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Constructing new Calabi–Yau 3-folds and their mirrors via conifold transitions

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Abstract

We construct a surprisingly large class of new Calabi–Yau 3-folds X with small Picard numbers and propose a construction of their mirrors X^* using smoothings of toric hypersurfaces with conifold singularities. These new examples are related to the previously known ones via conifold transitions. Our results generalize the mirror construction for Calabi–Yau complete intersections in Grassmannians and flag manifolds via toric degenerations. There exist exactly 198849 reflexive four-polytopes whose two-faces are only triangles or parallelograms of minimal volume. Every such polytope gives rise to a family of Calabi–Yau hypersurfaces with at worst conifold singularities. Using a criterion of Namikawa we found 30241 reflexive four-polytopes such that the corresponding Calabi–Yau hypersurfaces are smoothable by a flat deformation. In particular, we found 210 reflexive four-polytopes defining 68 topologically different

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Calabi–Yau 3-folds with $h_{11} = 1$. We explain the mirror construction and compute several new Picard–Fuchs operators for the respective one-parameter families of mirror Calabi–Yau 3-folds.

1 Introduction

Toric geometry provides simple and efficient combinatorial tools [1] for the construction of large classes of Calabi–Yau manifolds from generic hypersurfaces [2] and complete intersections [3,4] in toric varieties. An important additional benefit of this construction is its invariance under mirror symmetry. In particular, it enables the computation of quantum cohomology and instanton numbers using generalized hypergeometric functions [5, 6]. Calabi–Yau 3-folds obtained from hypersurfaces in four-dimensional (4D) toric varieties have been enumerated completely [7, 8]. Some large lists of Calabi–Yau 3-folds obtained from complete intersections have been compiled and analyzed in [9,10]. Their fibration structures [8,11] and torsion in cohomology [12] are of particular interest for applications to string theory [10,13]. Thus, toric constructions provide by far the largest number of known examples, but they are, nevertheless, quite special in the zoo of all Calabi–Yau 3-folds about which little is known. Even finiteness of topological types of Calabi–Yau 3-folds remains still an open question.

According to M. Reid [14] it is expected that an appropriate partial compactification of the moduli space of all Calabi–Yau 3-folds, which allows Calabi–Yau varieties with mild singularities, will be connected. Using this idea, we can try to get new examples of Calabi–Yau 3-folds by studying singular limits of Calabi–Yau 3-folds obtained by toric methods.

In the present work, we focus on Calabi–Yau 3-folds $\widehat{X}_f$ obtained from generic hypersurfaces $\overline{X}_f$ in 4D Gorenstein toric Fano varieties $\mathbb{P}_\Delta$ corresponding to 4D reflexive polytopes Δ because the complete list of these polytopes is known [7, 8]. We classify all hypersurfaces $\overline{X}_f$ with at most conifold singularities coming from the singularities of the ambient Gorenstein toric Fano variety $\mathbb{P}_\Delta$. Standard toric methods allow us to resolve the conifold singularities of $\overline{X}_f$ by a toric resolution of $\mathbb{P}_\Delta$ and to obtain a smooth Calabi–Yau 3-fold $\widehat{X}_f$. However, in this paper we are interested in smoothing $\overline{X}_f$ to a Calabi–Yau 3-fold Y by a flat deformation. Thus the two Calabi–Yau 3-folds $\widehat{X}_f$ and Y are connected by a so-called *conifold transition.* Moreover, a similar conifold transition exists for the mirrors $(\widehat{X}_f)^*$ and Y^*.

The above approach was motivated by the previous work [15, 16], which shows that all Grassmannians and Flag manifolds allow flat degenerations to Gorenstein toric Fano varieties $\mathbb{P}_\Delta$ having at worst conifold singularities in codimension 3. Therefore, smooth 3D Calabi–Yau complete intersections Y in Grassmannians and Flag manifolds can be regarded as smoothings of generic Calabi–Yau complete intersections $\overline{X}$ in the corresponding Gorenstein toric Fano variety $\mathbb{P}_\Delta$. It was observed in [15, 16] that the mirrors Y^* of Y are obtained by specializations of the complex structure of the mirrors $\widehat{X}^*$ coming from the already known toric construction.

In Section 2, we explain the construction and present our results by describing the lists of polytopes and Hodge data whose details are available on the internet. In Section 3, we discuss the conifold transition in the mirror family (which is related to the transitions studied in [17]). This enables the construction of non-toric mirror pairs and the computation of quantum cohomologies. In Section 4, we focus on one-parameter models for which we compute the topological data and also initiate the study of the mirror map by direct evaluation of the principal period, which allows us to find a number of new Picard–Fuchs operators. We conclude with a discussion of open problems, generalizations, and work to be done.

2 Reflexive polytopes and conifold transitions

Let $M \cong \mathbb{Z}^4$ and $N = \mathrm{Hom}(M, \mathbb{Z})$ be a dual pair of lattices of rank 4 together with the canonical pairing $\langle *, * \rangle \,:\, N \times M \to \mathbb{Z}$ and let $M_\mathbb{R} = M \otimes \mathbb{R}$, $N_\mathbb{R} = N \otimes \mathbb{R}$ be their real extensions. It is known [2] that the generic families of Calabi–Yau hypersurfaces $\overline{X}_f$ in 4D Gorenstein toric Fano varieties $\mathbb{P}_\Delta$ and their mirrors $\overline{X}_g \subset \mathbb{P}_{\Delta^\circ}$ are in one-to-one correspondence to the polar pairs $\Delta \subset M_\mathbb{R}$, $\Delta^\circ \subset N_\mathbb{R}$ of reflexive four-polytopes. By definition reflexivity of Δ and Δ° means that

$$\Delta^\circ = \{y \in N_\mathbb{R} \;:\; \langle y, x\rangle \geq -1 \;\forall x \in \Delta\} \tag{2.1}$$

and that both Δ and Δ° are lattice polytopes, i.e., all vertices of Δ (resp. Δ°) are elements of M (resp. N).

The hypersurfaces $\overline{X}_f \subset \mathbb{P}_\Delta$ and $\overline{X}_g \subset \mathbb{P}_{\Delta^\circ}$ are the closures of the affine hypersurfaces X_f and X_g defined by generic Laurent polynomials

$$f := \sum_{m \in \Delta \cap M} a_m t^m \quad \text{and} \quad g := \sum_{n \in \Delta^\circ \cap N} b_n t^n. \tag{2.2}$$

Denote by Σ (resp. by Σ°) the fan of cones over simplices in $\partial\Delta$ (resp. in $\partial\Delta^\circ$) in a maximal coherent triangulation of Δ (resp. Δ°) [18]. Then the projective toric variety $\mathbb{P}_{\Sigma^\circ}$ is a maximal partial projective crepant (MPPC) desingularization of $\mathbb{P}_\Delta$. Similarly, $\mathbb{P}_\Sigma$ is a MPPC desingularization of $\mathbb{P}_{\Delta^\circ}$. Denote by $\widehat{X}_f \subset \mathbb{P}_{\Sigma^\circ}$ and $\widehat{X}_g \subset \mathbb{P}_\Sigma$ the closures of X_f and X_g in $\mathbb{P}_{\Sigma^\circ}$ and $\mathbb{P}_\Sigma$, respectively. Then $\widehat{X}_f$ and $\widehat{X}_g$ are smooth Calabi–Yau 3-folds [2] and for the Hodge numbers $h^{1,1}$ and $h^{2,1}$ one has

$$h^{1,1}(\widehat{X}_f) = h^{2,1}(\widehat{X}_g) = l(\Delta^\circ) - 5 - \sum_{\operatorname{codim}(\theta^\circ)=1} l^*(\theta^\circ) + \sum_{\operatorname{codim}(\theta^\circ)=2} l^*(\theta^\circ) l^*(\theta), \tag{2.3}$$

$$h^{1,1}(\widehat{X}_g) = h^{2,1}(\widehat{X}_f) = l(\Delta) - 5 - \sum_{\operatorname{codim}(\theta)=1} l^*(\theta) + \sum_{\operatorname{codim}(\theta)=2} l^*(\theta) l^*(\theta^\circ), \tag{2.4}$$

where $l(\Delta)$ denotes the number of lattice points of Δ and $l^*(\theta)$ denotes the number of lattice points in the relative interior of θ. The faces $\theta \subset \Delta$ and $\theta^\circ \subset \Delta^\circ$ denote polar sets of points satisfying the inequality in equation (2.1), so that $\dim(\theta) + \dim(\theta^\circ) = 3$.

The smoothness of generic hypersurfaces $\widehat{X}_f$ and $\widehat{X}_g$ follows from the fact that the singularities of the MPPC resolutions $\mathbb{P}_{\Sigma^\circ}$ and $\mathbb{P}_\Sigma$ have codimension at least 4, i.e., singular points in the 4D ambient spaces $\mathbb{P}_{\Sigma^\circ}$ and $\mathbb{P}_\Sigma$ can generically be avoided by the hypersurface equations $f = 0$ and $g = 0$. For special values of the coefficients $\{a_m\}$ and $\{b_n\}$ in equation (2.2) the corresponding Calabi–Yau varieties $\widehat{X}_f$ and $\widehat{X}_g$ may of course be singular.

From now on we want to restrict the types of singularities of $\overline{X}_f \subset \mathbb{P}_\Delta$ under consideration and demand that all 2D faces θ° of the dual polytope Δ° are either unimodular triangles (i.e., spanned by a subset of a lattice basis) or parallelograms of minimal volume, whose two triangulations hence are unimodular. This implies that all nonisolated singularities of $\mathbb{P}_\Delta$ are one-parameter families of toric conifold singularities defined by an equation $u_1u_2 - u_3u_4 = 0$. These families $\mathbb{T}_\theta$ of singularities are in one-to-one correspondence to 1D faces $\theta \subset \Delta$ such that the dual face $\theta^\circ \subset \Delta^\circ$ is a parallelogram.

The morphism $\mathbb{P}_{\Sigma^\circ} \to \mathbb{P}_\Delta$ induces a small crepant resolution $\widehat{X}_f \to \overline{X}_f$, which replaces every conifold point in $\overline{X}_f$ by a copy of $\mathbb{P}^1$. We remark that every one-parameter family $\mathbb{T}_\theta \subset \mathbb{P}_\Delta$ (the 2D dual face $\theta^\circ \subset \Delta^\circ$ is a parallelogram) of toric conifold singularities has exactly $l(\theta) - 1$ distinct intersection points with the generic hypersurface $\overline{X}_f \subset \mathbb{P}_\Delta$. In order to analyse the possibility of smoothing $\overline{X}_f$ by a flat deformation we apply the following criterion of Namikawa (we formulate it in a simplified version):

Theorem 2.1 (see [19])**.** *Let X be a Calabi–Yau 3-fold with k-isolated conifold singularities*

$$\{p_1, \ldots, p_k\} = Sing\, X$$

and $f : Z \to X$ be a small resolution of these singularities such that $C_i := f^{-1}(p_i) \cong \mathbb{P}^1$ and f an isomorphism over $X \setminus \{p_1, \ldots, p_k\}$. Then X can be deformed to a smooth Calabi–Yau 3-folds if and only if the homology classes $[C_i] \in H_2(Z, \mathbb{C})$ satisfy a linear relation

$$\sum_{i=1}^{k} \alpha_i [C_i] = 0,$$

where $\alpha_i \neq 0$ for all i. It is easy to see that the last condition is equivalent to the fact that the subspace in $H_2(Z, \mathbb{C})$ generated by the homology classes $[C_1], \ldots, [C_k]$ coincides with the subspace generated by $\{[C_1], \ldots, [C_k]\} \setminus \{[C_i]\}$ for all $i = 1, \ldots, k$.

In our situation, we can choose Z to be $\widehat{X}_f$. Let $P(\Delta)$ be the set of all 1D faces θ of Δ such that the dual two-face θ° is a parallelogram of minimal volume. We set $k_\theta = l(\theta) - 1$. Then a generic Calabi–Yau hypersurface $\overline{X}_f \subset \mathbb{P}_\Delta$ contains exactly

$$k = \sum_{\theta \in P(\Delta)} k_\theta$$

conifold points. Let $\{v_1, \ldots, v_l\}$ be the set of all vertices of the dual polytope Δ°. Then it can be shown that the homology group $H_2(\widehat{X}_f, \mathbb{Q})$ can be identified with the subgroup $R_{\Delta^\circ} \subset \mathbb{Q}^l$ consisting of rational vectors $(\lambda_1, \ldots, \lambda_l) \in \mathbb{Q}^l$ such that

$$\sum_{i=1}^{l} \lambda_i v_i = 0.$$

In order to determine the homology class $[C_i] \in H_2(\widehat{X}_f, \mathbb{Q})$ we first remark that locally for each conifold point $p_i \in \overline{X}_f$, there exist exactly two different small resolutions f_i and f_i' of p_i. The corresponding homology classes of exceptional curves differ by their signs $[f_i^{-1}(p_i)] = -[(f_i')^{-1}(p_i)]$. The next step is to see that for any $\theta \in P(\Delta)$ all k_θ conifold points in the intersection $\mathbb{T}_\theta \cap \overline{X}_f$ define (up to signs) the same homology class in $R_{\Delta^\circ} \cong H_2(\widehat{X}_f, \mathbb{Q})$

coming from the linear relation

$$\rho_\theta : v_i + v_j - v_s - v_r = 0,$$

where v_i, v_j, v_s and v_r are vertices of the parallelogram θ° and $[v_i, v_j]$, $[v_s, v_r]$ are the two diagonals of θ°. Therefore, the homology classes $[C_1], \dots, [C_k] \in H_2(\widehat{X}_f, \mathbb{Q})$ coincide (up to signs) with the elements ρ_θ for $\theta \in P(\Delta)$. Each element $\pm\rho_\theta$ appears $k_\theta = l(\theta) - 1$ times in the sequence $[C_1], \dots, [C_k]$. Thus, the smoothing criterion of Namikawa can be formulated for $\overline{X}_f$ as follows:

Smoothing criterion. Under the above assumption on Δ°, a generic Calabi–Yau hypersurface $\overline{X}_f \subset \mathbb{P}_\Delta$ is smoothable to a Calabi–Yau 3-folds Y by a flat deformation if and only if for any 1D face $\theta \in P(\Delta)$ such that $k_\theta = 1$ the element ρ_θ is a linear combination of the remaining elements $\rho_{\theta'}$ with $\theta' \in P(\Delta)$, $\theta' \neq \theta$.

Using the classification of 4D reflexive polytopes, one can show that there exist exactly 198849 reflexive polytopes Δ such that all 2D faces of the dual polytope Δ° are either basic triangles, or parallelograms of minimal volume. Let $p := |P(\Delta)|$ and l be the number of vertices of Δ°. We define the matrix $\Lambda(\Delta)$ of size $p \times l$ whose rows are coefficients of the linear relation ρ_θ $(\theta \in P(\Delta))$. Then a generic Calabi–Yau hypersurface $\overline{X}_f$ is smoothable by a flat deformation if and only if for all $\theta \in P(\Delta)$ such that $k_\theta = 1$ the removing of the corresponding row $\Lambda_\theta(\Delta)$ from $\Lambda(\Delta)$ does not reduce the rank of the matrix. This smoothing condition reduces the number of relevant polytopes from 198849 to 30241 as detailed according to Picard numbers $h^{1,1}$ in table 1.

The Hodge numbers of the smoothed Calabi–Yau 3-folds Y can be computed by the well-known formula (see e.g. [20])

$$h^{1,1}(Y) = h^{1,1}(\widehat{X}_f) - \mathtt{rk}, \quad h^{1,2}(Y) = h^{1,2}(\widehat{X}_f) + \mathtt{dp} - \mathtt{rk}, \tag{2.5}$$

where $\mathtt{rk}$ is the rank of the matrix $\Lambda(\Delta)$ of linear relations and $\mathrm{dp} = k = \sum_{\theta \in P(\Delta)} k_\theta$ denotes the number of double points in $\overline{X}_f$. For the smoothable cases they are listed in table 2 and displayed as circles over the background of

Table 1: Numbers of polytopes for conifold Calabi–Yau spaces with Picard number $h^{1,1}$.

Picard number	1	2	3	4	5	6	7	8	9	10	11	12	15
Polytopes	8871	43080	74570	50863	17090	3540	646	124	41	17	2	4	1
Smoothable	210	3470	11389	10264	3898	815	140	35	9	8	1	1	1

Table 2: Hodge data $(h^{1,1}, h^{2,1})$ for the 30241 smoothed conifold Calabi–Yau spaces.

$h^{1,1}$	No.(Δ)	$h^{2,1}$
1	210	25,28–41,45,47,51,53,55,59,61,65,73,76,79,89,101,103,129
2	3470	26,28–60,62–68,70,72,74,76,77,78,80,82–84,86,88,90,96,100, 102,112,116,128
3	11389	25,27–73,75–79,81,83,85,87,89,91,93,95,99,101,103, 105,107,111,115
4	10264	24,28,30–76,78–82,84,86,88–98,100,102,104,106,112
5	3898	27,29,30–83,85–93,97
6	815	28,30–32,34–56,58–70,72–76,80,82
7	140	27,29–31,33–35,37–41,43,45,47,49–51,53,55,57,59,61,62,64,76
8	35	30,32–34,36,38,40,42,44,52
9	9	31,33,37
10	8	26,30,34,36
11	1	27
12	1	28
15	1	23

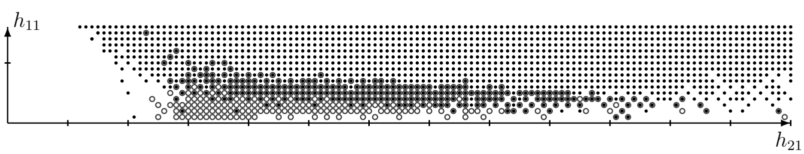

Figure 1: Smoothed conifolds (circles) and toric hypersurfaces (dots) with $h^{1,1} \leq 16$, $h^{2,1} \leq 130$.

toric hypersurface data in figure 1. The complete data, which was computed using the software package PALP [21], is available on the internet [22].

The enumeration of the polytopes Δ and of the Hodge data is, of course, only the first step and further work is required to compute the additional data like intersection form, Chern classes and quantum cohomology of Y. This program will be initiated for the 210 examples of Y with Picard number 1 in Section 4.

3 Mirror families

Let us discuss the explicit construction of mirrors of Calabi–Yau 3-folds Y from the previous section. As we have seen Y is obtained from $\widehat{X}_f$ by a conifold transition. It turns out that the mirror family Y^* of Y is also obtained by a conifold transition from the mirrors $\widehat{X}_g = (\widehat{X}_f)^*$.

For this we have to specialize the generic family of Laurent polynomials

$$g = \sum_{n \in \Delta^\circ \cap N} b_n t^n,$$

to a special one by imposing additional conditions on the coefficients b_n: for every $\theta \in P(\Delta)$ we demand

$$b_{v_i} b_{v_j} = b_{v_r} b_{v_s}, \tag{3.6}$$

where v_i, v_j, v_r and v_s are vertices of the parallelogram θ° satisfying the equation

$$v_i + v_j = v_s + v_r.$$

We denote the specialized Laurent polynomial by $\tilde{g}$.

Our main observation is that the hypersurfaces $\widehat{X}_{\tilde{g}}$ from the specialized family should have the same number $k = \sum_{\theta \in P(\Delta)} k_\theta$ of conifold singularities so that we could consider $\widehat{X}_{\tilde{g}}$ as a flat conifold degeneration of smooth Calabi–Yau 3-folds $\widehat{X}_g$.

Let us explain this observation in more detail. For any $\theta \in P(\Delta)$, we can choose a basis $e_1, \ldots, e_4$ of the lattice M and the dual basis $e_1^\circ, \ldots, e_4^\circ$ of the dual lattice N in such a way that $-e_3$ and $-e_3 - k_\theta e_4$ are vertices of the 1D face $\theta \subset \Delta$ and

$$e_3^\circ, \;\; e_3^\circ + e_1^\circ, \;\; e_3^\circ + e_2^\circ, \;\; e_3^\circ + e_2^\circ + e_1^\circ$$

are vertices of the dual (parallelogram) face $\theta^\circ \subset \Delta^\circ$. We put $w_j := -e_3 - je_4 \in \theta \cap M$ $(0 \leq j \leq k_\theta)$. Then

$$\theta \cap M = \{w_0, \ldots, w_{k_\theta}\}.$$

Every pair of lattice points w_j, w_{j-1} $(1 \leq j \leq k_\theta)$ generates a 2D cone σ_j in the fan Σ. Since e_1, e_2, w_j, w_{j-1} is a $\mathbb{Z}$-basis of M the cone σ_j defines an affine open torus invariant subset $U_{\sigma_j} \subset \mathbb{P}_\Sigma$ such that $U_{\sigma_j} \cong (\mathbb{C}^*)^2 \times \mathbb{C}^2$. Denote $w_3^{(j)} := (j-1)e_3^\circ - e_4^\circ$ and $w_4^{(j)} := -je_3^\circ + e_4^\circ$. Then $e_1^\circ, e_2^\circ, w_3^{(j)}, w_4^{(j)}$ is a $\mathbb{Z}$-basis of N dual to e_1, e_2, w_j, w_{j-1}. We use this basis in order to define the local coordinates $t_1, t_2, t_3^{(j)}, t_4^{(j)}$ on U_{σ_j}. Then the equation of $\widehat{X}_{\tilde{g}}$ in U_{σ_j}

can be written as follows:

$$\tilde{g}_j(t) = b + bt_1 + bt_2 + bt_1t_2 + t_3^{(j)}t_4^{(j)} \sum_{n\in A_j} b_n t^n \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{(j)}, t_4^{(j)}],$$

where $A_j \subset N$ is a finite number of lattice points $n = (n_1, n_2, n_3, n_4) \in N$ satisfying the conditions $n_3 \geq 0$, $n_4 \geq 0$. Therefore, the polynomial

$$\tilde{g}_j(t) = b(1+t_1)(1+t_2) + b_0 t_3^{(j)}t_4^{(j)} + t_3^{(j)}t_4^{(j)} \sum_{n\in A_j\setminus\{0\}} b_n t^n$$

has a conifold singularity at the point $q_j = (-1, -1, 0, 0) \in U_{\sigma_j}$ $(1 \leq j \leq k_\theta)$. By repeating this computation for every 1D face $\theta \in P(\Delta)$, we obtain $k = \sum_{\theta\in P(\Delta)} k_\theta$ conifold points in $\widehat{X}_{\tilde{g}}$.

Remark. Unfortunately, these above arguments do not show that we have found all singular points of $\widehat{X}_{\tilde{g}}$. We hope that this is true in many cases.

Now we are going to obtain the mirrors Y^* using small resolutions of singularities of $\widehat{X}_{\tilde{g}}$. By a result of Smith *et al.* [23, Theorem 2.9], in order that $\widehat{X}_{\tilde{g}}$ admits a projective small resolution, the homology classes $[L_1], \ldots, [L_k] \in H_3(\widehat{X}_g, \mathbb{Z})$ of the vanishing cycles $L_i \cong S^3$ must satisfy a linear relation

$$\sum_{i=1}^{k} c_i[L_i] = 0, \quad c_i \neq 0 \ \ \forall i.$$

This condition can be considered as “mirror” to the criterion of Namikawa.

In the case when $h^{1,1}(Y) = 1$ the specialization equations (3.6) show that we can put $b_0 = 1$ and $b_n = -z$ $\forall n \neq 0$, so that we obtain a one-parameter family of Laurent polynomials $\tilde{g}$ depending only on z.

4 One-parameter manifolds

We now focus on the list of the 210 polytopes that lead to one-parameter families with smoothable conifold singularities. According to a theorem by Wall [24] the diffeomorphism type of a Calabi–Yau is completely characterized by its Hodge numbers, intersection ring and second Chern class. For Calabi–Yau 3-folds with the Picard number one the latter two amount to the triple intersection number H^3 and the number Hc_2. The resulting 68 different topological types are listed in table 3.

Table 3: Topological data with multiplicities N_Δ of polytopes and N_{ϖ_0} of principal periods. The last column, denoted PF, refers to the Picard–Fuchs operator, either by equation number (in parentheses) or (in boldface) by the reference number in the tables of [27].

No.	h_{12}	H^3	c_2H	c_3	N_Δ	N_{ϖ_0}	No.	h_{12}	H^3	c_2H	c_3	N_Δ	N_{ϖ_0}	PF
1	**25**	79	94	−48	1	1	35	36	92	104	−70	5	3	
2	**28**	99	102	−54	1	1	36	36	107	110	−70	16	5	
3	28	104	104	−54	2	1	37	**37**	117	114	−72	12	1	(4.16)
4	**29**	74	92	−56	2	2	38	**38**	102	108	−74	2	1	
5	29	88	100	−56	1	1	39	**39**	96	108	−76	2	1	
6	29	93	102	−56	4	3	40	39	152	116	−76	1	1	(4.15)
7	29	98	104	−56	1	1	41	**40**	91	106	−78	2	1	
8	**30**	98	104	−58	5	4	42	**41**	86	104	−80	2	1	
9	30	103	106	−58	2	2	43	41	116	116	−80	13	1	(4.14)
10	30	108	108	−58	4	2	44	**45**	144	120	−88	2	2	(4.17) , **214**
11	**31**	78	96	−60	1	1	45	**47**	144	120	−92	2	1	**289**
12	31	83	98	−60	2	2	46	47	176	128	−92	3	1	(4.13)
13	31	98	104	−60	6	4	47	**51**	168	132	−100	1	1	**218**
14	31	103	106	−60	3	1	48	51	200	140	−100	3	2	(4.19) , (4.20)
15	31	108	108	−60	1	1	49	**53**	168	132	−104	2	1	**287**
16	31	118	112	−60	5	2	50	53	232	148	−104	4	1	(4.12)
17	31	124	112	−60	2	1	51	**55**	136	124	−108	1	1	**209**
18	**32**	83	98	−62	2	2	52	**59**	24	72	−116	2	1	**29**
19	32	98	104	−62	3	1	53	59	28	76	−116	3	1	**26**
20	32	108	108	−62	4	3	54	59	32	80	−116	4	1	**42**
21	32	113	110	−62	1	1	55	**61**	20	68	−120	1	1	**25**
22	32	118	112	−62	4	3	56	61	36	84	−120	4	1	**185**
23	**33**	78	96	−64	1	1	57	**65**	16	64	−128	1	1	**3**
24	33	97	106	−64	1	1	58	65	44	92	−128	1	1	(4.11)
25	33	108	108	−64	4	1	59	**73**	9	54	−144	1	1	**4**
26	**34**	97	106	−66	3	3	60	73	12	60	−144	2	1	**5**
27	34	102	108	−66	6	3	61	73	32	80	−144	1	1	**10**
28	34	123	114	−66	1	1	62	**76**	15	66	−150	2	1	**24**
29	**35**	87	102	−68	1	1	63	**79**	48	96	−156	1	1	**11**
30	35	92	104	−68	7	5	64	79	432	192	−156	1	1	**12**
31	35	97	106	−68	5	3	65	**89**	8	56	−176	2	1	**6**
32	35	102	108	−68	8	4	66	**101**	80	128	−200	1	1	**51**
33	35	112	112	−68	13	3	67	**103**	648	252	−204	1	1	**8**
34	**36**	82	100	−70	1	1	68	**129**	108	156	−256	1	1	**14**

The entries in table 3 have been computed combinatorially with the formulas

$$H^3 = \mathrm{Vol}(\Delta)/(\mathrm{Ind})^3, \quad c_2 \cdot H = (12\,|\partial\Delta \cap M|\,)/\mathrm{Ind}, \tag{4.7}$$

where Vol denotes the lattice volume, $|\partial\Delta \cap M|$ is the number of boundary lattice points of Δ and Ind is the index of the affine sublattice of M that is generated by the vertices of Δ. There is one exception to this rule, namely the convex hull of the Newton polytope of[1]

$$g = t_1 + t_1t_2^2 + t_1t_2^2t_3^4 + t_1t_2^2t_4^4 + t_1^{-3}t_2^{-2}t_3^{-4} + t_1^{-3}t_2^{-2}t_4^{-4} + t_1^{-3}t_2^{-2}, \tag{4.8}$$

which is one of the two polytopes that lead to the entry no. 65 with $h_{12} = 89$ in the table. For this variety the divisor H has multiplicity two so that effectively Ind has to be doubled in equation (4.7).

A glance at figure 1 shows that we constructed a surprisingly rich new set as compared to toric hypersurfaces, and also when compared to other know constructions of one-parameter examples [10,15,16,25]. Observe that in all instances in table 3 with given Hodge numbers each of the intersection numbers H^3 and $c_2 \cdot H$ uniquely determines the other.

Denote by Vert(Δ°) the set of vertices of the polytope Δ°. For the computation of the quantum cohomology we start with the computation of the principal period

$$\varpi_0(z) = \oint \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{dt_4}{t_4}\left(1 - z\sum_{v\in Vert(\Delta^\circ)} t^v\right)^{-1}, \tag{4.9}$$

where all coefficients of the relevant non-constant monomials can be set to $b_j = -z$ because mirror symmetry amounts to the restriction $b_{j_1}b_{j_2} = b_{j_3}b_{j_4}$ if the corresponding vertices form a parallelogram with $v_{j_1} + v_{j_2} = v_{j_3} + v_{j_4}$ and $h^{2,1}(\widehat{X}_g) - rk = 1$ implies that rescalings of the homogeneous coordinates can be used to identify the corresponding coefficients with the complex structure modulus $-z$. The function $\varpi_0(z)$ is the unique regular power series solution in the kernel of the Picard–Fuchs operator

$$\mathcal{O} = \theta^4 + \sum_{n=1}^{d} z^n \sum_{i=0}^{4} c_{ni}\theta^i, \quad \theta = z\frac{d}{dz}. \tag{4.10}$$

This operator can then be used to compute the periods with logarithmic singularities and the instanton numbers via the mirror map as explained,

[1] This is one of the five polytopes for which a facet of Δ has an interior point.

for example, in [6]. Our method for the computation of $\mathcal{O}$ is direct evaluation of the period by expansion of the last term in equation (4.9) in z up to (at least) degree $5d$ and determination of the coefficients c_{ni} from $\mathcal{O}(\varpi_0) = 0$ for the ansatz equation (4.10).

We have computed all Picard–Fuchs operators for the manifolds with $h_{12} \geq 45$, which are the cases No. 44–No. 68 in table 3. They have been determined independently by Duco van Straten [26] and Gert Almkvist *et al.* [27]. Most of these operators were known before [27]. Here, we only list three examples that are needed, in addition to equations (4.17), (4.19) and (4.20) below, as references in table 3:

No. 58 $\mathbf{X^{65}_{44,92}}$

$$\begin{aligned}&\theta^4 - 4\ z\ \theta(\theta+1)(2\theta+1)^2 - 32\ z^2\ (2\theta+1)(2\theta+3)(11\theta^2+22\theta+12)\\&\quad - 1200\ z^3\ (2\theta+1)(2\theta+3)^2(2\theta+5)\\&\quad - 4864\ z^4\ (2\theta+1)(2\theta+3)(2\theta+5)(2\theta+7)\end{aligned} \tag{4.11}$$

No. 50: $\mathbf{X^{53}_{232,148}}$

$$\begin{aligned}&\theta^4 - \frac{2}{29}\ z\ (1318\theta^4 + 2336\theta^3 + 1806\theta^2 + 638\theta + 87)\\&\quad - \frac{4}{841}\ z^2\ (90996\theta^4 + 744384\theta^3 + 1267526\theta^2 + 791584\theta + 168345)\\&\quad + \frac{100}{841}\ z^3\ (341720\theta^4 + 77256\theta^3 - 467701\theta^2 - 110403\theta - 36540)\\&\quad + \frac{10000}{841}\ z^4\ (2\theta+1)(68\theta^3 + 1842\theta^2 + 2899\theta + 1215)\\&\quad - \frac{5000000}{841}\ z^5\ (\theta+1)^2(2\theta+1)(2\theta+3).\end{aligned} \tag{4.12}$$

No. 46: $\mathbf{X^{47}_{176,128}}$

$$\begin{aligned}&\theta^4 - \frac{4}{11}\ z\ (432\theta^4 + 624\theta^3 + 477\theta^2 + 165\theta + 22)\\&\quad + \frac{32}{121}\ z^2\ (12944\theta^4 + 4736\theta^3 - 15491\theta^2 - 12914\theta - 2860)\end{aligned}$$

$$
\begin{aligned}
&- \frac{80}{121}\ z^3\ (106688\theta^4 - 1140048\theta^3 - 1591132\theta^2 - 830228\theta - 15455)\\
&- \frac{51200}{121}\ z^4\ (2\theta+1)(4\theta+3)(76\theta^2 + 189\theta + 125)\\
&+ \frac{2048000}{121}\ z^5\ (2\theta+1)(2\theta+3)(4\theta+3)(4\theta+5). \qquad (4.13)
\end{aligned}
$$

The calculations become quite expensive when the number of vertices of Δ° becomes large, as is mostly the case for manifolds with small h_{12}. Nevertheless we could determine, so far, the operators for three more examples:

No: 43: $\mathbf{X^{41}_{116,116}}$:

$$
\begin{aligned}
&\theta^4 + \frac{2}{29}\ z\ \theta(24\theta^3 - 198\theta^2 - 128\theta - 29)\\
&\quad - \frac{4}{841}\ z^2\ (44284\theta^4 + 172954\theta^3 + 248589\theta^2 + 172057\theta + 47096)\\
&\quad - \frac{4}{841}\ z^3\ (525708\theta^4 + 2414772\theta^3 + 4447643\theta^2 + 3839049\theta + 1275594)\\
&\quad - \frac{8}{841}\ z^4\ (1415624\theta^4 + 7911004\theta^3 + 17395449\theta^2 + 17396359\theta + 6496262)\\
&\quad - \frac{16}{841}\ z^5\ (\theta+1)(2152040\theta^3 + 12186636\theta^2 + 24179373\theta + 16560506)\\
&\quad - \frac{32}{841}\ z^6\ (\theta+1)(\theta+2)(1912256\theta^2 + 9108540\theta + 11349571)\\
&\quad - \frac{10496}{841}\ z^7\ (\theta+1)(\theta+2)(\theta+3)(5671\theta + 16301)\\
&\quad - \frac{24529152}{841}\ z^8\ (\theta+1)(\theta+2)(\theta+3)(\theta+4). \qquad (4.14)
\end{aligned}
$$

No: 40: $\mathbf{X^{39}_{152,116}}$:

$$
\begin{aligned}
&\theta^4 - \frac{1}{19}\ z\ (4333\theta^4 + 6212\theta^3 + 4778\theta^2 + 1672\theta + 228)\\
&\quad + \frac{1}{361}\ z^2\ (43074495\theta^4 + 76004844\theta^3 + 62164066\theta^2 + 28024244\theta + 530556)\\
&\quad - \frac{1}{361}\ z^3\ (937293669\theta^4 + 2133168000\theta^3 + 2360371196\theta^2\\
&\quad + 1257486612\theta + 25260804)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{4}{361}\, z^4\, (2408138800\theta^4 + 7785292000\theta^3 + 10414477759\theta^2 \\
&+ 6318028809\theta + 138510993) \\
&- \frac{1636}{361}\, z^5\, (\theta+1)(28513240\theta^3 + 100355160\theta^2 + 112212410\theta + 3481470) \\
&+ 6022116\, z^6\, (\theta+1)(\theta+2)(2\theta+1)(2\theta+5). \qquad (4.15)
\end{aligned}
$$

No: 37: $\mathbf{X^{37}_{117,114}}$:

$$
\begin{aligned}
\theta^4 &- \frac{1}{13}\, z\, \theta(56\theta^3 + 178\theta^2 + 115\theta + 26) \\
&- \frac{1}{169}\, z^2\, (284660\theta^4 + 1094420\theta^3 + 1656030\theta^2 + 1173380\theta + 32448) \\
&- \frac{1}{169}\, z^3\, (2331140\theta^4 + 12579060\theta^3 + 26228150\theta^2 + 24678420\theta + 872352) \\
&- \frac{1}{169}\, z^4\, (9895850\theta^4 + 68522980\theta^3 + 177379390\theta^2 + 199697540\theta + 8108448) \\
&- \frac{1}{169}\, z^5\, (\theta+1)(24589670\theta^3 + 180072870\theta^2 + 440475820\theta + 35386584) \\
&- \frac{9}{169}\, z^6\, (\theta+1)(\theta+2)(3931630\theta^2 + 25390290\theta + 4164444) \\
&- \frac{297}{169}\, z^7\, (\theta+1)(\theta+2)(\theta+3)(86830\theta + 34604) \\
&- \frac{55539}{13}\, z^8\, (\theta+1)(\theta+2)(\theta+3)(\theta+4). \qquad (4.16)
\end{aligned}
$$

The largest degree of a coefficient that we computed so far is degree 65, which we did for the conifolds Nos 17 and 28, so that in these cases the Picard–Fuchs operators would have at least degree 14. Further results will be put on our data supplement web page at [22] as they become available.

4.1 Fractional transformations and instanton numbers

Even though we do not yet know the Picard–Fuchs operators in many cases it can be seen already from the first terms in the power series expansion of the principal period, which polytopes will yield identical operators. In addition to the degeneracy of up to 13 different polytopes yielding the same Picard–Fuchs operator, we thus observe that the same diffeomorphism type can yield up to five different Picard–Fuchs operators, as indicated in table 3. Among the operators that we know this phenomenon occurs twice:

For no. 44: $\mathbf{X^{45}_{144,120}}$ we find

$$\begin{aligned}&\theta^4 - 2\ z\ (102\theta^4 + 204\theta^3 + 155\theta^2 + 53\theta + 7)\\&\quad + 4\ z^2\ (\theta+1)^2(396\theta^2 + 792\theta + 311)\\&\quad - 784\ z^3\ (\theta+1)(\theta+2)(2\theta+1)(2\theta+5),\end{aligned}\tag{4.17}$$

for $f_\Delta = \frac{t_1t_4}{t_3} + \frac{t_2t_4}{t_1} + \frac{t_1t_4}{t_2t_3} + t_1t_4 + \frac{t_2}{t_1} + \frac{1}{t_1t_4} + \frac{1}{t_1} + \frac{t_1}{t_2t_3} + \frac{t_1}{t_2t_4} + \frac{t_3}{t_1t_4} + \frac{t_2t_3}{t_1t_4} + \frac{t_1}{t_2} + \frac{t_2t_3}{t_1} + t_1$ and

$$\begin{aligned}&\theta^4 - 2\ z\ (90\theta^4 + 188\theta^3 + 141\theta^2 + 47\theta + 6)\\&\quad - 4\ z^2\ (564\theta^4 + 1520\theta^3 + 1705\theta^2 + 934\theta + 192)\\&\quad - 16\ z^3\ (2\theta+1)(286\theta^3 + 813\theta^2 + 851\theta + 294)\\&\quad - 192\ z^4\ (2\theta+1)(2\theta+3)(4\theta+3)(4\theta+5),\end{aligned}\tag{4.18}$$

for $f_\Delta = \frac{1}{t_1} + \frac{t_4}{t_1} + \frac{t_2t_4}{t_1} + \frac{t_2}{t_1} + \frac{t_2t_3}{t_1} + \frac{t_2t_3t_4}{t_1} + \frac{t_3t_4}{t_1} + \frac{t_3}{t_1} + \frac{t_1}{t_2t_4} + \frac{t_1}{t_2} + \frac{t_1}{t_4} + \frac{t_1}{t_3t_4} + \frac{t_1}{t_3} + \frac{t_1}{t_2t_3}$.

For no. 48: $\mathbf{X^{51}_{200,140}}$ the two operators, with three respective polytopes, are

$$\begin{aligned}&\theta^4 - z(113\theta^4 + 226\theta^3 + 173\theta^2 + 60\theta + 8) - 8\ z^2\ (\theta+1)^2(119\theta^2 + 238\theta + 92)\\&\quad - 484\ z^3\ (\theta+1)(\theta+2)(2\theta+1)(2\theta+5),\end{aligned}\tag{4.19}$$

for $f_\Delta = t_1t_2 + \frac{t_3}{t_2} + \frac{t_3t_4}{t_2} + \frac{t_2}{t_4} + \frac{1}{t_2} + \frac{t_2}{t_3} + \frac{t_4}{t_2} + \frac{t_3}{t_1t_2} + \frac{1}{t_1t_2} + \frac{t_4}{t_1t_2} + \frac{t_2}{t_3t_4} + \frac{t_2}{t_1t_3t_4}$ as well as for the Newton polytope of $f_\Delta = \frac{t_2t_4}{t_1} + \frac{t_2}{t_1} + \frac{t_3}{t_1} + \frac{t_3t_4}{t_1} + \frac{1}{t_1} + \frac{t_4}{t_1} + \frac{t_2t_3}{t_1} + \frac{t_1}{t_4} + \frac{t_1}{t_2t_3} + \frac{t_1}{t_2t_3t_4} + \frac{t_1}{t_2t_4} + \frac{t_1}{t_3}$ and

$$\begin{aligned}&\theta^4 - z\ (137\theta^4 + 258\theta^3 + 201\theta^2 + 72\theta + 10)\\&\quad + 4\ z^2\ (387\theta^4 + 1016\theta^3 + 1151\theta^2 + 642\theta + 135)\\&\quad - 4\ z^3\ (2\theta+1)(758\theta^3 + 2137\theta^2 + 2269\theta + 820)\\&\quad + 2000\ z^4\ (\theta+1)^2(2\theta+1)(2\theta+3),\end{aligned}\tag{4.20}$$

for $f_\Delta = \frac{t_1}{t_2t_3} + \frac{t_1t_4}{t_2t_3} + \frac{t_2t_3}{t_1} + \frac{t_2t_3}{t_1t_4} + \frac{t_2t_4}{t_1} + \frac{t_2}{t_1} + \frac{t_3}{t_1t_4} + \frac{1}{t_1} + \frac{1}{t_1t_4} + \frac{t_1}{t_2} + \frac{t_1}{t_2t_4} + \frac{t_1t_4}{t_3}$.

It is, of course, an interesting question whether the symplectic Gromov–Witten invariants can give a finer classification than the diffeomorphism type. We do, however, not know a single example of such a situation. We hence expect that the Picard–Fuchs operators (4.17 and 4.18) and (4.19 and

4.20) are related by rational changes of variables that do not change the instanton numbers (cf. appendix A of [10]). This is indeed the case.

For the diffeomorphism type $\mathbf{X^{45}_{144,120}}$ the differential operator (4.17) is transformed into (4.18) by the change of variables

$$z \to \frac{z}{1+4z}, \tag{4.21}$$

and the instanton numbers are, for both cases,

$$n^{(0)} = \{3744, 50112, 1656320, 77726016, 4505800320, \\ 298578230016, 21713403010176, \ldots\}. \tag{4.22}$$

For the diffeomorphism type $\mathbf{X^{51}_{200,140}}$ the differential operator (4.19) is transformed into (4.20) by the change of variables

$$z \to \frac{z}{1-4z}, \tag{4.23}$$

and the instanton numbers are, again for both cases,

$$n^{(0)} = \{2600, 25600, 530000, 15880000, 584279000, \\ 24562482400, 1132828485400, \ldots\}. \tag{4.24}$$

It will be interesting to check whether this phenomenon continues to hold for the cases with a smaller Hodge number $h^{1,2}$ for which there are up to five different Picard–Fuchs operators for the same diffeomorphism type.

5 Outlook

We have constructed a surprisingly rich set of new Calabi–Yau manifolds using conifold transitions from toric Calabi–Yau hypersurfaces. Small resolutions dual to the flat deformations of the conifold singularities have been used to construct the mirror families and to compute quantum cohomologies via mirror symmetry.

The Picard–Fuchs operators have been determined for 28 of the 68 different diffeomorphism types of one-parameter families. In addition to the (computationally expensive) completion of this calculation it will be interesting to also enumerate the diffeomorphism types for the large number of cases with $h_{11} > 1$ and to work out respective Picard–Fuchs operators. For this, generalizations of the combinatorial formulas for the intersection rings and for integral cohomologies should to be derived. It would also be interesting to extend the calculations of higher-genus topological string of [28] to our new one-parameter families and to check integrality of BPS states as a test for our proposed mirror construction.

Since already the case of toric hypersurfaces turned out to be a rich source of new Calabi–Yau 3-folds it would also be interesting to generalize our construction to complete intersections. Such transitions, however of a different type, have already be studied in [17], where a specialization of the quintic equation and a blow-up of the resulting conifold singularity was used to arrive at the two-parameter bi-degree $(4,1)(1,1)$ complete intersection in $\mathbb{P}^4 \times \mathbb{P}^1$. In this construction the conifold transition relates complete intersections of different codimension, but one stays in the realm of toric ambient spaces.[2]

Another interesting example of codimension two has been discussed in Appendix E.2 of [10], where a toric realization of the hypergeometric function related to degrees (2,12) with weights (1,1,1,1,4,6), as derived in [29], is found along a singular one-parameter subspace of the complex structure moduli space of a toric complete intersection. It is tempting to speculate that a flat deformation of that singularity may exist, which could define a smooth Calabi–Yau family with the desired monodromy.

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