbb{CP}^n_\theta)$ of coherent sheaves of right $\mathcal{O}_{\mathbb{CP}^n_\theta}$-modules on $\mathsf{Open}(\mathbb{CP}^n_\theta)$

with the abelian quotient category $\mathsf{gr}(\mathcal{A})/\mathsf{tor}(\mathcal{A})$. We denote by $\pi : \mathsf{gr}(\mathcal{A}) \to \mathsf{coh}(\mathbb{CP}^n_\theta)$ the canonical projection functor. Under this correspondence, the structure sheaf $\mathcal{O}_{\mathbb{CP}^n_\theta}$ of non-commutative $\mathbb{C}$-algebras on $\mathsf{Open}(\mathbb{CP}^n_\theta)$ is the image $\pi(\mathcal{A})$ of the homogeneous coordinate algebra itself, regarded as a free right $\mathcal{A}$-module of rank one. Throughout we abbreviate

$$\mathrm{Hom}(E,F) := \mathrm{Hom}_{\mathsf{coh}(\mathbb{CP}^n_\theta)}(E,F)$$

for $E, F \in \mathsf{coh}(\mathbb{CP}^n_\theta)$. By [5, Cor. 2.19], if $n \times n$ complex skew-symmetric matrices θ and θ' are related by $\theta'^{\,ij} = \theta^{ij} + \vartheta^i - \vartheta^j$ for some $\vartheta^1, \ldots, \vartheta^n \in \mathbb{C}$, then the abelian categories $\mathsf{coh}(\mathbb{CP}^n_\theta)$ and $\mathsf{coh}(\mathbb{CP}^n_{\theta'})$ are equivalent.

The abelian category $\mathsf{gr}(\mathcal{A})$ is equipped with a shift functor which is the autoequivalence sending a graded module $M = \bigoplus_{k\geq 0} M_k$ to the shifted module $M(l)$ defined by $M(l)_k = M_{l+k}$. The induced shift functor on the quotient category $\mathsf{coh}(\mathbb{CP}^n_\theta)$ sends a sheaf $E = \pi(M)$ to $E(k) := \pi(M(k))$. Since $\mathcal{A}$ is a noetherian regular algebra, the correspondence which sends a sheaf $E \in \mathsf{coh}(\mathbb{CP}^n_\theta)$ to the graded module

$$\Gamma(E) := \bigoplus_{k=0}^{\infty} \mathrm{Hom}\big(\mathcal{O}_{\mathbb{CP}^n_\theta}(-k), E\big)$$

defines a functor $\Gamma : \mathsf{coh}(\mathbb{CP}^n_\theta) \to \mathsf{gr}(\mathcal{A})$ such that $\pi \circ \Gamma$ is isomorphic to the identity functor of the category $\mathsf{coh}(\mathbb{CP}^n_\theta)$ [3, Sections 3–4].

Let $\mathsf{gr}_L(\mathcal{A})$ be the abelian category of finitely-generated graded *left* $\mathcal{A}$-modules. We will denote by $\pi_L : \mathsf{gr}_L(\mathcal{A}) \to \mathsf{coh}_L(\mathbb{CP}^n_\theta) := \mathsf{gr}_L(\mathcal{A})/\mathsf{tor}_L(\mathcal{A})$ the corresponding quotient projection, and by $\Gamma_L : \mathsf{coh}_L(\mathbb{CP}^n_\theta) \to \mathsf{gr}_L(\mathcal{A})$ its right inverse such that $\pi_L \circ \Gamma_L$ is isomorphic to the identity functor of $\mathsf{coh}_L(\mathbb{CP}^n_\theta)$. For any sheaf $E \in \mathsf{coh}(\mathbb{CP}^n_\theta)$, the graded left $\mathcal{A}$-module

$$\underline{\mathcal{H}\mathrm{om}}\left(E, \mathcal{O}_{\mathbb{CP}^n_\theta}\right) = \pi_L\left(\bigoplus_{k=0}^{\infty} \mathrm{Hom}\big(E, \mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big)\right)$$

is called the *dual sheaf* of E and is denoted $E^\vee \in \mathsf{coh}_L(\mathbb{CP}^n_\theta)$. The internal Hom-functor $\underline{\mathcal{H}\mathrm{om}}\,(-, \mathcal{O}_{\mathbb{CP}^n_\theta}) : \mathsf{coh}(\mathbb{CP}^n_\theta) \to \mathsf{coh}_L(\mathbb{CP}^n_\theta)$ is left exact and has corresponding right derived functors $\underline{\mathcal{E}\mathrm{xt}}^p(-, \mathcal{O}_{\mathbb{CP}^n_\theta})$ given by

$$\underline{\mathcal{E}\mathrm{xt}}^p\big(E, \mathcal{O}_{\mathbb{CP}^n_\theta}\big) = \pi_L\left(\bigoplus_{k=0}^{\infty} \mathrm{Ext}^p\big(E, \mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big)\right)$$

for $p \geq 0$, where $\mathrm{Ext}^p(E,F)$ is the p-th derived functor of the Hom-functor $\mathrm{Hom}(E,F)$ for $E, F \in \mathsf{coh}(\mathbb{CP}^n_\theta)$. Since $\mathcal{A}$ is a noetherian regular algebra,

there are isomorphisms [13]

$$\mathrm{Ext}^p(E,F) \cong \mathrm{Ext}^p_L(F^\vee,E^\vee) := \mathrm{Ext}^p_{\mathsf{coh}_L(\mathbb{CP}^n_\theta)}(F^\vee,E^\vee)$$

for any $p \geq 0$ and for any pair of torsion free sheaves $E, F \in \mathsf{coh}(\mathbb{CP}^n_\theta)$. The sheaves $E = \mathcal{O}_{\mathbb{CP}^n_\theta}(k)$, $k \in \mathbb{Z}$ are locally free, i.e., $\underline{\mathcal{E}\mathrm{xt}}^p(E, \mathcal{O}_{\mathbb{CP}^n_\theta}) = 0$ for all $p > 0$, with $\underline{\mathcal{H}\mathrm{om}}\,(\mathcal{O}_{\mathbb{CP}^n_\theta}(k), \mathcal{O}_{\mathbb{CP}^n_\theta}(l)) = \mathcal{O}_{\mathbb{CP}^n_\theta}(l-k)$ as sheaves of bimodules.

Under the conditions spelled out in [13, Prop. 6.4], a non-commutative version of the Beilinson spectral sequence can be developed following [25] by using a double Koszul bicomplex of the algebra $\mathcal{A}$. For this, split the left Koszul complex $\mathcal{K}^\bullet(\mathcal{A}) \cong \mathcal{A} \otimes (\mathcal{A}^!)^*$ of $\mathcal{A}$–$\mathcal{A}^!$-bimodules into finite-dimensional subcomplexes $\mathcal{K}^\bullet_{(p)}(\mathcal{A})$ for the total degree p. Then for any sheaf $F \in \mathsf{coh}(\mathbb{CP}^n_\theta)$, there is a spectral sequence with first term

$$E_1^{p,q} = \mathrm{Ext}^q(\mathcal{Q}^p, F) \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(-p) \Longrightarrow E^i_\infty = \begin{cases} F, & i = p+q = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

where $p = 0, 1, \dots, n$, and $\mathcal{Q}^p = \pi_L\big(\mathcal{K}^p_{(0)}(\mathcal{A})\big)^\vee$ is the sheaf on $\mathsf{Open}(\mathbb{CP}^n_\theta)$ corresponding to the cohomology of the truncated left Koszul complex for total degree zero given by

$$\mathcal{K}^p_{(0)}(\mathcal{A}) = \ker\left(\mathcal{A}(-p) \otimes \big(\mathcal{A}^!_p\big)^* \longrightarrow \bigoplus_{k=1}^{p} \mathcal{A}(k-p) \otimes \big(\mathcal{A}^!_{p-k}\big)^*\right). \tag{3.6}$$

For $p = 1$, it is shown in [13, Ex. 6.10] that the cohomology module $\mathcal{K}_1(\mathcal{A})$ of the left Koszul complex of $\mathcal{A}$ truncated at the first term can be naturally identified with the coherent sheaf

$$\Omega^1_{\mathbb{CP}^n_\theta} = \ker\big(\mu_{\mathcal{A}} : (\mathcal{A}^!_1)^* \otimes \mathcal{A} \to \mathcal{A}\big)$$

of Kähler differentials on $\mathsf{Open}(\mathbb{CP}^n_\theta)$. Here $\mu_{\mathcal{A}}$ denotes the product map on the algebra $\mathcal{A}$.

3.3 Invariants of torsion-free sheaves

A coherent sheaf E on $\mathsf{Open}(\mathbb{CP}^n_\theta)$ is torsion free if it embeds into a locally free sheaf (a bundle), or equivalently if the right $\mathcal{A}$-module $M = \Gamma(E)$ contains no finite-dimensional submodules. They have natural isomorphism invariants associated to them. The (Goldie) rank of E is the maximal number of non-zero direct summands of E, regarded as $\mathcal{A}$-submodules. This

agrees with the notion of rank given in [13, Def. 4.7] and is denoted $\mathrm{rank}(E)$. There is also a well-defined Euler characteristic

$$\chi(E) = \sum_{p\geq 0} (-1)^p \dim_{\mathbb{C}} \big(H^p(\mathbb{CP}^n_\theta, E)\big),$$

where $H^p(\mathbb{CP}^n_\theta, E) := \mathrm{Ext}^p(\mathcal{O}_{\mathbb{CP}^n_\theta}, E)$ are finite-dimensional vector spaces over $\mathbb{C}$ by the χ-condition of [13, Prop. 6.7], together with the Hilbert polynomial

$$h_E(s) = \chi\big(E(s)\big) \ \in \ \mathbb{Q}[s].$$

The first Chern class c_1 is defined by the requirement of additivity on short exact sequences together with

$$c_1\big(\mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big) = k. \tag{3.7}$$

This uniquely determines $c_1(E)$ upon using the ampleness property of [13, Prop. 6.7] to construct a resolution of E by shifts of the structure sheaf $\mathcal{O}_{\mathbb{CP}^n_\theta}$, and then applying additivity. With this definition, one has [36]

$$c_1\big(E(k)\big) = c_1(E) + k\ \mathrm{rank}(E). \tag{3.8}$$

In particular, for any ideal sheaf $I \in \mathsf{coh}(\mathbb{CP}^n_\theta)$, i.e., a torsion free sheaf of rank one on $\mathsf{Open}(\mathbb{CP}^n_\theta)$, there is a unique shift $I(k)$ of I which has $c_1 = 0$.

Proposition 3.9. *If $E \in \mathsf{coh}(\mathbb{CP}^n_\theta)$ is a torsion free sheaf, then $\underline{\mathcal{E}\mathrm{xt}}^n(E, \mathcal{O}_{\mathbb{CP}^n_\theta}) = 0$.*

Proof. This is a special case of [6, Prop. 2.0.6]. □

In our constructions of instanton moduli spaces, we shall need appropriate notions of stability.

Definition 3.10. A torsion free sheaf $E \in \mathsf{coh}(\mathbb{CP}^n_\theta)$ is said to be *μ-stable* (resp. *μ-semistable*) if for every proper non-trivial subsheaf $F \subset E$, one has

$$\frac{c_1(F)}{\mathrm{rank}(F)} < \frac{c_1(E)}{\mathrm{rank}(E)}$$

(resp. $\leq$).

3.4 Monads

We shall now describe a general construction of coherent sheaves, which will be instrumental in our analysis of instanton moduli spaces later on.

Definition 3.11. A *monad* on a (regular) non-commutative projective variety $X_\theta = X_\theta(\mathcal{I})$ is a complex

$$\underline{\mathcal{C}}^\bullet : 0 \longrightarrow \mathcal{C}_{-1} \xrightarrow{\sigma_{\underline{\mathcal{C}}^\bullet}} \mathcal{C}_0 \xrightarrow{\tau_{\underline{\mathcal{C}}^\bullet}} \mathcal{C}_1 \longrightarrow 0$$

of locally free sheaves on $\mathsf{Open}(X_\theta)$ which is exact at the first and last terms. The coherent sheaf $\mathcal{C} = H^0(\underline{\mathcal{C}}^\bullet) = \ker(\tau_{\underline{\mathcal{C}}^\bullet})/\mathrm{im}(\sigma_{\underline{\mathcal{C}}^\bullet})$ on $\mathsf{Open}(X_\theta)$ is called the *cohomology of the monad* $\underline{\mathcal{C}}^\bullet$. A *morphism of monads* is a homomorphism of complexes.

In this paper we are primarily interested in *linear monads* on the non-commutative projective spaces $\mathbb{CP}^n_\theta$, which are complexes of sheaves of free right $\mathcal{A}$-modules of the form

$$\begin{aligned}\underline{\mathcal{C}}^\bullet : 0 \longrightarrow\ & V_{-1} \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(-1) \xrightarrow{\sigma_w} V_0 \otimes \mathcal{O}_{\mathbb{CP}^n_\theta} \\ & \xrightarrow{\tau_w} V_1 \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(1) \longrightarrow 0 \end{aligned} \tag{3.12}$$

for finite-dimensional complex vector spaces V_{-1}, V_0 and V_1, where $\sigma_w \in V_{-1}^* \otimes V_0 \otimes \mathcal{A}_1$ (resp. $\tau_w \in V_0^* \otimes V_1 \otimes \mathcal{A}_1$) is an injective (resp. surjective) $\mathcal{A}$-module homomorphism such that $\tau_w \circ \sigma_w = 0$. Here $\mathcal{A}_1$ is the degree one component of the graded coordinate algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^n_\theta)$, i.e., the vector space spanned by the generators $w_1, \dots, w_{n+1}$. Note that $\mathcal{A}_1 \cong H^0(\mathbb{CP}^n_\theta, \mathcal{O}_{\mathbb{CP}^n_\theta}(1))$ by [13, Prop. 6.8]. This definition is also well-posed on any regular non-commutative projective variety $X_\theta(\mathcal{I})$.

Proposition 3.13. *If $E \in \mathsf{coh}(\mathbb{CP}^n_\theta)$ is the cohomology sheaf of a linear monad complex (*3.12*), then it has invariants*

$$\begin{aligned} \mathrm{rank}(E) &= \dim_{\mathbb{C}}(V_0) - \dim_{\mathbb{C}}(V_{-1}) - \dim_{\mathbb{C}}(V_1), \\ c_1(E) &= \dim_{\mathbb{C}}(V_{-1}) - \dim_{\mathbb{C}}(V_1), \\ \chi(E) &= \dim_{\mathbb{C}}(V_0) - (n+1)\,\dim_{\mathbb{C}}(V_1). \end{aligned}$$

Proof. From (3.12) and [6, Prop. 2.0.4 (1)] it follows that the kernel sheaf $\ker(\tau_w)$ is locally free, and there are short exact sequences of sheaves of $\mathcal{A}$-modules

$$0 \longrightarrow \ker(\tau_w) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{CP}^n_\theta} \xrightarrow{\tau_w} V_1 \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(1) \longrightarrow 0 \tag{3.14}$$

and

$$0 \longrightarrow V_{-1} \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(-1) \xrightarrow{\sigma_w} \ker(\tau_w) \longrightarrow E \longrightarrow 0. \tag{3.15}$$

We apply rank, c_1 and χ to these sequences using the fact that they are all additive in exact sequences [36]. The formula for the rank then follows immediately, while the formula for the first Chern class follows from (3.7). The expression for the Euler characteristic follows by [13, Prop. 6.8] which gives $\chi(\mathcal{O}_{\mathbb{CP}^n_\theta}) = \dim_{\mathbb{C}}(\mathcal{A}_0) = 1$, $\chi(\mathcal{O}_{\mathbb{CP}^n_\theta}(1)) = \dim_{\mathbb{C}}(\mathcal{A}_1) = n+1$, and $\chi(\mathcal{O}_{\mathbb{CP}^n_\theta}(-1)) = 0$. □

Corollary 3.16. *A linear monad complex* (3.12) *on* $\mathbb{CP}^n_\theta$ *exists only when*

$$\dim_{\mathbb{C}}(V_0) \;\geq\; \dim_{\mathbb{C}}(V_{-1}) + \dim_{\mathbb{C}}(V_1).$$

A fruitful feature of such sheaves E coming from linear monads is that the cohomology groups of the shifts $E(k)$ are qualitatively similar to those of the structure sheaf, in the sense that there is at most one non-trivial cohomology group, and only for degree shifts in the ranges $k \geq 0$ and $k \leq -n-1$, as in [13, Prop. 6.8].

Proposition 3.17. *If* $E \in \mathsf{coh}(\mathbb{CP}^n_\theta)$ *is the cohomology of a linear monad complex (*3.12*), then it has the following sheaf cohomology groups:*

(1) $H^p\big(\mathbb{CP}^n_\theta, E(k)\big) = 0$ *for all pairs of integers* $(p,k) = (0, k<0)$, $(1, k<-1)$, $(2 \leq p \leq n-2, k \in \mathbb{Z})$, $(n-1, k>-n)$ *and* $(n, k \geq -n)$*;*

(2) $H^1\big(\mathbb{CP}^n_\theta, E(-1)\big) = V_1$*; and*

(3) $\underline{\mathcal{E}\mathrm{xt}}^1\big(E, \mathcal{O}_{\mathbb{CP}^n_\theta}\big) = \mathrm{coker}(\sigma_w^*)$*, while* $\underline{\mathcal{E}\mathrm{xt}}^p\big(E(k), \mathcal{O}_{\mathbb{CP}^n_\theta}\big) = 0$ *for all* $p \geq 2$ *and for all* $k \in \mathbb{Z}$*.*

Proof. (1) The complex (3.12) of sheaves of free right $\mathcal{A}$-modules can be naturally extended by applying the degree k shift functor for any $k \in \mathbb{Z}$, whose cohomology coincides with the sheaf $E(k)$. This modifies the short exact sequences (3.14)–(3.15) to

$$0 \longrightarrow \big(\ker(\tau_w)\big)(k) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(k) \xrightarrow{\tau_w} V_1 \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(k+1) \longrightarrow 0$$

and

$$0 \longrightarrow V_{-1} \otimes \mathcal{O}_{\mathbb{CP}^n_\theta}(k-1) \xrightarrow{\sigma_w} \big(\ker(\tau_w)\big)(k) \longrightarrow E(k) \longrightarrow 0.$$

They induce long exact sequences in cohomology which contain the exact sequences

$$\begin{aligned} &H^p\big(\mathbb{CP}^n_\theta,\big(\ker(\tau_w)\big)(k)\big)\\ &\quad\longrightarrow\ V_0\otimes H^p\big(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big)\longrightarrow\ V_1\otimes H^p\big(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(k+1)\big)\\ &\quad\longrightarrow H^{p+1}\big(\mathbb{CP}^n_\theta,\big(\ker(\tau_w)\big)(k)\big) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &V_{-1}\otimes H^p\big(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(k-1)\big)\ \longrightarrow\ H^p\big(\mathbb{CP}^n_\theta,\big(\ker(\tau_w)\big)(k)\big)\\ &\quad\longrightarrow\ H^p\big(\mathbb{CP}^n_\theta,E(k)\big)\ \longrightarrow\ V_{-1}\otimes H^{p+1}\big(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(k-1)\big) \end{aligned} \tag{3.19}$$

for each $p\geq 0$, whose first arrows are both injections for $p=0$ and $H^{p+1}(\mathbb{CP}^n_\theta,-)=0$ for $p=n$ by [13, Prop. 6.8]. We use these two sequences and [13, Prop. 6.8] to find the values of (p,k) for which the isomorphisms $H^p(\mathbb{CP}^n_\theta,(\ker(\tau_w))(k))\xrightarrow{\approx}H^p(\mathbb{CP}^n_\theta,E(k))$ and $H^p(\mathbb{CP}^n_\theta,(\ker(\tau_w))(k))=0$ simultaneously hold, and the assertions follow.

(2) Set $k=-1$ in (3.19) and use $H^0(\mathbb{CP}^n_\theta,E(-1))=0$ by (1), together with

$$H^1\big(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(-2)\big)=H^2\big(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(-2)\big)=0$$

by [13, Prop. 6.8], to find the isomorphism $H^1(\mathbb{CP}^n_\theta,(\ker(\tau_w))(-1))\xrightarrow{\approx}H^1(\mathbb{CP}^n_\theta,E(-1))$. Setting $k=-1$ in (3.18) and using $H^0(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(-1))=H^1(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta}(-1))=0$ by [13, Prop. 6.8], there is an isomorphism $V_1\otimes H^0(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta})\xrightarrow{\approx}H^1(\mathbb{CP}^n_\theta,(\ker(\tau_w))(-1))$. The result now follows by [13, Prop. 6.8] which gives $H^0(\mathbb{CP}^n_\theta,\mathcal{O}_{\mathbb{CP}^n_\theta})\cong\mathcal{A}_0=\mathbb{C}$.

(3) We use $\mathcal{O}_{\mathbb{CP}^n_\theta}(l)^\vee\cong\mathcal{O}_{\mathbb{CP}^n_\theta}(-l)$ for any $l\in\mathbb{Z}$ (as sheaves of bimodules) and note that, by applying the internal Hom-functor $\underline{\mathcal{H}\mathrm{om}}\,(-,\mathcal{O}_{\mathbb{CP}^n_\theta})$ to the complex (3.12) using [13, Prop. 6.9], the cohomologies of the dual complex

$$\begin{aligned} \underline{\mathcal{C}}^\vee_\bullet:0\longrightarrow\ &\mathcal{O}_{\mathbb{CP}^n_\theta}(-1)\otimes V_1^*\ \xrightarrow{\tau_w^*}\ \mathcal{O}_{\mathbb{CP}^n_\theta}\otimes V_0^*\\ &\xrightarrow{\sigma_w^*}\ \mathcal{O}_{\mathbb{CP}^n_\theta}(1)\otimes V_{-1}^*\ \longrightarrow\ 0 \end{aligned} \tag{3.20}$$

coincide with $H^0(\underline{\mathcal{C}}^\vee_\bullet)=\underline{\mathcal{H}\mathrm{om}}\,(E,\mathcal{O}_{\mathbb{CP}^n_\theta})=E^\vee$ and $H^1(\underline{\mathcal{C}}^\vee_\bullet)=\underline{\mathcal{E}\mathrm{xt}}^1(E,\mathcal{O}_{\mathbb{CP}^n_\theta})$. We now use the fact that the $\mathcal{A}$-module $\ker(\tau_w)$ is locally free, together with

the dualizing sequence

$$0 \longrightarrow E^\vee(-k) \longrightarrow \big(\ker(\tau_w)\big)^\vee(-k) \xrightarrow{\sigma_w^*} \mathcal{O}_{\mathbb{CP}^n_\theta}(-k+1)\otimes V_{-1}^* \longrightarrow \\ \longrightarrow \underline{\mathcal{E}\mathrm{xt}}^1\big(E(k), \mathcal{O}_{\mathbb{CP}^n_\theta}\big) \longrightarrow 0,$$

and the result follows. □

It is also possible to algebraically characterize those linear monads whose cohomology sheaf is locally free or torsion free.

Proposition 3.21. *Let $E \in \mathsf{coh}(\mathbb{CP}^n_\theta)$ be the cohomology of a linear monad complex (3.12) and let $S = \Gamma_L(\mathrm{coker}(\sigma_w^*)) \in \mathsf{gr}_L(\mathcal{A})$. Then:*

(1) *E is locally free if and only if S is a finite-dimensional graded left $\mathcal{A}$-module.*

(2) *E is torsion free if and only if S has homological dimension $\leq n-2$ as a graded left $\mathcal{A}$-module.*

Proof. (1) By point (3) of Proposition 3.17 and [13, Prop. 6.9], E is locally free if and only if $\pi_L(S) = 0$, i.e., $\dim_{\mathbb{C}}(S) < \infty$.

(2) There is a spectral sequence

$$E_2^{p,q} = \underline{\mathcal{E}\mathrm{xt}}^q_L\big(\underline{\mathcal{E}\mathrm{xt}}^{-p}(E, \mathcal{O}_{\mathbb{CP}^n_\theta}), \mathcal{O}_{\mathbb{CP}^n_\theta}\big) \Longrightarrow E_\infty^i = \begin{cases} E, & i = p+q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

By point (3) of Proposition 3.17, $E_2^{p,q} = 0$ for all $p \leq -2$. By Serre duality [13, Prop. 6.7], one has

$$\mathrm{Ext}^q_L\big(\underline{\mathcal{E}\mathrm{xt}}^1(E, \mathcal{O}_{\mathbb{CP}^n_\theta}), \mathcal{O}_{\mathbb{CP}^n_\theta}(k)\big) \cong H^{n-q}_L\big(\mathbb{CP}^n_\theta, \underline{\mathcal{E}\mathrm{xt}}^1(E, \mathcal{O}_{\mathbb{CP}^n_\theta})(k+n+1)\big)^*$$

for all $k \in \mathbb{Z}$. The right-hand side vanishes for $q = 0, 1$ and for $k \gg 0$, by point (3) of Proposition 3.17 and our hypothesis which implies that the sheaves $\underline{\mathcal{E}\mathrm{xt}}^1(E, \mathcal{O}_{\mathbb{CP}^n_\theta})(k)$ in $\mathsf{coh}_L(\mathbb{CP}^n_\theta)$ have cohomological dimension $\leq n-2$ for k sufficiently large. Thus $E_2^{-1,q} = 0$ for $q = 0, 1$, and so the only $E_\infty^{p,-p}$ term, which might be non-zero is $E_\infty^{0,0}$. But the differentials coming into $E_k^{0,0}$ are always zero, so we get a sequence of inclusions

$$E = E_\infty^{0,0} \hookrightarrow \cdots \hookrightarrow E_3^{0,0} \hookrightarrow E_2^{0,0}.$$

The extremities imply injectivity of the canonical morphism $E \to E^{\vee\vee}$. By iterating the proof of point (3) of Proposition 3.17 and using [13, Prop. 6.9],

one shows that $E^\vee$ is locally free. By [6, Prop. 2.0.4 (2)], the sheaf $E^{\vee\vee}$ is also locally free. Hence E is torsion free.

For the converse statement, we use [6, Lem. 2.0.7] to choose an integer k large enough such that

$$H^0_L\big(\mathbb{CP}^n_\theta, \underline{\mathcal{E}\mathrm{xt}}^p(F, \mathcal{O}_{\mathbb{CP}^n_\theta})(k-n-1)\big) \cong H^{n-p}\big(\mathbb{CP}^n_\theta, F(-k)\big)^* \qquad (3.22)$$

for any coherent sheaf $F \in \mathsf{coh}(\mathbb{CP}^n_\theta)$ and for all $p \geq 0$. By [13, Prop. 6.8] it follows that the homological dimension of the graded left $\mathcal{A}$-module $\Gamma_L(\underline{\mathcal{E}\mathrm{xt}}^p(F, \mathcal{O}_{\mathbb{CP}^n_\theta}))$ is $\leq n-p$. If the cohomology sheaf E is torsion free, then there is an embedding $E \hookrightarrow \mathcal{E}$ into a locally free sheaf $\mathcal{E}$. By applying the functor $\underline{\mathcal{H}\mathrm{om}}(-, \mathcal{O}_{\mathbb{CP}^n_\theta})$ to the exact sequence $0 \to E \to \mathcal{E} \to \mathcal{E}/E \to 0$, one gets an isomorphism $\underline{\mathcal{E}\mathrm{xt}}^1(E, \mathcal{O}_{\mathbb{CP}^n_\theta}) \cong \underline{\mathcal{E}\mathrm{xt}}^2(\mathcal{E}/E, \mathcal{O}_{\mathbb{CP}^n_\theta})$. An application of the isomorphism (3.22) to $F = \mathcal{E}/E$ using point (3) of Proposition 3.17 then shows that the homological dimension of S is $\leq n-2$. □

3.5 Coequivariant sheaves

We would now like to regard the sheaves which are constructed as the cohomology of a linear monad as coequivariant sheaves E in the sense of [13, Section 4.2], i.e., as elements of the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules whose coactions are compatible with the coaction of $\mathcal{H}_\theta$ on $\mathcal{A}$. Generally, this occurs when the complex $\underline{\mathcal{C}}_\bullet$ in $\mathsf{coh}(X_\theta)$ of Definition 3.11 is also a complex in ${}^{\mathcal{H}_\theta}\mathscr{M}$, i.e., when the sheaves $\mathcal{C}_{-1}, \mathcal{C}_0, \mathcal{C}_1$ are objects of ${}^{\mathcal{H}_\theta}\mathscr{M}$ and the maps $\sigma_{\underline{\mathcal{C}}_\bullet}, \tau_{\underline{\mathcal{C}}_\bullet}$ are morphisms in ${}^{\mathcal{H}_\theta}\mathscr{M}$; it is easy to check that the cohomology of such a monad is a coequivariant sheaf. In contrast to the approach of [8,11], here we regard the set of all $\mathcal{A}$-module morphisms as a vector space, without further structure; our construction of instanton moduli spaces later on using the larger space of torsion free sheaves (rather than just the dense subset of vector bundles) will naturally use universal objects in this setting, and can be described by commutative parameter spaces and standard geometric invariant theory quotients.

For a linear monad (3.12), the first requirement is automatically satisfied due to the coaction (3.2) which lifts to all bundles $\mathcal{O}_{\mathbb{CP}^n_\theta}(k)$ for $k \in \mathbb{Z}$. The second requirement, on the other hand, restricts the allowed differentials. For this, we decompose the $\mathcal{A}$-module homomorphisms as

$$\sigma_w = \sum_{i=1}^{n+1} \sigma^i \otimes w_i, \quad \tau_w = \sum_{i=1}^{n+1} \tau^i \otimes w_i \qquad (3.23)$$

with $\sigma^i \in \mathrm{Hom}_{\mathbb{C}}(V_{-1}, V_0)$ and $\tau^i \in \mathrm{Hom}_{\mathbb{C}}(V_0, V_1)$ for $i = 1, \dots, n+1$.

Lemma 3.24. *The differentials σ_w and τ_w are morphisms in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ if and only if the vector spaces spanned by σ^i and τ^i for $i = 1, \dots, n+1$ are objects of ${}^{\mathcal{H}_\theta}\mathscr{M}$ with left $\mathcal{H}_\theta$-coactions given by*

$$\Delta_L\big(\sigma^i\big) = t_i^{-1} \otimes \sigma^i, \quad \Delta_L\big(\tau^i\big) = t_i^{-1} \otimes \tau^i, \quad i = 1, \dots, n,$$
$$\Delta_L\big(\sigma^{n+1}\big) = 1 \otimes \sigma^{n+1}, \quad \Delta_L\big(\tau^{n+1}\big) = 1 \otimes \tau^{n+1}.$$

Proof. The requisite $\mathcal{H}_\theta$-coequivariance conditions follow easily from (3.2). □

4 Noncommutative twistor geometry

4.1 Grassmannians $\mathbb{G}\mathrm{r}_{\boldsymbol{\theta}}(\boldsymbol{d};\boldsymbol{n})$

In [13, Section 5.3] we defined non-commutative Grassmann varieties $\mathbb{G}\mathrm{r}_\theta(d;V) \cong \mathbb{G}\mathrm{r}_\theta(d;n)$ associated to an $\mathcal{H}_\theta$-comodule V of dimension $n > d$. The homogeneous coordinate algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$ of the non-commutative Grassmannian $\mathbb{G}\mathrm{r}_\theta(d;n)$ is defined as a quotient of the algebra of a suitable projective space $\mathbb{CP}^N_\Theta \cong \mathbb{P}(\bigwedge^d_\theta V)$, with $N = \binom{n}{d} - 1$. The minors Λ^J which span the braided exterior algebra as in (2.5) are labelled by ordered d-multi-indices $J = (j_1 \cdots j_d)$, $1 \le j_\alpha \le n$. The non-commutativity relations between the minors are given by [13]

$$\Lambda^J \, \Lambda^K = \left(\prod_{\alpha,\beta=1}^{d} q^2_{j_\alpha k_\beta} \right) \, \Lambda^K \, \Lambda^J. \tag{4.1}$$

Regarding Λ^J as homogeneous coordinates in $\mathcal{A}(\mathbb{CP}^N_\Theta)$, the $N \times N$ non-commutativity matrix Θ of the projective space containing the embedding of $\mathbb{G}\mathrm{r}_\theta(d;n)$ is completely determined (mod 2π) from the $n \times n$ non-commutativity matrix θ of the Grassmannian as

$$\Theta^{JK} = \sum_{\alpha,\beta=1}^{d} \theta^{j_\alpha k_\beta}. \tag{4.2}$$

This is a necessary and sufficient condition for the existence of an embedding of the non-commutative Grassmannian $\mathbb{G}\mathrm{r}_\theta(d;n) \hookrightarrow \mathbb{CP}^N_\Theta$, with $N = \binom{n}{d} - 1$.

Given the non-commutative relations (4.1) between generators of the projective space, the next step is to exhibit non-commutative Plücker relations.

They generate a homogeneous ideal in the homogeneous coordinate algebra $\mathcal{A}(\mathbb{CP}^N_\Theta)$ of the projective space, and one defines the non-commutative quotient algebra as the graded homogeneous coordinate algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$ of the (embedding of the) non-commutative Grassmannian. The natural non-commutative version of Young symmetry relations takes into account the braided antisymmetry of the minors, and read [13]

$$\sum_{\gamma=1}^{d+1}\left(\prod_{\alpha=1}^{d} q_{i_\gamma i^\gamma_\alpha}\right)\left(\prod_{\beta=1}^{d-1} q_{i_\gamma j_\beta}\right)(-1)^\gamma\,\Lambda^{I\setminus i_\gamma}\,\Lambda^{i_\gamma\cup J}=0 \tag{4.3}$$

for all choices of $(d+1)$-multi-indices I and $(d-1)$-multi-indices J, where $i^\gamma_\alpha\in I\setminus i_\gamma$.

With respect to the non-commutative algebraic torus $T_\theta=(\mathbb{C}^\times_\theta)^n$, the coordinate algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$ is naturally an object of the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ by the left coaction

$$\Delta_L:\mathcal{A}\big(\mathbb{G}\mathrm{r}_\theta(d;n)\big)\;\longrightarrow\;\mathcal{H}_\theta\otimes\mathcal{A}\big(\mathbb{G}\mathrm{r}_\theta(d;n)\big),\quad \Delta_L(\Lambda^J)=t_J\otimes\Lambda^J \tag{4.4}$$

where $t_J:=t_{j_1}\cdots t_{j_d}$.

The tautological bundle $\mathcal{S}_\theta$ on $\mathsf{Open}(\mathbb{G}\mathrm{r}_\theta(d;n))$ is defined to be the subsheaf of elements of the free module $(f_1(\Lambda),\dots,f_n(\Lambda))\in\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))^{\oplus n}$ over the non-commutative Grassmannian, with each $f_k(\Lambda)$, $k=1,\dots,n$, a function on $\mathbb{G}\mathrm{r}_\theta(d;n)$, i.e., an element in $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$, which satisfy the equations

$$\sum_{\alpha=1}^{d+1}\left(\prod_{\beta=1}^{d} q_{j_\alpha j^\alpha_\beta}\right)(-1)^\alpha\,\Lambda^{J\setminus j_\alpha}\,f_{j_\alpha}=0 \tag{4.5}$$

for every ordered $(d+1)$-multi-index $J=(j_1\cdots j_{d+1})$ with $j_1<j_2<\cdots<j_{d+1}$, where the minors of order d obey the relations (4.1). We can use the Plücker map to regard the non-commutative minors $\Lambda^{J\setminus j_\alpha}$ as homogeneous coordinates in $\mathbb{P}(\bigwedge^d_\theta V)$. Then the quotient by the graded two-sided ideal generated by the set of homogeneous relations (4.5) defines the projection of the free module $\mathbb{P}(\bigwedge^d_\theta V)\otimes V\to\mathcal{S}_\theta$. In this case we have to consider the restriction of (4.5) to those elements Λ^J which also satisfy the Young symmetry relations (4.3). This gives the sheaf $\mathcal{S}_\theta$ the natural structure of a graded $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$-bimodule. In [13, Section 6.4] it is shown that the coherent sheaf of non-commutative Kähler differential forms on $\mathsf{Open}(\mathbb{G}\mathrm{r}_\theta(d;n))$ is

isomorphic to the braided tensor product

$$\Omega^1_{\mathbb{G}\mathrm{r}_\theta(d;n)} \;\cong\; \mathcal{S}_\theta \otimes_\theta \mathcal{Q}_\theta \tag{4.6}$$

as a bimodule algebra over $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(d;n))$, where $\mathcal{Q}_\theta$ is the orthogonal complement of the tautological bundle defined through the non-commutative Euler sequence

$$0 \longrightarrow \mathcal{Q}_\theta^\vee \longrightarrow \mathcal{A}\big(\mathbb{G}\mathrm{r}_\theta(d;n)\big) \otimes V \xrightarrow{\hat{\eta}} \mathcal{S}_\theta \longrightarrow 0. \tag{4.7}$$

Remark 4.8. For $d = 1$ this construction gives an alternative description of the homogeneous coordinate algebra of $\mathbb{CP}^{n-1}_\theta \cong \mathbb{G}\mathrm{r}_\theta(1;n)^*$; in this case $N = n-1$ and $\Theta = \theta$. The explicit mapping between the two homogeneous coordinate algebras, with relations (3.1) induced by the local fan construction of $\mathbb{CP}^{n-1}_\theta$ and (4.1) induced by the non-commutative deformation $\mathrm{GL}_\theta(n)$ of the general linear group, is described in [13, Section 5.2].

4.2 Klein quadric $\mathbb{G}\mathrm{r}_\theta(2;4)$

We explicitly work out the algebraic relations in the case $d = 2$ and $n = 4$. Consider the algebra projection $\mathcal{A}(\mathbb{CP}^5_\Theta) \to \mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$. Give to the multi-indices labelling the minors a lexicographic ordering, such that $1 = (1\,2)$, $2 = (1\,3)$, $3 = (1\,4)$, $4 = (2\,3)$, $5 = (2\,4)$, and $6 = (3\,4)$. Then the expression (4.2) for the skew-symmetric non-commutativity matrix Θ in terms of entries of θ is given by

$$\Theta = \begin{pmatrix} 0 & -\theta^{12}+\theta^{13}+\theta^{23} & -\theta^{12}+\theta^{14}+\theta^{24} & \theta^{12}+\theta^{13}+\theta^{23} & \theta^{12}+\theta^{14}+\theta^{24} & \theta^{13}+\theta^{14}+\theta^{23}+\theta^{24} \\ & 0 & -\theta^{13}+\theta^{14}+\theta^{34} & \theta^{12}+\theta^{13}-\theta^{23} & \theta^{12}+\theta^{14}-\theta^{23}+\theta^{34} & \theta^{13}+\theta^{14}+\theta^{34} \\ & & 0 & \theta^{12}+\theta^{13}-\theta^{24}-\theta^{34} & \theta^{12}+\theta^{14}-\theta^{24} & \theta^{13}+\theta^{14}-\theta^{34} \\ & & & 0 & -\theta^{23}+\theta^{24}+\theta^{34} & \theta^{23}+\theta^{24}+\theta^{34} \\ & & & & 0 & \theta^{23}+\theta^{24}-\theta^{34} \\ & & & & & 0 \end{pmatrix}.$$

It follows that in order to be interpreted as minors of a non-commutative matrix, the generators of $\mathcal{A}(\mathbb{CP}^5_\Theta)$ cannot have generic non-commutativity

relations. From the above matrix one sees that one must take for example

$$\Theta^{46} - \Theta^{56} = \Theta^{26} - \Theta^{36} = 2\,\theta^{34},$$

and hence it is easy to construct a matrix Θ parameterizing a projective space $\mathbb{CP}^5_\Theta$ that cannot contain any embedding of any Grassmannian $\mathbb{G}\mathrm{r}_\theta(2;4)$.

Let us consider the non-commutative Plücker relations (4.3). Various choices for the multi-indices I and J yield structure equations, i.e., the non-commutative relations among minors. For example, if we set $I = (1\,2\,3)$ and $J = (1)$, then the equation (4.3) reduces to a two-term relation (as the additional term contains $\Lambda^{(11)} = 0$) given by

$$-q_{21}\,q_{23}\,q_{21}\;\Lambda^{(13)}\,\Lambda^{(21)} + q_{31}\,q_{32}\;\Lambda^{(12)}\,\Lambda^{(31)} = 0.$$

Using the alternating properties [13, equation (2.30)] $\Lambda^{(21)} = -\Lambda^{(12)}$ and $\Lambda^{(31)} = -\Lambda^{(13)}$, this expression can be reordered as

$$\Lambda^{(12)}\,\Lambda^{(13)} = \;q_{12}^{-2}\,q_{23}^{2}\,q_{13}^{2}\;\Lambda^{(13)}\,\Lambda^{(12)},$$

which is exactly (4.1) for the two minors considered. One also arrives at (4.1) for all other choices of multi-indices which lead to a two-term Plücker equation. Thus from these "trivial" Plücker equations we can derive completely the non-commutativity relations (4.1).

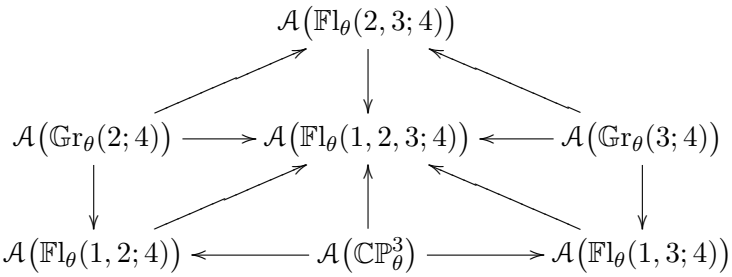
Now let us consider the only three-term Plücker equation, which is a non-commutative deformation of the well-known classical equation describing the Klein quadric $\mathbb{G}\mathrm{r}(2;4) \hookrightarrow \mathbb{CP}^5$. It comes from (4.3) with $I = (1\,2\,3)$ and $J = (4)$. After rearranging all indices labelling the minors in increasing order using antisymmetry, and the minors themselves using (4.1), we obtain

$$q_{31}\,q_{32}\,q_{34}\;\Lambda^{(12)}\,\Lambda^{(34)} - q_{21}\,q_{23}\,q_{24}\;\Lambda^{(13)}\,\Lambda^{(24)} + q_{12}\,q_{13}\,q_{14}\;\Lambda^{(23)}\,\Lambda^{(14)} = 0. \tag{4.9}$$

4.3 Twistor correspondences

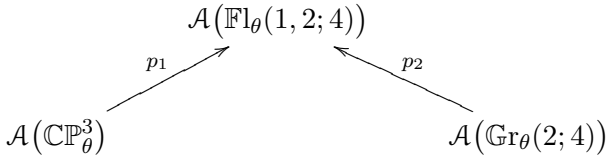
Noncommutative flag varieties associated to $\mathcal{H}_\theta$ comodules V of dimension n are defined in generality in [13, Section 5.4]. In this paper we will only need the flag varieties of GL(4); these are the ones which naturally appear in the double fibrations underlying the most important twistor correspondences [23]. All non-commutative double twistor fibrations are included

in the complete non-commutative flag variety for $n = 4$, according to the diagram



where all morphisms are subalgebra inclusions.

In this paper we will focus on the non-commutative correspondence diagram



(4.10)

which is a non-commutative deformation of the usual Penrose twistor correspondence, with $\mathcal{A}(\mathbb{C}\mathrm{P}^3_\theta)$ the "non-commutative twistor algebra". This diagram, as well as all double fibrations above, is an example of a "non-commutative correspondence" in the sense of [12, Section 5.2]. In the present case, the morphism $E \mapsto {p_2}^* \, p_{1*}(E)$ gives a map from sheaves in $\mathsf{coh}(\mathbb{C}\mathrm{P}^3_\theta)$ to sheaves in $\mathsf{coh}(\mathbb{G}\mathrm{r}_\theta(2;4))$, and it defines a non-commutative deformation of the usual Penrose–Ward twistor transform described in more detail in Section 4.4 below.

We will describe the homogeneous coordinate algebra of the non-commutative partial flag variety $\mathbb{F}\mathrm{l}_\theta(1,2;4)$ using the algebra projection [13]

$$\mathcal{A}\big(\mathbb{C}\mathrm{P}^3_\theta\big) \otimes_\theta \mathcal{A}\big(\mathbb{G}\mathrm{r}_\theta(2;4)\big) \longrightarrow \mathcal{A}\big(\mathbb{F}\mathrm{l}_\theta(1,2;4)\big).$$

The braided tensor product algebra $\mathcal{A}(\mathbb{C}\mathrm{P}^3_\theta) \otimes_\theta \mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$, as an object in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$, is generated by the homogeneous coordinate elements w_i with relations (4.1) for $d = 1$, i.e., $w_i \, w_j = q_{ij}^2 \, w_j \, w_i$ for $i, j = 1, 2, 3, 4$, and by the non-commutative 2×2 minors $\Lambda^{(j_1 j_2)}$, $1 \le j_1 < j_2 \le 4$, obeying (4.1) for $d = 2$ and the quadric relation (4.9). They also obey the structure equations

(relations among minors of different order, see [13, equation (2.29)])

$$w_i\,\Lambda^{(j_1j_2)} = q_{12}^{-2}\,q_{i\,j_1}^2\,q_{i\,j_2}^2\;\Lambda^{(j_1j_2)}\,w_i \tag{4.11}$$

for all $i=1,2,3,4$ and $1\le j_1<j_2\le 4$. The remaining non-commutative Plücker equations come from the Young symmetry relations of [13, equation (5.18)] with $d=2$ and $d'=1$. Working through all four increasing cyclic permutations of order three in S_4 for the multi-index I, by completely analogous calculations to those of Section 4.2 one finds the additional relations

$$\begin{aligned}
&q_{12}\,q_{13}\;\Lambda^{(23)}\,w_1 - q_{21}\,q_{23}\;\Lambda^{(13)}\,w_2 + q_{31}\,q_{32}\;\Lambda^{(12)}\,w_3 = 0,\\
&q_{12}\,q_{14}\;\Lambda^{(24)}\,w_1 - q_{21}\,q_{24}\;\Lambda^{(14)}\,w_2 + q_{41}\,q_{42}\;\Lambda^{(12)}\,w_4 = 0,\\
&q_{13}\,q_{14}\;\Lambda^{(34)}\,w_1 - q_{31}\,q_{34}\;\Lambda^{(14)}\,w_3 + q_{41}\,q_{43}\;\Lambda^{(13)}\,w_4 = 0,\\
&q_{23}\,q_{24}\;\Lambda^{(34)}\,w_2 - q_{32}\,q_{34}\;\Lambda^{(24)}\,w_3 + q_{42}\,q_{43}\;\Lambda^{(23)}\,w_4 = 0.
\end{aligned} \tag{4.12}$$

Other quantum deformations of these flag varieties in the context of twistor theory can be found in [7, 10, 18, 20, 25, 31].

4.4 Twistor transform

As mentioned in Section 4.3, the twistor transform of coherent sheaves on $\mathsf{Open}(\mathbb{CP}^3_\theta)$ is determined by the non-commutative correspondence diagram (4.10). It is the Fourier–Mukai-type transform $E\mapsto {p_2}^*\,p_{1*}(E)$ defined on $\mathsf{coh}(\mathbb{CP}^3_\theta)\to\mathsf{coh}(\mathbb{Gr}_\theta(2;4))$ as follows. Given a graded right $\mathcal{A}(\mathbb{CP}^3_\theta)$-module M in $\mathsf{gr}(\mathcal{A}(\mathbb{CP}^3_\theta))$, the push-forward

$$M' = p_{1*}(M) = M\otimes_{\mathcal{A}(\mathbb{CP}^3_\theta)}\mathcal{A}\big(\mathbb{Fl}_\theta(1,2;4)\big) \tag{4.13}$$

is a bigraded right module over $\mathcal{A}(\mathbb{Fl}_\theta(1,2;4))$, where on the right-hand side we regard the algebra $\mathcal{A}(\mathbb{Fl}_\theta(1,2;4))$ as an $\mathcal{A}(\mathbb{CP}^3_\theta)$-bimodule according to the action described in Section 4.3. The diagonal subspace of this module, in the sense of [25, Section 8], induces the push-forward functor $p_{1*}:\mathsf{coh}(\mathbb{CP}^3_\theta)\to\mathsf{coh}(\mathbb{Fl}_\theta(1,2;4))$. Similarly, one defines the push-forward functor $p_{2*}:\mathsf{coh}(\mathbb{Gr}_\theta(2;4))\to\mathsf{coh}(\mathbb{Fl}_\theta(1,2;4))$, which has a right adjoint functor denoted ${p_2}^*:\mathsf{coh}(\mathbb{Fl}_\theta(1,2;4))\to\mathsf{coh}(\mathbb{Gr}_\theta(2;4))$.

For the non-commutative twistor transforms of elementary bundles on $\mathsf{Open}(\mathbb{CP}^3_\theta)$ we have the following computation.

Lemma 4.14. *The non-commutative twistor transforms of the locally free sheaves* $\mathcal{O}_{\mathbb{CP}^3_\theta}(k)$ *for* $k \in \mathbb{Z}$ *are given by*

$$p_2{}^* p_{1*}\big(\mathcal{O}_{\mathbb{CP}^3_\theta}(k)\big) = \begin{cases} \mathrm{Sym}^k_\theta(\mathcal{S}_\theta), & k \geq 0, \\ 0, & k < 0, \end{cases}$$

where $\mathcal{S}_\theta$ *is the tautological bundle on* $\mathsf{Open}(\mathbb{G}\mathrm{r}_\theta(2;4))$ *defined in Section* 4.1 *and* $\mathrm{Sym}^k_\theta(\mathcal{S}_\theta)$ *is the bundle associated to the graded right module*

$$\Gamma(\mathcal{S}_\theta)^{\otimes k} \Big/ \big\langle s_1 \otimes s_2 - \Psi^\theta_{\Gamma(\mathcal{S}_\theta),\Gamma(\mathcal{S}_\theta)}(s_1 \otimes s_2)\big\rangle_{s_1,s_2\in\Gamma(\mathcal{S}_\theta)}$$

over $\mathcal{A}\big(\mathbb{G}\mathrm{r}_\theta(2;4)\big)$.

Proof. The direct image formula (4.13) gives

$$p_{1*}(\mathcal{A}(\mathbb{CP}^3_\theta)) = \mathcal{A}(\mathbb{F}\mathrm{l}_\theta(1,2;4)) = p_{2*}(\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))),$$

whence $p_2{}^* p_{1*}(\mathcal{O}_{\mathbb{CP}^3_\theta}) = p_2{}^* p_{2*}(\mathcal{O}_{\mathbb{G}\mathrm{r}_\theta(2;4)}) = \mathcal{O}_{\mathbb{G}\mathrm{r}_\theta(2;4)}$ and the result holds for $k = 0$. By the non-commutative Plücker equations (4.12), the image of the set of generators $p_2^{-1}\,(p_1(\mathcal{A}(\mathbb{CP}^3_\theta)_1))$ consists of elements $f_k = w_k$ satisfying the equations (4.5) for $d = 2$ and $n = 4$, which defines the tautological bundle over the non-commutative Grassmann variety $\mathbb{G}\mathrm{r}_\theta(2;4)$. It follows that $p_2{}^* p_{1*}(\mathcal{O}_{\mathbb{CP}^3_\theta}(1)) = p_2{}^* p_{2*}(\mathcal{S}_\theta) = \mathcal{S}_\theta$. The result for negative degree shifts is then clear, and for positive degree shifts follows by applying the same reasoning to $\mathcal{A}(\mathbb{CP}^3_\theta)_1^{\otimes k}$. □

4.5 Sphere S^4_θ

In order to introduce a $*$-algebra structure on the algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$ and a compatible twistor construction of instanton bundles to be considered later on, in the remainder of this section we consider a reduction from a six complex parameter deformation to a one real parameter deformation. Thus, in particular, we set

$$q_{12} = q_{21}^{-1} =: q, \quad q_{ij} = 1 \quad \text{otherwise}, \tag{4.15}$$

and in addition we assume that $q \in \mathbb{R}$, i.e., that the non-commutativity parameter $\theta := \theta^{12}$ is purely imaginary.

The compatible $*$-involution is defined on the generators of $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$ as

$$\begin{aligned}&\Lambda^{(13)\dagger} = q\,\Lambda^{(24)}, \quad \Lambda^{(14)\dagger} = -q^{-1}\,\Lambda^{(23)},\\ &\Lambda^{(12)\dagger} = \Lambda^{(12)}, \quad \Lambda^{(34)\dagger} = \Lambda^{(34)},\end{aligned} \tag{4.16}$$

and then extended to the whole of $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$ as a conjugate linear anti-homomorphism. When substituted in the Klein quadric equation (4.9) we obtain

$$q\,\Lambda^{(12)}\,\Lambda^{(34)} - q^{-1}\,\Lambda^{(13)}\,\Lambda^{(13)\dagger} - q\,\Lambda^{(14)}\,\Lambda^{(14)\dagger} = 0. \tag{4.17}$$

Our choice of deformation (4.15) and of the real structure (4.16) is distinguished by the fact that, in the classical case, the quadratic form in (4.17) has signature $(5,1)$ which is the correct signature to interpret the relation (4.17) as the equation of a four-sphere in a real slice of $\mathbb{CP}^5$. Other choices would lead to different signature and hence to other real embedded subvarieties, see e.g., [4, Chap. III.1].

We interpret the corresponding $*$-algebra as the coordinate algebra $\mathcal{A}(S^4_\theta)$ of a non-commutative four-sphere S^4_θ with a non-central "radius". It is rather different from the spheres considered e.g., in [14] or even in [9, 27]. To see this, let us redefine the generators by writing $\frac{q}{2}\,(\Lambda^{(12)} - \Lambda^{(34)}) =: X$ and $\frac{q}{2}\,(\Lambda^{(12)} + \Lambda^{(34)}) =: R$, where both elements R, X are Hermitean and commute with each other but not with the remaining minors. Simple algebra then transforms the relation (4.17) into

$$\Lambda^{(13)}\,\Lambda^{(13)\dagger} + q^2\,\Lambda^{(14)}\,\Lambda^{(14)\dagger} + X^2 = R^2. \tag{4.18}$$

This is a deformation of the equation of a four-sphere in homogeneous coordinates on a real slice of $\mathbb{CP}^5$, with a non-central radius R. Due to this, one is not allowed to fix it to some real number (typically $R = 1$ in the classical case). Furthermore, in the present non-commutative setting the homogeneous element R does not generate a right or left denominator set, i.e., powers of the corresponding fraction element would not close to an algebra. Thus it is not possible to consider Ore localization with respect to R, an operation that classically would correspond to "rescaling" the homogeneous coordinates to get a description of the sphere in affine coordinates of $\mathbb{R}^5$. Thus, for our non-commutative sphere there is no hope for any global affine description. However, we will now see that one can still use localization to go to local patches for the sphere.

Using the two Hermitean generators $\Lambda^{(12)}$ and $\Lambda^{(34)}$ we can define two localizations to affine subvarieties of the Grassmannian $\mathbb{G}\mathrm{r}_\theta(2;4)$, whose

"real slices" are interpreted as local patches of the sphere S^4_θ. The two generators $\Lambda^{(12)}$ and $\Lambda^{(34)}$ are rather different in nature, and hence so are the corresponding localizations, as $\Lambda^{(34)}$ is central while $\Lambda^{(12)}$ is not. From the commutation relations (4.1) with $d = 2$ and $n = 4$, together with the q-values (4.15), one finds indeed that $\Lambda^{(34)}$ is a central element of the homogeneous coordinate algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$.

Proposition 4.19. *The central non-commutative minor $\Lambda^{(34)}$ generates a right denominator set in $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$, and the degree zero subalgebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))[\Lambda^{(34)\,-1}]_0$ of the right Ore localization of $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$ with respect to $\Lambda^{(34)}$ is isomorphic to the $\mathbb{C}$-algebra generated by elements $\xi_i, \bar{\xi}_i$, $i = 1, 2$ with the relations*

$$\begin{aligned}
&\xi_1\,\bar{\xi}_1 = q^2\;\bar{\xi}_1\,\xi_1, \quad \xi_2\,\bar{\xi}_2 = q^{-2}\;\bar{\xi}_2\,\xi_2,\\
&\xi_1\,\xi_2 = q^2\;\xi_2\,\xi_1, \quad \bar{\xi}_1\,\bar{\xi}_2 = q^{-2}\;\bar{\xi}_2\,\bar{\xi}_1,\\
&\xi_1\,\bar{\xi}_2 = \bar{\xi}_2\,\xi_1, \quad \xi_2\,\bar{\xi}_1 = \bar{\xi}_1\,\xi_2.
\end{aligned}$$

Proof. As already noted, the minor $\Lambda^{(34)}$ is a central element of $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$, whence its non-negative powers forms a right (and left) denominator set. The degree zero subalgebra of $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))[\Lambda^{(34)\,-1}]$ is generated by the elements

$$\begin{aligned}
&\xi_1 = -\Lambda^{(14)}\,\Lambda^{(34)\,-1}, \quad \xi_2 = -\Lambda^{(24)}\,\Lambda^{(34)\,-1},\\
&\bar{\xi}_1 = \Lambda^{(23)}\,\Lambda^{(34)\,-1}, \quad \bar{\xi}_2 = \Lambda^{(13)}\,\Lambda^{(34)\,-1},
\end{aligned}$$

together with

$$\rho = q\;\Lambda^{(12)}\,\Lambda^{(34)\,-1}.$$

Since $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))[\Lambda^{(34)\,-1}]$ is a commutative localization, a straightforward calculation using (4.1) establishes the commutation relations among the generators $\xi_i, \bar{\xi}_i$, $i = 1, 2$, while the Plücker relation (4.9) becomes

$$\rho = \xi_1\,\bar{\xi}_1 - \bar{\xi}_2\,\xi_2,$$

showing that the generator ρ in the algebra $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))[\Lambda^{(34)\,-1}]_0$ is redundant. □

The $*$-involution of (4.16) gives on the generators $\xi_i, \bar{\xi}_i$, $i = 1, 2$ the relations

$$\xi_1^\dagger = q^{-1}\;\bar{\xi}_1, \quad \xi_2^\dagger = -q^{-1}\;\bar{\xi}_2, \tag{4.20}$$

from which it also follows that $\rho = q\ \xi_1\,\xi_1^\dagger + q^3\ \xi_2\,\xi_2^\dagger$. The corresponding $*$-algebra is dual to a non-commutative real variety denoted $\mathbb{R}^4_\theta$. This non-commutative space differs from the q-deformed euclidean space $\mathbb{R}^4_{q,\hbar=0}$ considered in [25, Section 9]; it is somewhat analogous to the quantum Minkowski space constructed in [18], although its origin is very different. The present deformation originates through the general categorical prescription of Section 2.2 via the natural action of the torus $T = (\mathbb{C}^\times)^2$ on $\mathbb{R}^4$ described by the left coaction

$$\Delta_L : \mathcal{A}\big(\mathbb{R}^4\big) \ \longrightarrow\ \mathcal{H} \otimes \mathcal{A}\big(\mathbb{R}^4\big)$$

with

$$\begin{aligned} &\Delta_L(\xi_1) = t_1 \otimes \xi_1, \quad \Delta_L(\xi_2) = t_2 \otimes \xi_2, \\ &\Delta_L(\bar\xi_1) = t_2 \otimes \bar\xi_1, \quad \Delta_L(\bar\xi_2) = t_1 \otimes \bar\xi_2, \end{aligned} \tag{4.21}$$

and extended as an algebra map. These relations can be directly obtained from the torus action on $\mathbb{G}\mathrm{r}(2;4)$ described in (4.4) for our choice of deformation (4.15), which amount to acting non-trivially only on the indices 1 and 2.

Geometrically, the space $\mathbb{R}^4_\theta$ is regarded as an open affine subvariety of the non-commutative four-sphere S^4_θ defined before. A second open affine subvariety of S^4_θ is obtained by the localization onto another affine subvariety of the Grassmannian $\mathbb{G}\mathrm{r}_\theta(2;4)$ via the non-central Hermitean minor $\Lambda^{(12)}$.

Proposition 4.22. *The non-central non-commutative minor $\Lambda^{(12)}$ generates a left denominator set in $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$, and the degree zero subalgebra ${}_0[\Lambda^{(12)\,-1}]\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$ of the left Ore localization of $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$ with respect to $\Lambda^{(12)}$ is isomorphic to the $\mathbb{C}$-algebra generated by elements $\zeta_i, \bar\zeta_i$, $i = 1, 2$ with the relations*

$$\begin{aligned} &\zeta_1\,\bar\zeta_1 = q^{-2}\ \bar\zeta_1\,\zeta_1, \quad \zeta_2\,\bar\zeta_2 = q^2\ \bar\zeta_2\,\zeta_2, \\ &\zeta_1\,\zeta_2 = q^{-2}\ \zeta_2\,\zeta_1, \quad \bar\zeta_1\,\bar\zeta_2 = q^2\ \bar\zeta_2\,\bar\zeta_1, \\ &\zeta_1\,\bar\zeta_2 = \bar\zeta_2\,\zeta_1, \quad \zeta_2\,\bar\zeta_1 = \bar\zeta_1\,\zeta_2. \end{aligned}$$

Proof. From the commutation relations (4.1) with $d = 2$, $n = 4$, together with the q-values (4.15), one has

$$\big(\Lambda^{(12)}\big)\,\mathcal{A}\big(\mathbb{G}\mathrm{r}_\theta(2;4)\big) = \mathcal{A}\big(\mathbb{G}\mathrm{r}_\theta(2;4)\big)\,\big(\Lambda^{(12)}\big)$$

as sets. Whence the element $\Lambda^{(12)}$ is left (and right) permutable in $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$, so the set of non-negative powers of $\Lambda^{(12)}$ is a left (and right) denominator set. The degree zero subalgebra of $[\Lambda^{(12)\,-1}]\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$ is generated by the elements

$$\begin{aligned}\zeta_1 &= -\Lambda^{(12)\,-1}\,\Lambda^{(14)}, \quad \zeta_2 = -\Lambda^{(12)\,-1}\,\Lambda^{(24)},\\ \bar{\zeta_1} &= \Lambda^{(12)\,-1}\,\Lambda^{(23)}, \quad \bar{\zeta_2} = \Lambda^{(12)\,-1}\,\Lambda^{(13)},\end{aligned}$$

together with

$$\tilde{\rho} = q\ \Lambda^{(12)\,-1}\,\Lambda^{(34)}.$$

A straightforward calculation using (4.1) together with the rules for non-commutative Ore localization establishes the commutation relations among the generators $\zeta_i, \bar{\zeta_i}$, $i = 1, 2$, while the Plücker relation (4.9) becomes

$$\tilde{\rho} = q^2\ \zeta_1\,\bar{\zeta_1} - q^2\ \bar{\zeta_2}\,\zeta_2 = \bar{\zeta_1}\ \zeta_1 - \zeta_2\ \bar{\zeta_2}$$

similarly to the previous case. □

The $*$-involution on the algebra generated by $\zeta_i, \bar{\zeta_i}$, $i = 1, 2$ induced from the real structure (4.16) reads

$$\zeta_1^\dagger = q\ \bar{\zeta_1}, \quad \zeta_2^\dagger = -q^{-3}\ \bar{\zeta_2}, \tag{4.23}$$

from which it also follows that $\tilde{\rho} = q\ \zeta_1\,\zeta_1^\dagger + q^3\ \zeta_2\,\zeta_2^\dagger$. The corresponding non-commutative real variety is denoted $\widetilde{\mathbb{R}}^4_\theta$. The counterpart of the coaction (4.21) is now given by the dual left coaction

$$\Delta_L : \mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big) \ \longrightarrow\ \mathcal{H}_\theta \otimes \mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big)$$

with

$$\begin{aligned}\Delta_L(\zeta_1) &= t_2^{-1} \otimes \zeta_1, \quad \Delta_L(\zeta_2) = t_1^{-1} \otimes \zeta_2,\\ \Delta_L(\bar{\zeta_1}) &= t_1^{-1} \otimes \bar{\zeta_1}, \quad \Delta_L(\bar{\zeta_2}) = t_2^{-1} \otimes \bar{\zeta_2},\end{aligned} \tag{4.24}$$

and again extended as an algebra map.

The intersection of the two open affine subvarieties $\mathbb{R}^4_\theta$ and $\widetilde{\mathbb{R}}^4_\theta$ is described by adjoining the element $\tilde{\rho}$ to $\mathcal{A}(\mathbb{R}^4_\theta)$ and ρ to $\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$, and computing the gluing automorphism between the two resulting algebras. This also defines the non-commutative real variety S^4_θ in an analogous manner as our generic non-commutative toric varieties.

Proposition 4.25. *The algebras $\mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,]$ and $[\rho]\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$ are isomorphic as $*$-algebras in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules.*

Proof. Define an algebra morphism

$$G : \mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,] \;\longrightarrow\; [\rho]\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$$

on generators by

$$\big(\xi_j, \bar\xi_j\,\big) \;\longmapsto\; \big(\rho\;\zeta_j, \rho\;\bar\zeta_j\,\big) \quad \text{for} \quad j = 1, 2.$$

The inverse map G^{-1} is then given by

$$\big(\zeta_j, \bar\zeta_j\,\big) \;\longmapsto\; \big(\tilde\rho\;\xi_j, \tilde\rho\;\bar\xi_j\,\big) \quad \text{for} \quad j = 1, 2,$$

and one can check that G is an algebra isomorphism. Moreover, one has $G(\xi_i^\dagger) = G(\xi_i)^\dagger$, and hence $G(a^\dagger) = G(a)^\dagger$ for all $a \in \mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,]$. Finally, using the coactions (4.21) and (4.24) we compute

$$\Delta_L(\rho) = t_1\, t_2 \otimes \rho, \quad \Delta_L(\tilde\rho\,) = (t_1\, t_2)^{-1} \otimes \tilde\rho,$$

from which one easily shows that the map G is coequivariant, i.e., $\Delta_L \circ G = (\mathrm{id} \otimes G) \circ \Delta_L$, and hence is a $*$-isomorphism in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$. □

Remark 4.26. The "geometric" interpretation of the map G in the proof of Proposition 4.25 is as follows. In the overlap of the two patches there are two sets of generators to describe "points", and G describes how to pass from the affine coordinates ξ_j to the affine coordinates ζ_j. It is indeed the identity map in terms of the homogeneous coordinates Λ^J on the Grassmannian, i.e., we do not "move" points, we just describe how the coordinates of the two patches are related.

4.6 Twistor fibration

As a particular case of the non-commutative projective spaces in Section 3.1, one has the *non-commutative twistor algebra* $\mathcal{A}^{\mathrm{tw}} = \mathcal{A}(\mathbb{CP}^3_\theta)$, the homogeneous coordinate algebra generated by w_i, $i = 1, 2, 3, 4$ and the relations

$$\begin{aligned} &w_i\, w_k = w_k\, w_i, \quad i = 1, 2, 3, 4\ ,\ k = 3, 4, \\ &w_1\, w_2 = q^2\; w_2\, w_1. \end{aligned} \tag{4.27}$$

It is dual to the (complex) twistor space of the non-commutative sphere S^4_θ. When $q \in \mathbb{R}$, there is a natural real structure on $\mathcal{A}^{\mathrm{tw}}$ such that

$$w_1^\dagger = w_2, \quad w_2^\dagger = w_1, \quad w_3^\dagger = w_4, \quad w_4^\dagger = w_3. \tag{4.28}$$

The restriction functor induced by Proposition 4.19 is denoted

$$j^\bullet : \mathsf{coh}\big(\mathbb{G}\mathrm{r}_\theta(2;4)\big) \;\longrightarrow\; \mathsf{coh}\big(\mathbb{R}^4_\theta\big).$$

At the level of the non-commutative correspondence algebra $\mathcal{A}(\mathbb{F}\mathrm{l}_\theta(1,2;4))$, regarded as an $\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4))$-bimodule, the Plücker equations (4.12) in the localized coordinate algebra $\mathcal{A}(\mathbb{F}\mathrm{l}_\theta(1,2;4))[\Lambda^{(34)\,-1}]_0$ now read

$$w_1 = -w_3\,\xi_1 - w_4\,\bar{\xi}_2, \quad w_2 = -w_3\,\xi_2 - w_4\,\bar{\xi}_1. \tag{4.29}$$

Using the commutation relations (4.11) together with the multiplication rule of non-commutative Ore localization [13, Section 1.3], one easily checks that the generators w_3, w_4 commute not only among themselves but also with $\xi_i, \bar{\xi}_i$, $i = 1, 2$, and it follows that

$$\mathcal{A}\big(\mathbb{F}\mathrm{l}_\theta(1,2;4)\big)\big[\Lambda^{(34)\,-1}\big]_0 \cong \mathcal{A}\big(\mathbb{R}^4_\theta\big) \otimes \mathcal{A}\big(\mathbb{C}\mathrm{P}^1\big), \tag{4.30}$$

with $\mathcal{A}(\mathbb{C}\mathrm{P}^1) = \mathbb{C}[w_3, w_4]$ the homogeneous coordinate algebra of a commutative projective line $\mathbb{C}\mathrm{P}^1$. This isomorphism implies that, in the image of the functor $j^\bullet$, the tautological bundle $\mathcal{S}_\theta$ obtained through the twistor transform via Lemma 4.14 restricts to the free right $\mathcal{A}(\mathbb{R}^4_\theta)$-module of rank two, spanned by w_3 and w_4.

The situation is somewhat different for the non-commutative localization described by Proposition 4.22. The Plücker equations (4.12) in ${}_0[\Lambda^{(12)\,-1}]\mathcal{A}(\mathbb{F}\mathrm{l}_\theta(1,2;4))$ are

$$w_3 = -q\;\bar{\zeta}_1\,w_1 + q^{-1}\;\bar{\zeta}_2\,w_2, \quad w_4 = q\;\zeta_2\,w_1 - q^{-1}\;\zeta_1\,w_2. \tag{4.31}$$

Now, however, the generators w_1, w_2 do not commute with $\zeta_i, \bar{\zeta}_i$, $i = 1, 2$ in general; one finds

$$\begin{aligned}
w_1\,\zeta_1 &= q^{-2}\;\zeta_1\,w_1, \quad & w_1\,\bar{\zeta}_1 &= \bar{\zeta}_1\,w_1,\\
w_1\,\zeta_2 &= \zeta_2\,w_1, \quad & w_1\,\bar{\zeta}_2 &= q^{-2}\;\bar{\zeta}_2\,w_1,\\
w_2\,\zeta_1 &= \zeta_1\,w_2, \quad & w_2\,\bar{\zeta}_1 &= q^{2}\;\bar{\zeta}_1\,w_2,\\
w_2\,\zeta_2 &= q^{2}\;\zeta_2\,w_2, \quad & w_2\,\bar{\zeta}_2 &= \bar{\zeta}_2\,w_2.
\end{aligned}$$

As a consequence, the localized coordinate algebra has the structure of the braided tensor product algebra

$$ {}_0\big[\Lambda^{(12)\,-1}\big]\mathcal{A}\big(\mathbb{F}\mathrm{l}_\theta(1,2;4)\big) \cong \mathcal{A}\big(\mathbb{CP}^1_\theta\big) \otimes_\theta \mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big), $$

where the non-commutative projective line $\mathbb{CP}^1_\theta$ has homogeneous coordinates w_1, w_2 subject to the relations (4.27). We will show in Proposition 5.2 below that coherent sheaves on $\mathbb{CP}^1_\theta$ can be functorially identified with sheaves on a commutative line $\mathbb{CP}^1$, and hence, in the image of the restriction functor $\tilde{j}^\bullet : \mathsf{coh}(\mathbb{G}\mathrm{r}_\theta(2;4)) \to \mathsf{coh}(\widetilde{\mathbb{R}}^4_\theta)$ induced by Proposition 4.22, the tautological bundle $\mathcal{S}_\theta$ restricts to the free right $\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$-module of rank two. The free modules $\mathcal{A}(\mathbb{R}^4_\theta) \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$ carry natural $*$-involutions induced by (4.20), (4.23) and (4.28), and by Propositions 4.19, 4.22 and 4.25 there is naturally an isomorphism

$$ G_2 : \mathcal{A}(\mathbb{R}^4_\theta)[\tilde{\rho}\,] \otimes \mathbb{C}^2 \;\longrightarrow\; \mathbb{C}^2 \otimes [\rho]\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta) $$

in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$, which is compatible with the $*$-structures and satisfies $G_2(v \triangleleft a) = G(a) \triangleright G_2(v)$ for all $a \in \mathcal{A}(\mathbb{R}^4_\theta)[\tilde{\rho}\,]$ and $v \in \mathcal{A}(\mathbb{R}^4_\theta)[\tilde{\rho}\,] \otimes \mathbb{C}^2$. This describes the twistor bundle over the non-commutative sphere S^4_θ.

5 Instanton counting on $\mathbb{CP}^2_\theta$

5.1 Framed modules

The non-commutative projective plane $\mathbb{CP}^2_\theta$ was described in [13, Section 3.3] using combinatorial data encoded in the fan of $\mathbb{CP}^2$, and in Section 3.1 above via the homogeneous coordinate algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^2_\theta)$ whose generators have relations

$$ w_1\, w_2 = q^2\; w_2\, w_1, \quad w_1\, w_3 = w_3\, w_1, \quad w_2\, w_3 = w_3\, w_2, \tag{5.1} $$

where $q = \exp(\frac{\mathrm{i}}{2}\,\theta)$ with $\theta \in \mathbb{C}$. Let $\mathcal{A}_\ell := \mathcal{A}/(\mathcal{A} \cdot w_3)$. We identify $\mathcal{A}_\ell = \mathcal{A}(\mathbb{CP}^1_\theta)$ as the homogeneous coordinate algebra dual to a *non-commutative projective line* $\mathbb{CP}^1_\theta$. The algebra projection $p : \mathcal{A} \to \mathcal{A}_\ell$ is dual to a closed embedding $\mathbb{CP}^1_\theta \hookrightarrow \mathbb{CP}^2_\theta$ of non-commutative projective varieties.

For any pair $\theta, \theta' \in \mathbb{C} \setminus \pi\,\mathbb{Z}$, the abelian categories of coherent sheaves $\mathsf{coh}(\mathbb{CP}^2_\theta)$ and $\mathsf{coh}(\mathbb{CP}^2_{\theta'})$ are equivalent, while $\mathsf{coh}(\mathbb{CP}^2_\theta) \ncong \mathsf{coh}(\mathbb{CP}^2)$ for any $\theta \notin \pi\,\mathbb{Z}$. On the other hand, the category $\mathsf{coh}(\mathbb{CP}^1_\theta)$ is independent of $\theta \in \mathbb{C}$;

the geometric structure of the algebra $\mathcal{A}_\ell$ thus agrees with general expectations that generic deformed curves in algebraic geometry are commutative (see e.g., [24, Cor. 5.3] and [2]).

Proposition 5.2. *For any $\theta \in \mathbb{C}$ there is a natural equivalence of abelian categories*

$$\mathsf{coh}\big(\mathbb{CP}^1_\theta\big) \cong \mathsf{coh}\big(\mathbb{CP}^1\big).$$

Proof. By [13, Thm. 5.4], the degree zero subalgebra of the left Ore localization of the non-commutative algebra $\mathcal{A}(\mathbb{CP}^1_\theta)$ on maximal cones is given by

$$\big(\mathcal{A} \,/\, \big(\mathcal{A} \cdot w_3\big)\big)[w_i^{-1}]_0 \cong \mathbb{C}[y_{i+1}],$$

where $y_{i+1} = w_i^{-1}\, w_{i+1}$ for $i = 1, 2 \pmod 2$. This algebra is the same as the corresponding localization of the commutative homogeneous coordinate algebra $\mathcal{A}(\mathbb{CP}^1) = \mathbb{C}[w_1, w_2]$, and by [13, Prop. 4.6] the result follows. □

This proposition will enable us to exploit the known cohomology of sheaves on the commutative projective line $\mathbb{CP}^1$, since the functors Ext^p, $\underline{\mathcal{E}\mathrm{xt}}^p$ and H^p all commute with the functorial equivalence. For example, a sheaf E on $\mathsf{Open}(\mathbb{CP}^1_\theta)$ is locally free if and only if it is locally free as a sheaf on $\mathbb{CP}^1$ under the equivalence of Proposition 5.2, and hence by the Birkhoff–Grothendieck theorem it is isomorphic to a finite direct sum of rank one bundles $E \cong \mathcal{O}_{\mathbb{CP}^1_\theta}(k_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{CP}^1_\theta}(k_s)$ for some integers $k_1, \dots, k_s$. The inclusion of algebras $i : \mathcal{A}_\ell \hookrightarrow \mathcal{A}$ induces a restriction functor $i^\bullet : \mathsf{coh}(\mathbb{CP}^2_\theta) \to \mathsf{coh}(\mathbb{CP}^1_\theta)$ defined on right $\mathcal{A}$-modules M by

$$i^\bullet\big(\pi(M)\big) = \pi\big(M \,/\, (M \triangleleft w_3)\big).$$

By [13, Prop. 4.6] and [6, Prop. 3.3.9 (1)], the functor $i^\bullet$ is exact and its right adjoint is the faithful, exact push-forward functor $i_\bullet = p^\bullet : \mathsf{coh}(\mathbb{CP}^1_\theta) \to \mathsf{coh}(\mathbb{CP}^2_\theta)$ induced by the algebra projection $p : \mathcal{A} \to \mathcal{A}_\ell$, and defined on $\mathcal{A}_\ell$-modules N by

$$p^\bullet\big(\pi(N)\big) = \pi(N)$$

with w_3 acting as 0 on the right-hand side.

Let us fix (isomorphism classes of) complex vector spaces V and W of dimensions k and r, respectively. The purpose of this section is to describe the following (set-theoretic) moduli space.

Definition 5.3. (1) A (W,V)-*framed sheaf* is a coherent torsion free sheaf E on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ such that there exists an isomorphism $H^1(\mathbb{CP}^2_\theta, E(-1)) \cong V$, together with an isomorphism $i^\bullet(E) \cong W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}$ called a *framing of* E *(of type* W*) at infinity.*

(2) A *morphism of* (W,V)*-framed sheaves* E and E' is a homomorphism of $\mathcal{A}$-modules $\xi : E \to E'$ which preserves the framing isomorphisms, i.e., there is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \xi\ } & E' \\ & {\scriptstyle i^\bullet}\searrow & \downarrow{\scriptstyle i^\bullet} \\ & & W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}. \end{array}$$

Definition 5.4. The *(instanton) moduli space* $\mathcal{M}_\theta(W,V)$ is the set of isomorphism classes $[E]$ of (W,V)-framed sheaves E. When bases have been fixed for the vector spaces $V \cong \mathbb{C}^k$ and $W \cong \mathbb{C}^r$, we will denote this moduli space by $\mathcal{M}_\theta(r,k)$.

Proposition 5.5. *For any* (W,V)*-framed sheaf* E *on* $\mathsf{Open}(\mathbb{CP}^2_\theta)$ *and for any* $k \in \mathbb{Z}$*, there is a canonical exact sequence*

$$0 \longrightarrow E(k-1) \xrightarrow{\cdot w_3} E(k) \longrightarrow p^\bullet\big(W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}(k)\big) \longrightarrow 0. \qquad (5.6)$$

Proof. Using Proposition 5.2, this is essentially a straightforward adaptation of the proofs of [25, Lem. 6.1] and [6, Prop. 3.3.9]. □

The framed sheaf cohomology of $\mathbb{CP}^2_\theta$ can be described as follows.

Proposition 5.7. *For any* (W,V)*-framed sheaf* E *on* $\mathsf{Open}(\mathbb{CP}^2_\theta)$*, one has:*

(1) $H^0\big(\mathbb{CP}^2_\theta, E(-1)\big) = 0 = H^0\big(\mathbb{CP}^2_\theta, E(-2)\big)$;

(2) $H^2\big(\mathbb{CP}^2_\theta, E(-1)\big) = 0 = H^2\big(\mathbb{CP}^2_\theta, E(-2)\big)$; *and*

(3) $H^1\big(\mathbb{CP}^2_\theta, E(-1)\big) = V = H^1\big(\mathbb{CP}^2_\theta, E(-2)\big)$.

Proof. Use Proposition 5.2 and Proposition 5.5, and repeat the proofs of [6, Lem. 4.2.12] and [25, Lem. 6.2]. □

Corollary 5.8. *A framed sheaf E with isomorphism class $[E] \in \mathcal{M}_\theta(r,k)$ has invariants*

$$\operatorname{rank}(E) = r, \quad c_1(E) = 0, \quad \chi(E) = r - k.$$

Proof. By [36, Lem. 6.1], the Hilbert polynomial of E can be expressed as

$$h_E(s) = \frac{1}{2}\operatorname{rank}(E)\, s\,(s+1) + (c_1(E) + \operatorname{rank}(E))\, s + \chi(E). \tag{5.9}$$

Using (5.9), Proposition 5.5 and [6, Prop. 3.3.9], one has $h_{i^\bullet(E)}(s) = h_E(s) - h_E(s-1)$ and hence

$$\operatorname{rank}(E) = \operatorname{rank}(i^\bullet(E)) = \operatorname{rank}(W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}) = r.$$

Using Proposition 5.7, and setting $s = -1$ and $s = -2$ in (5.9), we arrive at the respective equations

$$-k = -r - c_1(E) + \chi(E), \quad -k = -r - 2c_1(E) + \chi(E),$$

which together yield $c_1(E) = 0$ and $\chi(E) = r - k$. □

In Section 7 we will indeed demonstrate that the set $\mathcal{M}_\theta(r,k)$ is a (coarse) moduli space by constructing universal modules (co)representing the moduli functor of framed torsion free objects of $\mathsf{coh}(\mathbb{CP}^2_\theta)$. The construction will rely crucially on the following basic property.

Lemma 5.10. *Every (W,V)-framed sheaf on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ is μ-semistable.*

Proof. Let $[E] \in \mathcal{M}_\theta(W,V)$ and suppose that $0 \neq F \subsetneq E$ is a proper subsheaf of E. Without loss of generality we may assume that E/F is torsion free. The restriction functor $i^\bullet : \mathsf{coh}(\mathbb{CP}^2_\theta) \to \mathsf{coh}(\mathbb{CP}^1_\theta)$ is exact, so $i^\bullet(F) \subset i^\bullet(E)$. Then by additivity of the first Chern class and Corollary 5.8 it follows that

$$c_1(F) = c_1\big(i^\bullet(F)\big) \leq c_1\big(i^\bullet(E)\big) = c_1(E) = 0,$$

and thus F cannot de-semistabilize E. □

5.2 Noncommutative ADHM construction

We will now reduce the study of moduli of (W,V)-framed sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ to a problem of *linear* algebra, which defines a non-commutative

deformation of the standard ADHM data. For this, we consider triples

$$(B,I,J)\in\mathfrak{X}(W,V):=\mathrm{Hom}_{\mathbb{C}}\big(V,V\otimes(\mathcal{A}_\ell^!)_1\big)\oplus\mathrm{Hom}_{\mathbb{C}}\big(W,V\otimes(\mathcal{A}_\ell^!)_2\big)\\ \oplus\mathrm{Hom}_{\mathbb{C}}(V,W);\tag{5.11}$$

here $\mathcal{A}_\ell^!=\mathcal{A}^!/(\mathcal{A}^!\cdot\check{w}_3)$ is the Koszul dual of the homogeneous coordinate algebra of the non-commutative projective line $\mathbb{CP}^1_\theta$ (see (3.3)), with generators $\check{w}_1,\check{w}_2$ satisfying relations

$$\check{w}_1^2=0=\check{w}_2^2,\quad \check{w}_1\,\check{w}_2=-q^2\;\check{w}_2\,\check{w}_1,\tag{5.12}$$

while for its components one has isomorphisms $(\mathcal{A}_\ell^!)_1\cong(\mathcal{A}_\ell)_1^*$ and $(\mathcal{A}_\ell^!)_2\cong\bigwedge_\theta^2(\mathcal{A}_\ell)_1^*$ as vector spaces in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules. Given B as above, define a vector space morphism $B\wedge_\theta B:V\to V\otimes(\mathcal{A}_\ell^!)_2$ as the composition of maps

$$B\wedge_\theta B=\big(\mathrm{id}_V\otimes\mu_{\mathcal{A}_\ell^!}\big)\circ\big(B\otimes\mathrm{id}_{\mathcal{A}_\ell^!}\big)\circ B,\tag{5.13}$$

where $\mu_{\mathcal{A}_\ell^!}:\mathcal{A}_\ell^!\otimes\mathcal{A}_\ell^!\to\mathcal{A}_\ell^!$ is the multiplication map of the algebra $\mathcal{A}_\ell^!$.

Since $(\mathcal{A}_\ell^!)_1\cong\mathbb{C}\check{w}_1\oplus\mathbb{C}\check{w}_2$, we can write the map $B:V\to V\otimes(\mathcal{A}_\ell^!)_1$ in the form

$$B=B_1\otimes\check{w}_1+B_2\otimes\check{w}_2\tag{5.14}$$

with $B_1,B_2\in\mathrm{End}_{\mathbb{C}}(V)$. Using (5.12) we can then evaluate the morphism (5.13) as

$$B\wedge_\theta B=[B_1,B_2]_\theta\otimes\check{w}_1\,\check{w}_2$$

where $[B_1,B_2]_{-\theta}\in\bigwedge_\theta^\bullet\mathrm{End}_{\mathbb{C}}(V)$ is the *braided commutator*

$$[B_1,B_2]_{-\theta}:=\mu_{\mathrm{End}_{\mathbb{C}}(V)}\big(B_1\otimes B_2-\Psi^\theta_{\mathrm{End}_{\mathbb{C}}(V),\mathrm{End}_{\mathbb{C}}(V)}(B_1\otimes B_2)\big)\\ =B_1\,B_2-q^2B_2\,B_1.\tag{5.15}$$

The braiding here arises when we consider $\mathcal{H}_\theta$-coequivariant sheaves, as in Section 3.5, using the natural coactions of the cotwisted Hopf algebra $\mathcal{H}_\theta$ induced by the torus $T=(\mathbb{C}^\times)^2$ on the moduli spaces. Then by point (1) of Definition 5.3 the finite-dimensional vector spaces V and W are $\mathcal{H}_\theta$-comodules, i.e., objects of the category ${}^{\mathcal{H}_\theta}\mathscr{M}$, and hence so is the vector space $\mathrm{End}_{\mathbb{C}}(V)\cong V^*\otimes V\cong V\otimes V$. The explicit coaction on the triples (5.11) can be computed from the coaction (3.4), Lemma 3.24, and the

monadic parameterization (5.29)–(5.30) of framed sheaves below. In this way one finds that the triples are $\mathcal{H}_\theta$-coinvariants, i.e.

$$\Delta_L(B, I, J) = (1 \otimes B, 1 \otimes I, 1 \otimes J), \tag{5.16}$$

which using (5.14) implies

$$\Delta_L(B_i) = t_i \otimes B_i \quad \text{for} \quad i = 1, 2.$$

As mentioned in Section 3.5, here we only regard the parameter subspace of $\mathrm{End}_{\mathbb{C}}(V)$ as a vector space object of ${}^{\mathcal{H}_\theta}\mathscr{M}$, hence we do not deform the product on the endomorphism algebra in (5.15); the deformation in (5.15) follows from (2.3).

Definition 5.17. The variety $\mathfrak{M}_\theta^{\mathrm{ADHM}}(W, V)$ of *non-commutative complex ADHM data* is the locally closed subvariety of triples (5.11) subject to the following two conditions:

(1) The *non-commutative complex ADHM equation*:

$$B \wedge_\theta B + I \circ J = 0 \tag{5.18}$$

in $\mathrm{Hom}_{\mathbb{C}}(V, V \otimes (\mathcal{A}_\ell^!)_2)$; and

(2) The *stability condition*: There are no proper non-zero subspaces V' of V such that $B(V') \subset V' \otimes (\mathcal{A}_\ell^!)_1$ and $I(W) \subset V' \otimes (\mathcal{A}_\ell^!)_2$.

The general linear group $\mathrm{GL}(V)$ of automorphisms of the vector space V acts naturally on the variety $\mathfrak{M}_\theta^{\mathrm{ADHM}}(W, V)$ as

$$g \triangleright (B, I, J) = \big(\tilde{g} \circ B \circ g^{-1}, \tilde{g} \circ I, J \circ g^{-1}\big), \tag{5.19}$$

where $g \in \mathrm{GL}(V)$ and $\tilde{g} := g \otimes \mathrm{id}_{\mathcal{A}_\ell^!}$.

Lemma 5.20. *The natural action of* $\mathrm{GL}(V)$ *on* $\mathfrak{M}_\theta^{\mathrm{ADHM}}(W, V)$ *is free and proper.*

Proof. Suppose that the triple $(B, I, J) \in \mathfrak{M}_\theta^{\mathrm{ADHM}}(W, V)$ is fixed by $g \in \mathrm{GL}(V)$. Then $\tilde{g} \circ B \circ g^{-1} = B$ and $\tilde{g} \circ I = I$, which respectively imply that $V' = \ker(g - \mathrm{id}_V)$ is B-coinvariant and that $I(W) \subset V' \otimes (\mathcal{A}_\ell^!)_2$. By the stability condition of Definition 5.17, it follows that $g = \mathrm{id}_V$. Thus $\mathrm{GL}(V)$ acts freely on $\mathfrak{M}_\theta^{\mathrm{ADHM}}(W, V)$.

Suppose now that the $\mathrm{GL}(V)$-orbit of some triple $(B,I,J) \in \mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$ is not closed. Then there is a non-trivial one-parameter subgroup $\lambda : \mathbb{C}^\times \to \mathrm{GL}(V)$ such that the limit $(B_0,I_0,J_0) = \lim_{t\to 0}\, \lambda(t) \triangleright (B,I,J)$ exists but does not belong to the orbit $\mathrm{GL}(V) \triangleright (B,I,J)$. Let $V = \bigoplus_{m\in\mathbb{Z}} V_m$ be the weight decomposition of the vector space V with respect to the subgroup $\lambda(\mathbb{C}^\times)$. The existence of the limit (B_0,I_0,J_0) implies that $B(V_m) \subset \big(\bigoplus_{k\geq m} V_k\big) \otimes (\mathcal{A}_\ell^!)_1$ and $I(W) \subset \big(\bigoplus_{k\geq 0} V_k\big) \otimes (\mathcal{A}_\ell^!)_2$. Set $V' = \bigoplus_{k\geq 0} V_k$. Then $B(V') \subset V' \otimes (\mathcal{A}_\ell^!)_1$ and $I(W) \subset V' \otimes (\mathcal{A}_\ell^!)_2$. Since (B_0,I_0,J_0) does not belong to the orbit $\mathrm{GL}(V) \triangleright (B,I,J)$, we must have that $\det(\lambda(t)) = t^N$ for some $N < 0$. This implies that V' is a proper subspace of V, which contradicts the stability condition of Definition 5.17. Thus $\mathrm{GL}(V)$ acts properly on $\mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$. □

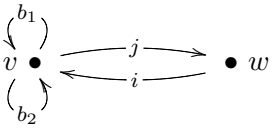
It follows from Lemma 5.20 that the quasi-projective variety of closed $\mathrm{GL}(V)$-orbits on the space $\mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$ is given by the geometric invariant theory quotient

$$\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(W,V) := \mathcal{M}_\theta^{\mathrm{ADHM}}(W,V) \,/\, \mathrm{GL}(V). \tag{5.21}$$

Our first characterization of the moduli space of Definition 5.4 is then as follows.

Theorem 5.22. *There is a natural (set-theoretic) bijection between the moduli space $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(W,V)$ of braided linear algebraic ADHM data and the moduli space $\mathcal{M}_\theta(W,V)$ of framed sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$.*

This theorem is proved in Section 5.3 below. In Section 7, we will analyze to what extent this bijection establishes an isomorphism of algebraic varieties; in particular, we will show that the parameterization (5.21) is a fine moduli space for (W,V)-framed sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ and that this isomorphism induces the bijection of Theorem 5.22. It shows that framed torsion free sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ are in a bijective correspondence with stable framed representations of the ADHM quiver



in the category of complex vector spaces, with a deformation of the usual relation specified by the $\mathbb{C}$-linear combination of paths

$$b_1\, b_2 - q^{-2}\, b_2\, b_1 + i\, j. \tag{5.23}$$

In the $\mathcal{H}_\theta$-coequivariant case, the bijection gives a T-equivariant isomorphism of algebraic spaces and this category can be replaced with the braided monoidal category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules.

5.3 Beilinson monads for framed sheaves

We will now prove Theorem 5.22. For this, we will mimic the classical approach. Thus we will exploit the Beilinson spectral sequence of Section 3.2 to obtain a monadic description of (W,V)-framed sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$, i.e., in terms of the cohomology of linear monads on $\mathbb{CP}^2_\theta$ as defined in Section 3.4. We will need the following vanishing lemma, analogous to Proposition 5.7.

Lemma 5.24. *For any (W,V)-framed sheaf E on $\mathsf{Open}(\mathbb{CP}^2_\theta)$, one has*

$$\mathrm{Hom}_L\big(E^\vee(1),\Omega^1_{\mathbb{CP}^2_\theta}(1)\big)=0=\mathrm{Ext}^2_L\big(E^\vee(1),\Omega^1_{\mathbb{CP}^2_\theta}(1)\big).$$

Proof. By (3.6) and [13, Ex. 6.10], the sheaf of Kähler differentials $\Omega^1_{\mathbb{CP}^2_\theta}$ is a bundle of bimodules which can be included in the non-commutative Euler sequence

$$0\longrightarrow\Omega^1_{\mathbb{CP}^2_\theta}\longrightarrow\mathcal{O}_{\mathbb{CP}^2_\theta}(-1)\otimes\mathcal{A}_1\longrightarrow\mathcal{O}_{\mathbb{CP}^2_\theta}\longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_L(E^\vee(1),-)$ to the degree one shift autoequivalence of this sequence in the category $\mathsf{coh}_L(\mathbb{CP}^2_\theta)$ of sheaves of left $\mathcal{A}$-modules, one induces a long exact sequence of Ext_L-modules which begins with the exact sequence

$$0\longrightarrow\mathrm{Hom}_L\big(E^\vee(1),\Omega^1_{\mathbb{CP}^2_\theta}(1)\big)\longrightarrow H^0\big(\mathbb{CP}^2_\theta,E(-1)\big)\otimes\mathcal{A}_1;$$

here we have used $\mathrm{Ext}^p_L(E^\vee(k),\mathcal{O}_{\mathbb{CP}^2_\theta}(l))\cong H^p(\mathbb{CP}^2_\theta,E(l-k))$ for $p\geq 0$ and $k,l\in\mathbb{Z}$. Whence by point (1) of Proposition 5.7, one has $\mathrm{Hom}_L\big(E^\vee(1),\Omega^1_{\mathbb{CP}^2_\theta}(1)\big)=0$. By the construction of the Koszul complex given in [13, Section 6.1] and Section 3.2 above, the sheaf $\Omega^1_{\mathbb{CP}^2_\theta}(1)$ can also be included in the exact sequence

$$0\longrightarrow\mathcal{O}_{\mathbb{CP}^2_\theta}(-2)\longrightarrow\mathcal{O}_{\mathbb{CP}^2_\theta}(-1)\otimes\big(\mathcal{A}^!_2\big)^*\longrightarrow\Omega^1_{\mathbb{CP}^2_\theta}(1)\longrightarrow 0,$$

with $(\mathcal{A}^!_2)^*$ the subspace of $\mathcal{A}_1\otimes\mathcal{A}_1$ spanned by the quadratic relations (5.1). By point (2) of [13, Prop. 6.8], one has $H^3(\mathbb{CP}^2_\theta,F)=0$ for all $F\in\mathsf{coh}(\mathbb{CP}^2_\theta)$,

and so by applying $\mathrm{Hom}_L(E^\vee(1),-)$ one induces a long exact sequence which terminates at the exact sequence

$$H^2\big(\mathbb{CP}^2_\theta, E(-2)\big)\otimes\big(\mathcal{A}^!_2\big)^* \longrightarrow \mathrm{Ext}^2_L\big(E^\vee(1),\Omega^1_{\mathbb{CP}^2_\theta}(1)\big) \longrightarrow 0.$$

By point (2) of Proposition 5.7, one thus finds $\mathrm{Ext}^2_L\big(E^\vee(1), \Omega^1_{\mathbb{CP}^2_\theta}(1)\big)=0$. □

Theorem 5.25. *Up to isomorphism, any (W,V)-framed sheaf on* Open $(\mathbb{CP}^2_\theta)$ *is the middle cohomology of a monad complex*

$$\underline{\mathcal{C}}^\bullet(W,V): 0\longrightarrow V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(-1) \xrightarrow{\sigma_w} \begin{array}{c} V\otimes\big(\mathcal{A}^!_\ell\big)_1\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}\\ \oplus\\ W\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}\end{array} \xrightarrow{\tau_w} V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(1)\longrightarrow 0.$$

Conversely, any linear monad $\underline{\mathcal{C}}^\bullet(W,V)$ on $\mathbb{CP}^2_\theta$ of this form, such that $S=\pi_L(\mathrm{coker}(\sigma_w^*))$ is a graded left artinian $\mathcal{A}$-module, defines an isomorphism class in $\mathcal{M}_\theta(W,V)$.

Proof. We use the Beilinson spectral sequence of Section 3.2. In the present case, the only non-vanishing sheaves $\mathcal{Q}^p$ appearing in (3.5) are given by

$$\begin{aligned}\mathcal{Q}^0&=\pi(\mathcal{A})^\vee \;=\; \mathcal{O}_{\mathbb{CP}^2_\theta},\\ \mathcal{Q}^1&=\pi\big(\ker(\mathcal{A}(-1)\otimes\mathcal{A}_1\to\mathcal{A})\big)^\vee \;=\; \Omega^1_{\mathbb{CP}^2_\theta}(1)^\vee,\\ \mathcal{Q}^2&=\pi\big(\mathcal{A}(-1)\big)^\vee \;=\; \mathcal{O}_{\mathbb{CP}^2_\theta}(1).\end{aligned}$$

The spectral sequence converges to $F\in\mathsf{coh}(\mathbb{CP}^2_\theta)$ concentrated in degree zero. We apply this sequence to the sheaf $F=E(-1)$ where $[E]\in\mathcal{M}_\theta(W,V)$. One has the sheaf cohomology groups

$$\begin{aligned}\mathrm{Ext}^q\big(\mathcal{O}_{\mathbb{CP}^2_\theta}(1),E(-1)\big)&=H^q\big(\mathbb{CP}^2_\theta,E(-2)\big),\\ \mathrm{Ext}^q\big(\mathcal{O}_{\mathbb{CP}^2_\theta},E(-1)\big)&=H^q\big(\mathbb{CP}^2_\theta,E(-1)\big),\end{aligned}$$

which both vanish for $q\neq 1$ by points (1) and (2) of Proposition 5.7. Likewise, one has

$$\mathrm{Ext}^q\big(\Omega^1_{\mathbb{CP}^2_\theta}(1)^\vee,E(-1)\big)=\mathrm{Ext}^q_L\big(E^\vee(1),\Omega^1_{\mathbb{CP}^2_\theta}(1)\big),$$

which is also trivial for $q\neq 1$ by Lemma 5.24. It follows that $E_1^{p,q}=0$ unless $p=0,1,2$ and $q=1$, and hence the spectral sequence (3.5) is simply

the three-term complex

$$\begin{aligned} 0 \longrightarrow\ & H^1\big(\mathbb{CP}^2_\theta, E(-2)\big) \otimes \mathcal{O}_{\mathbb{CP}^2_\theta}(-2) \\ \longrightarrow\ & \mathrm{Ext}^1_L\big(E^\vee(1), \Omega^1_{\mathbb{CP}^2_\theta}(1)\big) \otimes \mathcal{O}_{\mathbb{CP}^2_\theta}(-1) \\ \longrightarrow\ & H^1\big(\mathbb{CP}^2_\theta, E(-1)\big) \otimes \mathcal{O}_{\mathbb{CP}^2_\theta} \longrightarrow 0. \end{aligned} \tag{5.26}$$

Apply the degree one shift autoequivalence to write this complex in the generic monadic form of (3.12), i.e., with finite-dimensional vector spaces given by $V_{-1} = H^1(\mathbb{CP}^2_\theta, E(-2))$, $V_0 = \mathrm{Ext}^1_L(E^\vee(1), \Omega_{\mathbb{CP}^2_\theta}(1))$, and $V_1 = H^1(\mathbb{CP}^2_\theta, E(-1))$. By Proposition 3.13 with $n = 2$ and Corollary 5.8, one has $\dim_{\mathbb{C}}(V_{\pm 1}) = k$ and $\dim_{\mathbb{C}}(V_0) = 2k + r$, and by point (3) of Proposition 5.7 there are vector space isomorphisms $V_{\pm 1} \xrightarrow{\approx} V$.

It remains to identify the $\mathbb{C}$-vector space V_0 with $(V \otimes (\mathcal{A}^!_\ell)_1) \oplus W$. For this, we apply the exact restriction functor $i^\bullet$ to the short exact sequences (3.14)–(3.15). Since the cohomological dimension of the category $\mathsf{coh}(\mathbb{CP}^1_\theta)$ is one [13, Prop. 6.8 (1)], the corresponding induced long exact cohomology sequences (3.18)–(3.19) are non-trivial only for $p = 0, 1$. Since by [13, Prop. 6.8 (1)] one has

$$\begin{aligned} & H^0\big(\mathbb{CP}^1_\theta, \mathcal{O}_{\mathbb{CP}^1_\theta}(-1)\big) = H^1\big(\mathbb{CP}^1_\theta, \mathcal{O}_{\mathbb{CP}^1_\theta}(-1)\big) = 0, \\ & H^0\big(\mathbb{CP}^1_\theta, \mathcal{O}_{\mathbb{CP}^1_\theta}\big) = (\mathcal{A}_\ell)_0 = \mathbb{C}, \end{aligned}$$

and $i^\bullet(E) \cong W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}$, the sequence (3.19) gives isomorphisms

$$\begin{aligned} & H^0\big(\mathbb{CP}^1_\theta, i^\bullet(\ker(\tau_w))\big) \xrightarrow{\approx} H^0\big(\mathbb{CP}^1_\theta, i^\bullet(E)\big) \cong W, \\ & H^1\big(\mathbb{CP}^1_\theta, i^\bullet(\ker(\tau_w))\big) \xrightarrow{\approx} H^1\big(\mathbb{CP}^1_\theta, i^\bullet(E)\big) = 0. \end{aligned}$$

Using [13, Prop. 6.8 (1)] we identify the finite-dimensional complex vector space

$$H^0\big(\mathbb{CP}^1_\theta, \mathcal{O}_{\mathbb{CP}^1_\theta}(1)\big) = (\mathcal{A}_\ell)_1$$

with its Koszul dual $(\mathcal{A}^!_\ell)_1 \cong (\mathcal{A}_\ell)^*_1$. Then the exact sequence (3.18) truncates to the short exact sequence

$$0 \longrightarrow W \longrightarrow V_0 \longrightarrow V \otimes \big(\mathcal{A}^!_\ell\big)_1 \longrightarrow 0. \tag{5.27}$$

Since $i^\bullet(E)$ is locally free and its dual coincides with $H^0(i^\bullet(\underline{\mathcal{C}}_\bullet(W, V)^\vee))$, we can apply the same argument to the dual monad (3.20) to get the exact

sequence

$$0 \longrightarrow H^0\big(\mathbb{CP}^1_\theta, i^\bullet(\ker(\sigma_w^*))\big) \longrightarrow V_0^* \xrightarrow{\sigma^*} (\mathcal{A}_\ell)_1 \otimes V^* \longrightarrow 0\ ,$$

which implies that the map $\sigma : V \otimes (\mathcal{A}_\ell^!)_1 \to V_0$ is injective. Thus the sequence (5.27) splits, and we get the identification $V_0 \cong (V \otimes (\mathcal{A}_\ell^!)_1) \oplus W$ as desired. By Proposition 3.17, the vector space isomorphisms $V_{\pm 1} \xrightarrow{\approx} V$ and $V_0 \xrightarrow{\approx} (V \otimes (\mathcal{A}_\ell^!)_1) \oplus W$ can be uniquely extended to an isomorphism between the monad (5.26) and one of the form $\underline{\mathcal{C}}^\bullet(W,V)$ which is compatible with the framing isomorphisms (see [6, Lem. 4.2.15] and [25, Thm. 6.7]).

Conversely, if $\mathcal{C}$ is the middle cohomology sheaf of the monad $\underline{\mathcal{C}}^\bullet(W,V)$, then one has $H^1(\mathbb{CP}^2_\theta, \mathcal{C}(-1)) \cong V$ by point (2) of Proposition 3.17. Moreover, by Proposition 5.2 the restriction complex $i^\bullet(\underline{\mathcal{C}}^\bullet(W,V))$ is a complex of sheaves on $\mathsf{Open}(\mathbb{CP}^1_\theta)$ which is canonically quasi-isomorphic to $W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}$. The artinian condition on the module S implies that $\operatorname{coker}(\sigma_w^*)$ is an Artin sheaf in the sense of [6, Def. 2.0.8]. By point (2) of Proposition 3.21 with $n = 2$, it follows that $\mathcal{C}$ is a torsion free sheaf on $\mathsf{Open}(\mathbb{CP}^2_\theta)$.

It remains to prove that the construction above is independent of the choices of representatives for the respective isomorphism classes. By [6, Lem. 4.2.15], every isomorphism $\xi : E \to E'$ of (W,V)-framed sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ extends uniquely to an isomorphism between the corresponding monads $\underline{\mathcal{C}}^\bullet(W,V) \to \underline{\mathcal{C}}'^\bullet(W,V)$. The fact that the isomorphism ξ preserves the framing isomorphisms then forces the isomorphism between monads to be given by a commutative diagram

$$\begin{array}{ccccccccc}
\underline{\mathcal{C}}^\bullet(W,V):0 & \longrightarrow & V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(-1) & \xrightarrow{\sigma_w} & V_0\otimes\mathcal{O}_{\mathbb{CP}^2_\theta} & \xrightarrow{\tau_w} & V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g\otimes\mathrm{id}} & & \downarrow{\scriptstyle (\tilde g\oplus \mathrm{id}_W)\otimes\mathrm{id}} & & \downarrow{\scriptstyle g\otimes\mathrm{id}} & & \\
\underline{\mathcal{C}}'^\bullet(W,V):0 & \longrightarrow & V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(-1) & \xrightarrow[\sigma'_w]{} & V_0\otimes\mathcal{O}_{\mathbb{CP}^2_\theta} & \xrightarrow[\tau'_w]{} & V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(1) & \longrightarrow & 0
\end{array} \tag{5.28}$$

for some $g \in \mathrm{GL}(V)$. Conversely, any isomorphism of monads of the form (5.28) induces an isomorphism between the corresponding cohomology sheaves $E = H^0(\underline{\mathcal{C}}^\bullet(W,V))$ and $E' = H^0(\underline{\mathcal{C}}'^\bullet(W,V))$ which preserves the framing isomorphisms; the automorphism $(\tilde g \oplus \mathrm{id}_W) \otimes \mathrm{id}$ maps E onto E', regarded as submodules of $V_0 \otimes \mathcal{O}_{\mathbb{CP}^2_\theta}$. □

The proof of Theorem 5.22 is now completed by demonstrating that the ADHM moduli space (5.21) has the same monadic description as that in

Theorem 5.25. Given a triple of ADHM data $(B,I,J) \in \mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$, we define a canonical sheaf morphism

$$\begin{aligned}\sigma_{(B,I,J)} &= \begin{pmatrix} B\otimes w_3 + \mathrm{id}_V \otimes \big(q\,\check{w}_1\otimes w_1 + q^{-1}\,\check{w}_2\otimes w_2\big) \\ J\otimes w_3 \end{pmatrix} \\ &: V\otimes \mathcal{O}_{\mathbb{CP}^2_\theta}(-1) \;\longrightarrow\; \big((V\otimes(\mathcal{A}^!_\ell)_1)\oplus W\big)\otimes \mathcal{O}_{\mathbb{CP}^2_\theta}, \end{aligned} \tag{5.29}$$

where the homogeneous coordinates $\check{w}_i$ and w_j act by multiplication in $\mathcal{A}^!_\ell$ and $\mathcal{A}$, respectively. Similarly, define

$$\begin{aligned}\tau_{(B,I,J)} &= \big(\mathrm{id}_V\otimes \mu_{\mathcal{A}^!_\ell}\big)\circ\big(B\otimes \mathrm{id}_{\mathcal{A}^!_\ell}\big)\otimes w_3 \\ &\quad + \mathrm{id}_V\otimes\big(q^{-1}\,\check{w}_1\otimes w_1 + q\,\check{w}_2\otimes w_2\big) \quad I\otimes w_3 \\ &: \big((V\otimes(\mathcal{A}^!_\ell)_1)\oplus W\big)\otimes\mathcal{O}_{\mathbb{CP}^2_\theta} \;\longrightarrow\; \big(V\otimes(\mathcal{A}^!_\ell)_2\big)\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(1). \end{aligned} \tag{5.30}$$

Henceforth, we will use the natural vector space isomorphism $(\mathcal{A}^!_\ell)_2 \cong \mathbb{C}$. Then the maps (5.29)–(5.30) determine a chain of morphisms of coherent sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ given by

$$\begin{aligned}\underline{\mathcal{C}}_\bullet(B,I,J) : V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(-1) &\xrightarrow{\sigma_{(B,I,J)}} \begin{matrix} V\otimes\big(\mathcal{A}^!_\ell\big)_1\otimes\mathcal{O}_{\mathbb{CP}^2_\theta} \\ \oplus \\ W\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}\end{matrix} \\ &\xrightarrow{\tau_{(B,I,J)}} V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(1). \end{aligned} \tag{5.31}$$

We will show that the sequence (5.31) is a monad of the type considered in Theorem 5.25, and that every (W,V)-framed torsion free sheaf arises from this construction.

Theorem 5.32. *For any triple $(B,I,J)\in\mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$, the chain $\underline{\mathcal{C}}_\bullet(B,I,J)$ of morphisms is a monad complex which defines an isomorphism class in $\mathcal{M}_\theta(W,V)$. Conversely, any linear monad complex $\underline{\mathcal{C}}_\bullet(W,V)$ of the form given in Theorem* 5.25 *defines an isomorphism class in the moduli space $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(W,V)$.*

Proof. Apply the exact restriction functor $i^\bullet$ to (5.31) with

$$\begin{aligned} i^\bullet(\sigma_{(B,I,J)}) &= \begin{pmatrix} \mathrm{id}_V\otimes\big(q\,\check{w}_1\otimes w_1 + q^{-1}\,\check{w}_2\otimes w_2\big) \\ 0 \end{pmatrix}, \\ i^\bullet(\tau_{(B,I,J)}) &= \begin{pmatrix} \mathrm{id}_V\otimes\big(q^{-1}\,\check{w}_1\otimes w_1 + q\,\check{w}_2\otimes w_2\big) & 0\end{pmatrix}. \end{aligned}$$

It follows that the morphism $i^\bullet(\sigma_{(B,I,J)})$ is injective, $i^\bullet(\tau_{(B,I,J)})$ is surjective, while using (5.1) and (5.12) one easily finds $i^\bullet(\tau_{(B,I,J)})\circ i^\bullet(\sigma_{(B,I,J)}) = 0$. It

follows that $i^\bullet(\underline{\mathcal{C}}_\bullet(B,I,J))$ is a monad on $\mathbb{CP}^1_\theta$ whose cohomology is naturally isomorphic to $W\otimes\mathcal{O}_{\mathbb{CP}^1_\theta}$. More generally, an elementary calculation using (5.1) and (5.12)–(5.14) shows

$$\tau_{(B,I,J)}\circ\sigma_{(B,I,J)}=(B\wedge_\theta B+I\circ J)\otimes w_3^2,$$

from which it follows that (5.31) is a complex if and only if the triple (B,I,J) satisfies the non-commutative ADHM equations (5.18).

Arguing as in Proposition 5.5 using $i^\bullet(\ker(\sigma_{(B,I,J)}))=0$ from above, multiplication by w_3 yields isomorphisms of sheaves $\ker(\sigma_{(B,I,J)})(k-1)\cong\ker(\sigma_{(B,I,J)})(k)$ for all $k\in\mathbb{Z}$. By [25, Cor. 5.3], one has the trivial sheaf cohomology $H^p(\mathbb{CP}^2_\theta,\ker(\sigma_{(B,I,J)})(k))=0$ for $k\gg0$ and for all $p\geq1$. It follows that $\ker(\sigma_{(B,I,J)})$ is an Artin sheaf, which is locally free by [6, Prop. 2.0.4]. Hence $\ker(\sigma_{(B,I,J)})=0$ by [6, Prop. 2.0.9 (4)], and the map $\sigma_{(B,I,J)}$ is injective. The same argument shows that $\mathrm{coker}(\sigma^*_{(B,I,J)})$ is an Artin sheaf, and hence that the graded left $\mathcal{A}$-module $S=\pi_L(\mathrm{coker}(\sigma^*_{(B,I,J)}))$ has homological dimension zero. Analogously, $\mathrm{coker}(\tau_{(B,I,J)})$ is an Artin sheaf such that there are isomorphisms of sheaves $\mathrm{coker}(\tau_{(B,I,J)})(k-1)\cong\mathrm{coker}(\tau_{(B,I,J)})(k)$ for all $k\in\mathbb{Z}$, and we can mimic the proof of [6, Lem. 4.1.9]. For this, we note that by point (1) of [13, Prop. 6.8] we can write

$$\tau_{(B,I,J)}=\big(\mathrm{id}_V\otimes\mu_{\mathcal{A}}\big)\circ\big(H^0(\tau_{(B,I,J)})\otimes\mathrm{id}\big),$$

where $H^0(\tau_{(B,I,J)}):V_0\to V\otimes\mathcal{A}_1$ is the induced map on cohomology. Similarly, the canonical projection

$$\phi:V\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}(1)\ \longrightarrow\ \mathrm{coker}(\tau_{(B,I,J)})$$

induces a map $H^0(\phi)(-1):V\to H^0(\mathbb{CP}^2_\theta,\mathrm{coker}(\tau_{(B,I,J)}))$. Let $V'=\ker\big(H^0(\phi)(-1)\big)$. Then (5.30) and the composition $H^0(\phi\circ\tau_{(B,I,J)})=0$ imply $B(V')\subset V'\otimes(\mathcal{A}^!_\ell)_1$ and $I(W)\subset V'\otimes(\mathcal{A}^!_\ell)_2$. By the stability condition (2) of Definition 5.17, one has $V'=V$. Since ϕ is surjective, it follows that $\mathrm{coker}(\tau_{(B,I,J)})=0$ and hence $\tau_{(B,I,J)}$ is surjective. Putting everything together, the chain (5.31) is a monad whose cohomology sheaf is torsion free by point (2) of Proposition 3.21.

Conversely, given a monad $\underline{\mathcal{C}}_\bullet(W,V)$, we mimic the proof of [25, Thm. 6.7]. For this, we use (3.23) for $n=2$ to decompose the maps σ_w and τ_w into constant linear maps $\sigma^i:V\to V\otimes(\mathcal{A}^!_\ell)_1\oplus W$ and $\tau^i:V\otimes(\mathcal{A}^!_\ell)_1\oplus W\to V\otimes(\mathcal{A}^!_\ell)_2$ for $i=1,2,3$. Then the monadic condition $\tau_w\circ\sigma_w=0$

is equivalent to the system of vanishing morphism compositions

$$\begin{aligned} \tau^i\,\sigma^i &= 0, \quad i=1,2,3,\\ \tau^2\,\sigma^1 + q^2\;\tau^1\,\sigma^2 &= 0,\\ \tau^i\,\sigma^3 + \tau^3\,\sigma^i &= 0, \quad i=1,2. \end{aligned} \tag{5.33}$$

Since $i^\bullet(\sigma_w) = \sigma^1\otimes w_1 + \sigma^2\otimes w_2$ and $i^\bullet(\tau_w) = \tau^1\otimes w_1 + \tau^2\otimes w_2$ with cohomology $W\otimes\mathcal{O}_{\mathbb{CP}^1_\theta} \cong \ker(i^\bullet(\tau_w))/\mathrm{im}(i^\bullet(\sigma_w))$, from the proof of Theorem 5.25 it follows that, as in [25, Thm. 6.7], the composition $\tau^1\,\sigma^2 = -\tau^2\,\sigma^1$ is an isomorphism $V \xrightarrow{\approx} V\otimes(\mathcal{A}^!_\ell)_2$. With a suitable choice of basis for the vector space V we can set $\tau^1\,\sigma^2 = q^{-2}\;\mathrm{id}_V\otimes\check{w}_1\,\check{w}_2$. We can then choose a basis for W such that the first set of equations for $i=1,2$ and the second equation in (5.33) are solved by

$$\begin{aligned} \sigma^1 &= \begin{pmatrix} q\;\mathrm{id}_V\otimes\check{w}_1\\ 0\end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} q^{-1}\;\mathrm{id}_V\otimes\check{w}_2\\ 0\end{pmatrix},\\ \tau^1 &= \begin{pmatrix} q^{-1}\;\mathrm{id}_V\otimes\check{w}_1 & 0\end{pmatrix}, \quad \tau^2 = \begin{pmatrix} q\;\mathrm{id}_V\otimes\check{w}_2 & 0\end{pmatrix}. \end{aligned}$$

Then the third set of equations in (5.33) are solved by

$$\sigma^3 = \begin{pmatrix} B_1\otimes\check{w}_1 + B_2\otimes\check{w}_2\\ J\end{pmatrix}, \quad \tau^3 = \begin{pmatrix} B_1\otimes\check{w}_1 + B_2\otimes\check{w}_2 & I\end{pmatrix}$$

with arbitrary maps $B_1, B_2\in\mathrm{End}_{\mathbb{C}}(V)$, $J\in\mathrm{Hom}_{\mathbb{C}}(V,W)$, and $I\in\mathrm{Hom}_{\mathbb{C}}(W, V\otimes(\mathcal{A}^!_\ell)_2)$. Finally, the first equation in (5.33) with $i=3$ is equivalent to

$$\big(B_1\,B_2 - q^{-2}\;B_2\,B_1\big)\otimes\check{w}_1\,\check{w}_2 + I\circ J = 0,$$

which is just the ADHM equation (5.18). The morphisms (3.23) then take the forms given in (5.29)–(5.30).

It remains to check the stability condition (2) of Definition 5.17. Suppose that there exists a non-trivial proper subspace $V'\subset V$ such that

$$B_i(V')\subseteq V', \quad \mathrm{im}(I)\subseteq V'\otimes\big(\mathcal{A}^!_\ell\big)_2$$

for $i=1,2$. Let $B'_i = B_i\big|_{V'}$, $i=1,2$, and let $I' : W\to V'$ be the factorization of I through $V'\subset V$. Then there are natural inclusions

$$\mathrm{im}(I')\hookrightarrow V', \quad \mathrm{im}(B'_i)\hookrightarrow V'$$

for $i=1,2$. Set $V'_\perp := V/V'$, and consider the induced dual linear map on cohomology $H^0(\tau^*_w)(1) : V^*\to\mathcal{A}^*_1\otimes V^*_0$. Since $(V'_\perp)^*\subset\ker(I'^*)$, if $V'_\perp\neq$

0 then by (5.30) the induced map $H^0(\tau_w^*)(1)$ is not injective and hence τ_w cannot be surjective as a morphism of sheaves. This contradicts the assumption that $\underline{\mathcal{C}}_\bullet(W,V)$ is a monad. Therefore, a destabilizing proper subspace $V' \subset V$ as above cannot exist. Finally, any isomorphism of monads preserving the choices of bases for V and W made above is of the form (5.28), and coincides with the natural action of the group $\mathrm{GL}(V)$ on $\mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$. □

Remark 5.34. By point (3) of Proposition 3.17 and Theorem 5.25 it follows that a (W,V)-framed sheaf is locally free if and only if the dual map σ_w^* is surjective. From the proof of Theorem 5.32 this is true if and only if the corresponding dual set of ADHM data (B^*, J^*, I^*) satisfies the stability condition in $\mathcal{M}_\theta^{\mathrm{ADHM}}(V^*, W^*)$. The set of all such triples coincides with the inverse image under the affine map σ_w^* of the open subset of surjective vector space morphisms contained in $\mathcal{A}_1^* \otimes \mathrm{Hom}_{\mathbb{C}}(V^*, V_0^*)$, and hence is itself an open subset of $\mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$ in the Zariski topology. It follows that the moduli space of (W,V)-framed *vector bundles* form a dense open subset in $\mathcal{M}_\theta(W,V)$, and hence the space $\mathcal{M}_\theta(W,V)$ can be regarded as its (partial) compactification.

5.4 Self-conjugate monads on the twistor variety

We will now provide another characterization of the moduli space $\mathcal{M}_\theta(W,V)$ in terms of "real" linear algebraic data. In the sequel we shall assume that $q_{12} = q \in \mathbb{R}$, i.e., that the non-commutativity parameter θ is purely imaginary. There is a natural $*$-structure on the algebra $\mathcal{A}_\ell^!$ given by the $\mathbb{C}$-conjugate linear anti-algebra anti-involution $\mathscr{J} : \mathcal{A}_\ell^! \to (\mathcal{A}_\ell^!)^{\mathrm{op}}$ defined on generators by

$$\mathscr{J}(\check{w}_1, \check{w}_2) = (\check{w}_2, -\check{w}_1). \tag{5.35}$$

We will extend $\mathscr{J}$ as a morphism in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ of left $\mathcal{H}_\theta$-comodules; in particular, it is coequivariant

$$(\mathrm{id} \otimes \mathscr{J}) \circ \Delta_L = \Delta_L \circ \mathscr{J}\,.$$

As in Section 4.5, this real structure restricts the toric action, in this case to a coaction of the diagonal subgroup $\mathbb{C}^\times \subset T$ with $t_1 = t_2$.

Let us fix Hrmitean inner products on the complex vector spaces V and W. Then the space of complex ADHM data (5.11) also acquires a

natural anti-involution

$$\mathscr{J} : \mathcal{M}_\theta^{\mathrm{ADHM}}(W,V) \longrightarrow \mathcal{M}_\theta^{\mathrm{ADHM}}(W,V), \quad \mathscr{J}(B,I,J) = \big(B^\dagger, -J^\dagger, I^\dagger\big),$$

where we implicitly always use the vector space isomorphisms $(\mathcal{A}_\ell^!)_2 \cong \mathbb{C}$ and $(\mathcal{A}_\ell^!)^{\mathrm{op}} \cong \mathcal{A}_\ell^!$, and we set

$$B^\dagger := \mathscr{J}(B) = -B_2^\dagger \otimes \check{w}_1 + B_1^\dagger \otimes \check{w}_2 \tag{5.36}$$

with respect to the decomposition (5.14). With these definitions one has

$$\mathscr{J}(B \wedge_\theta B) = B^\dagger \wedge_\theta B^\dagger.$$

Given any pair of linear maps $B, B' \in \mathrm{Hom}_{\mathbb{C}}(V, V \otimes (\mathcal{A}_\ell^!)_1)$, we generalize the definition (5.13) to the morphism

$$B \wedge_\theta B' = \big(\mathrm{id}_V \otimes \mu_{\mathcal{A}_\ell^!}\big) \circ \big(B \otimes \mathrm{id}_{\mathcal{A}_\ell^!}\big) \circ B'$$

in $\mathrm{Hom}_{\mathbb{C}}(V, V \otimes (\mathcal{A}_\ell^!)_2)$. In general, this cannot be represented as in (5.15), but an explicit expression in terms of braided commutators is obtained for the sum

$$B \wedge_\theta B' + B' \wedge_\theta B = \big([B_1, B_2']_\theta - q^{-2}\,[B_2, B_1']_{-\theta}\big) \otimes \check{w}_1\,\check{w}_2.$$

We will demonstrate how to relate certain framed torsion free sheaves on $\mathbb{CP}_\theta^2$ to the following quotient of the space of complex ADHM data.

Definition 5.37. The variety $\mathcal{M}_\theta^{\mathrm{tw}}(W,V)$ of *non-commutative real ADHM data* is the subspace of $\mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$ consisting of triples (5.11) which satisfy, in addition to conditions (1) and (2) of Definition 5.17, the *non-commutative real ADHM equation*

$$B \wedge_\theta B^\dagger + B^\dagger \wedge_\theta B + I \circ I^\dagger - J^\dagger \circ J = 0 \tag{5.38}$$

in $\mathrm{End}_{\mathbb{C}}(V \otimes (\mathcal{A}_\ell^!)_2)$.

The natural $\mathrm{GL}(V)$-action on $\mathcal{M}_\theta^{\mathrm{ADHM}}(W,V)$ reduces on $\mathcal{M}_\theta^{\mathrm{tw}}(W,V)$ to an action of the group $\mathrm{U}(V)$ of unitary automorphisms of the vector space V. The corresponding space of stable orbits is denoted $\widehat{\mathcal{M}}_\theta^{\mathrm{tw}}(W,V)$. We will now provide a monadic description of this moduli space.

Consider the natural embedding of the non-commutative projective plane $\mathbb{CP}^2_\theta$ into the non-commutative twistor space $\mathbb{CP}^3_\theta$ with homogeneous coordinate algebra $\mathcal{A}^{\rm tw} = \mathcal{A}(\mathbb{CP}^3_\theta)$ described in Section 4.6. The homogeneous coordinate algebra of the original space $\mathbb{CP}^2_\theta$ is then recovered through $\mathcal{A} \cong \mathcal{A}^{\rm tw}/\langle w_4\rangle$. Let $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^{\rm tw}$ be the natural algebra inclusion. We will again denote by $i : \mathcal{A}_\ell \hookrightarrow \mathcal{A}^{\rm tw}$ the algebra inclusion of the non-commutative projective line $\mathbb{CP}^1_\theta$ with $\mathcal{A}_\ell \cong \mathcal{A}^{\rm tw}/\langle w_3, w_4\rangle$. Define a non-degenerate conjugate linear anti-involution $\mathscr{J} : \mathcal{A}^{\rm tw} \to (\mathcal{A}^{\rm tw})^{\rm op}$ acting on generators as

$$\mathscr{J}(w_1, w_2, w_3, w_4) = (w_2, -w_1, w_4, -w_3). \tag{5.39}$$

Consider a linear monad on $\mathbb{CP}^3_\theta$ of the form

$$\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V) : 0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{CP}^3_\theta}(-1) \xrightarrow{\sigma_w} V_0 \otimes \mathcal{O}_{\mathbb{CP}^3_\theta} \xrightarrow{\tau_w} V \otimes \mathcal{O}_{\mathbb{CP}^3_\theta}(1) \longrightarrow 0$$

with $V_0 = V \otimes (\mathcal{A}^!_\ell)_1 \oplus W$. Its restriction $i^\bullet(\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V))$ is again a monad on $\mathbb{CP}^1_\theta$ which is quasi-isomorphic to $W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}$. The anti-homomorphism $\mathscr{J}$ induces a functor between the categories of sheaves $\mathscr{J}^\bullet : \mathsf{coh}(\mathbb{CP}^3_\theta) \to \mathsf{coh}_L(\mathbb{CP}^3_\theta)$. Composing this functor with the dualizing functor $\underline{\mathcal{H}\rm om}\,(-, \mathcal{O}_{\mathbb{CP}^3_\theta})$ gives a functor $\mathsf{coh}(\mathbb{CP}^3_\theta) \to \mathsf{coh}(\mathbb{CP}^3_\theta)$, which we denote by $E \mapsto E^\dagger := \mathscr{J}^\bullet(E)^\vee$. This functor can be extended to the derived category of $\mathsf{coh}(\mathbb{CP}^3_\theta)$ and applied to a monad $\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)$ to give a monad $\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)^\dagger := \mathscr{J}^\bullet(\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)^\vee)$, with

$$\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)^\dagger : 0 \longrightarrow \overline{V}^* \otimes \mathcal{O}_{\mathbb{CP}^3_\theta}(-1) \xrightarrow{\sigma_w^\dagger} \overline{V_0}^* \otimes \mathcal{O}_{\mathbb{CP}^3_\theta} \xrightarrow{\tau_w^\dagger} \overline{V}^* \otimes \mathcal{O}_{\mathbb{CP}^3_\theta}(1) \longrightarrow 0.$$

Here the bars denote complex conjugation while $\sigma_w^\dagger := \mathscr{J}^\bullet(\sigma_w)^* \in \overline{V_0}^* \otimes \overline{V} \otimes (\mathcal{A}^{\rm tw}_1)^{\rm op}$ and $\tau_w^\dagger := \mathscr{J}^\bullet(\tau_w)^* \in \overline{V}^* \otimes \overline{V_0} \otimes (\mathcal{A}^{\rm tw}_1)^{\rm op}$. We will say that a monad of the form $\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)$ is *self-conjugate* if there is an isomorphism $\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)^\dagger \cong \underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)$ of complexes.

Theorem 5.40. *There is a natural (set-theoretic) bijective correspondence between isomorphism classes of self-conjugate linear monad complexes $\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)$ on $\mathbb{CP}^3_\theta$ and isomorphism classes in the moduli space $\widehat{\mathfrak{M}}^{\rm tw}_\theta(W,V)$ of braided real ADHM data.*

Proof. We argue exactly as we did in Section 5.3. Decompose the differentials as in (3.23), with constant linear maps $\sigma^i : V \to V_0$ and $\tau^i : V_0 \to V$ for

$i = 1, 2, 3, 4$. By suitable choices of bases for the vector spaces V and W, we can put these maps into the forms

$$\sigma^1 = \begin{pmatrix} q\ \mathrm{id}_V \otimes \check{w}_1 \\ 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} q^{-1}\ \mathrm{id}_V \otimes \check{w}_2 \\ 0 \end{pmatrix},$$
$$\sigma^3 = \begin{pmatrix} B_1 \otimes \check{w}_1 + B_2 \otimes \check{w}_2 \\ J \end{pmatrix}, \quad \sigma^4 = \begin{pmatrix} B_1' \otimes \check{w}_1 + B_2' \otimes \check{w}_2 \\ J' \end{pmatrix}$$

and

$$\tau^1 = \begin{pmatrix} q^{-1}\ \mathrm{id}_V \otimes \check{w}_1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} q\ \mathrm{id}_V \otimes \check{w}_2 & 0 \end{pmatrix},$$
$$\tau^3 = \begin{pmatrix} B_1 \otimes \check{w}_1 + B_2 \otimes \check{w}_2 & I \end{pmatrix}, \quad \tau^4 = \begin{pmatrix} B_1' \otimes \check{w}_1 + B_2' \otimes \check{w}_2 & I' \end{pmatrix},$$

with $B_1, B_2, B_1', B_2' \in \mathrm{End}_{\mathbb{C}}(V)$, $J, J' \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, and $I, I' \in \mathrm{Hom}_{\mathbb{C}}(W, V \otimes (\mathcal{A}_\ell^!)_2)$.

Self-conjugacy $\underline{\mathcal{C}}_\bullet^{\mathrm{tw}}(W, V)^\dagger \cong \underline{\mathcal{C}}_\bullet^{\mathrm{tw}}(W, V)$ is equivalent to the conditions $\sigma_w^\dagger = -\tau_w$ and $\tau_w^\dagger = \sigma_w$, or equivalently that $\sum_i \sigma^i \otimes w_i = \sum_i \tau^{i\,\dagger} \otimes \mathscr{J}(w_i)$ where $\tau^{i\,\dagger} := (\overline{\tau}^{\,i})^*$ are the adjoint linear maps with respect to the induced Hermitean metrics. Using (5.39), equating coefficients of the generators of $\mathcal{A}^{\mathrm{tw}}$ yields the relations

$$\sigma^1 = -\tau^{2\dagger}, \quad \sigma^2 = \tau^{1\dagger}, \quad \sigma^3 = -\tau^{4\dagger}, \quad \sigma^4 = \tau^{3\dagger},$$

which using (5.36) implies $(B_1', B_2', I', J') = (-B_2^\dagger, B_1^\dagger, -J^\dagger, I^\dagger)$, i.e.,

$$(B', I', J') = \mathscr{J}(B, I, J).$$

By using (5.14) the differentials may thus be expressed as

$$\sigma_w = \begin{pmatrix} B \otimes w_3 + B^\dagger \otimes w_4 + \mathrm{id}_V \otimes \big(q\,\check{w}_1 \otimes w_1 + q^{-1}\,\check{w}_2 \otimes w_2\big) \\ J \otimes w_3 + I^\dagger \otimes w_4 \end{pmatrix} \tag{5.41}$$

and

$$\tau_w = \Big((\mathrm{id}_V \otimes \mu_{\mathcal{A}_\ell^!}) \circ (B \otimes \mathrm{id}_{\mathcal{A}_\ell^!}) \otimes w_3 + (\mathrm{id}_V \otimes \mu_{\mathcal{A}_\ell^!}) \circ (B^\dagger \otimes \mathrm{id}_{\mathcal{A}_\ell^!}) \otimes w_4 + \mathrm{id}_V \otimes (q^{-1}\,\check{w}_1 \otimes w_1 + q\,\check{w}_2 \otimes w_2) \quad I \otimes w_3 - J^\dagger \otimes w_4 \Big). \tag{5.42}$$

By (5.12) and (4.27), we compute the composition of the maps (5.41)–(5.42) to get

$$\tau_w \circ \sigma_w = (B \wedge_\theta B + I \circ J) \otimes w_3^2 + \big(B^\dagger \wedge_\theta B^\dagger - J^\dagger \circ I^\dagger\big) \otimes w_4^2 + \big(B \wedge_\theta B^\dagger + B^\dagger \wedge_\theta B + I \circ I^\dagger - J^\dagger \circ J\big) \otimes w_3\, w_4.$$

The monadic condition $\tau_w \circ \sigma_w = 0$ is thus equivalent, respectively, to the complex ADHM equation (5.18), its image under the anti-involution $\mathscr{J}$, and the real ADHM equation (5.38). Stability follows similarly to the proof of Theorem 5.32, and is again equivalent to surjectivity of the map (5.42). Naturality also follows as before. □

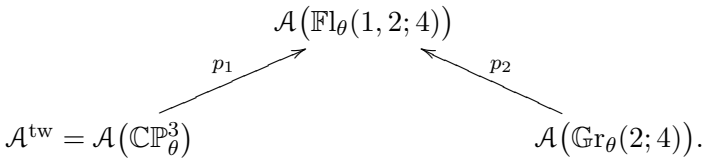
By repeating the arguments of Section 5.3, the cohomology of a self-conjugate monad complex of the form $\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)$, under the correspondence of Theorem 5.40, is a torsion free (W,V)-framed sheaf E on $\mathsf{Open}(\mathbb{CP}^3_\theta)$ obeying the reality condition $E^\dagger \cong E$, and with vanishing cohomology $H^1(\mathbb{CP}^3_\theta, E(-2)) = 0$ by point (1) of Proposition 3.17. Conversely, by adapting the proof of Theorem 5.25 using the Beilinson spectral sequence of Section 3.2, applied to $\mathbb{CP}^3_\theta$, any such sheaf E is the cohomology of a self-conjugate linear monad $\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V)$ on $\mathbb{CP}^3_\theta$, with $V = V_{-1} \cong \mathrm{Ext}^1_L\big(E^\vee(1), \Omega^1_{\mathbb{CP}^3_\theta}(2)\big)$ and the remaining monadic vector spaces analogous to those in the proof of Theorem 5.25. The restriction $\iota^\bullet(\underline{\mathcal{C}}^{\rm tw}_\bullet(W,V))$ of a self-conjugate monad on $\mathbb{CP}^3_\theta$ is a monad on $\mathbb{CP}^2_\theta$ which by Theorem 5.32 defines an isomorphism class in $\mathcal{M}_\theta(W,V)$. This gives a map of moduli spaces $\widehat{\mathcal{M}}^{\rm tw}_\theta(W,V) \to \mathcal{M}_\theta(W,V)$. At the level of non-commutative ADHM data, this map is just the natural inclusion of varieties $\mathcal{M}^{\rm tw}_\theta(W,V) \hookrightarrow \mathcal{M}^{\rm ADHM}_\theta(W,V)$. The virtue of this restriction is that this smaller class of torsion free sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ corresponds directly to a class of anti-selfdual connections on a canonically associated "instanton bundle", as we demonstrate in Section 6. We do not know if this construction is complete.

6 Construction of non-commutative instantons

6.1 Instanton bundles on S^4_θ

In this section we will use the monadic description of non-commutative real ADHM data on the twistor space $\mathbb{CP}^3_\theta$ to construct canonical bundles, called "instanton bundles", on the non-commutative sphere S^4_θ obtained in Section 4.5. This will be achieved via the twistor transform of framed torsion free sheaves E on $\mathsf{Open}(\mathbb{CP}^3_\theta)$ which satisfy the reality condition $E^\dagger \cong E$

of Section 5.4. It is determined by the non-commutative correspondence diagram



The generators w_i, $i = 1, 2, 3, 4$ and $\Lambda^{(ij)}$, $1 \leq i < j \leq 4$ of the homogeneous coordinate algebra of the non-commutative flag variety $\mathbb{F}\mathrm{l}_\theta(1, 2; 4)$ obey relations given by (4.27) and by (4.1) for $d = 2, n = 4$, the Plücker equations (4.9), and the relations (4.11) and (4.12), with the q-values (4.15) for $q \in \mathbb{R}$.

Using Lemma 4.14 we can apply the derived functor of the twistor transform $p_2{}^* p_{1*}$ to a self-conjugate linear monad complex $\underline{\mathcal{C}}{}^{\mathrm{tw}}_\bullet(W, V)$ on $\mathbb{CP}^3_\theta$ to get

$$p_2{}^* p_{1*}\big(\underline{\mathcal{C}}{}^{\mathrm{tw}}_\bullet(W,V)\big) : 0 \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{G}\mathrm{r}_\theta(2;4)} \xrightarrow{(\mathrm{id}_V \otimes \hat{\eta}) \circ \tau_w} V \otimes \mathcal{S}_\theta \longrightarrow 0, \quad (6.1)$$

where $\hat{\eta}$ is the rank two projector of (4.7) defining the tautological bundle over the Grassmann variety as $\mathcal{S}_\theta \cong \hat{\eta}\big(\mathcal{A}(\mathbb{G}\mathrm{r}_\theta(2;4)) \otimes \mathcal{A}^{\mathrm{tw}}_1\big)$, and $\mathcal{A}^{\mathrm{tw}}_1$ is the degree-one part of the algebra $\mathcal{A}^{\mathrm{tw}}$. It follows that the image of a cohomology sheaf $E = H^0\big(\underline{\mathcal{C}}{}^{\mathrm{tw}}_\bullet(W, V)\big)$ under the twistor transform is the sheaf on $\mathsf{Open}(\mathbb{G}\mathrm{r}_\theta(2; 4))$ given by

$$E' = p_2{}^* p_{1*}(E) = \ker\big((\mathrm{id}_V \otimes \hat{\eta}) \circ \tau_w\big).$$

We are interested in the restriction of this complex to the affine subvariety $\mathbb{R}^4_\theta$ of the non-commutative Grassmannian $\mathbb{G}\mathrm{r}_\theta(2; 4)$ described by Proposition 4.19 and the real structure (4.20).

It follows from (4.30) that the complex (6.1) restricts to the "Dirac operator"

$$\begin{aligned}\mathcal{D} := j^\bullet\,(\mathrm{id}_V \otimes \hat{\eta}) \circ \tau_w : \big(V \otimes (\mathcal{A}^!_\ell)_1 \oplus W\big) \otimes \mathcal{A}\big(\mathbb{R}^4_\theta\big)\\ \longrightarrow (V \oplus V) \otimes \mathcal{A}\big(\mathbb{R}^4_\theta\big),\end{aligned} \quad (6.2)$$

which can be written explicitly using (5.42) with (4.29), and the decompositions (5.14) and (5.36) as

$$\mathcal{D} = \begin{pmatrix} (\mathrm{id}_V \otimes \mu_{\mathcal{A}'_\ell}) \circ (B \otimes \mathrm{id}_{\mathcal{A}'_\ell}) \otimes 1 - \mathrm{id}_V \otimes (q^{-1}\,\check{w}_1 \otimes \xi_1 + q\,\check{w}_2 \otimes \xi_2) & I \\ (\mathrm{id}_V \otimes \mu_{\mathcal{A}'_\ell}) \circ (B^\dagger \otimes \mathrm{id}_{\mathcal{A}'_\ell}) \otimes 1 - \mathrm{id}_V \otimes (q^{-1}\,\check{w}_1 \otimes \bar{\xi}_2 + q\,\check{w}_2 \otimes \bar{\xi}_1) & -J^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} \big(B_1 - q^{-1}\,\xi_1\big) \otimes \check{w}_1 + \big(B_2 - q\,\xi_2\big) \otimes \check{w}_2 & I \\ \big(-B_2^\dagger - q^{-1}\,\bar{\xi}_2\big) \otimes \check{w}_1 + \big(B_1^\dagger - q\,\bar{\xi}_1\big) \otimes \check{w}_2 & -J^\dagger \end{pmatrix}. \tag{6.3}$$

By construction, $\mathcal{D}$ is a surjective morphism of free right $\mathcal{A}(\mathbb{R}^4_\theta)$-modules. Recall from the proof of Theorem 5.32 that surjectivity of the differential τ_w is equivalent to the stability condition (2) of Definition 5.17. For the same reason, and by point (3) of Proposition 3.17, the map $\tau_w^\dagger$ is injective, and hence $\mathcal{D}^\dagger$ is also injective. Here the †-involution is the tensor product of the real structures given by (4.20) and (5.35), and those of the chosen Hermitean structures on the vector spaces V and W.

Next we define the "Laplace operator"

$$\triangle := \mathcal{D} \circ \mathcal{D}^\dagger : (V \oplus V) \otimes \mathcal{A}\big(\mathbb{R}^4_\theta\big) \;\longrightarrow\; (V \oplus V) \otimes \mathcal{A}\big(\mathbb{R}^4_\theta\big). \tag{6.4}$$

Proposition 6.5. *The operator* $\triangle \in \mathrm{End}_{\mathcal{A}(\mathbb{R}^4_\theta)}\big((V \oplus V) \otimes \mathcal{A}(\mathbb{R}^4_\theta)\big)$ *is an isomorphism.*

Proof. Since $\mathcal{D}^\dagger \in \mathrm{Hom}_{\mathcal{A}(\mathbb{R}^4_\theta)}\big((V \oplus V) \otimes \mathcal{A}(\mathbb{R}^4_\theta), V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta)\big)$ is injective, it follows that $\mathrm{im}(\mathcal{D}^\dagger)$ is a free $\mathcal{A}(\mathbb{R}^4_\theta)$-submodule of $V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta)$. Its orthogonal complement with respect to the induced hermitean structures above can be naturally identified with the kernel of $\mathcal{D}$, and hence there is a decomposition

$$V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta) = \mathrm{im}(\mathcal{D}^\dagger) \oplus \ker(\mathcal{D}) \tag{6.6}$$

of $\mathcal{A}(\mathbb{R}^4_\theta)$-modules. In particular, $\mathrm{im}(\mathcal{D}^\dagger) \cap \ker(\mathcal{D}) = 0$, and hence $\triangle$ is injective since $\mathcal{D}^\dagger$ is injective.

Now let $v \in (V \oplus V) \otimes \mathcal{A}(\mathbb{R}^4_\theta)$. Since $\mathcal{D}$ is surjective, there exists $v_0 \in V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta)$ such that $\mathcal{D}(v_0) = v$. Using the decomposition (6.6) one has $v_0 = v'_0 + v''_0$ for some $v'_0 \in \mathrm{im}(\mathcal{D}^\dagger)$ and $v''_0 \in \ker(\mathcal{D})$. Setting $v'_0 = \mathcal{D}^\dagger(v')$ for $v' \in (V \oplus V) \otimes \mathcal{A}(\mathbb{R}^4_\theta)$, we then have $v = \mathcal{D}(v_0) = \mathcal{D}(v'_0) = \mathcal{D}\big(\mathcal{D}^\dagger(v')\big)$ and hence $\triangle$ is also surjective. □

The operators $\mathcal{D}$, $\mathcal{D}^\dagger$ and $\triangle$ are all $\mathcal{H}_\theta$-coequivariant morphisms with respect to the coactions given by (3.4), (4.21) and (5.16). The right $\mathcal{A}(\mathbb{R}^4_\theta)$-module

$$\mathcal{E} := \ker(\mathcal{D}) = \ker\big(j^\bullet\,(\mathrm{id}_V \otimes \hat{\eta}) \circ \tau_w\big) \tag{6.7}$$

is projective by (6.6). It is also finitely generated, and has rank $\dim_{\mathbb{C}}(W) = r$ since $\mathcal{D}$ is surjective. In fact, the isomorphism invariants of $\mathcal{E}$ are described by Corollary 5.8. Using Proposition 6.5 the corresponding projection can be given as

$$P := \mathrm{id}_{V_0} - \mathcal{D}^\dagger \circ \triangle^{-1} \circ \mathcal{D} \,:\, V_0 \otimes \mathcal{A}\big(\mathbb{R}^4_\theta\big) \;\longrightarrow\; \mathcal{E}, \tag{6.8}$$

with $P^2 = P = P^\dagger$ and trace $\mathrm{Tr}\,P = \dim_{\mathbb{C}}(V_0) - \dim_{\mathbb{C}}(V \oplus V) = r$. The module (6.7) is called an instanton bundle over $\mathbb{R}^4_\theta$; it defines an object of the category ${}^{\mathcal{H}_\theta}\mathscr{M}$. Using (5.28), one easily demonstrates that the isomorphism class of $\mathcal{E}$ depends only on the class of the non-commutative ADHM data (B, I, J) in the moduli variety $\mathcal{M}^{\mathrm{tw}}_\theta(W, V)$. Moreover, the reality condition $\mathcal{E}^\dagger \cong \mathcal{E}$ follows from the construction of Section 5.4.

Now we consider the restriction of the complex (6.1) to the affine subvariety $\widetilde{\mathbb{R}}^4_\theta$ of the non-commutative Grassmannian $\mathbb{G}\mathrm{r}_\theta(2;4)$ described by Proposition 4.22 and the real structure (4.23). Using (5.42) with (4.31), and the decompositions (5.14) and (5.36) we define a Dirac operator

$$\begin{aligned}\widetilde{\mathcal{D}} &:= \tilde{j}^{\,\bullet}\,(\mathrm{id}_V \otimes \hat{\eta}) \circ \tau_w \\ &= \begin{pmatrix} -q\,\big(B_1\,\bar{\zeta}_1 + B_2^\dagger\,\zeta_2 - q^{-2}\big) \otimes \check{w}_1 & -q\,\big(I\,\bar{\zeta}_1 + J^\dagger\,\zeta_2\big) \\ \quad -q\,\big(B_2\,\bar{\zeta}_1 - B_1^\dagger\,\zeta_2\big) \otimes \check{w}_2 & \\ & \\ q^{-1}\,\big(B_1\,\bar{\zeta}_2 + B_2^\dagger\,\zeta_1\big) \otimes \check{w}_1 & q^{-1}\,\big(I\,\bar{\zeta}_2 + J^\dagger\,\zeta_1\big) \\ +\,q^{-1}\,\big(B_2\,\bar{\zeta}_2 - B_1^\dagger\,\zeta_1 + q^2\big) \otimes \check{w}_2 & \end{pmatrix}\end{aligned} \tag{6.9}$$

in $\mathrm{Hom}_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\big(V_0 \otimes \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta), (V \oplus V) \otimes \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)\big)$. The operator $\widetilde{\mathcal{D}}$ is surjective by construction, and by repeating the arguments used above the module

$$\tilde{\mathcal{E}} := \ker\big(\widetilde{\mathcal{D}}\big) = \ker\big(\tilde{j}^{\,\bullet}\,(\mathrm{id}_V \otimes \hat{\eta}) \circ \tau_w\,\big)$$

is a projective module over $\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$, called an instanton bundle on $\widetilde{\mathbb{R}}^4_\theta$. Moreover, the coequivariant map

$$\widetilde{\triangle} := \widetilde{\mathcal{D}} \circ \widetilde{\mathcal{D}}^\dagger \;\in\; \mathrm{End}_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\big((V \oplus V) \otimes \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)\big) \tag{6.10}$$

is an isomorphism in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$, and the rank r projection $\widetilde{P}$ describing $\tilde{\mathcal{E}}$ is

$$\widetilde{P} := \mathrm{id}_{V_0} - \widetilde{\mathcal{D}}^\dagger \circ \widetilde{\triangle}^{-1} \circ \widetilde{\mathcal{D}} \ : \ V_0 \otimes \mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big) \ \longrightarrow \ \tilde{\mathcal{E}}$$

with $\widetilde{P}^2 = \widetilde{P} = \widetilde{P}^\dagger$ and $\mathrm{Tr}\,\widetilde{P} = r$.

It remains to determine the gluing automorphism between the two instanton bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$. For this, we use Proposition 4.25 to write the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0 \otimes \mathcal{A}\big(\mathbb{R}^4_\theta\big)[\tilde\rho\,] & \xrightarrow{\mathcal{D}} & (V\oplus V)\otimes \mathcal{A}\big(\mathbb{R}^4_\theta\big)[\tilde\rho\,] & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_{V_0}\otimes G} & & \Big\downarrow{\scriptstyle (\mathcal{G}\otimes 1)\circ(\mathrm{id}_{V\oplus V}\otimes G)} & & \\ 0 & \longrightarrow & V_0 \otimes [\rho]\mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big) & \xrightarrow[\widetilde{\mathcal{D}}]{} & (V\oplus V)\otimes [\rho]\mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big) & \longrightarrow & 0 \end{array} \qquad (6.11)$$

in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$, where the linear map $\mathcal{G} : V \oplus V \to (V \oplus V) \otimes \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$ is defined by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ \longmapsto \ \begin{pmatrix} -q\,\big(v_1\,\bar\zeta_1 - v_2\,\zeta_2\big) \\ q^{-1}\,\big(v_1\,\bar\zeta_2 - v_2\,\zeta_1\big) \end{pmatrix}.$$

The cohomology of the first row is $\mathcal{E}[\tilde\rho\,] := \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,]$, while the cohomology of the second row is $[\rho]\tilde{\mathcal{E}} := [\rho]\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta) \otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)} \tilde{\mathcal{E}}$. In this way the isomorphism of Proposition 4.25 induces the desired isomorphism of modules $G_{\mathcal{E}} : \mathcal{E}[\tilde\rho\,] \to [\rho]\tilde{\mathcal{E}}$ which is compatible with the $\dagger$-involutions and obeys $G_{\mathcal{E}}(\sigma \triangleleft a) = G(a) \triangleright G_{\mathcal{E}}(\sigma)$ for $\sigma \in \mathcal{E}[\tilde\rho\,]$, $a \in \mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,]$, thus defining an $\mathcal{H}_\theta$-coequivariant instanton bundle on the non-commutative sphere S^4_θ.

6.2 Instanton gauge fields

We will now construct canonical connections on the instanton bundles introduced in Section 6.1. For this, we first need to write down a canonical exterior differential algebra over the sphere S^4_θ as a deformation of its classical counterpart, following the general construction of Kähler differentials given in [13, Section 4.4]. We begin by constructing differential forms on the open subvariety $\mathbb{R}^4_\theta$.

Starting with the real affine variety $\mathbb{R}^4$, let $\Omega^\bullet_{\mathbb{R}^4} = \bigwedge^\bullet \Omega^1_{\mathbb{R}^4}$ be the usual classical differential calculus on the coordinate algebra $\mathcal{A}(\mathbb{R}^4)$, generated as

a differential graded algebra by degree zero elements $\xi_i, \bar{\xi}_i$ and degree one elements $\mathrm{d}\xi_i, \mathrm{d}\bar{\xi}_i$ satisfying the skew-commutation relations

$$\mathrm{d}\xi_i \wedge \mathrm{d}\xi_j = -\mathrm{d}\xi_j \wedge \mathrm{d}\xi_i, \quad \mathrm{d}\xi_i \wedge \mathrm{d}\bar{\xi}_j = -\mathrm{d}\bar{\xi}_j \wedge \mathrm{d}\xi_i, \quad \mathrm{d}\bar{\xi}_i \wedge \mathrm{d}\bar{\xi}_j = -\mathrm{d}\bar{\xi}_j \wedge \mathrm{d}\bar{\xi}_i$$

and the symmetric $\mathcal{A}(\mathbb{R}^4)$-bimodule structure

$$\xi_i \, \mathrm{d}\xi_j = \mathrm{d}\xi_j \, \xi_i, \quad \xi_i \, \mathrm{d}\bar{\xi}_j = \mathrm{d}\bar{\xi}_j \, \xi_i, \quad \bar{\xi}_i \, \mathrm{d}\bar{\xi}_j = \mathrm{d}\bar{\xi}_j \, \bar{\xi}_i$$

for $i, j = 1, 2$. The differential $\mathrm{d} : \Omega^0_{\mathbb{R}^4} := \mathcal{A}(\mathbb{R}^4) \to \Omega^1_{\mathbb{R}^4}$ is defined by $\xi_i \mapsto \mathrm{d}\xi_i$, $\bar{\xi}_i \mapsto \mathrm{d}\bar{\xi}_i$. It is extended uniquely to a map of degree one, $\mathrm{d} : \Omega^n_{\mathbb{R}^4} \to \Omega^{n+1}_{\mathbb{R}^4}$, using $\mathbb{C}$-linearity and the graded Leibniz rule

$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + (-1)^{\deg(\omega)} \, \omega \wedge \mathrm{d}\omega'$$

with $\mathrm{d}^2 = 0$, where the product is taken over $\mathcal{A}(\mathbb{R}^4)$ using the bimodule structure of $\Omega^\bullet_{\mathbb{R}^4}$. We will demand that the differential calculus on $\mathcal{A}(\mathbb{R}^4)$ is *coequivariant* for the torus coaction Δ_L given in (4.21). Then we can extend this coaction to a left coaction $\Delta_L : \Omega^\bullet_{\mathbb{R}^4} \to \mathcal{H} \otimes \Omega^\bullet_{\mathbb{R}^4}$ such that d is a left $\mathcal{H}$-comodule morphism and Δ_L is an $\mathcal{A}(\mathbb{R}^4)$-bimodule morphism.

Following the prescription of Section 2.2, we can now use the comodule cotwist to deform the differential structure in the same way as for the algebra itself in Proposition 4.19. One finds

$$\begin{aligned}
\xi_i \blacktriangleright_\theta \mathrm{d}\xi_j &= F_\theta(t_i, t_j) \, \xi_i \, \mathrm{d}\xi_j, \quad \bar{\xi}_i \blacktriangleright_\theta \mathrm{d}\bar{\xi}_j = F_\theta(t_{i+1}, t_{j+1}) \, \bar{\xi}_i \, \mathrm{d}\bar{\xi}_j, \\
\xi_i \blacktriangleright_\theta \mathrm{d}\bar{\xi}_j &= F_\theta(t_i, t_{j+1}) \, \xi_i \, \mathrm{d}\bar{\xi}_j, \quad \bar{\xi}_i \blacktriangleright_\theta \mathrm{d}\xi_j = F_\theta(t_{i+1}, t_j) \, \bar{\xi}_i \, \mathrm{d}\xi_j, \\
\mathrm{d}\xi_i \wedge_\theta \mathrm{d}\xi_j &= F_\theta(t_i, t_j) \, \mathrm{d}\xi_i \wedge \mathrm{d}\xi_j, \quad \mathrm{d}\bar{\xi}_i \wedge_\theta \mathrm{d}\bar{\xi}_j = F_\theta(t_{i+1}, t_{j+1}) \, \mathrm{d}\bar{\xi}_i \wedge \mathrm{d}\bar{\xi}_j, \\
\mathrm{d}\xi_i \wedge_\theta \mathrm{d}\bar{\xi}_j &= F_\theta(t_i, t_{j+1}) \, \mathrm{d}\xi_i \wedge \mathrm{d}\bar{\xi}_j, \quad \mathrm{d}\bar{\xi}_i \wedge_\theta \mathrm{d}\xi_j = F_\theta(t_{i+1}, t_j) \, \mathrm{d}\bar{\xi}_i \wedge \mathrm{d}\xi_j,
\end{aligned}$$

with indices read modulo 2. The braided exterior product $\wedge_\theta$ describes how the tensor product acts on the quotient of the tensor algebra

$$T_\theta\big(\Omega^1_{\mathbb{R}^4}\big) = \mathcal{A}\big(\mathbb{R}^4_\theta\big) \; \oplus \bigoplus_{n \geq 1} \big(\Omega^1_{\mathbb{R}^4}\big)^{\otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} n}$$

by the ideal generated by the braided skew-commutation relations (see (2.6)). The undeformed differential d is still a derivation (of degree one)

of the deformed product $\wedge_\theta$, as follows from general results of twist deformation theory [30] or simply by direct computation

$$\begin{aligned}\mathrm{d}(\omega_1 \wedge_\theta \omega_2) &= F_\theta\big(\omega_1^{(-1)}, \omega_2^{(-1)}\big)\ \mathrm{d}\big(\omega_1^{(0)} \wedge \omega_2^{(0)}\big) \\ &= F_\theta\big(\omega_1^{(-1)}, \omega_2^{(-1)}\big)\ \big(\mathrm{d}\omega_1^{(0)} \wedge \omega_2^{(0)} + (-1)^{\deg(\omega_1^{(0)})}\, \omega_1^{(0)} \wedge \mathrm{d}\omega_2^{(0)}\big) \\ &= \mathrm{d}\omega_1 \wedge_\theta \omega_2 + (-1)^{\deg(\omega_1)}\, \omega_1 \wedge_\theta \mathrm{d}\omega_2\end{aligned}$$

for homogeneous forms ω_1, ω_2, with the usual Sweedler notation $\Delta_L(\omega) = \omega^{(-1)} \otimes \omega^{(0)}$.

The above construction defines a canonical differential graded algebra $\Omega^\bullet_{\mathbb{R}^4_\theta} = \bigwedge^\bullet_\theta \Omega^1_{\mathbb{R}^4}$ for $\mathcal{A}(\mathbb{R}^4_\theta)$ with the same generators and the same differential d, but now subject to the braided skew-commutation relations

$$\begin{aligned}&\mathrm{d}\xi_i \wedge \mathrm{d}\xi_j = -q_{ij}^2\ \mathrm{d}\xi_j \wedge \mathrm{d}\xi_i, \quad \mathrm{d}\bar\xi_i \wedge \mathrm{d}\bar\xi_j \ = \ -q_{i+1\,j+1}^2\ \mathrm{d}\bar\xi_j \wedge \mathrm{d}\bar\xi_i, \\ &\mathrm{d}\xi_i \wedge \mathrm{d}\bar\xi_j = -q_{i\,j+1}^2\ \mathrm{d}\bar\xi_j \wedge \mathrm{d}\xi_i\end{aligned} \tag{6.12}$$

and the braided symmetric $\mathcal{A}(\mathbb{R}^4_\theta)$-bimodule structure

$$\begin{aligned}&\xi_i\ \mathrm{d}\xi_j = q_{ij}^2\ \mathrm{d}\xi_j\ \xi_i, \quad \bar\xi_i\ \mathrm{d}\bar\xi_j \ = \ q_{i+1\,j+1}^2\ \mathrm{d}\bar\xi_j\ \bar\xi_i, \\ &\xi_i\ \mathrm{d}\bar\xi_j = q_{i\,j+1}^2\ \mathrm{d}\bar\xi_j\ \xi_i, \quad \bar\xi_i\ \mathrm{d}\xi_j \ = \ q_{i+1\,j}^2\ \mathrm{d}\xi_j\ \bar\xi_i.\end{aligned}$$

Again we drop the explicit deformation symbols from the products. This also identifies $\Omega^\bullet_{\mathbb{R}^4_\theta}$ via the restriction

$$j^\bullet \Omega^1_{\mathbb{G}\mathrm{r}_\theta(2;4)} \cong \Omega^1_{\mathbb{R}^4_\theta}$$

of the bundle of non-commutative Kähler differentials (4.6). The real structure (4.20) extends to $\Omega^\bullet_{\mathbb{R}^4_\theta}$ by graded extension of the morphism $\xi_i \mapsto \xi_i^\dagger$.

We are now ready to construct a canonical connection on the right $\mathcal{A}(\mathbb{R}^4_\theta)$-module $\mathcal{E}$, i.e., a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \ \longrightarrow\ \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^1_{\mathbb{R}^4_\theta}$$

satisfying the Leibniz rule

$$\nabla(\sigma \triangleleft f) = (\nabla\sigma) \triangleleft f + \sigma \otimes \mathrm{d}f$$

for $f \in \mathcal{A}(\mathbb{R}^4_\theta)$ and $\sigma \in \mathcal{E}$. The connection ∇ extends to one-forms as a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^1_{\mathbb{R}^4_\theta} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^2_{\mathbb{R}^4_\theta}$$

satisfying

$$\nabla(\sigma \otimes \omega) = (\nabla\sigma) \otimes \omega + \sigma \otimes \mathrm{d}\omega$$

for $\omega \in \Omega^1_{\mathbb{R}^4_\theta}$ and $\sigma \in \mathcal{E}$. Two connections ∇ and ∇' are *gauge equivalent* if there exists an automorphism $g \in \mathrm{Aut}_{\mathcal{A}(\mathbb{R}^4_\theta)}(\mathcal{E})$ such that $\nabla = \tilde{g} \circ \nabla' \circ g^{-1}$, where $\tilde{g} = g \otimes \mathrm{id}$. The curvature $F_\nabla = \nabla \circ \nabla$ is defined by the composition

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^1_{\mathbb{R}^4_\theta} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^2_{\mathbb{R}^4_\theta}.$$

Since F_∇ is right $\mathcal{A}(\mathbb{R}^4_\theta)$-linear, it may be regarded as an element

$$F_\nabla \in \mathrm{Hom}_{\mathcal{A}(\mathbb{R}^4_\theta)}\big(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^2_{\mathbb{R}^4_\theta}\big).$$

If ∇ and ∇' are gauge equivalent via the gauge transformation $g \in \mathrm{Aut}_{\mathcal{A}(\mathbb{R}^4_\theta)}(\mathcal{E})$, then the corresponding curvatures are easily found to be related as $F_\nabla = \tilde{g} \circ F_{\nabla'} \circ g^{-1}$.

To define the instanton connection, let $\iota : \mathcal{E} \to V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta)$ denote the natural inclusion, and $\mathrm{d} : \mathcal{A}(\mathbb{R}^4_\theta) \to \Omega^1_{\mathbb{R}^4_\theta}$ the differential introduced above. We then take ∇ to be the Grassmann connection associated to the projection P, which is defined by the composition

$$\nabla : \mathcal{E} \xrightarrow{\iota} V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta) \xrightarrow{\mathrm{id}_{V_0} \otimes \mathrm{d}} V_0 \otimes \Omega^1_{\mathbb{R}^4_\theta} \xrightarrow{P \otimes \mathrm{id}} \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^1_{\mathbb{R}^4_\theta},$$

or $\nabla = P \circ \mathrm{d} \circ \iota$. It is easily seen to be compatible with the $\dagger$-involution, since

$$\nabla\big(\sigma^\dagger\big) = P\,\mathrm{d}\big(\sigma^\dagger\big) = P\,(\mathrm{d}\sigma)^\dagger = (P\,\mathrm{d}\sigma)^\dagger = \nabla(\sigma)^\dagger$$

for $\sigma \in \mathcal{E}$. Its curvature is given by

$$F_\nabla = \nabla^2 = P\,(\mathrm{d}P)^2.$$

Lemma 6.13. *The gauge equivalence class of the instanton connection $\nabla = P \circ \mathrm{d} \circ \iota$ depends only on the class of the non-commutative ADHM data (B, I, J) in the moduli variety $\mathcal{M}^{\mathrm{tw}}_\theta(W, V)$.*

Proof. Let $g_0 : \mathcal{E} \to \mathcal{E}'$ be the $\mathcal{A}(\mathbb{R}^4_\theta)$-module isomorphism induced by $\tilde{g} \oplus \mathrm{id}_W \in \mathrm{U}(V_0)$ in (5.28). Let $\iota' : \mathcal{E}' \to V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta)$ be the inclusion, and let $P' : V_0 \otimes \mathcal{A}(\mathbb{R}^4_\theta) \to \mathcal{E}'$ be the projection defined as in (6.8). Then $\iota' = g_0 \circ \iota \circ g_0^{-1}$ and $P' = g_0 \circ P \circ g_0^{-1}$. Thus

$$\begin{aligned}\nabla' &= P' \circ \mathrm{d} \circ \iota' \\ &= \big(g_0 \circ P \circ g_0^{-1}\big) \circ \mathrm{d}\big(g_0 \circ \iota \circ g_0^{-1}\big) \\ &= \big(g_0 \circ P \circ g_0^{-1}\big) \circ \big((\mathrm{d}g_0) \circ \iota \circ g_0^{-1} + g_0 \circ \mathrm{d}(\iota \circ g_0^{-1})\big) \\ &= \tilde{g}_0 \circ \big(P \circ \mathrm{d} \circ \iota\big) \circ g_0^{-1} \;=\; \tilde{g}_0 \circ \nabla \circ g_0^{-1},\end{aligned}$$

where we have used the Leibniz rule together with the fact that g_0 acts as the identity on $\mathcal{A}(\mathbb{R}^4_\theta)$ so that $\mathrm{d}g_0 = 0 = \mathrm{d}g_0^{-1}$. □

Similarly, the coequivariant differential calculus $\big(\Omega^\bullet_{\widetilde{\mathbb{R}}^4_\theta}, \mathrm{d}\big)$ over the affine variety $\widetilde{\mathbb{R}}^4_\theta$ is generated by $\zeta_i, \bar{\zeta}_i$ and $\mathrm{d}\zeta_i, \mathrm{d}\bar{\zeta}_i$ with $i = 1, 2$. The relations between $\zeta_i, \bar{\zeta}_i$ are given in Proposition 4.22 with the torus coaction in (4.24). By coequivariance, one has

$$\begin{aligned}\Delta_L(\mathrm{d}a) &= (\mathrm{id} \otimes \mathrm{d})\Delta_L(a), \\ a \blacktriangleright_\theta \Delta_L(\omega) = \Delta_L(a \blacktriangleright_\theta \omega), &\quad \Delta_L(\omega) \blacktriangleleft_\theta a = \Delta_L(\omega \blacktriangleleft_\theta a)\end{aligned}$$

for $a \in \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$ and $\omega \in \Omega^1_{\widetilde{\mathbb{R}}^4_\theta}$, where here $\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$ acts on $\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta) \otimes \Omega^1_{\widetilde{\mathbb{R}}^4_\theta}$ in the tensor product representation. Thus the differentials $\mathrm{d}\zeta_i, \mathrm{d}\bar{\zeta}_i$ for $i = 1, 2$ behave algebraically in exactly the same way as $\zeta_i, \bar{\zeta}_i$, and so we have the relations

$$\begin{aligned}\zeta_i\ \mathrm{d}\zeta_j &= q_{ij}^{-2}\ \mathrm{d}\zeta_j\ \zeta_i, \quad \bar{\zeta}_i\ \mathrm{d}\bar{\zeta}_j \;=\; q_{i+1\,j+1}^{-2}\ \mathrm{d}\bar{\zeta}_j\ \bar{\zeta}_i, \\ \zeta_i\ \mathrm{d}\bar{\zeta}_j &= q_{i\,j+1}^{-2}\ \mathrm{d}\bar{\zeta}_j\ \zeta_i, \quad \bar{\zeta}_i\ \mathrm{d}\zeta_j \;=\; q_{i+1\,j}^{-2}\ \mathrm{d}\zeta_j\ \bar{\zeta}_i, \\ \mathrm{d}\zeta_i \wedge \mathrm{d}\zeta_j &= -q_{ij}^{-2}\ \mathrm{d}\zeta_j \wedge \mathrm{d}\zeta_i, \quad \mathrm{d}\bar{\zeta}_i \wedge \mathrm{d}\bar{\zeta}_j \;=\; -q_{i+1\,j+1}^{-2}\ \mathrm{d}\bar{\zeta}_j \wedge \mathrm{d}\bar{\zeta}_i, \\ \mathrm{d}\zeta_i \wedge \mathrm{d}\bar{\zeta}_j &= -q_{i\,j+1}^{-2}\ \mathrm{d}\bar{\zeta}_j \wedge \mathrm{d}\zeta_i\end{aligned} \tag{6.14}$$

for $i, j = 1, 2$. The instanton connection $\widetilde{\nabla} = \widetilde{P} \circ \mathrm{d} \circ \tilde{\iota} : \tilde{\mathcal{E}} \to \Omega^1_{\widetilde{\mathbb{R}}^4_\theta} \otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)} \tilde{\mathcal{E}}$ on the projective left $\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$-module $\tilde{\mathcal{E}}$ is similarly defined as in the previous case, with curvature $F_{\widetilde{\nabla}} = \widetilde{P}\ (\mathrm{d}\widetilde{P})^2$ as an element of $\mathrm{Hom}_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\big(\tilde{\mathcal{E}}, \Omega^2_{\widetilde{\mathbb{R}}^4_\theta} \otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)} \tilde{\mathcal{E}}\big)$.

We need to check consistency between the connections $\nabla_{\tilde\rho}$ and ${}_\rho\widetilde{\nabla}$ on the adjoinment bundles $\mathcal{E}[\tilde\rho\,]$ and $[\rho]\tilde{\mathcal{E}}$, respectively, using the module isomorphism $G_{\mathcal{E}} : \mathcal{E}[\tilde\rho\,] \to [\rho]\tilde{\mathcal{E}}$ induced by the commutative diagram (6.11). For this, consider the diagram

$$\begin{array}{ccccccc}
\mathcal{E}[\tilde\rho\,] & \xrightarrow{\ \iota\ } & V_0\otimes\mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,] & \xrightarrow{\mathrm{id}_{V_0}\otimes\mathrm{d}} & V_0\otimes\Omega^1_{\mathbb{R}^4_\theta}[\tilde\rho\,] & \xrightarrow{\ P\ } & \mathcal{E}\otimes_{\mathcal{A}(\mathbb{R}^4_\theta)}\Omega^1_{\mathbb{R}^4_\theta}[\tilde\rho\,] \\
\downarrow{\scriptstyle G_{\mathcal{E}}} & & \downarrow{\scriptstyle \mathrm{id}_{V_0}\otimes G} & & \downarrow{\scriptstyle \mathrm{id}_{V_0}\otimes G^{(1)}} & & \downarrow{\scriptstyle G^{(1)}_{\mathcal{E}}} \\
[\rho]\tilde{\mathcal{E}} & \xrightarrow[\ \bar\iota\ ]{} & V_0\otimes[\rho]\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta) & \xrightarrow[\mathrm{id}_{V_0}\otimes\mathrm{d}]{} & V_0\otimes[\rho]\Omega^1_{\widetilde{\mathbb{R}}^4_\theta} & \xrightarrow[\ \bar P\ ]{} & [\rho]\Omega^1_{\widetilde{\mathbb{R}}^4_\theta}\otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\tilde{\mathcal{E}}
\end{array} \tag{6.15}$$

The commutativity of the first square follows from commutativity of the diagram in (6.11). Next, we can lift G to the bimodule isomorphism $G^{(1)} : \Omega^1_{\mathbb{R}^4_\theta}[\tilde\rho\,] \to [\rho]\Omega^1_{\widetilde{\mathbb{R}}^4_\theta}$ through the intertwining relation defined by commutativity of the second square. Finally, with this definition we can extend $G^{(1)}$ to the bundle isomorphism

$$G^{(1)}_{\mathcal{E}} : \mathcal{E}\otimes_{\mathcal{A}(\mathbb{R}^4_\theta)}\Omega^1_{\mathbb{R}^4_\theta}[\tilde\rho\,] \longrightarrow [\rho]\Omega^1_{\widetilde{\mathbb{R}}^4_\theta}\otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\tilde{\mathcal{E}}$$

by demanding that the third square be commutative; it is also compatible with the †-involutions. Then the instanton connections are related by the gauge transformation

$${}_\rho\widetilde{\nabla} = G^{(1)}_{\mathcal{E}}\circ\nabla_{\tilde\rho}\circ G^{-1}_{\mathcal{E}}, \tag{6.16}$$

as desired. This defines the instanton connection on the non-commutative sphere S^4_θ.

6.3 Anti-selfduality equations

We will now demonstrate that the Grassmann connections constructed in Section 6.2 define non-commutative instantons in analogy with the classical case, i.e., that they satisfy some form of anti-selfduality equations. For this, we need to construct suitable versions of the classical Hodge duality operator acting on two-forms. The euclidean metric on $\mathbb{R}^4$ induces a Hodge duality operator which on two-forms is the linear map $*:\Omega^2_{\mathbb{R}^4}\to\Omega^2_{\mathbb{R}^4}$ defined by

$$\begin{aligned}
&*(\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_2) = -\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_2, \quad *(\mathrm{d}\xi_1\wedge\mathrm{d}\xi_2) = \mathrm{d}\xi_1\wedge\mathrm{d}\xi_2,\\
&*(\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_1) = -\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_2, \quad *(\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_2) = -\mathrm{d}\xi_1\wedge\mathrm{d}\bar\xi_1,\\
&*(\mathrm{d}\bar\xi_1\wedge\mathrm{d}\bar\xi_2) = \mathrm{d}\bar\xi_1\wedge\mathrm{d}\bar\xi_2, \quad *(\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_1) = -\mathrm{d}\xi_2\wedge\mathrm{d}\bar\xi_1.
\end{aligned} \tag{6.17}$$

In contrast to the case of isospectral deformations [7, 8, 14, 28], the coaction (4.21) of the torus algebra $\mathcal{H} = \mathcal{A}((\mathbb{C}^\times)^2)$ on $\mathcal{A}(\mathbb{R}^4)$ is *not* isometric. However, it does coact by conformal transformations on $\mathcal{A}(\mathbb{R}^4)$, and hence preserves the Hodge operator. This is easily checked using (6.17) and (4.21) which shows that the operator $* : \Omega^2_{\mathbb{R}^4} \to \Omega^2_{\mathbb{R}^4}$ is coequivariant,

$$\Delta_L(*\omega) = (\mathrm{id} \otimes *)\Delta_L(\omega) \quad \text{for} \quad \omega \in \Omega^2_{\mathbb{R}^4},$$

and hence defines a morphism of the category ${}^{\mathcal{H}}\mathscr{M}$. Since the vector space $\Omega^2_{\mathbb{R}^4_\theta}$ coincides with its classical counterpart $\Omega^2_{\mathbb{R}^4}$, and since the quantization functor $\mathscr{F}_\theta : {}^{\mathcal{H}}\mathscr{M} \to {}^{\mathcal{H}_\theta}\mathscr{M}$ acts as the identity on objects and morphisms of ${}^{\mathcal{H}}\mathscr{M}$, we can define a Hodge duality operator $*_\theta : \Omega^2_{\mathbb{R}^4_\theta} \to \Omega^2_{\mathbb{R}^4_\theta}$ by the same formula (6.17), which by construction is a morphism in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$. In particular, it satisfies $*_\theta^2 = \mathrm{id}$ and the $\mathcal{H}_\theta$-coequivariant decomposition of the right $\mathcal{A}(\mathbb{R}^4_\theta)$-module

$$\Omega^2_{\mathbb{R}^4_\theta} = \Omega^{2,+}_{\mathbb{R}^4_\theta} \oplus \Omega^{2,-}_{\mathbb{R}^4_\theta}$$

into submodules, corresponding to eigenvalues ± 1 of $*_\theta$, is identical as a vector space to that of the classical case. Hence the eigenmodules of selfdual and anti-selfdual two-forms are given respectively by

$$\begin{aligned}
\Omega^{2,+}_{\mathbb{R}^4_\theta} &= \mathcal{A}\big(\mathbb{R}^4_\theta\big)\big\langle \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2, \mathrm{d}\bar\xi_1 \wedge \mathrm{d}\bar\xi_2, \mathrm{d}\xi_1 \wedge \mathrm{d}\bar\xi_1 - \mathrm{d}\xi_2 \wedge \mathrm{d}\bar\xi_2 \big\rangle, \\
\Omega^{2,-}_{\mathbb{R}^4_\theta} &= \mathcal{A}\big(\mathbb{R}^4_\theta\big)\big\langle \mathrm{d}\xi_1 \wedge \mathrm{d}\bar\xi_2, \mathrm{d}\xi_2 \wedge \mathrm{d}\bar\xi_1, \mathrm{d}\xi_1 \wedge \mathrm{d}\bar\xi_1 + \mathrm{d}\xi_2 \wedge \mathrm{d}\bar\xi_2 \big\rangle.
\end{aligned} \tag{6.18}$$

In a completely analogous way, by using the $\mathcal{H}$-coaction (4.24) one constructs a morphism $\tilde *_\theta : \Omega^2_{\widetilde{\mathbb{R}}^4_\theta} \to \Omega^2_{\widetilde{\mathbb{R}}^4_\theta}$ in the category ${}^{\mathcal{H}_\theta}\mathscr{M}$ with the same formula (6.17) but with affine coordinates ξ_i substituted with ζ_i, and with overall changes of sign reflecting the change of "orientation". There is an analogous $\mathcal{H}_\theta$-coequivariant decomposition

$$\Omega^2_{\widetilde{\mathbb{R}}^4_\theta} = \Omega^{2,+}_{\widetilde{\mathbb{R}}^4_\theta} \oplus \Omega^{2,-}_{\widetilde{\mathbb{R}}^4_\theta}$$

of two-forms into selfdual and anti-selfdual two-forms with

$$\begin{aligned}
\Omega^{2,+}_{\widetilde{\mathbb{R}}^4_\theta} &= \big\langle \mathrm{d}\zeta_1 \wedge \mathrm{d}\bar\zeta_2, \mathrm{d}\zeta_2 \wedge \mathrm{d}\bar\zeta_1, \mathrm{d}\zeta_1 \wedge \mathrm{d}\bar\zeta_1 + \mathrm{d}\zeta_2 \wedge \mathrm{d}\bar\zeta_2 \big\rangle \mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big), \\
\Omega^{2,-}_{\widetilde{\mathbb{R}}^4_\theta} &= \big\langle \mathrm{d}\zeta_1 \wedge \mathrm{d}\zeta_2, \mathrm{d}\bar\zeta_1 \wedge \mathrm{d}\bar\zeta_2, \mathrm{d}\zeta_1 \wedge \mathrm{d}\bar\zeta_1 - \mathrm{d}\zeta_2 \wedge \mathrm{d}\bar\zeta_2 \big\rangle \mathcal{A}\big(\widetilde{\mathbb{R}}^4_\theta\big).
\end{aligned} \tag{6.19}$$

The consistency condition for the corresponding morphisms on the adjoinment bimodules $*_\theta{}^{\tilde\rho} : \Omega^2_{\mathbb{R}^4_\theta}[\tilde\rho\,] \to \Omega^2_{\mathbb{R}^4_\theta}[\tilde\rho\,]$ and ${}^{\rho}\tilde *_\theta : [\rho]\Omega^2_{\widetilde{\mathbb{R}}^4_\theta} \to [\rho]\Omega^2_{\widetilde{\mathbb{R}}^4_\theta}$ are given

by lifting the isomorphism $G^{(1)}$ of (6.15) to the bimodule isomorphism $G^{(2)} : \Omega^2_{\mathbb{R}^4_\theta}[\tilde{\rho}] \to [\rho]\Omega^2_{\widetilde{\mathbb{R}}^4_\theta}$ through the intertwining relation defined by commutativity of the diagram

$$\begin{array}{ccc} \Omega^1_{\mathbb{R}^4_\theta}[\tilde{\rho}] & \xrightarrow{\ \mathrm{d}\ } & \Omega^2_{\mathbb{R}^4_\theta}[\tilde{\rho}] \\ {\scriptstyle \tilde{\mathscr{J}}_2 \circ G^{(1)}}\downarrow & & \downarrow{\scriptstyle G^{(2)}} \\ [\rho]\Omega^1_{\widetilde{\mathbb{R}}^4_\theta} & \xrightarrow[\ \mathrm{d}\ ]{} & [\rho]\Omega^2_{\widetilde{\mathbb{R}}^4_\theta} \end{array} ,$$

where $\tilde{\mathscr{J}}_2 : \Omega^\bullet_{\widetilde{\mathbb{R}}^4_\theta} \to \Omega^\bullet_{\widetilde{\mathbb{R}}^4_\theta}$ is induced by graded extension of the map $(\zeta_1, \zeta_2) \mapsto (\zeta_1, q^2\,\bar{\zeta}_2)$. One then has

$$^{\rho}\,\tilde{*}_\theta = G^{(2)} \circ *_\theta\, ^{\tilde{\rho}} \circ G^{(2)\,-1}, \tag{6.20}$$

which defines the $\mathcal{H}_\theta$-coequivariant Hodge operator on the non-commutative sphere S^4_θ.

Proposition 6.21. *The curvatures F_∇ and $F_{\widetilde{\nabla}}$ of the instanton connections are anti-selfdual, i.e., as two-forms they obey the anti-selfduality equations*

$$*_\theta F_\nabla = -F_\nabla, \quad \tilde{*}_\theta F_{\widetilde{\nabla}} = -F_{\widetilde{\nabla}}.$$

Proof. Using $\mathcal{D}(\sigma) = 0$ for $\sigma \in \mathcal{E}$, so that $\mathcal{D}(\mathrm{d}\sigma) = -(\mathrm{d}\mathcal{D})(\sigma)$ by the Leibniz rule, and $P(\mathrm{im}(\mathcal{D}^\dagger)) = 0$, the action of the curvature $F_\nabla = P\,(\mathrm{d}P)^2 \in \mathrm{Hom}_{\mathcal{A}(\mathbb{R}^4_\theta)}\big(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^2_{\mathbb{R}^4_\theta}\big)$ is given by

$$F_\nabla(\sigma) = P\left(\mathrm{d}(\mathrm{id}_{V_0} - \mathcal{D}^\dagger \circ \triangle^{-1} \circ \mathcal{D}) \wedge \mathrm{d}\sigma\right) = P\left(\mathrm{d}\mathcal{D}^\dagger \circ \triangle^{-1} \wedge \mathrm{d}\mathcal{D}(\sigma)\right) .$$

Using the commutation relations of Proposition 4.19 and (5.12), the real structures (4.20) and (5.35), and the ADHM equations (5.18) and (5.38), one finds that the Laplace operator (6.4) assumes the block diagonal form

$$\triangle = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix},$$

where the isomorphism $\delta \in \mathrm{End}_{\mathcal{A}(\mathbb{R}^4_\theta)}\big(V \otimes \mathcal{A}(\mathbb{R}^4_\theta)\big)$ is given by

$$\begin{aligned} \delta = {} & B_1\,B_1^\dagger + q^{-2}\,B_2\,B_2^\dagger + I\,I^\dagger + q^{-1}\,\rho - q^{-1}\,B_1\,\bar{\xi}_1 \\ & - B_1^\dagger\,\xi_1 + q^{-1}\,B_2\,\bar{\xi}_2 - B_2^\dagger\,\xi_2. \end{aligned}$$

It follows that F_∇ is proportional to $\mathrm{d}\mathcal{D}^\dagger \wedge \mathrm{d}\mathcal{D}$ as a two-form. The exterior derivative of the Dirac operator (6.3) and its adjoint are easily computed to be

$$\mathrm{d}\mathcal{D} = \begin{pmatrix} -q^{-1}\,\mathrm{d}\xi_1 \otimes \check{w}_1 - q\,\mathrm{d}\xi_2 \otimes \check{w}_2 & 0 \\ -q^{-1}\,\mathrm{d}\bar{\xi}_2 \otimes \check{w}_1 - q\,\mathrm{d}\bar{\xi}_1 \otimes \check{w}_2 & 0 \end{pmatrix},$$

$$\mathrm{d}\mathcal{D}^\dagger = \begin{pmatrix} -q^{-2}\,\mathrm{d}\bar{\xi}_1 \otimes \check{w}_1 & \mathrm{d}\xi_2 \otimes \check{w}_1 \\ \mathrm{d}\bar{\xi}_2 \otimes \check{w}_2 & -q^2\,\mathrm{d}\xi_1 \otimes \check{w}_2 \\ 0 & 0 \end{pmatrix}.$$

Using the commutation relations (6.12) we then find

$$\mathrm{d}\mathcal{D}^\dagger \wedge \mathrm{d}\mathcal{D} = \begin{pmatrix} -q^{-1}\left(\mathrm{d}\xi_1 \wedge \mathrm{d}\bar{\xi}_1 + \mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_2\right) & -\left(q + q^{-1}\right)\mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_1 & 0 \\ \left(q + q^{-1}\right)\mathrm{d}\xi_1 \wedge \mathrm{d}\bar{\xi}_2 & q^3\left(\mathrm{d}\xi_1 \wedge \mathrm{d}\bar{\xi}_1 + \mathrm{d}\xi_2 \wedge \mathrm{d}\bar{\xi}_2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Comparing with (6.18), we see that each entry of F_∇ belongs to the submodule $\Omega^{2,-}_{\mathbb{R}^4_\theta}$.

The anti-selfduality equation for the curvature $F_{\widetilde{\nabla}} = \tilde{P}\,(\mathrm{d}\tilde{P})^2 \in \mathrm{Hom}_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\left(\tilde{\mathcal{E}}, \Omega^2_{\widetilde{\mathbb{R}}^4_\theta} \otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)} \tilde{\mathcal{E}}\right)$ follows analogously. This time we use Proposition 4.22 and (4.23) to write the Laplace operator (6.10) in the block diagonal form

$$\tilde{\triangle} = \begin{pmatrix} \tilde{\delta} & 0 \\ 0 & \tilde{\delta} \end{pmatrix},$$

where the isomorphism $\tilde{\delta} \in \mathrm{End}_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\left(V \otimes \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)\right)$ is given by

$$\begin{aligned} \tilde{\delta} = {} & q^{-1}\left(B_1\,B_1^\dagger + q^{-2}\,B_2\,B_2^\dagger + I\,I^\dagger\right)\tilde{\rho} - B_1\,\bar{\zeta}_1 - q^{-1}\,B_1^\dagger\,\zeta_1 \\ & + q^{-2}\,B_2\,\bar{\zeta}_2 - q\,B_2^\dagger\,\zeta_2 + 1. \end{aligned}$$

The exterior derivative of the Dirac operator (6.9) and its adjoint have the form

$$\mathrm{d}\tilde{\mathcal{D}} = \begin{pmatrix} -q\left(B_1\,\mathrm{d}\bar{\zeta}_1 + B_2^\dagger\,\mathrm{d}\zeta_2\right) \otimes \check{w}_1 & -q\left(I\,\mathrm{d}\bar{\zeta}_1 + J^\dagger\,\mathrm{d}\zeta_2\right) \\ -q\left(B_2\,\mathrm{d}\bar{\zeta}_1 - B_1^\dagger\,\mathrm{d}\zeta_2\right) \otimes \check{w}_2 & \\ q^{-1}\left(B_1\,\mathrm{d}\bar{\zeta}_2 + B_2^\dagger\,\mathrm{d}\zeta_1\right) \otimes \check{w}_1 & q^{-1}\left(I\,\mathrm{d}\bar{\zeta}_2 + J^\dagger\,\mathrm{d}\zeta_1\right) \\ +\,q^{-1}\left(B_2\,\mathrm{d}\bar{\zeta}_2 - B_1^\dagger\,\mathrm{d}\zeta_1\right) \otimes \check{w}_2 & \end{pmatrix},$$

$$\mathrm{d}\tilde{\mathcal{D}}^\dagger = \begin{pmatrix} -\big(B_1^\dagger\,\mathrm{d}\zeta_1 - q^{-2}\,B_2\,\mathrm{d}\bar\zeta_2\big)\otimes\check w_1 & -\big(q^2\,B_1^\dagger\,\mathrm{d}\zeta_2 + B_2\,\mathrm{d}\bar\zeta_1\big)\otimes\check w_1 \\ -\big(B_2^\dagger\,\mathrm{d}\zeta_1 + q^{-2}\,B_1\,\mathrm{d}\bar\zeta_2\big)\otimes\check w_2 & -\big(q^2\,B_2^\dagger\,\mathrm{d}\zeta_2 - B_1\,\mathrm{d}\bar\zeta_1\big)\otimes\check w_2 \\ -I^\dagger\,\mathrm{d}\zeta_1 + q^{-2}\,J\,\mathrm{d}\bar\zeta_2 & -q^2\,I^\dagger\,\mathrm{d}\zeta_2 + J\,\mathrm{d}\bar\zeta_1 \end{pmatrix},$$

and using (6.14) we find

$$\begin{aligned}\mathrm{d}\tilde{\mathcal{D}}^\dagger\wedge\mathrm{d}\tilde{\mathcal{D}} = {}& q\,\mathcal{F}^{1,1}\,\big(\mathrm{d}\zeta_1\wedge\mathrm{d}\bar\zeta_1 - \mathrm{d}\zeta_2\wedge\mathrm{d}\bar\zeta_2\big)\\ &+\big(q+q^3\big)\,\mathcal{F}^{2,0}\,\mathrm{d}\zeta_1\wedge\mathrm{d}\zeta_2 + \big(q+q^{-1}\big)\,\mathcal{F}^{0,2}\,\mathrm{d}\bar\zeta_1\wedge\mathrm{d}\bar\zeta_2\end{aligned}$$

where

$$\mathcal{F}^{1,1} = \begin{pmatrix} B_1^\dagger B_1 + B_2 B_2^\dagger & B_1^\dagger B_2 - B_2 B_1^\dagger & B_1^\dagger I + B_2 J^\dagger \\ B_2^\dagger B_1 - B_1 B_2^\dagger & B_1 B_1^\dagger + B_2^\dagger B_2 & B_2^\dagger I - B_1 J^\dagger \\ I B_1 - J B_2^\dagger & I^\dagger B_2 + J B_1^\dagger & I^\dagger I - J J^\dagger \end{pmatrix},$$

$$\mathcal{F}^{2,0} = \begin{pmatrix} B_1^\dagger B_2^\dagger & -\big(B_1^\dagger\big)^2 & B_1^\dagger J^\dagger \\ \big(B_2^\dagger\big)^2 & -B_2^\dagger B_1^\dagger & B_2^\dagger J^\dagger \\ I^\dagger B_2^\dagger & -I^\dagger B_1^\dagger & I^\dagger J^\dagger \end{pmatrix},$$

$$\mathcal{F}^{0,2} = \begin{pmatrix} -B_2 B_1 & -B_2^2 & -B_2 I \\ B_1^2 & B_1 B_2 & B_1 I \\ J B_1 & J B_2 & J I \end{pmatrix}.$$

Comparing with (6.19), we see that $F_{\widetilde{\nabla}}$ belongs to $\Omega^{2,-}_{\mathbb{R}^4_\theta}$ as a two-form. □

It is easy to see that the anti-selfduality equations of Proposition 6.21 are consistent with the overlap relations (6.16) and (6.20). This defines the instanton equations on the non-commutative sphere S^4_θ.

7 Construction of instanton moduli spaces

7.1 Instanton moduli functors

In this section we will exploit the fact that our non-commutative variety $\mathbb{CP}^2_\theta$ is a member of one of the general classes of non-commutative projective planes considered in [15,36]; hence we can straightforwardly utilize their

moduli space constructions, and in the following we will freely borrow from their results. In the notation of [36], our $\mathbb{CP}^2_\theta$ is of type $\mathbf{S}_1$, associated to a curve $E \subset \mathbb{CP}^2$ which is isomorphic to a triangle (union of three lines $\mathbb{CP}^1$) such that each component is stabilized by an automorphism σ. Note that this is *very* different from the projective planes considered in [25], which are each of type $\mathbf{S}'_1$ with E isomorphic to the union of a line and a conic (and in particular are not given by toric non-commutative deformations of $\mathbb{CP}^2$). Recall from Section 2.3 that the non-commutative toric variety $\mathbb{CP}^2_\theta$ occurs in the (universal) flat family $\mathcal{A} = \mathcal{A}(\mathbb{CP}^2_\theta)$ parameterized by the commutative unital algebra $\mathcal{A}(\mathbb{C}^\times) = \mathbb{C}(q)$ over $\mathbb{C}$ dual to the smooth irreducible curve $\bigwedge^2 T \cong \mathbb{C}^\times$. This family includes the commutative polynomial algebra $\mathcal{A}(\mathbb{CP}^2) := \mathbb{C}[w_1, w_2, w_3]$ (for $q = 1$ or $\theta = 0$). The moduli spaces constructed by Nevins and Stafford in [36] all behave well in this family, and are $\mathbb{C}$-schemes in the usual sense; they are constructed as geometric invariant theory quotients of subvarieties of products of grassmannians.

In [36] it is shown that there exists a projective coarse moduli space $\mathcal{M}_{\mathbb{CP}^2_\theta}(r, c_1, \chi)$ for semistable torsion free modules in $\mathsf{coh}(\mathbb{CP}^2_\theta)$ of rank $r \geq 1$, first Chern class c_1, and Euler characteristic χ. This moduli space behaves well in families. In particular, there exists a quasi-projective T-scheme $\mathcal{M}_T(r, c_1, \chi) \to \bigwedge^2 T$ which is smooth over $\bigwedge^2 T$ and whose fibre over $q = \exp(\mathrm{i}\,\theta) \in \bigwedge^2 T$ is precisely the moduli space $\mathcal{M}_{\mathbb{CP}^2_\theta}(r, c_1, \chi)$; it follows that $\mathcal{M}_{\mathbb{CP}^2_\theta}(r, c_1, \chi)$ is smooth. We will show that the variety $\widehat{\mathcal{M}}^{\mathrm{ADHM}}_\theta(r, k)$ of non-commutative (complex) ADHM data is a fine moduli space for framed sheaves of rank r, first Chern class $c_1 = 0$, and Euler characteristic $\chi = r - k$ on $\mathsf{Open}(\mathbb{CP}^2_\theta)$. This isomorphism induces the bijections of Section 5. Recall that by a "fine" moduli space $\mathcal{M}$ here we mean that there exists a *universal* framed sheaf $\hat{\mathcal{E}}$, i.e., a family of framed sheaves parameterized by $\mathcal{M}$ such that for any other family of framed sheaves $\mathcal{E}$ parameterized by a $\mathbb{C}$-scheme S, there exists a unique morphism $\alpha : S \to \mathcal{M}$ and an isomorphism $\mathcal{E} \cong (\mathrm{id} \times \alpha^*)(\hat{\mathcal{E}})$ preserving the framing isomorphisms. In this case, $\mathcal{M}_{\mathbb{CP}^2_\theta}(r, 0, r-k)$ is a smooth quasi-projective $\mathbb{C}$-scheme for all $r \geq 1$ and for all $k \in \mathbb{N}_0$; when non-empty it has dimension

$$\dim_{\mathbb{C}} \big(\mathcal{M}_{\mathbb{CP}^2_\theta}(r, 0, r-k)\big) = 2\,r\,k - r^2 + 1$$

as in the classical case, and the tangent space at a point $[E]$ is the vector space

$$T_{[E]}\mathcal{M}_{\mathbb{CP}^2_\theta}(r, 0, r-k) = \mathrm{Ext}^1(E, E).$$

To construct our instanton moduli spaces, we look not for a set of objects as before but rather for a space which parameterizes those objects. For

this, we consider functors from the category Alg of commutative, unital noetherian $\mathbb{C}$-algebras to the category Set of sets. Let A be a unital noetherian $\mathbb{C}$-algebra. We write $\mathbb{CP}^n_{\theta,A}$ for the non-commutative variety dual to the algebra $\mathcal{A}(\mathbb{CP}^n_{\theta,A}) := A \otimes \mathcal{A}(\mathbb{CP}^n_\theta)$. We consider such *families* of algebras $\mathcal{A}(\mathbb{CP}^n_{\theta,A})$ in order to study the *global* structure of our moduli spaces, and endow them with geometric structure.

Definition 7.1. (1) A *family of (W,V)-framed sheaves parameterized by the algebra A* is an A-flat torsion free module $\mathcal{E}$ of rank r on $A \otimes \mathcal{A}$ such that $H^1\big(\mathbb{CP}^2_{\theta,A}, \mathcal{E}(-1)\big) \cong A \otimes V$ and $i^\bullet(\mathcal{E}) \cong W \otimes A \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}$.

(2) Two families of (W,V)-framed sheaves $\mathcal{E}$ and $\mathcal{E}'$ are *isomorphic* if they are isomorphic as (W,V)-framed modules on $A \otimes \mathcal{A}$.

For $A = \mathbb{C}$, i.e., for a family parameterized by a one-point space, this definition reduces to Definition 5.3.

Definition 7.2. (1) The *instanton moduli functor* is the covariant functor

$$\mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(W,V) : \mathsf{Alg} \longrightarrow \mathsf{Set}$$

that associates to every algebra A the set $\mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(W,V)(A)$ of isomorphism classes of families $\mathcal{E}$ of (W,V)-framed sheaves parameterized by A, and to every algebra morphism $f : A \to B$ associates the push-forward $\mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(W,V)(f)$ sending families parameterized by A to families parameterized by B. When bases have been fixed for the vector spaces $V \cong \mathbb{C}^k$ and $W \cong \mathbb{C}^r$, we will denote this moduli functor by $\mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(r,k)$.

(2) A *fine instanton moduli space* is a variety dual to a universal object representing the instanton moduli functor, i.e., a pair $(\hat{A}, \hat{\mathcal{E}})$, where $\hat{A}$ is an object of the category Alg and $\hat{\mathcal{E}} \in \mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(W,V)(\hat{A})$, such that for any pair $(A, \mathcal{E})$ with A an object of Alg and $\mathcal{E} \in \mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(W,V)(A)$ there exists a unique morphism $\alpha \in \mathrm{Hom}_{\mathsf{Alg}}(\hat{A}, A)$ such that $\mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(W,V)(\alpha)(\hat{\mathcal{E}}) \cong \mathcal{E}$.

By Yoneda's lemma, a fine instanton moduli space given by $(\hat{A}, \hat{\mathcal{E}})$ corresponds bijectively to a representation of the instanton moduli functor via an isomorphism of functors

$$\mathscr{H}_{\hat{A}} := \mathrm{Hom}_{\mathsf{Alg}}(\hat{A}, -) \xrightarrow{\approx} \mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(W,V),$$

with universal sheaf represented by $\hat{\mathcal{E}} = \mathscr{H}_{\hat{A}}(\mathrm{id}_{\hat{A}})$. For example, if $c_1 = 0$ and (r,k) are coprime integers, then $\mathcal{M}_{\mathbb{CP}^2_\theta}(r,0,r-k)$ is a fine moduli space (when non-empty) by [36, Prop. 7.15].

Proposition 7.3. *For any $r \geq 1$ and $k \geq 0$, the quotient $\widehat{\mathcal{M}}^{\mathrm{ADHM}}_\theta(r,k)$ is a fine instanton moduli space.*

Proof. We show that the coordinate algebra $\mathcal{A}\big(\widehat{\mathcal{M}}^{\mathrm{ADHM}}_\theta(r,k)\big)$ is a universal object in Alg which represents the instanton moduli functor $\mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(r,k)$. As detailed computations have been carried out in an analogous context in [36] (see [36, Prop. 8.13]), we only outline the main steps and skip many details. Given $(B,I,J) \in \mathcal{M}^{\mathrm{ADHM}}_\theta(r,k)$ and a commutative $\mathbb{C}$-algebra A, we use Theorem 5.32 to define a monad complex $\underline{\mathcal{C}}_\bullet(B,I,J)_A$ in $\mathsf{coh}(\mathbb{CP}^2_{\theta,A})$ as in (5.29)–(5.31). By Theorem 5.32 and [36, Thm. 5.8], the cohomology sheaf $H^0\big(\underline{\mathcal{C}}_\bullet(B,I,J)_A\big)$ is an A-flat family of torsion free sheaves in $\mathsf{coh}(\mathbb{CP}^2_{\theta,A})$ with $i^\bullet\big(H^0(\underline{\mathcal{C}}_\bullet(B,I,J)_A)\big) = H^0\big(i^\bullet(\underline{\mathcal{C}}_\bullet(B,I,J)_A)\big)$, and so $H^0\big(\underline{\mathcal{C}}_\bullet(B,I,J)_A\big)$ is (W,V)-framed. This defines a natural transformation of functors $\mathscr{H}_{\mathcal{A}(\widehat{\mathcal{M}}^{\mathrm{ADHM}}_\theta(r,k))} \to \mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(r,k)$, which is injective by (5.28) and [36, Lem. 5.11]. Applying Theorem 5.25 and Theorem 5.32 to A-flat families of (W,V)-framed torsion free objects of $\mathsf{coh}(\mathbb{CP}^2_{\theta,A})$ shows that this transformation is surjective. Finally, to demonstrate universality we use the proof of Lemma 5.20 to observe that the quotient map $\mathcal{M}^{\mathrm{ADHM}}_\theta(r,k) \to \widehat{\mathcal{M}}^{\mathrm{ADHM}}_\theta(r,k)$ is a principal $\mathrm{GL}(k)$-bundle in the étale topology. The group functor $\mathrm{GL}(k)$ acts on $\mathscr{H}_{\mathcal{A}(\mathcal{M}^{\mathrm{ADHM}}_\theta(r,k))}$, hence using Lemma 5.10 and [36, Thm. 4.3] one argues as in the proof of [36, Prop. 8.13] that the map of functors $\mathscr{H}_{\mathcal{A}(\mathcal{M}^{\mathrm{ADHM}}_\theta(r,k))} \to \mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(r,k)$ is also a principal $\mathrm{GL}(k)$-bundle, and hence that $\mathcal{A}\big(\widehat{\mathcal{M}}^{\mathrm{ADHM}}_\theta(r,k)\big)$ represents $\mathscr{M}^{\mathrm{inst}}_{\mathbb{CP}^2_\theta}(r,k)$. □

We may summarize the main consequences of the results described thus far in this section as follows.

Theorem 7.4. *Let $(r \geq 1, k \geq 0)$ be coprime integers. Then the instanton moduli space $\mathcal{M}_\theta(r,k)$, when non-empty, is a smooth quasi-projective variety of dimension*

$$\dim_{\mathbb{C}}\big(\mathcal{M}_\theta(r,k)\big) = 2\,r\,k.$$

The tangent space to $\mathcal{M}_\theta(r,k)$ at a point $[E]$ is canonically the vector space

$$T_{[E]}\mathcal{M}_\theta(r,k) = \mathrm{Ext}^1\big(E,E(-1)\big).$$

As an element of the Grothendieck group of the abelian category $\mathsf{coh}\big(\mathbb{CP}^2_{\theta,\mathcal{A}(\mathcal{M}_\theta(r,k))}\big)$*, the universal module* $\hat{\mathcal{E}}$ *on* $\mathcal{A}\big(\mathcal{M}_\theta(r,k)\big)\otimes\mathcal{A}(\mathbb{CP}^2_\theta)$ *is isomorphic to the virtual vector bundle whose fibre over* $[E]$ *is the virtual bundle in the K-theory of locally free coherent sheaves on* $\mathsf{Open}(\mathbb{CP}^2_\theta)$ *given by*

$$\hat{\mathcal{E}}_{[E]} = W\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}\ \oplus\ V\otimes\big((\mathcal{A}^!_\ell)_1\otimes\mathcal{O}_{\mathbb{CP}^2_\theta}\ \ominus\ \mathcal{O}_{\mathbb{CP}^2_\theta}(-1)\ \ominus\ \mathcal{O}_{\mathbb{CP}^2_\theta}(1)\big)$$

with $V = H^1\big(\mathbb{CP}^2_\theta, E(-1)\big)$ *and* $W = H^0\big(\mathbb{CP}^1_\theta, i^\bullet(E)\big)$.

Proof. By Lemma 5.10 the moduli space $\mathcal{M}_\theta(r,k)$ is an open subscheme of the moduli space $\mathcal{M}_{\mathbb{CP}^2_\theta}(r,0,r-k)$ which figures in [36, Cors. 8.3–8.4 and Lem. 8.5], and it represents the corresponding moduli functor. Via Proposition 5.2, its deformation theory may thus be embedded into the more general theory of framed modules developed by Huybrechts and Lehn [22]. The Zariski tangent space to $\mathcal{M}_\theta(r,k)$ at a point corresponding to a framed sheaf E on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ is isomorphic to the cohomology group $\mathrm{Ext}^1(E,E(-1))$, and there is an appropriate obstruction theory with values in $\mathrm{Ext}^2(E,E(-1))$. We will show that

$$\mathrm{Hom}\big(E,E(-1)\big) = 0 = \mathrm{Ext}^2\big(E,E(-1)\big). \tag{7.5}$$

For this, we use the short exact sequence (5.6) in the category $\mathsf{coh}(\mathbb{CP}^2_\theta)$ to induce long exact cohomology sequences for $k\in\mathbb{Z}$ which start at

$$0\ \longrightarrow\ \mathrm{Hom}\big(E,E(k-1)\big)\ \longrightarrow\ \mathrm{Hom}\big(E,E(k)\big)\ \longrightarrow\ \mathrm{Hom}\big(E,i_\bullet\,i^\bullet(E(k))\big).$$

Since $i^\bullet(E) = W\otimes\mathcal{O}_{\mathbb{CP}^1_\theta}$, the μ-semistability of E implies $\mathrm{Hom}\big(E,i_\bullet\,i^\bullet(E(k))\big) = 0$ and hence

$$\mathrm{Hom}\big(E,E(k-1)\big)\ \cong\ \mathrm{Hom}\big(E,E(k)\big)$$

for all $k\in\mathbb{Z}$. In particular, $\mathrm{Hom}(E,E(-1))\cong\mathrm{Hom}(E,E(-3))$ and the latter vector space is trivial by [36, Lem. 7.14]. Hence $\mathrm{Hom}(E,E(-1)) = 0$. By [36, Prop. 2.4 (2)], $\mathrm{Ext}^2(E,E(-1))$ is dual to the vector space $\mathrm{Hom}(E,E_\alpha(-2))$ for some automorphism $\alpha\in\mathrm{Aut}(\mathcal{A})$, where if $E=\pi(M)$ for a graded $\mathcal{A}$-module $M\in\mathsf{gr}(\mathcal{A})$, then $E_\alpha=\pi(M_\alpha)$ with M_α the right $\mathcal{A}$-module M twisted by α, i.e., the same underlying vector space $M_\alpha = M$ with right $\mathcal{A}$-module structure $v\triangleleft_\alpha f := v\triangleleft\alpha(f)$ for $f\in\mathcal{A}$ and $v\in M$. Since $\alpha(f)=0$ if and only if $f=0$, it is easy to see that the restriction functor $i^\bullet$ commutes with α-twisting, i.e., $i^\bullet(E_\alpha) = i^\bullet(E)_\alpha$, and hence so does framing. Thus one can apply the short exact sequence (5.6) to the

twisted sheaf E_α, and then repeat the previous argument verbatim to deduce $\mathrm{Hom}(E, E_\alpha(-2)) \cong \mathrm{Hom}(E, E_\alpha(-3))$, where the latter vector space is again trivial by [36, Lem. 7.14]. Hence $\mathrm{Ext}^2(E, E(-1)) = 0$.

Using (7.5) and [36, Cor. 6.2], the dimension of the tangent space is given by

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Ext}^1\big(E, E(-1)\big) = {}& \mathrm{rank}(E)\,\Big(\mathrm{rank}\big(E(-1)\big) - \chi\big(E(-1)\big)\Big) \\ & + c_1(E)\,\Big(3\,\mathrm{rank}\big(E(-1)\big) + c_1\big(E(-1)\big)\Big) \\ & - \chi(E)\,\mathrm{rank}\big(E(-1)\big).\end{aligned}$$

Using Corollary 5.8, together with $c_1(E(-1)) = -r$ by (3.8) and $\chi(E(-1)) = h_E(-1) = -k$ by (5.9), this number is equal to $2\,r\,k$. Finally, the expression for the virtual class of the universal sheaf $\hat{\mathcal{E}}$ follows from the proof of Proposition 7.3 and [13, Prop. 6.8 (1)]. □

To incorporate the construction of instanton gauge bundles and connections on the non-commutative sphere S^4_θ from Section 6, we may also wish to extend the definition of the instanton moduli functor according to the twistor construction of Section 5.4. Rather than going through all these details, which are not needed in this paper, we simply axiomatize the extension of the instanton moduli space that one finds. This extension appropriately reduces the dimension of the moduli space of Theorem 7.4.

Definition 7.6. A *family of instantons over S^4_θ parameterized by a unital $*$-algebra A* is a quintuple $(\mathcal{E}, \tilde{\mathcal{E}}, \nabla, \widetilde{\nabla}, G_{\mathcal{E}})$ consisting of:

(1) Finitely generated projective right and left modules $\mathcal{E}$ and $\tilde{\mathcal{E}}$ over the respective algebras $A \otimes \mathcal{A}(\mathbb{R}^4_\theta)$ and $A \otimes \mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)$ with dualizing anti-involutions which are compatible with the $*$-algebra structures;

(2) Connections

$$\begin{aligned}\nabla &: \mathcal{E} \longrightarrow \mathcal{E} \otimes_{A\otimes\mathcal{A}(\mathbb{R}^4_\theta)} \big(A \otimes \Omega^1_{\mathbb{R}^4_\theta}\big) \cong \mathcal{E} \otimes_{\mathcal{A}(\mathbb{R}^4_\theta)} \Omega^1_{\mathbb{R}^4_\theta}, \\ \widetilde{\nabla} &: \tilde{\mathcal{E}} \longrightarrow \big(A \otimes \Omega^1_{\widetilde{\mathbb{R}}^4_\theta}\big) \otimes_{A\otimes\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)} \tilde{\mathcal{E}} \cong \Omega^1_{\widetilde{\mathbb{R}}^4_\theta} \otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)} \tilde{\mathcal{E}}\end{aligned}$$

which are compatible with the dualizing anti-involutions and whose curvatures $F_\nabla = \nabla^2$ and $F_{\widetilde{\nabla}} = \widetilde{\nabla}^2$ obey the anti-selfduality equations

$$(\mathrm{id} \otimes *_\theta) F_\nabla = -F_\nabla, \quad (\mathrm{id} \otimes \tilde{*}_\theta) F_{\widetilde{\nabla}} = -F_{\widetilde{\nabla}}$$

in

$$\mathrm{Hom}_{A\otimes\mathcal{A}(\mathbb{R}^4_\theta)}\big(\mathcal{E},\mathcal{E}\otimes_{A\otimes\mathcal{A}(\mathbb{R}^4_\theta)}(A\otimes\Omega^2_{\mathbb{R}^4_\theta})\big)\cong\mathrm{Hom}_{A\otimes\mathcal{A}(\mathbb{R}^4_\theta)}\big(\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}(\mathbb{R}^4_\theta)}\Omega^2_{\mathbb{R}^4_\theta}\big)$$

and in

$$\mathrm{Hom}_{A\otimes\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\big(\tilde{\mathcal{E}},(A\otimes\Omega^2_{\widetilde{\mathbb{R}}^4_\theta})\otimes_{A\otimes\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\tilde{\mathcal{E}}\big)\cong\mathrm{Hom}_{A\otimes\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\big(\tilde{\mathcal{E}},\Omega^2_{\widetilde{\mathbb{R}}^4_\theta}\otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\tilde{\mathcal{E}}\big),$$

respectively; and

(3) An isomorphism

$$G_{\mathcal{E}}:\ \begin{array}{c}\mathcal{E}\otimes_{A\otimes\mathcal{A}(\mathbb{R}^4_\theta)}\big(A\otimes\mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,]\big)\\ \|\\ \mathcal{E}[\tilde\rho\,]\end{array}\ \longrightarrow\ \begin{array}{c}\big(A\otimes[\rho]\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)\big)\otimes_{A\otimes\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\tilde{\mathcal{E}}\\ \|\\ [\rho]\tilde{\mathcal{E}}\end{array}$$

which is compatible with the dualizing anti-involutions, and obeys $G_{\mathcal{E}}(\sigma\triangleleft a)=(\mathrm{id}\otimes G(a))\triangleright G_{\mathcal{E}}(\sigma)$ for $\sigma\in\mathcal{E}[\tilde\rho\,]$, $a\in\mathcal{A}(\mathbb{R}^4_\theta)[\tilde\rho\,]$ and ${}_\rho\widetilde{\nabla}\circ G_{\mathcal{E}}=G_{\mathcal{E}}\circ\nabla_{\tilde\rho}$.

Definition 7.7. Two families of instantons $(\mathcal{E},\tilde{\mathcal{E}},\nabla,\widetilde{\nabla},G_{\mathcal{E}})$ and $(\mathcal{E}',\tilde{\mathcal{E}}',\nabla',\widetilde{\nabla}',G'_{\mathcal{E}})$ are *equivalent* if there exist isomorphisms of projective modules $\varrho:\mathcal{E}\xrightarrow{\approx}\mathcal{E}'$ and $\tilde\varrho:\tilde{\mathcal{E}}\xrightarrow{\approx}\tilde{\mathcal{E}}'$ together with commutative diagrams

$$\begin{array}{ccc}\mathcal{E}&\xrightarrow{\nabla}&\mathcal{E}\otimes_{\mathcal{A}(\mathbb{R}^4_\theta)}\Omega^1_{\mathbb{R}^4_\theta}\\ \downarrow{\scriptstyle\varrho}&&\downarrow{\scriptstyle\varrho\otimes\mathrm{id}}\\ \mathcal{E}'&\xrightarrow[\nabla']{}&\mathcal{E}'\otimes_{\mathcal{A}(\mathbb{R}^4_\theta)}\Omega^1_{\mathbb{R}^4_\theta}\end{array}\quad,\qquad\begin{array}{ccc}\tilde{\mathcal{E}}&\xrightarrow{\widetilde{\nabla}}&\Omega^1_{\widetilde{\mathbb{R}}^4_\theta}\otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\tilde{\mathcal{E}}\\ \downarrow{\scriptstyle\tilde\varrho}&&\downarrow{\scriptstyle\mathrm{id}\otimes\tilde\varrho}\\ \tilde{\mathcal{E}}'&\xrightarrow[\widetilde{\nabla}']{}&\Omega^1_{\widetilde{\mathbb{R}}^4_\theta}\otimes_{\mathcal{A}(\widetilde{\mathbb{R}}^4_\theta)}\tilde{\mathcal{E}}'\end{array}\quad,$$

and

$$\begin{array}{ccc}\mathcal{E}[\tilde\rho\,]&\xrightarrow{G_{\mathcal{E}}}&[\rho]\tilde{\mathcal{E}}\\ \downarrow{\scriptstyle\varrho\otimes\mathrm{id}}&&\downarrow{\scriptstyle\mathrm{id}\otimes\tilde\varrho}\\ \mathcal{E}'[\tilde\rho\,]&\xrightarrow[G'_{\mathcal{E}}]{}&[\rho]\tilde{\mathcal{E}}'\end{array}\quad.$$

The corresponding moduli functor $\mathsf{Alg}\to\mathsf{Set}$ assigns to each unital $*$-algebra A the set of equivalence classes of families of instantons parameterized by A. We may also restrict the target of this functor to families

of instantons with gauge bundles of rank $r \geq 1$ and Euler characteristic $\chi = r - k$. The construction of Section 6 yields a universal object representing this functor. In Section 8 we shall also restrict the sources of these functors to the subcategory ${}^{\mathcal{H}_\theta}\mathsf{Alg}$ consisting of left $\mathcal{H}_\theta$-comodule algebras. Below we consider in detail some explicit instances of these moduli space constructions.

7.2 Rank 0 instantons

The case $r = 0$ ($W = 0$) is somewhat degenerate; it is not covered by the general analysis of Section 7.1 and must be dealt with separately. Sheaves E of rank 0 are given by $E = \pi(M)$ for some *torsion* module $M \in \mathsf{gr}(\mathcal{A})$, i.e., every element of M is annihilated by a non-zero element of the algebra $\mathcal{A} = \mathcal{A}(\mathbb{CP}^2_\theta)$. The deformation theory of [36, Sections 7–8] does not apply to the moduli space of such sheaves, which is only set-theoretic, i.e., it does not corepresent the instanton moduli functor of Section 7.1. Nevertheless, we will now demonstrate that the moduli space of instantons of rank 0 is still a coarse moduli space for some functor, which identifies it as the moduli space of finite-dimensional representations of an algebra dual to an affine non-commutative toric variety.

In terms of non-commutative ADHM data, in this case one has $I = J = 0$ and the braided ADHM equation (5.18) reduces to $B \wedge_\theta B = 0$, or equivalently

$$B_1\, B_2 = q^{-2}\; B_2\, B_1 . \tag{7.8}$$

Thus the datum $B \in \mathcal{M}_\theta^{\mathrm{ADHM}}(0,k)$ defines a k-dimensional representation of the affine coordinate algebra $\mathcal{A}(\mathbb{C}^2_\theta)$ of the complex algebraic Moyal plane $\mathbb{C}^2_\theta$ [13, Section 3.2], i.e., the polynomial algebra $\mathbb{C}[z_1, z_2]$ in two generators modulo the relation $z_1\, z_2 = q^2\, z_2\, z_1$. By the stability condition of Definition 5.17, this representation is irreducible. Thus by [21, Section 6.2], $\mathcal{M}_\theta^{\mathrm{ADHM}}(0,k)$ is the affine algebraic $\mathbb{C}$-scheme representing the moduli functor $\mathsf{Alg} \to \mathsf{Set}$ which sends a $\mathbb{C}$-algebra A to the set of simple $A \otimes \mathcal{A}(\mathbb{C}^2_\theta)$-module structures on $A^{\oplus k}$; for $A = \mathbb{C}$ this set consists of irreducible representations of $\mathcal{A}(\mathbb{C}^2_\theta)$ on $\mathbb{C}^k$. The natural $\mathrm{GL}(k)$-action corresponds to changes of basis. By [21, Prop. 6.3], the quotient $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(0,k)$ corepresents the moduli functor which sends a $\mathbb{C}$-scheme S to the set of isomorphism classes of S-families of simple k-dimensional $\mathcal{A}(\mathbb{C}^2_\theta)$-modules, i.e., locally free coherent $\mathcal{O}_S$-modules $\mathcal{F}_S$ of rank k, together with $\mathbb{C}$-algebra homomorphisms $\varrho_S : \mathcal{A}(\mathbb{C}^2_\theta) \to \mathrm{End}_{\mathsf{coh}(S)}(\mathcal{F}_S)$, such that $\mathcal{F}_S$ contains no proper subsheaves invariant under $\varrho_S(z_i)$ for $i = 1, 2$.

The classical case $\theta = 0$ is covered by [21, Prop. 6.4]. In this instance the scheme $\widehat{\mathcal{M}}^{\mathrm{ADHM}}_{\theta=0}(0,k)$ is isomorphic to the affine quotient $\big(\widehat{\mathcal{M}}^{\mathrm{ADHM}}_{\theta=0}(0,1)\big)^k/S_k$ since all simple modules over a commutative algebra are one-dimensional, and $\widehat{\mathcal{M}}^{\mathrm{ADHM}}_{\theta=0}(0,1) = \mathbb{C}^2$. It follows that the moduli space $\widehat{\mathcal{M}}^{\mathrm{ADHM}}_{\theta=0}(0,k)$ is canonically isomorphic to the k-th symmetric product $\mathsf{Sym}^k(\mathbb{CP}^2) := (\mathbb{C}^2)^k/S_k$ of the commutative affine plane; it parameterizes coherent sheaves on $\mathsf{Open}(\mathbb{CP}^2)$ which have zero-dimensional support of length k contained in $\mathbb{C}^2 \cong \mathbb{CP}^2 \setminus \mathbb{CP}^1$. Hence we write $\mathsf{Sym}^k_\theta(\mathbb{CP}^2_\theta) := \widehat{\mathcal{M}}^{\mathrm{ADHM}}_{\theta}(0,k)$ for all $\theta \in \mathbb{C}$. This construction generalizes [33, Prop. 2.10]. For generic deformation parameters $\theta \in \mathbb{C}$ this moduli space is relatively small; when $q \in \mathbb{C}$ is not a root of unity of order $2k$, the space $\mathsf{Sym}^k_\theta(\mathbb{CP}^2_\theta)$ only parameterizes representations of $\mathcal{A}(\mathbb{C}^2_\theta)$ wherein one of the matrices B_1 or B_2 is singular.

7.3 Rank 1 instantons

A torsion free sheaf $E \in \mathsf{coh}(\mathbb{CP}^2_\theta)$ has rank 1 if and only if $M = \Gamma(E) \in \mathsf{gr}(\mathcal{A})$ is isomorphic to a shift $\mathcal{I}(m)$ of a right ideal $\mathcal{I} \subset \mathcal{A}$ [13, Section 4.3]. By [36, Cor. 6.6 (1)], there exists a smooth, projective fine moduli space $\mathcal{M}_{\mathbb{CP}^2_\theta}(1,0,k)$ of dimension $2k$ for torsion free $\mathcal{A}$-modules in $\mathsf{coh}(\mathbb{CP}^2_\theta)$ with rank $r = 1$, first Chern class $c_1 = 0$, and Euler characteristic $\chi = 1 - k$. This moduli space also behaves well in the family $\mathcal{A} = \mathcal{A}(\mathbb{CP}^2_\theta)$, in the sense described in Section 7.1. In particular, $\mathcal{M}_{\mathbb{CP}^2_\theta}(1,0,k)$ is irreducible, hence connected, and is a *commutative* deformation of the Hilbert scheme of points $\mathsf{Hilb}^k(\mathbb{CP}^2)$ parameterizing zero-dimensional subschemes of degree k in $\mathbb{CP}^2$. We thus write $\mathsf{Hilb}^k_\theta(\mathbb{CP}^2_\theta) := \mathcal{M}_{\mathbb{CP}^2_\theta}(1,0,k)$ for all $\theta \in \mathbb{C}$; it is non-empty for all $k \geq 0$. As such sheaves are automatically (semi)stable, we may identify $\mathsf{Hilb}^k_\theta(\mathbb{CP}^2_\theta)$ with the instanton moduli space $\mathcal{M}_\theta(1,k)$ in this case.

De Naeghel and Van den Bergh [15] describe the deformation $\mathsf{Hilb}^k_\theta(\mathbb{CP}^2_\theta)$ as the scheme parameterizing torsion free graded $\mathcal{A}$-modules $\mathcal{I} = \bigoplus_{n\geq 0} \mathcal{I}_n$ of projective dimension one such that

$$\dim_{\mathbb{C}}(\mathcal{A}_n) - \dim_{\mathbb{C}}(\mathcal{I}_n) = k \quad \text{for} \quad n \gg 0. \tag{7.9}$$

In particular, it follows from [15, Lem. 3.3.1] that $\mathcal{I}$ has rank $r = 1$ as an $\mathcal{A}$-module and invariants $c_1(\mathcal{I}) = 0$, $\chi(\mathcal{I}) = 1 - k$. Thus $\mathcal{I}$ corresponds to an *ideal sheaf* in the sense of [13, Section 4.3], and hence corresponds to a closed subscheme of $\mathbb{CP}^2_\theta$ by [13, Thm. 4.10]. A stratification of $\mathsf{Hilb}^k_\theta(\mathbb{CP}^2_\theta)$ by Hilbert series is described in [15, Thm. 6.1]. There is a bijective correspondence between the set $\mathcal{D}_k$ of integer partitions of k with distinct parts and Hilbert series of objects in $\mathsf{Hilb}^k_\theta(\mathbb{CP}^2_\theta)$. Let $\mathcal{D} = \bigcup_{k\geq 0} \mathcal{D}_k$ be the set of

all integer partitions $\lambda = (\lambda_i)_{i\geq 1}$ with distinct parts. It is a classical result [1] that the generating function for elements of $\mathcal{D}$ is given by

$$Z_{\mathcal{D}}(Q) = \sum_{\lambda\in\mathcal{D}} Q^{|\lambda|} = \prod_{n=1}^{\infty} \big(1+Q^n\big), \tag{7.10}$$

where Q is a formal variable and $|\lambda| = \sum_i \lambda_i$ is the weight of the partition λ. This formula will be used in conjunction with instanton partition functions in Section 8.

In this case there is a bijection between the set of ideals of codimension k in the coordinate algebra $\mathcal{A}(\mathbb{C}^2_\theta)$ of the complex algebraic Moyal plane and the set of triples $(B_1, B_2, I) \in \mathrm{End}_{\mathbb{C}}(V)^{\oplus 2} \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V)$ satisfying (7.8) such that no proper B_i-invariant subspaces of V contain the image of I for $i = 1, 2$; the proof is essentially a step by step repetition of that in the classical case $\theta = 0$ [33, Thm. 1.9]. In the classical situation one shows that $J = 0$ in the rank 1 case [33, Prop. 2.9] and thus directly establishes an isomorphism between the Hilbert scheme of k points in $\mathbb{C}^2$ and the ADHM moduli space $\widehat{\mathcal{M}}^{\mathrm{ADHM}}_{\theta=0}(1,k)$. However, we will see directly below that the linear map $J \neq 0$ in general when $\theta \neq 0$, in agreement with what we found in Section 7.2. This reflects the fact that the non-commutative algebra $\mathcal{A}$ contains very few ideals, or equivalently that $\mathbb{CP}^2_\theta$ for generic θ has very few zero-dimensional non-commutative subschemes [13, Section 4.3]. We can regard the scheme $\mathsf{Sym}^k_\theta(\mathbb{CP}^2_\theta)$ of Section 7.2 and $\mathsf{Hilb}^k_\theta(\mathbb{CP}^2_\theta)$ simultaneously as a commutative deformation of the resolution of the singularity $\mathsf{Sym}^k(\mathbb{CP}^2)$ provided by the Hilbert–Chow morphism $\mathsf{Hilb}^k(\mathbb{CP}^2) \to \mathsf{Sym}^k(\mathbb{CP}^2)$, which sends an ideal to its support. Noncommutative deformations of this kind are constructed in [19, 26] using the covering by cotangent bundles T^*U provided by the symplectic resolution.

7.4 Charge 1 instantons

Consider now the case $k = \dim_{\mathbb{C}}(V) = 1$. The projective plane is rigid against commutative deformations. Hence there are no commutative deformations of the Hilbert scheme $\mathsf{Hilb}^1(\mathbb{CP}^2) = \mathbb{CP}^2$, and so

$$\mathsf{Hilb}^1_\theta\big(\mathbb{CP}^2_\theta\big) \;\cong\; \mathbb{CP}^2$$

for all $\theta \in \mathbb{C}$. To see this directly, we note that in this case the morphisms of the ADHM data (B_1, B_2, I, J) act via multiplication by scalars $(b_1, b_2, i, j) \in \mathbb{C}^4$. The stability condition of Definition 5.17 implies that

$i \neq 0$, while invariance under the action (5.19) of $\mathrm{GL}(1) = \mathbb{C}^\times$ means that we can rescale so that $i = 1$. The braided ADHM equation (5.18) can then be used to solve for $j \in \mathbb{C}$ as

$$j = \left(q^{-2} - 1\right)\, b_1\, b_2,$$

which for $\theta \neq 0$ is non-zero in general. Thus the moduli space $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(1,1)$ is coordinatized by the quadruples $(b_1, b_2, 1, (q^{-2}-1)\, b_1\, b_2) \in \mathbb{C}^4$, representing an affine patch $(b_1, b_2) \in \mathbb{C}^2$ of the projective plane $\mathbb{CP}^2$. Similarly, one has (set-theoretically)

$$\mathsf{Sym}_\theta^1\big(\mathbb{CP}^2_\theta\big) \;\cong\; \mathbb{C}^2$$

for all $\theta \in \mathbb{C}$ as the one-dimensional representations over $\mathbb{C}$ of the algebra relation (7.8) necessarily have either $B_1 = 0$ or $B_2 = 0$ when $\theta \neq 0$.

In the higher rank cases $r \geq 2$, we can regard the morphisms I and J as vectors

$$I = (i_1, \dots, i_r) \in W^*, \quad J = \begin{pmatrix} j_1 \\ \vdots \\ j_r \end{pmatrix} \in W$$

in a chosen orthonormal basis of $W \cong \mathbb{C}^r$. The braided ADHM equation (5.18) now defines a quadric in $\mathbb{C}^{2r+2}$ given by

$$\left(1 - q^{-2}\right)\, b_1\, b_2 + \sum_{l=1}^{r} i_l\, j_l = 0. \tag{7.11}$$

Stability is now equivalent to $i_l \neq 0$ for all $l = 1, \dots, r$, showing that the moduli space $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(r,1)$ is quasi-projective. An element $t \in \mathrm{GL}(1) = \mathbb{C}^\times$ acts trivially on b_1, b_2, and as multiplication by t on i_l and by t^{-1} on j_l for each $l = 1, \dots, r$. We can use this scaling symmetry to set $i_r = 1$, and then use (7.11) to eliminate $j_r \in \mathbb{C}$. This coordinatizes $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(r,1)$ as a patch $\mathbb{C}^2 \times \mathbb{C}^{r-1} \times (\mathbb{C}^\times)^{r-1}$. Again this construction is identical to that of the classical case $\theta = 0$, giving the charge 1 instanton moduli space

$$\mathcal{M}_\theta(r,1) \;\cong\; \mathbb{CP}^2 \times T^*\mathbb{CP}^{r-1}$$

for all $\theta \in \mathbb{C}$ and $r \geq 1$. For $k \geq 2$, the moduli spaces of non-commutative instantons are generically different from their classical counterparts at $\theta = 0$.

7.5 Instanton deformation complex

We will now give an alternative proof of Theorem 7.4 which provides a much more powerful description of the geometry of the instanton moduli spaces. For this, we will view the non-commutative ADHM equation (5.18) on the affine space of triples (5.11) as the zero locus of the map $\mu_c : \mathfrak{X}(W,V) \to \mathfrak{gl}(V)^* \otimes (\mathcal{A}_\ell^!)_2$ defined by

$$\mu_c(B,I,J) := B \wedge_\theta B + I \circ J. \tag{7.12}$$

The restriction of this map to stable elements of $\mathfrak{X}(W,V)$ (in the sense of Definition 5.17) is denoted $\tilde{\mu}_c$. Then the moduli space (5.21) can be represented as the reduction

$$\widehat{\mathfrak{M}}_\theta^{\mathrm{ADHM}}(W,V) = \tilde{\mu}_c^{-1}(0) \,/\, \mathrm{GL}(V).$$

Fixing bases of the complex vector spaces V and W naturally induces a basis for each fibre of the tangent bundle $T\mathfrak{X}(W,V)$, with dual basis denoted $(\mathrm{d}B, \mathrm{d}I, \mathrm{d}J)$ at a point $x = (B,I,J) \in \mathfrak{X}(W,V)$. The differential

$$\mathrm{d}\mu_c = \mathrm{d}B \wedge_\theta B + B \wedge_\theta \mathrm{d}B + \mathrm{d}I \circ J + I \circ \mathrm{d}J$$

is the linearization of the braided ADHM quiver relations (5.23) with

$$\mathrm{d}\mu_c(b,i,j) = b \wedge_\theta B + B \wedge_\theta b + i \circ J + I \circ j. \tag{7.13}$$

Let $\varphi : \mathrm{GL}(V) \to \mathfrak{X}(W,V)$ be the orbit map $g \mapsto g \triangleright (B,I,J)$ defined by (5.19). Its restriction to stable elements of $\mathfrak{X}(W,V)$ is denoted $\tilde{\varphi}$. The differential

$$\mathrm{d}\varphi(\xi) = \big([B,\xi], (\xi \otimes \mathrm{id}_{\mathcal{A}_\ell^!})\, I, -J\,\xi\big)$$

is the linearization of the action of the gauge group $\mathrm{GL}(V)$ on $\mathfrak{X}(W,V)$. It is easy to compute that $\mathrm{d}\mu_c \circ \mathrm{d}\varphi = 0$ for all $\theta \in \mathbb{C}$.

Theorem 7.14. *The tangent space $T_{[E]}\mathfrak{M}_\theta(r,k)$ to the instanton moduli space at a closed point $[E] = [(B,I,J)]$ is isomorphic to the cohomology*

group $H^1(\underline{\mathscr{I}}_\bullet(E))$ *of the complex*

$$\begin{aligned}\underline{\mathscr{I}}_\bullet(E) : 0 \longrightarrow \operatorname{End}_{\mathbb{C}}(V) \xrightarrow{\mathrm{d}\tilde{\varphi}} & \begin{array}{c}\big(\mathcal{A}_\ell^!\big)_1 \otimes \operatorname{End}_{\mathbb{C}}(V)\\ \oplus\\ \big(\mathcal{A}_\ell^!\big)_2 \otimes \operatorname{Hom}_{\mathbb{C}}(W,V)\\ \oplus\\ \operatorname{Hom}_{\mathbb{C}}(V,W)\end{array}\\ \xrightarrow{\mathrm{d}\tilde{\mu}_c} & \big(\mathcal{A}_\ell^!\big)_2 \otimes \operatorname{End}_{\mathbb{C}}(V) \longrightarrow 0.\end{aligned}$$

Proof. We know that $[E] \in \mathcal{M}_\theta(r,k)$ is quasi-isomorphic in the category $\mathsf{coh}(\mathbb{CP}^2_\theta)$ to the monad complex $\underline{\mathcal{C}}_\bullet(B,I,J)$ defined in (5.31). The tangent-obstruction complex of the moduli space $\mathcal{M}_\theta(r,k)$ can then be determined by a standard calculation in deformation theory, using the cohomology calculation of [13, Prop. 6.8]. In particular, the p-th hypercohomology group $\mathbb{H}^p$ of the complex $\underline{\mathcal{D}}_\bullet(E) := \mathcal{H}om(\underline{\mathcal{C}}_\bullet(E), \underline{\mathcal{C}}_\bullet(E(-1)))^\bullet$ computes $\operatorname{Ext}^p(E,E(-1))$, and the complex $\underline{\mathscr{I}}_\bullet(E)$ is the standard hypercohomology spectral sequence for $\underline{\mathcal{D}}_\bullet(E)$. The infinitesimal deformation space is $H^1(\underline{\mathscr{I}}_\bullet(E)) = \mathbb{H}^1(\underline{\mathcal{D}}_\bullet(E))$ and the obstruction space is $H^2(\underline{\mathscr{I}}_\bullet(E)) = \mathbb{H}^2(\underline{\mathcal{D}}_\bullet(E))$. We will show that

$$H^0\big(\underline{\mathscr{I}}_\bullet(E)\big) = 0 = H^2\big(\underline{\mathscr{I}}_\bullet(E)\big).$$

For this, we consider the dual of the differential $\mathrm{d}\tilde{\mu}_c : \underline{\mathscr{I}}_1(E) \to \underline{\mathscr{I}}_2(E)$ given by

$$\mathrm{d}\tilde{\mu}_c^* : \operatorname{End}_{\mathbb{C}}(V) \longrightarrow \begin{array}{c}\operatorname{End}_{\mathbb{C}}(V) \otimes \big(\mathcal{A}_\ell^!\big)_1\\ \oplus\\ \operatorname{Hom}_{\mathbb{C}}(W,V)\\ \oplus\\ \operatorname{Hom}_{\mathbb{C}}(V,W)\end{array}$$

with

$$\mathrm{d}\tilde{\mu}_c^*(\psi) = \big([\psi,B_1]_\theta \otimes \check{w}_1 + [B_2,\psi]_\theta \otimes \check{w}_2, \psi\, I, J\, \psi\big),$$

where we use the identification $(\mathcal{A}_\ell^!)_2 \cong \mathbb{C}$. Suppose that $\mathrm{d}\tilde{\mu}_c^*(\psi) = 0$, and consider the subspace $V' = \ker(\psi) \subset V$. Then it is easy to see that V' is B-invariant and contains the image of I, whence by the stability condition either $V' = 0$ or $V' = V$. But if $\psi \in \operatorname{End}_{\mathbb{C}}(V)$ is injective then $I = 0$, contradicting stability, and so $V' = V$. It follows that the differential $\mathrm{d}\tilde{\mu}_c$ is surjective, and thus $H^2(\underline{\mathscr{I}}_\bullet(E)) = 0$. The proof that the differential $\mathrm{d}\tilde{\varphi} : \underline{\mathscr{I}}_0(E) \to \underline{\mathscr{I}}_1(E)$ is injective, and hence that $H^0(\underline{\mathscr{I}}_\bullet(E)) = 0$, is carried out in exactly the same way. □

7.6 Braided symplectic reduction

By analogy with the classical case, it is natural to hope that the noncommutative deformation $\mathbb{CP}^2 \to \mathbb{CP}^2_\theta$ induces a (commutative) deformation $\mathsf{Hilb}^k(\mathbb{CP}^2) \to \mathsf{Hilb}^k_\theta(\mathbb{CP}^2_\theta)$ (as constructed in Section 7.3) that also carries a Poisson structure and a (holomorphic) symplectic structure. In particular, the Hilbert scheme $\mathsf{Hilb}^k(\mathbb{C}^2)$ has an algebraic symplectic structure induced by the hyper-Kähler metric. On the other hand, in [25, Section 9] it is pointed out that the braided ADHM equations (5.18) and (5.38) are not hyper-Kähler moment map equations, whence one cannot construct hyper-Kähler or even symplectic structures on the quotient space using standard hyper-Kähler or symplectic quotient techniques. We will now briefly describe the sense in which our instanton moduli spaces may be regarded as symplectic quotients.

For this, define a braided symplectic form on $\mathcal{X}(W,V)$ by

$$\omega\big((B,I,J),(B',I',J')\big) := \mathrm{Tr}\,(B \wedge_\theta B' + I \circ J' - I' \circ J),$$

where we use the identification $(\mathcal{A}^!_\ell)_2 \cong \mathbb{C}$, and Tr is the usual trace on $\mathrm{End}_{\mathbb{C}}(V)$ which agrees with the quantum trace Tr_q of [29, Prop. 9.3.5] for our class of deformations. We use the same notation for the form induced on the tangent bundle $T\mathcal{X}(W,V)$. We will study conditions under which the map μ_c defined by (7.12) is a braided (complex) moment map; this will restrict the allowed hamiltonian vector fields.

Firstly, it is easy to check that μ_c is $\mathrm{GL}(V)$-equivariant, i.e., $\mu_c(g \triangleright x) = \mathrm{Ad}^*_{g^{-1}}\mu_c(x)$. Secondly, we require compatibility with the braided symplectic form ω, i.e., for any $v \in T\mathcal{X}(W,V)$ and $\xi \in \mathfrak{gl}(V)$, we want

$$\big\langle \mathrm{d}\mu_c(v), \xi \big\rangle = \omega(\hat{\xi}, v) \tag{7.15}$$

with $\hat{\xi}$ the vector field in $T\mathcal{X}(W,V)$ associated to the linearized action of $\xi \in \mathfrak{gl}(V)$ on $\mathcal{X}(W,V)$. We embed the two-torus $T = (\mathbb{C}^\times)^2$ in $\mathrm{GL}(V)$; then the compatibility condition (7.15) can be formulated as a compatibility requirement between the "geometric" T-action and the "internal" action of the gauge group $\mathrm{GL}(V)$ on $\mathrm{Hom}_{\mathbb{C}}(V, V \otimes (\mathcal{A}^!_\ell)_1)$.

Proposition 7.16. *The map μ_c is compatible with the braided symplectic form ω for coequivariant morphisms $\xi \in \mathrm{End}_{\mathbb{C}}(V)$ with left $\mathcal{H}_\theta$-coaction of the form*

$$\Delta_L(\xi) = \xi^{(-1)} \otimes \xi,$$

where $\xi^{(-1)} \in \mathcal{H}_\theta$ *obeys*

$$F_\theta\big(t_1, \xi^{(-1)}\big) = q^{-1} = F_\theta\big(\xi^{(-1)}, t_2\big).$$

Proof. We compute both sides of (7.15) for $v = (v_1 \otimes \check{w}_1 + v_2 \otimes \check{w}_2, v_I, v_J)$ and $\hat{\xi} = (\hat{\xi}_1 \otimes \check{w}_1 + \hat{\xi}_2 \otimes \check{w}_2, \hat{\xi}_I, \hat{\xi}_J)$. Using (7.13) we find that the left-hand side of (7.15) is given by

$$\big\langle \mathrm{d}\mu_c(v), \xi \big\rangle = \ \mathrm{Tr}\,\big([v_1, B_2]_\theta \ + [B_1, v_2]_\theta + v_I\, J + I\, v_J\big)\, \xi.$$

This is equal to the right-hand side of (7.15) provided that the image of $\xi \in \mathfrak{gl}(V)$ under the tangent of the orbit map of $x = (B, I, J)$ is given by

$$\hat{\xi}_1 = [\xi, B_1]_\theta, \quad \hat{\xi}_2 = [\xi, B_1]_{-\theta}\ , \quad \hat{\xi}_I = (\xi \otimes \mathrm{id}_{\mathcal{A}^!_\ell})\, I, \quad \hat{\xi}_J = -J\, \xi. \tag{7.17}$$

The linearization of the $\mathrm{GL}(V)$-action (5.19), with $g = \mathrm{id}_V + \xi$, on B is given by

$$\big((\mathrm{id}_V + \xi) \otimes \mathrm{id}_{\mathcal{A}^!_\ell}\big)\, \big(B_1 \otimes \check{w}_1 + B_2 \otimes \check{w}_2\big)\, \big((\mathrm{id}_V - \xi) \otimes \mathrm{id}_{\mathcal{A}^!_\ell}\big),$$

which we compose using the braided tensor product

$$(B_i \otimes \check{w}_i)\, (\xi \otimes \mathrm{id}_{\mathcal{A}^!_\ell}) = B_i\, F_\theta^{-2}\big(\check{w}_i^{(-1)}, \xi^{(-1)}\big)\, \xi^{(0)} \otimes \check{w}_i^{(0)}$$

with the usual Sweedler notation $\Delta_L(\xi) = \xi^{(-1)} \otimes \xi^{(0)}$, and so on. Using the coaction (3.4) the conjugate action on B becomes

$$\big(\xi\, B_1 - B_1\, F_\theta^2(t_1, \xi^{(-1)})\, \xi^{(0)}\big) \otimes \check{w}_1 + \big(\xi\, B_2 - B_2\, F_\theta^{-2}(\xi^{(-1)}, t_2)\, \xi^{(0)}\big) \otimes \check{w}_2.$$

The requisite $\mathcal{H}_\theta$-coequivariance conditions now follow by comparing with (7.17). □

One can take, for example, $\xi^{(-1)} = t_1^{-1} + t_2^{-1}$ in Proposition 7.16; this condition is of course an identity for $\theta = 0$, in which case there is no restriction on the allowed hamiltonian vector fields. In a similar fashion, one can treat a real moment map $\mu_r : \mathfrak{X}(W, V) \to \mathfrak{gl}(V)^* \otimes (\mathcal{A}^!_\ell)_2$ whose zero locus is the non-commutative real ADHM equation (5.38).

8 Gauge theory partition functions

8.1 Torus actions

In instanton computations one is interested in equivariant characteristic classes with respect to the natural action of a torus $\widetilde{T}$ on the instanton moduli space [35]. A combinatorial formula for the instanton counting functions can be computed explicitly by classifying the $\widetilde{T}$-fixed points in the instanton moduli space $\mathcal{M}_\theta(r,k)$, and computing the Euler characteristic classes of the equivariant normal bundles to the fixed loci of the torus action. We first describe this torus action explicitly.

For $r \geq 1$, let $\widetilde{T} = T \times (\mathbb{C}^\times)^{r-1}$ with $T = (\mathbb{C}^\times)^2$ the "geometrical" torus used for the deformation of $\mathbb{CP}^2$ and the instanton moduli below. The canonical generators of the coordinate algebra of $\widetilde{T}$ are denoted

$$z = (t_1, t_2, \rho_1, \dots, \rho_r),$$

where we identify $(\mathbb{C}^\times)^{r-1}$ with the maximal torus of $\mathrm{SL}(r)$ given as the hypersurface

$$\prod_{l=1}^{r} \rho_l = 1$$

in $(\mathbb{C}^\times)^r$. Using this presentation of the torus $(\mathbb{C}^\times)^{r-1}$, we denote its characters by $m = \sum_l m_l\, g_l^* = (m_1, \dots, m_r) \in \mathbb{Z}^r$. For any $[E] \in \mathcal{M}_\theta(r,k)$, there is a natural coaction of the Hopf algebra $\mathcal{H}^{(r)} := \mathbb{C}(\rho_1, \dots, \rho_r)$ on the framing module $i^\bullet(E) \cong W \otimes \mathcal{O}_{\mathbb{CP}^1_\theta}$ obtained by fixing a basis $w_1, \dots, w_r$ for the complex vector space W and defining

$$\Delta_L^{(r)} \,:\, i^\bullet(E) \;\longrightarrow\; \mathcal{H}^{(r)} \otimes i^\bullet(E)$$

on $f = \sum_l w_l \otimes f_l \in i^\bullet(E)$ by

$$\Delta_L^{(r)}(f) = \sum_{l=1}^{r} \rho_l \otimes w_l \otimes f_l.$$

The coaction of the Hopf algebra $\mathcal{H}_\theta$ on the moduli space $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(W,V)$ is given by (5.16). To describe the coaction of $\mathcal{H}^{(r)}$, consider the corresponding dual basis $w_1^*, \dots, w_r^*$ for W^*. Similarly, introduce a basis $v_1, \dots, v_k$ for the complex vector space V with corresponding dual basis $v_1^*, \dots, v_k^*$ for V^*. With respect to these bases, we decompose the linear maps I and J

as $I = \sum_{a,l} v_a \otimes I_{a,l} \otimes w_l^*$ with $I_{a,l} \in \left(\mathcal{A}_\ell^!\right)_2$ and $J = \sum_{l,a} J_{l,a}\, w_l \otimes v_a^*$ with $J_{l,a} \in \mathbb{C}$. Then the left $\mathcal{H}^{(r)}$-coaction on the non-commutative ADHM data (B, I, J) is given by

$$\Delta_L^{(r)}(B,I,J) = \left(1 \otimes B, \sum_{a=1}^{k} \sum_{l=1}^{r} \rho_l^{-1} \otimes v_a \otimes I_{a,l} \otimes w_l^*, \right.$$
$$\left. \sum_{l=1}^{r} \sum_{a=1}^{k} J_{l,a}\, \rho_l \otimes w_l \otimes v_a^* \right).$$

By construction, the isomorphism $\mathcal{M}_\theta(W,V) \xrightarrow{\approx} \widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(W,V)$ is $\widetilde{T}$-coequivariant.

8.2 Torus fixed points

A fixed point $[E] \in \mathcal{M}_\theta(r,k)^{\widetilde{T}}$ is an isomorphism class of a coequivariant sheaf E (see Section 3.5) which is equipped with a natural coaction of the Hopf algebra $\widetilde{\mathcal{H}}_\theta := \mathcal{H}_\theta \otimes \mathcal{H}^{(r)}$; hence E decomposes into a finite direct sum of torsion free $\mathcal{A}$-modules graded by the character lattice of the torus $\widetilde{T}$ as

$$E = \bigoplus_{p \in L^*} \bigoplus_{m \in \mathbb{Z}^r} E(p,m). \tag{8.1}$$

The left $\widetilde{\mathcal{H}}_\theta$-coactions $\widetilde{\Delta}_L : E(p,m) \to \widetilde{\mathcal{H}}_\theta \otimes E(p,m)$ are given for $f \in E(p,m)$ by $\widetilde{\Delta}_L(f) = t^p \otimes \rho^m \otimes f$.

The coaction of $\widetilde{\mathcal{H}}_\theta$ on the moduli space naturally makes the vector spaces V and W into objects of the monoidal category ${}^{\widetilde{\mathcal{H}}_\theta}\mathscr{M}$. By definition, as $\widetilde{\mathcal{H}}_\theta$-comodules there are decompositions $W = \bigoplus_l W_l$ with $W_l = \mathbb{C}w_l$ and

$$V = \bigoplus_{p \in L^*} \bigoplus_{m \in \mathbb{Z}^r} V(p,m). \tag{8.2}$$

The maps (5.11) are morphisms in the category ${}^{\widetilde{\mathcal{H}}_\theta}\mathscr{M}$, i.e., there are commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{B} & V \otimes \left(\mathcal{A}_\ell^!\right)_1 \\ {\scriptstyle \widetilde{\Delta}_L}\downarrow & & \downarrow{\scriptstyle \widetilde{\Delta}_L} \\ \widetilde{\mathcal{H}}_\theta \otimes V & \xrightarrow[\mathrm{id}\otimes B]{} & \widetilde{\mathcal{H}}_\theta \otimes V \otimes \left(\mathcal{A}_\ell^!\right)_1 \end{array} \quad , \quad \begin{array}{ccc} W & \xrightarrow{I} & V \otimes \left(\mathcal{A}_\ell^!\right)_2 \\ {\scriptstyle \widetilde{\Delta}_L}\downarrow & & \downarrow{\scriptstyle \widetilde{\Delta}_L} \\ \widetilde{\mathcal{H}}_\theta \otimes W & \xrightarrow[\mathrm{id}\otimes I]{} & \widetilde{\mathcal{H}}_\theta \otimes V \otimes \left(\mathcal{A}_\ell^!\right)_2 \end{array} \quad ,$$

and

$$\begin{array}{ccc} V & \xrightarrow{\quad J \quad} & W \\ {\scriptstyle\widetilde{\Delta}_L}\downarrow & & \downarrow{\scriptstyle\widetilde{\Delta}_L} \\ \widetilde{\mathcal{H}}_\theta \otimes V & \xrightarrow[\mathrm{id}\otimes J]{} & \widetilde{\mathcal{H}}_\theta \otimes W \end{array} \quad .$$

From these commutative diagrams it follows that the only non-trivial components of the linear maps B_1, B_2, I, J with respect to the character decomposition (8.2) are given by

$$\begin{aligned} B_i(p,m) : V(p,m) &\longrightarrow V(p-e_i^*,m), \\ I_l : W_l &\longrightarrow V(-e_1^*-e_2^*,g_l^*)\otimes\big(\mathcal{A}_\ell^!\big)_2, \\ J_l : V(0,g_l^*) &\longrightarrow W_l \end{aligned} \tag{8.3}$$

for $i=1,2$, $l=1,\dots,r$, $p=p_1\,e_1^*+p_2\,e_2^*\in L^*$, and $m=\sum_l m_l\,g_l^*\in\mathbb{Z}^r$. In particular, we have

$$I(W) \subseteq \bigoplus_{l=1}^{r} V(-e_1^*-e_2^*,g_l^*)\otimes\big(\mathcal{A}_\ell^!\big)_2. \tag{8.4}$$

Moreover, from the braided ADHM equation (5.18) we have the relations

$$B_2(p-e_1^*,g_l^*)\circ B_1(p,g_l^*) = q^2\; B_1(p-e_2^*,g_l^*)\circ B_2(p,g_l^*) \tag{8.5}$$

in $\mathrm{Hom}_{\mathbb{C}}\big(V(p,g_l^*),V(p-e_1^*-e_2^*,g_l^*)\big)$ for any $p\neq 0$ and $l=1,\dots,r$.

Using the isomorphism $\big(\mathcal{A}_\ell^!\big)_2\cong\mathbb{C}$, we define the subspace $V'\subseteq V$ by

$$V' = \bigoplus_{(i_1,\dots,i_n)} B_{i_1\dots i_n}\big(\mathrm{im}(I)\big), \tag{8.6}$$

where the sum runs through all finite ordered collections $(i_1,\dots,i_n)$ of indices $i_a=1,2$ of arbitrary length, and $B_{i_1\dots i_n}:=B_{i_n}\,B_{i_{n-1}}\cdots B_{i_1}$; to the empty collection we assign $B_\emptyset:=\mathrm{id}_V$. Then $B_i(V')\subseteq V'$, $i=1,2$, and $\mathrm{im}(I)\subseteq V'$. From (8.3) and (8.4) it follows that

$$V' \subseteq \bigoplus_{p_1,p_2\leq -1}\;\bigoplus_{l=1}^{r} V(p,g_l^*).$$

Hence by the stability condition, $V(p,m)=0$ in (8.2) if $p_1 \geq 0$ or $p_2 \geq 0$ or $m \neq g_l^*$, for otherwise V' would be a proper destabilizing subspace of V. It follows from (8.3) that $J=0$ for any fixed point $[(B,I,J)] \in \widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(W,V)^{\widetilde{T}}$, and the character decomposition (8.2) truncates to

$$V = \bigoplus_{p_1,p_2 \leq -1} \bigoplus_{l=1}^{r} V_l(p) \tag{8.7}$$

with $V_l(p) := V(p,g_l^*)$ and $V=V'$. By Section 7.3, points of $\widehat{\mathcal{M}}_\theta^{\mathrm{ADHM}}(W,V)^{\widetilde{T}}$ correspond bijectively to ideals of codimension $k=\dim_{\mathbb{C}}(V)$ in the affine coordinate algebra $\mathcal{A}(\mathbb{C}^2_\theta)$. In the following we denote the finite set of lattice points $\lambda^l := \big\{(p_1,p_2) \in \mathbb{N}^2 \;\big|\; V_l(-p) \neq 0\big\}$ for each $l=1,\dots,r$.

Since $\dim_{\mathbb{C}}(W_l)=1$ for $l=1,\dots,r$, using (8.5) and $\mathbb{C}$-linearity the above argument also implies

$$V_l(0) = \mathbb{C}\, I_l(1), \quad V_l(-p) = \mathbb{C}\, B_{1,l}^{p_1} B_{2,l}^{p_2} I_l(1)$$

for any $p=(p_1,p_2) \in \lambda^l$ and $l=1,\dots,r$, where we have equated (8.6) with (8.7) and set

$$\begin{aligned} B_{1,l}^{p_1} &:= B_1\big(-p_1\, e_1^* - p_2\, e_2^*, g_l^*\big) \circ B_1\big(-(p_1-1)\, e_1^* - p_2\, e_2^*, g_l^*\big) \\ &\quad \circ \cdots \circ B_1\big(-e_1^* - p_2\, e_2^*, g_l^*\big), \\ B_{2,l}^{p_2} &:= B_2\big(-e_1^* - p_2\, e_2^*, g_l^*\big) \circ B_2\big(-e_1^* - (p_2-1)\, e_2^*, g_l^*\big) \\ &\quad \circ \cdots \circ B_2\big(-e_1^* - e_2^*, g_l^*\big). \end{aligned}$$

In particular, $\dim_{\mathbb{C}}\big(V_l(-p)\big)=1$ for all $p \in \lambda^l$ and $l=1,\dots,r$. Combined with the expression for the universal sheaf $\hat{\mathcal{E}}$ given by Theorem 7.4, it follows that the character decomposition (8.1) truncates to

$$E = \bigoplus_{l=1}^{r} \mathcal{I}_l,$$

where $\mathcal{I}_l = \bigoplus_{p \in L^*} E(p,g_l^*)$ for each $l=1,\dots,r$ is an $\mathcal{H}_\theta$-coequivariant torsion free sheaf of rank one on $\mathsf{Open}(\mathbb{CP}^2_\theta)$. It thus suffices to focus on the case $r=1$ and look for a combinatorial characterization of the finite lattice $\lambda \subset \mathbb{N}^2$; in this instance $\widetilde{T}=T=(\mathbb{C}^\times)^2$.

We first note that by applying B_i to vectors of the form $B_i^{p_i}\, I(1)$, we may conclude that

$$\dim_{\mathbb{C}}\big(V(-p_1,0)\big) \geq \dim_{\mathbb{C}}\big(V(-p_1-s_1,0)\big),$$
$$\dim_{\mathbb{C}}\big(V(0,-p_2)\big) \geq \dim_{\mathbb{C}}\big(V(-p_2-s_2)\big)$$

for all $s_1, s_2 \geq 1$, where these dimensions are all either zero or one. Given $p \in \lambda$, so that $V(-p) \cong \mathbb{C}$, consider now the (*not* commutative) diagram

$$\begin{array}{ccc} V(-p) & \xrightarrow{B_1} & V(-p-e_1^*) \\ {\scriptstyle B_2}\downarrow & & \downarrow{\scriptstyle B_2} \\ V(-p-e_2^*) & \xrightarrow[B_1]{} & V(-p-e_1^*-e_2^*) \end{array}$$

and use the braided commutation relation $B_2\, B_1(1) = q^2\, B_1\, B_2(1)$ to deduce the allowed configurations of non-trivial vector spaces around each such square. For example, the configurations with $V(-p-e_1^*) = 0$, $V(-p-e_2^*) \cong \mathbb{C} \cong V(-p-e_1^*-e_2^*)$ and $V(-p-e_2^*) = 0$, $V(-p-e_1^*) \cong \mathbb{C} \cong V(-p-e_1^*-e_2^*)$ are forbidden by the braided commutation relation. On the other hand, configurations with $V(-p-e_i^*) \cong \mathbb{C}$ for $i = 1, 2$ consistently allow for $V(-p-e_1^*-e_2^*)$ to have dimension zero or one. It follows from these conditions that the finite lattice $\lambda \subset \mathbb{N}^2$ defines a Young diagram oriented as in [33], i.e., if $p = (p_1, p_2) \in \lambda$, then $p' \in \lambda$ for all integral points $p' = (p_1', p_2')$ with $1 \leq p_1' \leq p_1$ and $1 \leq p_2' \leq p_2$. The total number of points in λ is denoted $|\lambda| = \sum_i \lambda_i$, where λ_i is the number of points in the i-th column of λ. In the general case $r \geq 1$, we have thus proven the following result.

Proposition 8.8. *The $\widetilde{T}$-fixed locus $\mathcal{M}_\theta(r,k)^{\widetilde{T}}$ is a finite set of points in bijective correspondence with length r sequences $\vec{\lambda} = (\lambda^1, \dots, \lambda^r)$ of Young diagrams λ^l of size $|\,\vec{\lambda}\,| = k$, where*

$$|\,\vec{\lambda}\,| := \sum_{l=1}^{r} |\lambda^l|.$$

This result coincides with that of the classical case $\theta = 0$ [34, Prop. 2.9]. It can be understood in terms of the non-commutative toric geometry as follows. By [13, Prop. 4.15], $\mathcal{H}_\theta$-coequivariant ideal sheaves on $\mathsf{Open}(\mathbb{CP}^2_\theta)$ are in bijective correspondence with L^*-graded subschemes of $\mathbb{CP}^2_\theta$, where L^* is the character lattice of $T = (\mathbb{C}^\times)^2$; in particular, irreducible subschemes correspond to prime ideals in the spectrum of the homogeneous coordinate algebra $\mathcal{A}$. Moreover, the $\mathcal{H}_\theta$-coinvariant ideals $\mathcal{I} \subset \mathcal{A}$ are *monomial* ideals. If $\mathcal{I}$ obeys the condition (7.9), then it determines a finite partition

$\lambda_{\mathcal{I}}(k)$ of k by considering lattice points corresponding to monomials *not* contained in $\mathcal{I}$.

8.3 Instanton partition functions

In the classical case, instanton partition functions of topologically twisted supersymmetric gauge theories are given by integrating suitable characteristic classes over the instanton moduli spaces. The equivariant partition function is then the generating function for the integral

$$Z_{\rm inst}(r;Q,z)=\sum_{k=0}^{\infty}Q^k\int_{\mathcal{M}_\theta(r,k)^{\widetilde{T}}}\omega(z),$$

where Q is a formal variable and $\omega(z)$ is an equivariant cohomology class depending on the canonical generators z of the coordinate algebra of $\widetilde{T}$. The integral is evaluated formally by applying the localization theorem in equivariant cohomology, hence $\int_{\mathcal{M}_\theta(r,k)^{\widetilde{T}}}\omega(z)$ is a rational function in the coordinate algebra $\mathcal{A}(\widetilde{T})$.

From Section 8.2 it follows that the equivariant characters of the $\widetilde{\mathcal{H}}_\theta$-comodules V and W are given by

$$\mathrm{ch}_{\widetilde{T}}(V)=\sum_{l=1}^{r}\sum_{p\in\lambda^l}\rho_l\,t_1^{1-p_1}\,t_2^{1-p_2},\quad \mathrm{ch}_{\widetilde{T}}(W)=\sum_{l=1}^{r}\rho_l,$$

as in the classical case [34]. By (3.4), the restriction $\underline{\mathscr{I}}_\bullet(E)\big|_{\vec{\lambda}}$ of the complex of Theorem 7.14 to a $\widetilde{T}$-fixed point $\vec{\lambda}$ is a complex in the category ${}^{\widetilde{\mathcal{H}}_\theta}\mathscr{M}$. By Proposition 8.8, the computation of the $\widetilde{T}$-equivariant character of the tangent bundle over the instanton moduli space thus proceeds exactly as in the classical case (see [17] and [34, Thm. 2.11]). At a fixed point parameterized by a length r sequence $\vec{\lambda}=(\lambda^1,\dots,\lambda^r)$ of Young diagrams with $|\,\vec{\lambda}\,|=k$, one has

$$\begin{aligned}\mathrm{ch}_{\widetilde{T}}\big(T_{\vec{\lambda}}\mathcal{M}_\theta(r,k)\big)=\sum_{l,l'=1}^{r}\rho_l^{-1}\,\rho_{l'}\,\Bigg(&\sum_{p\in\lambda^l}t_1^{\,p_1-(\lambda^{l'})^t_{p_2}}\,t_2^{\,\lambda^l_{p_1}-p_2+1}\\&+\sum_{p'\in\lambda^{l'}}t_1^{\,(\lambda^l)^t_{p'_2}-p'_1+1}\,t_2^{\,p'_2-\lambda^{l'}_{p'_1}}\Bigg)\end{aligned}$$

where λ^t_j denotes the number of points in the j-th row of the Young diagram λ. In particular, the corresponding equivariant Euler class of the normal bundle to the fixed point is given by

$$\bigwedge_{-1} T^*_{\vec\lambda}\mathcal{M}_\theta(r,k) = \prod_{l,l'=1}^{r} \prod_{p\in\lambda^l} \left(1-\rho_l\,\rho_{l'}^{-1}\,t_1^{(\lambda^{l'})^t_{p_2}-p_1}\,t_2^{p_2-\lambda^l_{p_1}-1}\right) \times \prod_{p'\in\lambda^{l'}} \left(1-\rho_l\,\rho_{l'}^{-1}\,t_1^{p'_1-(\lambda^l)^t_{p'_2}-1}\,t_2^{\lambda^{l'}_{p'_1}-p'_2}\right).$$

With these ingredients one can now write down the standard equivariant instanton partition functions for supersymmetric gauge theories (without matter fields). The equivariant cohomology class $\omega(z)$ is, by the localization theorem, given by the pullback of a class $\tilde\omega$ on $\mathcal{M}_\theta(r,k)$ which descends from an equivariant class, evaluated at the fixed point $\vec\lambda$, and divided by the Euler character $\bigwedge_{-1} T^*_{\vec\lambda}\mathcal{M}_\theta(r,k)$. For example, for $\tilde\omega=1$ we reproduce Nekrasov's partition function [35] for pure $\mathcal{N}=2$ gauge theory

$$Z^{\mathcal{N}=2}_{\rm inst}(r;Q,z) = \sum_{k=0}^{\infty} \sum_{\vec\lambda\,:\,|\,\vec\lambda\,|=k} \frac{Q^{|\,\vec\lambda\,|}}{\bigwedge_{-1} T^*_{\vec\lambda}\mathcal{M}_\theta(r,k)}.$$

Another standard example is obtained by taking $\tilde\omega$ to be the Euler class of the tangent bundle over $\mathcal{M}_\theta(r,k)$; in this case $\omega(z)=1$ (independently of the equivariant parameters z) and the localization integral simply counts the fixed points of the $\widetilde{T}$-action on the instanton moduli spaces. This results in the Vafa–Witten partition function [38] for $\mathcal{N}=4$ gauge theory

$$Z^{\mathcal{N}=4}_{\rm inst}(r;Q) = \sum_{\vec\lambda} Q^{|\,\vec\lambda\,|} = \prod_{n=1}^{\infty} \frac{1}{\left(1-Q^n\right)^r}. \tag{8.9}$$

For $r=1$, we can compare the "bosonic" partition function (8.9) enumerating torus fixed points with the "fermionic" partition function (7.10) which counts Hilbert series stratifications of the instanton moduli spaces. It would be interesting to combine these two partition functions in the non-equivariant case in order to truly capture the generic differences between the moduli spaces $\mathcal{M}_\theta(r,k)$ for $\theta=0$ and $\theta\neq0$ away from the torus fixed points.

Acknowledgments

We thank Roger Bielawski, Simon Brain, Ken Brown, Alastair Craw, Iain Gordon, Harald Grosse, Derek Harland, Chiara Pagani, Tony Pantev,

Christian Sämann and David Treumann for helpful discussions and correspondence. The work of GL was partially supported by the Italian Project "Cofin08–Noncommutative Geometry, Quantum Groups and Applications". The work of RJS was supported in part by grant ST/G000514/1 "String Theory Scotland" from the UK Science and Technology Facilities Council.

References

[1] G.E. Andrews, *The theory of partitions*, Cambridge University Press, 1998.

[2] M. Artin and J.T. Stafford, *Noncommutative graded domains with quadratic growth*, Invent. Math. **66** (1995), 231–276.

[3] M. Artin and J.J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), 228–287.

[4] M.F. Atiyah, *Geometry of Yang–Mills fields*, Accademia Nazionale dei Lincei, Pisa, 1979.

[5] D. Auroux, L. Katzarkov and D. Orlov, *Mirror symmetry for weighted projective planes and their non-commutative deformations*, Ann. Math. **167** (2008), 867–943.

[6] V. Baranovsky, V. Ginzburg and A. Kuznetsov, *Quiver varieties and a non-commutative* $\mathbb{P}^2$, Compos. Math. **134** (2002), 283–318.

[7] S.J. Brain and G. Landi, *Families of monads and instantons from a non-commutative ADHM construction*, Clay Math. Proc. **11** (2010), 55–84.

[8] S.J. Brain and G. Landi, *Moduli spaces of non-commutative instantons: gauging away non-commutative parameters*, Quart. J. Math. **63** (2012), 41–86.

[9] S.J. Brain and G. Landi, *Differential and twistor geometry of the quantum Hopf fibration*, preprint `arXiv:1103.0419 [math.QA]`.

[10] S.J. Brain and S. Majid, *Quantisation of twistor theory by cocycle twist*, Commun. Math. Phys. **284** (2008), 713–774.

[11] S.J. Brain and W.D. van Suijlekom, *The ADHM construction of instantons on non-commutative spaces*, Rev. Math. Phys. **23** (2011), 261–307.

[12] J. Brodzki, V. Mathai, J. Rosenberg and R.J. Szabo, *Noncommutative correspondences, duality and D-branes in bivariant K-theory*, Adv. Theor. Math. Phys. **13** (2009), 497–552.

[13] L. Cirio, G. Landi and R.J. Szabo, *Algebraic deformations of toric varieties I. General constructions*, `arXiv:1001.1242 [math.QA]`.

[14] A. Connes and G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations*, Commun. Math. Phys. **221** (2001), 141–159.

[15] K. De Naeghel and M. Van den Bergh, *Ideal classes of three-dimensional Artin–Schelter regular algebras*, J. Algebra **283** (2005), 399–429.

[16] B. Fang, C.-C.M. Liu, D. Treumann and E. Zaslow, *The coherent-constructible correspondence and homological mirror symmetry for toric varieties*, Adv. Lect. Math. **18** (2010), 3–37.

[17] R. Flume and R. Poghossian, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg–Witten prepotential*, Int. J. Mod. Phys. A **18** (2003), 2541–2563.

[18] I.B. Frenkel and M. Jardim, *Quantum instantons with classical moduli spaces*, Commun. Math. Phys. **237** (2003), 471–505.

[19] I. Gordon and J.T. Stafford, *Rational Cherednik algebras and Hilbert schemes*, Adv. Math. **198** (2005), 222–274.

[20] K.C. Hannabuss, *Noncommutative twistor space*, Lett. Math. Phys. **58** (2001), 153–166.

[21] M. Hauzer and A. Langer, *Moduli spaces of framed perverse instantons on* $\mathbb{P}^3$, Glasgow Math. J. **53** (2011), 51–96.

[22] D. Huybrechts and M. Lehns, *Framed modules and their moduli*, Int. J. Math. **6** (1995), 297–324.

[23] P.S. Howe and G.G. Hartwell, *A superspace survey*, Class. Quant. Grav. **12** (1995), 1823–1880.

[24] C. Ingalls, *Quantum toric varieties*, preprint available at `http://kappa.math.unb.ca/~colin/research/nctoric.pdf`.

[25] A. Kapustin, A. Kuznetsov and D. Orlov, *Noncommutative instantons and twistor transform*, Commun. Math. Phys. **221** (2001), 385–432.

[26] M. Kashiwara and R. Rouquier, *Microlocalization of rational Cherednik algebras*, Duke Math. J. **144** (2008), 525–573.

[27] G. Landi, C. Pagani and C. Reina, *A Hopf bundle over a quantum four-sphere from the symplectic group*, Commun. Math. Phys. **263** (2006), 65–88.

[28] G. Landi, C. Pagani, C. Reina and W.D. van Suijlekom, *Noncommutative families of instantons*, Int. Math. Res. Not. **12** (2008), 038.

[29] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, 1995.

[30] S. Majid and R. Oeckl, *Twisting of quantum differentials and the Planck scale Hopf algebra*, Commun. Math. Phys. **205** (1999), 617–655.

[31] S. Murray and C. Sämann, *Quantization of flag manifolds and their supersymmetric extensions*, Adv. Theor. Math. Phys. **12** (2008), 641–710.

[32] D. Nadler and E. Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. **22** (2009), 233–286.

[33] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, American Mathematical Society, 1999.

[34] H. Nakajima and K. Yoshioka, *Instanton counting on blowup I. 4-dimensional pure gauge theory*, Invent. Math. **162** (2005), 313–355.

[35] N.A. Nekrasov, *Seiberg–Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2004), 831–864.

[36] T.A. Nevins and J.T. Stafford, *Sklyanin algebras and Hilbert schemes of points*, Adv. Math. **210** (2007), 405–478.

[37] R.J. Szabo, *Crystals, instantons and quantum toric geometry*, Acta Phys. Polon. Suppl. **4** (2011), 461–494.

[38] C. Vafa and E. Witten, *A strong coupling test of S-duality*, Nucl. Phys. B **431** (1994), 3–77.