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Plane-symmetric spacetimes with positive cosmological constant. The case of stiff fluids

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Abstract

We consider plane-symmetric spacetimes satisfying Einstein's field equations with positive cosmological constant, when the matter is a fluid whose pressure is equal to its mass-energy density (i.e., a so-called stiff fluid). We study the initial-value problem for the associated Einstein equations and establish a global existence result. The late-time asymptotics of solutions is also rigorously derived, and we conclude that the spacetime approaches the de Sitter spacetime while the matter disperses asymptotically. A technical difficulty dealt with here lies in the fact that solutions may contain vacuum states as well as velocities approaching the speed of light, both possibilities leading to singular behavior in the evolution equations.

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1 Introduction

The study of global properties of cosmological spacetimes is a fundamental problem in mathematical relativity, as it provides a first step toward understanding fundamental issues such as the structure of singularities and the cosmic censorship conjecture. Such a study can be reduced to investigating the global existence and asymptotic behavior of solutions to the Einstein equations, possibly coupled to the equations of motion for a specific matter model. In the present paper, we treat the class of perfect fluids whose pressure p and mass-energy density $\mu \geq 0$ coincide. This is a limiting case ($\gamma = 2$) with the class of pressure laws $p = (\gamma - 1)\,\mu$, in which $\gamma \in [1, 2]$ is referred as the adiabatic exponent of the fluid. Our main result concerns the initial-value problem for the associated Einstein equations: we establish a global existence result and rigorously determine the late-time asymptotic behavior of solutions. This allows us to conclude that the spacetime is future geodesically complete and approaches the de Sitter spacetime whereas the matter asymptotically disperses.

Observe first that singularities generically arise in initially smooth solutions to the fluid equations, that is, shock waves in the general case $\gamma \in (1, 2]$ and shell-crossing singularities in the case $\gamma = 1$. This is true even when gravitational effects are taken into account [4]. If the solution is to be continued beyond shock waves, it is necessary to lower the regularity of initial data and search for weak solutions, as investigated by LeFloch (see the review [1] and the references therein).

On the other hand, existence of *smooth* solutions even in a long-time evolution can sometime be established in physically interesting situations. This is especially true when a cosmological constant is included, as we do in the present paper. Global-in-time solutions and the existence of future geodesically complete spacetimes can be established under a smallness condition on the initial data, as recognized by Tchapnda [10] for $\gamma = 1$ and under the assumption of plane symmetry and, later, without symmetry and for $\gamma \in (1, 4/3)$, by Rodnianski and Speck [5] and Speck [6, 7].

As far as the limiting case $\gamma = 2$ is concerned, plane symmetric spacetimes have been investigated by Tabensky and Taub [8] and LeFloch and Stewart [2]. In particular, Tabensky and Taub [8] rely on two different coordinate systems in their analysis, a comoving coordinate system in which the fluid is at rest, and a characteristic coordinate system. On the other hand, the work [1, 2] introduced the notion of weakly regular solutions to the Einstein equations.

In the present paper, we rely on areal coordinates, a coordinate system in which the time is defined to be the area–radius function determined by surfaces of symmetry. In these geometry-based coordinates, we prove a global-in-time existence theorem (in the future direction) for plane-symmetric solutions to the Einstein-stiff fluid equations with cosmological constant. Importantly, we also derive the leading asymptotic behavior of solutions and conclude with the future geodesic completeness of the constructed spacetime.

Our analysis relies on a change of fluid variables that allows us to write the fluid equations in a way analogous to the case of a massless scalar field, and then to take advantage of techniques for semi-linear hyperbolic equations. (A similar structure was observed in [11].) A specific technical difficulty overcome in this work originates in the fact that solutions may naturally contain vacuum states as well as velocities approaching the speed of light, both possibilities leading to singular behavior in the evolution equations.

Note finally that our results extend to compressible fluids the conclusions obtained by Tchapnda and Rendall [9] for the Vlasov equation of (collisionless) kinetic dynamics.

The outline of the paper is as follows. Section 2 is concerned with the derivation of the field equations for stiff fluids under plane-symmetry. Next, in Section 3 we develop the local existence and uniqueness theory and then, in Section 4, determine the global geometry and asymptotic behavior of the spacetimes under consideration.

2 Einstein-stiff fluid equations

2.1 Gravitational field equations

We consider spacetimes (M, g) such that the manifold has the topology $M = I \times \mathbb{T}^3$, where I is a real interval and $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ is the three-torus. The metric g and the matter fields are required to be invariant under the action of the Euclidean group E_2 on the universal cover. It is also required that the spacetime has an E_2-invariant Cauchy surface of constant areal time. In such conditions the metric can be expressed in the form

$$ds^2 = -e^{2\eta(t,x)}dt^2 + e^{2\lambda(t,x)}dx^2 + t^2(dy^2 + dz^2), \tag{2.1}$$

where the time variable describes $t > 0$ and the spatial variable the interval $x \in [0, 1]$, while the variables y and z range in $[0, 2\pi]$; the metric coefficients η and λ are periodic in x with period 1. The Einstein equations read

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta}, \tag{2.2}$$

where $G^{\alpha\beta}$ is the Einstein tensor, $T^{\alpha\beta}$ the energy-momentum tensor and Λ is the cosmological constant which we assume to be positive. We also introduce the notation

$$\rho = e^{2\eta} T^{00}, \quad j = e^{\lambda+\eta} T^{01}, \quad S = e^{2\lambda} T^{11}, \quad p = t^2 T^{22},$$

which defines the fluid variables of interest.

After a tedious computation in the above coordinates, (2.2) take the form of the following evolution and constraint equations (where the subscripts t, x denote partial differentiation):

$$e^{-2\eta}(2t\lambda_t + 1) - \Lambda t^2 = 8\pi t^2 \rho, \tag{2.3}$$

$$e^{-2\eta}(2t\eta_t - 1) + \Lambda t^2 = 8\pi t^2 S, \tag{2.4}$$

$$\eta_x = -4\pi t e^{\lambda+\eta} j, \tag{2.5}$$

$$e^{-2\lambda}(\eta_{xx} + \eta_x(\eta_x - \lambda_x)) - e^{-2\eta}\left(\lambda_{tt} + (\lambda_t - \eta_t)\left(\lambda_t + \frac{1}{t}\right)\right) + \Lambda = 8\pi p. \tag{2.6}$$

2.2 Stiff fluid equations

The so-called stiff fluid under consideration is an isentropic perfect fluid with energy density $\mu > 0$ equal to its pressure, that is, $p = \mu$. The four-velocity vector U^α of the fluid is normalized to be of unit length: $U^\alpha U_\alpha = -1$. The plane symmetry allows us to set $U^\alpha := \xi(e^{-\eta}, e^{-\lambda}u, 0, 0)$, where $\xi = (1 - u^2)^{-1/2}$ is the relativistic factor and u is the scalar velocity satisfying $|u| < 1$. The energy momentum tensor for the stiff fluid is

$$T^{\alpha\beta} = \mu(2U^\alpha U^\beta + g^{\alpha\beta}),$$

that is

$$\begin{aligned} T^{00} &= e^{-2\eta}\frac{1+u^2}{1-u^2}\mu =: e^{-2\eta}\rho, \quad T^{01} = e^{-\lambda-\eta}\frac{2u\mu}{1-u^2} =: e^{-\lambda-\eta}j, \\ T^{11} &= e^{-2\lambda}\frac{1+u^2}{1-u^2}\mu =: e^{-2\lambda}S, \quad T^{22} = T^{33} = t^{-2}\mu, \end{aligned} \tag{2.7}$$

while, due to the above assumptions, all the other components vanish identically.

The stiff fluid equations read

$$\nabla_\alpha T^{\alpha\beta} = 0. \tag{2.8}$$

We can assume that the components $T^{\alpha 2}$ and $T^{\alpha 3}$ vanish identically, while by computing the remaining two components we arrive at the two evolution equations

$$\begin{aligned}
\rho_t + e^{\eta-\lambda} j_x &= -2\lambda_t \rho - 2\eta_x e^{\eta-\lambda} j - \frac{2}{t}(\rho + \mu), \\
j_t + e^{\eta-\lambda} \rho_x &= -2\lambda_t j - 2\eta_x e^{\eta-\lambda} \rho - \frac{2}{t} j.
\end{aligned} \tag{2.9}$$

The equations may be put into a simpler form, as follows. Observe that the first-order principal part of (2.9) is a strictly hyperbolic system of two equations associated with the two distinct speeds $\pm e^{\eta-\lambda}$. Introducing the Riemann invariants

$$r := \frac{1+u}{1-u}\mu = \rho + j, \quad s := \frac{1-u}{1+u}\mu = \rho - j, \tag{2.10}$$

and the directional derivatives

$$D^+ := \partial_t + e^{\eta-\lambda}\partial_x, \quad D^- := \partial_t - e^{\eta-\lambda}\partial_x,$$

and then combining the equations in (2.9) together, we obtain

$$\begin{aligned}
D^+ r &= -2\left(\lambda_t + \eta_x e^{\eta-\lambda} + \frac{1}{t}\right) r - \frac{2}{t}\sqrt{rs}, \\
D^- s &= -2\left(\lambda_t - \eta_x e^{\eta-\lambda} + \frac{1}{t}\right) s - \frac{2}{t}\sqrt{rs}.
\end{aligned} \tag{2.11}$$

Finally, the expressions for λ_t and η_x taken from (2.3) and (2.5) can be plugged in (2.11), and by setting $X = e^\eta \sqrt{r}$ and $Y = e^\eta \sqrt{s}$ we arrive at

$$\begin{aligned}
D^+ X &= -\Lambda t e^{2\eta} X - \frac{1}{t} Y, \\
D^- Y &= -\frac{1}{t} X - \Lambda t e^{2\eta} Y,
\end{aligned} \tag{2.12}$$

which we will refer to as the *stiff fluid equations* for the unknowns r and s.

2.3 Basic properties

It is easily checked that (2.6) is a *consequence* of equations (2.3)–(2.5), (2.9). One can also check that (2.5) is a constraint equation, that is, it is automatically satisfied for all times once it is satisfied on an initial Cauchy hypersurface. Therefore, we will work with (2.3), (2.4) and (2.12) for the unknowns η, λ, r and s. Observe that by definition r and s must be non-negative. From the definition we see that $S = \rho = (r+s)/2$.

We will solve the initial-value problem with data prescribed on the hypersurface $t = 1$. Observe that once the fluid variables have been determined, the metric coefficient η is obtained by integrating (2.4) in the time direction, i.e.,

$$e^{-2\eta(t,x)} = \frac{e^{-2\overline{\eta}(x)}}{t} + \frac{1}{t}\int_1^t \tau^2(\Lambda - 4\pi(r+s)(\tau,x))\,d\tau \tag{2.13}$$

with $\overline{\eta} := \eta(1,\cdot)$. Next, η being known, the following equation (obtained from (2.3) and (2.4)),

$$\lambda_t(t,x) = \eta_t(t,x) + \Lambda t e^{2\eta} - \frac{1}{t}, \tag{2.14}$$

is integrated in time to yield the second metric coefficient

$$\lambda(t,x) = \overline{\lambda}(1,x) + \int_1^t \lambda_t(\tau,x)\,d\tau \tag{2.15}$$

with $\overline{\lambda} = \lambda(1,\cdot)$. Therefore, it will be enough to concentrate on the stiff fluid equations (2.12) together with the metric equation (2.13), that determine an evolution system for the unknowns η, r, s.

Observe that there exists some $T^* > 1$ such that the right-hand side term in (2.13) is positive on $[1, T^*) \times [0,1]$. Estimates for r and s can easily be derived as follows. The expressions for λ_t and η_x taken from (2.3) and (2.5) can be plugged in (2.11) to yield

$$\begin{aligned} D^+ r &= -\left(8\pi t e^{2\eta} s + \Lambda t e^{2\eta} + \frac{1}{t}\right) r - \frac{2}{t}\sqrt{rs}, \\ D^- s &= -\left(8\pi t e^{2\eta} r + \Lambda t e^{2\eta} + \frac{1}{t}\right) s - \frac{2}{t}\sqrt{rs}. \end{aligned} \tag{2.16}$$

Using the fact that r and s are positive, this implies

$$D^+ r \le -t^{-1} r, \quad D^- s \le -t^{-1} s,$$

and integrating this along the characteristic curves associated with the operators $D^{\pm}$ implies that

$$r \leq r(1,\cdot)t^{-1}, \quad s \leq s(1,\cdot)t^{-1}. \tag{2.17}$$

As a consequence, if η is bounded then so are X and Y.

A straightforward computation leads to the following result.

Lemma 2.1. *Set*

$$\begin{aligned} b_1 &= (\lambda - \eta)_x e^{\eta-\lambda} - \Lambda t e^{2\eta}, \quad b_2 = -2\Lambda t \eta_x e^{2\eta}, \\ b_3 &= (\eta - \lambda)_x e^{\eta-\lambda} - \Lambda t e^{2\eta}, \quad b = -\frac{1}{t}. \end{aligned}$$

If X and Y solve (2.12) then X_x and Y_x satisfy

$$\begin{aligned} D^+ X_x &= b_1 X_x + b Y_x + b_2 X, \\ D^- Y_x &= b X_x + b_3 Y_x + b_2 Y. \end{aligned} \tag{2.18}$$

The following result will be used to obtain bounds on derivatives of X and Y.

Lemma 2.2. *Set*

$$\begin{aligned} K(t) &= \sup\{(X+Y)(t,x) \mid x \in [0,1]\}, \\ A(t) &= \sup\{(|X_x| + |Y_x|)(t,x) \mid x \in [0,1]\}, \\ v(t) &= \sup\{|(\lambda-\eta)_x| e^{\eta-\lambda} + \Lambda t e^{2\eta} + \frac{1}{t} \mid x \in [0,1]\}, \\ h(t) &= 2\Lambda t \sup\{|\eta_x| e^{2\eta} \mid x \in [0,1]\}. \end{aligned}$$

If (X,Y) and (X_x,Y_x) solve (2.12) and (2.18), respectively, with $X_x(1) = (e^{\overline{\eta}}\sqrt{\overline{r}})_x$ and $Y_x(1) = (e^{\overline{\eta}}\sqrt{\overline{s}})_x$, then

$$A(t) \leq A(1) + \int_1^t \big(v(\tau)A(\tau) + h(\tau)K(\tau)\big)\, d\tau. \tag{2.19}$$

Proof. Equations (2.18) can be written in the form

$$\frac{d}{dt}X_x(t,\gamma_1(t)) = \big(b_1X_x + bY_x + b_2X\big)(t,\gamma_1(t)),$$
$$\frac{d}{dt}Y_x(t,\gamma_2(t)) = \big(bX_x + b_3Y_x + b_2Y\big)(t,\gamma_2(t)),$$

where γ_1 and γ_2 are the integral curves corresponding to D^+ and D^- respectively.

Integrating this over $[1,t]$, taking the absolute value in each equation, adding the resulting inequalities and taking the supremum of each term yields (2.19). □

3 Local existence theory

3.1 Main statement of this section

We are interested in regular solutions, defined as follows.

Definition 3.1. A regular solution to the plane-symmetric Einstein-stiff fluid equations consists of two metric coefficients η, λ and Riemann invariants r, s given as continuously differentiable functions defined on $[1,T]\times[0,1]$ and periodic in space.

We pose the initial-value problem by choosing some functions $\overline{\eta}, \overline{\lambda}, \overline{r}, \overline{s}$ as periodic functions on $[0,1]$ satisfying the constraint

$$\overline{\eta}_x = -2\pi t e^{\overline{\lambda}+\overline{\eta}}(\overline{r}-\overline{s}), \tag{3.1}$$

and, on the initial hypersurface $t=1$, we impose

$$(\eta,\lambda,r,s)(1,\cdot) = (\overline{\eta},\overline{\lambda},\overline{r},\overline{s}). \tag{3.2}$$

Theorem 3.1 (Local existence and uniqueness theory in the Riemann variables)**.** *Given periodic, continuously differentiable data $\overline{\eta}, \overline{\lambda}, \overline{r}, \overline{s}$ prescribed on the initial hypersurface $t=1$ and satisfying the constraint* (3.1)*, there exists a future development which consists of continuously differentiable functions η, λ, r, s defined on some time interval $[1,T)$ (with $T\in(1,\infty]$)) that are periodic in space and satisfy the stiff fluid equations* (2.12)*, together with the evolution equations* (2.3) *and* (2.4)*.*

Once the Riemann invariants r and s are known, the primary fluid variables μ and u can be determined from equations (2.7) and (2.10):

$$\mu = \sqrt{rs}, \quad u = \frac{\sqrt{r} - \sqrt{s}}{\sqrt{r} + \sqrt{s}}.$$

By construction, the Riemann invariants are *bounded*, and this property is equivalent to the following restriction in the fluid variables:

$$\frac{1 \pm u}{1 \mp u}\, \mu \lesssim 1. \tag{3.3}$$

Observe that this condition allows the density to vanish, and the velocity component u to approach ± 1, which is the normalized light-speed. The condition is equivalent to saying

$$0 \leq \mu \lesssim 1 - |u|^2. \tag{3.4}$$

Theorem 3.2 (Local existence and uniqueness theory in the fluid variables)**.** *Under the assumptions of Theorem* 3.1*, the problem with initial data satisfying the uniform bound* (3.4) *admits a local-in-time solution which is unique in the following (generalized) sense: if μ_1, u_1 and μ_2, u_2 denote fluid solutions to the same initial-value problem, then*

$$\begin{aligned} &\textit{either } \mu_1 = \mu_2 > 0 \quad \textit{and } \; u_1 = u_2, \\ &\textit{or } \mu_1 = \mu_2 = 0 \;\; \textit{and } \; u_1, u_2 \;\; \textit{are arbitrary.} \end{aligned}$$

3.2 Proof of the local existence result

We rely on an iterative argument and define a sequence (η_n, r_n, s_n) in the following way.

(1) For $t \in [1, +\infty)$ and $x \in [0,1]$, we set $(\eta_0, r_0, s_0)(t,x) := (\overline{\eta}, \overline{r}, \overline{s})(x)$, $T_0 = +\infty$.

(2) If η_{n-1}, r_{n-1}, s_{n-1} are regular on $[1, T_{n-1}) \times [0,1]$ with $T_{n-1} \leq \infty$, then we define T_n to be supremum of all $t' \in (1, T_{n-1})$ such that

$$\frac{e^{-2\overline{\eta}(x)}}{t} + \frac{1}{t}\int_1^t \tau^2(\Lambda - 4\pi(r_{n-1} + s_{n-1})(\tau, x))\, d\tau > 0,$$

for all $x \in [0,1]$ and $t \in [1, t']$, and we then set

$$e^{-2\eta_n(t,x)} = \frac{e^{-2\overline{\eta}(x)}}{t} + \frac{1}{t}\int_1^t \tau^2(\Lambda - 4\pi(r_{n-1} + s_{n-1})(\tau, x))\, d\tau. \tag{3.5}$$

(3) We define r_n and s_n such that $X_n = e^{\eta_n}\sqrt{r_n}$, and $Y_n = e^{\eta_n}\sqrt{s_n}$ are solutions of the system

$$
\begin{aligned}
D^+_{n-1} X_n &= a_{n-1} X_{n-1} + b Y_{n-1}, \\
D^-_{n-1} Y_n &= b X_{n-1} + a_{n-1} Y_{n-1},
\end{aligned}
\tag{3.6}
$$

where $a_{n-1} = -\Lambda t e^{2\eta_{n-1}}$, $b = -\frac{1}{t}$. $D^\pm_n$ is the $D^\pm$-operator corresponding to the nth iterate. We prescribe the same initial data (3.2) for all n.

Observe that $T_n \geq T^*$ for all n, so that all the iterates are well defined and regular on the fixed time interval $[1, T^*)$.

In order to prove that the sequence of iterates converges to a regular solution, we establish uniform bounds on the iterates as well as their time and space derivatives, and we prove their uniform convergence. This is done in a series of lemmas.

In the sequel we denote by $\| \ \|$ the sup-norm on the function space of interest, C denotes a constant that may change at each occurrence.

Lemma 3.1. *The sequences η_n, X_n, Y_n, r_n, s_n and $(\eta_n)_t$ are uniformly bounded in n, in the sup-norm by a continuous function of t, on a time interval $[1, T^{(1)}]$.*

Proof. Set

$$
\begin{aligned}
P_n(t) &:= \sup\{e^{2\eta_n(t,x)} \ | x \in [0,1]\}, \\
K_n(t) &:= \sup\{(X_n + Y_n)(t,x) \ | x \in [0,1]\}.
\end{aligned}
$$

Using equations (3.6), we apply the same argument used in the proof of Lemma 2.2 and obtain

$$
K_n(t) \leq K_0 + \int_1^t m_{n-1}(\tau) K_{n-1}(\tau) \, d\tau, \tag{3.7}
$$

with

$$
\begin{aligned}
m_n(t) &= \sup\left\{ \Lambda t e^{2\eta_n} + \frac{1}{t}; \ x \in [0,1] \right\} \\
&\leq t(1+\Lambda)(1 + P_n(t)),
\end{aligned}
$$

so that

$$
K_n(t) \leq K_0 + (1+\Lambda) \int_1^t \tau(1 + P_{n-1}(\tau)) K_{n-1}(\tau) \, d\tau. \tag{3.8}
$$

On the other hand, equation (3.5) implies

$$(\eta_n)_t = \frac{1}{2t} - \frac{\Lambda}{2} t e^{2\eta_n} + 2\pi t e^{2\eta_n - 2\eta_{n-1}} (X_{n-1}^2 + Y_{n-1}^2), \tag{3.9}$$

and since $e^{-2\eta_{n-1}} \le \frac{e^{-2\overline{\eta}} + \Lambda t^3}{t} \le C(1+\Lambda)t^2$, it follows that

$$P_n(t) \le \|e^{2\overline{\eta}}\| + C(1+\Lambda) \int_1^t \tau^3 (1 + K_{n-1}(\tau))^2 (1 + P_n(\tau))^2 \, d\tau. \tag{3.10}$$

Now defining $Q_n(t) := \sup\{K_m(t) + P_m(t); m \le n\}$ and adding (3.7) and (3.10) we arrive at

$$Q_n(t) \le K_0 + \|e^{2\overline{\eta}}\| + C(1+\Lambda) \int_1^t \tau^3 (1 + Q_n(\tau))^4 \, d\tau. \tag{3.11}$$

Let $[1, T^{(1)})$ (with $T^{(1)} \in (1, T^*]$) be the maximal interval of existence for the solution z_1 of the integral equation

$$z_1(t) = K_0 + \|e^{2\overline{\eta}}\| + C(1+\Lambda) \int_1^t \tau^3 (1 + z_1(\tau))^4 \, d\tau, \quad z_1(1) = K_0 + \|e^{2\overline{\eta}}\|.$$

Then $Q_n(t) \le z_1(t)$, for all $n \in \mathbb{N}$ and $t \in (1, T^{(1)})$. The same is true for K_n and P_n. It follows that η_n, X_n, Y_n, and then r_n, s_n and $(\eta_n)_t$ are uniformly bounded. To bound $(\eta_n)_t$, we use (3.9). □

Lemma 3.2. *The sequences $(\eta_n)_x$, $(X_n)_x$, $(Y_n)_x$, $(X_n)_t$, $(Y_n)_t$, $(r_n)_x$, $(s_n)_x$, $(r_n)_t$ and $(s_n)_t$ are uniformly bounded in n, the sup-norm by a continuous function of t on a time interval $[1, T^{(2)}]$.*

Proof. Set

$$\begin{aligned} A_n(t) &:= \sup\{|(X_n)_x| + |(Y_n)_x|(t,x) \ |x \in [0,1]\}, \\ A_0 &:= \sup\{(|\overline{X}_x| + |\overline{Y}_x|)(x) \ |x \in [0,1]\}, \\ B_n(t) &:= \sup\{|(e^{-2\eta_n(t,x)})_x| \ |x \in [0,1]\}. \end{aligned}$$

Then taking the spatial derivative in (3.6) gives the following equations:

$$\begin{aligned} D_{n-1}^+ (X_n)_x &= (\lambda_{n-1} - \eta_{n-1})_x e^{\eta_{n-1} - \lambda_{n-1}} (X_n)_x - 2\Lambda t (\eta_{n-1})_x e^{2\eta_{n-1}} X_n \\ &\quad - \Lambda t e^{2\eta_{n-1}} (X_{n-1})_x - \frac{1}{t} (Y_{n-1})_x, \\ D_{n-1}^- (Y_n)_x &= (\eta_{n-1} - \lambda_{n-1})_x e^{\eta_{n-1} - \lambda_{n-1}} (Y_n)_x - 2\Lambda t (\eta_{n-1})_x e^{2\eta_{n-1}} Y_n \\ &\quad - \Lambda t e^{2\eta_{n-1}} (Y_{n-1})_x - \frac{1}{t} (X_{n-1})_x. \end{aligned}$$

But, using Lemma 3.1, we have

$$\begin{aligned}|(\lambda_{n-1} - \eta_{n-1})_x(s)| &= |(\overline{\lambda} - \overline{\eta})_x + 2\Lambda \int_1^s \tau(\eta_{n-1})_x e^{2\eta_{n-1}} \, d\tau| \\ &\le Cs^2(1 + B_{n-1}(s)),\end{aligned}$$

so that applying the same argument as in Lemma 2.2 and using Lemma 3.1 again, we obtain

$$A_n(t) \le A_0 + C \int_1^t \tau^2(1 + B_{n-1}(\tau))(1 + A_{n-1}(\tau) + A_n(\tau))) \, d\tau. \tag{3.12}$$

On the other hand, we have

$$(e^{-2\eta_n(t,x)})_x = \frac{-2\overline{\eta}_x e^{-2\overline{\eta}}}{t} - \frac{4\pi}{t} \int_1^t \tau^2(r_{n-1} + s_{n-1})_x(\tau, x)) \, d\tau, \tag{3.13}$$

which implies

$$B_n(t) \le 2\|\overline{\eta}_x e^{-2\overline{\eta}}\| + C \int_1^t \tau^2(A_{n-1} + B_{n-1})(\tau) \, d\tau. \tag{3.14}$$

We have used the fact that

$$\begin{aligned}|(r_n + s_n)_x| &= |(e^{-2\eta_n})_x(X_n^2 + Y_n^2) + 2e^{-2\eta_n}\big(X_n(X_n)_x + Y_n(Y_n)_x\big)| \\ &\le C(A_n + B_n)(t).\end{aligned}$$

Now defining $E_n(t) := \sup\{A_m(t) + B_m(t); m \le n\}$ and adding (3.12) and (3.14) we arrive at

$$E_n(t) \le A_0 + 2\|\overline{\eta}_x e^{-2\overline{\eta}}\| + C \int_1^t \tau^2(1 + E_n(\tau))^2 \, d\tau. \tag{3.15}$$

Let $[1, T^{(2)})$ (with $T^{(2)} \le T^{(1)}$) be the maximal interval of existence for the solution z_2 of the integral equation

$$\begin{aligned}z_2(t) &= A_0 + 2\|\overline{\eta}_x e^{-2\overline{\eta}}\| + C \int_1^t \tau^2(1 + z_2(\tau))^2 \, d\tau, \\ z_2(1) &= A_0 + 2\|\overline{\eta}_x e^{-2\overline{\eta}}\|.\end{aligned}$$

Then $E_n(t) \le z_2(t)$, for all $n \in \mathbb{N}$ and $t \in (1, T^{(2)})$. The same is true for A_n and B_n. It follows that $(\eta_n)_x$, $(X_n)_x$, $(Y_n)_x$, $(X_n)_t$, $(Y_n)_t$, $(r_n)_x$, $(s_n)_x$ and then $(r_n)_t$, $(s_n)_t$ and $(\eta_n)_{tx}$ are uniformly bounded. $\square$

Lemma 3.3. *The sequences* (η_n), (X_n) *and* (Y_n) *converge uniformly on* $[1, T^{(3)}]$ *for all* $T^{(3)}$ *less than* $T^{(2)}$.

Proof. For $t \in [1, T^{(3)}]$, define

$$\begin{aligned}
\theta_n(t) &:= \sup\{|X_{n+1} - X_n|(t,x) + |Y_{n+1} - Y_n|(t,x); x \in [0,1]\},\\
\alpha_n(t) &:= \sup\{\|(\eta_{n+1} - \eta_n)(s)\| + \|(X_{n+1} - X_n)(s)\|\\
&\qquad + \|(Y_{n+1} - Y_n)(s)\|; s \in [1,t]\},\\
\tilde{X}_n &:= X_{n+1} - X_n, \quad \tilde{Y}_n := Y_{n+1} - Y_n.
\end{aligned}$$

Combining equations (3.6) written for $n+1$ and n gives

$$\begin{aligned}
D_n^+ \tilde{X}_n &= a_n \tilde{X}_{n-1} + b\tilde{Y}_{n-1} + F_n,\\
D_n^- \tilde{Y}_n &= b\tilde{X}_{n-1} + a_n \tilde{Y}_{n-1} + G_n,
\end{aligned} \tag{3.16}$$

with

$$\begin{aligned}
F_n &= -(e^{2\eta_n} - e^{2\eta_{n-1}})\Lambda t X_{n-1} - (e^{\eta_n - \lambda_n} - e^{\eta_{n-1} - \lambda_{n-1}})(X_n)_x,\\
G_n &= -(e^{2\eta_n} - e^{2\eta_{n-1}})\Lambda t Y_{n-1} + (e^{\eta_n - \lambda_n} - e^{\eta_{n-1} - \lambda_{n-1}})(Y_n)_x.
\end{aligned}$$

Reasoning as in the proof of Lemma 2.2, we have

$$\theta(t) \le \int_1^t \big(m_n(\tau)\theta_{n-1} + \sup\{|F_n(\tau,x)| + |G_n(\tau,x)|; x \in [0,1]\}\big)\, d\tau, \tag{3.17}$$

and this implies that

$$|\tilde{X}_n| + |\tilde{Y}_n| \le C \int_1^t \alpha_{n-1}(\tau)\, d\tau, \tag{3.18}$$

we have used the mean value theorem to handle the terms $e^{2\eta_n} - e^{2\eta_{n-1}}$ and $e^{\eta_n - \lambda_n} - e^{\eta_{n-1} - \lambda_{n-1}}$, and the previous lemmas.

On the other hand, equation (3.9) implies

$$(\eta_{n+1}-\eta_n)_t = -\frac{\Lambda}{2}t(e^{2\eta_{n+1}}-e^{2\eta_n}) + 2\pi t e^{2\eta_{n+1}-2\eta_n}\big((X_{n+1}^2-X_n^2) \\ + (Y_{n+1}^2-Y_n^2)\big) + 2\pi t(e^{2\eta_{n+1}-2\eta_n}-e^{2\eta_n-2\eta_{n-1}})(X_n^2+Y_n^2),$$

and using Lemma 3.1 and the mean value theorem it follows after integration in time that

$$|\eta_{n+1}-\eta_n| \leq C\int_1^t (|\eta_{n+1}-\eta_n| + |\eta_n-\eta_{n-1}| + |X_{n+1}-X_n| \\ + |Y_{n+1}-Y_n|)(\tau)\, d\tau,$$

so that

$$|\eta_{n+1}-\eta_n| \leq C\int_1^t (\alpha_n+\alpha_{n-1})(\tau)\, d\tau. \tag{3.19}$$

Combining (3.18) and (3.19) leads to

$$\alpha_n(t) \leq C\int_1^t (\alpha_n+\alpha_{n-1})(\tau)\, d\tau,$$

which, by Gronwall's inequality, implies

$$\alpha_n(t) \leq C\int_1^t \alpha_{n-1}(\tau)\, d\tau,$$

and by induction

$$\alpha_n(t) \leq \frac{C^{n+1}}{n!},$$

and so $\alpha_n \to 0$ as $n\to\infty$. This establishes the uniform convergence of η_n, X_n and Y_n. □

It follows from (3.9) that the sequence $(\eta_n)_t$ converges uniformly as well. In the following lemma, the uniform convergence of other iterates derivatives is proven.

Lemma 3.4. *The sequences $(\eta_n)_x$, $(X_n)_x$, $(Y_n)_x$, $(X_n)_t$ and $(Y_n)_t$ converge uniformly on $[1, T^{(4)}]$, where $[1,T^{(4)}]\subset[1,T^{(3)}]$.*

Proof. We set

$$\beta_n(t) := \sup\Big\{\|(\eta_{n+1}-\eta_n)_x(s)\| + \|(X_{n+1}-X_n)_x(s)\| \\ + \|(Y_{n+1}-Y_n)_x(s)\|; s\in[1,t]\Big\}.$$

Taking the space derivative in equations (3.6) gives

$$D_n^+(X_{n+1})_x = \tilde{C}_n, \quad D_n^-(Y_{n+1})_x = \tilde{D}_n \tag{3.20}$$

with

$$\tilde{C}_n = (\lambda_n-\eta_n)_x e^{\eta_n-\lambda_n}(X_{n+1})_x - 2\Lambda t(\eta_n)_x e^{2\eta_n}X_n - \Lambda t e^{2\eta_n}(X_n)_x - \frac{1}{t}(Y_n)_x,$$
$$\tilde{D}_n = (\eta_n-\lambda_n)_x e^{\eta_n-\lambda_n}(Y_{n+1})_x - 2\Lambda t(\eta_n)_x e^{2\eta_n}Y_n - \Lambda t e^{2\eta_n}(Y_n)_x - \frac{1}{t}(X_n)_x.$$

Let γ_n^1 and γ_n^2 be the integral curves corresponding to D_n^+ and D_n^- respectively, that start from the point (s,x) that is, for each n,

$$(\gamma_n^1)_t = e^{\eta_n-\lambda_n},\ (\gamma_n^2)_t = -e^{\eta_n-\lambda_n},\ \gamma_n^1(s)=\gamma_n^2(s)=x. \tag{3.21}$$

Integrating the first equation in (3.20) along γ_n^1, the second one along γ_n^2 yields after subtraction

$$\begin{aligned}(X_{n+1}-X_n)(s) &= \int_1^s \left(\tilde{C}_n(\tau,\gamma_n^1(\tau)) - \tilde{C}_{n-1}(\tau,\gamma_{n-1}^1(\tau))\right)\,d\tau,\\ (Y_{n+1}-Y_n)(s) &= \int_1^s \left(\tilde{D}_n(\tau,\gamma_n^1(\tau)) - \tilde{D}_{n-1}(\tau,\gamma_{n-1}^1(\tau))\right)\,d\tau.\end{aligned} \tag{3.22}$$

But we have

$$|\tilde{C}_n(\tau,\gamma_n^1(\tau)) - \tilde{C}_{n-1}(\tau,\gamma_{n-1}^1(\tau)| \\ \leq |\tilde{C}_n(\tau,\gamma_n^1(\tau)) - \tilde{C}_n(\tau,\gamma_{n-1}^1(\tau)| + |(\tilde{C}_n-\tilde{C}_{n-1})(\tau)|. \tag{3.23}$$

Given now any $\varepsilon>0$, we find, for any sufficiently large n,

$$|\tilde{C}_n(\tau,\gamma_n^1(\tau)) - \tilde{C}_n(\tau,\gamma_{n-1}^1(\tau)| \leq C\varepsilon, \tag{3.24}$$

we have used the uniform convergence of η_n, the uniform continuity of $\tilde{C}_n$ over the compact set $[1,T^{(4)}]\times\big(\gamma_n^1([1,T^{(4)}])\cup\gamma_{n-1}^1([1,T^{(4)}])\big)$, and the following inequality which follows from (3.21)

$$|\gamma_n^1-\gamma_{n-1}^1|(\tau) \leq C\,\sup\Big\{\|(e^{2\eta_n}-e^{2\eta_n-1})(t)\|\ ;\ t\in[1,T^{(4)}]\Big\}. \tag{3.25}$$

For the second term of the right-hand side in (3.23) we have

$$
\begin{aligned}
&\tilde{C}_n - \tilde{C}_{n-1} \\
&\quad = \big((\lambda_n - \eta_n)_x - (\lambda_{n-1} - \eta_{n-1})_x\big) e^{\eta_n - \lambda_n} (X_{n+1})_x + (\lambda_{n-1} - \eta_{n-1})_x \\
&\qquad \times \Big(e^{\eta_n - \lambda_n} (X_{n+1} - X_n)_x + (e^{\eta_n - \lambda_n} - e^{\eta_{n-1} - \lambda_{n-1}})(X_{n+1})_x \Big) \\
&\qquad - 2\Lambda t (\eta_n - \eta_{n-1})_x e^{2\eta_n} X_n - 2\Lambda t (\eta_{n-1})_x \\
&\qquad \times \Big(e^{2\eta_n} (X_n - X_{n-1}) + (e^{2\eta_n} - e^{2\eta_{n-1}})(X_{n-1}) \Big) \\
&\qquad - \Lambda t (e^{2\eta_n} - e^{2\eta_{n-1}})(X_n)_x - \Lambda t e^{2\eta_{n-1}} \\
&\qquad \times \big((X_n)_x - (X_{n-1})_x\big) - \frac{1}{t}\big((Y_n)_x - (Y_{n-1})_x\big),
\end{aligned}
$$

and

$$
(\lambda_n - \eta_n)_x = (\overline{\lambda} - \overline{\eta})_x + 2\Lambda \int_1^t t (\eta_n)_x e^{2\eta_n} \, d\tau,
$$

so that

$$
|(\lambda_n - \eta_n)_x - (\lambda_{n-1} - \eta_{n-1})_x| \leq C\varepsilon + C \sup\{\|(\eta_n - \eta_{n-1})_x(t)\|; t \in [1, T^{(4)}]\}.
$$

Thus, for n sufficiently large,

$$
\|(\tilde{C}_n - \tilde{C}_{n-1})(\tau)\| \leq C\varepsilon + C(\beta_n + \beta_{n-1})(\tau). \tag{3.26}
$$

It then follows from (3.22)–(3.24) and (3.26) that for n sufficiently large,

$$
\begin{aligned}
|(X_{n+1} - X_n)_x|(s) &\leq C\varepsilon + C \int_1^s (\beta_n + \beta_{n-1})(\tau) \, d\tau, \\
|(Y_{n+1} - Y_n)_x|(s) &\leq C\varepsilon + C \int_1^s (\beta_n + \beta_{n-1})(\tau) \, d\tau.
\end{aligned} \tag{3.27}
$$

On the other hand, taking the spatial derivative in (3.9), subtracting the resulting equations written for $n+1$ and n gives

$$
\begin{aligned}
&(\eta_{n+1} - \eta_n)_{tx} \\
&\quad = -\Lambda t (\eta_{n+1} - \eta_n)_x e^{2\eta_{n+1}} - \Lambda t (\eta_n)_x (e^{2\eta_{n+1}} - e^{2\eta_n}) \\
&\qquad + 4\pi t (\eta_{n+1} - \eta_n)_x e^{2(\eta_{n+1} - \eta_n)} (X_n^2 + Y_n^2) \\
&\qquad - 4\pi t (\eta_n - \eta_{n-1})_x e^{2(\eta_n - \eta_{n-1})} (X_{n-1}^2 + Y_{n-1}^2) \\
&\qquad + 4\pi t e^{2(\eta_{n+1} - \eta_n)} \big((X_n - X_{n-1})_x X_n + (Y_n - Y_{n-1})_x Y_n \big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\pi t e^{2(\eta_{n+1}-\eta_n)}\big((X_{n-1})_x X_n + (Y_{n-1})_x Y_n\big) \\
&- 4\pi t e^{2(\eta_n-\eta_{n-1})}\big((X_{n-1})_x X_{n-1} + (Y_{n-1})_x Y_{n-1}\big),
\end{aligned}
$$

from this and the previous lemmas, it follows that, for n sufficiently large,

$$
|(\eta_{n+1}-\eta_n)_x|(s) \le C\varepsilon + C\int_1^s (\beta_n + \beta_{n-1})(\tau)\, d\tau. \tag{3.28}
$$

Combining (3.27) and (3.28), and taking the supremum over $s \in [1,t]$ yields, for n sufficiently large,

$$
\beta_n(t) \le C\varepsilon + C\int_1^t (\beta_n + \beta_{n-1})(\tau)\, d\tau,
$$

and by Gronwall's lemma it follows that, for n sufficiently large and $t \in [1, T^{(4)}]$,

$$
\delta_n(t) \le C\varepsilon,
$$

where $\delta_n(t) := \sup\{\beta_m, m \le n\}$. The uniform convergence of $(\eta_n)_x$, $(X_n)_x$, $(Y_n)_x$, $(X_n)_t$ and $(Y_n)_t$ follows. $\square$

Lemmas 3.3 and 3.4 allow us to pass to the limit in (3.5) and (3.6) and obtain a regular solution (η, X, Y) to our system on a time interval $[1, T)$. It is easily checked that this solution is unique. Namely, let (η_i, X_i, Y_i), $i = 1, 2$, be two regular solutions of the Cauchy problem for the same initial data $(\overline{\eta}, \overline{X}, \overline{Y})$ at $t = 1$. Using the same argument as in the proof of iterates convergence leads to

$$
\alpha(t) \le C\int_1^t \alpha(\tau)\, d\tau,
$$

where $\alpha(t) = \sup\{\|(\eta_1-\eta_2)(s)\| + \|(X_1-X_2)(s)\| + \|(Y_1-Y_2)(s)\|\ s \in [1,t]\}$. It follows that $\alpha(t) = 0$, for $t \in [1, T)$ i.e., the solution is unique.

We have thus established the existence of a unique, local-in-time regular solution (η, λ, r, s) to the Cauchy problem for the plane symmetric Einstein-stiff fluid equations written in areal coordinates.

4 Global existence theory and asymptotics

4.1 Global existence

We are now in a position to establish the following main result, which takes advantage of our assumption $\Lambda > 0$.

Theorem 4.1 (Global existence theory and asymptotics)**.** *Under the assumptions in Theorem* 3.1*, the solution constructed therein is defined up to $T = +\infty$, the spacetime is future geodesically complete, and the following asymptotic properties hold at late times:*

$$\begin{aligned}
&\eta = -\ln t(1 + O\big((\ln t)^{-1}\big)), \quad \lambda = \ln t(1 + O\big((\ln t)^{-1}\big)), \\
&r = O(t^{-1}), \quad s = O(t^{-1}), \\
&\eta_t = -\frac{1}{t}(1 + O(t^{-1})), \quad \lambda_t = \frac{1}{t}(1 + O(t^{-1})), \\
&\eta_x = O(1).
\end{aligned} \tag{4.1}$$

Consequently, the generalized Kasner exponents associated with this spacetime (*cf.* (4.11)*, below*) *tend to* $1/3$*:*

$$\lim_{t\to\infty} \frac{\kappa_1^1(t,x)}{\kappa(t,x)} = \lim_{t\to\infty} \frac{\kappa_2^2(t,x)}{\kappa(t,x)} = \lim_{t\to\infty} \frac{\kappa_3^3(t,x)}{\kappa(t,x)} = \frac{1}{3},$$

where $\kappa = \kappa_i^i$ denotes the trace of the second fundamental form κ_i^j.

In particular, this shows that the spacetime approaches the *de Sitter spacetime* asymptotically. To establish this global result, we begin with a continuation criterion, based on the same notation as in the previous section.

Lemma 4.1. *Let $[1, T)$ be the maximal interval of existence of solutions to the system under consideration. If* $\sup\{|\eta(t,x)| \; |x \in [0,1], t \in [1,T)\} < +\infty$ *then $T = +\infty$.*

Proof. It suffices to prove that under the assumption that η is bounded on $[1, T)$, the same is true for η_x, η_t, X, Y, X_x, Y_x, X_t and Y_t. First of all, by definition we have $X = e^\eta \sqrt{r}$ and $Y = e^\eta \sqrt{s}$ and it follows from the decay inequalities (2.17) that X and Y are bounded. Next, recalling that $\eta_x = -2\pi t e^{\eta+\lambda}(r - s)$ and $\eta_t = \frac{1}{2t} + 2\pi t(X^2 + Y^2) - \frac{\Lambda t}{2} e^{2\eta}$, we find that η_x and η_t are bounded as well. Here, we have used the fact that

$$\lambda(t,x) = (\overline{\lambda} - \overline{\eta})(x) + \eta(t,x) - \ln t + \Lambda \int_1^t \tau e^{2\eta}(\tau, x) \, d\tau.$$

Taking the spatial derivative in this equation implies

$$(\lambda - \eta)_x(t,x) = (\overline{\lambda} - \overline{\eta})(x) + 2\Lambda \int_1^t \tau(\eta_x e^{2\eta})(\tau, x) \, d\tau,$$

so that $v(t)$, defined in Lemma 2.2, is bounded. Rewriting (2.19)

$$A(t) \leq A(1) + \int_1^t \big(v(\tau)A(\tau) + h(\tau)K(\tau)\big)\ d\tau,$$

and using the fact that h and K are bounded, Gronwall's lemma allows us to conclude that A, and then X_x and Y_x are bounded. Bounds on X_t and Y_t then follow from (2.12). □

We now prove that η is bounded in order to conclude that $T = +\infty$.

Lemma 4.2. *The function η satisfies*

$$\sup\big\{|\eta(t,x)|/x \in [0,1], t \in [1,T)\big\} < +\infty.$$

Proof. We can deduce from (2.13) that $e^{-2\eta(t,x)} \leq \frac{e^{-2\overline{\eta}(x)}+\Lambda t^3}{t}$, i.e.,

$$e^{2\eta(t,x)} \geq \frac{t}{C + \Lambda t^3}, \tag{4.2}$$

which provides a (negative, say) lower bound on η. Now, let us prove that

$$\int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx \leq Ct^{-4},\ t \in [1,T),\ x \in [0,1], \tag{4.3}$$

which will eventually lead us to an upper bound for η.

Using the equations (2.3), (2.4) and (2.9), after some computations we find

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx\right) &= \int_0^1 e^{\eta+\lambda}\rho\left(-\frac{1}{t} - \Lambda t e^{2\eta}\right)\ dx \\ &\quad - \int_0^1 e^{2\eta}\big(j_x + 2\eta_x j\big)\ dx - \int_0^1 \frac{2}{t}e^{\eta+\lambda}\mu\ dx.\end{aligned}$$

Since $\mu \geq 0$ and

$$\int_0^1 e^{2\eta}\big(j_x + 2\eta_x j\big)\ dx = \int_0^1 \big(e^{2\eta}j\big)_x\ dx = 0,$$

it follows that

$$\frac{d}{dt}\left(\int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx\right) \leq \frac{1}{t}\int_0^1 e^{\eta+\lambda}\rho\big(-1 - \Lambda t^2 e^{2\eta}\big)\ dx. \tag{4.4}$$

Thanks to (4.2), we have

$$-\Lambda t^2 e^{2\eta} \le \frac{-\Lambda t^3}{C + \frac{\Lambda}{3} t^3} \le -3 + \frac{9C}{\Lambda} t^{-3},$$

so that (4.4) implies

$$\begin{aligned}
&\frac{d}{dt}\left(t^4 \int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx \right) \\
&= 4t^3 \int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx + t^4 \frac{d}{dt}\left(\int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx \right) \\
&\le 4t^3 \int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx + t^3 \int_0^1 e^{\eta+\lambda}\rho \left(-4 + \frac{9C}{\Lambda t^3} \right)\ dx \\
&\le \frac{9C}{\Lambda t^4}\, t^4 \int_0^1 (e^{\eta+\lambda}\rho)(t,x)\ dx,
\end{aligned}$$

from which we deduce (4.3) by integration.

We are now in a position to make use of the integral estimate (4.3). Recalling that $\eta_x = -4\pi t e^{\eta+\lambda} j$ and $0 \le j \le \rho$, we control the spatial oscillation of η at each time, as follows:

$$\begin{aligned}
\left| \eta(t,x) - \int_0^1 \eta(t,\tau)\ d\tau \right| &= \left| \int_0^1 \int_\tau^x \eta_x(t,\sigma)\ d\sigma\ d\tau \right| \le \int_0^1 \int_0^1 |\eta_x(t,\sigma)|\ d\sigma\ d\tau \\
&\le 4\pi t \int_0^1 (e^{\eta+\lambda} j)(t,\sigma)\ d\sigma \le 4\pi t \int_0^1 (e^{\eta+\lambda}\rho)(t,\sigma)\ d\sigma,
\end{aligned}$$

and thanks to (4.3), this implies that

$$\left| \eta(t,x) - \int_0^1 \eta(t,\tau)\, d\tau \right| \le C t^{-3}, \quad t \in [1,T),\ x \in [0,1]. \tag{4.5}$$

We will have the desired upper bound on η, provided we can control its integral. Recalling that $\eta_t - \lambda_t = \frac{1}{t} - \Lambda t e^{2\eta}$ and using (4.2) gives

$$\frac{\partial}{\partial t} e^{\eta-\lambda} = (\eta_t - \lambda_t) e^{\eta-\lambda} \le e^{\eta-\lambda} \left(\frac{1}{t} - \frac{\Lambda t^2}{C + \frac{\Lambda}{3} t^3} \right),$$

and, after integration,

$$e^{\eta-\lambda} \le C \frac{t}{C + \frac{\Lambda}{3} t^3} \le C t^{-2}, \quad t \in [1,T),\ x \in [0,1]. \tag{4.6}$$

Next, using (2.4), (4.2), (4.3) and (4.6), we have

$$\begin{aligned}\int_0^1 \eta(t,x)\ dx &= \int_0^1 \overline{\eta}(x)\ dx + \int_1^t \int_0^1 \eta_t(s,x)\ dxds \\ &\le C + \int_1^t \frac{1}{2s} \int_0^1 \left(1 + e^{2\eta}(8\pi s^2\rho - \Lambda s^2)\right)\ dxds \\ &\le C + \frac{1}{2}\ln t + 4\pi \int_1^t \int_0^1 s e^{\eta-\lambda} e^{\eta+\lambda}\rho\ dxds \\ &\quad - \int_1^t \int_0^1 \frac{\Lambda}{2} s e^{2\eta}\ dxds,\end{aligned}$$

thus

$$\begin{aligned}\int_0^1 \eta(t,x)\ dx &\le C + \frac{1}{2}\ln t + C\int_1^t s^{-5}\ ds - \frac{1}{2}\int_1^t \frac{\Lambda s^2}{C + \frac{\Lambda}{3}s^3}\ ds \\ &\le C + \frac{1}{2}\ln\left(\frac{\Lambda t}{C + \frac{\Lambda}{3}t^3}\right).\end{aligned}$$

It then follows from (4.5) that

$$\eta(t,x) \le C(1 + t^{-3}) + \frac{1}{2}\ln\left(\frac{\Lambda t}{C + \frac{\Lambda}{3}t^3}\right),$$

which leads to an upper bound for η, i.e.,

$$e^{2\eta(t,x)} \le Ct^{-2}, \quad t \in [1,T),\ x \in [0,1],$$

and the proof is complete. □

4.2 Late-time asymptotics

We determine now the explicit leading asymptotic behavior of r, s, η, λ, λ_t, η_t and η_x, and then check that each of the generalized Kasner exponents tends to 1/3. We have proven that (see equation (2.17))

$$r = O(t^{-1}), \quad s = O(t^{-1}), \tag{4.7}$$

and equation (2.13) implies

$$(te^{-2\eta})_t = \Lambda t^2 - 4\pi t^2(r+s).$$

Integrating over $[1, t]$ and using (4.7), we obtain

$$\left| te^{-2\eta} - \frac{\Lambda}{3} t^3 \right| \leq Ct^2,$$

that is, $e^{-2\eta} = (\Lambda/3)t^2(1 + O(t^{-1}))$, so that

$$e^{\eta} = \sqrt{\frac{3}{\Lambda}} t^{-1}(1 + O(t^{-1})).$$

In view of $\eta_t = (1/2t) - (\Lambda/2)te^{2\eta} + 2\pi te^{2\eta}(r+s)$, one has

$$\eta_t = -\frac{1}{t}(1 + O(t^{-1})), \tag{4.8}$$

and, after integration over $[1, t]$, $\eta = -\ln t(1 + O\big((\ln t)^{-1}\big))$.

Since $\lambda_t = \eta_t + \Lambda te^{2\eta} - (1/t)$, one also has

$$\lambda_t = \frac{1}{t}(1 + O(t^{-1})), \tag{4.9}$$

and integrating over$[1, t]$ gives $\lambda = \ln t(1 + O\big((\ln t)^{-1}\big))$. This implies $e^{\lambda} = O(t)$, and recalling that $\eta_x = -2\pi te^{\lambda+\eta}(r - s)$ one deduces that

$$\eta_x = O(1). \tag{4.10}$$

Consider the generalized Kasner exponents which take the following form for the metric under consideration (see for instance [3]):

$$\frac{\kappa_1^1(t,x)}{\kappa(t,x)} = \frac{t\lambda_t}{t\lambda_t + 2}, \quad \frac{\kappa_2^2(t,x)}{\kappa(t,x)} = \frac{\kappa_3^3(t,x)}{\kappa(t,x)} = \frac{1}{t\lambda_t + 2}, \tag{4.11}$$

where $\kappa(t, x) = \kappa_i^i(t, x)$ is the trace of the second fundamental form $\kappa_{ij}(t, x)$ of the metric. It follows from (4.9) that as t tends to ∞, each of these quantities tends to 1/3, uniformly in x.

4.3 Future geodesic completeness

The late-time asymptotic expansion above allows us to establish that the spacetime is future geodesically complete, as follows. Let $\tau \mapsto \big(\gamma^\alpha\big)(\tau)$ (with $t = \gamma^0(\tau)$) be a future-directed causal geodesic defined on an interval $[1, \tau_+)$ with τ_+ maximal, and normalized so that $\gamma^0(\tau_0) = t(\tau_0) = 1$ for some $\tau_0 \in [1, \tau_+)$. We are going to prove that $\tau_+ = +\infty$.

Since γ is causal and future directed, we have

$$g_{\alpha\beta}\gamma_\tau^\alpha\gamma_\tau^\beta = -m^2, \quad \gamma_\tau^0 > 0,$$

where $m = 0$ if γ is null, and $m \neq 0$ if γ is timelike. Since $\frac{dt}{d\tau} = \gamma_\tau^0 > 0$, the geodesic can be parameterized by the coordinate time t. With respect to this coordinate time the geodesic exists on the whole interval $[1, +\infty)$ since on each bounded interval of t the Christoffel symbols are bounded and the right-hand sides of the geodesic equation (written in coordinate time) are linearly bounded in γ_τ^1, γ_τ^2 and γ_τ^3.

Along the geodesic we define

$$w := e^\lambda\gamma_\tau^1, \quad F := t^4\left((\gamma_\tau^2)^2 + (\gamma_\tau^2)^3\right).$$

Using the geodesic equation it is easily checked that

$$\frac{dw}{d\tau} = -\lambda_t\gamma_\tau^0 w - e^{2\eta-\lambda}\eta_x(\gamma_\tau^0)^2, \quad \frac{dF}{d\tau} = 0.$$

The relation between coordinate time and proper time is then given by

$$\frac{d\tau}{dt} = (\gamma_\tau^0)^{-1} = \frac{e^\eta}{\sqrt{m^2 + w^2 + F/t^2}}. \tag{4.12}$$

We will now exhibit a lower bound for $d\tau/dt$ by a function with divergent integral on $[1, +\infty)$ and, to this end, an estimate on w as a function of the coordinate time is needed.

Assume that $w(t) > 0$ for some $t \geq 1$. Then, as long as $w(s) > 0$, we have

$$\begin{aligned}\frac{dw}{ds} &= -\lambda_t w - e^{\eta-\lambda}\eta_x\sqrt{m^2 + w^2 + F/s^2} \\ &= 4\pi s e^{2\eta}(j\sqrt{m^2 + w^2 + F/s^2} - \rho w) + \frac{1}{2t}w - \frac{\Lambda}{2}se^{2\eta}w.\end{aligned} \tag{4.13}$$

Using the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ and equation (4.2), we obtain

$$\frac{dw}{ds} \leq 4\pi s e^{2\eta}(|j| - \rho)w + \frac{1 - \Lambda s^2 e^{2\eta}}{2s} w + 4\pi s e^{2\eta} |j| \sqrt{m^2 + F/s^2}.$$

We can drop the first two terms which are negative since $|j| \leq \rho$ and

$$1 - \Lambda s^2 e^{2\eta} \leq \frac{C}{\Lambda} s^{-3} - 2 < 0, \quad s \text{ sufficiently large},$$

and we estimate the third term by Cs^{-2} (since $|j| \leq Cs^{-1}$ and $e^{2\eta} \leq Cs^{-2}$). It then follows that

$$\frac{dw}{ds} \leq Cs^{-2}. \tag{4.14}$$

Let $t_0 \in [1, t)$ be the smallest time such that $w(s) > 0$ for all $s \in [t_0, t)$. Then integrating (4.14) over $[t_0, t]$ gives

$$w(t) \leq C.$$

For the case $w(t) < 0$, it follows from (4.13) that, as long as $w(s) < 0$

$$\begin{aligned}
\frac{dw}{ds} &\geq 4\pi s e^{2\eta}(-\rho\sqrt{m^2 + w^2 + F/s^2} - \rho w) + \frac{1 - \Lambda s^2 e^{2\eta}}{2s} w \\
&\geq -4\pi s e^{2\eta} \rho \sqrt{m^2 + F/s^2} + 8\pi s e^{2\eta} \rho w \\
&\geq Cs^{-2}(-1 + w),
\end{aligned}$$

we have used the fact that $|j| \leq \rho$, $\frac{1-\Lambda s^2 e^{2\eta}}{2s} < 0$ for large s and the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. Therefore we have

$$\frac{1}{1-w} \frac{d(1-w)}{ds} \leq Cs^{-2}. \tag{4.15}$$

Let $t_1 \in [1, t)$ be the smallest time such that $w(s) < 0$ for all $s \in [t_1, t)$. Then integrating (4.15) over $[t_1, t]$ implies

$$-w(t) \leq C.$$

In either case, we arrive at

$$|w(t)| \leq C, \quad t \geq 1.$$

On the other hand, equation (4.2) implies that

$$e^{\eta} \geq Ct^{-1}, \quad t \geq 1,$$

so we then deduce from (4.12) that

$$\frac{d\tau}{dt} \geq \frac{Ct^{-1}}{\sqrt{m^2 + C + F}},$$

and since the integral of the right-hand side over $[1, +\infty)$ diverges, it follows that $\tau_+ = +\infty$ and the proof of future geodesic completeness is completed.

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