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Symmetries of massless vertex operators in $AdS_5 \times S^5$

Andrei Mikhailov[1,2]

[1]California Institute of Technology 452-48, Pasadena, CA 91125, USA
[2]Institute for Theoretical and Experimental Physics,
117259, Bol. Cheremushkinskaya 25, Moscow, Russia
andrei@theory.caltech.edu

Abstract

The worldsheet sigma-model of the superstring in $AdS_5 \times S^5$ has a one-parameter family of flat connections parametrized by the spectral parameter. The corresponding Wilson line is not BRST invariant for an open contour, because the BRST transformation leads to boundary terms. These boundary terms define a cohomological complex associated to the endpoint of the contour. We study the cohomology of this complex for Wilson lines in some infinite-dimensional representations. We find that for these representations the cohomology is nontrivial at the ghost number 2. This implies that it is possible to define a BRST invariant open Wilson line. The central point in the construction is the existence of massless vertex operators transforming exactly covariantly under the action of the global symmetry group. In flat space massless vertices transform covariantly only up to adding BRST exact terms. But in AdS we show that it is possible to define vertices so that they transform exactly covariantly.

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1 Introduction

Nonlocal conserved charges play the central role in quantum integrability [1]. For the superstring in $AdS_5 \times S^5$ their existence was proven in the classical sigma-model in [2] using the Green–Schwarz–Metsaev–Tseytlin formalism, and in [3–5] using the pure spinor formalism. The existence of the nonlocal conserved charges at the quantum level was proven in [5, 6].

1.1 Open Wilson lines on the worldsheet

1.1.1 Wilson lines and BRST operator

The most important feature of the string worldsheet σ-model (besides the conformal invariance) is the existence of the BRST structure. The physically meaningful constructions should respect the action of the BRST operator Q_{BRST}.

The nonlocal conserved charges of [3–5] are only BRST invariant up to the boundary terms. To be more precise, we need to introduce the transfer matrix that is the generating function of the nonlocal conserved charges. For an open contour C_A^B connecting points A and B on the string worldsheet:

we define the transfer matrix:

$$T_\rho[C_A^B] = P\exp\left(-\int_A^B J[z]\right), \tag{1.1}$$

where C_A^B is a contour connecting points A and B on the string worldsheet, and ρ is a representation of $\mathbf{g} = psu(2,2|4)$. The currents $J[z]$ depend on the spectral parameter z and the representation ρ. The transfer matrix is a function of the *spectral parameter* z. It is the generating function of the conserved charges. It can be also though of as the Wilson line operator corresponding to the flat connection $J[z]$ on the string worldsheet. In this paper, we will use both expressions: "transfer matrix" and "Wilson line", and understand them as synonyms.

The BRST variation of the Wilson line $T_\rho[C_A^B]$ results in the boundary terms. Using the notations of [7]:

$$\varepsilon Q_{\mathrm{BRST}} T[C_A^B](z) = \left(\frac{1}{z}\varepsilon\lambda_3(B) + z\varepsilon\lambda_1(B)\right) T[C_A^B](z) - T[C_A^B](z)\left(\frac{1}{z}\varepsilon\lambda_3(A) + z\varepsilon\lambda_1(A)\right). \tag{1.2}$$

This equation is very interesting. Nontrivial boundary terms in (1.2) allow to "bootstrap" at least at the classical level the structure of r—s matrices, see Section 7 of [7]. On the other hand, these boundary terms present a problem: the nonlocal conserved charges are not physical quantities, at least not in an obvious sense. (Because they are not in the kernel of Q_{BRST}.) What should we do with them?

1.1.2 The plugs

A natural thing to try is to find an operator which when inserted at the endpoint of the Wilson line would make it Q_{BRST}-closed. This is somewhat analogous to the open Wilson line in QCD. The expression $P\exp\int_A^B A_\mu\, dx^\mu$ is unphysical, because it is not gauge invariant. The physical quantity is:

$$\overline{\psi}(B)\left(P\exp\int_A^B A_\mu dx^\mu\right)\psi(A). \tag{1.3}$$

We may call $\psi(A)$ and $\overline{\psi}(B)$ "the plugs" because they fix "leaking boundary terms" in gauge tranformations, or in BRST transformations. Can we find similar plugs for the Wilson line on the string worldsheet in $AdS_5 \times S^5$? In this paper, we will report a progress in this direction.

1.1.3 Wilson lines in infinite-dimensional representations

Remember that the Wilson line depends on a choice of representation; we have to choose a representation ρ of $psu(2,2|4)$. Consider the space of states of the linearized supergravity multiplet in $AdS_5 \times S^5$. It splits into the direct sum of infinitely many infinite-dimensional irreducible representations of $psu(2,2|4)$, each corresponding to a BPS state. We will argue that when ρ is one of those infinite-dimensional BPS representations, then there is a suitable plug of the ghost number 2. This is closely related to the vertex operators for the massless states in $AdS_5 \times S^5$. In fact, we will relate the BRST cohomology complex corresponding to the endpoint of the Wilson line to the BRST complex corresponding to the vertex operators. This is essentially an example of the Frobenius reciprocity. The main nontrivial point is the construction of the vertex operators transforming *strictly covariantly* under the global supersymmetries of $AdS_5 \times S^5$. This is different from flat space where massless vertices transform covariantly only up to BRST exact terms.

Wilson lines in infinite-dimensional representations played an important role in the integrable context in [8] in the construction of the Q-operator. They also played an important role in the AdS/CFT context in [9] for the interpretation of the YM Feynman diagramms in the string worldsheet theory.

1.2 Representation theory interpretation of the SUGRA spectrum

Consider an infinite-dimensional irreducible representation $\mathcal{H}$ of $psu(2,2|4)$. It is natural to ask the question:

When does $\mathcal{H}$ appear in the decomposition of the space of solutions of the linearlized Type IIB SUGRA equations in $AdS_5 \times S^5$? (1.4)

The results of our paper imply that this answer can be answered directly in terms of the structure of $\mathcal{H}$, as a representation of $psu(2,2|4)$. Namely, given $\mathcal{H}$, we consider the following complex:

$$\ldots \longrightarrow \mathcal{H}' \otimes_{\mathbf{g}_{\bar{0}}} \mathcal{P}^n \stackrel{Q_{\text{endpoint}}}{\longrightarrow} \mathcal{H}' \otimes_{\mathbf{g}_{\bar{0}}} \mathcal{P}^{n+1} \longrightarrow \ldots \tag{1.5}$$

defined entirely in terms of $\mathcal{H}$ — see Section 7.2.2. Then, we claim that the multiplicity of $\mathcal{H}$ in the space of linearized SUGRA solutions is equal to the dimension of the second cohomology $H^2(Q_{\text{endpoint}}, \mathcal{H}' \otimes_{\mathbf{g}_{\bar{0}}} \mathcal{P}^\bullet)$ of this complex. (We only claim this when $\mathcal{H}$ has "high enough" spin on S^5; see the end of Section 4.5.)

1.3 The plan of the paper

Most of the paper is about massless vertex operators in $AdS_5 \times S^5$. In Section 2, we give a geometrical definition of vertex operators using the representation of $AdS_5 \times S^5$ as the coset space G/H. In Section 3, we explain what it means for the vertex to be strictly covariant, and then in Sections 4 and 5 prove the existence of such covariant vertices. In Section 6, we discuss the flat space limit of our construction. (Although it is impossible to construct the strictly covariant vertex in the flat space, but nevertheless the construction in $AdS_5 \times S^5$ has a well-defined flat space limit, which does transforms strictly covariantly, but only under a subgroup $SO(1,4) \times SO(5) \subset SO(1,9)$.) In Section 7, we explain how the covariant vertex plugs the endpoint of the Wilson line. In Section 8, we present some consequences of our construction; we explain how to prepare the vertex operator depending on the spectral parameter.

1.4 Notations

The algebra of supersymmetries of $AdS_5 \times S^5$ has a $\mathbf{Z}_4$ grading:

$$psu(2,2|4) = \mathbf{g} = \mathbf{g}_{\bar{0}} + \mathbf{g}_{\bar{1}} + \mathbf{g}_{\bar{2}} + \mathbf{g}_{\bar{3}}. \tag{1.6}$$

We denote $\mathcal{U}\mathbf{g}$ the universal enveloping algebra of $\mathbf{g}$.

Let g denote the group element of $\mathcal{U}\mathbf{g}$, i.e., an element of the group $PSU(2,2|4)$:

$$g = \mathrm{e}^{\omega}\, \mathrm{e}^{\theta_L + \theta_R}\, \mathrm{e}^{x}, \tag{1.7}$$

where $\omega \in \mathbf{g}_{\bar{0}}$, $\theta_{\mathrm{L}} \in \mathbf{g}_{\bar{3}}$, $\theta_{\mathrm{R}} \in \mathbf{g}_{\bar{1}}$ and $x \in \mathbf{g}_{\bar{2}}$. So defined $\theta_{\mathrm{L,R}}$ and x are coordinates of the super-$AdS_5 \times S^5$. The generators of $\mathbf{g} = psu(2,2|4)$ are the same as in [7, 23]:

$$t^3_\alpha \in \mathbf{g}_{\bar{3}}, \quad t^1_{\dot{\alpha}} \in \mathbf{g}_{\bar{1}}, \quad t^2_\mu \in \mathbf{g}_{\bar{2}}, \quad t^0_{[\mu\nu]} \in \mathbf{g}_{\bar{0}}. \tag{1.8}$$

For a vector space L we will denote L' the dual vector space. In particular, the space of states is denoted $\mathcal{H}$ and the space of linear functionals on the states is denoted $\mathcal{H}'$. We will mostly consider the linear functionals, which are the values of some supergravity fields (such as the Ramond–Ramond field strength) at a fixed point in $AdS_5 \times S^5$. These could be also thought of as non-normalizable elements of $\mathcal{H}$, "delta-functions type of states" in $\mathcal{H}$.

For an even vector space L we denote $\Lambda^n L$ the space of antisymmetric tensors. For an odd vector space $\Lambda^n L$ will stand for symmetric tensors.

Example: $\mathbf{g}_{\bar{3}}$ is an odd vector space, and $\mathbf{g}'_{\bar{3}}$ is the dual space. Therefore $\Lambda^5 \mathbf{g}'_{\bar{3}}$ is identified with the fifth-order polynomials of some bosonic spinor variable λ^α.

2 Massless vertex operators as functions on the group manifold

Massless vertex operators in $AdS_5 \times S^5$ were introduced in [10].

Using the group theory language, we can define the vertex operator as a collection of functions $V_{\alpha\beta}(g)$, $V_{\alpha\dot\beta}(g)$ and $V_{\dot\alpha\dot\beta}(g)$ of $g \in PSU(2,2|4)$ subject to the condition of $\mathbf{g}_0$-covariance, which says that for any $h \in SO(1,4) \times SO(5)$ we should get:

$$V_{\alpha\beta}(hg) = h_\alpha^{\alpha'} h_\beta^{\beta'} V_{\alpha'\beta'}(g), \tag{2.1}$$

$$V_{\alpha\dot\beta}(hg) = h_\alpha^{\alpha'} h_{\dot\beta}^{\dot\beta'} V_{\alpha'\dot\beta'}(g), \tag{2.2}$$

$$V_{\dot\alpha\dot\beta}(hg) = h_{\dot\alpha}^{\dot\alpha'} h_{\dot\beta}^{\dot\beta'} V_{\dot\alpha'\dot\beta'}(g). \tag{2.3}$$

Here $h_\alpha^{\alpha'}$ and $h_{\dot\beta}^{\dot\beta'}$ are the matrix elements of $h \in G_0$ acting on $\mathbf{g}_3$ and $\mathbf{g}_1$, respectively.

For a tensor field $\varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(g)$ we introduce the covariant derivatives:

$$\begin{aligned}
\mathcal{T}_\alpha^{\bar 3}\varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(g) &= \left.\frac{d}{ds}\right|_{s=0} \varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(e^{-st_\alpha^3}g)\\
\mathcal{T}_m^{\bar 2}\varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(g) &= \left.\frac{d}{ds}\right|_{s=0} \varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(e^{-st_m^2}g)\\
\mathcal{T}_{\dot\alpha}^{\bar 1}\varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(g) &= \left.\frac{d}{ds}\right|_{s=0} \varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(e^{-st_{\dot\alpha}^1}g)\\
\mathcal{T}_{[mn]}^{\bar 0}\varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(g) &= \left.\frac{d}{ds}\right|_{s=0} \varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(e^{-st_{[mn]}^0}g).
\end{aligned} \tag{2.4}$$

Collectively:

$$\mathcal{T}_A^{\overline{n}}\varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(g) = \left.\frac{d}{ds}\right|_{s=0} \varphi_{\alpha_1\ldots\alpha_m\ \dot\beta_1\ldots\dot\beta_n}(\mathrm{e}^{-st_A^{\overline{n}}}g). \tag{2.5}$$

Note that the covariant derivatives $\mathcal{T}_A^{\bar n}$ for $\bar n \neq \bar 0$ satisfy the condition of $\mathbf{g}_0$-covariance, for example:

$$\mathcal{T}_\alpha^3 V_{\beta\gamma}(hg) = h_\alpha^{\alpha'} h_\beta^{\beta'} h_\gamma^{\gamma'} \mathcal{T}_{\alpha'}^3 V_{\beta'\gamma'}(g). \tag{2.6}$$

Also, the $\mathbf{g}_0$-covariance condition (2.1)–(2.3) can be formulated as the following explicit expression for the covariant derivative $\mathcal{T}_{[\mu\nu]}$ along $\mathbf{g}_0$:

$$\mathcal{T}^0_{[\mu\nu]} V_{\beta\gamma}(g) = \frac{d}{ds}\bigg|_{s=0} V_{\beta\gamma}(\mathrm{e}^{-st^0_{[\mu\nu]}} g) = f_{[\mu\nu]\beta}{}^{\beta'} V_{\beta'\gamma}(g) + f_{[\mu\nu]\gamma}{}^{\gamma'} V_{\beta\gamma'}(g). \tag{2.7}$$

The condition that the vertex operator is $\mathcal{Q}$-closed can be written as follows:

$$\mathcal{T}^3_{(\alpha} V_{\beta\gamma)} = f_{(\alpha\beta}{}^m S_{\gamma)m}, \tag{2.8}$$

$$\mathcal{T}^1_{(\dot\alpha} V_{\dot\beta\dot\gamma)} = f_{(\dot\alpha\dot\beta}{}^m S_{\dot\gamma)m}, \tag{2.9}$$

$$\mathcal{T}^3_{(\alpha} V_{\beta)\dot\gamma} + \mathcal{T}^1_{\dot\gamma} V_{\alpha\beta} = f_{\alpha\beta}{}^m A_{m\dot\gamma}, \tag{2.10}$$

$$\mathcal{T}^1_{(\dot\alpha|} V_{\gamma|\dot\beta)} + \mathcal{T}^3_{\gamma} V_{\dot\alpha\dot\beta} = f_{\dot\alpha\dot\beta}{}^m A_{\gamma m}, \tag{2.11}$$

where A and S are defined by these equations.

The gauge transformations are:

$$\delta_{\Phi,\widetilde\Phi} V_{\alpha\beta} = \mathcal{T}^3_{(\alpha}\Phi_{\beta)}, \tag{2.12}$$

$$\delta_{\Phi,\widetilde\Phi} V_{\dot\alpha\dot\beta} = \mathcal{T}^1_{(\dot\alpha}\widetilde\Phi_{\dot\beta)}, \tag{2.13}$$

$$\delta_{\Phi,\widetilde\Phi} V_{\alpha\dot\beta} = \mathcal{T}^3_{\alpha}\widetilde\Phi_{\dot\beta} + \mathcal{T}^1_{\dot\beta}\Phi_{\alpha}, \tag{2.14}$$

where Φ and $\widetilde\Phi$ are the parameters of the gauge transformations.

The BRST operator is:

$$\mathcal{Q} = \mathcal{Q}_L + \mathcal{Q}_R = \lambda^\alpha \mathcal{T}^3_\alpha + \tilde\lambda^{\dot\alpha}\mathcal{T}^1_{\dot\alpha}. \tag{2.15}$$

Our definition of the vertex is slightly weaker than the definition of [10]. The definition of [10] requires that:

$$V_{\alpha\beta} = 0 \ \text{ and } \ V_{\dot\alpha\dot\beta} = 0 \tag{2.16}$$

the only nonzero component of the vertex remains $V_{\alpha\dot\beta}$. In fact, $V_{\alpha\beta}$ and $V_{\dot\alpha\dot\beta}$ are always Q-exact. Therefore the condition (2.16) can always be satisfied by adding to the vertex something $\mathcal{Q}_{\mathrm{BRST}}$-exact. In this sense the components $V_{\alpha\beta}$ and $V_{\dot\alpha\dot\beta}$ can always be "gauged away". But we want the *covariant* vertex operator. We suspect that it might be impossible to gauge away $V_{\alpha\beta}$ and $V_{\dot\alpha\dot\beta}$ in a covariant way. This is the reason why we prefer to allow these components in the definition of the vertex operator.

3 Covariant vertex: the definition

3.1 Vertices and states

In string theory vertex operators represent states. Vertex operators are functions $V(x, \theta, \lambda)$. The global symmetries act on x and θ, and therefore act on vertex operators.

On the other hand, the global symmetries act on the space of states. Therefore the action of the global symmetry group on states should agree with the action on vertex operators. Naively, this would imply that if $V_\Psi(x, \theta, \lambda)$ is a vertex operator corresponding to the state Ψ then

$$V_{g\Psi}(x, \theta, \lambda) = V_\Psi(gx, g\theta, \lambda). \tag{3.1}$$

But in fact this formula, generally speaking, holds only up to BRST-trivial corrections (terms which are Q_{BRST} of something).

Note that $V_\Psi(x, \theta, \lambda)$ is not defined unambiguously, because we could add to it BRST-exact terms and get physically equivalent vertex. We will prove that in $AdS_5 \times S^5$ it is possible to use this freedom in the definition of V_Ψ and choose V_Ψ so that it transforms covariantly, as in (3.1). In our proof, we will use the fact that vertices corresponding to supergravity states *exist*. This was proven in [10]. Given the existence of the vertex, we will prove that it is always possible to correct it by a BRST-exact expression, if necessary, to get a *covariant* vertex.

3.2 Examples of covariant vertices: AdS radius and β-deformation

The first example of the covariant vertex was given in [11]. It was shown that the *zero mode* dilaton vertex is given by the expression[1] which is independent of x and θ:

$$V(x, \theta, \lambda) = \mathrm{Str}(\lambda_3 \lambda_1). \tag{3.2}$$

This operator plays the central role in [11]. The corresponding marginal deformation of the action changes the radius of $AdS_5 \times S^5$. The radius is invariant under the global symmetries, therefore in this case covariance means invariance; the vertex (3.2) is invariant under the global symmetries.

[1] Using the notations of [11]: $\eta_{\alpha\hat{\alpha}} \lambda^\alpha \hat{\lambda}^{\hat{\alpha}}$.

This is related to the fact that the action of the pure spinor superstring in $AdS_5 \times S^5$ is exactly invariant under the global symmetries (while the action in flat space is invariant only up to adding a total derivative). Our construction can be considered a generalization of (3.2) for fields with non-trivial dependence on x and θ.

The second example[2]

$$V_{ab}^{\text{beta}} = (g^{-1}\varepsilon(\lambda_3 - \lambda_1)g)_a\ (g^{-1}\varepsilon'(\lambda_3 - \lambda_1)g)_b \tag{3.3}$$

Here the indices a and b enumerate the adjoint representation of $psu(2,2|4)$. Note that (3.3) is antisymmetric under the exchange of a and b. Therefore this vertex transforms in the antisymmetric product of two adjoint representations of $psu(2,2|4)$. This antisymmetric product splits into two irreducible components. The first component is the adjoint representation. But the part of (3.3) belonging to the adjoint representation is actually Q_{BRST}-exact:

$$\begin{aligned} f^{ab}{}_c V_{ab}^{\text{beta}} &= [(g^{-1}\varepsilon(\lambda_3 - \lambda_1)g), (g^{-1}\varepsilon'(\lambda_3 - \lambda_1)g)]_c \\ &= \varepsilon Q_{\text{BRST}}(g^{-1}\varepsilon'(\lambda_3 + \lambda_1)g)_c. \end{aligned} \tag{3.4}$$

(Note that this formula played an important role in Section 6 of [5].)

The deformation of the action corresponding to (3.3) follows from the standard descent procedure. Let us denote:

$$\Lambda_a(\varepsilon) = (g^{-1}\varepsilon(\lambda_3 - \lambda_1)g)_a. \tag{3.5}$$

This is the ghost number 1 cocycle corresponding to the local conserved currents, see also Appendix A. It corresponds to the local conserved currents in the following sense:

$$d\Lambda_a(\varepsilon) = \varepsilon Q(j_a), \tag{3.6}$$

where $j_{a\pm}(\tau^+, \tau^-)$ is the density of the local conserved charge corresponding to the global symmetries. Therefore:

$$d(\Lambda_{[a}(\varepsilon)\Lambda_{b]}(\varepsilon')) = 2\varepsilon Q j_{[a}\Lambda_{b]}(\varepsilon') \tag{3.7}$$

and

$$d(j_{[a}\Lambda_{b]}(\varepsilon)) = -\frac{1}{2}\varepsilon Q(j_{[a} \wedge j_{b]}). \tag{3.8}$$

[2]Note in revised version: a more detailed discussion of V_{ab}^{beta} will be presented in the forthcoming paper with Bedoya *et al.*

We conclude that for any constant antisymmetric matrix B^{ab} we can infinitesimally deform the worldsheet action as follows:

$$S \to S + B^{ab} \int j_{[a} \wedge j_{b]}. \tag{3.9}$$

Consider for example B_{ab} in the directions of S^5. We get:

$$S \to S + B^{[kl][mn]} \left(\int X_{[k} dX_{l]} \wedge X_{[m} dX_{n]} + \cdots \right), \tag{3.10}$$

where X_j describes the embedding of S^5 into $\mathbf{R}^6$ and dots denote θ-dependent terms. These θ-dependent terms appear because j_a includes θ. Equation (3.10) corresponds to the marginal deformations of the $\mathcal{N} = 4$ Yang–Mills known as *β-deformations* [12], as follows from their quantum numbers.

The subspace $\mathbf{g} \subset \mathbf{g} \wedge \mathbf{g}$ corresponds to B of the following form:

$$B^{[kl][mn]} = \delta^{km} A^{ln} - \delta^{lm} A^{kn} + \delta^{ln} A^{km} - \delta^{kn} A^{lm}, \tag{3.11}$$

where A^{mn} is antisymmetric matrix; then the corresponding deformation of the Lagrangian is a total derivative $d(A^{mn} X_m dX_n)$. The complementary space has real dimension 90, it corresponds to the representation $\mathbf{45_C}$ of $so(6)$. This is the expected quantum numbers of the linearized β-deformation, cp. Section 3.1 of [13] and references therein. It was observed in [13] that some of these deformations are obstructed when we pass from the linearized supergravity equations to the nonlinear equations. Not all of the deformations (3.9) can be extended to the solutions of the nonlinear supergravity equations *as solutions constant in AdS_5 directions*, but only those which satisfy some nonlinear equations on B^{ab}. If these nonlinear equations are not satisfied, then the nonlinear solutions will have "resonant terms" and because of these resonant terms will not be periodic[3] in the global time of AdS_5.

[3]Similar phenomenon for "fast moving strings" was discussed in [14, 15]. Generally speaking, deviations from periodicity in the global time of AdS correspond to something like anomalous dimension. In this case it shows that the beta function of the deformed theory is actually nonzero at the higher order in the deformation, unless if additional (cubic) constraints are imposed on the deformation parameter.

3.3 Universal vertex

Suppose that we are looking at the *massless* states transforming in some representation $\mathcal{H}$ of the global symmetry group $PSU(2,2|4)$. For every state $\Psi \in \mathcal{H}$ we have the corresponding vertex operator $\mathcal{V}(\Psi)$. As we discussed, $\mathcal{V}(\Psi)$ consists of the components: $\mathcal{V}_{\alpha\beta}(\Psi)$, $\mathcal{V}_{\alpha\dot{\beta}}(\Psi)$ and $\mathcal{V}_{\dot{\alpha}\dot{\beta}}(\Psi)$. We write $\mathcal{V}$ instead of V to stress that $\mathcal{V}$ is a function of Ψ. Mathematically it would be more appropriate to call it "a linear operator from the space of states to the space of vertex operators". We will call $\mathcal{V}$ the "universal vertex" for the representation $\mathcal{H}$ because it is a uniform definition of vertex operators for all states in $\mathcal{H}$:

$$\mathcal{V} : \mathcal{H} \longrightarrow (\text{functions of } x, \theta, \lambda).$$

The global symmetry group $G = PSU(2,2|4)$ acts on both space of states and space of vertex operators. It acts on the space of states by definition, because it is the global symmetry group of the theory. It also acts on the space of vertex operators. The action on the space of vertex operators may seem obvious, but we would like to spell it out explicitly because we feel that some confusion is possible. A vertex operator has components $V_{\alpha\beta}$, $V_{\alpha\dot{\beta}}$ and $V_{\dot{\alpha}\dot{\beta}}$ which are all functions of the group element g, i.e., $V_{\alpha\beta}(g)$, $V_{\alpha\dot{\beta}}(g)$ and $V_{\dot{\alpha}\dot{\beta}}(g)$ satisfying the conditions of $\mathbf{g}_0$-covariance (2.1)–(2.3). Then, the action of the global symmetry transformation $g' \in PSU(2,2|4)$ is defined as follows:

$$\begin{aligned} (g'.V_{\alpha\beta})(g) &= V_{\alpha\beta}(gg'), \\ (g'.V_{\alpha\dot{\beta}})(g) &= V_{\alpha\dot{\beta}}(gg'), \\ (g'.V_{\dot{\alpha}\dot{\beta}})(g) &= V_{\dot{\alpha}\dot{\beta}}(gg'). \end{aligned} \tag{3.12}$$

Because g' hits g on the right, this action of the global symmetries is manifestly consistent with the conditions of $\mathbf{g}_0$-covariance (2.1)–(2.3) and also commutes with the covariant derivatives (2.4).

This defines the action of G on $\mathcal{V}(\Psi)$ for any fixed Ψ; the expression:

$$g'.(\mathcal{V}(\Psi)) \tag{3.13}$$

is defined by (3.12):

$$(g'.(\mathcal{V}(\Psi)))(g) = (\mathcal{V}(\Psi))(gg'). \tag{3.14}$$

3.4 Covariant universal vertex

It is natural to ask, if it is true that (3.13) is equal to this:

$$\mathcal{V}(g'\Psi). \tag{3.15}$$

In other words, if it is true or not that:

$$(\mathcal{V}(\Psi)\,)(gg') \stackrel{?}{=} (\mathcal{V}(g'\Psi)\,)(g). \tag{3.16}$$

This is not automatically true. What *is* automatically true[4] is this statement:

$$(\mathcal{V}(\Psi)\,)(gg') = (\mathcal{V}(g'\Psi)\,)(g) \;+\; \mathcal{Q}_{\mathrm{BRST}}(\mathrm{smth}). \tag{3.17}$$

Remember that vertex operators are defined modulo BRST-exact expressions. The question is, can we choose a representative for $\mathcal{V}(\Psi)$ in the equivalence class of $\mathcal{V}(\Psi) \simeq \mathcal{V}(\Psi) + \mathcal{Q}_{\mathrm{BRST}}(\mathrm{smth})$, "uniformly in Ψ", so that (3.16) is true?

The answer to this question is "no" in flat space, but "yes" in AdS. It turns out that in $AdS_5 \times S^5$ it is possible to choose the vertex operator to be covariant.

Let us introduce the notation for the action of the global symmetries (compare to (2.5)):

$$t^{\overline{n}}_A f(g) = \left.\frac{d}{ds}\right|_{t=0} f(g\,\mathrm{e}^{st^{\overline{n}}_A}). \tag{3.18}$$

<u>Definition</u>: The covariant vertex is a superfield

$$\mathcal{V}(\Psi) = \lambda^\alpha\lambda^\beta\mathcal{V}_{\alpha\beta}(\Psi) + \lambda^\alpha\tilde{\lambda}^{\dot\beta}\mathcal{V}_{\alpha\dot\beta}(\Psi) + \tilde{\lambda}^{\dot\alpha}\tilde{\lambda}^{\dot\beta}\mathcal{V}_{\dot\alpha\dot\beta}(\Psi) \tag{3.19}$$

depending linearly on the state Ψ, and such that:

1. It is annihilated by $\mathcal{Q}$:

$$(\lambda^\alpha\mathcal{T}_\alpha + \tilde{\lambda}^{\dot\alpha}\mathcal{T}_{\dot\alpha})\mathcal{V}(\Psi) = 0 \tag{3.20}$$

   and is not $\mathcal{Q}$-exact.

[4] "Automatically true" means true under the assumption that the vertex operators exist. The existence was proven in [10].

2. The action of the global symmetry on $\mathcal{V}$ as a function of g agrees with the action of the global symmetry on the space of states:

$$t^{\bar{n}}.\mathcal{V}(\Psi) = \mathcal{V}(t^{\bar{n}}\Psi). \tag{3.21}$$

4 Taylor series for the vertex

Let us study vertex operators for states, which are not necessarily normalizable. In other words, let us forget about the boundary conditions near the boundary of AdS, and study the supergravity states which are not necessarily normalizable. Moreover, let us pick a point in $AdS_5 \times S^5$ and consider the Taylor expansion of the supergravity fields around this point. Let us not worry about the convergence of the Taylor series. Just study the supergravity equations, BRST cohomology *etc.* on formal Taylor series. The question of convergence, and the question of the behaviour at spacial infinity, can be studied later.

For the study of the Taylor series the mathematical notion of the *coinduced representation* is useful.

4.1 A review of coinduced representations

Let us study the supergravity fields around a point in $AdS_5 \times S^5$ corresponding to the unit $\mathbf{1} \in G$. If we do not insist on convergence, then the space of supergravity fields around a point can be replaced by a more algebraic notion, the so-called *coinduced representation* [16].

For a representation V of G_0, we define the *coinduced representation* $\mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} V$, in the following way:

Definition:

the space of linear functions f from the universal enveloping algebra $\mathcal{U}\mathbf{g}$ to V, which satisfy the condition of $\mathbf{g}_0$-invariance:

$$f(x\xi) = \rho(x) f(\xi) \text{ for any } x \in \mathbf{g}_0,\ \xi \in \mathcal{U}\mathbf{g}$$

is called the *coinduced representation* and denoted $\mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} V$.

The mathematical notation for such functions is:

$$f \in \mathrm{Hom}_{\mathbf{g}_0}(\mathcal{U}\mathbf{g}, V). \tag{4.1}$$

Here "Hom(A,B)" means the space of linear maps from A to B, and the subindex $\mathbf{g}_0$ means $\mathbf{g}_0$-invariant functions: $f(x\xi) - \rho(x)f(\xi) = 0$.

The action of $\mathbf{g}$ on this space is defined by the formula:

$$x.f(\xi) = f(\xi x), \text{ for } x \in \mathbf{g},\ \xi \in \mathcal{U}\mathbf{g}. \tag{4.2}$$

To summarize:

$$\boxed{\mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} V = \mathrm{Hom}_{\mathbf{g}_0}(\mathcal{U}\mathbf{g}, V)}.$$

We will now explain that $\mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} V$ encodes the Taylor series of various tensor fields on $AdS_5 \times S^5$, where V is the representation of $\mathbf{g}_0$ corresponding to the type of the tensor.

Let us start with the trivial representation $V = \mathbf{C}$ of $\mathbf{g}_{\bar{0}}$. In this case we should get scalar fields on AdS. The correspondence between the elements of $\mathrm{Hom}_{\mathbf{g}_0}(\mathcal{U}\mathbf{g}, \mathbf{C})$ and the scalar fields on AdS goes as follows. Given a scalar field $\phi(g)$, the corresponding element $f \in \mathrm{Hom}_{\mathbf{g}_0}(\mathcal{U}\mathbf{g}, \mathbf{C})$ is given by the formula:

$$f(x_1 x_2 \cdots x_n) = x_1.x_2 \ldots x_n.\phi(\mathbf{1}), \tag{4.3}$$

where

$$x_1.x_2 \ldots x_n.\phi(g) = \left.\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n}\right|_{t_1 = \ldots = t_n = 0} \phi(g\, \mathrm{e}^{t_1 x_1}\, \mathrm{e}^{t_2 x_2} \cdots \mathrm{e}^{t_n x_n}).$$

In this case the $\mathbf{g}_0$-invariance condition says that $f(x\xi) = 0$ for any $x \in \mathbf{g}_0$, and this is indeed satisfied for f defined in (4.3) because $\phi(\mathrm{e}^x) = \phi(\mathbf{1})$ for any $x \in \mathbf{g}_0$ because ϕ is well defined on G/G_0.

There is also a map going in the opposite direction. Namely, given f a linear function from $\mathcal{U}\mathbf{g}$ to $\mathbf{C}$ we define the corresponding scalar field $\phi(g)$ as follows:

$$\phi(g) = f(g). \tag{4.4}$$

Note that on the right-hand side we treat g as a *group element*[5] of $\mathcal{U}\mathbf{g}$.

We have just explained why for $V = \mathbf{C}$ the space $\mathrm{Hom}_{\mathbf{g}_0}(\mathcal{U}\mathbf{g}, V)$ encodes the Taylor coefficients of the scalar function on AdS; for general V a similar

[5] The $\xi \in \mathcal{U}\mathbf{g}$ is called *group element* if it is of the form $\xi = \mathrm{e}^x$ for some $x \in \mathbf{g}$.

construction shows that $\mathrm{Hom}_{\mathbf{g}_0}(\mathcal{U}\mathbf{g}, V)$ encodes the Taylor coefficients of the tensor field with indices transforming in the representation V of $\mathbf{g}_0$.

4.2 Pure spinors

To describe the linearized SUGRA in $AdS_5 \times S^5$ we need two pure spinor variables $\lambda_3 \in \mathbf{g}_3$ and $\lambda_1 \in \mathbf{g}_1$ satisfying the constraints:

$$\{\lambda_3, \lambda_3\} = \{\lambda_1, \lambda_1\} = 0. \tag{4.5}$$

We will consider various types of vertex operators, which are homogeneous polynomials in λ_3 and λ_1. Note that (4.5) are invariant under the action of $\mathbf{g}_0$. Therefore the polynomials of λ_3 and λ_1 form a representation of $\mathbf{g}_0$. We will introduce the notation for such polynomials:

Definition: We denote $\mathcal{P}^{(m,n)}$ the space of polynomials of λ_3 and λ_1, which have the degree m in λ_3 and n in λ_1.

We will define the polynomials of λ by specifying their coefficients, which are elements of $\Lambda^m \mathbf{g}_3' \otimes \Lambda^n \mathbf{g}_1'$ (see Section 1.4 for notations). We have to "discard" those polynomials that are identically zero because of the pure spinor constraints (4.5).

As a trivial example, let us consider the quadratic polynomials of λ_3. The coefficients belong to $\Lambda^2 \mathbf{g}_3'$. Let us denote t_3^α the basis vectors of $\mathbf{g}_3'$, such that:

$$\langle t_3^\alpha, t_\beta^3 \rangle = \delta_\beta^\alpha.$$

The space $\Lambda^2 \mathbf{g}_3'$ consists of expressions of the form $U_{\alpha\beta} t_3^\alpha \otimes t_3^\beta$, where $U_{\alpha\beta} = U_{\beta\alpha}$. As an example of the polynomial which is identically zero, take $f_{\alpha\beta}^\mu t_3^\alpha \otimes t_3^\beta$, where $f_{\alpha\beta}^\mu$ is the structure constants defined by $\{t_\alpha^3, t_\beta^3\} = f_{\alpha\beta}^\mu t_\mu^2$. Such a polynomial is identically zero because of the pure spinor constraint (4.5):

$$f_{\alpha\beta}^\mu \lambda_3^\alpha \lambda_3^\beta = 0.$$

To summarize

$$\mathcal{P}^{(m,n)} = (\Lambda^m \mathbf{g}_3' / (\Lambda^m \mathbf{g}_3')_{\mathrm{null}}) \otimes (\Lambda^n \mathbf{g}_1' / (\Lambda^n \mathbf{g}_1')_{\mathrm{null}}), \tag{4.6}$$

where $(\Lambda^m \mathbf{g}_3')_{\mathrm{null}}$ denotes a subspace of $\Lambda^m \mathbf{g}_3'$ corresponding to those polynomials on $\mathbf{g}_3$ which vanish identically on λ_3 because of the pure spinor constraint.

We will also introduce:

$$\mathcal{P}^l = \bigoplus_{m+n=l} \mathcal{P}^{(m,n)}. \tag{4.7}$$

Note that $\mathcal{P}^{(m,n)}$ and $\mathcal{P}^l$ are representations of $\mathbf{g}_0$, but not of $\mathbf{g}$. The construction of coinduced representation is used to build the representations of $\mathbf{g}$ from these spaces.

4.3 Vertex as an element of $\mathbf{Coind}_{\mathbf{g_0}}^{\mathbf{g}} \boldsymbol{\mathcal{P}^2}$

In this section, we consider the Taylor series of the vertex operator and do not bother about the convergence and the behaviour near the boundary. Then the vertex operator can be considered an element of the coinduced representation:

$$\mathcal{V}(\Psi) \in \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2. \tag{4.8}$$

We would like to discuss vertex operators "uniformly" for all vectors $\Psi \in \mathcal{H}$. We will therefore introduce the "universal" vertex operator:

$$\mathcal{V} \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2). \tag{4.9}$$

In other words, we have a linear function on the Hilbert space $\mathcal{H}$ which to every vector $\Psi \in \mathcal{H}$ associates the corresponding vertex operator:

$$\mathcal{V} \ : \ \Psi \mapsto \mathcal{V}(\Psi) \quad \in \quad \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2. \tag{4.10}$$

Given a state $\Psi \in \mathcal{H}$ we get $\mathcal{V}(\Psi)$ — an element of $\mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2$. This means, by definition, that for every Ψ, the object $\mathcal{V}(\Psi)$ is a linear map from $\mathcal{U}\mathbf{g}$ to $\mathcal{P}^2$ satisfying the $\mathbf{g}_0$-invariance condition:

$$\mathcal{V}(\Psi)(x\xi) = \rho(x)\ \mathcal{V}(\Psi)(\xi), \ \text{ for any } \ x \in \mathbf{g}_0,\ \xi \in \mathcal{U}\mathbf{g}. \tag{4.11}$$

Given such $\mathcal{V}(\Psi)$, how do we construct the "usual" vertex operator? As an element of $\mathrm{Coind}|_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2$ our $\mathcal{V}(\Psi)$ is a function of $\xi \in \mathcal{U}\mathbf{g}$ with values in $\mathcal{P}^2$. Let us evaluate this function on a group element $\xi = g = \mathrm{e}^x$, where $x \in \mathbf{g}$. We get $\mathcal{V}(\Psi)(g)$ — an element from $\mathcal{P}^2$, i.e., a quadratic polynomial in λ_3 and λ_1. The "usual" vertex operator is just the evaluation of this polynomial:

$$V_\Psi(g, \lambda) = \mathcal{V}(\Psi)(g)(\lambda). \tag{4.12}$$

4.4 Action of the BRST operator

The BRST complex is:

$$\dots \stackrel{Q_{\mathrm{BRST}}}{\longrightarrow} \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^n) \stackrel{Q_{\mathrm{BRST}}}{\longrightarrow} \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^{n+1}) \stackrel{Q_{\mathrm{BRST}}}{\longrightarrow} \dots . \tag{4.13}$$

The BRST operator acts on the universal vertex $\mathcal{V}(\Psi)$ in the following way:

$$(Q_{\mathrm{BRST}}\mathcal{V})(\Psi)(\xi)(\lambda) = \mathcal{V}(\Psi)(\lambda_3\xi + \lambda_1\xi)(\lambda). \tag{4.14}$$

Note that $Q_{\mathrm{BRST}}\mathcal{V}$ is an element of $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^3)$. In terms of the "usual" vertex $V_\Psi(g, \lambda)$ defined by (4.12) we get:

$$Q_{\mathrm{BRST}} V_\Psi(g, \lambda) = \left.\frac{d}{dt}\right|_{t=0} V_\Psi(\mathrm{e}^{t(\lambda_3+\lambda_1)} g, \lambda). \tag{4.15}$$

4.5 Covariant universal vertex

Statement of covariance. Note that in Equation (4.9) we use the notation $\mathrm{Hom}_{\mathbf{C}}$ rather than $\mathrm{Hom}_{\mathbf{g}}$. There is no *a priori* reason why $\mathcal{V}$ would respect the action of $\mathbf{g}$. But in the next section we will see that under some conditions on $\mathcal{H}$, it is possible to choose the universal vertex operator, which does respect the global symmetry. We will call it the *covariant* universal vertex:

$$\mathcal{V} \in \mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2). \tag{4.16}$$

Given Equation (4.2) this implies:

$$\mathcal{V}(\Psi)(\xi x) + \mathcal{V}(x\Psi)(\xi) = 0. \tag{4.17}$$

Condition on $\mathcal{H}$: sufficiently high spin. The conditions on $\mathcal{H}$ are the following. Consider $\mathcal{H}$ as a representation of $so(6) \subset \mathbf{g}$ — the symmetry algebra of S^5. As a representation of $so(6)$, $\mathcal{H}$ is the direct sum of infinitely many finite-dimensional representations of $so(6)$. We request that the minimal value of the quadratic Casimir of $so(6)$ on $\mathcal{H}$ be sufficiently high.

5 Existence of the covariant vertex

In this section, we will use some facts about the Lie algebra cohomology, which we learned mostly from [16–18]. See Chapter 3 Section 6 of [19] for a very brief summary.

5.1 Brief summary

The physical states correspond to the cohomology of Q_{BRST} at the ghost number 2, therefore:

$$\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, H^2(Q_{\mathrm{BRST}}, \mathrm{Coind}|^{\mathbf{g}}_{\mathbf{g}_0}\mathcal{P}^\bullet)) = \mathbf{C}^{d(\mathcal{H})}, \tag{5.1}$$

where $d(\mathcal{H})$ is the multiplicity of $\mathcal{H}$ (how many times $\mathcal{H}$ enters in the SUGRA spectrum on $AdS_5 \times S^5$). We will argue that the second cohomology of the BRST operator can be calculated using the covariant subcomplex. In other words,

$$\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, H^2(Q_{\mathrm{BRST}}, \mathrm{Coind}|^{\mathbf{g}}_{\mathbf{g}_0}\mathcal{P}^\bullet)) = H^2(Q_{\mathrm{BRST}}, \mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}^{\mathbf{g}}_{\mathbf{g}_0}\mathcal{P}^\bullet)) \tag{5.2}$$

(Notice that $\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}^{\mathbf{g}}_{\mathbf{g}_0}\mathcal{P}^\bullet)$ is the covariant subcomplex.) To prove Eq. (5.2) we rewrite it in the following form:

$$\begin{aligned} &H^0(\mathbf{g},\ \mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\ H^2(Q_{\mathrm{BRST}}, \mathrm{Coind}|^{\mathbf{g}}_{\mathbf{g}_0}\mathcal{P}^\bullet))) \\ &\quad = H^2(Q_{\mathrm{BRST}},\ H^0(\mathbf{g},\ \mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\ \mathrm{Coind}^{\mathbf{g}}_{\mathbf{g}_0}\mathcal{P}^\bullet))). \end{aligned} \tag{5.3}$$

Here we have used the fact that for any representation L of the Lie algebra $\mathbf{g}$ the zeroth cohomology group $H^0(\mathbf{g}, L)$ equals the space of invariants $\mathrm{Inv}_{\mathbf{g}} L$. In particular, for two representations $\mathcal{A}$ and $\mathcal{B}$, $H^0(\mathbf{g},\ \mathrm{Hom}_{\mathbf{C}}(\mathcal{A}, \mathcal{B})) = \mathrm{Hom}_{\mathbf{g}}(\mathcal{A}, \mathcal{B})$.

The idea of the proof of (5.3) is to note that the left and the right-hand side of (5.3) are two different second approximations to calculating the cohomology of the "total" differential $Q_{\mathrm{BRST}} + Q_{\mathrm{Lie}}$. Therefore the equality of the left-hand side and the right-hand side follows if we prove that the second approximation is actually exact. To prove that we will need several vanishing theorems. These vanishing theorems essentially follow from the fact that as a representation of $so(6)$ (the rotations of S^5) $\mathcal{H}$ is a direct sum of finite-dimensional representations. This can be seen from the explicit description of the supergravity solutions in [20].

5.2 Bicomplex and spectral sequence

Let us start by fixing some universal vertex (not necessarily covariant):

$$\mathcal{V}\ :\ \mathcal{H} \to \mathrm{Coind}^{\mathbf{g}}_{\mathbf{g}_0}\mathcal{P}^2. \tag{5.4}$$

At this point we do not require that this vertex is covariant; it is *a priori* an element of $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2)$ rather than $\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2)$. We will introduce the Lie algebra BRST operator of $\mathbf{g}$. For each generator t_i of $\mathbf{g}$ we introduce the corresponding ghost c^i, and define:

$$Q_{\mathrm{Lie}} = c^i t_i - \frac{1}{2} f_{ij}^k c^i c^j \frac{\partial}{\partial c^k}. \tag{5.5}$$

We will consider the action of Q_{Lie} on expressions polynomial in c^i. The polynomials of c^i are specified by their coefficients; in degree l the coefficients live in $\Lambda^l \mathbf{g}^*$. Therefore Q_{Lie} acts on the vertex operator as follows:

$$\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2) \xrightarrow{Q_{\mathrm{Lie}}} \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^2) \otimes \mathbf{g}. \tag{5.6}$$

We will consider the bicomplex with the differential Q_{tot}:

$$Q_{\mathrm{tot}} = Q_{\mathrm{BRST}} + Q_{\mathrm{Lie}}. \tag{5.7}$$

To prove the existence of the covariant vertex we will consider the spectral sequence computing the cohomology of this bicomplex. There are two ways to construct the spectral sequence. One can first calculate the cohomology of Q_{BRST} and then consider Q_{Lie} as a perturbation. The other way is to first calculate the cohomology of Q_{Lie} and then consider Q_{BRST} as a perturbation. These two ways of calculating the cohomology of Q_{tot} should give the same result. We will see that this implies the existence of the covariant vertex. We will now consider the two methods in turn.

First Q_{BRST} then Q_{Lie} The first term of the spectral sequence has:

$$E_1^{p,q} = H^q_{Q_{\mathrm{BRST}}}(\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^\bullet)) \otimes \Lambda^p \mathbf{g}', \quad d_1 : \; E_1^{p,q} \to E_1^{p+1,q}, \tag{5.8}$$

where $\Lambda^p \mathbf{g}'$ stands for the c-ghosts; an element of $E_1^{p,q}$ is schematically $\lambda^q c^p$. The differential in the first term is $d_1 = Q_{\mathrm{Lie}}$. The second term is:

$$E_2^{p,q} = H^p(\mathbf{g}, \; H^q_{Q_{\mathrm{BRST}}}(\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^\bullet))), \quad d_2 : \; E_2^{p,q} \to E_2^{p+2,q-1} \tag{5.9}$$

The higher differentials are of the type $d_r : \; E_r^{p,q} \to E_r^{p+r,q+1-r}$.

First Q_{Lie} then Q_{BRST} The first term is:

$$\tilde{E}_1^{p,q} = H^p(\mathbf{g}, \; \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^q)), \tilde{d}_1 : \tilde{E}_1^{p,q} \to \tilde{E}_1^{p,q+1}, \tag{5.10}$$

where $\tilde{d}_1 = Q_{\mathrm{BRST}}$. The higher differentials are of the type $\tilde{d}_r : \; \tilde{E}_r^{p,q} \to \tilde{E}_r^{p+1-r,q+r}$.

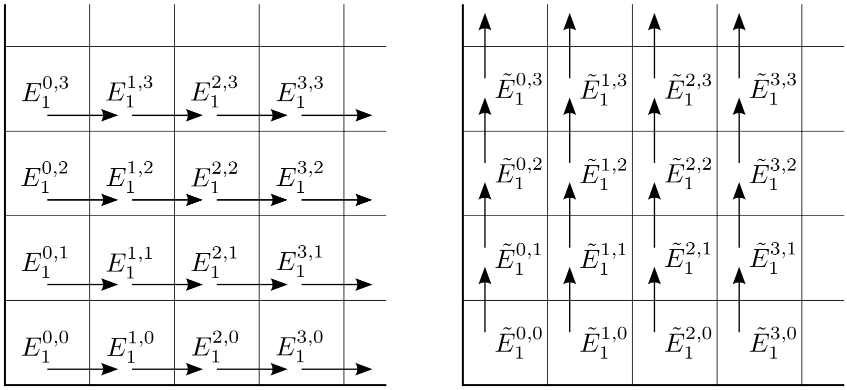

Figure 1: The first page of E and $\tilde{E}$; arrows denote d_1 and $\tilde{d}_1$. The Lie algebra ghost number (the number of c's) increases in the horizontal direction, while the BRST ghost number (the number of λ_3 plus the number of λ_1) in the vertical direction.

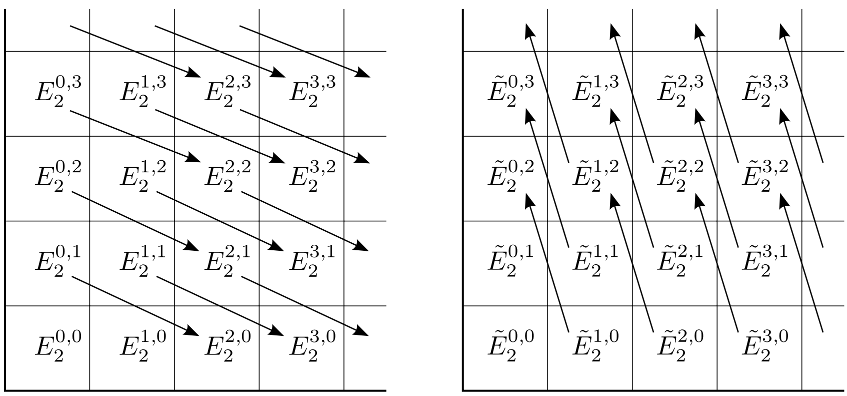

Figure 2: The second page of E and $\tilde{E}$; arrows denote d_2 and $\tilde{d}_2$.

Existence of the covariant vertex. First of all, we want to show that $E_\infty^{0,2} = E_1^{0,2}$. The first observation is that by definition $d_1 : E_1^{0,2} \to E_1^{1,2}$ is zero. This is because the vertex is *covariant up to BRST-exact correction* (see equation (3.17)). Therefore $E_1^{0,2} = E_2^{0,2}$. Also, we will show (for $\mathcal{H}$ with large-enough spin) that $E_2^{2,1} = E_2^{3,0} = 0$. This implies that $E_\infty^{0,2} = E_1^{0,2}$.

Then we remember the relation between $E_\infty^{0,2}$ and $H^2(Q_{\rm tot})$, which is the following. The space $H^2(Q_{\rm tot}) = E^2$ has a filtration, corresponding to the number of the c-ghosts. Namely, $F^p E^2$ consists of expressions containing at least p c-ghosts. Then $E_\infty^{0,2} = E^2/F^1E^2 = H^2(Q_{\rm tot})/F^1H^2(Q_{\rm tot})$.

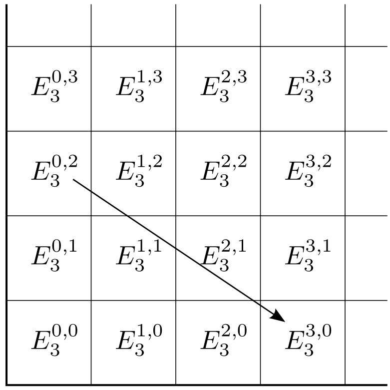

Figure 3: The differential d_3 of E_3.

To summarize:

$$E_1^{0,2} = [\text{unintegrated vertices}], \tag{5.11}$$

$$E_1^{0,2} = H^2(Q_{\text{tot}})/F^1 H^2(Q_{\text{tot}}). \tag{5.12}$$

On the other hand, we will show that $\tilde{E}_1^{1,1} = \tilde{E}_1^{2,0} = 0$ (see equation (5.25), (5.28)) and also that $\tilde{E}_1^{1,0} = 0$ (similar to (5.25)). This implies that:

$$H^2(Q_{\text{tot}}) = \frac{\text{Ker } \tilde{d}_1 : \ \tilde{E}_1^{0,2} \to \tilde{E}_1^{0,3}}{\text{Im } \tilde{d}_1 : \ \tilde{E}_1^{0,1} \to \tilde{E}_1^{0,2}}. \tag{5.13}$$

We are now ready to prove the existence of the covariant vertex. Notice that $\tilde{E}_1^{0,q} = H^0(\mathbf{g}, \text{Hom}_{\mathbf{C}}(\mathcal{H}, \text{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^q))$ is the space of functions $f_\Psi(x, \theta, \lambda)$, parametrized by $\Psi \in \mathcal{H}$, transforming covariantly under $\mathbf{g}$. This means that $\tilde{E}_1^{0,\bullet}$ is the *covariant subcomplex* of the BRST complex. (The subspace consisting of the covariant expressions.) Equation (5.13) shows that:

$$H^2(Q_{\text{tot}}) \text{ is the second cohomology of the covariant subcomplex.} \tag{5.14}$$

Now the comparison of (5.11), (5.12) and (5.14) shows that the cohomology of Q_{BRST} can be calculated using the covariant subcomplex. In fact, if the representation $\mathcal{H}$ has large-enough momentum in S^5, then $F^1 E^2$ is zero (because already $E_2^{1,1}$ and $E_2^{2,0}$ are zero). This means that the factor space on the right-hand side of (5.12) is just $H^2(Q_{\text{tot}})$.

This means that there is a covariant choice of the vertex. In the rest of this section, we will prove the required vanishing theorems and explain explicitly how the noncovariant vertex can be modified into the covariant one.

Gauge transformations. It is also true that $\tilde{E}_1^{1,0} = 0$, and therefore $\tilde{E}_2^{1,0} = 0$. This is proven similarly to (5.25). This implies that $\tilde{d}_2 : \tilde{E}_2^{1,0} \to \tilde{E}_2^{0,2}$ is zero. This means that when considering the gauge transformations of the covariant vertices it is enough to consider the gauge transformations with the covariant parameters; if a covariant vertex is BRST trivial, then it is a BRST variation of a covariant expression.

5.3 The descent

Since the vertex transforms covariantly up to BRST-exact terms, we must have:

$$Q_{\mathrm{Lie}}\mathcal{V} = Q_{\mathrm{BRST}}\mathcal{W}, \tag{5.15}$$

where

$$\mathcal{W} \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}}\mathcal{P}^1) \otimes \mathbf{g}. \tag{5.16}$$

Note that $Q_{\mathrm{Lie}}\mathcal{W}$ is Q_{BRST}-closed and has ghost number 1:

$$Q_{\mathrm{Lie}}\mathcal{W} \in \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}}\mathcal{P}^1) \otimes \Lambda^2\mathbf{g}. \tag{5.17}$$

We want to argue that there exists $\mathcal{U}$ such that $Q_{\mathrm{Lie}}\mathcal{W} = Q_{\mathrm{BRST}}\mathcal{U}$. More precisely: note that we are free to add to $\mathcal{W}$ something in the kernel of Q_{BRST}; we want to prove that it is possible to use this freedom and choose $\mathcal{W}$ so that there exists $\mathcal{U}$ such that $Q_{\mathrm{Lie}}\mathcal{W} = Q_{\mathrm{BRST}}\mathcal{U}$. An obstacle to this would be a nonzero $d_2\mathcal{V}$ where

$$\begin{aligned} d_2 : H^0(\mathbf{g},\ & H^2_{Q_{\mathrm{BRST}}}(\mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\ \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}}\mathcal{P}^\bullet))) \\ & \to H^2(\mathbf{g},\ H^1_{Q_{\mathrm{BRST}}}(\mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\ \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}}\mathcal{P}^\bullet))). \end{aligned} \tag{5.18}$$

We want to argue that the space $H^2(\mathbf{g},\ H^1_{Q_{\mathrm{BRST}}}(\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}}\mathbf{C})))$ on the right-hand side is zero. Note that the BRST cohomology in ghost number 1 corresponds to local conserved charges. But the only conserved charges are the global symmetries $psu(2,2|4)$, and those transform in the adjoint representation[6] of $\mathbf{g}$. This means that on the right-hand side of (5.18) we have:

$$H^2(\mathbf{g},\ \mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\mathbf{g})). \tag{5.19}$$

This cohomology group is zero. Indeed, we can compute it using the Serre–Hochschild spectral sequence of $\mathbf{g}_{\mathrm{even}} \subset \mathbf{g}$. Already the first term of this spectral sequence consists of the following spaces, which are all zero:

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{g}_{\mathrm{even}}}(\Lambda^2\mathbf{g}_{\mathrm{odd}},\ \mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\mathbf{g})),\quad H^1(\mathbf{g}_{\mathrm{even}},\ \mathrm{Hom}_{\mathbf{C}}(\mathbf{g}_{\mathrm{odd}}\otimes\mathcal{H},\mathbf{g})),\\ &H^2(\mathbf{g}_{\mathrm{even}},\mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\mathbf{g})).\end{aligned} \tag{5.20}$$

Note that $\mathbf{g}_{\mathrm{even}} = \mathbf{g}_A \oplus \mathbf{g}_S$ where $\mathbf{g}_A = so(2,4)$ and $\mathbf{g}_S = so(6)$. Consider the corresponding Casimir operators Δ_A and Δ_S. For the cohomology to be nonzero, we need both of them zero, but Δ_S is positive definite at least for $\mathcal{H}$ with large enough momenta. (Note also that $\mathcal{H}$ is an infinite-dimensional irreducible representation of $\mathbf{g}$, so there are no invariants in its tensor product with powers of $\mathbf{g}$.)

Therefore $Q_{\mathrm{Lie}}\mathcal{W} = Q_{\mathrm{BRST}}\mathcal{U}$ for some $\mathcal{U}$. In other words $d_2\mathcal{V}$ is zero, and we can proceed with computing d_3.

Consider $\mathcal{Z} = Q_{\mathrm{Lie}}\mathcal{U}$. Note that $\mathcal{Z}$ has zero pure spinor ghost number, and $Q_{\mathrm{BRST}}\mathcal{Z} = 0$. Since $\mathcal{Z}$ is of ghost number 0, this implies that $\mathcal{Z}$ is a constant; it does not contain any x or θ. Also, we could have added a constant to $\mathcal{U}$ without affecting $Q_{\mathrm{BRST}}\mathcal{U}$; therefore $\mathcal{Z}$ by itself is not very well defined by our construction. What is well defined is $\mathcal{Z}$ modulo the image of Q_{Lie}:

$$[\mathcal{Z}] \in H^3_{Q_{\mathrm{Lie}}}(\mathcal{H}'\otimes\Lambda^{\bullet}\mathbf{g}) = H^3(\mathbf{g},\mathcal{H}'). \tag{5.21}$$

5.4 The ascent

But the Lie algebra cohomology group $H^3(\mathbf{g},\mathcal{H}')$ is zero:

$$H^3(\mathbf{g},\mathcal{H}') = 0. \tag{5.22}$$

One can see that it is zero from the Serre–Hochschild spectral sequence corresponding to $\mathbf{g}_{\mathrm{even}} \subset \mathbf{g}$. Already the first term of this spectral sequence

[6] We calculate the *covariant* cohomology of $H^1(Q_{\mathrm{BRST}}, \mathrm{Coind}^{\mathbf{g}}_{\mathbf{g}_0}\mathbf{C})$ at the ghost number 1 in Appendix A. However we did not calculate this cohomology without the assumption of covariance. But we know from physics that every cohomology class at the ghost number 1 corresponds to a local conserved current in the pure spinor sigma model. And we know the classification of the local conserved currents, they transform in the adjoint of $\mathbf{g}$.

consists of the following spaces, which are all zero:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{g}_{\mathrm{even}}}(\Lambda^3\mathbf{g}_{\mathrm{odd}},\mathcal{H}') &= H^1(\mathbf{g}_{\mathrm{even}},\mathrm{Hom}_{\mathbf{C}}(\Lambda^2\mathbf{g}_{\mathrm{odd}},\mathcal{H}'))\\ &= H^2(\mathbf{g}_{\mathrm{even}},\mathrm{Hom}_{\mathbf{C}}(\mathbf{g}_{\mathrm{odd}},\mathcal{H}')) = H^3(\mathbf{g}_{\mathrm{even}},\mathcal{H}') = 0. \qquad (5.23)\end{aligned}$$

The vanishing of these cohomologies can be proven as follows. Note that $\mathbf{g}_{\mathrm{even}}$ splits into $\mathbf{g}_A = so(2,4)$ and $\mathbf{g}_S = so(6)$. For the cohomology to be nontrivial, both Δ_A and Δ_S should be zero. But $-\Delta_S$ is positive definite.

Therefore we can remove $\mathcal{Z}$ by modifying $\mathcal{U}$, adding to $\mathcal{U}$ a constant term $-\Delta\mathcal{U}$ so that the modified $\mathcal{U}-\Delta\mathcal{U}$ has $Q_{\mathrm{Lie}}(\mathcal{U}-\Delta\mathcal{U}) = 0$. (Note that adding the constant term does not change the image of $\mathcal{U}$ under Q_{BRST}.) Is it possible to find such $\mathcal{U}'$ that $\mathcal{U}-\Delta\mathcal{U} = Q_{\mathrm{Lie}}\mathcal{U}'$? The answer is "yes", because

$$H^2(\mathbf{g},\mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}}\mathbf{C})) = 0. \qquad (5.24)$$

This can be proven using the Shapiro's lemma (Proposition 6.8 and Theorem 6.9 from [16]; see Appendix B for a review):

$$\begin{aligned}&H^2(\mathbf{g},\mathrm{Hom}_{\mathbf{C}}(\mathcal{H},\mathrm{Coind}|_{\mathbf{g}_0}^{\mathbf{g}}\mathbf{C}))\\ &\quad= \mathrm{Ext}^2_{\mathbf{g}}(\mathcal{H},\mathrm{Coind}|_{\mathbf{g}_0}^{\mathbf{g}}\mathbf{C})\\ &\quad= \mathrm{Ext}^2_{\mathbf{g}_0}(\mathcal{H}|_{\mathbf{g}_0},\mathbf{C})\\ &\quad= H^2(\mathbf{g}_0,\mathcal{H}'|_{\mathbf{g}_0}). \qquad (5.25)\end{aligned}$$

Note that $\mathbf{g}_0 = so(1,4)\oplus so(5)$. We want to prove that $H^2(\mathbf{g}_0,\mathcal{H}'|_{\mathbf{g}_0}) = 0$. The space $\mathcal{H}'$ consists of functionals on the space of states. Since we work in the vicinity of the fixed point $x_0 \in AdS_5\times S^5$ our $\mathcal{H}'$ is generated by the values of various supergravity fields at the point x_0. For example, the Ramond–Ramond field strength $H_{ijk}(x_0)$ and its derivatives. Under the action of $so(1,4)\oplus so(5)$ this space splits into infinitely many finite-dimensional representations. For example, $\partial_i H_{kjl}(x_0)$ lives in $(\mathrm{Vect}\otimes\Lambda^3\mathrm{Vect})_0$, where Vect is is the vector representation of $so(1,4)\oplus so(5)$ and index 0 means that the contraction $g^{ij}\partial_i H_{jkl}$ is zero[7] . It follows from the general theory of Lie algebra cohomology that H^2 of $so(1,4)\oplus so(5)$ with coefficients in any finite-dimensional representation is zero.

[7]This is an over-simplification; in fact one has to add also the expression of the form $g^{pp'}g^{qq'}F_{iklpq}H_{jp'q'}$, for the contraction to be zero; the Ramond–Ramond 5-form F_{iklpq} is nonzero in the AdS background.

These arguments imply that $\mathcal{U}'$ is in the image of Q_{Lie}. We can modify $\mathcal{W}$ by adding to it:

$$\Delta\mathcal{W} = Q_{\mathrm{BRST}} Q_{\mathrm{Lie}}^{-1} \mathcal{U}'. \tag{5.26}$$

Then we have:

$$Q_{\mathrm{Lie}}(\mathcal{W} + \Delta\mathcal{W}) = 0. \tag{5.27}$$

Now we use:

$$\begin{aligned} &H^1(\mathbf{g}, \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathbf{g}_{\mathrm{odd}})) \\ &\quad = H^1(\mathbf{g}_0,\ \mathcal{H}'|_{\mathbf{g}_0} \otimes_{\mathbf{C}} \mathbf{g}_{\mathrm{odd}}) = 0. \end{aligned} \tag{5.28}$$

Therefore $\mathcal{W} + \Delta\mathcal{W}$ is in the image of Q_{Lie}.

Now the *modified vertex:*

$$\mathcal{V} + Q_{\mathrm{BRST}} Q_{\mathrm{Lie}}^{-1}(\mathcal{W} + \Delta\mathcal{W}) \tag{5.29}$$

is covariant.

Our procedure could perhaps be summarized as follows:

$$\begin{aligned} \mathcal{V}_{\mathrm{covariant}} &= \mathcal{V} + Q_{\mathrm{BRST}} Q_{\mathrm{Lie}}^{-1}(\mathcal{W} + Q_{\mathrm{BRST}} Q_{\mathrm{Lie}}^{-1}(\mathcal{U} - Q_{\mathrm{Lie}}^{-1}\mathcal{Z})), \\ &\quad \text{where} \\ \mathcal{W} &= Q_{\mathrm{BRST}}^{-1} Q_{\mathrm{Lie}} \mathcal{V}, \\ \mathcal{U} &= Q_{\mathrm{BRST}}^{-1} Q_{\mathrm{Lie}} \mathcal{W}, \\ \mathcal{Z} &= Q_{\mathrm{Lie}} \mathcal{U} \text{ (does not depend on } x, \theta). \end{aligned} \tag{5.30}$$

6 How the descent procedure works in the flat space limit

In flat space it is impossible to choose a covariant vertex, because of the nontrivial cohomology

$$\mathcal{Z} \in H^3(\text{super-Poincare algebra}, \mathbf{C}), \tag{6.1}$$

which represents the NSNS 3-form field strength.

But one can satisfy a weaker covariance condition. Note that in flat space the generators of the Lorentz subalgebra $so(1,9)$ of the Poincare algebra can not be obtained as commutators of other generators. Therefore it is consistent to require the covariance under all translations and supersymmetries,

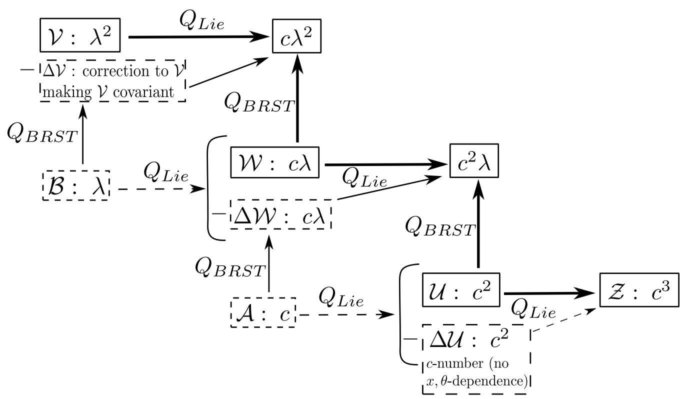

Figure 4: Adjustment of the vertex operator.

but only some rotations. In particular, it turns out that we can choose a vertex covariant under:

$$\mathbf{SP}_{\text{small}} = \{\text{translations, supersymmetries, and } so(1,4) \oplus so(5) \subset so(1,9)\}. \tag{6.2}$$

This is a subalgebra of the super-Poincare algebra:

$$\mathbf{SP}_{\text{small}} \subset \mathbf{SP} \tag{6.3}$$

corresponding to the split of the space-time:

$$\mathbf{R}^{1+9} = \mathbf{R}^{1+4}_A \times \mathbf{R}^5_S. \tag{6.4}$$

We will say that the ten spacetime directions split into $1+4$ A-directions and 5 S-directions (the letters A and S stand for the AdS and the sphere).

Let us now explain how the diagram of figure 4 works in flat space.

6.1 Maxwell field

Instead of considering figure 4 literally let us study the similar diagram for the supersymmetric Maxwell field (rather than supergravity). This is a toy model; the supersymmetric Maxwell field in flat space is “one half of the supergravity field”. The “usual” (noncovariant) vertex operator is of the form:

$$V(x,\theta) = (\lambda\Gamma^\mu\theta)a_\mu + (\lambda\Gamma^\mu\theta)(\psi\Gamma^\mu\theta) - \frac{1}{4}(\theta\Gamma^{\mu\nu\rho}\theta)(\lambda\Gamma^\rho\theta)\partial_{[\mu}a_{\nu]} + \cdots, \tag{6.5}$$

where $a_\mu = a_\mu(x)$ and $\psi = \psi(x)$ are the vector potential and the photino.

6.1.1 Action of the Poincare algebra

Let us first try to understand if it is possible to choose the vertex covariant under the even Poincare algebra. The vertex operator (6.5) involves the gauge field a_μ. Because of the gauge invariance the gauge field is not in one-to-one correspondence with the physical states. The physical states are described by $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, not by a_μ. To describe a_μ in terms of $f_{\mu\nu}$, let us break the translational symmetries by choosing a point 0 in spacetime. Then we can write, in the vicinity of the chosen point:

$$a = \iota_E \frac{1}{\mathcal{L}_E} f, \tag{6.6}$$

where $E = x^\mu \frac{\partial}{\partial x^\mu}$ and ι_E and $\mathcal{L}_E$ are the corresponding ι and Lie derivative. For example, $\mathcal{L}_E(xdx) = 2xdx$ and $\iota_E dx = x$. Note that equation (6.6) is one particular way to choose a vector potential with the field strength f.

Let us therefore replace a with $\iota_E \frac{1}{\mathcal{L}_E} f$. This breaks the translation symmetry, since the gauge (6.6) depended on a choice of point $x = 0$. Can we restore the translational symmetry? Let us introduce the operator Q_{Lie} acting on the physical vertex operators in the following way:

$$Q_{\mathrm{Lie}} V^I = c^\mu (t_\mu . V^I - t^I_{\mu J} V^J). \tag{6.7}$$

Here the index I runs over an infinite set enumerating the basis vectors of $\mathcal{H}$, and t_μ are the generators of translations $\frac{\partial}{\partial x^\mu}$. Fermionic parameters c^μ are the Lie-algebraic ghosts of the translation algebra. This operator Q_{Lie} measures the deviation of the vertex operator from transforming covariantly under the action of the global shift. We observe that $Q_{\mathrm{Lie}} V$ is d of something:

$$\begin{aligned} Q_{\mathrm{Lie}} \left(\iota_E \frac{1}{\mathcal{L}_E} f \right) &= \left[\mathcal{L}_c,\ \iota_E \frac{1}{\mathcal{L}_E} \right] f = \left(\iota_c \frac{1}{\mathcal{L}_E} - \iota_E \frac{1}{\mathcal{L}_E} \mathcal{L}_c \frac{1}{\mathcal{L}_E} \right) f \\ &= d \left(\frac{1}{\mathcal{L}_E(\mathcal{L}_E + 1)} \iota_E \iota_c f \right). \end{aligned} \tag{6.8}$$

Let us calculate $Q_{\rm Lie}$ of this "something":

$$\begin{aligned} Q_{\rm Lie}\left(\frac{1}{\mathcal{L}_E(\mathcal{L}_E+1)}\iota_E\iota_c f\right) &= \left\{\mathcal{L}_c,\ \frac{1}{\mathcal{L}_E(\mathcal{L}_E+1)}\iota_E\iota_c\right\} f \\ &= \frac{1}{(\mathcal{L}_E+1)(\mathcal{L}_E+2)}\mathcal{L}_c\ \iota_E\ \iota_c\ f + \frac{1}{\mathcal{L}_E(\mathcal{L}_E+1)}\ \iota_E\ \iota_c\ \mathcal{L}_c\ f \\ &= \frac{1}{(\mathcal{L}_E+1)(\mathcal{L}_E+2)}\iota_c^2\ f + \frac{2}{\mathcal{L}_E(\mathcal{L}_E+1)(\mathcal{L}_E+2)}\iota_E\iota_c\mathcal{L}_c f. \end{aligned} \tag{6.9}$$

Expanding f in Taylor series around $x=0$ and taking into account that $df=0$, we can see that (6.9) is equal to:

$$\frac{1}{2}\iota_c^2 f(0). \tag{6.10}$$

We should stress that (6.9) is equal to the *constant* (independent of x) expression (6.10). In other words, the only term in the Taylor expansion of (6.9) around the point $x=0$ is the constant term. One can see it, for example, because the Lie derivative $\mathcal{L}_E$ of (6.9) vanishes. This can be seen from the identity $\mathcal{L}_E\iota_c^2 f + 2\iota_E\iota_c\mathcal{L}_c f = 0$, which follows from $df=0$.

Note that if we started with some other point x_0 (not the origin), then (6.10) would change by Q_{Lie} of something. For example, an infinitesimal shift by y would change (6.10) by the $Q_{\rm Lie}$-exact expression:

$$\frac{1}{2}\iota_c^2 y^\rho\partial_\rho f(0) = y^\rho c^\mu c^\nu \partial_\rho f_{\mu\nu}(0) = -c^\mu c^\nu\partial_\mu f_{\nu\rho}y^\rho = Q_{\rm Lie}(f_{\nu\rho}c^\nu y^\rho) \tag{6.11}$$

(This is a manifestation of the general fact, that a Lie algebra acts trivially in its cohomology.)

The $Q_{\rm Lie}$-cohomology class of:

$$\iota_c^2 f(0) = f_{\mu\nu}(0)c^\mu c^\nu \tag{6.12}$$

is the obstacle for defining a such that $f=da$ in a covariant way.

We have so far discussed only the action of shifts. The expression (6.12) as we defined it represents the cohomology class of the algebra of translations $\mathbf{R}^{1+9}$. But we can also think of it as a cocycle of the Poincare algebra. Indeed, $f_{\mu\nu}$ transforms covariantly under rotations and boosts and therefore (6.12) is closed under the Q_{Lie} of the full Poincare algebra.

The $Q_{\rm Lie}$ of the full Poincare algebra is the sum of $Q_{\rm Lie}^{\rm translations}$ of translations $\mathbf{R}^{1+9}$ and $Q_{\rm Lie}^{\rm Lorentz}$ of rotations and boosts. Expression (6.12) is in the kernel of $Q_{\rm Lie}^{\rm translations}$ by our construction, and more explicitly because f is a closed form. But it is also in the kernel of $Q_{\rm Lie}^{\rm Lorentz}$ because f transforms covariantly under rotations and boosts.

Another question is whether or not (6.12) is exact. One can see that this is not exact as a cocycle of the full Poincare algebra, in the following way. Let $\mathbf{P}$ stand for the Poincare algebra. We have:

$$f_{\mu\nu}(0)c^\mu c^\nu \in H^2(\mathbf{P}, \mathcal{H}'). \tag{6.13}$$

Note that the space of states of the gauge field contains a proper subspace closed under the action of the Poincare algebra. (In other words, it is not an irreducible representation.) This subspace consists of those gauge fields which have a constant field strength: $f_{\mu\nu}(x) = f_{\mu\nu}(0)$. Let us call this subspace $\mathcal{H}_{\text{zero-modes}}$:

$$\mathcal{H}_{\text{zero-modes}} \subset \mathcal{H} \tag{6.14}$$

Therefore there is a projection

$$\mathcal{H}' \to (\mathcal{H}_{\text{zero-modes}})'. \tag{6.15}$$

This projection naturally acts on the cocycles of $\mathbf{P}$ with values in $\mathcal{H}'$, and therefore on the cohomology groups:

$$\begin{aligned} &\mathcal{H}' \otimes \Lambda^\bullet \mathbf{P} \to (\mathcal{H}_{\text{zero-modes}})' \otimes \Lambda^\bullet \mathbf{P}, \\ &H^\bullet(\mathbf{P}, \mathcal{H}') \to H^\bullet(\mathbf{P}, (\mathcal{H}_{\text{zero-modes}})'). \end{aligned} \tag{6.16}$$

It is straightforward to see that the projection of (6.12) to $(\mathcal{H}_{\text{zero-modes}})' \otimes \Lambda^\bullet \mathbf{P}$ is automatically a nonzero cohomology class.

Indeed $\mathcal{H}'_{\text{zero-modes}}$ transforms as antisymmetric rank 2 tensor of the Lorentz algebra, and trivially under translations. Therefore the cohomology complex of the Poincare algebra with coefficients in $\mathcal{H}'_{\text{zero-modes}}$ is equivalent to the cohomology complex of the Lorentz algebra with coefficients in $\Lambda^2 \mathbf{R}^{1+9} \otimes \Lambda^\bullet \mathbf{R}^{1+9}$; the projection of (6.12) is in $H^0(\text{Lorentz}, (\Lambda^2\mathbf{R}^{1+9} \otimes \Lambda^2\mathbf{R}^{1+9})_{\text{inv}})$.

This implies that (6.12) represents a nontrivial cohomology class in $H^2(\mathbf{P}, \mathcal{H}')$. This is what prevents us from choosing the vertex covariant with respect to the Poincare algebra.

6.1.2 The obstacle (6.12) vanishes after we break P to $\mathbf{P}_{\mathbf{small}}$

Let us start with introducing some notations. For any vector v^μ we denote $\overline{v}$ the vector with the components:

$$\overline{v}^\mu = \begin{cases} v^\mu \text{ if } \mu \in \{0,1,\ldots,4\}, \\ -v^\mu \text{ if } \mu \in \{5,\ldots,9\}. \end{cases} \tag{6.17}$$

Also introduce:

$$2\Delta_S = \partial_\mu \overline{\partial}_\mu - \partial_\mu \partial_\mu = 2 \sum_{i\in\{5,\ldots,9\}} \left(\frac{\partial}{\partial x^i}\right)^2 . \tag{6.18}$$

What happens if we do not require the invariance under the full Poincare algebra $\mathbf{P}$, but only under the $\mathbf{P}_{\text{small}}$ of (6.2)? Then we can restrict ourselves to the subspace of $\mathcal{H}$ where $-\Delta_S$ is a fixed positive number. On this subspace, it is possible to express a_μ in terms of $f_{\mu\nu}$ in a $\mathcal{P}_{\text{small}}$-covariant way. Let us choose the covariant gauge:

$$\partial^\mu a_\mu = 0 \tag{6.19}$$

and fix the residual gauge transformations with the additional "axial" gauge gauge condition:

$$\overline{\partial}^\mu a_\mu = 0, \tag{6.20}$$

where $\overline{\partial}$ is introduced as in (6.17). In the gauge (6.20) we can express the gauge field a_μ in terms of the gauge field strength $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$:

$$a_\mu = \frac{1}{2\Delta_S} \overline{\partial}_\nu f_{\mu\nu}. \tag{6.21}$$

Now we have two different expressions for the vector potential, equations (6.6) and (6.21). The difference between these two expressions is a gauge transformation. Let us use a "diacritical" mark to distinguish (6.21) from (6.6):

$$\acute{a}_\mu = \frac{1}{2\Delta_S} \overline{\partial}_\nu f_{\mu\nu} \text{ versus } a_\mu = \left(\iota_E \frac{1}{\mathcal{L}_E} f\right)_\mu . \tag{6.22}$$

Note that $\acute{a}$ is $\mathbf{P}_{\text{small}}$-covariant:

$$Q_{\text{Lie}} \acute{a} = 0, \tag{6.23}$$

while a is not — see equation (6.8). Also, for every gauge field we can calculate $\acute{a}(0)$. This is a functional of the gauge field, i.e., an element of $\mathcal{H}'$.

If we calculate its Q_{Lie} as a cochain with values in $\mathcal{H}'$ we get:

$$Q_{\text{Lie}}(\acute{a}_\mu(0)) = -(\mathcal{L}_c \acute{a}_\mu)(0). \tag{6.24}$$

Now let us return to equations (6.8), (6.9) and (6.10). On the subspace $-\Delta_S = \text{const} > 0$ the cohomology class of (6.10) trivializes:

$$c^\mu c^\nu f_{\mu\nu}(0) = Q_{Lie}(\iota_c \acute{a}(0)). \tag{6.25}$$

This is analogous to equation (5.22) of Section 5.3. Therefore the same arguments as we presented in Section 5.3 should imply that

$$\frac{1}{\mathcal{L}_E(\mathcal{L}_E+1)} \iota_E \iota_c f(x) - \iota_c \acute{a}(0) = Q_{\text{Lie}} \text{ (something)}. \tag{6.26}$$

Indeed we have:

$$\frac{1}{\mathcal{L}_E(\mathcal{L}_E+1)} \iota_E \iota_c f(x) - \iota_c \acute{a}(0) = Q_{Lie}\left(-\iota_E\left(\acute{a}(0) + \frac{1}{\mathcal{L}_E}(\acute{a} - \acute{a}(0))\right)\right). \tag{6.27}$$

Also note that:

$$d\left(\iota_E\left(\acute{a}(0) + \frac{1}{\mathcal{L}_E}(\acute{a} - \acute{a}(0))\right)\right) = \acute{a} - a. \tag{6.28}$$

This means that the correction of the vector potential:

$$a \to \acute{a} \tag{6.29}$$

is completely analogous to the correction of the vertex operator described in Section 5.3. It turns the non-covariant expression $a_\mu dx^\mu$ into the covariant expression $\acute{a}_\mu dx^\mu$.

6.1.3 Action of the supersymmetry

Let us now study the action of the supersymmetry generators. Let us consider the part of the Q_{Lie} involving the supersymmetry generators. We have:

$$Q_{\text{Lie}} = Q_{\text{Lie}}^{(x,\theta)} + Q_{\text{Lie}}^{\mathcal{H}}, \tag{6.30}$$

$$Q_{\text{Lie}}^{(x,\theta)} = \xi^\alpha \frac{\partial}{\partial \theta^\alpha} - \xi^\alpha \Gamma^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu}, \tag{6.31}$$

$$Q^{\mathcal{H}}_{\mathrm{Lie}} = \xi^\alpha t_\alpha \tag{6.32}$$

$$Q_{\mathrm{BRST}} = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha} + \lambda^\alpha \Gamma^\mu_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^\mu}. \tag{6.33}$$

Here t_α is the supersymmetry transformation acting on the space of states $\mathcal{H}$ and therefore (after fixing the gauge!) on a_μ and ψ. We write only the part of Q_{Lie} corresponding to the super-translations; ξ^α are the bosonic ghosts corresponding to the super-translations. To make t_α act on a and ψ, we have to choose the gauge.

With this notation, let us first of all present $Q^{(x,\theta)}_{\mathrm{Lie}}$ acting on the vertex operator (6.5) in the following form:

$$\begin{aligned}
\varepsilon' Q^{(x,\theta)}_{\mathrm{Lie}} \varepsilon V &= (\varepsilon\lambda\Gamma^\mu\varepsilon'\xi)a_\mu - (\varepsilon'\xi\Gamma^\nu\theta)(\varepsilon\lambda\Gamma^\mu\theta)\partial_\nu a_\mu \\
&\quad - \frac{1}{2}(\varepsilon'\xi\Gamma^{\mu\nu\rho}\theta)(\varepsilon\lambda\Gamma_\rho\theta)\partial_{[\mu}a_{\nu]} - \frac{1}{4}(\theta\Gamma^{\mu\nu\rho}\theta)(\varepsilon\lambda\Gamma_\rho\varepsilon'\xi)\partial_{[\mu}a_{\nu]} \\
&\quad + (\varepsilon\lambda\Gamma^\mu\varepsilon'\xi)(\psi\Gamma_\mu\theta) + (\varepsilon\lambda\Gamma^\mu\theta)(\psi\Gamma_\mu\varepsilon'\xi) + \cdots \\
&= -\frac{3}{2}(\varepsilon\lambda\Gamma^\mu\theta)\left((\varepsilon'\xi\Gamma_\mu\psi) + \frac{1}{2\Delta_S}\partial_\mu(\varepsilon'\xi\overline{\Gamma}^\rho\partial_\rho\psi)\right) \\
&\quad - \frac{2}{3}(\varepsilon\lambda\Gamma^\rho\theta)(\varepsilon'\xi\Gamma^{\mu\nu}\Gamma_\rho\theta)\partial_{[\mu}a_{\nu]} + \cdots \\
&\quad + \varepsilon Q_{\mathrm{BRST}}\left(\frac{1}{2}(\theta\Gamma^\mu\varepsilon'\xi)(\psi\Gamma_\mu\theta) + \frac{3}{2}\frac{1}{2\Delta_S}(\varepsilon'\xi\overline{\Gamma}^\mu\partial_\mu\psi)\right. \\
&\quad \left. + (\theta\Gamma^\mu\varepsilon'\xi)a_\mu - \frac{1}{12}(\theta\Gamma^{\mu\nu\rho}\theta)(\theta\Gamma_\rho\varepsilon'\xi)\partial_{[\mu}a_{\nu]} + \cdots\right).
\end{aligned} \tag{6.34}$$

This implies that

$$\varepsilon' Q^{\mathcal{H}}_{\mathrm{Lie}} a_\mu = -\frac{3}{2}\left((\varepsilon'\xi\Gamma_\mu\psi) + \partial_\mu\frac{1}{2\Delta_S}(\varepsilon'\xi\overline{\Gamma}^\rho\partial_\rho\psi)\right), \tag{6.35}$$

$$\varepsilon' Q^{\mathcal{H}}_{\mathrm{Lie}} \psi = -\frac{2}{3}\varepsilon'\xi\Gamma^{\mu\nu}\partial_{[\mu}a_{\nu]}. \tag{6.36}$$

Therefore we have indeed:

$$\varepsilon' Q_{\mathrm{Lie}} \varepsilon V = \varepsilon Q_{\mathrm{BRST}}\left((\theta\Gamma^\mu\varepsilon'\xi)a_\mu + \frac{3}{2}\frac{1}{2\Delta_S}(\varepsilon'\xi\overline{\Gamma}^\mu\partial_\mu\psi) + \cdots\right). \tag{6.37}$$

On the right-hand side $Q_{\rm BRST}$ is taken of the expression which is $Q_{\rm Lie}$-exact:

$$\begin{aligned}&(\theta\Gamma^{\mu}\varepsilon'\xi)a_{\mu}+\frac{3}{2}\frac{1}{2\Delta_S}(\varepsilon'\xi\overline{\Gamma}^{\mu}\partial_{\mu}\psi)+\cdots\\&\quad=\varepsilon' Q_{\rm Lie}\left(\frac{3}{2}\frac{1}{2\Delta_S}(\theta\overline{\Gamma}^{\mu}\partial_{\mu}\psi)-\frac{1}{2}\frac{1}{2\Delta_S}(\theta\Gamma^{\mu\rho\sigma}\theta)\overline{\partial}_{\mu}\partial_{\rho}a_{\sigma}+\cdots\right).\end{aligned}\tag{6.38}$$

Then we have:

$$\begin{aligned}&Q_{\rm BRST}\left(\frac{3}{2}\frac{1}{2\Delta_S}(\theta\overline{\Gamma}^{\mu}\partial_{\mu}\psi)-\frac{1}{2}\frac{1}{2\Delta_S}(\theta\Gamma^{\mu\rho\sigma}\theta)\overline{\partial}_{\mu}\partial_{\rho}a_{\sigma}+\cdots\right)\\&\quad=\frac{1}{2\Delta_S}\left(\frac{3}{2}(\lambda\overline{\Gamma}^{\mu}\partial_{\mu}\psi)+\frac{3}{2}(\theta\Gamma^{\nu}\lambda)(\theta\overline{\Gamma}^{\mu}\partial_{\mu}\partial_{\nu}\psi)-(\lambda\Gamma^{\mu\rho\sigma}\theta)\overline{\partial}_{\mu}\partial_{\rho}a_{\sigma}+\cdots\right).\end{aligned}\tag{6.39}$$

This means that the following vertex operator:

$$\tilde{V}=-\frac{3}{2}\frac{1}{2\Delta_S}(\lambda\overline{\Gamma}^{\mu}\partial_{\mu}\psi)+(\lambda\Gamma^{\mu}\theta)a_{\mu}+\frac{1}{2\Delta_S}(\lambda\Gamma^{\mu\rho\sigma}\theta)\partial_{\rho}\overline{\partial}_{\mu}a_{\sigma}+\cdots\tag{6.40}$$

transforms covariantly under the odd shifts. This can be also understood as follows:

$$\tilde{V}=-\frac{3}{2}\frac{1}{2\Delta_S}(\lambda\overline{\Gamma}^{\mu}\partial_{\mu}\psi)+(\lambda\Gamma^{\rho}\Gamma^{\mu\nu}\theta)\frac{1}{2\Delta_S}\overline{\partial}_{\rho}\partial_{[\mu}a_{\nu]}+\cdots.\tag{6.41}$$

Now we recognize what it is

$$\tilde{V}(x,\theta,\lambda)=\frac{1}{2\Delta_S}\lambda^{\alpha}\overline{\Gamma}^{\mu}_{\alpha\beta}\partial_{\mu}W^{\beta}(x,\theta),\tag{6.42}$$

where $W^{\alpha}(x,\theta)$ is the superfield[8] related to the Maxwell superfield $A_{\alpha}(x,\theta)$ by the chain of transformations:

$$\mathcal{T}_{(\alpha}A_{\beta)}=\frac{1}{2}\Gamma^{\mu}_{\alpha\beta}A_{\mu},\tag{6.43}$$

$$\mathcal{T}_{\alpha}A_{\mu}-\mathcal{T}_{\mu}A_{\alpha}=g_{\mu\nu}\Gamma^{\nu}_{\alpha\beta}W^{\beta}.\tag{6.44}$$

See [21] for a recent discussion of W^{α}.

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[8] I want to thank Yuri Aisaka for discussions about W^{α}.

6.2 Supergravity field

In flat space, the supergravity fields split into the product of the left and the right component; the left and right components are essentially free Maxwell fields. The bispinor field is defined as follows:

$$P^{\alpha\dot{\beta}} = W_L^{\alpha} W_R^{\dot{\beta}}, \tag{6.45}$$

where W is the field strength superfield of the free Maxwell theory; the $\theta = 0$ component of W^{α} is the gaugino ψ^{α}.

This bispinor field is a linear combination of the RR field strengths contracted with the gamma matrices [22]:

$$\begin{aligned} P^{\alpha\dot{\beta}} &= F_{\underline{lmnpq}} e^{\underline{l}}_a e^{\underline{m}}_b e^{\underline{n}}_c e^{\underline{p}}_d e^{\underline{q}}_e (\Gamma^{abcde})^{\alpha\dot{\beta}}, \\ &+ a_3 F_{\underline{lmn}} e^{\underline{l}}_a e^{\underline{m}}_b e^{\underline{n}}_c (\Gamma^{abc})^{\alpha\dot{\beta}} + a_1 F_{\underline{l}} e^{\underline{l}}_a (\Gamma^{a})^{\alpha\dot{\beta}}, \end{aligned} \tag{6.46}$$

where a_3 and a_1 are some numeric coefficients, which we do not need. The supersymmetry variations of P is given by this equation:

$$t^3_{\alpha} P^{\beta\dot{\beta}} = \delta^{\beta}_{\alpha} C^{\dot{\beta}} + (\Gamma_{mn})^{\beta}_{\alpha} C^{\dot{\beta}mn}, \tag{6.47}$$

$$t^1_{\dot{\alpha}} P^{\beta\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} C^{\beta} + (\Gamma_{mn})^{\dot{\beta}}_{\dot{\alpha}} C^{\beta mn}, \tag{6.48}$$

where C^{β} is a combination of the left dilatino χ, and the left gravitino field strength $\partial_{[m}\psi_{n]}$, and $C^{\dot{\beta}}$ is a combination of the corresponding right fields $\tilde{\chi}$ and $\partial_{[m}\tilde{\psi}_{n]}$.

The $\mathbf{SP}_{\text{small}}$-covariant vertex in flat space is the product of two expressions of the form (6.42):

$$\tilde{V} = (\lambda\overline{\Gamma}^{\mu}\ \Delta_S^{-2}\partial_{\mu}\partial_{\nu}P\ \overline{\Gamma}^{\nu}\tilde{\lambda}). \tag{6.49}$$

This is BRST equivalent to[9] :

$$\tilde{V}' = (\lambda\overline{\Gamma}^{\mu}\Delta_S^{-1}P\Gamma_{\mu}\tilde{\lambda}). \tag{6.50}$$

[9]I want to thank N. Berkovits for suggesting this simplified form, and many other useful comments.

6.3 Relation between the covariant vertex in $AdS_5 \times S^5$ and the flat space expressions (6.49), (6.50)

The construction of Section 5 implies that the $PSU(2,2|4)$-covariant vertex operator exists in $AdS_5 \times S^5$. This construction gives (6.49) when applied in the flat space limit. (We have demonstrated this for the Maxwell field, but the free supergravity vertex in flat space is just the product of "left" and "right" Maxwell vertices.) Therefore both (6.49) and the BRST-equivalent (6.50) should be the flat space limits of some covariant vertices in $AdS_5 \times S^5$.

Notice that (6.50) reduces to (3.2) in the zero momentum limit, except for the overall normalization factor $\frac{1}{\Delta_S}$, which becomes singular on the zero mode.

However, we were not able to write explicit expressions in terms of the supergravity fields in AdS space, which would explicitly generalize the flat space formula (6.49) or (6.50).

7 Covariant vertex and the endpoint of the Wilson line

7.1 The BRST complex of the endpoint

Consider the Wilson line operator corresponding to a semi-infinite contour going from infinity to some point B on the string worldsheet:

in some representation ρ of $\mathbf{psu}(2,2|4)$. Consider the action of $Q_{\rm BRST}$ on this operator. If we neglect the contribution of the boundary terms at infinity, then the BRST variation is [3–5]:

$$\varepsilon Q_{\rm BRST} T[C^B_\infty](z) = \left(\frac{1}{z}\varepsilon\lambda_3^\alpha(B)\rho(t^3_\alpha) + z\varepsilon\lambda_1^{\dot\alpha}(B)\rho(t^1_{\dot\alpha})\right) T[C^B_\infty](z) \qquad (7.1)$$

(See Section 2.2 of [23] and Section 7 of [7] for a discussion of this formula.)

Let us fix some vector ψ in the representation ρ "at infinity"; then this expression:

$$T[C^B_\infty](z)\psi \qquad (7.2)$$

is a vector in the representation space of ρ. Pick a vector v in the dual space, and evaluate it on (7.2):

$$v\left(T[C^B_\infty](z)\psi\right) \in \mathbf{C}. \qquad (7.3)$$

This gives a number. Consider vectors v depending on the pure spinors λ_3, λ_1 and the spectral parameter z. Then equation (7.1) can be regarded as defining the action of Q_{BRST} on v:

$$Q_{\mathrm{endpoint}} v = \left(\frac{1}{z} \lambda_3^{\alpha} \rho(t_{\alpha}^{3}) + z \lambda_1^{\dot{\alpha}} \rho(t_{\dot{\alpha}}^{1}) \right) v. \tag{7.4}$$

This defines the BRST complex of the endpoint. The n-cochains of this complex are elements

$$v \in \mathrm{Hom}_{\mathbf{g}_{\bar{0}}} \left(\left[\begin{array}{l} \text{linear space of the} \\ \text{representation } \rho \text{ in which} \\ \text{we evaluate Wilson line} \end{array} \right], \ \mathcal{P}^n \right),$$

where $\mathcal{P}^n$ is defined in Section 4.2, and the differential is (7.4). The "plugs" which we introduced in Section 1.1.2 are the cohomologies of this complex. Unfortunately we do not know a general classification of the cohomologies of this complex for a general representation ρ.

7.2 The BRST complex of the Wilson line endpoint is isomorphic to the BRST complex of covariant vertices

We will now consider the special case where ρ is the representation of $psu(2,2|4)$ on the space of states $\mathcal{H}$ of the BPS multiplet. In this case we will relate the BRST complex of the endpoint (7.4) to the BRST complex of covariant supergravity vertices.

7.2.1 Frobenius reciprocity

Let us remember the general structure of the covariant vertex from Section 4.5:

$$\mathcal{V} \in \mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^n) = \mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Hom}_{\mathbf{g}_{\bar{0}}}(\mathcal{U}\mathbf{g}, \mathcal{P}^n)).$$

Here $n = 2$ for the supergravity vertex, but we want to consider the whole BRST complex so we keep n. Let us evaluate $\mathcal{V}$ on the unit of the group:

$$\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \underbrace{\mathrm{Hom}_{\mathbf{g}_{\bar{0}}}(\mathcal{U}\mathbf{g}, \mathcal{P}^n)}_{\text{evaluate on} \mathbf{1} \in \mathcal{U}\mathbf{g}}). \tag{7.5}$$

This defines a correspondence between covariant vertices $\mathcal{V}$ and vectors in $\mathrm{Hom}_{\mathbf{g}_{\bar{0}}}(\mathcal{H}, \mathcal{P}^n)$:

$$\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^n) \ni \mathcal{V} \stackrel{\text{evaluate on } \mathbf{1}}{\longrightarrow} v \in \mathrm{Hom}_{\mathbf{g}_{\bar{0}}}(\mathcal{H}, \mathcal{P}^n). \tag{7.6}$$

Note that $\mathcal{V}$ is a function of x, θ while v is essentially its value at $x = \theta = 0$. Nevertheless, the correspondence (7.6) is a one-to-one correspondence between the elements of $\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_{\bar{0}}}^{\mathbf{g}} \mathcal{P}^n)$ and the elements of $\mathrm{Hom}_{\mathbf{g}_{\bar{0}}}(\mathcal{H}, \mathcal{P}^n)$. Indeed, the symmetry under $\mathbf{g}$:

$$\mathrm{Hom}\underbrace{\mathbf{g}}_{\text{this } \mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} \mathcal{P}^n),$$

allows us to relate $\mathcal{V}(\Psi)(g)$ to $\mathcal{V}(g^{-1}\Psi)(\mathbf{1})$, see equation (4.17). In other words, if we know the value of the covariant vertex at the point $x = \theta = 0$ *for all states* Ψ, then because of the global symmetry we know the covariant vertex everywhere (for arbitrary x and θ). This construction is an example of the *Frobenius reciprocity*:

$$\boxed{\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}} L) \simeq \mathrm{Hom}_{\mathbf{g}_{\bar{0}}}(\mathcal{H}|_{\mathbf{g}_{\bar{0}}}, L)},$$

which is true for any representation L of $\mathbf{g}_{\bar{0}}$; in our case $L = \mathcal{P}^n$.

To summarize, given the covariant vertex $\mathcal{V}$, we define $v \in \mathrm{Hom}_{\mathbf{g}_{\bar{0}}}(\mathcal{H}, \mathcal{P}^n)$ by saying that the value of v on $\Psi \in \mathcal{H}$ is:

$$v(\Psi) = \mathcal{V}(\Psi)(\mathbf{1}). \tag{7.7}$$

7.2.2 The action of $Q_{\mathbf{BRST}}$

Note that both the left-hand side and the right-hand side of (7.7) are elements of $\mathcal{P}^n$, i.e., polynomials of λ_3 and λ_1. We can evaluate them on λ:

$$v(\Psi)(\lambda_3, \lambda_1) = \mathcal{V}(\Psi)(\mathbf{1})(\lambda_3, \lambda_1). \tag{7.8}$$

This is a quadratic polynomial in λ_3 and λ_1. Equations. (4.17) and (4.14) imply that the action of Q_{BRST} on the covariant vertex corresponds to the following action on v:

$$(Q_{\mathrm{BRST}} v)(\Psi)(\lambda_3, \lambda_1) = -v(\lambda_3 \Psi + \lambda_1 \Psi)(\lambda_3, \lambda_1). \tag{7.9}$$

This formula for Q_{BRST} can be interpreted in the following way. The space $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathcal{P}^2)$ is obviously a representation of $\mathbf{g} = psu(2, 2|4)$, just because $\mathcal{H}$ is by definition a representation of $\mathbf{g}$. (The $\mathcal{P}^2$ part just "goes along for the ride".) Let us denote this representation ρ (the action of $x \in \mathbf{g}$ on v is $\rho(x)v$). Then (7.9) implies that the action of $\mathcal{Q}_{\mathrm{BRST}}$ on

$\mathrm{Hom}_{\mathbf{g}}(\mathcal{H}, \mathrm{Coind}_{\mathbf{g}_0}^{\mathbf{g}}\mathcal{P}^2)$ corresponds to the action of the nilpotent operator Q_{endpoint} on v defined by this formula:

$$Q_{\mathrm{BRST}} v(\lambda_3, \lambda_1) = (\lambda_3^{\alpha}\rho(t^3_{\alpha}) + \lambda_1^{\dot\alpha}\rho(t^1_{\dot\alpha}))v(\lambda_3, \lambda_1). \tag{7.10}$$

This is identical to Q_{endpoint} of (7.4) at $z = 1$. We conclude that:

> v represents a cohomology class $H^2(Q_{\mathrm{endpoint}})$ of the following complex:

$$\ldots \longrightarrow \mathcal{H}' \otimes_{\mathbf{g}_{\bar 0}} \mathcal{P}^n \stackrel{Q_{\mathrm{endpoint}}}{\longrightarrow} \mathcal{H}' \otimes_{\mathbf{g}_{\bar 0}} \mathcal{P}^{n+1} \longrightarrow \cdots . \tag{7.11}$$

In other words, the BRST complex on covariant massless vertices (independent of derivatives) is equivalent to the endpoint complex $\mathrm{Hom}_{\mathbf{g}_0}(\mathcal{H}, \mathcal{P}^{\bullet})$.

7.2.3 Including the spectral parameter z corresponds to rescaling the pure spinors

We have demonstrated that the complex (7.10) is equivalent to (7.4) at $z = 1$. But in fact (7.4) at $z = 1$ is equivalent to (7.4) at $z \neq 1$ by rescaling of λ_3 and λ_1. In other words, the map

$$\begin{aligned} &v \mapsto v' \\ &v'(\lambda_3, \lambda_1) = v(z^{-1}\lambda_3, z\lambda_1) \end{aligned} \tag{7.12}$$

is the equivalence of the complex (7.4) at $z = 1$ and the same complex at $z \neq 1$.

7.3 Endpoint BRST complex and the Lie algebra cohomology

There is a relation between the endpoint cohomology and the cohomology of the positive-frequency part of the loop algebra of $psl(4|4)$.

Consider the algebra formed by the positive frequency $\mathbf{Z}_4$-twisted loops with values in $psl(4|4)$. We will denote is $L_+\mathbf{g}$. The cohomology complex is generated by the ghosts c^a_{-k}, where $k \in \{1, 2, 3, \ldots\}$ and a the enumerates the adjoint representation of $psl(4|4)$. We have c^{α}_{-1}, c^m_{-2}, $c^{\dot\alpha}_{-3}$, $c^{[mn]}_{-4}$, c^{α}_{-5}, etc. The "energy" operator L_0 counts the lower indices, for example:

$$L_0 c^{\dot\alpha}_{-3} = -3c^{\dot\alpha}_{-3}, \quad L_0 c^{\alpha}_{-1}c^{[mn]}_{-4} = -5c^{\alpha}_{-1}c^{[mn]}_{-4}.$$

Note that L_0 is a symmetry of the cohomology complex. Another symmetry is the c-ghost number (the number of letters c). Let $H^p_q(L_+\mathbf{g}, \mathbf{C})$ denote the cohomology group with $L_0 = q$ and ghost number p. The first cohomology group $H^1(L_+\mathbf{g}, \mathbf{C})$ is generated by c^α_{-1} (and therefore has $L_0 = -1$). Some of other nontrivial cohomology groups are:

$$H^2_{-2}(L_+\mathbf{g}, \mathbf{C}) : X_{\alpha\beta} c^\alpha_{-1} c^\alpha_{-1}, \quad f_m{}^{\alpha\beta} X_{\alpha\beta} = 0, \tag{7.13}$$

$$H^3_{-3}(L_+\mathbf{g}, \mathbf{C}) : X_{\alpha\beta\gamma} c^\alpha_{-1} c^\beta_{-1} c^\gamma_{-1}, \quad f_m{}^{\alpha\beta} X_{\alpha\beta\gamma} = 0. \tag{7.14}$$

Generally speaking, H^k_{-k} is generated by the expressions of the form:

$$H^k_{-k}(L_+\mathbf{g}, \mathbf{C}) : \quad X_{\alpha_1\cdots\alpha_k} c^{\alpha_1}_{-1} \cdots c^{\alpha_k}_{-1}, \quad f_m{}^{\alpha_1\alpha_2} X_{\alpha_1\alpha_2\cdots\alpha_k} = 0. \tag{7.15}$$

But H^k_{-k} is not all of the cohomology, for example there is nontrivial[10] H^2_{-4}. The pure spinor cohomology should be identified with the part of the $L_+\mathbf{g}$ cohomology with "energy" equal to minus the ghost number, i.e., H^k_{-k}.

8 Application: vertex operators depending on the spectral parameter

8.1 How to introduce the spectral parameter into the vertex operator

In flat infinite space massless vertex operators have the form:

$$V_k(x, \theta) = p(\lambda, \theta)\, \mathrm{e}^{\mathrm{i}kX} \tag{8.1}$$

where p is some polynomial of λ and θ. We can write

$$X(\tau^+, \tau^-) = X_L(\tau^+) + X_R(\tau^-), \tag{8.2}$$

where

$$X_L = \int_\infty^{(\tau^+,\tau^-)} d\tau^+ \partial_+ X \text{ and } X_R = \int_\infty^{(\tau^+,\tau^-)} d\tau^- \partial_- X. \tag{8.3}$$

Therefore there is a generalization of (8.1):

$$V_{k_L,k_R}(x, \theta) = p(\lambda, \theta) e^{ik_L X_L + ik_R X_R}. \tag{8.4}$$

This generalization is only well defined in flat space on those worldsheets which do not have handles, or in toroidal compactifications with appropriate

[10] This can be seen from the comparison of the character of the $H^\bullet(L_+\mathbf{g}, \mathbf{C})$ with the character of [24].

integrality conditions on k_L and k_R. We can formally consider (8.4), for example on an infinite worldsheet without handles, if we neglect boundary effects.

We will now argue that there is a partial analogue of (8.4) in $AdS_5 \times S^5$.

Given a state $\Psi \in \mathcal{H}$ we can prepare a nonlocal z-dependent covariant vertex operator, in the following way. Consider the transfer matrix $T_{\infty}^{(\tau^+,\tau^-)}(z)$ from infinity to the point (τ^+,τ^-) on the worldsheet. Let us fix a vector $\Psi^{(\infty)} \in \mathcal{H}$, and consider:

$$V_{\Psi_\infty}(\tau^+,\tau^-|z) = v\left(T_{(\infty)}^{(\tau^+,\tau^-)}(z)\Psi^{(\infty)}\right)(z^{-1}\lambda_3(\tau^+,\tau^-),\ z\lambda_1(\tau^+,\tau^-)). \tag{8.5}$$

So defined $V_{\Psi_\infty}(\tau^+,\tau^-|z)$ is analogous to:

$$\mathrm{e}^{\mathrm{i}kz^{-1}X_L+\mathrm{i}kzX_R}. \tag{8.6}$$

In particular, for $z=1$ we get $T_{(\infty)}^{(\tau^+,\tau^-)}(z) = g(\tau^+,\tau^-)g(\infty)^{-1}$ and therefore:

$$V_{\Psi_\infty}(\tau^+,\tau^-|1) = v\left(g(\tau^+,\tau^-)g(\infty)^{-1}\Psi^{(\infty)}\right)(\lambda(\tau^+,\tau^-)). \tag{8.7}$$

This formula gives us back the covariant vertex for the state Ψ if we identify:

$$g(\infty)^{-1}\Psi^{(\infty)} = \Psi \tag{8.8}$$

8.2 Is there a two-point vertex operator?

For any $u \in \mathcal{H}'$, we define $u^\dagger$ as a non-normalizable vector in the space of states, characterized by the formula:

$$u(\Psi) = (u^\dagger, \Psi), \quad \text{for any } \ \Psi \in \mathcal{H}, \tag{8.9}$$

where $(,)$ is the Hermitean scalar product in $\mathcal{H}$. Note that $u^\dagger$ strictly speaking does not belong to $\mathcal{H}$ because it is not normalizable. For example, one-dimensional quantum mechanics has $\mathcal{H} = L^2(\mathbf{R})$ — the space of square integrable functions of one variable, with the norm $||f||^2 = \int dx|f(x)|^2$. The dual space $\mathcal{H}'$ is the space of generalized functions; if $u \in \mathcal{H}'$ is defined by the formula $u(f) = f(0)$ then $u^\dagger$ is a delta-function $\delta(x)$.

Using these notations we can define the two-point vertex operator:

$$V^{2pt}((\tau_1^+,\tau_1^-),(\tau_2^+,\tau_2^-)) = v\left(T_{\tau_2}^{\tau_1} v^\dagger(\lambda(\tau_2^+,\tau_2^-))\right)(\lambda(\tau_1^+,\tau_1^-)). \quad (8.10)$$

However, we conjecture that this two-point vertex operator is in fact BRST exact. Indeed, although we have not checked it explicitly, it should be true that the derivative of $V^{2pt}((\tau_1^+,\tau_1^-),(\tau_2^+,\tau_2^-))$ with respect to τ_1 is Q_{BRST}-exact. Therefore up to BRST-exact terms this vertex is independent of τ_1 and τ_2. On the other hand, when $\tau_1 \to \tau_2$ we get a local vertex operator of the ghost number 4. There is no $psu(2,2|4)$-invariant cohomology at the ghost number 4. This implies that (8.10) is BRST exact.

9 Conclusions

In this paper, we introduced a family of z-dependent vertex operators (8.5) parametrized by a choice of the BPS representation of $psu(2,2|4)$. Schematically, these vertex operators have a form:

$$V_{\Psi_\infty}(\tau^+,\tau^-|z) = \left\langle \mathrm{plug}(\tau^+,\tau^-|z) \left| P\exp\left(-\int_\infty^{(\tau^+,\tau^-)} J[z]\right)\right| \Psi^\infty \right\rangle. \quad (9.1)$$

This expression is strictly speaking not BRST invariant, because of the boundary term at infinity. Indeed we have put Ψ^∞ an arbitrary vector from $\mathcal{H}$, and this is generally speaking not a valid plug. We assume that we can neglect this boundary term because it is at infinity[11]. We can consider $V_{\Psi_\infty}(\tau^+,\tau^-|z)$ locally near the point (τ^+,τ^-). Note that $\langle\mathrm{plug}(\tau^+,\tau^-)|$ is a λ-dependent vector in the dual space to $\mathcal{H}$. (In fact $\langle\mathrm{plug}(\tau^+,\tau^-)|$ depends on τ^+ and τ^- through $\lambda(\tau^+,\tau^-)$.) We can also think of $\langle\mathrm{plug}(\tau^+,\tau^-)|$ as an element of $\mathcal{H}$, but then we have to remember that it is not normalizable; it is a δ-function type of state, rather than a proper wave packet. Note that for a fixed λ, our $\langle\mathrm{plug}|$ is a *fixed* vector in $\mathcal{H}'$. In other words, for every BPS representation $\mathcal{H}$ we have a map, which takes a pair of pure spinors and transforms them into a vector in the space of BPS states:

pure spinors λ_3, λ_1	$\longmapsto$	a non-normalizable vector in $\mathcal{H}$ which we call $\langle\mathrm{plug}|$

(9.2)

[11] An attempt to bring the second endpoint from infinity is described in Section 8.2.

It would be interesting to describe this map explicitly. The non-normalizable vector in $\mathcal{H}$ on the right-hand side of (9.2) is obviously not invariant under $psu(2,2|4)$ (it belongs to an irreducible representation). But it transforms covariantly under $so(1,4) \oplus so(5) \subset psu(2,2|4)$, in the sense that the action of $so(1,4) \oplus so(5)$ on the right-hand side of (9.2) agrees with the action of $so(1,4) \oplus so(5)$ on the left-hand side of (9.2).

Another way of thinking about $\langle\text{plug}|$ is in terms of the cohomology of the operator:

$$\frac{1}{z}\lambda_3^{\alpha} t_{\alpha}^3 + z\lambda_1^{\dot{\alpha}} t_{\dot{\alpha}}^1 \tag{9.3}$$

acting on the BPS representation $\mathcal{H}$ (more precisely, the $\mathbf{g}_{\bar{0}}$-invariant tensor product of $\mathcal{H}'$ with the space of polynomials of λ_3, λ_1). Our results imply that the second cohomology of this operator is nontrivial[12], represented by the cocycle (9.2).

Note that this provides a purely representation-theoretic characterization of the linearized SUGRA spectrum on $AdS_5 \times S^5$. Indeed, the question of the existence of the excitation transforming in the representation $\mathcal{H}$ is reduced to the calculation of the cohomology of the operator (9.3), which is defined in terms of the generators t_a of the representation $\mathcal{H}$.

There is also another example of a plug, a plug of the ghost number 1. Consider the Wilson line in the adjoint representation. The cohomology of (9.3) in the adjoint representation is nontrivial and is represented by:

$$\frac{1}{z}\lambda_3^{\alpha} t_{\alpha}^3 - z\lambda_1^{\dot{\alpha}} t_{\dot{\alpha}}^1. \tag{9.4}$$

This is obviously a λ-dependent vector in the adjoint representation, of the ghost number 1. One can verify that this is annihilated by (9.3); note the relative minus sign of the second term in (9.4). Therefore we can take (9.4) as a plug, and consider:

$$V(\tau^+,\tau^-|z) = \mathrm{Str}\left(\left(\frac{1}{z}\lambda_3^{\alpha} t_{\alpha}^3 - z\lambda_1^{\dot{\alpha}} t_{\dot{\alpha}}^1\right) P\exp\left(-\int_{\infty}^{(\tau^+,\tau^-)} J[z]\right)\Psi^{(\infty)}\right).$$

At $z=1$ the corresponding integrated vertex operator is the density of the local conserved charge $\mathrm{Str}((j_+ d\tau^+ - j_- d\tau^-)\Psi^{(\infty)})$. We will prove in Appendix A that (9.4) is the only example of the endpoint cohomology at ghost number 1. In particular, there is no nontrivial cohomology for representations other than the adjoint.

[12] A similar (but different) cohomology problem was considered in [25–27].

With these notations the two-point vertex (8.10) reads:

$$V^{2-pt}((\tau_2^+,\tau_2^-),(\tau_1^+,\tau_1^-)|z) \\ = \left\langle \mathrm{plug}(\tau_2^+,\tau_2^-) \middle| P\exp\left(-\int_{(\tau_1^+,\tau_1^-)}^{(\tau_2^+,\tau_2^-)} J[z]\right) \middle| \mathrm{plug}(\tau_1^+,\tau_1^-) \right\rangle . \quad (9.5)$$

(But as we discussed at the end of Section 8.2 this must be BRST exact.)

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Appendix A Cohomology at ghost number one

In this section, we will prove that the only cohomology at ghost number 1 are the global $psu(2,2|4)$ conserved charges.

A.1 Global conserved charges and BRST cohomology

The conserved charges are the descendants of the cohomology classes of the ghost number 1. The "standard" local conserved charges correspond to the global symmetries $PSU(2,2|4)$. They descend from the following operator:

$$\mathrm{Ad}(g).(\lambda_3^{\alpha} t_{\alpha}^3 - \lambda_1^{\dot\alpha} t_{\dot\alpha}^1). \tag{A.1}$$

In other words, we have the following cohomology class of the ghost number one in the adjoint representation of $\mathbf{g}$:

$$\lambda_3^{\alpha} t_{\alpha}^3 - \lambda_1^{\dot\alpha} t_{\dot\alpha}^1. \tag{A.2}$$

In this section, we will prove that there are no nontrivial cohomology classes of the ghost number 1 in the *covariant complex*, in representations other than the adjoint.

A.2 Cohomology classes at ghost number 1: the defining equations

Fix a representation $\mathcal{F}$ of $\mathbf{g} = psu(2,2|4)$. We will assume two things about $\mathcal{F}$:

- as a representation of $\mathbf{g}$ it is irreducible;
- as a representation of $\mathbf{g}_{even}$ it is completely reducible, i.e., decomposes into the direct sum of irreducible representations

We will write a representative of the cohomology class in the following way:

$$\lambda_3^{\alpha} V_{\alpha} + \lambda_1^{\dot\alpha} \tilde{V}_{\dot\alpha}. \tag{A.3}$$

The condition of $\mathbf{g}_0$-invariance says that V_α and $\tilde{V}_{\dot\alpha}$ should define intertwining operators of $\mathbf{g}_0$:

$$V \in \mathrm{Hom}_{\mathbf{g}_0}(\mathbf{g}_3, \mathcal{F}), \tag{A.4}$$

$$\tilde{V} \in \mathrm{Hom}_{\mathbf{g}_0}(\mathbf{g}_1, \mathcal{F}). \tag{A.5}$$

In other words:

$$t^0_{[\rho\sigma]} V_\alpha = f_{[\rho\sigma]\alpha}{}^{\beta} V_\beta, \tag{A.6}$$

$$t^0_{[\rho\sigma]} V_{\dot\alpha} = f_{[\rho\sigma]\dot\alpha}{}^{\dot\beta} V_{\dot\beta}, \tag{A.7}$$

— the conditions of $\mathbf{g}_0$-covariance. The conditions for being annihilated by Q are:

$$t^3_\alpha V_\beta + t^3_\beta V_\alpha = f_{\alpha\beta}{}^\mu A_\mu, \tag{A.8}$$

$$t^1_{\dot\alpha} \tilde V_{\dot\beta} + t^1_{\dot\beta} \tilde V_{\dot\alpha} = f_{\dot\alpha\dot\beta}{}^\mu \tilde A_\mu, \tag{A.9}$$

$$t^3_\alpha \tilde V_{\dot\beta} + t^1_{\dot\beta} V_\alpha = 0. \tag{A.10}$$

For example, the class (A.2) is represented by:

$$\begin{aligned} V_\alpha &= t^3_\alpha, \\ \tilde V_{\dot\alpha} &= -t^1_{\dot\alpha}, \\ A_\mu = -\tilde A_\mu &= 2t^2_\mu. \end{aligned} \tag{A.11}$$

We consider the solutions of (A.8) trivial if they are of the form:

$$V_\alpha = t^3_\alpha \Phi, \;\; V_{\dot\alpha} = t^1_{\dot\alpha}\Phi, \tag{A.12}$$

where $t_0^{[\mu\nu]}\Phi = 0$.

We want to prove the following:

Theorem. *Nontrivial solutions to equations* (A.8)*–*(A.10) *exist only when* $\mathcal{F}$ *is the adjoint representation of* $\mathbf{g}$, *and are given by* (A.11) *up to adding a trivial solution. There are no other nontrivial solutions.*

We will now proceed to prove this.

A.3 Cohomology classes at ghost number 1: consequences of the defining equations

Acting on (A.8) by $t^1_{\dot\beta}$ we get:

$$(f_{\dot\beta\alpha}{}^{[\rho\sigma]} t^0_{[\rho\sigma]} V_\beta - t^3_\alpha t^1_{\dot\beta} V_\beta) + (\alpha \leftrightarrow \beta) = f_{\alpha\beta}{}^\mu t^1_{\dot\beta} A_\mu. \tag{A.13}$$

This with equations (A.6) and (A.10) implies:

$$(f_{\dot\beta\alpha}{}^{[\rho\sigma]} f_{[\rho\sigma]\beta}{}^\gamma V_\gamma + (\alpha \leftrightarrow \beta)) + f_{\alpha\beta}{}^\mu t^2_\mu \tilde V_{\dot\beta} = f_{\alpha\beta}{}^\mu t^1_{\dot\beta} A_\mu. \tag{A.14}$$

This with the Jacobi identity for ff implies:

$$f_{\mu\dot\beta}{}^\alpha V_\alpha = t^2_\mu \tilde V_{\dot\beta} - t^1_{\dot\beta} A_\mu. \tag{A.15}$$

Similarly we have:

$$f_{\mu\beta}{}^{\dot\alpha}\tilde V_{\dot\alpha} = t^2_\mu V_\beta - t^3_\beta \tilde A_\mu. \tag{A.16}$$

Let us act on (A.15) by $f^{\dot\gamma\dot\beta}{}_\nu t^1_{\dot\gamma}$:

$$\begin{aligned} f^{\dot\gamma\dot\beta}{}_\nu f_{\mu\dot\beta}{}^\alpha t^1_{\dot\gamma} V_\alpha &= f^{\dot\gamma\dot\beta}{}_\nu t^1_{\dot\gamma} t^2_\mu \tilde V_{\dot\beta} - f^{\dot\gamma\dot\beta}{}_\nu t^1_{\dot\gamma} t^1_{\dot\beta} A_\mu \\ &= f^{\dot\gamma\dot\beta}{}_\nu f_{\dot\gamma\mu}{}^\alpha t^3_\alpha \tilde V_{\dot\beta} + \frac{1}{2} f^{\dot\gamma\dot\beta}{}_\nu f_{\dot\gamma\dot\beta}{}^\lambda t^2_\mu \tilde A_\lambda - \frac{1}{2} f^{\dot\gamma\dot\beta}{}_\nu f_{\dot\gamma\dot\beta}{}^\lambda t^2_\lambda A_\mu. \end{aligned} \tag{A.17}$$

This and (A.10) implies:

$$t^2_\mu \tilde A_\nu - t^2_\nu A_\mu = 0. \tag{A.18}$$

Let us denote: $B_\mu = A_\mu + \tilde A_\mu$. We have:

$$t^2_{[\mu} B_{\nu]} = 0. \tag{A.19}$$

Note that the gauge transformation

$$\delta V_\alpha = t^3_\alpha \Phi\ , \quad \delta \tilde V_{\dot\alpha} = t^1_{\dot\alpha}\Phi. \tag{A.20}$$

where Φ is $\mathbf{g}_{\bar 0}$-invariant leads to:

$$\delta B_\mu = t^2_\mu \Phi. \tag{A.21}$$

Therefore we should think of B_μ as an element of $H^1(\mathbf{g}_{\mathrm{even}}, \mathbf{g}_{\bar 0}, \mathcal{F})$. But this cohomology group is zero because $H^1(\mathbf{g}_{\mathrm{even}}, \mathcal{F}) = 0$ (note that the Serre–Hochschild spectral sequence for $\mathbf{g}_{\mathrm{even}} \subset \mathbf{g}$ has $E_2^{p,0} = H^p(\mathbf{g}, \mathbf{g}_{\mathrm{even}}, \mathcal{F})$ and d_2 acts from $E_2^{p,q}$ to $E_2^{p+2,q-1}$). Therefore we should be able to gauge away B_μ. Let us therefore assume:

$$A_\mu = -\tilde A_\mu. \tag{A.22}$$

Note that this equation and (A.18) implies:

$$t^2_\mu A_\nu + t^2_\nu A_\mu = 0. \tag{A.23}$$

Now we can rewrite (A.15) and (A.16) as follows:

$$f_{\mu\dot\beta}{}^\alpha V_\alpha = t^2_\mu \tilde V_{\dot\beta} - t^1_{\dot\beta} A_\mu, \tag{A.24}$$

$$f_{\mu\beta}{}^{\dot\alpha} \tilde V_{\dot\alpha} = t^2_\mu V_\beta + t^3_\beta A_\mu. \tag{A.25}$$

Let us define $F_{\mu\nu}$ by the following equation:

$$F_{\rho\sigma} = t^2_\rho A_\sigma. \tag{A.26}$$

Equation (A.23) implies that $F_{\rho\sigma}$ is antisymmetric: $F_{\rho\sigma} = -F_{\sigma\rho}$. We get:

$$\begin{aligned} t^2_\lambda F_{\mu\nu} &= t^2_\lambda t^2_\mu A_\nu = f_{\lambda\mu}{}^{[\rho\sigma]} t^0_{[\rho\sigma]} A_\nu + t^2_\mu t^2_\lambda A_\nu \\ &= f_{\lambda[\mu|}{}^{[\rho\sigma]} f_{[\rho\sigma]|\nu]}{}^{\kappa} A_\kappa - t^2_{[\mu} t^2_{\nu]} A_\lambda \\ &= -\frac{1}{2} f_{\mu\nu}{}^{[\rho\sigma]} f_{[\rho\sigma]\lambda}{}^{\kappa} A_\kappa - \frac{1}{2} f_{\mu\nu}{}^{[\rho\sigma]} f_{[\rho\sigma]\lambda}{}^{\kappa} A_\kappa \\ &= -f_{\mu\nu}{}^{[\rho\sigma]} f_{[\rho\sigma]\lambda}{}^{\kappa} A_\kappa. \end{aligned} \tag{A.27}$$

Therefore $F_{\mu\nu}$ can be expressed in terms of $G_{[\mu\nu]}$ and $M_{\mu\nu}$, which are defined by this equation:

$$F_{\mu\nu} = f_{\mu\nu}{}^{[\rho\sigma]} G_{[\rho\sigma]} + M_{\mu\nu}, \tag{A.28}$$

where $M_{\mu\nu} = -M_{\nu\mu}$ is nonzero only when μ is tangent to AdS_5 and ν is tangent to S_5, or vice versa, and:

$$t^2_\lambda M_{\mu\nu} = 0, \tag{A.29}$$

$$t^2_\lambda G_{[\mu\nu]} = f_{\lambda[\mu\nu]}{}^{\kappa} A_\kappa. \tag{A.30}$$

Then the covariance under $\mathbf{g}_{\bar{0}}$ implies that $M_{\mu\nu} = 0$. This means that the linear space formed by A_μ and $G_{[\mu\nu]}$ is closed under the action of $\mathbf{g}_{\rm even}$, and is in fact the adjoint representation of $\mathbf{g}_{\rm even}$ (where A_μ corresponds to $2t_\mu$ and $G_{[\mu\nu]}$ corresponds to $2t_{[\mu\nu]}$). This is already close to what we wanted to prove. But we have to also tame the expressions of this form:

$$t^3_\alpha t^1_{\dot\beta} t^1_{\dot\gamma} t^3_\delta A_\mu, \quad t^1_{\dot\alpha} t^3_\beta t^3_\gamma G_{[\mu\nu]}, \quad \text{etc.} \tag{A.31}$$

For this purpose, let us use (A.24) and (A.25) in this expression:

$$\begin{aligned} t^3_\alpha t^1_{\dot\beta} A_\mu - t^1_{\dot\beta} t^3_\alpha A_\mu &= t^3_\alpha (t^2_\mu \tilde{V}_{\dot\beta} - f_{\mu\dot\beta}{}^{\gamma} V_\gamma) + t^1_{\dot\beta}(t^2_\mu V_\alpha - f_{\mu\alpha}{}^{\dot\gamma} \tilde{V}_{\dot\gamma}) \\ &= -f_{\mu\alpha}{}^{\dot\gamma}(t^1_{\dot\gamma}\tilde{V}_{\dot\beta} + t^1_{\dot\beta}\tilde{V}_{\dot\gamma}) - f_{\mu\dot\beta}{}^{\gamma}(t^3_\alpha V_\gamma + t^3_\gamma V_\alpha) \\ &= f_{\dot\beta\mu}{}^{\gamma} f_{\alpha\gamma}{}^{\nu} A_\nu - f_{\alpha\mu}{}^{\dot\gamma} f_{\dot\gamma\dot\beta}{}^{\nu} A_\nu. \end{aligned} \tag{A.32}$$

On the other hand, the combination $t^3_\alpha t^1_{\dot\beta} A_\mu + t^1_{\dot\beta} t^3_\alpha A_\mu$ can be calculated using the $\mathbf{g}_0$-invariance. This implies:

$$t^3_\alpha t^1_{\dot\beta} A_\mu = f_{\dot\beta\mu}{}^{\gamma} f_{\alpha\gamma}{}^{\nu} A_\nu. \tag{A.33}$$

We will also use this:

$$\begin{aligned} t^1_{\dot\alpha}F_{\mu\nu} &= t^1_{\dot\alpha}t^2_\mu A_\nu = \frac{1}{c}f_\mu{}^{\gamma\delta}t^1_{\dot\alpha}t^3_\gamma t^3_\delta A_\nu \\ &= \frac{1}{c}f_\mu{}^{\gamma\delta}f_{\dot\alpha\gamma}{}^{[\rho\sigma]}t^0_{[\rho\sigma]}t^3_\delta A_\nu - \frac{1}{c}f_\mu{}^{\gamma\delta}t^3_\gamma t^1_{\dot\alpha}t^3_\delta A_\nu \\ &= \frac{1}{c}(fff)(t^3A). \end{aligned} \tag{A.34}$$

Here we used the schematic notation (fff) for a product of three structure constants with some indices contracted, and c is determined from $f_\mu{}^{\alpha\beta}f_{\alpha\beta}{}^\nu = c\delta^\nu_\mu$. The subspace of $\mathcal{F}$ generated by acting on A and G by finitely many t^3 and t^1 is finite-dimensional. Indeed, using (A.33) and (A.34) we can prove that it is generated as a linear space by expressions of the form:

$$t^1_{[\dot\alpha_1}\cdots t^1_{\dot\alpha_k]}A_\mu,\ \ t^3_{[\alpha_1}\cdots t^3_{\alpha_k]}A_\mu \quad (k\geq 0), \tag{A.35}$$

$$\text{and } G_{[\mu\nu]} \tag{A.36}$$

where the square brackets stand for the antisymmetrization of the indices (for example $t^3_{[\alpha}t^3_{\beta]}A_\mu$ stands for $(t^3_\alpha t^3_\beta - t^3_\beta t^3_\alpha)A_\mu$). Equations (A.33) and (A.34) imply that this subspace is closed under the action of $\mathbf{g}$. Because $\mathcal{F}$ is assumed to be irreducible, we conclude that $\mathcal{F}$ is generated by (A.35), (A.36). Because of the antisymmetrization of the indices of t^3 and t^1 there are only finitely many linearly independent expressions of the form (A.35). This proves that $\mathcal{F}$ is a finite-dimensional space.

The subspace in $\mathcal{F}$ generated by:

$$A_\mu,\ \ G_{[\mu\nu]},\ \ t_\alpha A_\mu,\ \ t_{\dot\alpha}A_\mu \tag{A.37}$$

is closed under $\mathbf{g}_{even}$. Let us denote this space $\mathcal{L}$. Obviously $\mathcal{L}\subset\mathcal{F}$.

Theorem. $\mathcal{L}=\mathcal{F}$.

Proof. One can see that for any element v of $\mathcal{H}$ (i.e., a finite linear combination of expressions of the form (A.35)) there is a number p such that for any $q>p$ and any q elements $\xi_1,\ldots,\xi_q$ of $\mathbf{g}_{\text{even}}$ we get:

$$\xi_1\cdots\xi_q\, v\in\mathcal{L}. \tag{A.38}$$

Indeed, let us consider for example acting by $\xi\in\mathbf{g}_{\bar 2}$ on expressions of the form $t^3_{[\alpha_1}\cdots t^3_{\alpha_k]}A_\mu$. Let us define the degree of such an expression by the following formula:

$$\deg t^3_{[\alpha_1} \cdots t^3_{\alpha_k]} A_\mu = \deg t^1_{[\dot\alpha_1} \cdots t^1_{\dot\alpha_k]} A_\mu = k. \tag{A.39}$$

More precisely, we introduce a filtration of $\mathcal{H}$ saying that $F^k\mathcal{H}$ consists of all the elements of $\mathcal{H}$, which can be written as linear combinations of the expressions of the form (A.35) of the degree less or equal k. Using equations (A.33) and (A.34) we derive that for $k > 1$:

$$t^2_\mu F^k \mathcal{H} \subset F^{k-1}\mathcal{H}. \tag{A.40}$$

This implies (A.38). Because of the assumption that $\mathcal{F}$ is completely reducible as a representation of $\mathbf{g}_{even}$, equation (A.38) implies that $\mathcal{L} = \mathcal{F}$.

We conclude that $\mathcal{F}$ is in fact generated by expressions (A.37). Note that the linear space generated by (A.37) consists of the even subspace generated by A_μ and $G_{[\mu\nu]}$, and odd subspace generated by $t_\alpha A_\mu, t_{\dot\alpha} A_\mu$. The even subspace is the same as in the adjoint representation. Therefore the odd space should be also the same.

This proves that $\mathcal{H}$ is the adjoint representation of $\mathbf{g} = psu(2,2|4)$.

A.4 Relaxing the requirement that $\mathcal{F}$ is irreducible

We have argued that the subspace generated by (A.35) in fact coincides with $\mathcal{F}$, based on $\mathcal{F}$ being an irreducible representation of $\mathbf{g}$. This requirement can be replaced with the requirement that $H^1(\mathbf{g}, \mathcal{F}) = 0$. Suppose that (A.35) generate a smaller subspace $\mathcal{F}_{A+G} \subset \mathcal{F}$. Let us denote v and $\tilde v$ the projections of V and $\tilde V$ on $\mathcal{F}/\mathcal{F}_{A+G}$:

$$v = V \mod \mathcal{F}_{A+G}. \tag{A.41}$$

Then (A.9) implies:

$$t^3_\alpha v_\beta + t^3_\beta v_\alpha \;=\; t^1_{\dot\alpha} \tilde v_{\dot\beta} + t^1_{\dot\beta} \tilde v_{\dot\alpha} \;=\; t^3_\alpha \tilde v_{\dot\beta} + t^1_{\dot\beta} v_\alpha \;=\; 0. \tag{A.42}$$

These equations imply that $(v_\alpha, \tilde v_{\dot\beta})$ form the spinor representation of $\mathbf{g}_{even}$:

$$t_m v_\alpha = f_{m\alpha}{}^{\dot\beta} \tilde v_{\dot\beta}, \tag{A.43}$$

$$t_m \tilde v_{\dot\alpha} = f_{m\dot\alpha}{}^{\beta} v_\beta. \tag{A.44}$$

We will now explain, using (A.43) and (A.44), that v can be gauged away if $H^1(\mathbf{g}, \mathbf{g}_{\text{even}}, \mathcal{F}) = 0$. We will also explain that $H^1(\mathbf{g}, \mathbf{g}_{\text{even}}, \mathcal{F}) = 0$ if $H^1(\mathbf{g}, \mathcal{F}) = 0$.

More generally, let us consider the relative Lie algebra cohomology complex $C^\bullet(\mathbf{g}, \mathbf{g}_{\text{even}}, \mathcal{F})$. The cochains are tensors with spinor indices satisfying:

$$t^2_m v_{\alpha_1\dots\alpha_p\,\dot\beta_1\dots\dot\beta_q} = p f_{m(\alpha_1}{}^{\dot\alpha_1} v_{\alpha_2\dots\alpha_p)\,\dot\alpha_1\dot\beta_1\dots\dot\beta_q} + q f_{m(\dot\beta_1|}{}^{\beta_1} v_{\beta_1\alpha_1\dots\alpha_p|\,\dot\beta_2\dots\dot\beta_q)}. \tag{A.45}$$

The differential in relative cohomology is:

$$(Qv)_{\alpha_1\cdots\alpha_p\dot\beta_1\cdots\dot\beta_q} = t^3_{(\alpha_1} v_{\alpha_2\cdots\alpha_p)\dot\beta_1\cdots\dot\beta_q} + t^1_{(\dot\beta_1|} v_{\alpha_1\cdots\alpha_p|\dot\beta_2\cdots\dot\beta_q)}. \tag{A.46}$$

This can be thought of as a distant relative of the pure spinor BRST complex. The difference is that a stronger covariance condition is imposed ($\mathbf{g}_{\text{even}} \supset \mathbf{g}_{\bar 0}$) and also no constraints on the ghost variables. We will now explain that the BRST cohomology of the $\mathbf{g}_{\text{even}}$-covariant complex is zero for large enough quantum numbers, unlike the cohomology of the "normal" BRST complex (which is only $\mathbf{g}_{\bar 0}$-covariant).

Indeed, this relative cohomology is related to $H^\bullet(\mathbf{g}, \mathcal{F})$ by the Serre-Hochschild spectral sequence. Namely

$$H^p(\mathbf{g}, \mathbf{g}_{\text{even}}, \mathcal{F}) = E_2^{p,0}. \tag{A.47}$$

The differential d_r acts from $E_r^{p,q}$ to $E_r^{p+r,q+1-r}$. In particular, $E_r^{1,0}$ is related to $E_r^{1+r,1-r}$.

$$E_r^{1-r,-1+r} \xrightarrow{d_r} E_r^{1,0} \xrightarrow{d_r} E_r^{1+r,1-r}. \tag{A.48}$$

This means that $E_2^{1,0}$ cannot cancel with anything and therefore $H^1(\mathbf{g}, \mathcal{F}) = 0$ implies that $H^1(\mathbf{g}, \mathbf{g}_{\text{even}}, \mathcal{F}) = 0$.

Similarly, vanishing of $H^2(\mathbf{g}, \mathcal{F})$ and $H^1(\mathbf{g}_{\text{even}}, \mathcal{F})$ implies vanishing of $H^2(\mathbf{g}, \mathbf{g}_{\text{even}}, \mathcal{F})$. Indeed, the action of d_2 is:

$$E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \xrightarrow{d_2} (0 = E_2^{4,-1}), \tag{A.49}$$

where

$$E_2^{2,0} = H^2(\mathbf{g}, \mathbf{g}_{\text{even}}, \mathcal{F}), \tag{A.50}$$

$$E_1^{0,1} = H^1(\mathbf{g}_{\text{even}}, \mathcal{F}). \tag{A.51}$$

Therefore

$$F^2 H^2(\mathbf{g}, \mathcal{F}) = E_2^{2,0}/\mathrm{Im}\ (d_2 : E_2^{0,1} \to E_2^{2,0}) \tag{A.52}$$

Appendix B Shapiro's lemma

This section is a brief review of the Shapiro's lemma applied to Lie superalgebras:

$$H^n(\mathbf{g},\ \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathrm{Hom}_{\mathcal{U}\mathbf{h}}(\mathcal{U}\mathbf{g}, A))) = H^n(\mathbf{h},\ \mathrm{Hom}_{\mathbf{C}}(\mathcal{H}|_{\mathbf{h}}, A)). \tag{B.1}$$

We will follow [16].

B.1 Relation between cohomology and Ext

The proof starts with pointing out the relation between the cohomology and the Ext group:

$$H^n(\mathbf{g}, \mathrm{Hom}_{\mathbf{C}}(M, N)) = \mathrm{Ext}^n_{\mathcal{U}\mathbf{g}}(M, N). \tag{B.2}$$

This is proven as follows. By definition $\mathrm{Ext}^n_{\mathcal{U}\mathbf{g}}(M, N)$ is computed using the projective resolution $\cdots \to P^1_M \to P^0_M \to M \to 0$ of M, as a module over $\mathcal{U}\mathbf{g}$. Given such a projective resolution, $\mathrm{Ext}^n_{\mathcal{U}\mathbf{g}}(M, N)$ is identified with the nth cohomology group of the complex:

$$\ldots \mathrm{Hom}_{\mathcal{U}\mathbf{g}}(P^{n-1}_M, N) \to \mathrm{Hom}_{\mathcal{U}\mathbf{g}}(P^n_M, N) \to \mathrm{Hom}_{\mathcal{U}\mathbf{g}}(P^{n+1}_M, N) \to \cdots . \tag{B.3}$$

This is the complex of *vector spaces*, the spaces of invariants in the modules $\mathrm{Hom}_{\mathbf{C}}(P^n_M, N)$. But in fact $\mathrm{Hom}_{\mathbf{C}}(P^n_M, N)$ are injective $\mathcal{U}\mathbf{g}$-modules. Therefore the following complex is an injective resolution of $\mathrm{Hom}_{\mathbf{C}}(M, N)$:

$$\ldots \mathrm{Hom}_{\mathbf{C}}(P^{n-1}_M, N) \to \mathrm{Hom}_{\mathbf{C}}(P^n_M, N) \to \mathrm{Hom}_{\mathbf{C}}(P^{n+1}_M, N) \to \cdots . \tag{B.4}$$

Therefore the cohomologies of (B.3) are identified with the Lie algebra cohomologies $H^n\,(\mathbf{g}, \mathrm{Hom}_{\mathbf{C}}(M, N))$. It remains to prove that $\mathrm{Hom}_{\mathbf{C}}(P^n_M, N)$ are injective $\mathcal{U}\mathbf{g}$-modules. It turns out that if P is projective, then $\mathrm{Hom}_{\mathbf{C}}(P, N)$

is injective. This is equivalent to the statement that the following contravariant functor:

$$W \mapsto \mathrm{Hom}_{\mathcal{U}\mathbf{g}}(W, \mathrm{Hom}_{\mathbf{C}}(P, N)) \tag{B.5}$$

is exact. Note that there is a canonical isomorphism:

$$\mathrm{Hom}_{\mathcal{U}\mathbf{g}}(W, \mathrm{Hom}_{\mathbf{C}}(P, N)) \simeq \mathrm{Hom}_{\mathcal{U}\mathbf{g}}(P, \mathrm{Hom}_{\mathbf{C}}(W, N)), \tag{B.6}$$

$$(f : W \to \mathrm{Hom}_{\mathbf{C}}(P, N)) \mapsto (g : P \to \mathrm{Hom}_{\mathbf{C}}(W, N)), \tag{B.7}$$

where:

$$g(p)(w) = (-)^{\bar{p}\bar{w}} f(w)(p). \tag{B.8}$$

We have to verify that so defined g indeed belongs to $\mathrm{Hom}_{\mathcal{U}\mathbf{g}}(P, \mathrm{Hom}_{\mathbf{C}} (W, N))$; what has to be verified is the invariance of g under $\mathbf{g}$. For $\xi \in \mathbf{g}$, taking into account that used that $\bar{f} = \bar{g}$, we get:

$$\begin{aligned}
(\xi.g)(p)(w) &= \rho_N(\xi)(g(p)(w)) - (-)^{\bar{\xi}\,\overline{g(p)}} g(p)(\rho_W(\xi)w) \\
&\quad - (-)^{\bar{\xi}\,\bar{g}} g(\rho_P(\xi)p)(w) \\
&= (-)^{\bar{p}\bar{w}} \rho_N(\xi)(f(w)(p)) - (-)^{\bar{\xi}\,\overline{g(p)} + \bar{p}(\bar{\xi}+\bar{w})} f(\rho_W(\xi)w)(p) \\
&\quad - (-)^{\bar{\xi}\,\bar{g} + \bar{w}(\bar{p}+\bar{\xi})} f(w)(\rho_P(\xi)p) \\
&= (-)^{\bar{p}\bar{w}} (\rho_N(\xi)(f(w)(p)) - (-)^{\bar{\xi}\,\bar{f}} f(\rho_W(\xi)w)(p) -) \\
&\quad - (-)^{\bar{\xi}\,\overline{f(w)}} f(w)(\rho_P(\xi)p)).
\end{aligned} \tag{B.9}$$

This is zero because of the covariance condition on f. It can be similarly verified that the map $f \mapsto g$ commutes with the action of $\mathcal{U}\mathbf{g}$. Therefore (B.5) is naturally equivalent to:

$$W \mapsto \mathrm{Hom}_{\mathcal{U}\mathbf{g}}\left(P, \mathrm{Hom}_{\mathbf{C}}(W, N)\right), \tag{B.10}$$

which is a composition of the exact functors $W \mapsto \mathrm{Hom}_{\mathbf{C}}(W, N)$ and $V \mapsto \mathrm{Hom}_{\mathcal{U}\mathbf{g}}(P, V)$. This means that (B.5) is an exact functor, and therefore $\mathrm{Hom}_{\mathbf{C}}(P, N)$ is an injective $\mathcal{U}\mathbf{g}$-module. This concludes the proof of (B.2).

B.2 Shapiro's lemma for Ext

$$\mathrm{Ext}^n_{\mathcal{U}\mathbf{g}}(M, \mathrm{Hom}_{\mathcal{U}\mathbf{h}}(\mathcal{U}\mathbf{g}, A)) = \mathrm{Ext}^n_{\mathcal{U}\mathbf{h}}(M|_{\mathbf{h}}, A). \tag{B.11}$$

This is proved by noticing that:

$$\mathrm{Hom}_{\mathcal{U}\mathbf{g}}\left(P^i_M, \mathrm{Hom}_{\mathcal{U}\mathbf{h}}(\mathcal{U}\mathbf{g}, A)\right) = \mathrm{Hom}_{\mathcal{U}\mathbf{h}}\left(P^i_M\big|_{\mathcal{U}\mathbf{h}},\ A\right) \tag{B.12}$$

and that $P^i_M|_{\mathcal{U}\mathbf{h}}$ is a projective resolution for the restriction of M to $\mathcal{U}\mathbf{h}$, because $\mathcal{U}\mathbf{g}$ is projective (in fact free) as an $\mathcal{U}\mathbf{h}$ module[13].

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[13]Therefore the restriction of a projective module from $\mathcal{U}\mathbf{g}$ to $\mathcal{U}\mathbf{h}$ is a projective module over $\mathcal{U}\mathbf{h}$; to see this observe that projective modules are the same as free summands of free modules.

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