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2D and 3D topological field theories for generalized complex geometry

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Abstract

Using the Alexandrov–Kontsevich–Schwarz–Zaboronsky (AKSZ) prescription we construct 2D and 3D topological field theories associated to generalized complex manifolds. These models can be thought of as 2D and 3D generalizations of A- and B-models. Within the BV framework we show that the 3D model on a two-manifold cross an interval can be reduced to the 2D model.

1 Introduction

Recently, generalized complex geometry (GCG) has attracted considerable interest both in the physics and mathematics communities. GCG has been introduced by Hitchin [17] and further developed by Gualtieri [15] as a notion

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which unifies symplectic and complex geometries. At the same time GCG can be thought of as a complex analogue of the Dirac geometry introduced by Courant and Weinstein in [11, 12].

In this work we discuss the Batalin–Vilkovisky (BV) formulation [3] of 2D and 3D topological sigma models with target a generalized complex manifold. GCG has a simple description [14] in the language of graded manifolds. This will enable us to use the Alexandrov–Kontsevich–Schwarz–Zaboronsky (AKSZ) prescription [2] for the construction of solutions to the classical master equation. We study the relation between 3D and 2D models within the BV framework. Naturally, our results have a wider interpretation in the context of general 3D and 2D AKSZ models. This work contains only the construction of the models; issues such as gauge fixing, localization and the calculation of correlators are left for another more technical paper [9].

Let us comment on the literature and on the relations between our and others' work. Different 2D and 3D versions of topological sigma models for generalized complex structures were discussed previously within the BV formalism. To mention some, there are the 2D Zucchini model [36], the 3D Ikeda models [19, 20] and the Pestun model [27]. These models are interesting on their own. Our main intention here is to show that the powerful AKSZ framework produces the simple and unique 2D and 3D models associated to GCG. Moreover, 2D and 3D models are related to each other in a rather canonical way.

The article is organized as follows: Section 2 contains a brief review of the AKSZ construction of solutions to the classical master equation. In Section 3 we review the AKSZ models with target a symplectic graded manifold of degree 1 or 2. Section 4 recalls the description of GCG in terms of graded manifolds. This enables us to construct 2D and 3D AKSZ models. In Section 5 we discuss the relation between these models. The main idea is to use Losev's trick [24], the partial integration of a subsector of the theory. Section 6 gives a summary and provides an outlook to forthcoming work. At the end of the paper we present two technical appendices with the explicit formulas describing GCG in the language of graded manifolds.

2 The AKSZ-BV formalism

The BV formalism [3] is a powerful tool in the quantization of an action functional that is degenerate (e.g., due to gauge equivalence). This procedure embeds the space of fields into the so-called BV manifold, which is equipped with an odd symplectic structure and thereby an odd BV bracket $\{\cdot, \cdot\}$.

The original action is enlarged to a new action S satisfying the so-called master equation $\{S, S\} = 0$. One then chooses a Lagrangian submanifold inside the BV manifold; the original path integral is now replaced by the integration of S over this Lagrangian submanifold. The geometrical essence of this procedure was expounded by Schwarz [32], which reformulated the BV formalism as the "PQ-structure" on a supermanifold. The P-structure is just the symplectic structure and the Q-structure is a nilpotent vector field Q that corresponds to $\{S, \cdot\}$ in the BV case.

In this section we review the AKSZ construction [2] of solutions of the classical master equation within BV formalism. We closely follow the presentation given in [31] and use the language of graded manifolds which are sheaves of $\mathbb{Z}$-graded commutative algebras over a smooth manifold; for further details the reader may consult [34]. We consider both the real and complex cases and treat them formally on equal footing. However, in the complex case additional care is required (see [2] for further details).

The AKSZ solution of the classical master equation is defined starting from the following data:

The source: A graded manifold $\mathcal{N}$ endowed with a homological vector field D and a measure $\int_{\mathcal{N}} \mu$ of degree $-n-1$ for some positive integer n such that the measure is invariant under D.

The target: A graded symplectic manifold $(\mathcal{M}, \omega)$ with $\deg(\omega) = n$ and a homological vector field Q preserving ω. We require that Q is Hamiltonian, i.e., there exists a function Θ of degree $n+1$ such that $Q = \{\Theta, -\}$. Therefore Θ satisfies the following Maurer–Cartan equation:

$$\{\Theta, \Theta\} = 0.$$

Introduce the (infinite-dimensional) graded manifold $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ of maps from $\mathcal{N}$ to $\mathcal{M}$. Its body is the manifold of morphisms from $\mathcal{N}$ to $\mathcal{M}$ (i.e., sheaf morphisms of the sheaves describing the two graded manifolds). A soul is added to allow for morphisms parameterized by other graded manifolds: namely, $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ is uniquely characterized by the property that morphisms from $\mathcal{P} \times \mathcal{N}$ to $\mathcal{M}$ are the same as morphisms from $\mathcal{P}$ to $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ for any graded manifold $\mathcal{P}$. By abuse of language we will often speak of maps from $\mathcal{N}$ to $\mathcal{M}$ and write $\mathcal{N} \longrightarrow \mathcal{M}$ when referring to constructions involving $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$.

With our choices for $\mathcal{N}$ and $\mathcal{M}$, $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ is naturally equipped with an odd symplectic structure; moreover, D and Q can be interpreted as homological vector fields on $\mathrm{Maps}(\mathcal{N}, \mathcal{M})$ that preserve this odd symplectic structure. The AKSZ solution $S_{\rm BV}$ is the Hamiltonian for the homological vector field $D+Q$ on Maps$(\mathcal{N},\mathcal{M})$ and thus it satisfies the classical master equation automatically.

Let us provide some details for this elegant construction. We denote by Σ and M the underlying smooth manifolds to $\mathcal{N}$ and $\mathcal{M}$, respectively. We choose a set of coordinates $X^A=\{x^\mu;\psi^m\}$ on the target $\mathcal{M}$, where $\{x^\mu\}$ are the coordinates for an open $U\subset M$ and $\{\psi^m\}$ are the coordinates in the formal directions. We also choose coordinates $\{\xi^\alpha;\theta^a\}$ on the source $\mathcal{N}$, where $\{\xi^\alpha\}$ are the local coordinates on Σ and $\{\theta^a\}$ are the coordinates in the formal directions of $\mathcal{N}$. We then collect local coordinates on Maps$(\mathcal{N},\mathcal{M})$ into the superfield Φ,

$$\Phi^A=\Phi_0^A(u)+\theta^a\Phi_a^A(u)+\frac{1}{2}\theta^{a_2}\theta^{a_1}\Phi^A_{a_1a_2}(u)+\cdots, \tag{2.1}$$

where Φ_0^A, Φ_a^A, $\Phi^A_{a_1a_2}$, …(the coordinates on Maps$(\mathcal{N},\mathcal{M})$), are functions on $\Phi_0^{-1}(U)$. They are assigned a degree such that Φ^A has degree equal to the degree of X^A.

The symplectic form ω of degree n on $\mathcal{M}$ can be written in Darboux coordinates as $\omega=dX^A\omega_{\rm AB}dX^B$. Using this form we define the symplectic form of degree -1 on Maps$(\mathcal{N},\mathcal{M})$ as

$$\omega_{\rm BV}=\frac{1}{2}\int_{\mathcal{N}}\mu\quad\delta\Phi^A\,\omega_{\rm AB}\,\delta\Phi^B. \tag{2.2}$$

Thus the space of maps Maps$(\mathcal{N},\mathcal{M})$ is naturally equipped with the odd Poisson bracket $\{,\}$. Since the space Maps$(\mathcal{N},\mathcal{M})$ is infinite-dimensional we cannot define the BV Laplacian properly. We can only talk about the naive odd Laplacian. However on Maps$(\mathcal{N},\mathcal{M})$ we can discuss the solutions of the classical master equation. Assuming that ω admits a Liouville form Ξ the AKSZ action then reads

$$S_{\rm BV}[\Phi]=S_{\rm kin}[\Phi]+S_{\rm int}[\Phi]=\int_{\mathcal{N}}\mu\ \left(\Xi_A(\Phi)D\Phi^A+(-1)^{n+1}\Phi^*(\Theta)\right) \tag{2.3}$$

and it solves the classical master equation $\{S_{\rm BV},S_{\rm BV}\}=0$ with respect to the bracket defined by the symplectic structure (2.2). Since the measure μ is invariant under D, $S_{\rm kin}$ depends only on ω, not a concrete choice of Ξ. In particular, using the Darboux coordinates the first term in (2.3) can be written

$$S_{\rm kin}[\Phi] = \int_{\mathcal{N}} \mu \ \frac{1}{2}\Phi^A \omega_{\rm AB} D\Phi^B. \tag{2.4}$$

Action (2.3) is invariant under all orientation preserving diffeomorphisms of Σ and thus defines a topological field theory. The solutions of the classical field equations of (2.3) are graded differentiable maps $(\mathcal{N}, D) \to (\mathcal{M}, Q)$, i.e., maps which commute with the homological vector fields.

The standard choice for the source is the odd tangent bundle $\mathcal{N} = T[1]\Sigma_{n+1}$, for any smooth manifold Σ of dimension $n+1$, with $D = d$ the de Rham differential over Σ and the canonical coordinate measure $\mu = d^{n+1}\xi \ d^{n+1}\theta \equiv d^{n+1}z$

$$S_{\rm BV}[\Phi] = \int_{T[1]\Sigma_{n+1}} d^{n+1}z \ \big(\Xi_A(\Phi) D\Phi^A + (-1)^n \Phi^*(\Theta)\big). \tag{2.5}$$

For the rest of the paper we consider only the case when the source is $T[1]\Sigma_{n+1}$. However, the more exotic situations are possible, e.g., the holomorphic part of an odd tangent bundle, etc. see [28]. Next, we consider the case when Σ_{n+1} has a boundary. For this we need to impose certain boundary conditions within AKSZ prescription, see [7] for details. In particular, the BV classical master equation for (2.5) is only satisfied up to total derivative terms

$$\begin{aligned} \{S_{\rm BV}, S_{\rm BV}\} &= \int_{T[1]\Sigma_{n+1}} d^{n+1}z \ D(\Xi_A(\Phi) D\Phi^A + (-1)^n \Phi^*(\Theta)) \\ &= \int_{T[1]\partial\Sigma_{n+1}} d^{n}z \ (\Xi_A(\Phi) D\Phi^A + (-1)^n \Phi^*(\Theta)). \end{aligned} \tag{2.6}$$

Thus a natural choice for the boundary condition[1] is

$$\Phi : T[1]\partial\Sigma_{n+1} \to \mathcal{L} \subset \mathcal{M}, \tag{2.7}$$

where $\mathcal{L}$ is a Lagrangian submanifold of the target $\mathcal{M}$ such that

$$\Xi|_{\mathcal{L}} = 0, \quad \Theta|_{\mathcal{L}} = 0. \tag{2.8}$$

Now with these additional conditions the solution $S_{\rm BV}$ is the Hamiltonian for homological vector field $D + Q$ on $\mathrm{Maps}(T[1]\Sigma_{n+1} \to \mathcal{M}, T[1]\partial\Sigma_{n+1} \to \mathcal{L})$ and thus it satisfies automatically the classical master equation.

[1]Throughout the paper, for the sake of clarity we assume that $\partial\Sigma_{n+1}$ has a single component. The generalization beyond this case is quite obvious.

Let us make a few concluding remarks. The advantage of the AKSZ construction is that it converts complicated questions into a simple geometrical framework. For example, the analysis of the classical observables is straightforward. The homological vector field Q on $\mathcal{M}$ defines a complex on $C^\infty(\mathcal{M})$ whose cohomology we denote $H_Q(\mathcal{M})$. Take $f \in C^\infty(\mathcal{M})$ and expand $\Phi^* f$ in the formal variables on $\mathcal{N}$

$$\Phi^* f = O^{(0)}(f) + \theta^a O^{(1)}_a(f) + \frac{1}{2}\theta^{a_2}\theta^{a_1} O^{(2)}_{a_1 a_2}(f) + \cdots .$$

We denote by $\delta_{\rm BV}$ the Hamiltonian vector field for $S_{\rm BV}$, which is homological as a consequence of the classical master equation. The action of $\delta_{\rm BV}$ on $\Phi^* f$ is given by the following expression:

$$\delta_{\rm BV}(\Phi^* f) = \{S_{\rm BV}, \Phi^* f\} = D\Phi^* f + \Phi^* Q f.$$

Thus if $Qf = 0$ and μ_k is a D-invariant linear functional on the functions of $\mathcal{N}$ (e.g., a representative of an homology class of Σ), then $\mu_k(O^{(k)}(f))$ is $\delta_{\rm BV}$-closed and can serve as a classical observable. Therefore $H_Q(\mathcal{M})$ naturally defines a set of classical observables in the theory. The classical action (2.3) can be deformed to first order by

$$\int_{\mathcal{N}} \mu \; O^{(n+1)}(f)$$

with $f \in H_Q(\mathcal{M})$.

The gauge fixing in the BV framework corresponds to the choice of a Lagrangian submanifold in the space of fields. For a given Lagrangian submanifold we can choose the adapted coordinates with the odd symplectic form written as follows:

$$\omega_{\rm BV} = \int_{\mathcal{N}} \mu \; \delta\Phi^a \; \delta\Phi^+_a, \tag{2.9}$$

such that the Lagrangian is defined by the condition $\Phi^+ = 0$. We expand a master action formally into a power series Φ^+

$$S_{\rm BV}[\Phi, \Phi^+] = S_{\rm GF}(\Phi) + Q^a(\Phi)\Phi^+_a + \frac{1}{2}\sigma^{ab}(\Phi)\Phi^+_a\Phi^+_b + \cdots ,$$
$$\{S_{\rm BV}, S_{\rm BV}\} = 0 \Rightarrow Q^a \frac{\partial}{\partial\Phi^a} S_{\rm GF}(\Phi) = 0; \quad [Q, Q]^a = 2\sigma^{ba}\frac{\partial}{\partial\Phi^b} S_{\rm GF}(\Phi).$$

Hence the gauge fixed action $S_{\rm GF}(\Phi)$ has Becchi, Rouet, Stora and Tyutin (BRST) symmetry Q which is nilpotent on shell. Due to this simple observation it is very easy to analyze the BRST symmetries of the gauge fixed action.

The AKSZ prescription is algebraic in its nature and thus it can be generalized even further, see for example [5].

3 AKSZ for symplectic GrMfld of degree 1 and 2

In this section we review the relevant facts about symplectic graded manifolds (GrMfld) of degree 1 and 2 with nilpotent Hamiltonians of degree 2 and 3, respectively. The symplectic target of degree 1 with nilpotent Hamiltonian of degree 2 leads to the AKSZ construction of the Poisson sigma model [7] while the symplectic manifold of degree 2 with nilpotent Hamiltonian of degree 3 leads to the AKSZ construction of the Courant sigma model [31].

3.1 Symplectic GrMfld of degree 1 and 2

Here we review the basic facts about symplectic GrMflds of degree 1 and 2. In particular, we consider some specific examples which are relevant for our further discussion. Our review is somewhat informal and we refer the reader for further details to [29, 30].

When $\mathcal{M}$ is of degree 1, we denote the coordinates x, η with degree 0, 1. The local patches are glued through degree preserving transition functions. Degree preserving means that the transition function for the degree 1 coordinate η^A must be linear in η and the coefficient of linearity may depend on the degree zero coordinate x (since we assume that there is no negatively graded coordinate). One immediately sees that degree 1 GrMflds are exhausted by $L[1]$, where $L \to M$ is a vector bundle. A degree 1 vector field on such a manifold must have the form

$$Q = 2\eta^A A^\mu_A(x)\frac{\partial}{\partial x^\mu} - f^A_{BC}(x)\eta^B\eta^C\frac{\partial}{\partial \eta^A}. \tag{3.1}$$

Requiring $Q^2 = 0$ puts constraint on the coefficients

$$\begin{aligned} &A^\nu_{[A}\partial_\nu A^\mu_{B]} = A^\mu_C f^C_{AB}, \\ &A^\mu_A \partial_\mu f^D_{BC} + f^D_{AX} f^X_{BC} + \text{cyclic in } ABC = 0, \end{aligned} \tag{3.2}$$

where we use the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$. In fact, these data give rise to a Lie algebroid structure [33]: if one pick a basis ℓ_A for the sections of L, then we can define an anchor map $\pi : L \to TM : \pi(\ell_A) \equiv A^\mu_A \partial_\mu$, and the structure function f^A_{BC} defines the Lie bracket for the sections of L: $[\ell_B, \ell_C] = f^A_{BC}\ell_A$ satisfying the extra condition $[\ell_A, f\ell_B] = f[\ell_A, \ell_B] + (\pi(\ell_A)f)\ell_B$. The second of the equations (3.2) is the condition for the Jacobi identity for the Lie bracket, while the first says that the anchor π is a homomorphism between the Lie bracket of L and the Lie bracket of TM. It is also easy to see that Q acts on the functions $f(x, \eta)$ as the Lie algebroid differential d_L on $\Gamma(\wedge^\bullet L^*)$. If one is further restricted to symplectic degree 1 GrMflds, then one can utilize the symplectic structure to identify the degree 1 coordinate η with the fiber coordinate of T^*M. In other words degree 1 symplectic manifolds are exhausted by $T^*[1]M$ with symplectic structure

$$\omega = d\eta_\mu dx^\mu,$$

where x^μ is coordinate of degree 0 on M and η_ν is the fiber coordinate of degree 1. The Hamiltonian of degree 2 is given by the following expression:

$$\Theta = \alpha^{\mu\nu}(x)\eta_\mu\eta_\nu, \tag{3.3}$$

where $\alpha = \alpha^{\mu\nu}\partial_\mu \wedge \partial_\nu$ is bivector on M. $\{\Theta, \Theta\} = 0$ if and only if α is Poisson structure. The homological vector field on $T^*[1]M$ is

$$Q = 2\alpha^{\mu\nu}\eta_\nu \frac{\partial}{\partial x^\mu} + \partial_\mu \alpha^{\nu\rho}\eta_\nu\eta_\rho \frac{\partial}{\partial \eta_\mu}, \tag{3.4}$$

which gives rise to the Lie algebroid structure on T^*M associated to a Poisson structure on M.

Example 1 (Lie algebroid)**.** Consider L being a vector bundle over M with the Lie algebroid structure described above. Then the dual bundle L^* considered as a total manifold is equipped with a Poisson structure

$$\alpha(x, \lambda) = f^C_{AB}(x)\lambda_C \; \frac{\partial}{\partial \lambda_A} \wedge \frac{\partial}{\partial \lambda_B} + 2A^\mu_A(x) \; \frac{\partial}{\partial \lambda_A} \wedge \frac{\partial}{\partial x^\mu}, \tag{3.5}$$

where the x's are coordinates on M and the λ's are coordinates on the fiber of L^*. Both x and λ are of degree zero. The corresponding symplectic manifold of degree 1 is $T^*[1]L^*$ with the symplectic structure of degree 1

$$\omega = d\eta_\mu dx^\mu + dj^A d\lambda_A,$$

where η, j are the coordinates of degree 1. From (3.5) it follows that the nilpotent Hamiltonian of degree 2 is given

$$\Theta = f_{AB}^C(x)\lambda_C j^A j^B + 2A_A^\mu(x) j^A \eta_\mu. \tag{3.6}$$

Thus, Lie algebroid structure on L can be encoded in terms of $(T^*[1]L^*, \omega, \Theta)$.

Now let us discuss the graded symplectic manifolds of degree 2. The symplectic (nonnegatively) graded symplectic manifold $\mathcal{M}$ of degree 2 corresponds to vector bundle E over M with a fiberwise nondegenerate symmetric inner product $\langle\ ,\ \rangle$ (it can be of arbitrary signature). For a given E, $\mathcal{M}$ is a symplectic submanifold of $T^*[2]E[1]$ corresponding to the isometric embedding $E \hookrightarrow E \oplus E^*$ with respect to the canonical pairing on $E \oplus E^*$, i.e., $e^a \to (e^a, g_{ab}e^b)$, where g_{ab} is the constant fiber metric $\langle\ ,\ \rangle$ written in a local basis of sections for E. Indeed $\mathcal{M}$ is a minimal symplectic realization of $E[1]$. In local Darboux coordinates (x^μ, p_μ, e^a) of degree 0, 2 and 1, respectively, the symplectic structure is

$$\omega = dp_\mu dx^\mu + \frac{1}{2} de^a g_{ab} de^b. \tag{3.7}$$

Any degree 3 function would have the following general form:

$$\Theta = p_\mu A_a^\mu(x) e^a + \frac{1}{6} f_{abc}(x) e^a e^b e^c. \tag{3.8}$$

As it has been shown in [30] the solutions of the equation $\{\Theta, \Theta\} = 0$ correspond to Courant algebroid structures on $(E, \langle\ ,\ \rangle)$ with the Courant–Dorfman bracket given by $[\cdot,\cdot] = \{\{\cdot, \Theta\}, \cdot\}$, where $\{\ ,\ \}$ stands for the Poisson bracket on the symplectic manifold $\mathcal{M}$. In expression (3.8) the quantity A_a^μ and $-g^{da} f_{dbc}$ are interpreted as the anchor and the structure function for Courant algebroid E, respectively. We refer the reader to [30] for detailed discussion of degree 2 symplectic GrMflds and its relation to Courant algebroids.

Example 2 ($TM \oplus T^*M$ Courant algebroid)**.** The standard example of a Courant algebroid is the tangent plus cotangent bundle $TM \oplus T^*M$ of a smooth manifold M. In this case the corresponding symplectic manifold of degree 2 is $\mathcal{M} = T^*[2]T^*[1]M$. The degree 0,1 subspace in this case is $T^*[1]M \oplus T[1]M$. Pick local coordinates $p_\mu, v^\mu, q_\mu, x^\mu$ with degree 2,1,1 and

0 with the metric induced from the natural pairing between TM and T^*M. The Hamiltonian function of degree 3 is $\Theta = P_\mu v^\mu$ which induces a vector field corresponding to the Hamiltonian lift of the de Rham differential

$$Q = \{\Theta, \cdot\} = v^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial q_\mu}.$$

If there is a closed three form H, then there exists another Hamiltonian function of degree 3

$$\Theta = p_\mu v^\mu + \frac{1}{6} H_{\mu\nu\rho} v^\mu v^\nu v^\rho, \tag{3.9}$$

which gives rise to the twisted Courant structure on $TM \oplus T^*M$.

Example 3 (Lie bialgebroid)**.** Consider a Lie algebroid L and assume that the dual bundle L^* is equipped with a Lie algebroid structure. The pair (L, L^*) is called bialgebroid if d_L is a derivation of the Schouten bracket on $\Gamma(\wedge^\bullet L^*)$. For any bialgebroid (L, L^*) the vector bundle $E = L \oplus L^*$ is naturally equipped with the structure of Courant algebroid [23]. Thus we can apply the previous considerations. The graded manifold $T^*[2]L[1]$ is equipped with the symplectic structure of degree 2

$$\omega = dp_\mu dx^\mu + d\ell_A d\ell^A, \tag{3.10}$$

where ℓ^A are fiber coordinates[2] of degree 1 for L and ℓ_A are fiber coordinates of degree 1 for L^*. The Hamiltonian of degree 3 has the same form as in (3.8), but written in the basis adapted to the bialgebroid splitting $E = L \oplus L^*$.

Example 4 (Lie algebroid)**.** Take a Lie algebroid L, then the vector bundle $E = L \oplus L^*$ can be regarded as bialgebroid with the trivial bracket and zero anchor on L^*. Thus $E = L \oplus L^*$ is equipped with Courant algebroid structure. The corresponding graded symplectic manifold is $T^*[2]L[1]$ with the symplectic structure (3.10). The corresponding Hamiltonian of degree 3 is

$$\Theta = 2p_\mu A^\mu_A(x)\ell^A - f^A_{BC}(x)\ell_A \ell^B \ell^C.$$

The vector field $Q = \{\Theta, \cdot\}$ acts as the Lie algebroid differential on functions $f(x, \ell^A)$. In general, it has an interpretation related to the adjoint representation of L [1].

[2]Through the paper we adapt the same notation for the fiber coordinates of a vector bundle and the sections of the dual bundle.

3.2 2D AKSZ model

We may apply the AKSZ approach to the 2D case when the source manifold $\mathcal{N} = T[1]\Sigma_2$ with Σ_2 being a 2D manifold. The target $\mathcal{M} = T^*[1]M$ is a symplectic manifold of degree 1 equipped with the Hamiltonian (3.3). The space of fields defined as

$$\mathrm{Maps}(T[1]\Sigma_2, T^*[1]M)$$

with odd symplectic structure

$$\omega_{\mathrm{BV}} = \int_{T[1]\Sigma_2} d^2\xi d^2\theta\ \delta\boldsymbol{\eta}_\mu \delta\mathbf{X}^\mu. \tag{3.11}$$

The corresponding BV action is written as

$$S_{\mathrm{BV}} = \int_{T[1]\Sigma_2} d^2\xi d^2\theta \left(\boldsymbol{\eta}_\mu D\mathbf{X}^\mu + \alpha^{\mu\nu}(\mathbf{X})\boldsymbol{\eta}_\mu\boldsymbol{\eta}_\nu\right), \tag{3.12}$$

where we use bold letters for the superfields corresponding to the coordinates on $T^*[1]M$, η and x. This action is the BV formulation of the Poisson sigma model corresponding to the Poisson manifold (M, α). Action (3.12) is a solution of a classical master equation if $\partial\Sigma_2 = \emptyset$, [7]. If $\partial\Sigma_2 \neq \emptyset$ then the following boundary conditions can be imposed:

$$T[1]\partial\Sigma_2 \longrightarrow N^*[1]C,$$

where C is a coisotropic submanifold of M. With these boundary conditions the requirements (2.7) and (2.8) are satisfied [8].

In particular, we are interested in the situation when the Poisson manifold is the dual bundle of a Lie algebroid L, see Example 1. In this case the space of fields is

$$T[1]\Sigma_2 \longrightarrow T^*[1]L^*$$

with the odd symplectic structure

$$\omega_{\mathrm{BV}} = \int_{T[1]\Sigma_2} d^2\xi d^2\theta \left(\delta\boldsymbol{\eta}_\mu \delta\mathbf{X}^\mu + \delta\mathbf{j}^A \delta\boldsymbol{\lambda}_A\right). \tag{3.13}$$

The corresponding BV action is

$$S_{\mathrm{BV}} = \int_{T[1]\Sigma_2} d^2\xi d^2\theta (\boldsymbol{\eta}_\mu D\mathbf{X}^\mu + \mathbf{j}^A D\boldsymbol{\lambda}_A + f_{AB}^C(\mathbf{X})\boldsymbol{\lambda}_C \mathbf{j}^A \mathbf{j}^B + 2A_A^\mu(\mathbf{X})\mathbf{j}^A \boldsymbol{\eta}_\mu), \tag{3.14}$$

where we use the obvious correspondence between superfields and the coordinates on $T^*[1]L^*$. Action (3.14) satisfies the classical master equation if $\partial\Sigma_2 = \emptyset$. If $\partial\Sigma_2 \neq \emptyset$ then the following boundary conditions can be imposed:

$$T[1]\partial\Sigma_2 \longrightarrow N^*[1]K^\perp,$$

where K is a subalgebroid of L and $K^\perp \subset K^*$ is the annihilator of K:

$$K_x^\perp = \{\alpha \in K_x^* : \alpha(v) = 0 \ \forall v \in K_x\}.$$

Let us remind that a Lie subalgebroid K of L is a morphism of Lie algebroids $F : K \to L$, $f : C \to M$. such that F and f are injective immersions. It is easy to see that for boundary conditions labeled by subalgebroid of L the conditions (2.7) and (2.8) are satisfied, see [4] for similar analysis.

3.3 3D AKSZ model

Having the general description of a graded symplectic manifold of degree 2 and the Hamiltonian function Θ of degree 3 we can write the AKSZ action for a Courant algebroid E. The space of fields is defined as

$$T[1]\Sigma_3 \longrightarrow \mathcal{M},$$

where $\mathcal{M}$ is a symplectic submanifold of $T^*[2]E[1]$ which provides a minimal symplectic realization of $E[1]$. The odd symplectic structure on the space of maps is

$$\omega_{\mathrm{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \left(\delta\mathbf{P}_\mu \delta\mathbf{X}^\mu + \frac{1}{2}\delta\mathbf{e}^a g_{ab}\delta\mathbf{e}^b\right). \tag{3.15}$$

The BV action is

$$S_{\mathrm{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \left(\mathbf{P}_\mu D\mathbf{X}^\mu + \frac{1}{2}\mathbf{e}^a g_{ab} D\mathbf{e}^b - \mathbf{P}_\mu A_a^\mu(\mathbf{X})\mathbf{e}^a - \frac{1}{6}f_{abc}(\mathbf{X})\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c\right), \tag{3.16}$$

where we identify the superfields and the coordinates on $\mathcal{M}$ in an obvious way. Action (3.16) satisfies the classical master equation if $\partial\Sigma_3 = \emptyset$. If $\partial\Sigma_3 \neq \emptyset$ then the additional boundary conditions should be imposed

$$T[1]\partial\Sigma_3 \;\longrightarrow\; \mathcal{L},$$

where $\mathcal{L}$ is a submanifold of $N^*[2]K[1]$ corresponding to the isometric embedding $E \hookrightarrow E \oplus E^*$ (see the previous discussion) and K is a Dirac structure supported on a submanifold C [6]. A Dirac structure supported on a submanifold $i\colon C \hookrightarrow M$ is defined as a subbundle $K \subset i^*E = E|_C$ such that $K_x \subset E_x$ is maximally isotropic for all $x \in C$, K is compatible with the anchor (i.e., $A(K) \subset TC$) and $[e_1, e_2]|_C \in \Gamma(K)$ for any sections e_1, e_2 of E such that $e_1|_C, e_2|_C \in \Gamma(K)$.

Let us illustrate the general construction with a few concrete examples. We start with the Courant algebroid structure over $TM \oplus T^*M$. The space of fields is described as follows:

$$T[1]\Sigma_3 \;\longrightarrow\; T^*[2]T^*[1]M$$

and the BV action is

$$S_{\mathrm{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \left(\mathbf{P}_\mu D\mathbf{X}^\mu + \frac{1}{2}\mathbf{v}^\mu D\mathbf{q}_\mu + \frac{1}{2}\mathbf{v}^\mu D\mathbf{q}_\mu \right.$$
$$\left. -\mathbf{P}_\mu \mathbf{v}^\mu - \frac{1}{6} H_{\mu\nu\rho}(\mathbf{X})\mathbf{v}^\mu\mathbf{v}^\nu\mathbf{v}^\rho\right), \tag{3.17}$$

where we use the notations adapted to Example 2. If $\partial\Sigma_3 \neq \emptyset$ then the possible boundary conditions would be

$$T[1]\partial\Sigma_3 \;\longrightarrow\; N^*[2]T[1]C,$$

where C is a submanifold and $H|_C = 0$. In this case $K = N^*C \oplus TC$ is an example of Dirac structure supported on C. There is another way to construct a Dirac structure supported on C. Let us choose a two form on C, $B \in \Omega^2(C)$. Then applying the B-transform to $N^*C \oplus TC$ we obtain another bundle $e^B(N^*C \oplus TC)$ over C. It is easy to show that this gives rise to a Dirac structure with support over C if $H|_C = dB$. The pair (C, B) with the condition $H|_C = dB$ has been discussed by Gualtieri [15, 16], under the name of generalized submanifold. Using the local coordinates from Example 2 adapted to submanifold C we have the following description of the

Lagrangian submanifold $\mathcal{L}$ in $T^*[2]T^*[1]M$:

$$x^n = 0, \quad v^n = 0, \quad q_i = B_{ij}(x)v^j, \quad p_i = -\frac{1}{2}\partial_i B_{jk} v^j v^k,$$

where n stands for the normal directions and i, j, k for the tangential directions for C. One can easily check that the Liouville form $\Xi = p_\mu dx^\mu + v^\mu dq_\mu$ and the Hamiltonian Θ (3.9) vanish when restricted to $\mathcal{L}$ provided that $H|_C = dB$. Thus the appropriate boundary condition for the BV model would be

$$T[1]\partial\Sigma_3 \longrightarrow \mathcal{L},$$

where $\mathcal{L}$ corresponds to a pair (C, B) in a way described above.

Now let us discuss the BV theory corresponding to Example 4. Again this is just a special case of Courant sigma model. Consider the space of fields described as

$$T[1]\Sigma_3 \longrightarrow T^*[2]L[1],$$

with the odd symplectic structure

$$\omega_{\text{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \left(\delta \mathbf{P}_\mu \delta \mathbf{X}^\mu + \delta \boldsymbol{\ell}_A \delta \boldsymbol{\ell}^A\right) \tag{3.18}$$

and the BV action given by the following expression:

$$S_{\text{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \bigg(\mathbf{P}_\mu D\mathbf{X}^\mu + \frac{1}{2}\boldsymbol{\ell}_A D\boldsymbol{\ell}^A + \frac{1}{2}\boldsymbol{\ell}^A D\boldsymbol{\ell}_A \\ - 2\mathbf{P}_\mu A^\mu_A(\mathbf{X})\boldsymbol{\ell}^A + f^A_{BC}(\mathbf{X})\boldsymbol{\ell}_A\boldsymbol{\ell}^B\boldsymbol{\ell}^C\bigg), \tag{3.19}$$

where our notations are adopted to Example 4. If $\partial\Sigma_3 \neq \emptyset$ then the following boundary conditions should be imposed

$$T[1]\partial\Sigma_3 \longrightarrow N^*[2]K[1],$$

where K is a subalgebroid of L. It is straightforward to see that conditions (2.7) and (2.8) are satisfied for this choice.

4 AKSZ for generalized complex manifolds (GCM)

In this section we discuss the description of GCG in terms of graded symplectic manifolds. We apply this to the construction of the AKSZ action in 2D and 3D cases.

4.1 Graded geometry for GCM

Consider the Courant algebroid E and associated to it the symplectic graded manifold $\mathcal{M}$ of degree 2. The Courant structure on E is defined through the Hamiltonian function of degree 3

$$\mathcal{S} = p_\mu A^\mu_a(x) e^a + \frac{1}{6} f_{abc}(x) e^a e^b e^c, \tag{4.1}$$

where from now on we use $\mathcal{S}$ to denote this concrete Hamiltonian. Consider the following function of degree 2 independent from p:

$$J = \frac{1}{2} J_{ab}(x) e^a e^b,$$

where by construction $J_{ab} = -J_{ba}$. In [14] it has been observed that the function $\mathcal{S}$ and J satisfy the relation

$$\{J, \{J, \mathcal{S}\}\} = -\mathcal{S} \tag{4.2}$$

if and only if $J^a_{\ b} = g^{ac} J_{cb}$ defines the splitting of $E \otimes \mathbb{C} = L \oplus \bar{L}$ where L is a maximally isotropic subbundle closed under the Courant bracket. One of the condition which follows from (4.2) is $J^a_{\ b} J^b_{\ c} = -\delta^a_b$ and thus the subbundle L is defined as +i eigenbundle of $J^a_{\ b}$, thus $L^* = \bar{L}$. Although the interpretation of (4.2) has been presented in [14] we prefer to give the details in Appendix A. We find a number of useful formulas while investigating this relation. If we choose the Courant algebroid $E = T^*M \oplus TM$, then the splitting $(TM \oplus T^*M) \otimes \mathbb{C} = L \oplus \bar{L}$ with L being a maximally isotropic involutive subbundle defines a generalized complex structure (GCS) [15, 16]. The Courant bracket restricted to L becomes a Lie bracket $[\cdot, \cdot]$ and thus L is a complex Lie algebroid.

Next we observe that, if on $\mathcal{M}$ we have the functions $\mathcal{S}$ and J with property (4.2), then we can construct the function of degree 3

$$\Theta_{(\alpha,\beta)} = \alpha \mathcal{S} + \beta \{J, \mathcal{S}\} \tag{4.3}$$

which satisfies $\{\Theta, \Theta\} = 0$ for arbitrary constants α and β. If α and β are real numbers, then there exists a symplectomorphism on $\mathcal{M}$ which connects

$(\alpha^2+\beta^2)^{1/2}\mathcal{S}$ with $\Theta_{(\alpha,\beta)}$. Namely, J gives rise to the flow

$$\partial_t\Theta(t) = \{J, \Theta(t)\},$$

which has the following explicit solution:

$$\Theta(t) = (\alpha^2+\beta^2)^{1/2}\left(\cos t\ \mathcal{S} + \sin t\ \{J,\mathcal{S}\}\right).$$

At $t=0$, $\Theta(t)$ corresponds to $(\alpha^2+\beta^2)^{1/2}\mathcal{S}$. On the other hand if we choose t to be such that $\cos t = \alpha(\alpha^2+\beta^2)^{-1/2}$ and $\sin t = \beta(\alpha^2+\beta^2)^{-1/2}$ then $\Theta(t)$ coincides with (4.3). Therefore, we do not get a new Hamiltonian of degree 3 if we deal with the real coefficients in (4.3). Indeed any function of degree 2 generates a symplectomorphism of $\mathcal{M}$ [30] and our particular function J realizes the $U(1)$ action on $\mathcal{M}$.

To get something nontrivial we have to complexify our graded symplectic manifold $\mathcal{M}$ and allow complex coefficients in (4.3). One can be easily convinced that, up to equivalence, the only nontrivial complex nilpotent Hamiltonians are

$$\Theta = \mathcal{S} + \mathrm{i}\{J,\mathcal{S}\} \tag{4.4}$$

and its complex conjugate. All other complex combinations of $\mathcal{S}$ and $\{J,\mathcal{S}\}$ do not give rise to anything new (it can be seen through the appropriate redefinitions). It is natural to choose the coordinates adapted to the splitting $E = L\oplus\bar{L}$. The manifold $\mathcal{M}$ with the symplectic structure (3.7) is symplectomorphic to $T^*[2]\bar{L}[1]$ with the symplectic structure

$$\omega = d\tilde{p}_\mu dx^\mu + d\bar{\ell}_A d\ell^A,$$

where $\bar{\ell}_A$ are odd coordinate along a fiber of $\bar{L}$ and ℓ^A along a fiber of L. Moreover the Hamiltonian (4.4) written in new coordinates becomes

$$\Theta = 2\tilde{p}_\mu A^\mu_A(x)\ell^A - f^A_{BC}(x)\bar{\ell}_A\ell^B\ell^C. \tag{4.5}$$

The proof of these statements and further technical details are presented in Appendix B. In the new coordinates the function J looks particular simple

$J = \mathrm{i}\bar{\ell}_A \ell^A$. The manifold $T^*[2]\bar{L}[1]$ has two natural gradings described by

$$\epsilon_1 = \tilde{p}_\mu \frac{\partial}{\partial \tilde{p}_\mu} + \bar{\ell}_A \frac{\partial}{\partial \bar{\ell}_A}, \tag{4.6}$$

$$\epsilon_2 = \tilde{p}_\mu \frac{\partial}{\partial \tilde{p}_\mu} + \ell^A \frac{\partial}{\partial \ell^A}, \tag{4.7}$$

where $\epsilon_1 + \epsilon_2$ corresponds to the original grading and $\epsilon_1 - \epsilon_2$ is generated by $\mathrm{i}J$. Thus the homological vector field for the Hamiltonian (4.4) comes from the splitting of $\{\mathcal{S}, \cdot\}$ according to the grading defined by $\epsilon_1 - \epsilon_2$.

Here we have discussed the complex case. However if in relation (4.2) the sign minus on the right-hand side is replaced by plus then this corresponds to a real bialgebroid $E = L \oplus L^*$. A similar discussion with a few minor changes can be repeated for this case.

4.2 AKSZ for 3D σ-model on GCM

Using the discussion from the previous subsection, it is straightforward to construct the appropriate BV master action. Starting from the manifold $\mathcal{M}$ we define the space of maps as

$$T[1]\Sigma_3 \longrightarrow \mathcal{M}.$$

Using the complex Hamiltonian (4.4) on $\mathcal{M}$, we construct the master action

$$\begin{aligned} S_{\mathrm{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \bigg(&\mathbf{P}_\mu D\mathbf{X}^\mu + \frac{1}{2}\mathbf{e}^a g_{ab} D\mathbf{e}^b \\ &-(\delta^a_b - \mathrm{i}J^a_{\ b}(\mathbf{X}))\mathbf{e}^b A^\mu_a(\mathbf{X})\mathbf{P}_\mu - \frac{1}{6} f_{abc}(\mathbf{X})\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \\ &+\frac{\mathrm{i}}{2} J^d_{\ a}(\mathbf{X}) f_{dbc}(\mathbf{X})\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c + \frac{\mathrm{i}}{2} A^\mu_c(\mathbf{X})\partial_\mu J_{ab}(\mathbf{X})\mathbf{e}^a\mathbf{e}^b\mathbf{e}^c \bigg), \end{aligned} \tag{4.8}$$

where we use the notations adapted to our discussion of geometry of $\mathcal{M}$. The corresponding odd symplectic structure is (3.15). Action (4.8) satisfies the classical master equation if $\partial\Sigma_3 = \emptyset$. If $\partial\Sigma_3 \neq \emptyset$ then we have to impose the additional boundary conditions on the fields. Recall from Section 3.3 that the boundary conditions for the Courant sigma model are specified by the Dirac structure K supported on a submanifold C; K gives rise to a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{M}$ and $\mathcal{S}|_{\mathcal{L}} = 0$. Now we have to see when $\{J, \mathcal{S}\}|_{\mathcal{L}} = 0$. The simplest way to get it is to require that $J|_{\mathcal{L}} = 0$. This follows from a simple property of symplectic geometry: if two functions vanish on a given Lagrangian then their bracket also vanishes on this Lagrangian.[3] The simplest way to achieve this is to require that $K_x \subset E_x$ is preserved under action of $J^a_{\ b}$ for all $x \in C$. Since K_x is maximally isotropic it would imply that $J|_{\mathcal{L}} = 0$. To summarize, boundary conditions for action (4.8) are labeled by Dirac structures K supported on C which are invariant under of the action of $J^a_{\ b}$.

As an illustration let us consider $E = TM \oplus T^*M$. In this case a solution of equation (4.2) gives rise a the GCS. Action (4.8) can be easily rewritten for this case. As we discussed in Section 3.3 for a submanifold C and two form $B \in \Omega^2(C)$, there exists a Dirac structure supported over C which we denoted $K = e^B(N^*C \oplus TC)$. As discussed above $e^B(N^*C \oplus TC)$ gives rise to the correct boundary condition if we require that it is invariant under the action of the GCS. This corresponds exactly to the definition of generalized complex submanifold suggested by Gualtieri [15]. Thus the boundary conditions for 3D AKSZ model are labeled by generalized complex submanifolds.

The manifold $\mathcal{M}$ is symplectomorphic to $T^*[2]\bar{L}[1]$. This induces a symplectomorphism at the level of fields. Namely, the symplectic structure (3.15) can be mapped to

$$\omega_{\mathrm{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \left(\delta\tilde{\mathbf{P}}_\mu \delta\mathbf{X}^\mu + \delta\bar{\boldsymbol{\ell}}_A \delta\boldsymbol{\ell}^A\right), \tag{4.9}$$

which is defined over the space of maps

$$T[1]\Sigma_3 \longrightarrow T^*[2]\bar{L}[1].$$

The explicit formulas for the redefinitions of fields can be obtained from those given in Appendix B. Moreover by using the explicit manipulations in the appendix, action (4.8) is recast into the following:

$$\begin{aligned} S_{\mathrm{BV}} = \int_{T[1]\Sigma_3} d^3\xi d^3\theta \Big(&\tilde{\mathbf{P}}_\mu D\mathbf{X}^\mu + \frac{1}{2}\bar{\boldsymbol{\ell}}_A D\boldsymbol{\ell}^A + \frac{1}{2}\boldsymbol{\ell}^A D\bar{\boldsymbol{\ell}}_A \\ &-2\tilde{\mathbf{P}}_\mu A^\mu_A(\mathbf{X})\boldsymbol{\ell}^A + f^A_{BC}(\mathbf{X})\bar{\boldsymbol{\ell}}_A\boldsymbol{\ell}^B\boldsymbol{\ell}^C\Big), \end{aligned} \tag{4.10}$$

where we kept all boundary terms arising in the redefinition. This is the complex version of 3D AKSZ theory corresponding to the Lie algebroid. In this formulation we analyze the boundary conditions as at the end of Section 3.3. Moreover, this discussion will be naturally compatible with the way we

[3]The simplest way to prove it is to perform a calculation of a bracket in the coordinates adapted to a Lagrangian submanifold.

described boundary conditions as Dirac structure supported over C which are invariant under $J^a_{\ b}$.

4.3 AKSZ for 2D σ-model on GCM

Since the GCS gives rise to a complex Lie algebroid L we can apply the construction from Section 4.1. The space of fields is defined as

$$T[1]\Sigma_2 \longrightarrow T^*[1]\bar{L},$$

where we regard L as a formal complex Poisson manifold. The BV action is

$$\begin{aligned} S_{\mathrm{BV}} = \int_{T[1]\Sigma_2} d^2\xi d^2\theta (&\boldsymbol{\eta}_\mu D\mathbf{X}^\mu + \mathbf{j}^A D\boldsymbol{\lambda}_A \\ &+ f_{AB}^C(\mathbf{X})\boldsymbol{\lambda}_C \mathbf{j}^A \mathbf{j}^B + 2A_A^\mu(\mathbf{X})\mathbf{j}^A \boldsymbol{\eta}_\mu), \end{aligned} \tag{4.11}$$

with the anchor and structure constants for L (see appendices for the explicit expressions). Obviously, this model can be written for the case of $TM \oplus T^*M$. The boundary conditions in this model corresponds to Lie subalgebroids of L. If we consider the case of GCS then the generalized complex submanifold C gives rise to a Lie algebroid over C, which can be interpreted as a Lie subalgebroid of L; see [16] for the details. Thus, generalized complex submanifolds give rise to the correct boundary conditions for this model.

5 Reduction from 3D to 2D

In this section we discuss the relation between the 3D and 2D models introduced above. The consistent reduction is done through the following observation.

5.1 Separation of ultraviolet (UV) and infrared (IR) (Losev's trick)

Losev [24] suggested a framework for dealing with effective theories within the BV framework. This idea was further developed and used in [10, 25, 26]. Here we apply the idea of effective theory to the dimensional reduction of 3D AKSZ theory down to 2D AKSZ theory. We believe that this is the correct conceptual framework for the discussion of the dimensional reduction within BV formalism.

The idea is essentially very simple; let us sketch it. Assume that the BV manifold is of a product structure $V = V_{\mathrm{UV}} \times V_{\mathrm{IR}}$, and the odd Laplacian is also decomposed $\Delta = \Delta_{\mathrm{UV}} + \Delta_{\mathrm{IR}}$. Let S_{BV} be a BV action satisfying the quantum master equation $\Delta \mathrm{e}^{-S} = 0$ on V. We shall refer to V_{UV} as UV degrees of freedom and to V_{IR} as IR degrees of freedom. We can "integrate out" the UV degrees of freedom and get an "effective action" just as one would do for a normal quantum field theory. More concretely, pick a Lagrangian submanifold $\mathcal{L} \hookrightarrow V_{\mathrm{UV}}$ and define the effective action on V_{IR} as

$$\mathrm{e}^{-S_{\mathrm{eff}}} = \int_{\mathcal{L}} \mathrm{e}^{-S_{\mathrm{BV}}}. \tag{5.1}$$

One can check that S_{eff} satisfies the quantum master equation on V_{IR}

$$\Delta_{\mathrm{IR}} \mathrm{e}^{-S_{\mathrm{eff}}} = \int_{\mathcal{L}} \Delta_{\mathrm{IR}} \mathrm{e}^{-S} = \int_{\mathcal{L}} \Delta \mathrm{e}^{-S} - \int_{\mathcal{L}} \Delta_{\mathrm{UV}} \mathrm{e}^{-S} = 0, \tag{5.2}$$

where the first term vanishes due to the master equation and the second term due to the integration over a Lagrangian submanifold of a Δ_{UV}-exact term. Furthermore, assume that two choices of Lagrangian submanifolds $\mathcal{L} \hookrightarrow V_{\mathrm{UV}}$ and $\mathcal{L}' \hookrightarrow V_{\mathrm{UV}}$ are related by a gauge fixing fermion Ψ such that $\Delta\Psi = 0$, then

$$\begin{aligned} \int_{\mathcal{L}'} \mathrm{e}^{-S} - \int_{\mathcal{L}} \mathrm{e}^{-S} = \int_{\mathcal{L}} \{\Psi, \mathrm{e}^{-S}\} = \int_{\mathcal{L}} -\Delta(\Psi \mathrm{e}^{-S}) - \Delta(\Psi)\mathrm{e}^{-S} \\ + \Psi\Delta(\mathrm{e}^{-S}) = -\Delta_{\mathrm{IR}} \int_{\mathcal{L}} \Psi \mathrm{e}^{-S}. \end{aligned}$$

Thus the change of the gauge fixing in UV-sector leads to change in $\mathrm{e}^{-S_{\mathrm{eff}}}$ up to Δ_{IR}-exact term. These manipulations are well defined if the BV manifold is finite dimensional. For the infinite-dimensional manifold this construction is formal. For further details of construction the reader may consult [26].

Using this trick, we start from some solution to the classical master equation and integrate out certain degrees of freedom. The remaining effective action will also satisfy the classical master equation.

5.2 3D AKSZ model on $\Sigma_3 = \Sigma_2 \times I$

Using the idea of partial gauge fixing and integration in the UV-sector we will show that the 3D theory on $\Sigma_2 \times I$ corresponding to the Lie algebroid defined by (3.18) and (3.19) is equivalent to the 2D theory on Σ_2 corresponding to the same Lie algebroid defined by (3.13) and (3.14). This works provided the following boundary conditions:

$$T[1]\partial\Sigma_3 \longrightarrow L[1] \tag{5.3}$$

are imposed on the 3D theory.

Let us present the details of the derivation. We take the 3D source to be $T[1]\Sigma_3 = T[1](\Sigma_2 \times I)$, and name the even and odd coordinates along the interval I as (θ^t, t). Expand all superfields according to $\Phi(t, \theta^t) = \Phi(t) + \theta^t \Phi_t(t)$ and perform explicitly the θ_t-integration. The odd symplectic structure (3.18) becomes

$$\omega_{\mathrm{BV}} = -\int d^2\xi d^2\theta dt(-\delta\mathbf{P}_\mu \delta\mathbf{X}_t + \delta\boldsymbol{\ell}_A \delta\boldsymbol{\ell}_t^A + \delta\mathbf{P}_{t\mu}\delta\mathbf{X}^\mu + \delta\boldsymbol{\ell}_{tA}\delta\boldsymbol{\ell}^A) \tag{5.4}$$

and the BV action (3.19) is

$$\begin{aligned} S_{\mathrm{BV}} = \int d^2\xi d^2\theta dt(&\mathbf{P}_\mu \partial_t \mathbf{X}^\mu + \mathbf{P}_{t\mu}\mathrm{D}\mathbf{X}^\mu - \mathbf{P}_\mu \mathrm{D}\mathbf{X}_t^\mu - \frac{1}{2}\boldsymbol{\ell}_A \partial_t \boldsymbol{\ell}^A \\ &+ \frac{1}{2}\boldsymbol{\ell}_{tA}\mathrm{D}\boldsymbol{\ell}^A - \frac{1}{2}\boldsymbol{\ell}^A \partial_t \boldsymbol{\ell}_A + \frac{1}{2}\boldsymbol{\ell}_t^A \mathrm{D}\boldsymbol{\ell}_A + \frac{1}{2}\boldsymbol{\ell}_A \mathrm{D}\boldsymbol{\ell}_t^A + \frac{1}{2}\boldsymbol{\ell}^A \mathrm{D}\boldsymbol{\ell}_{tA} \\ &- 2\mathbf{P}_{t\mu}A_A^\mu(\mathbf{X})\boldsymbol{\ell}^A - 2\mathbf{P}_\mu(\partial_\nu A_A^\mu(\mathbf{X}))\mathbf{X}_t^\nu \boldsymbol{\ell}^A - 2\mathbf{P}_\mu A_A^\mu(\mathbf{X})\boldsymbol{\ell}_t^A \\ &+ \partial_\mu(f_{BC}^A(\mathbf{X}))\mathbf{X}_t^\mu \boldsymbol{\ell}_A \boldsymbol{\ell}^B \boldsymbol{\ell}^C + f_{BC}^A(\mathbf{X})\boldsymbol{\ell}_{tA}\boldsymbol{\ell}^B\boldsymbol{\ell}^C + 2f_{BC}^A(\mathbf{X})\boldsymbol{\ell}_A \boldsymbol{\ell}^B \boldsymbol{\ell}_t^C), \end{aligned} \tag{5.5}$$

where now D stands for the de Rham along $T[1]\Sigma_2$. Since the symplectic structure (5.4) decomposes in two separate pieces we can choose the UV sector to correspond to $(\mathbf{X}_t^\mu, \mathbf{P}_\mu, \boldsymbol{\ell}_t^A, \boldsymbol{\ell}_A)$. Next we choose the Lagrangian $\mathcal{L}$ in the UV sector as follows: $\mathbf{X}_t^\mu = 0$, $\boldsymbol{\ell}_t^A = 0$. We get

$$\begin{aligned} S_{\mathrm{BV}}|_{\mathcal{L}} \;=\; \int d^2\xi d^2\theta dt \Big(&\mathbf{P}_\mu \partial_t \mathbf{X}^\mu + \mathbf{P}_{t\mu} D\mathbf{X}^\mu - \frac{1}{2}\boldsymbol{\ell}_A \partial_t \boldsymbol{\ell}^A + \frac{1}{2}\boldsymbol{\ell}_{tA} D\boldsymbol{\ell}^A \\ &- \frac{1}{2}\boldsymbol{\ell}^A \partial_t \boldsymbol{\ell}_A + \frac{1}{2}\boldsymbol{\ell}^A D\boldsymbol{\ell}_{tA} - 2\mathbf{P}_{t\mu}A_A^\mu(\mathbf{X})\boldsymbol{\ell}^A + f_{BC}^A(\mathbf{X})\boldsymbol{\ell}_{tA}\boldsymbol{\ell}^B\boldsymbol{\ell}^C\Big), \end{aligned}$$

We are now left with the integration over the remaining fields in the UV-sector: $\mathbf{P}_\mu$ and $\boldsymbol{\ell}_A$. Thus, we end up with the following IR action:

$$\begin{aligned} S_{\mathrm{BV}}^{\mathrm{IR}} = \int d^2\xi d^2\theta dt(&\mathbf{P}_{t\mu} D\mathbf{X}^\mu + \boldsymbol{\ell}_{tA} D\boldsymbol{\ell}^A \\ &+ 2A_A^\mu(\mathbf{X})\boldsymbol{\ell}^A \mathbf{P}_{t\mu} + f_{BC}^A(\mathbf{X})\boldsymbol{\ell}_{tA}\boldsymbol{\ell}^B\boldsymbol{\ell}^C), \end{aligned} \tag{5.6}$$

where the integration over $\mathbf{P}$ implements the condition $\partial_t \mathbf{X}^\mu = 0$ and the integration over $\boldsymbol{\ell_A}$ implements the condition $\partial_t \boldsymbol{\ell}^A = 0$. There is a subtlety in the present integration over $\mathbf{P}_\mu$ and $\boldsymbol{\ell}_A$. Namely, if $\mathbf{P}_\mu$ and $\boldsymbol{\ell}_A$ had zero modes (t-independent pieces) then we would not be able to integrate them completely. This is why we need boundary conditions that imply the absence of zero modes: namely,

$$P_\mu|_{T[1]\partial\Sigma_3} = \boldsymbol{\ell}_A|_{T[1]\partial\Sigma_3} = 0,$$

which are also the correct conditions from the points of view of the AKSZ construction. The fields $\mathbf{X}^\mu$ and $\boldsymbol{\ell}^A$ are t-independent and the other fields $\mathbf{P}_{t\mu}$ and $\boldsymbol{\ell}_{tA}$ enter the action (5.6) linearly. Therefore upon the following identification

$$\mathbf{X}^\mu = \mathbf{X}^\mu, \quad \boldsymbol{\eta}_\mu = \int_I dt\ \mathbf{P}_{t\mu}, \quad \mathbf{j}^A = \boldsymbol{\ell}^A, \quad \boldsymbol{\lambda}_A = \int_I dt\ \boldsymbol{\ell}_{tA}, \tag{5.7}$$

the 3D action (5.6) collapses to the 2D theory given by (3.14).

We have shown that the 3D AKSZ theory for a Lie algebroid on $\Sigma_2 \times I$ can be reduced to the 2D AKSZ theory for the same Lie algebroid, provided that the specific boundary conditions are imposed. We would like to stress that for the case $\Sigma_2 \times S^1$ the reduction would not work properly due to the presence of zero modes.

Here we have discussed the reduction for the real model. The reduction for the complex Lie algebroid works in exactly the same way and all expressions remain true modulo notations.

6 Summary

A Lie algebroid L can be encoded by saying that $L[1]$ is equipped with a homological vector field of degree 1. We considered two possible Hamiltonian lifts of this vector field, for the symplectic manifold $T^*[1]L^*$ of degree 1 and for the symplectic manifold $T^*[2]L[1]$ of degree 2. Using the AKSZ construction, these two lifts give rise to 2D and 3D topological field theories, respectively. We also discussed the allowed boundary conditions for these theories. Moreover, we have shown that the 3D theory on $\Sigma_2 \times I$ reduces to the 2D theory on Σ_2, upon specific boundary conditions.

A GCS structure is a complex Lie algebroid L with the additional property that $\bar{L} = L^*$. Thus, all our formal considerations are equally applicable to the case of a GCS. One can show that our 2D theory with GCS corresponding to an ordinary complex structure is, upon gauge fixing, equivalent to the

B-model [35], while the 2D theory for a symplectic structure is equivalent to the A-model [35]. The more general 2D models on a generalized Kähler manifold should correspond to a topological twist of the $N = (2, 2)$ nonlinear sigma model [18, 21]. We will present the detailed analysis of the gauge fixing for these models in a forthcoming work [9].

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Appendix A Integrability of GCS

Grabowski [14] suggested a description of GCG in terms of graded manifolds. Here we review this and provide a number of useful relations.

Following Roytenberg [30], we describe a Courant algebroid E in terms of a graded symplectic manifold $\mathcal{M}$ of degree 2 with Hamiltonian $\mathcal{S}$ of degree 3. The manifold $\mathcal{M}$ is the minimal symplectic realization of $E[1]$ and in local Darboux coordinates (x^μ, p_μ, e^a) of degree 0, 2 and 1 respectively the symplectic structure is

$$\omega = dp_\mu dx^\mu + \frac{1}{2} de^a g_{ab} de^b, \tag{A.1}$$

where g_{ab} is the constant fiber metric $\langle\ ,\ \rangle$ written in a local basis of sections for E: e^a are the fiber odd coordinates on E which transform as sections of E^*. We use the metric g_{ab} to raise and lower the indies, thus relating E and E^*. Any degree 3 function will have the following general form:

$$\Theta = p_\mu A^\mu_a(x) e^a + \frac{1}{6} f_{abc}(x) e^a e^b e^c. \tag{A.2}$$

As it has been shown in [30], the solutions of the equation $\{\Theta, \Theta\} = 0$ correspond to Courant algebroid structure on $(E, \langle\ ,\ \rangle)$ with the Courant–Dorfman bracket given by $[\cdot, \cdot] = \{\{\cdot, \Theta\}, \cdot\}$, where $\{\ ,\ \}$ stands for the Poisson bracket on the symplectic manifold $\mathcal{M}$. In expression (A.2) the quantity A^μ_a and $-g^{da}f_{dbc}$ are interpreted as the anchor and the structure functions for the Courant algebroid E, respectively. Equivalently, we can discuss the Courant algebroid structure on E^*. The Courant–Dorfman brackets for coordinates are given by the "structure functions" ${f^{ab}}_c$ to be $[e^a, e^b] = \{\{e^a, \mathcal{S}\}, e^b\} = -{f^{ab}}_c e^c$ or equivalently $\langle [e^a, e^b], e^c\rangle = \{\{\{e^a, \mathcal{S}\}, e^b\}, e^c\} = -f^{abc}$. Next, we define a function of degree 2 which is independent of p as follows $J = \frac{1}{2}J_{ab}(x)e^a e^b$, such that $J^a_{\ b}(x) = g^{ac}J_{cb}(x) : E \to E$ is interpreted as an endomorphism of E and ${J_a}^b(x) = J_{ac}(x)g^{cb} : E^* \to E^*$ as an endomorphism of E^*. We want to study the relation $\{J, \{J, \mathcal{S}\}\} = -\mathcal{S}$. Expand out the brackets

$$
\begin{aligned}
\{J, \mathcal{S}\} &= \{\frac{1}{2}J_{ab}(x)e^a e^b, p_\mu A^\mu_a(x)e^a + \frac{1}{6}f_{abc}e^a e^b e^c\}\\
&= -\frac{1}{2}(\partial_\mu J_{ab}(x))e^a e^b e^c A^\mu_c(x) + e^c J_{ca}(x)\\
&\quad \times \left[A^\mu_b(x)p_\mu + \frac{1}{2}f_{bde}(x)e^d e^e\right] g^{ab},\\
\{J, \{J, \mathcal{S}\}\} &= -(\partial_\mu J_{ad}(x))e^a e^d e^c {J_c}^b(x)A^\mu_b(x)\\
&\quad - (e^f {J_f}^h(x))(\partial_\mu J_{hk}(x))e^k e^a A^\mu_a(x)\\
&\quad + e^f {J_f}^h(x){J_h}^b(x)\left[A^\mu_b(x)p_\mu + \left(\frac{1}{6} + \frac{1}{3}\right) f_{bde}(x)e^d e^e\right]\\
&\quad - e^f {J_f}^h(x)e^c {J_c}^b(x)f_{bhe}(x)e^e.
\end{aligned}
$$

Thus the condition $\{J, \{J, \mathcal{S}\}\} = -\mathcal{S}$ requires

$$
{J_a}^b(x){J_b}^c(x) = -\delta^c_a, \tag{A.3}
$$

$$
\begin{aligned}
N(J)_{abc} \equiv &-(\partial_\mu J_{ab}(x)){J_c}^d(x)A^\mu_d(x) - {J_a}^d(x)(\partial_\mu J_{db}(x))A^\mu_c(x)\\
&- \frac{1}{3}f_{abc}(x) - {J_a}^e(x){J_b}^d(x)f_{dec}(x) + \text{cyclic in}(abc) = 0,
\end{aligned} \tag{A.4}
$$

where by construction $N_{abc} \in \wedge^3 E$. Condition (A.3) implies that E can decomposed into the sum of two maximally isotropic spaces, namely $E \otimes \mathbb{C} = L \oplus \bar{L}$. We introduce the projection operators

$$
\Pi^a_{\pm b}(x) = \frac{1}{2}\,(\delta^a_b \pm iJ^a_{\ b}(x)),
$$

such that Π_- projects L and Π_+ its complex conjugate, $\bar{L}$. By construction L (and $\bar{L}$) are maximally isotropic with respect to g, since $J_{ab}(x) = -J_{ba}(x)$.

Next we show that (A.4) gives the integrability condition which states that L is involutive under the Courant–Dorfman bracket $[\cdot,\cdot]$, that is

$$\Pi_+[\Pi_- e^a, \Pi_- e^b] = 0.$$

The real part of this expression gives

$$[e^a, e^b] - [J^a{}_c(x)e^c, J^b{}_d(x)e^d] + J[e^a, J^b{}_d(x)e^d] + J[J^a{}_c(x)e^c, e^b] = 0, \quad \text{(A.5)}$$

where by J we understand the endomorphism of E. Using the relations

$$\begin{aligned}
[e^a, J^b{}_d(x)e^d] &= \{\{e^a, \mathcal{S}\}, J^b{}_d(x)e^d\} \\
&= A^{a\mu}(x)(\partial_\mu J^b{}_d(x))e^d + J^b{}_d(x)[e^a, e^d], \\
\left[J^a{}_c(x)e^c, e^b\right] &= \{\{J^a{}_c(x)e^c, \mathcal{S}\}, e^b\} \\
&= -A^{b\mu}(x)(\partial_\mu J^a{}_c(x))e^c - J^a{}_c(x)[e^b, e^c] \\
&\quad + e^d A^\mu_d(x)(\partial_\mu J^{ab}(x))
\end{aligned}$$

and the "structure constant" $f^{ab}{}_c$ expression (A.5) can be rewritten as

$$\begin{aligned}
&- f^{ab}{}_c e^c + J^d_c(A^{a\mu}\partial_\mu J^b{}_d)e^c - J^b{}_d f^{ad}{}_e J^e_c e^c - J^d_c e^c A^{b\mu}(\partial_\mu J^a{}_d) \\
&+ J^a_c f^{bc}{}_d J^d_e e^e + J^d_c A^\mu_d \partial_\mu J^{ab} e^c.
\end{aligned}$$

This can be further massaged into

$$\begin{aligned}
&A^{a\mu}(\partial_\mu J^b{}_d)J^{dc} + J^{dc}A^\mu_d(\partial_\mu J^{ab}) + (\text{cyclic in a,b,c}) \\
&\quad - f^{abc} - J^{[b}{}_d f^{a]d}{}_e J^{ec} - J^b{}_d J^a{}_e f^{dec} = 0
\end{aligned}$$

and thus the integrability condition becomes

$$\begin{aligned}
&A^\mu_a(\partial_\mu J_{bd})J^d_c + J^d{}_c A^\mu_d(\partial_\mu J_{ab}) + \left[-\frac{1}{3}f_{abc} + J^d_b J^e{}_c f_{ade}\right] \\
&\quad + (\text{cyclic in } a, b, c) = 0,
\end{aligned}$$

which coincides with (A.4). Since the sections of L are now involutive under the Courant bracket (which is antisymmetric when restricted to L), L and likewise $\bar{L}$ defines a *Lie algebroid* structure.

Example 5. Consider the standard Courant algebroid structure on $TM \oplus T^*M$ and the corresponding graded symplectic manifold $T^*[2]T^*[1]M$ from

Example 2. Consider the function of degree 2 of the form

$$J = J^{\mu}_{\ \nu}(x) q_\mu v^\nu.$$

Condition (A.3) says that $J^{\mu}_{\ \nu}$ is almost complex structure and condition (A.4) becomes

$$J_\rho^{\ \sigma}\partial_\sigma J_\mu^{\ \nu} + J_\mu^{\ \sigma}\partial_\sigma J^\nu_{\ \rho} + J_\mu^{\ \sigma}\partial_\rho J_\sigma^{\ \nu} + J^\nu_{\ \sigma}\partial_\mu J^\sigma_{\ \rho} = 0, \tag{A.6}$$

which we recognize as the standard Nijenhuis tensor: $J^\sigma_{\ [\rho}\partial_\sigma J^\nu_{\ \mu]} - J^\nu_{\ \sigma}\ \partial_{[\rho} J^\sigma_{\ \mu]} = 0$. Thus, we end up with the standard complex structure on M.

Example 6. Take another example of function of degree 2 on $T^*[2]T^*[1]M$

$$J = \frac{1}{2}(\omega_{\mu\nu}(x) v^\mu v^\nu + \omega^{\mu\nu}(x) q_\mu q_\nu).$$

Condition (A.3) implies that $\omega_{\mu\nu}\omega^{\nu\lambda} = \delta^\lambda_\mu$ and condition (A.4) becomes $\omega^{\sigma[\mu}\partial_\sigma\omega^{\nu\rho]} = 0$. Thus, $\omega_{\mu\nu}$ is a closed nondegenerate two form, symplectic structure.

Example 7. On $T^*[2]T^*[1]M$ the general form of function of degree 2 independent from p is

$$J = \frac{1}{2}L_{\mu\nu}(x) v^\mu v^\nu + J^\mu_{\ \nu}(x) q_\mu v^\nu + \frac{1}{2}P^{\mu\nu}(x) q_\mu q_\nu.$$

By plugging this into (A.3) and (A.4) we get the conditions for a GCS on $TM \oplus T^*M$ which were analyzed in [22, 13]. We would like to stress that the language of graded symplectic manifolds allows one to obtain those complicated conditions by performing rather simple calculations.

We have another useful observation regarding the integrability of $J^a_{\ b}$. Define ∂^+_c by

$$\partial^+_c V^a \equiv A^\mu_c \partial_\mu V^a + \frac{1}{3} f^a_{\ bc} V^b.$$

Then N_{abc} is in fact the (3,0)+(0,3) part of $\partial^+_{[c}J_{ab]} \in \wedge^3 E$. Let us check this explicitly

$$
\begin{aligned}
&(\partial^+_{c'}J_{a'b'})\Pi^{a'}_{-a}\Pi^{b'}_{-b}\Pi^{c'}_{-c} + (\text{cyc in abc})\\
&\quad = -(\partial^+_{c'}J^{a'}_{\ a})\Pi_{-a'b}\Pi^{c'}_{-c} + (\text{cyc in abc})\\
&\quad = -\frac{1}{4}\left[\partial^+_c J_{ba} - J^{c'}_{\ c}(\partial^+_{c'}J^{a'}_{\ a})J_{a'b}\right] + \frac{\mathrm{i}}{4}\left[J_{a'b}\partial^+_c J^{a'}_{\ a} + J^{c'}_{\ c}\partial^+_{c'}J_{ba}\right]\\
&\qquad + (\text{cyc in abc})\\
&\quad = -\frac{1}{4}\left[\partial^+_c J_{ba} + J_c^{\ d}J_a^{\ f}(\partial^+_d J_{fb})\right] + \frac{\mathrm{i}}{4}\left[J_a^{\ d}\partial^+_c J_{db} + J_c^{\ d}\partial^+_d J_{ab}\right]\\
&\qquad + (\text{cyc in abc})\\
&\quad = \frac{1}{4}J_c^{\ d}N_{abd} - \frac{\mathrm{i}}{4}N_{abc}.
\end{aligned}
$$

Thus, the integrability condition says that $\partial^+_{[c}J_{ab]}$ is of type (2,1)+(1,2). This reinterpretation of integrability is analogous to the description of the integrability of the almost complex structure J on the Hermitian manifold (J, g). The almost complex structure is integrable if and only if $d\omega$ is of a type (2,1)+(1,2), where $\omega = gJ$.

Appendix B Change of coordinates

Consider the symplectic graded manifold $\mathcal{M}$ of degree 2 associated to a vector bundle E with fiberwise nondegenerate symmetric inner product $\langle\ ,\ \rangle$. In local Darboux coordinates the symplectic structure is given by (A.1). Now assume that we have endomorphisms $J^a_{\ b}$ of E such that $J^a_{\ b}J^b_{\ c} = -\delta^a_c$ and $J_{ab} = g_{ac}J^c_{\ b} = -J_{ba}$. Thus, this endomorphism defines a splitting of E into two maximally isotropic subbundles, $E \otimes \mathbb{C} = L \oplus \bar{L}$. Then the symplectic manifold $\mathcal{M}$ is symplectomorphic to $T^*[2]L[1] = T^*[2]\bar{L}[1]$. Let us show this explicitly.

Introduce the vielbein F^a_A which can simply be understood as the i-eigenvector of the endomorphism $J^a_{\ b}$ labeled by index A. We lower and raise the Euclidean indices with the pairing g_{ab} and its inverse g^{ab}. The following properties of the vielbein follow from their interpretation as the eigenvectors of $J^a_{\ b}$:

$$
\begin{aligned}
J^a_{\ b}F^b_A &= \mathrm{i}F^a_A; &\quad J^a_{\ b}F^{Ab} &= -iF^{Aa} &&\text{definition},\\
F^A_a F_{Bb}g^{ab} &= \delta^A_B; &\quad F^A_a F^B_b g^{ab} &= F^a_A F^b_B g_{ab} = 0 &&\text{orthonormality},\\
F^a_A F^A_b &= \Pi^a_{-b}, &\quad F^{Aa}F_{Ab} &= \Pi^a_{+b} &&\text{completeness}.
\end{aligned}
$$

We introduce the following new coordinates:

$$\ell^A = F^A_a e^a; \quad \bar{\ell}_A = F_{Aa} e^a,$$
$$\tilde{p}_\mu = p_\mu + \frac{1}{2}(\partial_\mu F_{Aa}) F^a_B \ell^A \ell^B + \frac{1}{2}(\partial_\mu F^A_a) F^{Ba} \bar{\ell}_A \bar{\ell}_B - F^A_a (\partial_\mu F^a_B) \bar{\ell}_A \ell^B,$$

which are adopted to the splitting $E \otimes \mathbb{C} = L \oplus \bar{L}$. The symplectic form (A.1) goes to $\omega = d\tilde{p}_\mu dx^\mu + d\bar{\ell}_A d\ell^A$. The Liouville form changes as follows:

$$\Xi = p_\mu dX^\mu + \frac{1}{2} e^a g_{ab} de^b \Rightarrow \tilde{p}_\mu dx^\mu + \frac{1}{2} \bar{\ell}_A d\ell^A + \frac{1}{2} \ell^A d\bar{\ell}_A.$$

Next, assume that the endomorphism $J^a_{\ b}$ is integrable in sense we have discussed in the previous appendix. We can give the integrability condition more concisely in this new basis. Recall that $N_{abc} = (\partial^+_{[a} J_{bc]})^{(3,0)+(0,3)}$, we have

$$\begin{aligned} 0 &= \frac{1}{2} F^a_A F^b_B F^c_C (\partial^+_{[a} J_{bc]}) = F^a_A F^b_B F^c_C (\partial_a J_{bc} - \frac{1}{3} f_{a\ b}^{\ d} J_{dc} + \frac{1}{3} f_{a\ c}^{\ d} J_{db}) \\ &\quad + \text{cyc in } ABC \\ &= 2i((\partial_A F^c_C) F_{Bc} + \frac{1}{3} F^a_A F^b_B F^c_C f_{abc}) + \text{cyc in } ABC, \end{aligned} \tag{B.1}$$

where we used the following notations: $\partial_A := F^a_A A^\mu_a \partial_\mu$, $\partial^A := F^A_a A^{a\mu} \partial_\mu$. Using this identity we can show that

$$\begin{aligned} [\ell^A, \ell^B] &= \{\{p_\mu A^\mu_a e^a + \frac{1}{6} f_{abc} e^a e^b e^c, F^A_a e^a\}, F^B_b e^b\} \\ &= \ell^C \big(F^a_C (\partial_{[a} F^A_{c]}) F^{Bc} + F^b_C \partial^A F^B_b + F^c_C F^{Bb} F^{Aa} f_{cba} \big) \end{aligned}$$

and similarly the other combination

$$[\bar{\ell}_A, \bar{\ell}_B] = \bar{\ell}_C \big(F^{Ca} (\partial_{[a} F_{c]A}) F^c_B + F^C_c \partial_A F^c_B + F^{Cc} F^b_B F^a_A f_{cba} \big).$$

In this new basis the corresponding Hamiltonian of degree 3 takes very simple form

$$\mathcal{S} + i\{J, \mathcal{S}\} \quad = \quad 2(\Pi_- e)^a A^\mu_a p_\mu + \frac{1}{3} (\Pi_- e)^a f_{abc} e^b e^c - \frac{\mathrm{i}}{2} (\partial^+_c J_{ab}) e^a e^b e^c$$

By using the fact $\partial^+_{[c} J_{ab]}$ is (2,1) and (1,2), we can simplify the last term in this expression.

Thus, we can finally rewrite the Hamiltonian in the following form:

$$\mathcal{S} + i\{J, \mathcal{S}\} = 2\ell^C A^\mu_C \tilde{p}_\mu - \ell^A \ell^B \bar{\ell}_C f^C_{AB},$$

where we use the data for the Lie algebroid L

$$A^\mu_C = F^a_C A^\mu_a,$$

$$f^C_{AB} = F^{Ca}(\partial_{[a} F_{c]A}) F^c_B + F^C_c \partial_A F^c_B + F^{Cc} F^b_B F^a_A f_{cba}.$$

Here we performed the calculations for the complex bialgebroid $(L, \bar{L})$. The generalization for the real bialgebroid is a straightforward modifications of the present calculation.

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