

Perturbative study of the transfer matrix on the string worldsheet in $\text{AdS}_5 \times S^5$

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Abstract

Quantum non-local charges are central to the quantum integrability of a sigma model. In this paper we study the quantum consistency and UV finiteness of non-local charges of string theory in $\text{AdS}_5 \times S^5$. We use the pure spinor formalism. We develop the near-flat space expansion of the transfer matrix and calculate the one-loop divergences. We find that the logarithmic divergences cancel at the level of one loop. This gives strong support to the quantum integrability of the full string theory. We develop a calculational setup for the renormalization group analysis of Wilson line type of operators on the string worldsheet.

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1 Introduction

The integrability of $\mathcal{N} = 4$ SYM and string theory on $\text{AdS}_5 \times S^5$ has emerged as a major tool in testing the AdS/CFT correspondence. Despite the impressive progress that the assumption of integrability has enabled, relatively little is known about the integrable structure that underlies both theories. Understanding this would be desirable for a number of reasons: the showpiece of AdS/CFT at present is undoubtedly the asymptotic S -matrix [1], based on the earlier works [2–6], which seems to correctly capture the

scattering in infinite volume for both strings and SYM. Applicability of factorized scattering assumes the existence of quantum non-local conserved charges [7, 8]. Furthermore, the S -matrix alone does not describe the full spectrum, as it assumes scattering in infinite volume and thus fails to capture finite-size corrections. This shortcoming was made explicit in the string theory in [9] and for $\mathcal{N} = 4$ SYM in [10]. A possible way to fix this problem is to apply the thermodynamic Bethe-ansatz procedure, as suggested in [11], which however may be limited to very specific, low-lying states. Alternatively, a systematic procedure, based on the Baxter Q -operator, was proposed in [12, 13], which in the case of sinh-Gordon theory enables to compute the finite-size effects for all states. Key to this analysis is, however, the knowledge of the quantum symmetries of the problem — which in the case of AdS/CFT remains to be uncovered.

In this paper we would like to take a step in the direction of a better understanding of the quantum transfer matrix and the associated non-local charges of string theory in $\text{AdS}_5 \times S^5$. We will use the pure spinor formalism. One of our motivations is to see how far the computational feasibility of the pure spinors extends beyond flat space. Furthermore, the pure spinor string in $\text{AdS}_5 \times S^5$ [14–17] has various features that make it a natural framework for quantum computations. The theory is conformal on the worldsheet [18, 19] and, as we shall see, quantum computations can be performed without choosing a specific gauge that breaks the global symmetries. In view of integrability, an interesting combination of integrable and conformal structure of the worldsheet theory emerges. The existence of classical local conserved charges was established in [20]. Integrability of the quantum theory was anticipated in [18, 21], where an argument showing BRST-invariance of the non-local charges was put forward.

In this paper we examine the UV finiteness of the transfer matrix. We analyse the short-distance singularities of the currents. We perform a perturbative expansion around flat space, determine the OPE of the currents to the leading order in curvature corrections and study the logarithmic divergences of the path ordered integrals of currents. We find that the logarithmic divergences of the transfer matrix cancel at the level of one loop. This gives support to the claim that the quantum string in $\text{AdS}_5 \times S^5$ is quantum integrable.

Important related works in the WZW literature are [22, 23]. Those papers discussed the short-distance singularities of loop operators in boundary and bulk WZW models.

The plan of this paper is as follows: in Section 2 we begin with a very brief summary of the pure spinor string in $\text{AdS}_5 \times S^5$, discussing the

zero-curvature formulation of the equations of motion, and the classical transfer matrix. In Section 3 we discuss the near-flat space expansion of the action. We study the short-distance singularities in the OPE of the currents and calculate the field renormalization, which is necessary in our formalism. Flat space limit was recently discussed in [17]. Section 4 contains the general discussion of the renormalization of the Wilson line type of operators. In Sections 5 and 6 we calculate the logarithmic divergences of the transfer matrix at the one-loop level, and find that they cancel. As a consistency check of our calculational framework, we show explicitly in Section 7 that logarithmic divergences cancel in the global symmetry charge. Section 8 is a conclusion. Appendix A provides details on the algebra $\mathfrak{psu}(2, 2|4)$. In Appendix B we discuss linear divergences; they would be important in the higher loop calculations.

2 Brief introduction to pure spinors

2.1 Action functional and “capital currents” J_{\pm}

The target space $\text{AdS}_5 \times S^5$ is the coset space

$$\frac{\text{PSU}(2, 2|4)}{\text{SO}(4, 1) \times \text{SO}(5)}.$$

A point of this space can be parametrized by the group element $g \in \text{PSU}(2, 2|4)$ modulo the left shift by $h \in \text{SO}(4, 1) \times \text{SO}(5)$:

$$g \equiv hg, \quad h \in \text{SO}(4, 1) \times \text{SO}(5). \quad (2.1)$$

The action in the pure spinor formalism [14–17] is constructed out of the “capital” currents¹

$$J_{\pm} = -\partial_{\pm} g g^{-1}, \quad g \in \frac{\text{PSU}(2, 2|4)}{\text{SO}(4, 1) \times \text{SO}(5)}, \quad (2.2)$$

and bosonic ghosts $(\lambda^{\alpha}, w_{\pm\alpha})$. The Lie-algebra $\mathfrak{psu}(2, 2|4)$ has a \mathbb{Z}_4 grading

$$\mathfrak{psu}(2, 2|4) = \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3. \quad (2.3)$$

This \mathbb{Z}_4 grading has a clear physical meaning. It depends on the choice of a point x_0 in $\text{AdS}_5 \times S^5$. This same point will be used in our flat space

¹We call these currents “capital” because there are also “small case” currents; see Section 2.4. Capital currents are invariant under the global symmetry $\text{PSU}(2, 2|4)$. Small case currents are invariant under the gauge transformations (2.1).

expansion. The flat space limit is the limit when the string is localized near this point x_0 . In this limit the target space $\text{AdS}_5 \times S^5$ is approximated by flat space, which is the tangent space $T_{x_0}(\text{AdS}_5 \times S^5)$. In the flat space type IIB superstring there are two supersymmetries, ϵ_L and ϵ_R , which are both Majorana–Weyl spinors. One comes from the left sector on the string worldsheet and the other from the right sector, so they are physically distinguished. We say that the Killing spinor on $\text{AdS}_5 \times S^5$ is in $\mathfrak{g}_{\bar{1}}$ if it becomes ϵ_L in the flat limit near x_0 , and in $\mathfrak{g}_{\bar{3}}$ if it becomes ϵ_R . This is a grading, because the anticommutator of two left supersymmetries gives us a translation, while the anticommutator of left and right supersymmetry gives zero in the flat space limit.

It is useful to label the generators of $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$ using this flat space picture. The bosonic generators in the flat space limit are boosts and rotations $t_{[\mu\nu]}^0$, and also the translations t_{μ}^2 ; here μ and ν are the vector indices of the tangent space. The fermionic generators are the left supersymmetries t_{α}^3 and the right supersymmetries $t_{\dot{\alpha}}^1$. Here α and $\dot{\alpha}$ are both the spinor indices of the Majorana–Weyl spinors; they have the same chirality (left Majorana–Weyl spinors). In other words, α versus $\dot{\alpha}$ do not indicate different chiralities but the grading (3 and 1, respectively). To summarize, we have the following set of generators:

$$t = \{t_{[\mu\nu]}^0, t_{\dot{\alpha}}^1, t_{\mu}^2, t_{\alpha}^3\}, \tag{2.4}$$

where $\mu = 0, \dots, 9$ and $\alpha, \dot{\alpha} = 1, \dots, 16$. The bosonic subalgebra is $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{2}}$, where $\mathfrak{g}_{\bar{0}}$ corresponds to the denominator algebra $\mathfrak{so}(4, 1) \oplus \mathfrak{so}(5)$.

The bosonic ghosts (λ_3, w_{1+}) and (λ_1, w_{3-}) take values in $\mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{3}}$ and satisfy the pure spinor condition

$$\lambda_1 \Gamma^{\mu} \lambda_1 = \lambda_3 \Gamma^{\mu} \lambda_3 = 0, \tag{2.5}$$

where $\Gamma_{\alpha\beta}^{\mu}$ are the $\text{SO}(9, 1)$ gamma matrices. The solution space to the pure spinor constraint is 11 complex dimensional and is parametrized by the coset space $\text{SO}(10)/U(5)$.

The action functional is [14, 24]

$$S = \frac{R^2}{\pi} \int d^2 z \text{Str} \left(\frac{1}{2} J_{2+} J_{2-} + \frac{3}{4} J_{1+} J_{3-} + \frac{1}{4} J_{3+} J_{1-} \right. \\ \left. + w_{1+} \partial_- \lambda_3 + w_{3-} \partial_+ \lambda_1 + N_{0+} J_{0-} + N_{0-} J_{0+} - N_{0+} N_{0-} \right), \tag{2.6}$$

where the ghost currents

$$N_{0+} = -\{w_{1+}, \lambda_3\}, \quad N_{0-} = -\{w_{3-}, \lambda_1\} \quad (2.7)$$

can be seen to couple non-trivially to the physical fields. The currents are contracted with the generators of $\mathfrak{psu}(2, 2|4)$:

$$J_{0+} = J_{0+}^{[\mu\nu]} t_{[\mu\nu]}^0, \quad J_{2+} = J_{2+}^\mu t_\mu^2, \quad J_{3+} = J_{3+}^\alpha t_\alpha^3, \quad J_{1+} = J_{1+}^{\dot{\alpha}} t_{\dot{\alpha}}^1, \quad (2.8)$$

which are chosen in a finite-dimensional representation. The ‘‘Str’’ in the action is the supertrace in the fundamental representation $4|4$ of $\mathfrak{su}(4|4)$; it defines the invariant bilinear form on $\mathfrak{psu}(2, 2|4)$. The physical spectrum is obtained as the cohomology of the BRST-operator. The classical BRST-transformation is generated by

$$Q = \int \text{Str} (\lambda_1 J_{3-} d\tau^- + \lambda_3 J_{1+} d\tau^+). \quad (2.9)$$

Using the pure spinor constraint (2.5) it follows that Q is nilpotent up to a gauge transformation. Quantum BRST and conformal invariance of the action were established in [18, 19].

It is probably not very easy to define the classical limit, due to the pure spinor ghosts; it is not obvious that just setting $\lambda = w = 0$ would yield the correct classical theory. But the main point in favour of the pure spinor approach is that it is possible to do perturbative calculations in the quantum theory without introducing the light cone gauge, and thereby maintaining the full $\mathfrak{psu}(2, 2|4)$ symmetry.

The only serious obstacle in this direction is lack of experience with the curved $\beta\gamma$ systems; for some recent progress in this direction see [25] and references therein.

2.2 Equations of motion and zero-curvature equations

Classical integrability was established first for the GSMT action [26] by rewriting the equations of motion in terms of zero-curvature equations [27]. Classical integrability for the pure spinor string can be established likewise. In the following we will review the analysis in [18, 20, 21].² The zero-curvature conditions on the currents J are

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0. \quad (2.10)$$

²Further discussion of the classical dynamics have appeared in [28, 29].

Let us introduce the D_0 covariant derivative:

$$D_0 = d + J_0.$$

This is just the standard Levi–Civita metric connection in the tangent space to $\text{AdS}_5 \times S^5$. (While the “full” covariant derivative $d + J$ can sometimes be identified as the “long” connection modified by the Ramond–Ramond five-form field strength; it is roughly speaking $d + \frac{1}{2}\omega^{ab}\Gamma_{ab} + d\hat{x}F^{abcde}\Gamma_{abcde}$.) The tangent space to the space of solutions of (2.10) is parametrized by ξ :

$$\delta_\xi J = d\xi + [J, \xi]. \quad (2.11)$$

When $\xi = \xi_3$ we obtain the equations for J_1 :

$$D_{0+}J_{1-} + [J_{3+}, J_{2-}] + [J_{2+}, J_{3-}] - [N_{0+}, J_{1-}] + [J_{1+}, N_{0-}] = 0, \quad (2.12)$$

$$D_{0-}J_{1+} + [J_{1-}, N_{0+}] - [N_{0-}, J_{1+}] = 0. \quad (2.13)$$

When $\xi = \xi_1$ we obtain the equations for J_3 :

$$D_{0+}J_{3-} - [N_{0+}, J_{3-}] + [J_{3+}, N_{0-}] = 0, \quad (2.14)$$

$$D_{0-}J_{3+} - [N_{0+}, J_{3-}] + [J_{3+}, N_{0-}] - [J_{2+}, J_{1-}] - [J_{1+}, J_{2-}] = 0. \quad (2.15)$$

When $\xi = \xi_2$ we obtain the equations for J_2 :

$$D_{0+}J_{2-} + [J_{3+}, J_{3-}] - [N_{0+}, J_{2-}] + [J_{2+}, N_{0-}] = 0, \quad (2.16)$$

$$D_{0-}J_{2+} - [J_{1+}, J_{1-}] - [N_{0+}, J_{2-}] + [J_{2+}, N_{0-}] = 0. \quad (2.17)$$

The pure spinors do not change the condition that

$$\partial_+ J_{0-} - \partial_- J_{0+} + [J_{0+}, J_{0-}] + [J_{2+}, J_{2-}] + [J_{3+}, J_{1-}] + [J_{1+}, J_{3-}] = 0. \quad (2.18)$$

This is a “geometrical condition” on the worldsheet connection. In deriving the equations of motion for λ we have to take into account that

$$\text{Str}\{w_{1+}, \lambda_3\}J_{0-} = \text{Str} w_{1+}[\lambda_3, J_{0-}], \quad (2.19)$$

because w and λ are both odd elements of the superalgebra. For odd a and b we have $\text{Str} ab = -\text{Str} ba$. Therefore the equations of motion for λ are:

$$D_{0-}\lambda_3 - [N_{0-}, \lambda_3] = 0, \quad (2.20)$$

$$D_{0+}\lambda_1 - [N_{0+}, \lambda_1] = 0, \quad (2.21)$$

$$[\lambda_3, N_{0+}] = [\lambda_1, N_{0-}] = 0, \quad (2.22)$$

where the last equation is kinematical. The equations of motion for w are:

$$D_{0-}w_{1+} - [N_{0-}, w_{1+}] = 0, \quad (2.23)$$

$$D_{0+}w_{3-} - [N_{0+}, w_{3-}] = 0. \quad (2.24)$$

This also implies that

$$D_{0-}N_{0+} - [N_{0-}, N_{0+}] = 0, \quad (2.25)$$

$$D_{0+}N_{0-} - [N_{0+}, N_{0-}] = 0. \quad (2.26)$$

It is useful to introduce the combined current, which will play the role of the Lax pair,

$$J_+(z) = J_{0+} - N_{0+} + \frac{1}{z}J_{3+} + \frac{1}{z^2}J_{2+} + \frac{1}{z^3}J_{1+} + \frac{1}{z^4}N_{0+}, \quad (2.27)$$

$$J_-(z) = J_{0-} - N_{0-} + zJ_{1-} + z^2J_{2-} + z^3J_{3-} + z^4N_{0-}, \quad (2.28)$$

where z is the spectral parameter. The equations of motion can then be written as zero-curvature conditions:

$$[\partial_+ + J_+(z), \partial_- + J_-(z)] = 0. \quad (2.29)$$

An important point to notice is that the Lax pair in the pure spinor formulation is different from the one in [27] based on the Metsaev–Tseytlin action [26]. This is true even after dropping the ghost terms. The Lax connection in the Metsaev–Tseytlin formulation is

$$J_+^{\text{MT}}(z) = J_{0+} + zJ_{1+} + \frac{1}{z}J_{3+} + \frac{1}{z^2}J_{2+}, \quad (2.30)$$

and the resulting equations are different in the matter sector. Nevertheless, the theories are of course classically equivalent, which follows by choosing the specific gauge in the Metsaev–Tseytlin formulation $J_{1+} = 0$ and $J_{3-} = 0$.

Now let us notice that the equations of motion (2.20)–(2.22) are equivalent to the statement that the coefficients:

- of z^{-5} in $[\partial_+ + J_+(z), z^{-1}\lambda_3]$,
- of z in $[\partial_+ + J_+(z), z\lambda_1]$,
- of z^5 in $[\partial_- + J_-(z), z\lambda_1]$,
- of z^{-1} in $[\partial_- + J_-(z), z^{-1}\lambda_3]$

are all zero. Therefore the BRST transformation is given by this formula:

$$[\epsilon Q, J_{\pm}(z)] = D_{\pm}^{(z)}(\epsilon\lambda(z)), \tag{2.31}$$

where $\lambda(z) = \frac{1}{z}\lambda_3 + z\lambda_1$. This means that Q acts as an infinitesimal dressing transformation.³

2.3 The transfer matrix

The transfer matrix is defined as the path ordered exponential:

$$\Omega_{\tau_r}^{\tau_l}(z) = P \exp \left[- \int_{\tau_r}^{\tau_l} (J_+(z)d\tau^+ + J_-(z)d\tau^-) \right]. \tag{2.32}$$

The zero-curvature equations (2.29) are equivalent to the flatness of the connection $J(z)$; this implies that $\Omega(z)$ does not depend on the choice of the contour. Classically, with the periodic boundary conditions, the expansion coefficients of $\text{Str } \Omega(z)$ in z around $0, \infty$ yield an infinite family of local charges [31], and the expansion around $z = 1$ results in an infinite set of non-local charges. Classically, these charges are all in involution (their Poisson brackets vanish).

Here we will study the logarithmic divergences of $\Omega(z)$ by explicitly computing the short-distance expansion of the currents.

2.4 “Small case” currents

The “small case currents” j_1, j_2, j_3 are defined as follows:

$$j_a = g^{-1} J_a g. \tag{2.33}$$

We also define j_0 :

$$j_0 = g^{-1} N g. \tag{2.34}$$

The most important property of these small case currents is that they are gauge invariant under (2.1). They do not have a definite grading; it is not

³Dressing transformations are the gauge transformations of the Lax connection preserving the analytical structure of the connection as a function of the spectral parameter [30].

true that j_1 belongs to \mathfrak{g}_1 . The global conserved charges corresponding to the Killing vectors and spinors of $\text{AdS}_5 \times S^5$ are given by linear combinations:

$$q_{\text{global}} = \int *(4j_0 + 3j_1 + 2j_2 + j_3). \quad (2.35)$$

Lüscher used the small case currents to construct the Yangian conserved charge in the $O(n)$ non-linear sigma model [8]. In his approach the Yangian charge was constructed from the ordered double integral of the small case currents:

$$q_{2,\text{Yangian}} = \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_1 j(\tau_2)j(\tau_1) + \int_{-\infty}^{+\infty} d\tau k(\tau). \quad (2.36)$$

It should be possible to generalize his arguments and prove the finiteness of the higher conserved charges for the string in $\text{AdS}_5 \times S^5$. The logarithmic divergences were present (and in fact played an important role) in [8, 32–34]. They appeared because of the collisions $\tau_2 \rightarrow \tau_1$ in (2.36), see Section 4.2. But they were all proportional to algebraic structures like $f_a^{bc}[t_b, t_c]$, which would vanish for the algebra $\mathfrak{psu}(2, 2|4)$ because the adjoint Casimir is zero in this case.

In this paper we will take a more pedestrian approach and calculate the divergences explicitly using the capital currents expanded around flat space. Our main reason is the following. We feel that the transfer matrix itself should play a fundamental role, rather than the non-local conserved charges which are obtained by expanding around $z = 1$. In [8] only the bilocal charges were explicitly studied (notice that the higher non-local charges could be obtained by calculating the Poisson brackets of the bilocal charges). It was important that the only source of logarithmic divergences were double collisions of j . In fact, in the expansion of the transfer matrix, triple collisions contribute to the divergence already at the one-loop level. And quadruple and higher order collisions would contribute at higher loops. This does not affect the bilocal charge, but the argument based on generating the higher order charges by applying the Poisson brackets to the bilocal charges is rather indirect.

The bilocal charge does not really “probe” the structure of the Lax equation (2.27)–(2.29). One could imagine that the bilocal conserved charge is given by an expression of the form (2.36), but this does not yet imply that the conserved charge can be obtained from $\Omega(z)$ given by (2.32) with J given by (2.27) and (2.28). In the pure spinor formalism the expressions (2.27) and (2.28) for the Lax connection appear rather artificial, and it is useful to

verify that this is a sensible construction in the quantum theory, by explicit calculations.

Note that in [21] the BRST-invariance of $q_{2, \text{Yangian}}$ was proven. The argument relied on the vanishing of the ghost-number +1 BRST-cohomology class.

3 Action and OPE in the near-flat space limit

The flat space limit of $\text{AdS}_5 \times S^5$ requires taking the large radius limit $R \rightarrow \infty$ and rescaling the bosonic and fermionic fields with $\frac{1}{R}$ so that the action becomes a quadratic expression like $(\partial x)^2 + (\partial \theta)^2$ plus non-linear terms proportional to powers of $\frac{1}{R}$. We will now explain how to do this rescaling.

3.1 Viel-bein

A viel-bein is a choice of basis in the tangent space to $\text{AdS}_5 \times S^5$. It is equivalent to the choice of the lift $g \in \text{PSU}(2, 2|4)$ for each point in $\frac{\text{PSU}(2, 2|4)}{\text{SO}(4, 1) \times \text{SO}(5)}$. We will parametrize the points of the space $\text{AdS}_5 \times S^5$ by $(x^\mu, \vartheta_L^\alpha, \vartheta_R^{\dot{\alpha}})$ as in [26], so that the viel-bein is

$$g(\vartheta, x) = \exp\left(\frac{1}{R}\vartheta_L^\alpha t_\alpha^3 + \frac{1}{R}\vartheta_R^{\dot{\alpha}} t_{\dot{\alpha}}^1\right) \exp\left(\frac{1}{R}x^\mu t_\mu^2\right). \tag{3.1}$$

Let us introduce the shorthand notations:

$$x = x^\mu t_\mu^2, \quad \vartheta_L = \vartheta_L^\alpha t_\alpha^3 \quad \text{and} \quad \vartheta_R = \vartheta_R^{\dot{\alpha}} t_{\dot{\alpha}}^1.$$

Note that x, ϑ_L and ϑ_R are all even elements of the Lie superalgebra $\mathfrak{psu}(2, 2|4)$. With this notation the viel-bein can be written as follows:

$$g = e^{\frac{1}{R}(\vartheta_L + \vartheta_R)} e^{\frac{1}{R}x}. \tag{3.2}$$

If $g = \exp \xi^a t_a$ then

$$\partial g g^{-1} = \partial \xi - \frac{1}{2}[\partial \xi, \xi] + \frac{1}{6}[[\partial \xi, \xi], \xi] + \dots \tag{3.3}$$

This implies the expansion for the currents in terms of the flat space fields x and ϑ :

$$\begin{aligned}
 -J_{2+} &= \frac{1}{R} \partial_+ x + \frac{1}{2R^2} [\vartheta_L, \partial_+ \vartheta_L] + \frac{1}{2R^2} [\vartheta_R, \partial_+ \vartheta_R] + \frac{1}{6R^3} [x, [x, \partial_+ x]] \\
 &\quad + \frac{1}{2R^3} [\vartheta_R, [\vartheta_L, \partial_+ x]] + \frac{1}{2R^3} [\vartheta_L, [\vartheta_R, \partial_+ x]] + O\left(\frac{1}{R^4}\right), \\
 -J_{3+} &= \frac{1}{R} \partial_+ \vartheta_L + \frac{1}{R^2} [\vartheta_R, \partial_+ x] \\
 &\quad + \frac{1}{2R^3} [\vartheta_L, [x, \partial_+ x]] + \frac{1}{6R^3} [\vartheta_R, [\vartheta_L, \partial_+ \vartheta_L]] + \frac{1}{6R^3} [\vartheta_L, [\vartheta_R, \partial_+ \vartheta_L]] \\
 &\quad + \frac{1}{6R^3} [\vartheta_L, [\vartheta_L, \partial_+ \vartheta_R]] + \frac{1}{6R^3} [\vartheta_R, [\vartheta_R, \partial_+ \vartheta_R]] + O\left(\frac{1}{R^4}\right), \\
 -J_{1+} &= \frac{1}{R} \partial_+ \vartheta_R + \frac{1}{R^2} [\vartheta_L, \partial_+ x] \\
 &\quad + \frac{1}{2R^3} [\vartheta_R, [x, \partial_+ x]] + \frac{1}{6R^3} [\vartheta_L, [\vartheta_L, \partial_+ \vartheta_L]] + \frac{1}{6R^3} [\vartheta_R, [\vartheta_R, \partial_+ \vartheta_L]] \\
 &\quad + \frac{1}{6R^3} [\vartheta_R, [\vartheta_L, \partial_+ \vartheta_R]] + \frac{1}{6R^3} [\vartheta_L, [\vartheta_R, \partial_+ \vartheta_R]] + O\left(\frac{1}{R^4}\right), \\
 -J_{0+} &= \frac{1}{2R^2} [x, \partial_+ x] + \frac{1}{2R^2} [\vartheta_R, \partial_+ \vartheta_L] + \frac{1}{2R^2} [\vartheta_L, \partial_+ \vartheta_R] \\
 &\quad + \frac{1}{2R^3} [\vartheta_L, [\vartheta_L, \partial_+ x]] + \frac{1}{2R^3} [\vartheta_R, [\vartheta_R, \partial_+ x]] + O\left(\frac{1}{R^4}\right). \tag{3.4}
 \end{aligned}$$

3.2 Global symmetries

Global symmetries act as constant right shifts:

$$g(\vartheta, x) \mapsto g(\vartheta, x)g_0 = h(\vartheta, x; g_0)g(S_{g_0} \cdot (\vartheta, x)), \tag{3.5}$$

where g_0 is a constant element of $\mathfrak{psu}(2, 2|4)$. We see that in the gauge (3.1) the action of the global symmetry g_0 corresponds to the “shift” of ϑ, x (which we denoted $S_{g_0} \cdot (\vartheta, x)$) and a gauge transformation with some parameter h which is a function of g_0 and x (this is sometimes referred to as compensating transformation). For example, the translation corresponds to $g_0 = e^{\frac{1}{R}\xi}$ with $\xi \in \mathfrak{g}_2$; the corresponding S_{g_0} and h are:

$$S_{g_0} x = x + \xi + \frac{1}{3R^2} [x, [x, \xi]] + \dots, \tag{3.6}$$

$$S_{g_0}\vartheta = \vartheta + \frac{1}{R^2}[\vartheta, [x, \xi]] + \dots, \quad (3.7)$$

$$h(\vartheta, x; e^\xi) = \exp\left(\frac{1}{2R^2}[x, \xi] + \dots\right). \quad (3.8)$$

Therefore in our gauge the global isometries act on the currents as gauge transformations:

$$S_{g_0}.J = -dh h^{-1} + hJh^{-1}, \quad (3.9)$$

where $h = h(\vartheta, x; g_0)$.

3.3 Action

The action without the ghost terms is

$$S = \frac{1}{\pi} \int d^2v \left(\frac{1}{2} C_{\mu\nu} J_{2+}^\mu J_{2-}^\nu + \frac{1}{4} C_{\alpha\dot{\beta}} J_{3+}^\alpha J_{1-}^{\dot{\beta}} + \frac{3}{4} C_{\dot{\alpha}\beta} J_{1+}^{\dot{\alpha}} J_{3-}^\beta \right). \quad (3.10)$$

Here $C_{\mu\nu}$, $C_{\alpha\dot{\beta}}$ and $C_{\dot{\alpha}\beta}$ are the invariant tensors, see Appendix A.

The path integral is the sum over histories of e^{-S} .

The near-flat space expansion of the currents yields

$$S = \frac{1}{\pi} \int d^2v \left(\frac{1}{2} C_{\mu\nu} \partial_+ x^\mu \partial_- x^\nu + C_{\dot{\alpha}\beta} \partial_+ \vartheta_R^{\dot{\alpha}} \partial_- \vartheta_L^\beta \right. \\ \left. - \frac{1}{2R} f_{\mu\alpha\beta} \partial_+ x^\mu \vartheta_L^\alpha \partial_- \vartheta_L^\beta - \frac{1}{2R} f_{\mu\dot{\alpha}\beta} \partial_- x^\mu \vartheta_R^{\dot{\alpha}} \partial_+ \vartheta_R^\beta + \dots \right). \quad (3.11)$$

Further we will need to know the order $1/R^2$ terms as well. These are

$$\frac{1}{\pi} \int d^2v \frac{1}{R^2} \text{Str } \mathcal{L}_2, \quad (3.12)$$

where

$$\mathcal{L}_2 = -\frac{1}{6} [x, \partial_+ x] [x, \partial_- x] \\ - \frac{1}{4} [\vartheta_R, \partial_+ x] [\vartheta_L, \partial_- x] + \frac{1}{4} [\vartheta_L, \partial_+ x] [\vartheta_R, \partial_- x] \\ - \frac{1}{8} [\vartheta_R, \partial_+ \vartheta_L] [x, \partial_- x] - \frac{1}{8} [\vartheta_L, \partial_- \vartheta_R] [x, \partial_+ x] \\ - \frac{3}{8} [\vartheta_L, \partial_+ \vartheta_R] [x, \partial_- x] - \frac{3}{8} [\vartheta_R, \partial_- \vartheta_L] [x, \partial_+ x]$$

$$\begin{aligned}
 & -\frac{1}{24}[\vartheta_L, \partial_+\vartheta_L][\vartheta_L, \partial_-\vartheta_L] - \frac{1}{24}[\vartheta_R, \partial_-\vartheta_R][\vartheta_R, \partial_+\vartheta_R] \quad (3.13) \\
 & + \frac{1}{24}[\vartheta_L, \partial_+\vartheta_L][\vartheta_R, \partial_-\vartheta_R] - \frac{1}{8}[\vartheta_L, \partial_-\vartheta_L][\vartheta_R, \partial_+\vartheta_R] \\
 & - \frac{1}{6}[\vartheta_R, \partial_+\vartheta_L][\vartheta_R, \partial_-\vartheta_L] - \frac{1}{6}[\vartheta_L, \partial_-\vartheta_R][\vartheta_L, \partial_+\vartheta_R] \\
 & - \frac{1}{12}[\vartheta_R, \partial_+\vartheta_L][\vartheta_L, \partial_-\vartheta_R] - \frac{1}{4}[\vartheta_L, \partial_+\vartheta_R][\vartheta_R, \partial_-\vartheta_L].
 \end{aligned}$$

3.4 The OPEs of the elementary fields

From the quadratic part of the action we have

$$\langle x^\mu(w, \bar{w})x^\nu(0) \rangle = -\pi C^{\mu\nu}(\partial_w\partial_{\bar{w}})^{-1}, \quad (3.14)$$

$$\langle \vartheta_L^\alpha(w, \bar{w})\vartheta_R^\beta(0) \rangle = -\pi C^{\alpha\beta}(\partial_w\partial_{\bar{w}})^{-1}, \quad (3.15)$$

$$\langle \vartheta_R^\alpha(w, \bar{w})\vartheta_L^\beta(0) \rangle = -\pi C^{\alpha\beta}(\partial_w\partial_{\bar{w}})^{-1}. \quad (3.16)$$

Note that $\frac{\partial}{\partial\bar{w}}\frac{1}{w} = \pi\delta^2(w, \bar{w})$. Therefore

$$\langle x^\mu(w, \bar{w})x^\nu(0) \rangle = -C^{\mu\nu} \log|w|^2. \quad (3.17)$$

3.5 A useful “symmetry”

Notice that all the formulas are valid if J_+ is exchanged with J_- , dotted spinor indices exchanged with undotted and ϑ_L with ϑ_R . In other words

$$(J_{0+}^{[\mu\nu]} \leftrightarrow J_{0-}^{[\mu\nu]}, J_{1+}^{\dot{\alpha}} \leftrightarrow J_{3-}^{\alpha}, J_{2+}^{\mu} \leftrightarrow J_{2-}^{\mu}, J_{3+}^{\alpha} \leftrightarrow J_{1-}^{\dot{\alpha}}). \quad (3.18)$$

3.6 OPE of currents

The near-flat space expansion of the currents and the OPE of x and ϑ imply the OPE of the currents to order $1/R^3$

$$J_{1+}^{\dot{\alpha}}(w_1)J_{2+}^{\mu}(w_2) = \frac{1}{R^3} \frac{\partial_+\vartheta_L^\gamma}{w_1 - w_2} f_{\gamma}^{\dot{\alpha}\mu} + O\left(\frac{1}{R^4}\right), \quad (3.19)$$

$$\begin{aligned}
 J_{3+}^{\alpha}(w_3)J_{2+}^{\mu}(w_2) &= \frac{2}{R^3} \frac{\partial_+\vartheta_R^{\dot{\beta}}}{w_3 - w_2} f_{\dot{\beta}}^{\alpha\mu} + \frac{1}{R^3} \frac{\bar{w}_3 - \bar{w}_2}{(w_3 - w_2)^2} \partial_-\vartheta_R^{\dot{\gamma}} f_{\dot{\gamma}}^{\alpha\mu} \\
 &+ O\left(\frac{1}{R^4}\right), \quad (3.20)
 \end{aligned}$$

$$J_{1+}^{\dot{\alpha}}(w_a)J_{1+}^{\dot{\beta}}(w_b) = -\frac{1}{R^3} \frac{\partial_+ x^\mu}{w_a - w_b} f_\mu^{\dot{\alpha}\dot{\beta}} + O\left(\frac{1}{R^4}\right), \quad (3.21)$$

$$J_{3+}^\alpha(w_a)J_{3+}^\beta(w_b) = -\frac{2}{R^3} \frac{\partial_+ x^\mu}{w_a - w_b} f_\mu^{\alpha\beta} - \frac{1}{R^3} \frac{\bar{w}_a - \bar{w}_b}{(w_a - w_b)^2} \partial_- x^\mu f_\mu^{\alpha\beta} + O\left(\frac{1}{R^4}\right), \quad (3.22)$$

$$J_{1+}^{\dot{\alpha}}(w_1)J_{3+}^\alpha(w_3) = -\frac{1}{R^2} \frac{1}{(w_1 - w_3)^2} C^{\dot{\alpha}\alpha} + O\left(\frac{1}{R^4}\right), \quad (3.23)$$

$$J_{2+}^\mu(w_m)J_{2+}^\nu(w_n) = -\frac{1}{R^2} \frac{1}{(w_m - w_n)^2} C^{\mu\nu} + O\left(\frac{1}{R^4}\right), \quad (3.24)$$

$$J_{0+}^{[\mu\nu]}(w_0)J_{1+}^{\dot{\alpha}}(w_1) = -\frac{1}{2R^3} \left(\frac{\vartheta_R^{\dot{\beta}}(w_0)}{(w_0 - w_1)^2} + \frac{\partial_+ \vartheta_R^{\dot{\beta}}(w_0)}{(w_0 - w_1)} \right) f_{\dot{\beta}}^{\dot{\alpha}[\mu\nu]} + O\left(\frac{1}{R^4}\right), \quad (3.25)$$

$$J_{0+}^{[\mu\nu]}(w_0)J_{3+}^\alpha(w_3) = -\frac{1}{2R^3} \left(\frac{\vartheta_L^\beta(w_0)}{(w_0 - w_3)^2} + \frac{\partial_+ \vartheta_L^\beta(w_0)}{(w_0 - w_3)} \right) f_\beta^{\alpha[\mu\nu]} + O\left(\frac{1}{R^4}\right), \quad (3.26)$$

$$J_{0+}^{[\mu\nu]}(w_0)J_{2+}^\lambda(w_2) = -\frac{1}{2R^3} \left(\frac{x^\kappa(w_0)}{(w_0 - w_2)^2} + \frac{\partial_+ x^\kappa(w_0)}{(w_0 - w_2)} \right) f_\kappa^{\lambda[\mu\nu]} + O\left(\frac{1}{R^4}\right). \quad (3.27)$$

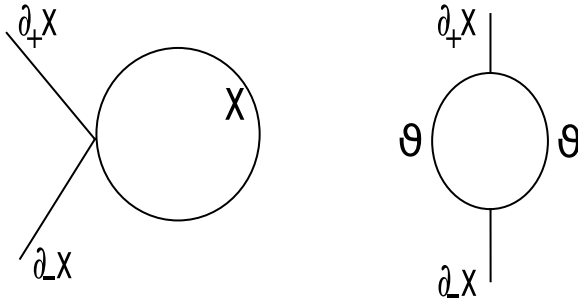
The OPEs are computed by evaluating all the Feynman diagrams J_+J_+ , including the non-linear terms in the action, and thus differ from just evaluating the OPE of J_+J_+ using the expansion in (3.4) and the free field OPEs of x and ϑ . This in particular leads to the interesting terms involving \bar{w} and the left-moving fields, which are however not unexpected in an interacting theory, where the decoupling of left- and right-moving modes is generically not possible. There are analogous OPEs of J_-J_- and J_+J_- . We will not write the complete table here. We will compute the necessary singularities in the following sections as we need them.

The OPEs of the currents were computed also in [35] using the background field method. Our OPEs are in agreement⁴ with [35].

⁴There was a mistake in the original version of our paper. The mistake was in the coefficients of the singularities of J_1J_2 and J_3J_2 , which we are not using in our further calculations. Notice the difference of notations: \mathbf{g}_1 of our paper corresponds to \mathbf{g}_3 of [35].

3.7 Field renormalization

The worldsheet theory for the pure spinor superstring in $\text{AdS}_5 \times S^5$ is believed to be UV finite. But this does not preclude log divergences, if these divergences could be absorbed into the field redefinition. It turns out that there is a renormalization of the field x , because there are logarithmic divergences of the type $\partial_+ x \partial_- x$. There exist two sorts of contribution to such divergences, the normal ordering of the quartic vertices and the fish diagram from the contraction of the two cubic vertices.



Quartic vertices: The contribution comes from the interaction terms in the action of the form $x x \partial x \partial x$ and $\vartheta \vartheta \partial x \partial x$. These terms come from the $J_{2+}J_{2-}$, $J_{3+}J_{1-}$ and $J_{1+}J_{3-}$ terms in the action. Our definition of the group element $e^{R^{-1}(\vartheta_L + \vartheta_R)}e^{R^{-1}x}$ is such that the terms in $J = -dgg^{-1}$ not containing ∂x do not contain x at all. Taking this into account, the terms in J_2 relevant to the wave function renormalization are:

$$\begin{aligned}
 J_2 = & -dx - \frac{1}{2}[\vartheta_L, d\vartheta_L] - \frac{1}{2}[\vartheta_R, d\vartheta_R] - \frac{1}{6}[x, [x, dx]] - \frac{1}{2}[\vartheta_L, [\vartheta_R, dx]] \\
 & - \frac{1}{2}[\vartheta_R, [\vartheta_L, dx]] + \dots
 \end{aligned}
 \tag{3.28}$$

The quartic terms in the Lagrangian leading to the log divergences of the type $\partial_+ x \partial_- x$ are:

$$\begin{aligned}
 & \frac{1}{6}(\partial_+ x, [x, [x, \partial_- x]]) + \frac{1}{2}(\partial_+ x, [\vartheta_L, [\vartheta_R, \partial_- x]]) + \frac{1}{2}(\partial_+ x, [\vartheta_R, [\vartheta_L, \partial_- x]]) \\
 & - \frac{1}{4}(\partial_+ x, [\vartheta_R, [\vartheta_L, \partial_- x]]) - \frac{3}{4}(\partial_+ x, [\vartheta_L, [\vartheta_R, \partial_- x]]).
 \end{aligned}$$

The log divergences in the terms with fermions cancel, and the $x x \partial x \partial x$ vertex leads to the log divergence

$$-\frac{1}{6} \log \epsilon^2(\partial_+ x, [t^\mu, [t^\mu, \partial_- x]])$$

which contributes to the renormalization of the field x .

Fish diagram: The cubic vertices are $-\frac{1}{2}([\partial_+x, \vartheta_L], \partial_- \vartheta_L)$ and $-\frac{1}{2}([\partial_-x, \vartheta_R], \partial_+ \vartheta_R)$. The log divergence in the fish is effectively the same as the log divergence of the expression $-\frac{1}{2}([\partial_+x, \vartheta_L], [\partial_-x, \vartheta_R])$.

Therefore the total log divergence from the quartic vertices and from the fish is

$$-\frac{1}{6} \frac{1}{R^2} \log \epsilon^2(\partial_+x, [t_\mu^2, [t_\mu^2, \partial_-x]]) - \frac{1}{2} \frac{1}{R^2} \log \epsilon^2 C^{\alpha\dot{\beta}}(\partial_+x, \{t_\alpha^3, [t_\beta^1, \partial_-x]\}). \tag{3.29}$$

This means that we should replace x with the renormalized x , which is

$$\begin{aligned} x &= x^{\text{ren}} + \frac{1}{6} \frac{1}{R^2} \log \epsilon^2[t_\mu^2, [t_\mu^2, x^{\text{ren}}]] + \frac{1}{2} \frac{1}{R^2} \log \epsilon^2 C^{\alpha\dot{\beta}}\{t_\alpha^3, [t_\beta^1, x^{\text{ren}}]\} \\ &= x^{\text{ren}} - \frac{1}{3R^2} \log |\epsilon|^2 C_2 \cdot x^{\text{ren}}. \end{aligned} \tag{3.30}$$

In other words, the renormalization of dx is such that this expression

$$dx + \frac{1}{6}[x, [x, dx]] + \frac{1}{2}[\vartheta_L, [\vartheta_R, dx]] \tag{3.31}$$

remains finite. One can see that this is the same as saying that J_2 is finite. (We should stress that our analysis is only valid to the order R^{-3} .)

Equation (3.30) can be checked against the formula (3.6) for the global shift. The expression for $S_{g_0} x^{\text{ren}}$ as a function of x^{ren} is the same as the expression for $S_{g_0} x$ as a function of x :

$$S_{g_0} x^{\text{ren}} = x^{\text{ren}} + \xi + \frac{1}{3R^2} : [x^{\text{ren}}, [x^{\text{ren}}, \xi]] : + \dots \tag{3.32}$$

if we take into account that $[x, [x, \xi]] = : [x, [x, \xi]] : - \log |\epsilon|^2 C_2 \cdot \xi + \dots$. This agrees with the non-renormalization of R .

4 Renormalization of the Wilson line type of operators

4.1 Logarithmic divergences

We consider a non-local operator of the form

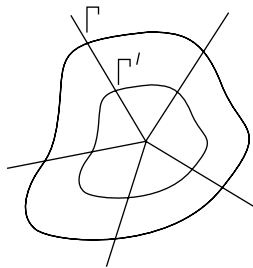
$$\Omega[\Gamma] = P \exp \left(- \int_\Gamma J \right), \tag{4.1}$$

where Γ is a contour and J has a regular expansion in powers of R^{-1} , starting with the leading term of the order R^{-1} . We assume that J is a one-form and an element of the Lie superalgebra (in our case $\mathfrak{psu}(2, 2|4)$). We expand the path ordered exponential in powers of R^{-1} and get an infinite series of terms of the type

$$\int_{\tau_1 < \tau_2 < \dots < \tau_n} J(\tau_1) \dots J(\tau_n). \tag{4.2}$$

When we compute the expectation value of this operator we typically encounter linear and logarithmic divergences. Linear divergences depend on the regularization scheme. But logarithmic divergences should not depend on the regularization scheme.

A Wilson loop type of operator cannot be conformally invariant, and therefore cannot be independent of the choice of the contour, if logarithmic divergences are present. Indeed, let us consider two contours Γ and Γ' which are related by a dilatation.



This means that $\Gamma' = \lambda\Gamma$ pointwise, where λ is a real number (the dilatation parameter). Both $\Omega[\Gamma]$ and $\Omega[\Gamma']$ have UV divergences, therefore we should regularize them. Let us use the contour split prescription, as in Appendix B. We should use the same regularization for both contours, with the same parameter ϵ . The renormalized Wilson line Ω^{ren} is equal to the regularized Ω_ϵ plus counterterms:

$$\Omega^{\text{ren}}[\Gamma] = \lim_{\epsilon \rightarrow 0} (\Omega_\epsilon[\Gamma] + C_\epsilon[\Gamma]), \tag{4.3}$$

where $C_\epsilon[\Gamma]$ includes all the counterterms. If it is true that the Wilson line does not depend on the choice of the contour, then we should have

$$\Omega^{\text{ren}}[\Gamma'] = \Omega^{\text{ren}}[\Gamma]. \tag{4.4}$$

But let us apply a dilatation to (4.3). Conformal invariance implies

$$\Omega_\epsilon[\Gamma] = \Omega_{\lambda\epsilon}[\Gamma']. \tag{4.5}$$

By definition $\Omega^{\text{ren}}[\Gamma]$ does not depend on ϵ . If it were true that $C_{\lambda\epsilon}[\Gamma'] = C_\epsilon[\Gamma]$ then (4.5) would imply (4.4). But in fact, if there are logarithmic divergences, then it is not true that $C_{\lambda\epsilon}[\Gamma'] = C_\epsilon[\Gamma]$. Indeed, the logarithmic divergences are of the form

$$\int d\tau^+ \partial_{+x} f(x) \log \epsilon, \quad (4.6)$$

and this expression is not invariant under conformal transformations (because ϵ has weight 1). Note that the linear divergences, which are of the form

$$\int \frac{d\tau^+}{\epsilon^+} f(x), \quad (4.7)$$

are invariant under conformal transformations, but the logarithmic divergences (4.6) are not. Therefore the independence of the Wilson line on the contour should imply the absence of logarithmic divergences. It is not clear to us whether the converse is also true.

There are several possible sources of logarithmic divergences. The currents A are typically composite objects and could therefore get “internal” logarithmic divergences.⁵ Even if J were elementary fields, in an interacting theory they could get logarithmic divergences because of field renormalization. Also, logarithmic divergences arise when two or more points on the integration contour collide, for example $\tau_1 \rightarrow \tau_2$ or $\tau_1 \rightarrow \tau_2 \rightarrow \tau_3$.

The divergences that we analyse in this paper are the “internal” or “field renormalization” divergences, and the divergences due to double or triple collisions on the contour. Let us first discuss the logarithmic divergences due to the double collisions.

4.2 Double collisions

4.2.1 Double collisions with the second-order pole a c -number

We will now discuss the log divergences in the double collisions of the type $j_+ \leftarrow j_+$. There are also logarithmic divergences in the collisions $j_+ \leftarrow j_-$

⁵For example, in the free theory the “composite” vertex operator $e^{ik\phi}$ gets the anomalous dimension $\simeq k^2$ due to its “internal” divergence.

and $j_- \longleftrightarrow j_+$. Suppose that we have the currents $j_+^a(w)$ with the OPE:

$$j_+^a(w)j_+^b(0) = C^{ab} \frac{1}{w^2} + F_c^{ab} k_+^c(0) \frac{1}{w} + \tilde{F}_c^{ab} \frac{\bar{w}}{w^2} k_-^c(0) + \dots, \tag{4.8}$$

where k_+ are some other currents, $F_c^{ab} = -F_c^{ba}$ are the coefficients of the singular term in the OPE and dots denote finite terms. We will assume for a moment that C^{ab} are c -numbers, then the second-order pole $\frac{1}{w^2}$ does not contribute to the logarithmic divergence. Only the simple pole contributes. Let us consider the path ordered contour integral:

$$\int_{-\infty}^{\infty} dw_1 j_+^a(w_1) t_a \int_{-\infty}^{w_1} dw_2 j_+^b(w_2) t_b, \tag{4.9}$$

where $[t_a, t_b] = f_{ab}^c t_c$. When $w_1 \rightarrow w_2$ we obtain the logarithmic divergence of the form

$$-\frac{1}{2} \ln \epsilon \int_{-\infty}^{\infty} (dw j_+^a F_a^{bc} + d\bar{w} j_-^a \tilde{F}_a^{bc}) f_{bc}^e t_e. \tag{4.10}$$

4.2.2 Double collisions with field-dependent second-order pole

The coefficient of the second-order pole may depend on w, \bar{w} , as in (3.25)–(3.27). In this case there are two ways of presenting the singularity:

$$j_+^a(w_1)j_+^b(w_2) = \frac{C^{ab}(w_1, \bar{w}_1)}{(w_1 - w_2)^2} + F_c^{ab} k_+^c \frac{1}{w_1 - w_2} + \tilde{F}_c^{ab} k_-^c \frac{\bar{w}_1 - \bar{w}_2}{(w_1 - w_2)^2} + \dots \tag{4.11}$$

$$\begin{aligned} &= \frac{C^{ab}(w_2, \bar{w}_2)}{(w_1 - w_2)^2} + (F_c^{ab} k_+^c + \partial C^{ab}) \frac{1}{w_1 - w_2} \\ &+ \frac{C^{ab}(w_2, \bar{w}_2)}{(w_1 - w_2)^2} + (\tilde{F}_c^{ab} k_-^c + \bar{\partial} C^{ab}) \frac{\bar{w}_1 - \bar{w}_2}{(w_1 - w_2)^2} + \dots \end{aligned} \tag{4.12}$$

The path ordered double integral $\int_{w_1 > w_2} j_+(w_1)dw_1 j_+(w_2)dw_2$ is logarithmically divergent because of the collision $w_1 \rightarrow w_2$. The logarithmic divergence depends on the order of evaluation of the integrals. Suppose that we first integrate over w_2 , and then over w_1 . Then, using the first formula (4.11) we obtain the log divergence:

$$-\frac{1}{2} \int \log \epsilon [t_a, t_b] (F_c^{ab} k_+^c(w, \bar{w})dw + \tilde{F}_c^{ab} k_-^c(w, \bar{w})d\bar{w}). \tag{4.13}$$

On the other hand, if we first integrate over w_1 and then over w_2 , then using the second formula (4.12) we obtain

$$-\frac{1}{2} \int \log \epsilon[t_a, t_b] (F_c^{ab} k_+^c(w, \bar{w}) dw + \tilde{F}_c^{ab} k_-^c(w, \bar{w}) d\bar{w} + dC^{ab}). \tag{4.14}$$

Notice that the difference between (4.13) and (4.14) is a total derivative. It can be integrated and contributes only through the contact terms. To understand these contact terms, consider for example the ordered integral of three currents:

$$\int_{w_1 > w_2 > w_3} dw_1 j_+^a(w_1) t_a \quad dw_2 j_+^b(w_2) t_b \quad dw_3 j_+^c(w_3) t_c. \tag{4.15}$$

Consider the log divergence coming from the collision of w_1 with w_2 . The difference between (4.13) and (4.14) is equal to $[t_a, t_b] dC^{ab}$. This is a total derivative, it almost integrates to zero except for contact term arising from the condition $w_2 > w_3$; a similar contact term arises when we first collide w_2 and w_3 , and together they give

$$\int dw_3 [[t_a, t_b] C^{ab}(w_3), j_+^c(w_3) t_c]. \tag{4.16}$$

But in fact the double pole $1/w^2$ in the product $j^a j^b$ leads to an additional log divergence from the *triple collision* $j^a j^b j^c$. This will be explained in the next subsection. If the coefficient of the double pole is not a constant, then the log divergence of the triple collision will also depend on the order of integrations. And the difference in the log divergence of the triple collision calculated with two different orders of integration will precisely cancel (4.16).

4.3 Triple collisions

4.3.1 Triple collisions when the second-order pole is a c -number

Triple collisions lead to logarithmic divergences in the case when there is a second-order pole in the OPE $j_+ j_+$. Let us first assume that the second-order pole comes with the c -number coefficient:

$$j_+^a(w) j_+^b(0) = -\frac{1}{R^2} \frac{1}{w^2} C^{ab} + \dots \tag{4.17}$$

The simple pole does not contribute to the logarithmic divergence in the triple collisions, and therefore in our discussion of the triple collisions we

can assume that the singularity in the product of two currents is just the second-order pole, as if $j_+^a = \partial_+ \phi^a$ with free fields ϕ^a :

$$j_+^a(w)j_+^b(0) = -\frac{1}{R^2} \frac{1}{w^2} C^{ab} + : j_+^a(w)j_+^b(0) :$$

Just to understand how the log divergence appears in triple collisions, let us first consider the situation where n constant matrices X_1, \dots, X_n are inserted at the positions a_1, \dots, a_n on the contour:

$$W = P \left[X_1(a_1) \cdots X_n(a_n) \exp \left(\int j_+ d\tau^+ \right) \right]. \tag{4.18}$$

We assume that X_j are constant c -number matrices, and the notation $X_j(a_j)$ just means that X_j is inserted at the point a_j on the contour. This is only needed to specify the right order of multiplication of matrices. Wick's theorem implies

$$W = P \left[X_1(a_1) \cdots X_n(a_n) \exp \left(-\frac{1}{2R^2} \int d\tau_1^+ \int d\tau_2^+ \frac{C^{ab} t^a(\tau_1^+) t^b(\tau_2^+)}{(\tau_1^+ - \tau_2^+)^2} \right) : \right. \\ \left. \times \exp \left(\int j_+ d\tau^+ \right) : \right].$$

Again, t^a are constant c -number matrices (the generators of the algebra), but we use the notation $t^a(\tau)$ to indicate that t^a is inserted at the point τ on the contour. This is important because of the path ordering of the product of the matrices. Let us consider the effect of just one contraction:

$$W = P \left[X_1(a_1) \cdots X_n(a_n) \left(-\frac{1}{2R^2} \int d\tau_1^+ \int d\tau_2^+ \frac{C^{ab} t^a(\tau_1^+) t^b(\tau_2^+)}{(\tau_1^+ - \tau_2^+)^2} \right) : \right. \\ \left. \times \exp \left(\int j_+ d\tau^+ \right) : \right].$$

Let us first illustrate the main point by focusing on four insertions. There are three types of terms. The first type has both t^a in the same interval between two consecutive X -insertions:

$$X_i \longleftrightarrow t^a \longleftrightarrow t^b \longleftrightarrow X_{i+1} : \int_{a_i+2\epsilon}^{a_{i+1}-\epsilon} d\tau_2 \int_{a_i+\epsilon}^{\tau_2-\epsilon} d\tau_1 \frac{1}{(\tau_1 - \tau_2)^2} = + \log \epsilon, \tag{4.19}$$

where we again dropped all subleading terms in ϵ . The second possibility is that they are between consecutive insertions

$$\begin{aligned}
 & X_i \longleftrightarrow t^a \longleftrightarrow X_{i+1} \longleftrightarrow t^b \longleftrightarrow X_{i+2} : \\
 & \times \int_{a_i+\epsilon}^{a_{i+1}-\epsilon} d\tau_1 \int_{a_{i+1}+\epsilon}^{a_{i+2}-\epsilon} d\tau_2 \frac{1}{(\tau_1 - \tau_2)^2} = -\log \epsilon. \tag{4.20}
 \end{aligned}$$

Finally, all further separated insertions do not contribute to the $\log(\epsilon)$ terms:

$$\begin{aligned}
 & X_i \leftrightarrow t^a \leftrightarrow X_{i+1} \cdots X_{i+k} \leftrightarrow t^b \leftrightarrow X_{i+k+1} : \\
 & \times \int_{a_i+\epsilon}^{a_{i+1}-\epsilon} d\tau_1 \int_{a_{i+k}+\epsilon}^{a_{i+k+1}-\epsilon} d\tau_2 \frac{1}{(\tau_1 - \tau_2)^2} = \text{finite}. \tag{4.21}
 \end{aligned}$$

With this rule, we can compute the contribution from $J_{2+}J_{2+}$:

$$\begin{aligned}
 & P \left(\prod_{j=1}^n X_j(a_j) \int \int \frac{C^{ab} t^a(\tau_1) t^b(\tau_2)}{(\tau_1^+ - \tau_2^+)^2} \right) \\
 & = \log(\epsilon) C^{ab} \sum_j \cdots X_{j-2} X_{j-1} (t^a t^b X_j - t^a X_j t^b) X_{j+1} X_{j+3} \cdots \\
 & = \sum_j \cdots X_{j-2} X_{j-1} \left(\frac{1}{2} \log \epsilon C^{ab} [t^a, [t^b, X_j]] \right) X_{j+1} X_{j+2} \cdots. \tag{4.22}
 \end{aligned}$$

Straightforward generalization of this argument yields that to lowest order in $\frac{1}{R^2}$ the logarithmic divergence due to triple collisions from bosonic currents, in particular $j_+ = J_{2+}$, is given by

$$\begin{aligned}
 & P \left[X_1(a_1) \cdots X_n(a_n) \exp \left(\int j_+ d\tau^+ \right) \right] \\
 & \longrightarrow -\frac{1}{2} \frac{1}{R^2} \log \epsilon \left(\sum_{k=1}^n P \left[X_1 \cdots C^{ab} [t^a, [t^b, X_k]] \cdots X_n \exp \left(\int j_+ d\tau^+ \right) \right] \right. \\
 & \left. + P \left[X_1 \cdots X_n \int d\tau^+ C^{ab} [t^a, [t^b, j_+]] \exp \left(\int j_+ d\tau^+ \right) \right] \right). \tag{4.23}
 \end{aligned}$$

Similarly the effect of fermionic currents j_+ can be analysed. The OPE has leading order, which is again a c -number

$$j_+^\alpha(w) j_+^{\dot{\beta}}(0) = -\frac{1}{R^2} \frac{1}{w^2} C^{\alpha\dot{\beta}} + \dots \tag{4.24}$$

Assuming that the c -number insertions $X_i(a_i)$ are bosonic the logarithmic divergence is

$$\begin{aligned}
 &P \left[X_1(a_1) \cdots X_n(a_n) \exp \left(\int j_+ d\tau^+ \right) \right] \\
 &\longrightarrow -\frac{1}{2} \frac{1}{R^2} \log \epsilon \left(\sum_{k=1}^n P \left[X_1 \cdots C^{\alpha\beta} \{t^\alpha, [t^\beta, X_k]\} \cdots X_n \exp \left(\int j_+ d\tau^+ \right) \right] \right. \\
 &\left. + P \left[X_1 \cdots X_n \int d\tau^+ C^{\alpha\beta} [t^\alpha, \{t^\beta, j_+\}] \exp \left(\int j_+ d\tau^+ \right) \right] \right). \tag{4.25}
 \end{aligned}$$

Similar expressions hold with suitably altered commutators/anti-commutators for fermionic insertions $X_i(a_i)$.

There are three types of sources for triple-collisions: $J_{2+}^\mu J_{2+}^\nu$, $J_{3+}^\alpha J_{1+}^\beta$ and $J_{1+}^\beta J_{3+}^\alpha$. For $X \in \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{3}}$ equation (A.13) implies vanishing of the logarithms. So the only contributions arise from bosonic insertions X . In this section we will be interested in the transfer matrix itself and thus not discuss any insertions. Then the non-trivial contributions are the second lines of (4.23) and (4.25).

4.3.2 Triple collisions with field-dependent second-order pole

So far we considered the case when the most singular double pole term in the OPE is field independent, just a c -number. But in fact we will encounter more general situations, for example in the computation of the divergence proportional to $[x, dx]$ in the triple collision $J_{2+}(0) \leftrightarrow J_{2+}(w_2) \leftrightarrow J_{0+}(w_0)$. After evaluating the OPE $J_{2+}(w_2)J_{0+}(w_0)$ one needs to integrate $x(w_0)/(w_2 - w_0)^2$ with respect to w_2 and w_0 . This is the type of triple collision that we will now discuss. Just to have a simple example, consider again the system of “free currents:”

$$j_+^a(w)j_+^b(0) = -\frac{1}{R^2} \frac{1}{w^2} C^{ab+} : j_+^a(w)j_+^b(0) :$$

Imagine we have some function $\Phi^{ab}(w, \bar{w})$ and consider the path ordered double integral:

$$\int_{w_1 > w_2} (t_a \Phi^{ac}(w_1, \bar{w}_1) j_+^c(w_1, \bar{w}_1) dw_1) (t_b j_+^b(w_2, \bar{w}_2) dw_2).$$

It is best to think of $\Phi^{ab}(w, \bar{w})$ as a c -number valued function on the world-sheet. Notice that in our application $J_{0+} = \frac{1}{2}[\partial_+ x, x]$ and Φ is actually a

field: $\Phi^{ab} = (\text{ad}(x))^{ab}$. But for understanding what is going on, it is enough to consider the example where Φ is a c -number valued function of w and \bar{w} .

The OPE between Φj_+ and j_+ contains a field-dependent second-order pole

$$\Phi^{ac}(w_1)j_+^c(w_1) j_+^b(w_2) = -\frac{1}{R^2} \frac{\Phi^{ac}(w_1)C^{cb}}{(w_1 - w_2)^2} + \Phi^{ac}(w_1) : j_+^c(w)j_+^b(w_2) : \tag{4.26}$$

As we explained in Section 4.2.2, the log divergence from the double collisions depends in this situation on the order of taking integrals. The log divergence from the triple collisions also depends on the order of integrations, so that the *sum* of the divergences in double and triple collisions is independent of the order of integrations.

Let us agree that having the singularity of the form (4.26), we first integrate over w_2 and then over w_1 . Then there is no log divergence from the double collision, but the triple collisions do contribute to the log divergence. Let us consider the log divergences coming from the following three integrals in triple collisions.

First integral:

$$\begin{aligned} X_{i+1} &\longleftrightarrow t^b \Phi^{ba} \longleftrightarrow t^a \longleftrightarrow X_i : \\ &\times \int_{a_i+2\epsilon}^{a_{i+1}-\epsilon} d\tau_2 \int_{a_i+\epsilon}^{\tau_2-\epsilon} d\tau_1 \frac{\Phi^{ba}(\tau_2)}{(\tau_1 - \tau_2)^2} = \log \epsilon \Phi^{ba}(a_i). \end{aligned} \tag{4.27}$$

Second integral:

$$\begin{aligned} X_{i+1} &\longleftrightarrow t^a \longleftrightarrow t^b \Phi^{ba} \longleftrightarrow X_i : \\ &\times \int_{a_i+2\epsilon}^{a_{i+1}-\epsilon} d\tau_2 \int_{a_i+\epsilon}^{\tau_2-\epsilon} d\tau_1 \frac{\Phi^{ba}(\tau_1)}{(\tau_1 - \tau_2)^2} = \log \epsilon \Phi^{ba}(a_{i+1}). \end{aligned} \tag{4.28}$$

Note the difference of (4.27) and (4.28): $\log \epsilon \Phi(a_i)$ versus $\log \epsilon \Phi(a_{i+1})$. In (4.27) we first integrated over τ_1 and then over τ_2 , and in (4.28) first over τ_2 and then over τ_1 .

Third integral:

$$\begin{aligned} X_i &\leftrightarrow t^b \Phi^{ba} \leftrightarrow X_{i+1} \leftrightarrow t^a \leftrightarrow X_{i+2} : \\ &\times \int_{a_{i+1}+\epsilon}^{a_i-\epsilon} d\tau_2 \int_{a_{i+2}+\epsilon}^{a_{i+1}-\epsilon} d\tau_1 \frac{\Phi^{ba}(\tau_1)}{(\tau_1 - \tau_2)^2} = -\log \epsilon \Phi^{ba}(a_{i+1}). \end{aligned} \tag{4.29}$$

There is also a contribution from the collision $X_i \longleftrightarrow t^a \longleftrightarrow X_{i+1} \longleftrightarrow t^b \Phi^{ba} \longleftrightarrow X_{i+2}$ which is also proportional to $-\log \epsilon \Phi^{ba}(a_{i+1})$. The field-dependent triple collision is therefore

$$\begin{aligned}
 &P \left[X_1(a_1) \cdots X_n(a_n) \exp \left(\int j_+^a t^a d\tau^+ + \int \Phi^{ab} j_+^b t^a d\tau^+ \right) \right] \\
 &\longrightarrow -\frac{1}{R^2} \log \epsilon \left(\sum_{k=1}^n P \left[X_1 \cdots [\Phi^{ab} t^a, [t^b, X_k]] \cdots X_n \right. \right. \\
 &\quad \times \exp \left(\int j_+ d\tau^+ + \int \Phi^{ab} j_+^b t_a d\tau^+ \right) \left. \left. \right] \right. \\
 &\quad \left. + P \left[X_1 \cdots X_n \int d\tau^+ [\Phi^{ab} t^a, [t^b, j_+]] \exp \left(\int j_+ d\tau^+ \right) \right] \right). \tag{4.30}
 \end{aligned}$$

4.4 Divergences due to the interaction terms in the action

We will explain this type of divergences using a simplified model. Consider a couple of scalar fields ϕ_1 and ϕ_2 with the action

$$\frac{1}{\pi} \int d\tau^+ d\tau^- \left(\partial_+ \phi_1 \partial_- \phi_2 + \frac{1}{R} n_+ \phi_1 \partial_- \phi_2 \right), \tag{4.31}$$

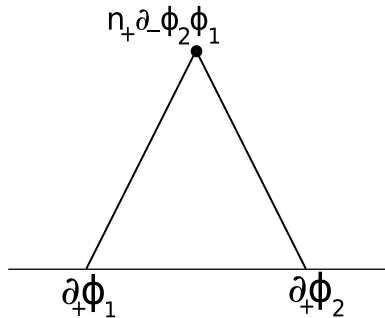
where n_+ is some function of τ^+, τ^- ; we could treat it as a classical source.

4.4.1 Effect of the interaction on the double collisions

1. $(++) \rightarrow (+)$: Consider the ordered integral

$$\int \int_{w_1 > w_2} dw_1 \partial_{w_1} \phi_1 \quad dw_2 \partial_{w_2} \phi_2. \tag{4.32}$$

This integral is logarithmically divergent because of the interaction term in the action.



Notice that the divergence can be calculated as the divergence of the “shifted” expression

$$\int \int_{w_1 > w_2} dw_1 \left(\partial_{w_1} \phi_1 - \frac{1}{R} n_{w_1} \phi_1 \right) dw_2 \partial_{w_2} \phi_2, \tag{4.33}$$

in the free theory. This is just a convenient way of representing the contraction of $\partial\phi_1$ with the interaction vertex $\int n\phi_1\bar{\partial}\phi_2$ in the action. Therefore the divergence is equal to

$$\frac{1}{R} \log \epsilon \int dw n_w. \tag{4.34}$$

2. $(+-) \rightarrow (+)$: This is is very similar,

$$\int \int_{\tau_1 > \tau_2} dw_1 \partial_{w_1} \phi_1 d\bar{w}_2 \partial_{\bar{w}_2} \phi_2. \tag{4.35}$$

The divergence is equal to

$$\frac{1}{R} \log \epsilon \int dw n_w. \tag{4.36}$$

3. $(-+) \rightarrow (+)$:

$$\int \int_{\tau_1 > \tau_2} d\bar{w}_1 \partial_{\bar{w}_1} \phi_1 dw_2 \partial_{w_2} \phi_2. \tag{4.37}$$

The divergence is the same as in $(+-) \rightarrow (+)$:

$$\frac{1}{R} \log \epsilon \int dw n_w. \tag{4.38}$$

4. $(--)\rightarrow (+)$: Consider the integral with two $d\tau^-$,

$$\int \int_{w_1 > w_2} d\bar{w}_1 \partial_{\bar{w}_1} \phi_1 d\bar{w}_2 \partial_{\bar{w}_2} \phi_2. \tag{4.39}$$

We were slightly surprised to find that this integral has a divergence proportional to $\int d\tau^+ n_+$. To calculate this divergence we have to

evaluate the integral over the position of the interaction vertex, with two contractions:

$$\int \int_{w_1 > w_2} \left(-\frac{1}{\pi}\right) \int d^2v n_w(v) \frac{\partial}{\partial \bar{w}_1} \frac{\partial}{\partial \bar{v}} \log |w_1 - v|^2 \frac{\partial}{\partial \bar{w}_2} \log |w_2 - v|^2 \quad (4.40)$$

$$= \int d\bar{w}_1 d\bar{w}_2 \frac{w_1 - w_2}{(\bar{w}_1 - \bar{w}_2)^2} n_w = -\frac{1}{R} \log \epsilon \int dw n_w. \quad (4.41)$$

4.4.2 Effect of the interaction on composite currents like $[x, \partial x]$

Example when the interaction vertex does not lead to a logarithmic divergence: Consider the contour integral of the “composite” operator

$$\int d\bar{w} \phi_2 \partial_{\bar{w}} \phi_1. \quad (4.42)$$

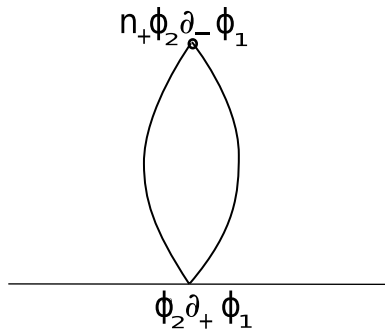
The integral over the interaction vertex becomes

$$\int d^2v \frac{1}{(\bar{v} - \bar{w})^2} \log |v - \epsilon|^2. \quad (4.43)$$

This is convergent.

Example when the integration vertex does lead to a log divergence:

$$\int dw \phi_2 \partial_w \phi_1. \quad (4.44)$$



The log divergence can be calculated by the following trick. Replace

$$\phi_2 \partial_w \phi_1 \rightarrow \phi_2 \left(\partial_w \phi_1 - \frac{1}{R} n_w \phi_1 \right), \quad (4.45)$$

and calculate the OPEs as in the free field theory. The logarithmic divergence is

$$\frac{1}{R} \log |\epsilon|^2 \int dw n_w. \tag{4.46}$$

Therefore,

$$\int dw \phi_1 \partial_w \phi_2 \mapsto -\frac{1}{R} \log |\epsilon|^2 \int dw n_w. \tag{4.47}$$

4.5 Algebraicity

Notice that one-loop logarithmic divergences are always of the form $\int \Phi$ where Φ is a one-form composed of the elementary fields and an element of $\mathfrak{psu}(2, 2|4)$. For example we get the expressions like $\int \log \epsilon d\tau^+ [\vartheta, \partial_+ x]$ but we never get something like $\int \log \epsilon d\tau^+ \{\vartheta, \partial_+ x\}$; the difference is that the anticommutator $\{\vartheta, \partial_+ x\}$ would not belong to the Lie algebra $\mathfrak{psu}(2, 2|4)$.

5 Calculation of logarithmic divergences

In this section we will apply the technique developed in Section 4 and calculate the one-loop logarithmic divergences in the transfer matrix. We will classify the divergences according to their dependence on the field and the power of the spectral parameter z .

5.1 Divergences proportional to J_{2+}

5.1.1 Coefficient of z^{-6}

There are two contributions. One comes from the double collision of $J_{1+}^{\dot{\alpha}}(w_a)$ $J_{1+}^{\dot{\beta}}(w_b)$:

$$\frac{1}{z^6} \frac{1}{2} \frac{1}{R^3} \log \epsilon \partial x^\mu C^{\dot{\alpha}\alpha} \{t_{\dot{\alpha}}^1, [t_{\dot{\alpha}}^3, t_\mu^2]\} = \frac{1}{z^6} \frac{1}{4} \frac{1}{R^3} \log \epsilon C_{\text{odd}} \cdot \partial x. \tag{5.1}$$

The other one comes from the triple collisions, as described in Section 4.3:

$$-\frac{1}{z^6} \frac{1}{2} \frac{1}{R^3} \log \epsilon (C_{\text{odd}} + C_{\dot{2}}) \cdot \partial x. \tag{5.2}$$

These two contributions cancel because of (A.25). Notice that this cancellation requires an interaction term $-\frac{1}{2} \frac{1}{R} f_{\mu\alpha\beta} \partial_+ x^\mu \vartheta_L^\alpha \partial_- \vartheta_L^\beta$ in the action.

If this term was zero, for example, we would have a coefficient 2 in (3.21), as in (3.22), and this would destroy the balance.

5.1.2 Coefficient of z^{-2}

As we have seen in Section 3.7 there are no internal log divergences in J_{2+} , at the order R^{-3} . Therefore we are left with the contributions from:

Double collision $J_{0+}J_{2+}$:

$$\frac{1}{2}z^{-2} \log \epsilon \in C_{\bar{2}} \cdot \partial_+ x. \tag{5.3}$$

Double collision $J_{3+}J_{3+}$:

$$\frac{1}{2}z^{-2} \log \epsilon \in C_{\text{odd}} \cdot \partial_+ x. \tag{5.4}$$

Double collision $J_{1+}J_{1-}$ does not contribute.

We conclude that the log divergence proportional to $z^{-2}\partial_+ x$ is

$$-\frac{1}{2}z^{-2} \log \epsilon \in C_{\bar{2}} \cdot \partial_+ x. \tag{5.5}$$

It looks like we got a non-zero log divergence, but it turns out that this divergence combines with the divergence proportional to $z^{-2}\partial_- x$ to a total derivative. We will explain this in Section 5.1.4.

5.1.3 Coefficient of z^2

There are contributions from the triple collisions $J_{1-}J_{3-}J_{2+}$ and $J_{2-}J_{2-}J_{2+}$ which are proportional both to z^2 . This should cancel against the double collision $J_{0+}J_{2-}$ and $J_{3-}J_{3+}$, which are also proportional to z^2 . Let us verify this. The contributions from the triple collisions are as usual:

$$\begin{aligned} & -\frac{1}{2R^3}z^2 \log \epsilon \left(C^{\dot{\alpha}\dot{\alpha}}\{t_{\dot{\alpha}}^1, [t_{\dot{\alpha}}^3, t_{\dot{\mu}}^2]\} + C^{\alpha\dot{\alpha}}\{t_{\alpha}^3, [t_{\dot{\alpha}}^1, t_{\dot{\mu}}^2]\} + C^{\kappa\lambda}[t_{\kappa}^2, [t_{\lambda}^2, t_{\dot{\mu}}^2]] \right) \partial_+ x^{\dot{\mu}} \\ & = -\frac{1}{R^3} \frac{1}{2}z^2 \log \epsilon (C_{\text{odd}} + C_{\bar{2}}) \cdot \partial_+ x. \end{aligned} \tag{5.6}$$

Now let us evaluate the contribution of $J_{0+}J_{2-}$. We obtain

$$J_{0+}^{[\mu\nu]}(w_0)J_{2-}^{\lambda}(w_2) = -\frac{1}{2R^3} \frac{\partial_+ x^{\kappa}(w_0)}{(\bar{w}_0 - \bar{w}_2)} f_{\kappa}^{\lambda[\mu\nu]}. \tag{5.7}$$

This gives the logarithmic contribution:

$$\frac{1}{R^3} \frac{1}{2} z^2 \log \epsilon C_{\bar{2}} \cdot \partial_+ x. \tag{5.8}$$

There is also a z^2 contribution from $J_{3-} J_{3+}$:

$$J_{3-}^\beta(w_b) J_{3+}^\alpha(w_a) = -\frac{1}{R^3} \frac{1}{\bar{w}_b - \bar{w}_a} \partial_+ x^\nu f_\nu^{\beta\alpha}. \tag{5.9}$$

This leads to the divergence

$$\frac{1}{R^3} \frac{1}{2} z^2 \log \epsilon C_{\text{odd}} \cdot \partial_+ x. \tag{5.10}$$

One can see that (5.6) + (5.8) + (5.10) = 0. But there is also a contribution from the double collision $J_{1-} J_{1-}$, because of (4.40). Because of the interaction the OPE of $J_{1-} J_{1-}$ contains $\partial_+ x$:

$$J_{1-}^{\dot{\alpha}}(w_a) J_{1-}^{\dot{\beta}}(w_b) = -\partial x^\mu f_\mu^{\dot{\alpha}\dot{\beta}} \frac{w_a - w_b}{(\bar{w}_a - \bar{w}_b)^2} + \dots \tag{5.11}$$

Therefore as explained in Section 4.4.1 double collision $J_{1-} J_{1-}$ contributes

$$\frac{1}{4} z^2 \log \epsilon C_{\text{odd}} \cdot \partial_+ x d\tau^+. \tag{5.12}$$

Therefore the total *log divergence proportional to $z^2 \partial_+ x$ is*

$$-\frac{1}{2} z^2 \log \epsilon C_{\bar{2}} \cdot \partial_+ x d\tau^+. \tag{5.13}$$

5.1.4 When the log divergence is a total derivative

Logarithmic divergences with the coefficient z^{-2} apparently start at the order R^{-3} . But in fact they are total derivatives and therefore contribute only through contact terms, as we discussed in Section 4.2.2. Indeed, (5.13) and the symmetry ($+ \leftrightarrow -$), ($z \leftrightarrow z^{-1}$) imply that there is the divergence:

$$-\frac{1}{2} z^{-2} \log \epsilon C_{\bar{2}} \cdot \partial_- x d\tau^-, \tag{5.14}$$

from the double collision $J_{3+} J_{3+}$. This and (5.5) implies that the total divergence proportional to z^{-2} at the order R^{-3} is the total derivative:

$$-\frac{1}{2} z^{-2} \log \epsilon C_{\bar{2}} \cdot dx. \tag{5.15}$$

Similarly the divergent term (5.13) and the order z^2 contribution to the divergence proportional to ∂_-x gives

$$-\frac{1}{2}z^2 \log \epsilon \in C_{\bar{2}}.dx. \tag{5.16}$$

Integration of (5.15) gives us the boundary terms. Some of these boundary terms are of the form

$$\frac{1}{R^4} \log \epsilon \left(-\frac{1}{2z^3}[\partial_+\vartheta_L, C_{\bar{2}}.x] \quad \text{or} \quad -\frac{1}{2z^4}[\partial_+x, C_{\bar{2}}.x] \quad \text{or} \right. \\ \left. -\frac{1}{2z^5}[\partial_+\vartheta_R, C_{\bar{2}}.x] \right). \tag{5.17}$$

These terms are of the order R^{-4} and of the same structure as the logarithmic divergences arising from the double collisions and internal divergences in the order R^{-4} . For example, consider the term

$$-\frac{1}{R^4} \frac{1}{2z^5} \log \epsilon [\partial_+\vartheta_R, x]. \tag{5.18}$$

A divergent term of the same structure appears in the double collision $J_{1+}J_{2+}$. Indeed, there is a term in the action of the form $\frac{1}{R^2}\partial_+\vartheta_R\partial_-x\vartheta_Lx$. This term leads to the divergence of the form (5.18) from the double collision $\frac{1}{R^2}\partial_+\vartheta_R(w)\partial_+x(0)$, $w \rightarrow 0$. Another possibility for the boundary terms is

$$\frac{1}{R^4} \log \epsilon \left(-\frac{1}{2}z[\partial_-\vartheta_L, x] \quad \text{or} \quad -\frac{1}{2}[\partial_-x, x] \quad \text{or} \quad -\frac{1}{2z}[\partial_-\vartheta_R, x] \right). \tag{5.19}$$

They also interfere with various double collisions, just like the terms in (5.17). There are also the boundary terms of the form

$$-\frac{1}{R^5} \log \epsilon \frac{1}{2z^2} \left(1 - \frac{1}{z^4} \right) [x, N_+]. \tag{5.20}$$

They “interfere” with the logarithmic divergence which appears in the double collision $J_{2+}N_+$, because of the non-linear term $J_{0-}N_+$ in the action.

In Section 5.4 we will explicitly verify the cancellation of the contact terms from integrating the total derivative dx in $\int \frac{1}{z^2} \log \epsilon \, dx \int \frac{1}{z^2} d\tau^+ \partial x$ against other divergences proportional to $z^{-4}[x, \partial_+x]$.

We see that the left $\mathfrak{g}_{\bar{2}}$ divergences with the coefficient z^{-2} actually belong to the order R^{-4} , rather than the order R^{-3} . Integrating the total derivative can be understood as a z -dependent gauge transformation of $J(z)$, see Section 5.5.

5.2 Divergences proportional to J_{3+} and J_{1+}

The divergences proportional to J_{3+} and J_{1+} at the order R^{-3} are zero because of the identity (A.13). Let us explain this for J_{3+} . Coefficient of z^{-1} could come from the following sources:

- “internal” anomalous dimension of J_{3+} from the interaction term $-\frac{1}{2} \frac{1}{R} f_{\mu\dot{\alpha}\dot{\beta}} \partial_- x^\mu \vartheta_R^{\dot{\alpha}} \partial_+ \vartheta_R^{\dot{\beta}}$ in the action;
- the double collision $J_{0+} J_{3+}$;
- double collisions $J_{1+} J_{2-}$ and $J_{2+} J_{1-}$, because of the interaction.

All these contributions are zero, being proportional to one of the expressions in (A.13). The coefficient of z^{-5} is zero for the same reason. There are the following potential contributions:

- double collisions $J_{1+} J_{2+}$,
- triple collisions,

but they are all zero because of (A.13). There could be also terms proportional to z^3 , from the double collisions $J_{0+} J_{3-}$ and from the triple collisions of the “wrapping” type $J_- J_{3+} J_-$; they are zero for the same reason.

5.3 Divergences proportional to N_+

5.3.1 Ghosts

The terms in the action containing ghosts are

$$S_{\text{ghosts}} = \frac{1}{\pi} \int d^2v \text{Str} (w_{1+} (\partial_- \lambda_3 + [J_{0-}, \lambda_3]) + \tilde{w}_{3-} (\partial_+ \tilde{\lambda}_1 + [J_{0+}, \tilde{\lambda}_1]) - N_+ N_-). \tag{5.21}$$

The ghost current is defined as

$$N_+^{[\mu\nu]} = -\frac{1}{R^2} \{w_+, \lambda\} = -\frac{1}{R^2} w_+^{\dot{\alpha}} \lambda^\beta f_{\dot{\alpha}\beta}^{[\mu\nu]}, \tag{5.22}$$

and has OPE

$$N_+^{[\mu_1\nu_1]}(v) N_+^{[\mu_2\nu_2]}(0) = \frac{1}{R^2} \frac{1}{v} f_{[\mu_3\nu_3]}^{[\mu_1\nu_1][\mu_2\nu_2]} N_+^{[\mu_3\nu_3]} + \frac{1}{R^4} \frac{c}{v^2} + \dots, \tag{5.23}$$

where c is a c -number. This c -number would play a role in the logarithmic divergences, but at higher orders.

5.3.2 Logarithmic divergence proportional to N_+

Let us consider the renormalization of the coefficient of N_+ . The potentially divergent expressions arise in the order R^{-4} . For the expressions proportional to the matter fields, we verified the cancellation of the logarithmic divergences up at the order R^{-3} . But for the expressions containing ghosts we will calculate all the potentially divergent terms in the order R^{-4} . There are the following sources of the logarithmic divergence proportional to N_+ :

Double collisions N_+N_+ : The first source of the anomalous dimension is the N_+N_+ collision. The corresponding contribution to the anomalous dimension is

$$\frac{1}{2} \frac{1}{R^2} \left(1 - \frac{1}{z^4}\right)^2 \log \epsilon C_0.N_+. \quad (5.24)$$

Triple collisions $J_+N_+J_+$: The second source of the logarithmic divergence is the “wrapping” of $t^a \otimes t^a$ and $t^\alpha \otimes t_\alpha$ and $t_\alpha \otimes t^\alpha$ around $t_{\mu\nu}$. It is proportional to

$$-\frac{1}{2} \frac{1}{R^2} \frac{1}{z^4} \left(1 - \frac{1}{z^4}\right) \log \epsilon (C - C_0).N_+. \quad (5.25)$$

Mixing with J_{0+} : Another contribution comes from the mixing of J_{0+} into N_+ caused by the term $\text{str } N_+J_{0-}$ in the action, as described in Section 4.4.2:

$$-J_{0+} \rightarrow \frac{1}{2} \frac{1}{R^2} \log |\epsilon|^2 (C - C_0).N_+. \quad (5.26)$$

Double collisions $J_{1+}J_{3+}$ and $J_{2+}J_{2+}$: For example, the double collision $J_{2+}J_{2+}$ leads to the log divergence $\simeq N_+$ which can be effectively described as the log divergence of this collision:

$$\left(\partial_+x + \frac{1}{2}[N_+, x]\right) \longleftrightarrow \left(\partial_+x + \frac{1}{2}[N_+, x]\right).$$

The total contribution from the double collision $J_{2+}J_{2+}$ and $J_{1+}J_{3+}$ is

$$-\frac{1}{2} \frac{1}{R^2} \frac{1}{z^4} \log \epsilon (C - C_0).N_+. \quad (5.27)$$

Triple collision $J_-N_+J_-$: These triple collisions contribute

$$-\frac{1}{2} \frac{1}{R^2} z^4 \left(1 - \frac{1}{z^4}\right) \log \epsilon (C - C_0).N_+. \quad (5.28)$$

Mixing with J_{0-} : There is no such mixing, see Section 4.4.2.

Double collisions J_-J_+ : Their contribution can be effectively calculated by evaluating in the free theory the N_+ -singularity in the $J_{2-}J_{2+}$ collision:

$$(\partial_-x + [N_-, x]) \longleftrightarrow (\partial_+x + [N_+, x]), \tag{5.29}$$

and similar $J_{1-}J_{3+}$ and $J_{3-}J_{1+}$ collisions. The result is

$$-\frac{1}{R^2} \log \epsilon(C - C_0).N_+. \tag{5.30}$$

Double collisions J_-J_- : These collisions contribute because of (4.40):

$$\frac{1}{2} \frac{1}{R^2} z^4 \log \epsilon(C - C_0).N_+. \tag{5.31}$$

The total result is that the logarithmic divergences proportional to N_+ add up to zero: (5.24) + (5.25) + (5.26) + (5.27) + (5.28) + (5.30) + (5.31) = 0.

5.4 Divergences of the type $x \partial_+x$

5.4.1 The coefficient of $z^{-4}x \partial_+x$

First let us calculate the coefficient of $\frac{1}{z^4}[\partial_+x, x]$. Notice that in the expansion of the transfer matrix we obtain the term $\int(-\frac{1}{2}[dx, x])$ which is proportional to z^0 , but we do not have classically any terms which would be proportional to z^4 or z^{-4} . There are the following divergent contributions:

Triple collisions: There are triple collisions of the form

$$J_{2+} \longleftrightarrow J_{2+} \longleftrightarrow \left(-\frac{1}{2}[\partial_+x, x]\right) \quad \text{and} \quad J_{3+} \longleftrightarrow J_{1+} \longleftrightarrow \left(-\frac{1}{2}[\partial_+x, x]\right).$$

As we discussed in Section 4.3 the contribution of the triple collisions is due to the second-order terms that appear in the OPE of ∂_+x with ∂_+x , or in the OPE of $\partial_+\vartheta_R$ with $\partial_+\vartheta_L$. Let us first consider the triple collisions with two J_{2+} . One contribution comes from the contraction of ∂_+x in two J_{2+} ;

the divergence is:

$$\frac{1}{4} \log \epsilon C^{\mu\nu} [t_\mu^2, [t_\nu^2, [\partial_+ x, x]]]. \tag{5.32}$$

The other contribution comes from the contraction of $\partial_+ x$ in J_{2+} with $\partial_+ x$ in $[\partial_+ x, x]$, this gives the following divergence:

$$\frac{1}{2} \log \epsilon C^{\mu\nu} [[t_\mu^2, x], [t_\nu^2, \partial_+ x]]. \tag{5.33}$$

The total of contributions from $J_{2+} \longleftrightarrow J_{2+} \longleftrightarrow (-\frac{1}{2}[\partial_+ x, x])$ collisions is

$$\frac{1}{2} \log \epsilon [\partial_+ x, C_{\bar{2}.x}]. \tag{5.34}$$

Now let us consider the collision with J_{1+} and J_{3+} . The result is

$$\begin{aligned} & \frac{1}{4} \log \epsilon (C^{\alpha\dot{\alpha}} \{t_\alpha^3, [t_{\dot{\alpha}}^1, [\partial_+ x, x]]\} + C^{\dot{\alpha}\alpha} \{t_{\dot{\alpha}}^1, [t_\alpha^3, [\partial_+ x, x]]\}) \\ &= -\frac{1}{2} \log \epsilon [\partial_+ x, C_{\bar{2}.x}]. \end{aligned} \tag{5.35}$$

Therefore *the total contribution from the triple collisions* is zero: (5.34) + (5.35) = 0.

Double collisions: One possible double collision is $J_{2+} \longleftrightarrow J_{2+}$. But in fact this double collision does not contribute to $[\partial_+ x, x]$. (Let us prove that $J_{2+} J_{2+}$ does not contribute. The terms in J_{2+} which could contribute are $-\partial_+ x - \frac{1}{6}[x, [x, \partial_+ x]]$. There is a contribution from the collision $\partial_+ x \longleftrightarrow \frac{1}{6}[x, [x, \partial_+ x]]$ but it cancels with the contribution from the collision $\partial_+ x \longleftrightarrow \partial_+ x$ which arises because there is the term $-\frac{1}{6}[\partial_+ x, x][\partial_- x, x]$ in the action.)

But there is another double collision $J_{1+} \longleftrightarrow J_{3+}$, and it does give a non-zero contribution. Let us calculate the contribution of $J_{1+} \longleftrightarrow J_{3+}$ to the log divergence proportional to $[\partial_+ x, x]$. The relevant terms in the expansion of the currents are:

$$-J_{1+} = \partial_+ \vartheta_R - \frac{1}{2} [\vartheta_R, [\partial_+ x, x]] + \dots, \tag{5.36}$$

$$-J_{3+} = \partial_+ \vartheta_L - \frac{1}{2} [\vartheta_L, [\partial_+ x, x]] + \dots. \tag{5.37}$$

The relevant terms in the action are:

$$-\frac{1}{8} [\vartheta_L, \partial_- \vartheta_R][x, \partial_+ x] - \frac{3}{8} [\vartheta_R, \partial_- \vartheta_L][x, \partial_+ x]. \tag{5.38}$$

This means that the divergence is the same as if we collided:

$$\left(\partial_+\vartheta_R + \left(-\frac{1}{2} + \frac{3}{8}\right) [\vartheta_R, [\partial_+x, x]]\right) \longleftrightarrow \left(\partial_+\vartheta_L + \left(-\frac{1}{2} + \frac{1}{8}\right) [\vartheta_L, [\partial_+x, x]]\right),$$

in the free theory. This gives the *contribution from double collisions*:

$$-\frac{1}{4} \log \epsilon C_{\text{odd}} \cdot [\partial_+x, x] = \frac{1}{2} \log \epsilon [\partial_+x, C_{\bar{2}} \cdot x]. \tag{5.39}$$

Contribution from the total derivative: There is a contribution from (5.17):

$$-\frac{1}{2} \log \epsilon [\partial_+x, C_{\bar{2}} \cdot x]. \tag{5.40}$$

We see that the contribution from the boundary terms cancels the contribution from the double collisions, and therefore the total log divergence of the type $z^{-4}x \partial_+x$ is zero.

5.4.2 Coefficient of $z^0x \partial_+x$

We did not calculate this coefficient. But we have seen that the coefficient to $z^{-4}x \partial_+x$ is zero, and we will see that the coefficient of $z^4x \partial_+x$ is also zero. We know there should not be any log divergence at $z = 1$. Therefore the log divergence proportional to $z^0x \partial_+x$ should be zero.

5.4.3 Coefficient of z^4

Contribution from triple collisions: Triple collisions of the type $J_{1-}J_{3-}J_{0+}$ and $J_{2-}J_{2-}J_{0+}$ contribute to the divergence of the form $z^4[\partial_+x, x]$ and their contribution is equal to

$$\frac{1}{4} \log \epsilon (C_{\text{odd}} + C_{\bar{2}}) \cdot [\partial_+x, x] = -\frac{3}{8} \log \epsilon [\partial_+x, C_{\bar{2}} \cdot x]. \tag{5.41}$$

Contribution from double collisions: There is a contribution from $J_{2-}J_{2-}$ and a contribution from $J_{1-}J_{3-}$. Let us first consider the contribution from $J_{2-}J_{2-}$. The relevant interaction term in the action is $-\frac{1}{6}[\partial_+x, x][\partial_-x, x]$.

To get the divergence of the form $[\partial_+x, x]$ we should have ∂_-x in the interaction vertex contracted with a ∂_-x in one of the J_{2-} . We obtain

$$\begin{aligned} & -\frac{1}{6} \frac{1}{\pi} \int d^2v \frac{1}{(\bar{w}_L - \bar{v})^2(\bar{w}_R - \bar{v})} (\text{str}(t_\nu^2[t_\mu^2, [x, \partial_+x]]) \\ & \quad + \text{str}(t_\nu^2[x, [t_\mu^2, \partial_+x]])) [t_\nu^2, t_\mu^2] \\ & = -\frac{1}{6} \log \epsilon \left([[t_\mu^2, [x, \partial_+x]], t_\mu^2] + [[x, [t_\mu^2, \partial_+x]], t_\mu^2] \right) \\ & = -\frac{1}{8} \log \epsilon [\partial_+x, C_{\bar{2}.}x]. \end{aligned} \tag{5.42}$$

Now let us consider the double collision $J_{1-}J_{3-}$. The relevant interaction vertices are

$$-\frac{1}{8} [\vartheta_L, \partial_- \vartheta_R][x, \partial_+x] - \frac{3}{8} [\vartheta_R, \partial_- \vartheta_L][x, \partial_+x].$$

These two vertices give the same contribution and add up to

$$\begin{aligned} & -\frac{1}{2} \frac{1}{\pi} \int d^2v \frac{1}{(\bar{w}_L - \bar{v})^2(\bar{w}_R - \bar{v})} C^{\alpha\dot{\beta}} C^{\gamma\dot{\alpha}} \text{str}(t_{\dot{\beta}}^1[t_\gamma^3, [x, \partial_+x]]) \{t_{\dot{\alpha}}^3, t_{\dot{\alpha}}^1\} \\ & = -\frac{1}{4} \log \epsilon C_{\text{odd}.}[\partial_+x, x] = \frac{1}{2} \log \epsilon [\partial_+x, C_{\bar{2}.}x]. \end{aligned} \tag{5.43}$$

The total contribution $(5.41) + (5.42) + (5.43) = 0$. Therefore there is no logarithmic divergence of the type $z^4[\partial_+x, x]$.

5.5 Bulk divergences and divergences associated to the boundary

In this section we have collected evidence that the logarithmic divergences of the transfer matrix at one loop are zero modulo the total derivative. The total derivative was described in Section 5.1.4. This suggests that the logarithmic divergences of the transfer matrix have the following form:

$$\Omega(z) = f(\epsilon, z) \Omega(z)^{\text{finite}} f(\epsilon, z)^{-1}, \tag{5.44}$$

where $f(z, \epsilon)$ is a z -dependent gauge transformation

$$f(z, \epsilon) = \exp \left(-\frac{1}{2R^2} \left(z^2 + \frac{1}{z^2} \right) \log \epsilon C_{\bar{2}.}x + \dots \right). \tag{5.45}$$

Dots denote terms of the higher power in $\frac{1}{R}$. In this sense, we can say that the divergences are absorbed in a z -dependent gauge transformation.

But notice that when we compute the path ordered exponential of $-\int_C J$ over an *open* contour C , we obtain additional divergences associated to the endpoints.

We want to investigate the following question: can we distinguish between the bulk divergences, which are total derivative “propagating” to the endpoint, and the boundary divergences that are “inherent to the endpoint”? Instead of considering open contour with endpoints, it is more convenient to consider a closed contour and insert a constant matrix X at some point $\tau_0 = (\tau_0^+, \tau_0^-)$ inside the contour:⁶

$$P \left[X(\tau_0) \exp \left(- \int J(z) \right) \right]. \tag{5.46}$$

It seems that there are two different types of divergences associated with the insertion of X :

- (1) the divergences of $f^{-1} X f$ which arise because f is divergent, see (5.45); this is the effect of the divergences in the bulk of the contour, which are total derivatives and therefore “propagate” to the insertion point;
- (2) the divergences of diagrams localized near the insertion of X , as in Section 4.3.

But in fact the difference between these two types of divergences is a matter of convention. Indeed, let us return to Section 5.1.2 and remember how we calculated the divergence proportional to $z^{-2} \partial x$. One of the contributions to the divergence was from the collision $J_{2+} \longleftrightarrow J_{0+}$ which in the leading order was $-\partial_+ x(w_2) \longleftrightarrow \frac{1}{2} [\partial_+ x, x](w_0)$. The ambiguity arises when we decide whether to first integrate over w_2 and then over w_0 , or the other way around. We agreed in Section 4.3.2 to integrate first over w_2 , and followed this prescription in Section 5.1.2. It was more convenient because with this prescription only the first-order pole (from the contraction of $\partial_+ x(w_2)$ with $x(w_0)$) contributes to the log divergence. If we integrated first over w_0 , we would have a contribution to the log divergence from the second-order pole. But, of course, the divergences “associated to the insertion of X ” also depend on the order of integration. For example, the collision $X \longleftrightarrow [\partial_+ x, x] \longleftrightarrow \partial_+ x$ will not contribute to the log divergence “of the insertion X ” if we first integrate over the position of $\partial_+ x$, but will contribute if we first integrate over the position of $[\partial_+ x, x]$.

⁶Such an object is not gauge invariant with respect to the \mathfrak{g}_0 gauge transformations. But let us fix the gauge as in (3.1) and consider this expression in the fixed gauge, just as an example.

The lesson is that if we agreed on the order of integration in the bulk of the contour, we should use it consistently also when computing the log divergences associated to the boundary. A different arrangement of the order of integrations will lead to the different distribution of the log divergences between the bulk total derivative terms and the collisions with the boundary. In other words, when the contour is open, *there is no good distinction between the log divergences which come from the total derivative divergences in the bulk and the log divergences coming from the boundary effects.*

If the log divergences in the bulk of the contour are total derivatives, this means that it is possible to choose a prescription for the order of integrations such that the bulk divergence is zero. Of course, if we want to compute for the open contour the anomalous dimension of the endpoints, or the anomalous dimension of some insertion, then we have to consistently follow the same prescription calculating the boundary divergences.

6 Logarithmic divergences and global symmetries

Global symmetries described in Section 3.2 impose very strong constraints on the divergences, and actually imply that the cancellation of the one-loop logarithmic divergences follows from the cancellation of the simplest possible divergent expressions, those proportional to $\partial_{\pm}x$ and $\partial_{\pm}\vartheta$. This section consists of two parts. In the first part we will show that the global symmetries together with the results of the previous section imply that the one-loop logarithmic divergences vanish. In the second part we demonstrate the consistency of the short-distance singularities in the product $J_{2+}J_{2+}$ with the global symmetries.

6.1 Vanishing of the one-loop logarithmic divergences

Let us start with the log divergences proportional to $x\partial x$. In the previous section we demonstrated by explicit calculations that there are no such divergences. But in fact the cancellation of this type of divergences automatically follows from the cancellation of the log divergences of the form ∂x and the invariance under the global symmetries described in Section 3.2.

In the previous section we have shown that the divergent terms in the bulk of the form $\log \epsilon z^{4k-2}\partial_{\pm}x$ and $\log \epsilon z^{2k-1}\partial_{\pm}\vartheta$ are all total derivatives. Let us make a z -dependent gauge transformation eliminating these total derivatives. After such a z -dependent gauge transformation there are no log divergent terms of the form $\log \epsilon z^{4k-2}\partial_{\pm}x$ and $\log \epsilon z^{2k-1}\partial_{\pm}\vartheta$.

Let us first prove that the cancellation of the one-loop divergences of the form $\log \epsilon z^{4k-2} \partial_+ x$ implies the cancellation of the one-loop divergences of the form $\log \epsilon z^{4k} (\alpha [x, \partial_+ x] d\tau^+ + \beta [x, \partial_- x] d\tau^-)$. Indeed the shift of this expression by ξ would be

$$\log \epsilon z^{4k} (\alpha [\xi, \partial_+ x] d\tau^+ + \beta [\xi, \partial_- x] d\tau^-), \tag{6.1}$$

plus higher order terms.⁷ This contradicts the shift invariance unless $\alpha = \beta$, in which case (6.1) is the total derivative $z^{4k} \alpha [\xi, dx]$. But even if $\alpha = \beta$, the total derivative being integrated by parts would hit $-\int z^{-2} J_{2+} d\tau^+$ and give

$$\int z^{4k-2} \log \epsilon [\partial_+ x, \alpha [\xi, x]] d\tau^+, \tag{6.2}$$

which cannot be cancelled by anything. Indeed, we have shown that there are no counterterms of the form $z^{4k-2} \log \epsilon \partial_+ x$. The possible counterterms of the form $z^{4k-2} \log \epsilon [x, [x, \partial_+ x]]$ would have the variation

$$z^{4k-2} \log \epsilon ([\xi, [x, \partial_+ x]] + [x, [\xi, \partial_+ x]]), \tag{6.3}$$

which has a different structure from (6.2). In other words, the expression (6.2) cannot be represented as S_ξ of something⁸ of the type $xx\partial_+ x$. This argument shows that there are no divergences of the form $z^{4k} [x, \partial_\pm x]$, and also no divergences of the form $z^{4k-2} [x, [x, \partial_\pm x]]$.

Let us now rule out the divergences of the type $z^{2k} [\vartheta, \partial_\pm \vartheta]$. At the one-loop level we could only have divergences proportional to one of these expressions:

$$\begin{aligned} & z^{\pm 8} [\vartheta_L, \partial_\pm \vartheta_R], z^{\pm 8} [\vartheta_R, \partial_\pm \vartheta_L], z^{\pm 6} [\vartheta_L, \partial_\pm \vartheta_L], z^{\pm 6} [\vartheta_R, \partial_\pm \vartheta_R], z^{\pm 4} [\vartheta_L, \partial_\pm \vartheta_R], \\ & z^{\pm 4} [\vartheta_R, \partial_\pm \vartheta_L], z^{\pm 2} [\vartheta_L, \partial_\pm \vartheta_L], z^{\pm 2} [\vartheta_R, \partial_\pm \vartheta_R], [\vartheta_L, \partial_\pm \vartheta_R], [\vartheta_R, \partial_\pm \vartheta_L]. \end{aligned} \tag{6.4}$$

For example, there are no divergences of the form $\log \epsilon z^{-10} [\vartheta_L, \partial_\pm \vartheta_L]$, because such divergences would require colliding more than three currents; at the one-loop level there are no log divergences coming from the multiple collisions of the order higher than double and triple. Also, there are no divergences of the form $z^{-9} [\vartheta_R, \partial_\pm x]$ or $z^{-9} [\partial_\pm \vartheta_R, x]$. By counting the powers of z , such divergences could only appear in a triple collision $J_{1+} J_{1+} J_{1+}$, but

⁷Note that α and β are typically not c -numbers, but contain Casimir operators C_0 acting on $[x, \partial x]$.

⁸But if there was a divergent term of the form $z^{-2} \int \log \epsilon \partial_+ x d\tau^+$, then the variation of such a term, because of the higher order terms in (3.6), would be of the form $\frac{1}{3} z^{-2} \log \epsilon ([\partial_+ x, [x, \xi]] + [x, [\partial_+ x, \xi]])$, and this could combine with (6.3) to cancel (6.2).

there are no suitable contractions. Therefore the potential divergent terms with the highest negative power of z are $z^{-8}[\vartheta_L, \partial_{\pm}\vartheta_R]$ and $z^{-8}[\partial_{\pm}\vartheta_L, \vartheta_R]$.

We use the invariance under the shifts and the supershifts:

$$\delta x = \xi + \dots, \quad \delta\vartheta_{L,R} = \zeta_{L,R} + \dots.$$

Let us first rule out the possible divergence with the highest negative power of z :

$$\int \log \epsilon (\alpha z^{-8}[\vartheta_L, \partial_+\vartheta_R]d\tau^+ + \beta z^{-8}[\vartheta_L, \partial_-\vartheta_R]d\tau^-).$$

For the variation to be a total derivative it is necessary to have $\alpha = \beta$, and then we get the variation $\int \log \epsilon z^{-8}[\zeta_L, d\vartheta_R]$. Integrating this expression we would hit (for example) the classical term $\int d\tau^+ z^{-3}\partial_+\vartheta_R$ and get the contact term $\int d\tau^+ z^{-11}[\partial_+\vartheta_R, [\zeta_L, \vartheta_R]]$. Notice that this contact term could be cancelled by the variation of the divergent term proportional to $z^{-11}[\partial_+\vartheta_R, [\vartheta_L, \vartheta_R]]$ under $\delta\zeta_L$, if there is such a divergent term. But such a divergent term would also have a non-zero variation under $\delta\zeta_R$. The only way to match the variation under $\delta\zeta_R$ is to have also the divergence proportional to $z^{-8}[d\vartheta_L, \vartheta_R]$, which combines together with $z^{-8}[\vartheta_L, d\vartheta_R]$ to the total divergent term $z^{-8}d[\vartheta_L, \vartheta_R]$. This can be gauged away by a z -dependent gauge transformation.

Other possible divergences from the list (6.4) and also divergences of the type $z^{2k+1}[x, \partial_{\pm}\vartheta]$ and $z^{2k+1}[\partial_{\pm}x, \vartheta]$ could be ruled out by essentially the same arguments, first those proportional to z^{-7} , then z^{-6} and so on. Suppose that we have a divergent term of the form $z^{-k} \log \epsilon \int Y$, where Y is a one-form quadratic in the elementary fields, for example $Y = [x, d\vartheta]$. We should have $\delta Y = dZ$, and then the variation will give many contact terms including this one: $z^{-k-3}[\partial_+\vartheta_R, Z]$. This should be cancelled by δ of some divergent term cubic in the elementary fields, therefore we should have $Z = \delta X$. We have $\delta(Y - dX) = 0$ and this implies $Y = dX$ because $Y - dX$ is quadratic in elementary fields but contains only one derivative.

Therefore, the near-flat space expansion of the logarithmic divergences should start with the terms of the form $\int \log \epsilon \Phi(x, \vartheta)$ where $\Phi(x, \vartheta)$ is a worldsheet one-form composed of three or more x and ϑ , for example $\Phi = z^{-2}[\vartheta_R, [\vartheta_L, *dx]]$. Invariance of the log divergences under shifts requires that $\delta\Phi = d\Psi$, where Ψ is some expression composed of at least two ϑ or x and one $\delta\vartheta$ or δx . It turns out that this implies⁹ $\Phi = dG$, and therefore the log divergence can be gauged away by a z -dependent gauge transformation.

⁹This is because $\delta\Phi = d\Psi$ implies $\delta d\Phi = 0$, and since $d\Phi$ is a two-form composed of at least three x and ϑ this implies that $d\Phi = 0$. And $d\Phi = 0$ implies that $\Phi = dG$. Indeed,

This proves the absence of the one-loop logarithmic divergences of the form $F(x, \vartheta)dx$ and $F(x, \vartheta)d\vartheta$. Another possibility would be the logarithmic divergences of the form $F(x, \theta)N_{\pm}$. But in Section 5.3.2 we have shown that there are no log divergences proportional to N_{\pm} without x and θ . Therefore the lowest order terms in the near-flat space expansion would contain at least one x or θ . Such terms cannot be invariant under global shifts, and therefore should cancel.

It is not surprising that global symmetries relate the log divergences at finite x to the log divergences at $x = 0$, and therefore it is enough to prove that there are no divergences proportional to $\partial_{\pm}x$ and $\partial_{\pm}\vartheta$. It should be possible to reach the same conclusion using the background field method [35].

6.2 Singularity in the product $J_{2+}J_{2+}$ and global shifts

We have $J_{2+} = -\partial_+x + \dots$ and $J_{2+}^{\mu}(w_L)J_{2+}^{\nu}(w_R) = -\frac{1}{R^2} \frac{C^{\mu\nu}}{(w_L - w_R)^2} + \dots$. Let us consider the variation of J_{2+} under the global shift. According to (3.8) we have $\delta_{\xi}J_{2+} = \frac{1}{2}[[x, \xi], J_{2+}]$. Consider the variation of the product:

$$\begin{aligned} \delta_{\xi}(J_{2+}^{\mu}(w_L)J_{2+}^{\nu}(w_R)) &= \frac{1}{2}[[x, \xi], \partial_+x]^{\mu}(w_L)\partial_+x^{\nu}(w_R) \\ &\quad + \frac{1}{2}\partial_+x^{\mu}(w_L)[[x, \xi], \partial_+x]^{\nu}(w_R) \\ &= -\frac{1}{2} \frac{1}{(w_L - w_R)^2} [[x(w_L), \xi], t^{2\nu}]^{\mu} \\ &\quad + \frac{1}{2} \frac{1}{(w_L - w_R)} [[t^{2\nu}, \xi], \partial_+x]^{\mu} \\ &\quad + \dots + (w_L \leftrightarrow w_R, \mu \leftrightarrow \nu) \\ &= \frac{1}{2} \frac{\bar{w}_L - \bar{w}_R}{(w_L - w_R)^2} \text{str}([\partial_-x, \xi][t^{2\mu}, t^{2\nu}]) + \dots, \end{aligned} \tag{6.5}$$

where $t^{2\mu} = t_{\nu}^2 C^{\nu\mu}$. On the other hand from Sections 5.4.1 and 5.4.3 and the ‘‘symmetry’’ (3.18) we know that

$$J_{2+}^{\mu}(w_L)J_{2+}^{\nu}(w_R) [t_{\mu}^2, t_{\nu}^2] = -\frac{1}{4} \frac{\bar{w}_L - \bar{w}_R}{(w_L - w_R)^2} [\partial_-x, C_{\bar{2}}.x] + \dots$$

Φ contains at least three elementary fields, e.g., $x\vartheta * dx$. To the leading order in R^{-2} we can think of Φ as the charge density in the free field theory, because to the leading order $\partial\bar{\partial}x = \partial\bar{\partial}\vartheta = 0$. But local conserved charges in a free field theory are all quadratic in the free fields, there are no local conserved charges cubic or of higher order. Therefore, $d\Phi = 0$ implies that Φ is an exact form.

These arguments would not work if Φ was quadratic in the elementary fields. For example for $\delta[x, dx] = d[\delta x, x] + \dots$ where dots are higher order terms, because the leading term in $\delta x = \xi$ is constant. But $[x, dx]$ is not a total derivative, not even a closed form.

This formula is in agreement with (6.5) because for any $\xi^2 \in \mathfrak{g}_2$ and $\eta^2 \in \mathfrak{g}_2$ we have

$$\text{str}([\xi^2, \eta^2][t^{2\mu}, t^{2\nu}][t_\mu^2, t_\nu^2] = -\frac{1}{2}[\xi^2, C_2 \cdot \eta^2].$$

7 Infinite line

7.1 Transfer matrix on the infinite line

We will define the transfer matrix on the infinite line as the limit:

$$\lim_{\substack{c\tau_l \rightarrow +\infty \\ \tau_r \rightarrow -\infty}} (\Omega_{\tau_r}^{\tau_l}(z = 1))^{-1} \Omega_{\tau_r}^{\tau_l}(z). \tag{7.1}$$

This can be expressed through the “small case currents”:

$$\begin{aligned} P \exp \int_{-\infty}^{\infty} & [((1 - z^{-1})j_{3+} + (1 - z^{-2})j_{2+} + (1 - z^{-3})j_{1+} \\ & + (1 - z^{-4})j_{0+}) d\tau^+ + ((1 - z)j_{1-} + (1 - z^2)j_{2-} + (1 - z^3)j_{3-} \\ & + (1 - z^4)j_{0-}) d\tau^-]. \end{aligned} \tag{7.2}$$

The definition of the transfer matrix on the infinite line is such that the power of z does not correlate with the \mathbb{Z}_4 grading. This is because we divide by $\Omega(z = 1)$.

7.2 Global symmetry charge

It is useful to check our formalism by showing that the global Lorentz charge is finite. We will verify that there are no divergences proportional to $\partial_+ x$, $[\partial_+ x, x]$ and to N_+ . Consider the charge corresponding to boosts and rotations around the point $x = 0$. This charge can be computed by expanding the transfer matrix on the infinite line in $z = 1 + \zeta$ to the first order in ζ :

$$q_{\text{global}} = \int *g_{z=1}^{-1} \left[\frac{\partial}{\partial z} \Big|_{z=1} J(z) \right] g_{z=1}. \tag{7.3}$$

Note that we will omit the powers of $1/R$, since these are obvious (each x and ϑ comes with one power of $1/R$) and would only clutter the formulas.

Divergences proportional to ∂_+x : The terms responsible for the logarithmic divergences of the global current proportional to j_+ are:

$$\begin{aligned} &\left(1 - \frac{1}{z^2}\right) (g^{-1}J_{2+}g)_{\bar{2}} + \left(1 - \frac{1}{z}\right) ([\vartheta_L, \partial_+\vartheta_L] - [\vartheta_L, [\partial_+x, \vartheta_R]]) \\ &+ \left(1 - \frac{1}{z^3}\right) ([\vartheta_R, \partial_+\vartheta_R] - [\vartheta_R, [\partial_+x, \vartheta_L]]) + \dots \end{aligned}$$

Therefore the ∂_+x -part of the log divergence of j_+ is the same as the log divergence of:

$$\begin{aligned} &2J_{2+} - ([x, [x, \partial_+x]] + [\vartheta_L, [\vartheta_R, \partial_+x]] + [\vartheta_R, [\vartheta_L, \partial_+x]]) \\ &+ [\vartheta_L, (\partial_+\vartheta_L - [\partial_+x, \vartheta_R])] + 3[\vartheta_R, (\partial_+\vartheta_R - [\partial_+x, \vartheta_L])] + \dots \end{aligned} \quad (7.4)$$

Notice that $[\vartheta_L, \partial_+\vartheta_L]$ does not contribute to the log divergence, while $3[\vartheta_R, \partial_+\vartheta_R]$ contributes the same amount as $3[\vartheta_R, [\partial_+x, \vartheta_L]]$. The contribution of $[\vartheta_L, [\vartheta_R, \partial_+x]]$ is minus the contribution of $[x, [x, \partial_+x]]$. Therefore at the order R^{-3} the ∂_+x -piece of the log divergence in j_+ is the same as the ∂_+x -piece of the log divergence in $2J_{2+}$. But we have seen in Section 3.7 that $2J_{2+}$ to the order R^{-3} is finite. This shows that the log divergence of j_+ proportional to ∂_+x is zero at the order R^{-3} .

Divergences proportional to $[\partial_+x, x]$: We will split the calculation into two parts, first identifying the contribution of bosons, and then the contribution of fermions. The contribution of bosons comes from

$$-2\text{Ad}(e^{-x}).J_{2+} = 2\text{Ad}(e^{-x}).\left(\partial_+x^{\text{ren}} + \frac{1}{6} : [x^{\text{ren}}, [x^{\text{ren}}, \partial_+x^{\text{ren}}]] : \right). \quad (7.5)$$

We have taken into account that the coefficient of ∂_+x in J_{2+} is not renormalized, and therefore we should skip the contractions of x^{ren} with x^{ren} in $[x^{\text{ren}}, [x^{\text{ren}}, \partial_+x^{\text{ren}}]]$, as denoted by the double dots. Therefore, the log divergence is that of the expression

$$-2[x, \partial_+x^{\text{ren}}] - \frac{1}{3}[x, [x, [x, \partial_+x]]] - \frac{1}{3}[x, : [x, [x, \partial_+x]] :]. \quad (7.6)$$

The relation between x and x^{ren} is given by (3.30). Let us take only the term generated by the bosons:

$$x = x^{\text{ren}} + \frac{1}{6} \frac{1}{R^2} \log \epsilon^2 [t_\mu^2, [t_\mu^2, x^{\text{ren}}]] \quad (7.7)$$

$$+ \text{terms generated by fermions.} \quad (7.8)$$

The result is

$$2[\partial_+ x^{\text{ren}}, x^{\text{ren}}] - \frac{1}{2} \log \epsilon^2 [\partial_+ x, C_{\frac{1}{2}} \cdot x]. \tag{7.9}$$

But the expression $2[\partial_+ x^{\text{ren}}, x^{\text{ren}}]$ itself has an internal log divergence, because of the interaction vertex $-\frac{1}{6}[\partial_+ x, x][\partial_- x, x]$ in the action. The log divergence of $2[\partial_+ x^{\text{ren}}, x^{\text{ren}}]$ is the same as of the expression $\frac{2}{3}[x, : [x, [x, \partial_+ x]] :]$ and equals to $\frac{1}{2} \log \epsilon^2 [\partial_+ x, C_{\frac{1}{2}} \cdot x]$. This cancels $-\frac{1}{2} \log \epsilon^2 [\partial_+ x, C_{\frac{1}{2}} \cdot x]$ in (7.9) and gives *the total of zero* from bosons.

Now let us evaluate the contribution of fermions. One source of contribution is

$$-2\text{Ad}(e^{-x}e^{-\vartheta}) \cdot J_{2+}. \tag{7.10}$$

The relevant terms are

$$2[\partial_+ x^{\text{ren}}, x] - [x, [\vartheta_L, [\vartheta_R, \partial_+ x]]] - [x, [\vartheta_R, [\vartheta_L, \partial_+ x]]]. \tag{7.11}$$

Here we want to pick the fermionic contribution to the renormalization of x :

$$\begin{aligned} x = x^{\text{ren}} &+ \frac{1}{2} \frac{1}{R^2} \log \epsilon^2 C^{\alpha\beta} \{t_\alpha^3, [t_\beta^1, x^{\text{ren}}]\} \\ &+ \text{contribution of bosons.} \end{aligned} \tag{7.12}$$

Substitution of this formula into (7.11) gives the total contribution from (7.10) equal to

$$\log \epsilon^2 [\partial_+ x, C_{\frac{1}{2}} \cdot x]. \tag{7.13}$$

The other source of fermionic contributions is

$$-\text{Ad}(e^{-x}e^{-\vartheta}) \cdot (3J_{1+} + J_{3+}). \tag{7.14}$$

The relevant terms are

$$\text{Ad}(e^{-x}e^{-\vartheta}) \cdot \left(3\partial_+ \vartheta_R + 3[\vartheta_L, \partial_+ x] + \frac{3}{2}[\vartheta_R, [x, \partial_+ x]] \right) \tag{7.15}$$

$$+ \partial_+ \vartheta_L + [\vartheta_R, \partial_+ x] + \frac{1}{2}[\vartheta_L, [x, \partial_+ x]] \tag{7.16}$$

The log divergence of this is the same as of the expression

$$3[\partial_+ \vartheta_R, \vartheta_L] + [\partial_+ \vartheta_L, \vartheta_R] + [\partial_+ x, [\vartheta_R, [\vartheta_L, x]]],$$

and is equal to

$$-\log \epsilon^2 [\partial_+ x, C_2 \cdot x]. \tag{7.17}$$

Here we have taken into account that $4[\partial_+ \vartheta_R, \vartheta_L]$ mixes into $[\partial_+ x, x]$ because of the following interaction vertices in the action:

$$-\frac{1}{8} [\vartheta_L, \partial_- \vartheta_R] [x, \partial_+ x] - \frac{3}{8} [\vartheta_R, \partial_- \vartheta_L] [x, \partial_+ x].$$

We see that the total contribution from fermions (7.13) + (7.17) is also zero, and therefore the Lorentz current is not renormalized.

Divergences proportional to N_+ : The terms responsible for the logarithmic divergence proportional to N_+ are:

$$\begin{aligned} & \left(1 - \frac{1}{z^4}\right) N_+ + \frac{1}{2} \frac{1}{R^2} \left(1 - \frac{1}{z^4}\right) ([x, [x, N_+]] + [\vartheta_L, [\vartheta_R, N_+]] + [\vartheta_R, [\vartheta_L, N_+]]) \\ & + \frac{1}{R^2} \left(\frac{1}{z^2} - 1\right) [\partial_+ x, x] + \frac{1}{R^2} \left(\frac{1}{z} - 1\right) [\partial_+ \vartheta_L, \vartheta_R] \\ & + \frac{1}{R^2} \left(\frac{1}{z^3} - 1\right) [\partial_+ \vartheta_R, \vartheta_L] + \dots \end{aligned}$$

When we expand this in powers of ζ , the linear term is

$$\begin{aligned} & 4N_+ + \frac{2}{R^2} ([x, [x, N_+]] + [\vartheta_L, [\vartheta_R, N_+]] + [\vartheta_R, [\vartheta_L, N_+]]) \\ & - \frac{2}{R^2} \left(\frac{1}{2} [\partial_+ \vartheta_L, \vartheta_R] + [\partial_+ x, x] + \frac{3}{2} [\partial_+ \vartheta_R, \vartheta_L]\right) + \dots \end{aligned} \tag{7.18}$$

It follows from (A.17) that the log divergence of the second line is equal to the log divergence of:

$$-\frac{2}{R^2} ([\partial_+ \vartheta_L, \vartheta_R] + [\partial_+ x, x] + [\partial_+ \vartheta_R, \vartheta_L]) = -4J_{0+}. \tag{7.19}$$

Equation (5.26) shows that the mixing of $-4J_{0+}$ into N_+ cancels the log divergence of the first line in (7.18). This shows that the log divergence of j_+ proportional to N_+ is zero at the order R^{-4} .

Of course, finiteness of j_+ is guaranteed by the quantum worldsheet theory being invariant under the global symmetries.

8 Summary and conclusions

We have shown that the logarithmic divergences of the transfer matrix in the pure spinor superstring in $\text{AdS}_5 \times S^5$ vanish at the one-loop level. The Lax operator in the pure spinor string in $\text{AdS}_5 \times S^5$, although it looks somewhat cumbersome, seems to work beautifully in the quantum theory. It is reasonable to conjecture that the path ordered exponential of the Lax connection defines a sensible quantum transfer matrix. Notice that in the previously studied examples of massive integrable systems the transfer matrix had logarithmic divergences, while in our case it seems that only the linear divergences are present.

It would be very interesting to investigate the quantum commutation relations of the components of this transfer matrix, and see if they could be encoded in the form of the RTT relations. In principle this could be done in perturbation theory, in the near-flat space limit.

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Appendix A The algebra $\mathfrak{psu}(2, 2|4)$

A.1 Structure constants and invariant tensor

Consider the quadratic Casimir operator:

$$C = C^{\dot{\alpha}\alpha}(t_{\dot{\alpha}}^1 \otimes t_{\alpha}^3 - t_{\alpha}^3 \otimes t_{\dot{\alpha}}^1) + C^{\mu\nu} t_{\mu}^2 \otimes t_{\nu}^2 + C^{[\mu_1\mu_2][\nu_1\nu_2]} t_{[\mu_1\mu_2]}^0 \otimes t_{[\nu_1\nu_2]}^0. \quad (\text{A.1})$$

We also define C_0 :

$$C_0 = C^{[\mu_1\mu_2][\nu_1\nu_2]} t_{[\mu_1\mu_2]}^0 \otimes t_{[\nu_1\nu_2]}^0. \quad (\text{A.2})$$

We define $C^{\alpha\dot{\alpha}}$ and C with lower indices as follows:

$$C^{\alpha\dot{\alpha}} = -C^{\dot{\alpha}\alpha}, \quad C^{\alpha\dot{\beta}}C_{\dot{\beta}\gamma} = \delta_{\gamma}^{\alpha}. \tag{A.3}$$

The structure constants with upper indices are defined as

$$f_{\dot{\gamma}}^{\alpha\mu} = f_{\dot{\gamma}\dot{\alpha}}^{\mu} C^{\dot{\alpha}\alpha}, \quad f_{\gamma}^{\dot{\alpha}\mu} = f_{\gamma\alpha}^{\mu} C^{\alpha\dot{\alpha}}. \tag{A.4}$$

Similarly, the vector indices are raised by $C^{\mu\nu}$. Notice that $C^{\mu\nu}$ is a symmetric tensor. If we identify t_{μ}^2 with the Killing vectors on $\text{AdS}_5 \times S^5$ then $C^{\mu\nu} = g^{\mu\nu}$ should be identified with the metric. We have

$$f_{\dot{\gamma}}^{\alpha\mu} = -f_{\dot{\gamma}}^{\mu\alpha}.$$

We will normalize the supertrace so that

$$\text{str}(t_{\mu}^2 t_{\nu}^2) = g_{\mu\nu}, \quad \text{str}(t_{\dot{\alpha}}^1 t_{\dot{\beta}}^3) = C_{\dot{\alpha}\dot{\beta}}, \quad \text{str}(t_{\dot{\beta}}^3 t_{\dot{\alpha}}^1) = C_{\dot{\beta}\dot{\alpha}}. \tag{A.5}$$

A.2 Matrix realization

The algebra $\mathfrak{sl}(4|4)$ can be realized by the $(4|4) \times (4|4)$ -matrices of the form

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}. \tag{A.6}$$

The \mathbb{Z}_4 automorphism is $M \mapsto \Omega M$ where:

$$\Omega M = \begin{pmatrix} JA^t J & -JY^t J \\ JX^t J & JB^t J \end{pmatrix}, \tag{A.7}$$

where J is an antisymmetric matrix:

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{A.8}$$

Notice that $\Omega^4 M = M$. To get $\mathfrak{su}(2, 2|4)$ from $\mathfrak{sl}(4|4)$ we impose the reality condition $M^\dagger = -M$, where

$$M^\dagger = \begin{pmatrix} \Sigma A^\dagger \Sigma & -i\Sigma Y^\dagger \\ -iX^\dagger \Sigma & B^\dagger \end{pmatrix}. \tag{A.9}$$

The subspaces $\mathfrak{g}_{\bar{a}}$ are defined as eigenspaces of Ω :

$$\Omega\xi = i^a\xi \quad \text{for } \xi \in \mathfrak{g}_{\bar{a}}. \tag{A.10}$$

Notice that $\mathfrak{g}_{\bar{a}} \cap \mathfrak{su}(2, 2|4)$, $\bar{a} \in \{0, 1, 2, 3\}$ are real subspaces of $\mathfrak{psu}(2, 2|4)$. (And therefore Ω does not really act on $\mathfrak{su}(2, 2|4)$ but only on $\mathfrak{sl}(4|4)$.) The AdS superalgebra $\mathfrak{psu}(2, 2|4)$ is a factor algebra of $\mathfrak{su}(2, 2|4)$ by the centre, which is generated by the unit matrix.

The invariant bilinear form on the superalgebra $\mathfrak{psu}(2, 2|4)$ can be defined using the supertrace in the *fundamental* representation:

$$C(\xi, \eta) = \text{str } \xi\eta. \tag{A.11}$$

A.3 Some algebraic identities

Here we collect some useful algebraic identities. Note that

$$C^{\alpha\dot{\alpha}}\{t_{\alpha}^3, t_{\dot{\alpha}}^1\} = 0. \tag{A.12}$$

First identity: If X is a spinor (an element of $\mathfrak{g}_{\bar{1}}$ or $\mathfrak{g}_{\bar{3}}$) then

$$\begin{aligned} C^{\alpha\dot{\alpha}}[t_{\dot{\alpha}}^1, \{t_{\alpha}^3, X\}] &= C^{\alpha\dot{\alpha}}[t_{\alpha}^3, \{t_{\dot{\alpha}}^1, X\}] = C^{\mu\nu}[t_{\mu}^2, [t_{\nu}^2, X]] \\ &= C^{[\mu_1\nu_1][\mu_2\nu_2]}[t_{[\mu_1\nu_1]}^0, [t_{[\mu_2\nu_2]}^0, X]] = 0 \quad \text{if } X \in \mathfrak{g}_{\bar{1}} + \mathfrak{g}_{\bar{3}}. \end{aligned} \tag{A.13}$$

The total adjoint Casimir of $\mathfrak{psu}(2, 2|4)$ is zero. Therefore, it is enough to prove that these four expressions in (A.13) are equal to each other. Note that

$$C^{\alpha\dot{\alpha}}\{t_{\alpha}^3, t_{\dot{\alpha}}^1\} = 0, \tag{A.14}$$

because this would be in the centre of \mathfrak{g}_0 , but the centre of \mathfrak{g}_0 is trivial. This implies the equality of the first two expressions in (A.13). The other

equalities can be demonstrated as follows:

$$C^{\alpha\dot{\alpha}}[t_\alpha^3, \{t_\alpha^1, t_\beta^3\}] = -[t_\alpha^3, f_\beta^{\alpha[\mu\nu]} t_{[\mu\nu]}^0] = C^{[\mu_1\nu_1][\mu_2\nu_2]}[t_{[\mu_1\nu_1]}^0, [t_{[\mu_2\nu_2]}^0, t_\beta^3]], \tag{A.15}$$

$$C^{\dot{\alpha}\alpha}[t_\alpha^1, \{t_\alpha^3, t_\beta^3\}] = -[t_\alpha^1, f_\beta^{\dot{\alpha}\mu} t_\mu^2] = C^{\mu\nu}[t_\mu^2, [t_\nu^2, t_\beta^3]]. \tag{A.16}$$

Second identity: If $X \in \mathfrak{g}_2$ then

$$\begin{aligned} C^{\dot{\alpha}\alpha}[t_\alpha^1, \{t_\alpha^3, X\}] &= C^{\alpha\dot{\alpha}}[t_\alpha^3, \{t_\alpha^1, X\}] = -C^{\mu\nu}[t_\mu^2, [t_\nu^2, X]] \\ &= -C^{[\mu_1\nu_1][\mu_2\nu_2]}[t_{[\mu_1\nu_1]}^0, [t_{[\mu_2\nu_2]}^0, X]] \quad \text{if } X \in \mathfrak{g}_2. \end{aligned} \tag{A.17}$$

Note that in this case, when $X \in \mathfrak{g}_2$, these three expressions are equal to each other, but not zero.

Third identity: We define C_0 as follows:

$$C_0 = C^{[\mu_1\nu_1][\mu_2\nu_2]} t_{[\mu_1\nu_1]}^0 \otimes t_{[\mu_2\nu_2]}^0. \tag{A.18}$$

If $X \in \mathfrak{g}_0$ then

$$C_0 \cdot X = C^{[\mu_1\nu_1][\mu_2\nu_2]} [t_{[\mu_1\nu_1]}^0, [t_{[\mu_2\nu_2]}^0, X]] = \begin{cases} -6X & \text{if } X \in \mathfrak{so}(1,4), \\ 6X & \text{if } X \in \mathfrak{so}(5). \end{cases} \tag{A.19}$$

Casimir identities: If $\xi_2 \in \mathfrak{g}_2$ then the adjoint Casimir satisfies:

$$(C_{\bar{0}} + C_{\bar{2}}) \cdot [\xi_2, \eta_2] = [\xi_2, (C_{\bar{0}} + C_{\bar{2}}) \cdot \eta_2], \tag{A.20}$$

$$C_{\bar{0}} \cdot \xi_2 = C_{\bar{2}} \cdot \xi_2, \tag{A.21}$$

$$C_{\bar{2}} \cdot [\xi_2, \eta_2] = \frac{1}{2} [\xi_2, C_{\bar{2}} \cdot \eta_2], \tag{A.22}$$

$$C_{\bar{0}} \cdot [\xi_2, \eta_2] = \frac{3}{2} [\xi_2, C_{\bar{2}} \cdot \eta_2]. \tag{A.23}$$

We will also introduce

$$C_{\text{odd}} = C_{\bar{1}} + C_{\bar{3}}, \quad C_{\text{even}} = C_{\bar{0}} + C_{\bar{2}}. \tag{A.24}$$

Notice that for $\xi_2 \in \mathfrak{g}_2$:

$$C_{\text{odd}} \cdot \xi_2 = -2C_{\bar{2}} \cdot \xi_2. \tag{A.25}$$

Additional identities:

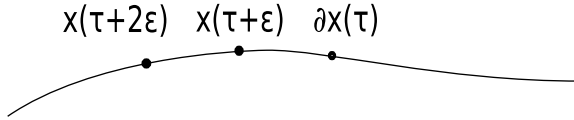
$$\begin{aligned} C^{\alpha\dot{\alpha}}(t_\alpha^3 t_\mu^2 t_\alpha^1 + t_\alpha^1 t_\mu^2 t_\alpha^3) &= 0, \\ [t_\mu^2, C^{\alpha\dot{\alpha}} t_\alpha^3 t_\alpha^1] &= 0. \end{aligned} \tag{A.26}$$

Appendix B Contour-split regularization and linear divergences

B.1 Regularization by splitting along the contour

We treat x, ϑ, λ and w as elementary fields. The capital currents J_{\pm} are composite operators constructed from these elementary fields. When calculating the Wilson loop, we assume that the composite operators are regularized by splitting *along the contour*. For example, the expression $[x, [x, \partial_+ x]]$ will be understood as follows:

$$\begin{aligned}
 &x(\tau + 2\epsilon)x(\tau + \epsilon)\partial_+x(\tau) - 2x(\tau + 2\epsilon)\partial_+x(\tau + \epsilon)x(\tau) \\
 &+ \partial_+x(\tau + 2\epsilon)x(\tau + \epsilon)x(\tau).
 \end{aligned}
 \tag{B.1}$$



Here $\tau + \epsilon$ denotes the shift of the point τ by the amount ϵ along the contour. This regularization depends on the parametrization of the contour. The good thing about this regularization is that it preserves the property that the transfer matrix is a total derivative when $z = 1$. This regularization can be understood as introducing the “dot-product:”

$$(\phi_1 \cdot \phi_2)(\tau) = \phi_1(\tau)\phi_2(\tau - \epsilon).
 \tag{B.2}$$

The integrated dot-product is associative:

$$\int d\tau \phi_1 \cdot (\phi_2 \cdot \phi_3) = \int d\tau (\phi_1 \cdot \phi_2) \cdot \phi_3.
 \tag{B.3}$$

This property implies that the regularized transfer matrix is a total derivative when $z = 1$, just like it was in the classical theory:

$$P \exp \left(- \int_{\tau_1}^{\tau_2} J^{[z=1]} \right) = g(\tau_1)g(\tau_2)^{-1}.
 \tag{B.4}$$

Here $g(\tau)$ is, schematically, $e^{x(\tau)} = 1 + x + \frac{x \cdot x}{2} + \frac{x \cdot x \cdot x}{6} + \dots$. We also introduce the split commutator $[[,]]$:

$$[[\phi_1, \phi_2]] = \phi_1 \cdot \phi_2 - \phi_2 \cdot \phi_1.
 \tag{B.5}$$

B.2 Example: Wilson line in the $O(2)$ non-linear sigma model

The $O(2)$ NLSM is a free field theory, a single boson ϕ with the action $\int d^2\tau \partial_+ \phi \partial_- \phi$. We want to define the Wilson line

$$P \exp \int_0^L \left[\frac{1}{z^2} \partial_+ \phi d\tau^+ + z^2 \partial_- \phi d\tau^- \right]. \tag{B.6}$$

This is very easy to calculate, if we allow ourselves to split ϕ into holomorphic and antiholomorphic parts:

$$\phi(\tau^+, \tau^-) = \phi_h(\tau^+) + \phi_a(\tau^-). \tag{B.7}$$

The transfer matrix is equal to

$$e^{\frac{1}{z^2} \phi_h(L) + z^2 \phi_a(L)} e^{-\frac{1}{z^2} \phi_h(0) - z^2 \phi_a(0)}. \tag{B.8}$$

But we want to insist on using our contour-split regularization. Therefore we should calculate the transfer matrix as $P \exp(-\int J^{[z]})$, where

$$J^{[z]} = -d \exp \left[\frac{1}{R} \left(\frac{1}{z^2} \phi_h + z^2 \phi_a \right) \right] \exp \left[-\frac{1}{R} \left(\frac{1}{z^2} \phi_h + z^2 \phi_a \right) \right]. \tag{B.9}$$

We should expand in powers of $\frac{1}{R}$ and use our dot product for ϕ , for example replace $\phi^2(\tau)$ with $\phi(\tau)\phi(\tau - \epsilon)$. One of the terms we get is

$$\frac{1}{R^2} \frac{1}{z^4} \frac{1}{2} \llbracket d\phi_h, \phi_h \rrbracket = -\frac{1}{R^2} \frac{1}{z^4} \frac{1}{\epsilon^+} \llbracket \llbracket d\tau^+ \rrbracket. \tag{B.10}$$

Here $\llbracket \rrbracket$ is a ‘‘placeholder,’’ for example $\phi \llbracket \phi = \phi(\tau + 2\epsilon)\phi(\tau)$ and $\phi \llbracket \llbracket \phi = \phi(\tau + 3\epsilon)\phi(\tau)$; the placeholder becomes important when we study the collisions. A more complicated example is

$$\frac{1}{R^3} \frac{1}{z^6} \frac{1}{6} \llbracket \llbracket d\phi_h, \phi_h \rrbracket, \phi_h \rrbracket \tag{B.11}$$

$$= \frac{1}{R^3} \frac{1}{z^6} \frac{1}{6} d\tau^+ \left(\frac{3}{\epsilon^+} \llbracket \phi_h, \llbracket \llbracket \rrbracket - \log \epsilon \llbracket \llbracket \partial_+ \phi, \llbracket \llbracket \rrbracket + 2 \log 2 \llbracket \partial_+ \phi \llbracket \rrbracket \right). \tag{B.12}$$

Note that this expression is almost expressed in terms of the derivatives of ϕ , except for the term $\frac{3}{\epsilon^+} \llbracket \phi_h, \llbracket \llbracket \rrbracket$. This term can be expressed through the derivatives of ϕ in the bulk; in the boundary terms ϕ_h enters either through explicit contractions, or as a derivative. In fact, we should probably think of any commutator with the placeholder as a total derivative, because it plays a role only in the boundary terms.

B.3 Linear divergences

Since J -currents are built on both x and ∂x , there are linear divergences in them. The coefficient of the linear divergence is either a c -number or a function of x (but no derivatives of x). For example, J_{0+} has a term $\frac{1}{2}[\partial_+x, x]$, which leads to the linear divergence:

$$\frac{1}{2}[[\partial_+x, x]] = -\frac{1}{\epsilon^+} C^{\mu\nu} t_\mu^2 t_\nu^2 + \dots . \tag{B.13}$$

Therefore we have to add the counterterms to the currents, to make the transfer finite, for example:

$$J_{0+}^{\text{with c.t.}} = J_{0+} + \frac{1}{\epsilon^+} (C^{\mu\nu} t_\mu^2 t_\nu^2 + C^{\alpha\dot{\alpha}} t_\alpha^3 t_{\dot{\alpha}}^1 + C^{\dot{\alpha}\alpha} t_{\dot{\alpha}}^1 t_\alpha^3) . \tag{B.14}$$

Note that these counterterms do not belong to the Lie algebra. Similarly, J_{1+} , J_{2+} and J_{3+} have linear divergences, as composite operators. In the expansion of J_{2+} , the terms responsible for the linear divergence are

$$J_2 = -dx - \frac{1}{6}[[x, [x, dx]]] + \dots . \tag{B.15}$$

This leads to the following linear divergence in J_{2+} : $-\frac{1}{2}\frac{1}{\epsilon^+}[[x, C^{\mu\nu} t_\mu^2 t_\nu^2]]$. But this linear divergence in fact cancels with the linear divergence arising in the $J_{0+}J_{2+}$ collision. Therefore there is no linear counterterm to J_{2+} .

The double collisions $J_{2+}J_{2+}$ and $J_{1+}J_{3+}$ give non-zero linear divergences which should be cancelled by the counterterm proportional to z^{-4} . There is no classical J_{4+} current, but we need to introduce such a counterterm to cancel the linear divergence:

$$\frac{1}{z^4} J_{4+}^{\text{c.t.}} = -\frac{1}{z^4} \frac{1}{R^2} \frac{1}{\epsilon^+} (C^{\mu\nu} t_\mu^2 t_\nu^2 + C^{\alpha\dot{\alpha}} t_\alpha^3 t_{\dot{\alpha}}^1 + C^{\dot{\alpha}\alpha} t_{\dot{\alpha}}^1 t_\alpha^3) . \tag{B.16}$$

There is a linear divergence in the collisions $J_{0+}J_{1+}$ and $J_{3+}J_{2+}$, which cancels with the internal linear divergence of J_{1+} . Similarly, the linear divergence in the collisions $J_{0+}J_{3+}$ and $J_{1+}J_{2+}$ cancels with the internal linear divergence of J_{3+} .

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