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Classification of free actions on complete intersections of four quadrics

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Abstract

In this paper we classify all free actions of finite groups on Calabi–Yau complete intersection of four quadrics in $\mathbb{P}^7$, up to projective equivalence. We get some examples of smooth Calabi–Yau three-folds with large non-abelian fundamental groups. We also observe the relation between some of these examples and moduli of polarized abelian surfaces.

1 Introduction

The original motivation of this paper is to generalize Beauville's construction of Calabi–Yau manifolds with a non-abelian fundamental group [1]. As one result of this paper, we construct many new examples of Calabi–Yau manifolds with non-abelian fundamental groups. In particular, we construct five families of Calabi–Yau three-folds with fundamental groups of order 64. All these families are related to pencils of certain abelian surfaces. Three of these families have been previously studied in [6, 8]. The new examples

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are constructed as free quotients of small resolutions of singular complete intersections of four quadrics in $\mathbb{P}^7$ that contain a pencil of (2,4) polarized abelian surfaces (Theorem 7.3).

We also classify all families of complete intersections of four quadrics in $\mathbb{P}^7$ with a free finite group action and at most ordinary double points (ODP) singularities. The key idea is to use Holomorphic Lefschetz formula to obtain restriction on possible group actions. This paper is quite elementary, the reasoning is sometimes very explicit and is never very deep. Calculations of this paper can be generalized to other complete intersections in projective spaces or in products of projective spaces.

The paper is organized as follows. In Section 2, we review the construction of a smooth Calabi–Yau three-fold with quaternion group H_8 acting freely on it due to Beauville. We will see how the character theory of H_8 and holomorphic Lefschetz formula make this the only possible family of complete intersections with H_8 action. We also see that no linear action of the dihedral group D_8 could lead to any similar examples. In Section 3, we give a brief review about projective representations of finite groups and define the terminology of allowable actions, semi-allowable actions and Lefschetz condition. Section 4 contains a scheme of the algorithm of classifying (semi-)allowable actions on complete intersections of four quadrics in $\mathbb{P}^7$. As an application we make several tables in the next section, listing all the (semi-)allowable actions with groups of order from 2 to 64. In Section 6 we compute the cut out equations of families of Calabi–Yau three-folds with order 64 semi-allowable actions. There are two such families with five different order 64 semi-allowable actions. In the last section we prove the existence of equivariant small resolutions (Sections 6.1 and 6.2). We also explain the relations between these Calabi–Yau three-folds and moduli of polarized abelian surfaces.

All the group-theoretic calculations are done in GAP[5]. The software package Macaulay 2[9] is also very useful to us in checking smoothness. I am grateful to my advisor Lev Borisov, who gave many important ideas for this project.

2 Beauville's example

In this section we will first review Beauville's example of a free action of quaternion group H_8 on a nine-dimensional family of smooth complete intersections of four quadrics in $\mathbb{P}^7$(see [1]). Additionally, we will explain why there is no such family with free action of the dihedral group D_8. In the process we will see how holomorphic Lefschetz formula leads to restriction on possible free group actions.

The quaternion group H_8 is the group of order 8 with elements $\pm 1, \pm i, \pm j$, $\pm k$ and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. By a character calculation, H_8 has 4 one-dimensional irreducible representations and 1 two-dimensional irreducible representation. We denote them by $V_1, \dots, V_4$ and W. The regular representation V has decomposition $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus W^{\oplus 2}$. The induced representation on the second symmetric product of V has decomposition $\mathrm{Sym}^2(V) = V_1^{\oplus 5} \oplus V_2^{\oplus 5} \oplus V_3^{\oplus 5} \oplus V_4^{\oplus 5} \oplus W^{\oplus 8}$. Pick four generic quadrics $q_1, \dots, q_4$ such that q_i belongs to V_i. For generic choice of q_i, Beauville showed that the complete intersection X in $\mathbb{P}(V^*)$, given by $q_1 = \dots = q_4 = 0$ is smooth and action of H_8 on X has no fixed points. As a consequence the quotient variety X/H_8 is a smooth Calabi–Yau manifold with fundamental group H_8.

The following theorem is a special case of the standard holomorphic Lefschetz formula:

Theorem 2.1. *Let X be a smooth algebraic variety over $\mathbb{C}$ and $f : X \to X$ be a holomorphic automorphism of finite order with no fixed points. For a linearized coherent sheaf $\mathcal{F}$, the Lefschetz number*

$$\Lambda(f, \mathcal{F}) := \sum_{q=0}^{m} (-1)^q \mathrm{Tr}(f^*; H^q(X, \mathcal{F}))$$

is zero, where Tr stands for the trace.

Holomorphic Lefschetz formula explains why Beauville needed to pick this particular representation V and these particular choices of quadrics q_i. We identify the vector space V with $H^0(X, \mathcal{O}(1))$. By Kodaira's vanishing theorem and holomorphic Lefschetz formula, $\mathrm{Tr}(g, H^0(X, \mathcal{O}(1))) = 0$ for any g non-identity. The quaternion group H_8 has five conjugacy classes represented by $\{(1), (i), (j), (-1), (k)\}$. By computing traces of each conjugacy class, we obtain the trace vector $[8, 0, 0, 0, 0]$ for V, which means it must be the regular representation. The induced representation $\mathrm{Sym}^2(V)$ has trace vector $[36, 0, 0, 4, 0]$. By Lefschetz formula $H^0(X, \mathcal{O}(2))$ has trace vector $[32, 0, 0, 0, 0]$. Their difference $[4, 0, 0, 4, 0]$ is the trace vector for the space of four quadrics. This is an actual group character for H_8. More precisely, $[4, 0, 0, 4, 0]$ is the sum of characters of the 4 one-dimensional irreducible representations $V_1, \dots, V_4$. This is why Beauville picked q_i from the direct sum of copies of V_i in $\mathrm{Sym}^2(V)$.

The only other non-abelian group of order 8 is the dihedral group D_8. It is natural to ask that whether D_8 acts freely on any smooth complete intersections of four quadrics in $\mathbb{P}^7$. Dihedral group D_8 is presented by $\{a, b | a^4 = 1, b^2 = 1; ab = ba^3\}$. It has five conjugacy classes $\{(1), (b), (ab),$

$(a^2), (a)\}$. Again, we identify V with $H^0(X, \mathcal{O}(1))$, and we assume $\mathcal{O}(1)$ can be linearized so that D_8 acts on V. If D_8 acts freely on X, the trace vector of V should be $[8, 0, 0, 0, 0]$, i.e., V must be the regular representation. The trace vector for $\mathrm{Sym}^2(V)$ is then $[36, 4, 4, 4, 0]$. Subtracting $[32, 0, 0, 0, 0]$, we obtain $[4, 4, 4, 4, 0]$. It is *not* a character of D_8. So D_8 cannot act linearly on any smooth complete intersection of four quadrics in $\mathbb{P}^7$.

For any group G of order bigger than eight, $\mathcal{O}(1)$ cannot be G-linearized. Because otherwise the holomorphic Lefschetz formula shows that the character of the action on $V = H^0(X, \mathcal{O}(1))$ is a fractional multiple of the character of the regular representation, which leads to a contradiction. Hence instead of linear representations we should look for projective representations. In next section, we will give a brief review on projective representations of finite groups. We will see how holomorphic Lefschetz formula puts restriction on these projective representations.

3 Preliminaries of projective representations

In the first part of this section we recall some facts about projective representations of finite groups. Our notations follow [3]. After that we define the notion of *allowable action* of a subgroup of $\mathbb{PGL}(8, \mathbb{C})$.

Definition 3.1. Let G be a finite group. A triple (Γ, f, A) is called a central extension of G if Γ is a group, $A \subseteq Z(\Gamma)$ and f is a homomorphism of Γ onto G such that $\ker f = A$. A central extension (Γ, f, A) is called Schur Cover of G if A equals the second group homology $H_2(G, \mathbb{Z})$; this homology group is called Schur multiplier of G.

Theorem 3.1 ([3]). *If (Γ, f, A) is a Schur cover of G, then every projective representation P of G lifts to a linear representation of Γ. Conversely, any linear representation of Γ where A acts by scalar matrices is a lift of a projective representation of G.*

Remark 3.1. Schur multiplier is an invariant of G while the Schur cover is not uniquely defined. But by last theorem, given a Schur cover of G, all the projective representations of G can be realized by linear representations of Γ.

Definition 3.2. Two projective representations of G are called projective equivalent if they are conjugated in $\mathbb{PGL}(n, \mathbb{C})$.

Any projective representation of G is given by a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathbb{GL}(8,\mathbb{C}) & \longrightarrow & \mathbb{PGL}(8,\mathbb{C}) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \end{array} \tag{3.1}$$

where Γ is a Schur cover of G. Usually the map τ is not injective. Consider the short exact sequence:

$$0 \longrightarrow K/\mathrm{Ker}(\tau) \longrightarrow \Gamma/\mathrm{Ker}(\tau) \longrightarrow G \longrightarrow 1.$$

Here $K/\mathrm{Ker}(\tau)$ is a cyclic group. By Theorem 3.1, projective representations of G are in one-to-one correspondence with linear representations of $\Gamma/\mathrm{Ker}(\tau)$.

Definition 3.3. We say that a finite group $G \subset \mathbb{PGL}(8,\mathbb{C})$ has an *allowable action* if G acts freely on some smooth complete intersection X of four quadrics in $\mathbb{P}^7$. We will call the correspondent G-action linear allowable action if G can be lifted to a subgroup of $\mathbb{GL}(8,\mathbb{C})$. Similarly, if the variety X is singular with ordinary double points, we say that G has a *semi-allowable action.*

Proposition 3.1. *If G has an allowable or semi-allowable action then $|G|$ divides 256.*

Proof. In [4], Browder and Katz proved a general theorem about free action of finite groups on projective varieties:

Theorem 3.2 ([4])**.** *Let X be a projective variety in $\mathbb{P}^n$ and G is a finite subgroup of $\mathbb{PGL}(n+1,\mathbb{C})$. If G acts freely on X then, $|G|$ divides the square of the degree of X.*

We are considering complete intersections of four quadrics X in $\mathbb{P}^7$, which have degree 16. By theorem of Browder and Katz, if G acts freely on X then $|G|$ divides 256. □

Remark 3.2. Later we are going to argue the maximal order of G is 64.

If G has an allowable action on X, then $H^0(X, \mathcal{O}(1))$ becomes a projective representation of G. We denote this vector space by V. By Theorem 3.1, the group $\Gamma/\mathrm{Ker}(\tau)$ acts linearly on V with the cyclic subgroup $K/\mathrm{Ker}(\tau)$

acting by scalar matrices. By Holomorphic Lefschetz formula, those elements in Γ but not in $K/\mathrm{Ker}(\tau)$ have trace zero. If we fix a generator σ of $K/\mathrm{Ker}(\tau)$ of order 2^d, then it should act on V as a scalar matrix ξI where ξ is a primitive 2^dth root of unity and I stands for identity matrix. Let us denote the trace vector of Γ for a given representation V by t_V^Γ. All the entries in t_V^Γ are zero except those corresponding to the conjugacy classes $\{(\sigma^k), k = 0, 1, \ldots, 2^d - 1\}$. These conjugacy classes have trace $8\xi^k$. Similarly entries of $t^\Gamma_{H^0(X,\mathcal{O}(2))}$ are $32\xi^{2k}$ for conjugacy classes $\{(\sigma^k), k = 0, 1, \ldots, 2^d - 1\}$ and zero otherwise. We can also compute the trace vector of the induced representation $\mathrm{Sym}^2(V)$ and denote it by $t^\Gamma_{\mathrm{Sym}^2(V)}$. The difference vector $v = t^\Gamma_{\mathrm{Sym}^2(V)} - t^\Gamma_{H^0(X,\mathcal{O}(2))}$ is the trace vector for the subrepresentation spanned by the four quadrics. The assumption that G acts freely on X will force t_V^Γ and v to be group characters.

Definition 3.4. We say a central extension

$$0 \longrightarrow K/\mathrm{Ker}(\tau) \longrightarrow \Gamma/\mathrm{Ker}(\tau) \longrightarrow G \longrightarrow 1$$

satisfies *Lefschetz condition* if the trace vectors t_V^Γ and v defined above are both group characters.

Proposition 3.2. *If G has a semi-allowable action then it satisfies Lefschetz condition.*

Proof. Apply holomorphic Lefschetz formula to $\pi^*(\mathcal{O}(1))$ and $\pi^*(\mathcal{O}(2))$ on the resolution $\pi : \widehat{X} \to X$. □

Remark 3.3. A priori, Lefschetz condition is only necessary but not sufficient for G to have allowable action. We still need to check the fixed loci of G in $\mathbb{P}^7$ do not intersect with X in order to verify the freeness. However, in our cases it turns out that all the groups satisfying Lefschetz condition are allowable when $|G| < 64$. When $|G| = 64$ the necessity of Lefschetz condition follows from the fact that ordinary double points are rational singularities. Details are left to the readers.

4 Classification algorithm

Our target is to classify the allowable and semi-allowable actions on complete intersections of four quadrics in $\mathbb{P}^7$ up to projective equivalence. In this section we describe the scheme of our algorithm.

Recall that every projective representation gives a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathbb{GL}(8,\mathbb{C}) & \longrightarrow & \mathbb{PGL}(8,\mathbb{C}) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \tau} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \end{array} \tag{4.1}$$

where Γ is a Schur cover of G and K is its Schur multiplier. Generally K is quite big but the exponent of K is controlled by order of G by the following lemma.

Lemma 4.1. *Let G be a finite group and K be its Schur multiplier. Denote exponent of K by e. Then e^2 divides $|G|$.*

Proof. See [3]. □

This lemma tells us the cyclic group $K/\mathrm{Ker}(\tau)$ in the central extension

$$0 \longrightarrow K/\mathrm{Ker}(\tau) \longrightarrow \Gamma/\mathrm{Ker}(\tau) \longrightarrow G \longrightarrow 1$$

has order at most eight. By Theorem 3.1, given a group G of order less or equal to 64, all projective representations of G can be lift to a linear representation of $\Gamma/\mathrm{Ker}(\tau)$.

Now we will describe the algorithm for $|G| = 64$. Lower order groups are handled similarly.

Lemma 4.2. *If $|G| = 64$ and G acts freely on X then $|K/\mathrm{Ker}(\tau)| \geq 4$.*

Proof. If $K/\mathrm{Ker}(\tau)$ has order 2 then the sheaf $\mathcal{O}(2)$ must be G linearizable, i.e., $\dim(H^0(X, \mathcal{O}(2)))$ must be divisible by 64. But $H^0(X, \mathcal{O}(2))$ has dimension 32. □

Following this lemma, it suffices to consider projective representations of a 64 group G given by the following two types of central extensions:

(1) A group H of order 256 with a subgroup $\mathbb{Z}/4$ acting as diagonal matrix $\xi_8^2 I$.
(2) A group H of order 512 with a subgroup $\mathbb{Z}/8$ acting as diagonal matrix $\xi_8 I$.

Again I represents the 8×8 identity matrix and ξ_8 is a primitive 8th root of unity.

Now we can summarize our algorithm step by step.

Step I: Check the Lefschetz condition for the central extensions

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow H \longrightarrow G \longrightarrow 1$$

and

$$0 \longrightarrow \mathbb{Z}/8 \longrightarrow H \longrightarrow G \longrightarrow 1.$$

Let H go over all groups of order 256 and 512 and produce all G that satisfy Lefschetz condition. We use the GAP([5]) library of finite groups of small order. There are $56,092$ different groups of order 256 and $10,494,213$ order 512 groups.

Step II: For each group G that appears in Step I, compute all possible extensions of G of the form:

$$0 \longrightarrow K/\mathrm{Ker}(\tau) \longrightarrow \Gamma/\mathrm{Ker}(\tau) \longrightarrow G \longrightarrow 1$$

for a fixed Schur cover Γ. This can be done by computing kernels of all the group characters of K. By Theorem 3.1, such extensions are in one-to-one correspondence with non-equivalent projective representations.

Step III: Check Lefschetz condition on extensions above and output those that satisfy it.

Step IV: Check the fixed loci of the group actions obtained above and show they do not intersect X.

Step V: Check that the generic complete intersection has at most ODP singularities in the semi-allowable case or is smooth in the allowable case.

The final output of the algorithm is a list of projective representations of groups with allowable or semi-allowable actions. The same group might appear on this list for several times with different projective representations. Computer algebra system involving in our algorithm are GAP ([5]) and MACAULAY ([**?**]). The results of these calculations are presented in the next section.

5 Results

In this section we present the results of the algorithm of the last section.

Remark 5.1. Many group-theoretic computations in this paper are done in GAP. It has a small group library where all groups of given order less than 2000 are listed. For instance, the quaternion group H_8 is represented by $(8, 4)$ in GAP library, where 8 for its order and 4 for its index in GAP library.

There are eight non-trivial groups of order less and equal to 8. We will see all of them have allowable actions except the dihedral group D_8. Further all the order 8 allowable action are linear.

There are 14 (resp. 51) non-isomorphic 16-groups (resp. 32-groups). In the following tables we list all the allowable groups by their indices, together with the extension $\Gamma/\mathrm{Ker}(\tau)$ representing the correspondent projective representation. We also give number of allowable actions up to projective equivalence.

When the order of the group is less than 64, the generic element of the family with allowable action is a smooth complete intersection of four quadrics in $\mathbb{P}^7$. However this is no longer true for 64 groups.

There are 267 different groups of order 64. In these 267 groups there are five groups that are semi-allowable.

Remark 5.2. We want to explore a little more about these five 64 groups because it turns out the geometry of them are particularly interesting. The group $(64, 2)$ is the abelian group $\mathbb{Z}/8 \times \mathbb{Z}/8$. Its Schur cover is the Heisenberg group $(\mathbb{Z}/8)^2 \ltimes \mathbb{Z}/8$. The group $(64, 3)$ is a semi-direct product of two copies of $\mathbb{Z}/8$ and $(64, 179)$ is a semi-direct product of quaternion group H_8 and $\mathbb{Z}/8$. These first three groups all contain a maximal abelian subgroup $\mathbb{Z}/4 \times \mathbb{Z}/8$, which has GAP index $(32, 3)$. It was observed in [**?**] that these three 64 groups act on the same family. This is a two-dimensional subfamily of the three-dimensional family with $(32, 3)$ action, which is invariant under certain involution.

The other two groups $(64, 68)$ and $(64, 72)$ do not have obvious semi-direct product structures. Both of them contain a maximal abelian subgroup $(\mathbb{Z}/4)^2 \times \mathbb{Z}/2$, which has GAP index $(32, 21)$. These two groups act on a different two-dimensional family (see Theorem 6.2).

Remark 5.3. All groups listed in Table 1 are subgroups of these five 64 groups with only two exceptions: $(32, 4)$ and $(32, 5)$. In $(32, 2)$ case, we are not sure whether both projective representations are induced from representations of 64 groups. It turns out all the actions for $|G| \leq 32$ in Table 1 are allowable. When $|G| = 64$, they are semi-allowable.

Table 1: (Semi-)allowable action of order 2–64

Groups	Extension	Schur multiplier
$\mathbb{Z}/2$	$\mathbb{Z}/2$	id group
$\mathbb{Z}/4$	$\mathbb{Z}/4$	id group
$\mathbb{Z}/2\times\mathbb{Z}/2$	$\mathbb{Z}/2\times\mathbb{Z}/2$	id group
$\mathbb{Z}/8$	$\mathbb{Z}/8$	id group
$\mathbb{Z}/2\times\mathbb{Z}/4$	$\mathbb{Z}/2\times\mathbb{Z}/4$,(16,3)	$\mathbb{Z}/2$
$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
H_8	H_8	id group
(16,2)	(64,18)	$\mathbb{Z}/4$
(16,4)	(32,14)	$\mathbb{Z}/2$
(16,5)	(32,5)	$\mathbb{Z}/2$
(16,10)	(32,22)	$(\mathbb{Z}/2)^3$
(16,12)	(32,29)	$(\mathbb{Z}/2)^2$
(32,2)	(64,18),(64,23)	$(\mathbb{Z}/2)^3$
(32,3)	(128,6)	$\mathbb{Z}/4$
(32,4)	(64,28)	$\mathbb{Z}/2$
(32,5)	(64,4)	$(\mathbb{Z}/2)^2$
(32,13)	(64,46)	$\mathbb{Z}/2$
(32,21)	(128,462)	$(\mathbb{Z}/2)^2\times\mathbb{Z}/4$
(32,35)	(64,182)	$(\mathbb{Z}/2)^2$
(32,47)	(64,224)	$(\mathbb{Z}/2)^5$
(64,2)	$(\mathbb{Z}/8)^2\rtimes\mathbb{Z}/8$	$\mathbb{Z}/8$
(64,3)	(256,321)	$\mathbb{Z}/4$
(64,68)	(256,4235)	$\mathbb{Z}/2\times\mathbb{Z}/4$
(64,72)	(256,4222),(256,4233)	$(\mathbb{Z}/2)^2\times\mathbb{Z}/4$
(64,179)	(256,6447)	$\mathbb{Z}/4$

Remark 5.4. The readers might observe that the 32-group $(32,2)$ and the 64-group $(64,72)$ have two different projective representations, i.e., there are two non-conjugated embeddings of these finite groups into $\mathbb{PGL}(8,\mathbb{C})$. Recall that projective representations are one-to-one correspondent with central extensions. They are quotient groups of some Schur cover. It is a natural question to ask that whether these two representations can be identified by some outer automorphism of the group. It turns out that the two different projective representations of $(64,72)$ are identified by some outer automorphism of $(64,72)$. In other words, these are two different ways of parameterizing the same subgroup of $\mathbb{PGL}(8,\mathbb{C})$ (see Section 6).

By Proposition 3.1 the maximal order of allowable action we can get is 256. Suppose there is an order 128 semi-allowable group. Then all its 64 subgroups must be semi-allowable. By a GAP calculation we check that there are no 128 groups, all of whose order 64 subgroups are among $\{(64,2),(64,3),(64,68),(64,72),(64,179)\}$. Hence there is no (semi-)allowable group of order bigger than 64.

6 Complete intersection varieties

In the last section we found five semi-allowable 64 groups. The following two theorems show that three of them act freely on a two-dimensional family of complete intersections of four quadrics in $\mathbb{P}^7$, and the other two groups act freely on a different dimension two family.

Theorem 6.1. *Let X be complete intersection of four quadrics:*

$$
\begin{aligned}
q_1 &= t_1(x_1^2+x_5^2)+t_2(x_2x_8+x_4x_6)+t_3x_2x_7,\\
q_2 &= t_1(x_2^2+x_6^2)+t_2(x_3x_1+x_5x_7)+t_3x_3x_8,\\
q_3 &= t_1(x_3^2+x_7^2)+t_2(x_4x_2+x_6x_8)+t_3x_4x_1,\\
q_4 &= t_1(x_4^2+x_8^2)+t_2(x_5x_3+x_7x_1)+t_3x_5x_2.
\end{aligned}
$$

There are three groups G_1, G_2, G_3 contained in $\mathbb{PGL}(8,\mathbb{C})$. Group G_1 is generated by τ and σ where $\sigma=(12345678)$ is permutation of the coordinates x_i and $\tau(x_i)=\xi^{i-1}x_i$ with ξ a primitive 8th root of unity. Group G_2 generated by τ and $\sigma_1=(18325476)$. Then G_2 is a non-abelian group isomorphic to a semi-direct product of two copies of $\mathbb{Z}/8$. Group G_3 is generated by τ and the permutations $\sigma_2=(1357)(2468)$ and $\sigma_3=(1256)(4387)$. It is a non-abelian group isomorphic to a semi-direct product of normal subgroup $\mathbb{Z}/8\mathbb{Z}$ generated by τ and the quaternion group H_8 generated by σ_2 and σ_3. As in Remark 6.3, $G_1=(64,2)$, $G_2=(64,3)$ and $G_3=(64,179)$. They act on X without fixed points.

Proof. See [?]. □

Now we introduce the other two groups of order 64 having semi-allowable actions. Define groups G_4, G_5, G_5' as following subgroups of $\mathbb{GL}(8,\mathbb{C})$. Group G_4 generated by coordinates transformations $\sigma_1,\sigma_2,\sigma_3$, where

$$
\begin{aligned}
\sigma_1 &: (x_1,\dots,x_8)\mapsto(\xi x_7,\xi x_8,\xi^3x_5,\xi^3x_6,-\xi x_3,-\xi x_4,\xi^3x_1,\xi^3x_2),\\
\sigma_2 &: (x_1,\dots,x_8)\mapsto(-x_2,ix_1,-x_4,-ix_3,-ix_6,x_5,ix_8,x_7),\\
\sigma_3 &: (x_1,\dots,x_8)\mapsto(\xi^3x_5,-\xi^3x_6,-\xi x_7,\xi x_8,\xi^3x_1,-\xi^3x_2,-\xi x_3,\xi x_4).
\end{aligned}
$$

Group G_5 is generated by $\sigma_3, \sigma_4, \sigma_5$ where

$$\sigma_4 : (x_1, \ldots, x_8) \mapsto (\xi x_7, \xi x_8, -\xi^3 x_5, \xi^3 x_6, -\xi x_3, -\xi x_4, \xi^3 x_1, -\xi^3 x_2),$$
$$\sigma_5 : (x_1, \ldots, x_8) \mapsto (\xi^3 x_6, \xi^3 x_5, -\xi x_8, \xi x_7, \xi^3 x_2, \xi^3 x_1, \xi x_4, -\xi x_3).$$

Group G'_5 is generated by $\sigma_3, \sigma_4, \xi\sigma_5$.

These three groups G_4, G_5, G'_5 all have order 256. Their indices in GAP are respectively (256,4235), (256,4222) and (256,4233). The corresponding projective linear groups in $\mathbb{PGL}(8, \mathbb{C})$ are $(64, 68)$ and $(64, 72)$. The last two groups G_5, G'_5 have the same projective group (64,72).

Remark 6.1. The two groups G_5, G'_5 lead to two non-equivalent projective representations of $(64, 72)$(see 1). However, the corresponding projective subgroups of $\mathbb{PGL}(8, \mathbb{C})$ are the same, i.e., these two representations only differ by an outer automorphism of $(64, 72)$.

By abusing notations, from now on we denote the projectivizations by G_4 and G_5.

Theorem 6.2. *Let X be a complete intersection of four quadrics in $\mathbb{P}^7$ cut out by:*

$$q_1 = t_1(x_1^2 + x_2^2) - t_2(x_3^2 + x_4^2) + t_1(x_5^2 + x_6^2) + t_2(x_7^2 + x_8^2),$$
$$q_2 = -t_2(x_1^2 + x_2^2) + t_1(x_3^2 + x_4^2) + t_2(x_5^2 + x_6^2) + t_1(x_7^2 + x_8^2),$$
$$q_3 = s_1(x_1^2 - x_2^2) - s_2(x_3^2 - x_4^2) + s_1(x_5^2 - x_6^2) + s_2(x_7^2 - x_8^2),$$
$$q_4 = -s_2(x_1^2 - x_2^2) + s_1(x_3^2 - x_4^2) + s_2(x_5^2 - x_6^2) + s_1(x_7^2 - x_8^2).$$

The groups G_4 and G_5 introduced above act freely on X.

Proof. We only prove the theorem for G_4. The argument for G_5 is completely analogous. Consider central extension

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow (256, 4235) \longrightarrow G_4 \longrightarrow 1$$

The group $(256, 4235)$ has 46 irreducible representations, indexed by $X_1, \ldots, X_{46}$. In particular, $X_1, \ldots, X_{16}$ are one-dimensional irreducible representations, $X_{17}, \ldots, X_{44}$ are two-dimensional irreducible representations and X_{45}, X_{46} are eight-dimensional irreducible representations. We identify V with $H^0(X, \mathcal{O}(1))$. Holomorphic Lefschetz formula force V to be the irreducible representation X_{45}. The second symmetric product of V has decomposition:

$$\begin{aligned}\mathrm{Sym}^2(V) = (\oplus_{i\in I} X_i) \oplus X_{35}^{\oplus 2} \oplus X_{36}^{\oplus 2} \text{ for}\\ I = \{19, 20, 21, 22, 25, 26, 27, 28, 33, 34, 41, 42, 43, 44\}\end{aligned}$$

The subrepresentation spanned by the four quadrics has decomposition $X_{35} \oplus X_{36}$, again follow from holomorphic Lefschetz formula. Pick a basis $(x_1, \ldots, x_8)$ for $V = X_{45}$. We get an induced basis for $\mathrm{Sym}^2(V)$. They are homogenous quadratic polynomials in $x_1, \ldots, x_8$. In particular,

$$\begin{aligned}X_{35}^{\oplus 2} &= \mathrm{Span}\{x_1^2 + x_2^2 + x_5^2 + x_6^2, x_3^2 + x_4^2 + x_7^2 + x_8^2\}\\ &\oplus \mathrm{Span}\{x_7^2 + x_8^2 - x_3^2 - x_4^2, x_5^2 + x_6^2 - x_1^2 - x_2^2\}.\end{aligned}$$

Respectively,

$$\begin{aligned}X_{36}^{\oplus 2} &= \mathrm{Span}\{x_1^2 - x_2^2 + x_5^2 - x_6^2, x_3^2 - x_4^2 + x_7^2 - x_8^2\}\\ &\oplus \mathrm{Span}\{x_7^2 - x_8^2 - x_3^2 + x_4^2, x_5^2 - x_6^2 - x_1^2 + x_2^2\}.\end{aligned}$$

These polynomials give the cut out equations (see Theorem 6.2). It is clear from these equations that parameter space of this two-dimensional family is a subset of $\mathbb{P}^1 \times \mathbb{P}^1$ where $(t_1 : t_2)$ and $(s_1 : s_2)$ are homogeneous coordinates of each $\mathbb{P}^1$.

To show G_4 acts without fixed points, we need to check the intersection of the fix loci of all conjugacy classes of G_4 with X are empty. It is easy to see this is the case for generic choice of t_1, t_2, s_1, s_2. □

Remark 6.2. We have mentioned $(64, 72)$ has two different projective representations $(256, 4222)$ and $(256, 4233)$. A calculation shows both of them act freely on this family.

7 Resolutions of singularities

We will investigate more about the geometry of these two families. Let X be a complete intersection of four quadrics cut out by equations in Theorem 6.1. We have seen in last section three 64 groups G_1, G_2 and G_3 act freely on X. This family was first discovered by Gross and Popescu. In [7], they studied the birational geometry of X, including the resolution of singularities. They have proved the following theorem in the case of G_1.

Theorem 7.1. *The singular Calabi–Yau three-fold X has an equivariant small projective resolution $\widetilde{X}$, i.e., $\widetilde{X}$ is a smooth projective Calabi–Yau three-fold with free actions by G_1, G_2 and G_3. The resolution $\widetilde{X}$ has Hodge numbers $h^{1,1} = 2$ and $h^{1,2} = 2$. Furthermore, $\widetilde{X}$ contains a pencil of abelian surfaces with polarization $(1, 8)$.*

In this section we obtain a similar result for the family in Theorem 6.2. We will prove the generic element X in this family also has an equivariant small projective resolution. Recall X is cut out by equations:

$$\begin{aligned}
q_1 &= t_1(x_1^2 + x_2^2) - t_2(x_3^2 + x_4^2) + t_1(x_5^2 + x_6^2) + t_2(x_7^2 + x_8^2),\\
q_2 &= -t_2(x_1^2 + x_2^2) + t_1(x_3^2 + x_4^2) + t_2(x_5^2 + x_6^2) + t_1(x_7^2 + x_8^2),\\
q_3 &= s_1(x_1^2 - x_2^2) - s_2(x_3^2 - x_4^2) + s_1(x_5^2 - x_6^2) + s_2(x_7^2 - x_8^2),\\
q_4 &= -s_2(x_1^2 - x_2^2) + s_1(x_3^2 - x_4^2) + s_2(x_5^2 - x_6^2) + s_1(x_7^2 - x_8^2).
\end{aligned}$$

The jacobian matrix of it is

$$\begin{pmatrix}
t_1x_1 & t_1x_2 & -t_2x_3 & -t_2x_4 & t_1x_5 & t_1x_6 & t_2x_7 & t_2x_8\\
-t_2x_1 & -t_2x_2 & t_1x_3 & t_1x_4 & t_2x_5 & t_2x_6 & t_1x_7 & t_1x_8\\
s_1x_1 & -s_1x_2 & -s_2x_3 & s_2x_4 & s_1x_5 & -s_1x_6 & s_2x_7 & -s_2x_8\\
-s_2x_1 & s_2x_2 & s_1x_3 & -s_1x_4 & s_2x_5 & -s_2x_6 & s_1x_7 & -s_1x_8
\end{pmatrix}.$$

A point on X is singular if and only if this matrix is degenerated.

Lemma 7.1. *A point $P \in X$ is singular if and only if exactly four coordinates out of $(x_1, \ldots, x_8)$ are zero.*

Proof. Let $P = (x_1 : \ldots : x_8)$ be a point on X. Observe that P has at most four zeros in coordinates because otherwise P cannot sit on X with generic choices of t_i and s_i. We first prove if P has four zeros then it must be a singular point. Let μ be a subset of four distinct numbers in $\{1, \ldots, 8\}$. Denote its complement by $\bar{\mu}$. Let P_μ be a point with $\{x_i = 0 | i \in \mu\}$ and $J_{\bar{\mu}}$ be the four by four minor of the jacobian matrix by picking the $\bar{\mu}$th columns. Since all equations $q_1, \ldots, q_4$ consist of square terms, the jacobian matrix J is equivalent with the coefficient matrix up to elementary transformation. So $J_{\bar{\mu}}$ degenerates if and only if $q_1, \ldots, q_4$ has non-zero solution of the form $\{(x_1, \ldots, x_8) | x_i \neq 0 \text{ for } i \in \mu\ x_i = 0 \text{ for } i \in \bar{\mu}\}$. This proves the first direction.

If P is a singular point, we pick a μ such that $\{x_i \neq 0 | i \in \mu\}$. Since P is singular the jacobian matrix J evaluated at P degenerates. In particular,

J_μ degenerates. If the other coordinates $\{x_i | i \in \bar{\mu}\}$ do not vanish simultaneously, then we obtain one parameter family of solutions along which the jacobian matrix degenerates. However the singular loci has dimension zero generally. So all the other coordinates must be zero. □

Given (1468), (1367), (1457), (2467), (2357), (2458), (1358) and (2368) as subsets of $\{1, \ldots, 8\}$. The corresponding points P_μ for μ equals any of these eight sets are singular points of X. Since X is cut out by degree two equations, there are exactly eight solutions for a given set μ. Hence each combinations give eight singular points. These 64 points form group orbits, for both G_4 and G_5. We will see later these 64 singularities are ordinary double points. Let us fix a set, say (1468). The corresponding singular points are $(0 : y_2 : y_3 : 0 : y_5 : 0 : y_7 : 0)$. Plug $y_1 = y_6 = y_4 = y_8 = 0$ into equations given in Theorem 6.2, we obtain:

$$\begin{aligned}
q_1 &= t_1 y_2^2 - t_2 y_3^2 + t_1 y_5^2 + t_2 y_7^2 = 0, \\
q_2 &= -t_2 y_2^2 + t_1 y_3^2 + t_2 y_5^2 + t_1 y_7^2 = 0, \\
q_3 &= -s_1 y_2^2 - s_2 y_3^2 + s_1 y_5^2 + s_2 y_7^2 = 0, \\
q_4 &= s_2 y_2^2 + s_1 y_3^2 + s_2 y_5^2 + s_1 y_7^2 = 0.
\end{aligned}$$

Solving t_1, t_2, s_1, s_2 by y_i, we rewrite the original equations as:

$$\begin{aligned}
q_1 = &(y_3^2 - y_7^2)(x_1^2 + x_2^2) - (y_2^2 + y_5^2)(x_3^2 + x_4^2) \\
&+ (y_3^2 - y_7^2)(x_5^2 + x_6^2) + (y_2^2 + y_5^2)(x_7^2 + x_8^2), \\
q_2 = &-(y_2^2 + y_5^2)(x_1^2 + x_2^2) + (y_3^2 - y_7^2)(x_3^2 + x_4^2) \\
&+ (y_2^2 + y_5^2)(x_5^2 + x_6^2) + (y_3^2 - y_7^2)(x_7^2 + x_8^2), \\
q_3 = &(y_3^2 - y_7^2)(x_1^2 - x_2^2) - (y_5^2 - y_2^2)(x_3^2 - x_4^2) \\
&+ (y_3^2 - y_7^2)(x_5^2 - x_6^2) + (y_5^2 - y_2^2)(x_7^2 - x_8^2), \\
q_4 = &-(y_5^2 - y_2^2)(x_1^2 - x_2^2) + (y_3^2 - y_7^2)(x_3^2 - x_4^2) \\
&+ (y_5^2 - y_2^2)(x_5^2 - x_6^2) + (y_3^2 - y_7^2)(x_7^2 - x_8^2).
\end{aligned}$$

Additionally y_2, y_3, y_5, y_7 satisfy a degree four relation $y_3^4 - y_7^4 = y_2^4 + y_5^4$.

These computations show that positions of the 64 singular points uniquely determine the family of complete intersections.

No we will describe the explicit equivariant crepant resolution for X for G_4.

Theorem 7.2. *There exist G-equivariant small resolutions $\widetilde{X} \longrightarrow X$ by blowing up a smooth G-invariant abelian surface in X for $G = G_4$ and $G = G_5$.*

Proof. To construct such a small resolution, we need to find a Weil divisor passing through the 64 ordinary double points, and invariant under the action of G_4. Such a divisor is never Cartier since it is locally cut out by more than one equation. By blowing up this divisor we obtain the projective small resolution. Consider the codimension one subscheme cut out by the following two equations:

$$f_1 = r_1x_1x_2 - r_2x_3x_4 + r_1x_5x_6 + r_2x_7x_8,$$
$$f_2 = -r_2x_1x_2 + r_1x_3x_4 + r_2x_5x_6 + r_1x_7x_8.$$

Notice equations $x_1x_2 + x_5x_6$ and $x_3x_4 + x_7x_8$ span the two-dimensional irreducible representation X_{33} and $x_1x_2 - x_5x_6$ and $x_3x_4 - x_7x_8$ span the two-dimensional irreducible representation X_{34}. And f_1, f_2 are two generic elements in $X_{33} \oplus X_{34}$. These two elements together with $q_1, \ldots, q_4$ cut out an G_4 invariant surface in X. We denote it by S_{r_1,r_2}. Under generic choices of coefficients r_1 and r_2, this is a smooth abelian surface. If we pick any one of f_1 and f_2 we will obtain unions of two abelian surfaces. Hence S_{r_1,r_2} is a Weil divisor but not Cartier. The abelian surface S_{r_1,r_2} has arithmetic genus $p_a = -1$, i.e., it is of degree 16 in $\mathbb{P}^7$. By varying r_1 and r_2 any two such surfaces intersect at the 64 singular points of X. It also follows from the form of equations that they are ordinary double points. By blowing up S_{r_1,r_2}, we obtain a smooth projective Calabi–Yau three-fold $\widetilde{X}$. Since S_{r_1,r_2} is G_4-invariant $\widetilde{X}$ also carries with a free G_4-action. □

Remark 7.1. In the case of G_5, we need to blow up a different Weil divisor cut out by equations:

$$f_1 = r_1x_1x_5 - r_2x_2x_6 + r_1x_3x_7 - r_2x_4x_8,$$
$$f_2 = -r_1x_1x_5 + r_2x_2x_6 + r_1x_3x_7 - r_2x_4x_8.$$

Recall that there are two different allowable actions of G_5, lifted to $G_5 = (256, 4222)$ and $G_5' = (256, 4233)$. Both of them act on this surface, i.e., they have the same equivariant resolutions.

Corollary 7.1. *The quotient variety $\widetilde{X}_{G_4}/G_4$(resp. $\widetilde{X}_{G_5}/G_5$) is a smooth projective Calabi–Yau threefold with fundamental group G_4(resp. G_5).*

Similar to [**?**] and [**?**], this family X also carries a fibration structure of abelian surfaces.

Proposition 7.1. *The equations f_1, f_2 form a sublinear system of dimension one of $\mathcal{O}(2)$ with 64 base points exactly at the 64 ordinary double points.*

Proof. We need to show $\phi : x \mapsto (f_1(x) : f_2(x))$ is a rational map defined outside the 64 ordinary double points. It is obvious that ϕ is defined at $X \setminus S_{r_1,r_2}$. For any points on S_{r_1,r_2} that are not the 64 ordinary double points, f_1 and f_2 have a common divisor, i.e., S_{r_1,r_2} is cut out locally just one equation. By dividing out the common divisor we extend ϕ everywhere except the 64 ordinary double points. □

Remark 7.2. Consider the space of quadrics spanned by $q_1, \ldots, q_4$ together with f_1, f_2. These equations cut out a $(2, 4)$ polarized abelian surfaces in $\mathbb{P}^7$ (see [2] for more about this abelian surface). Any four linear independent equations of these six cut out a Calabi–Yau complete intersection with 64 ordinary double points. However only a two-dimensional subfamily has free actions of G_4 and G_5.

Remark 7.3. Let X be the Calabi–Yau three-fold cut out by equations in Theorem 6.2. It contains a pencil of $(2, 4)$ polarized abelian surfaces [2]. Give the small resolution $\widetilde{X}$ in Theorem 7.2. The Calabi–Yau three-fold $\widetilde{X}$ has Hodge number $h^{1,1} = 10$ and $h^{1,2} = 10$. As we stated in the last remark, only a two-dimensional subfamily in this ten-dimensional family has free actions of G_4 and G_5. A similar argument to Remark 4.11 in [7] can be applied to compute the Hodge number of the quotient variety $\widetilde{X}/G$. We expect the quotient to have Hodge number $h^{1,1} = 2, h^{1,2} = 2$.

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