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General properties of the boundary renormalization group flow for supersymmetric systems in 1 + 1 dimensions

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Abstract

We consider the general supersymmetric one-dimensional quantum system with boundary, critical in the bulk but not at the boundary. The renormalization group (RG) flow on the space of boundary conditions is generated by the boundary beta functions $\beta^a(\lambda)$ for the boundary coupling constants λ^a. We prove a gradient formula $\partial \ln z/\partial\lambda^a = -g^S_{ab}\beta^b$ where $z(\lambda)$ is the boundary partition function at given temperature $T = 1/\beta$, and $g^S_{ab}(\lambda)$ is a certain positive-definite metric on the

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space of supersymmetric boundary conditions. The proof depends on canonical ultraviolet behavior at the boundary. Any system whose short distance behavior is governed by a fixed point satisfies this requirement. The gradient formula implies that the boundary energy, $-\partial \ln z/\partial\beta = -T\beta^a\partial_a \ln z$, is nonnegative. Equivalently, the quantity $\ln z(\lambda)$ decreases under the RG flow.

1 Introduction

In this paper we consider the renormalization group (RG) flow for supersymmetric one-dimensional quantum systems with boundary which are critical in the bulk but not critical on the boundary. First, we give a brief overview of what is known without the assumption of supersymmetry. There are many condensed matter applications for such systems, such as quantum impurities and quantum Hall edge excitations (see, e.g., [1] for a review). We expect that supersymmetric bulk-critical one-dimensional systems with boundaries — and junctions — can be realized in practice. Such supersymmetric quantum circuits might be useful for large-scale quantum computing [2].

Consider a bounded system of length L at low temperature $T = 1/\beta$. Let H_L be the hamiltonian[1] of the bounded system. The partition function is $Z_L = \mathrm{tr}\left(\mathrm{e}^{-\beta H_L}\right)$. There are two boundaries, one at each end. In the limit $L \to \infty$, the two boundaries decouple and the partition function of the whole system factorizes into a bulk contribution and two boundary contributions:

$$Z_L \sim \mathrm{e}^{\pi c L/6\beta} z z'. \tag{1.1}$$

Here c is the central charge of the conformal field theory describing the bulk critical system, $-\pi c/6\beta^2$ is the universal free energy density of the bulk conformal field theory, and z and z' are the L-independent contributions of the boundaries. For a unitary theory the sign of z can be fixed so that z is positive. The quantity z is the boundary partition function. It is a function $z(\lambda, \mu\beta)$ depending on the boundary coupling constants λ^a that parametrize the boundary condition and on the temperature $T = 1/\beta$ (in dimensionless units of the energy scale μ).

[1] We are considering 1d quantum mechanical systems so we can assume unitarity: the hamiltonian is a self-adjoint operator acting on a Hilbert space of states. By Wick rotation, our results apply equally well to 2d statistical systems that satisfy reflection positivity.

The boundary partition function has no representation of the form $z = \mathrm{tr}\left(\mathrm{e}^{-\beta h}\right)$ so there is no reason to believe that the boundary thermodynamic functions constructed from z will satisfy the usual thermodynamic principles. Nevertheless, it can be proved [3] that the boundary entropy

$$s = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln z \tag{1.2}$$

does decrease monotonically with temperature. That is, the boundary satisfies the second law of thermodynamics. We emphasize that this was not necessarily to be expected. The entropy of the whole system behaves as

$$S_L \sim s + s' + \frac{c\pi L}{3\beta} \tag{1.3}$$

as $L \to \infty$. The total entropy S_L decreases monotonically with temperature, but so does the bulk term. The subtraction of the bulk term precludes a straightforward derivation of the second law of thermodynamics for the boundary entropy s.

The RG equation is

$$\mu \frac{\partial \ln z}{\partial \mu} = \beta^a \frac{\partial \ln z}{\partial \lambda^a}, \tag{1.4}$$

where the $\beta^a(\lambda)$ are the boundary beta functions. The critical boundary conditions are described by the fixed points, $\beta^a = 0$. The boundary partition function at a fixed point is a number, scale invariant and therefore independent of temperature, traditionally denoted $z = g$. The number g was introduced as an invariant of critical boundary systems by Affleck and Ludwig [4], who called it the universal noninteger ground state degeneracy. They conjectured [4,5] that, for two critical boundary conditions connected by an RG trajectory, the value of g at the infrared fixed point is always smaller than the value at the ultraviolet fixed point. Affleck and Ludwig's conjecture follows from the second law of boundary thermodynamics, because $s = \ln g$ at each of the fixed points, and the scale μ can be traded for the temperature.

The second law of boundary thermodynamics is a consequence of yet a stronger statement, the boundary gradient formula proved in [3]:

$$\frac{\partial s}{\partial \lambda^a} = -g_{ab}\beta^b, \tag{1.5}$$

where g_{ab} is a certain positive-definite metric on the space of boundary couplings. Since $\ln z$ and s depend on the dimensionless product $\mu\beta$, the

RG equation for s can be written

$$\mu\frac{\partial s}{\partial\mu} = \beta\frac{\partial s}{\partial\beta} = \beta^a\frac{\partial s}{\partial\lambda^a}. \tag{1.6}$$

Contracting (1.5) with β^a gives

$$\beta\frac{\partial s}{\partial\beta} = \beta^a\frac{\partial s}{\partial\lambda^a} = -\beta^a g_{ab}\beta^b \leq 0 \tag{1.7}$$

which says that s decreases as the temperature decreases. The boundary second law thus follows from the gradient formula.

The proof of the gradient formula given in [3] used the euclidean description of the finite-temperature quantum system. The metric in equation (1.5) is

$$g_{ab} = \beta\int_0^\beta d\tau\ \left[1-\cos\left(2\pi\tau/\beta\right)\right]\ \langle\phi_a(\tau)\phi_b(0)\rangle_{\rm c}, \tag{1.8}$$

where $\langle\cdots\rangle_{\rm c}$ stand for the connected thermal correlation functions. The one-dimensional system with a single boundary is described by a two-dimensional euclidean field theory with spatial coordinate x, $0\leq x<\infty$, and euclidean time τ. The boundary is at $x=0$. The euclidean time τ is periodic with period β. The euclidean space–time is the semi-infinite cylinder with coordinates (x,τ). The boundary coupling constants λ^a couple to boundary operators $\phi_a(\tau)$, localized at $x=0$, so that

$$\frac{\partial\ln z}{\partial\lambda^a} = \int_0^\beta d\tau\ \langle\phi_a(\tau)\rangle = \beta\langle\phi_a\rangle. \tag{1.9}$$

An alternative proof of the gradient formula (1.5) using real-time methods was presented in [6]. There, the metric g_{ab} was expressed via response functions. The proof of the gradient formula (1.5) relies on the assumption that the two-point correlation functions of the boundary operators $\phi_a(\tau)$ with themselves and with the stress-energy tensor and $T_{\mu\nu}(x,\tau)$ behave canonically at short distance. This assumption is valid if the ultraviolet limit is governed by a fixed point, because then the boundary operators $\phi_a(\tau)$ must be relevant at the fixed point. It is interesting to note that no assumption of this kind is needed to prove Zamolodchikov's c-theorem [7], which establishes the monotonic decrease of the c-function under the RG flow in the space of bulk 2d field theories.

Now we specialize to supersymmetric one-dimensional systems with boundary. In supersymmetric systems the thermodynamic energy $-\partial\ln Z/\partial\beta$ is always nonnegative, because the hamiltonian is of the form $H=\hat{Q}^2$, where $\hat{Q}$ is the supercharge operator. However, it is not obvious that the boundary energy in such a supersymmetric system should be nonnegative. Consider again a finite system of length L. For the whole system, certainly $-\partial \ln Z_L/\partial\beta \geq 0$, but

$$-\frac{\partial \ln Z_L}{\partial \beta} = -\frac{\partial \ln z}{\partial \beta} - \frac{\partial \ln z'}{\partial \beta} + \frac{\pi c L}{6\beta^2} \tag{1.10}$$

as $L \to \infty$. The positivity of the large bulk energy prevents us from concluding that the boundary energy is positive.

In this paper we prove the positivity of the boundary energy by deriving a new gradient formula for the supersymmetric boundary RG flow

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}\beta^b, \tag{1.11}$$

where g^S_{ab} is a certain positive-definite metric on the space of supersymmetric boundary conditions (not the same metric as in the general gradient formula). Contracting with β^a gives

$$-\frac{\partial \ln z}{\partial \beta} = T\beta^a g^S_{ab}\beta^b \geq 0 \tag{1.12}$$

which proves that the boundary energy is nonnegative.

As in the case of the general gradient formula (1.5), which was to a large extent inspired by work done in string theory [8–12], the existence of a different gradient formula for the supersymmetric boundary RG flow was anticipated in the string theory literature [13–15]. It was conjectured in [13–15] that z is a potential function for such a gradient formula.[2] In [15] the expression for the metric g^{S}_{ab} was put forward, which we will show to be correct, but a proof of the gradient formula was still lacking. In this paper we give two different proofs of (1.11). In Section 3 we give a proof using the formalism of euclidean quantum field theory. In Section 4 we use real-time methods. The two proofs are compared in Section 5. In the euclidean

[2] In string theory, one wants a gradient formula for the beta function, such as (1.11), in order to have a space–time action principle. In string theory it is z rather than $\ln z$ that is a natural potential function (a string field theory action). The link between (1.5) and its stringy version requires special treatment of the tachyon zero mode [3]. The stringy version of the supersymmetric gradient formula (1.11) is trivially obtained by multiplying both sides by z.

approach the metric is written

$$g^S_{ab} = 2\pi \int_0^\beta d\tau \ \sin(\pi\tau/\beta)\, \langle \hat\phi_a(\tau)\hat\phi_b(0)\rangle, \tag{1.13}$$

where the $\hat\phi_a(\tau)$ are the fermionic superpartners[3] of the bosonic boundary operators $\phi_a(\tau)$. In the real-time approach, the same metric is written in terms of real-time response functions of the $\hat\phi_a(t)$,[4]

$$g^S_{ab} = \pi \int_{-\infty}^{\infty} dt\; e^{-\pi|t|/\beta} \langle\, \{\hat\phi_b(t),\, \hat\phi_a(0)\}\,\rangle. \tag{1.14}$$

Like the general gradient formula, formula (1.11) is proved under the condition of canonical short distance behavior at the boundary, now for the correlation functions $\langle\hat\phi_a(\tau)\hat\phi_b(\tau')\rangle$, $\langle\hat\phi_a(\tau)\hat\theta(\tau')\rangle$, and $\langle G_{\mu r}(\tau,x)\hat\phi_b(\tau')\rangle$ where $G_{\mu r}$ is the bulk supersymmetry current and $\hat\theta$ is its boundary part. Again, the condition is satisfied if the extreme UV limit is described by a fixed point (which would necessarily be supersymmetric). Then the UV scaling dimension of $\hat\phi_a$ is at most 1/2 and the bulk supercurrent $G_{\mu r}$ has canonical scaling dimension 3/2. At present, we see only technical reasons for the gradient formulas to depend on canonical UV behavior at the boundary.

The metric $g^S_{ab}(\lambda)$, like the bosonic metric $g_{ab}(\lambda)$, is covariant under change of coordinates λ^a in the space of boundary conditions. This follows from formulas (1.8) and (1.13) where the metrics are defined by expressions that are insensitive to possible contact terms in the two-point functions.

However both metrics may fail to be invariant under the RG flow. RG invariance is the condition that change of scale is equivalent to flow under the RG,

$$\mu\frac{\partial g_{ab}}{\partial\mu} = (\mathcal{L}_\beta g)_{ab} = \beta^c\frac{\partial g_{ab}}{\partial\lambda^c} + \frac{\partial\beta^c}{\partial\lambda^a} g_{cb} + g_{ac}\frac{\partial\beta^c}{\partial\lambda^b}. \tag{1.15}$$

RG invariance means that the metric, although it is defined at a certain temperature (scale), in fact does not depend on the arbitrary choice of scale. The metric depends only on the running coupling constants at the temperature at

[3] The one-point functions $\langle\phi_a(\tau)\rangle$ which appear on the left-hand side of the gradient formula can be non-vanishing because the global supersymmetry is spontaneously broken at non-zero temperature.

[4] We abuse notation in writing $\hat\phi_b(\tau)$ when we are discussing physics in euclidean time, and $\hat\phi_b(t)$ when discussing real-time physics. To be consistent, we should write either $\hat\phi_b(\tau)$ and $\hat\phi_b(\mathrm{i}t)$ or $\hat\phi_b(-\mathrm{i}t)$ and $\hat\phi_b(t)$. We are perhaps also abusing terminology when we refer to *response functions* of fermionic operators.

which it is measured. Without RG invariance, the metric depends on more than the running coupling constants at the physical temperature. There are many different gradient formulas, one for each temperature, all satisfied. We suppose that this unsatisfactory situation might be alleviated by introduction of some auxiliary couplings.

The problem with RG invariance of the metric is that the local fields need only transform covariantly under the RG flow up to total derivative operators,

$$\mu \frac{\partial \phi_a(\tau)}{\partial \mu} = \frac{\partial \beta^b}{\partial \lambda^a} \phi_b(\tau) + \partial_\tau \chi_a(\tau). \tag{1.16}$$

Such admixtures do not affect such quantities as $\partial s/\partial \lambda^a$ and $\partial \ln z/\partial \lambda^a$ but do affect local correlators such as are used in the definition of the metric (1.8). The transformation law (1.16) is consistent with our UV assumptions as long as the UV scaling dimension of the field χ_a is zero. Such fields can exist if the UV fixed point theory has multiple — degenerate — ground states. This is in the ultraviolet limit, not in the infrared, so there is no physical pathology. Note that the left-hand side of the gradient formula is RG invariant, so the right-hand side, $g_{ab}\beta^b$, must also be RG invariant. This puts constraints on the correlators of the $\chi_a(\tau)$. For supersymmetric theories, the scale transformation of the metric g^{S}_{ab} is affected by analogous admixtures in the RG transformation law for the fermionic boundary fields,

$$\mu \frac{\partial \hat{\phi}_a(\tau)}{\partial \mu} = \frac{\partial \beta^b}{\partial \lambda^a} \hat{\phi}_b(\tau) + \{\hat{Q}, \chi_a(\tau)\}. \tag{1.17}$$

It would be desirable both to find explicit examples where the metric is not RG invariant and also to get a deeper general understanding of such situations.

In an isolated supersymmetric system, the ground state energy E_0 is zero if and only if the supersymmetry is unbroken in the ground state. The low-temperature limit of the partition function is therefore a definitive diagnostic of spontaneous supersymmetry breaking in the ground state. When the supersymmetry is broken, then $\ln Z$ decreases as $-\beta E_0$, with no lower bound. When the supersymmetry is unbroken, the partition function Z decreases to a lower bound, the ground state degeneracy, so $\ln Z \geq 0$. In supersymmetric boundary systems, the low-temperature limit of $\ln z$ is more problematic. The gradient formula we prove here, equation (1.11), implies that the boundary thermodynamic energy is nonnegative,

$$e(\beta) = -\frac{\partial \ln z}{\partial \beta} \geq 0. \tag{1.18}$$

The general gradient formula implies the second law for the boundary,

$$\frac{\partial e}{\partial \beta} = -\frac{\partial^2 \ln z}{\partial \beta^2} = \frac{1}{\beta}\frac{\partial s}{\partial \beta} \leq 0. \tag{1.19}$$

So the boundary energy is nonnegative and decreases monotonically as $\beta \to \infty$. Therefore it must have a nonnegative limit

$$\lim_{\beta\to\infty} e(\beta) = e_0 \geq 0. \tag{1.20}$$

The bulk superconformal invariance implies that there is no bulk ground state energy, so all the ground state energy must be localized in the boundary. The total ground state energy is e_0. Therefore the supersymmetry is spontaneously broken if and only if $e_0 > 0$. Certainly, if $e_0 > 0$ then $\ln z$ goes as $-\beta e_0$ for large β. When the supersymmetry is unbroken, $e_0 = 0$, we can ask if $\ln z \geq 0$ as $\beta \to \infty$, as for an isolated supersymmetric system. The elementary proof does not work, as before, because in the finite system

$$\ln Z_L \sim \ln z + \ln z' + \frac{\pi c L}{6\beta} \tag{1.21}$$

so $\ln z$ is the difference of two positive numbers.[5] In fact, an example of supersymmetric critical boundary with $\ln z < 0$ has been given in [16] (the boundary condition labeled '0' there).

We cannot even say whether or not $\ln z$ is bounded below as $\beta \to \infty$, in general. There seems to be a parallel with the question of a lower bound on the boundary entropy s in the general, non-supersymmetric case. Unlike ordinary entropy, s can be negative. There are many examples. We cannot prove a universal lower bound on s, or a lower bound for a given bulk conformal field theory. We cannot even prove that s is bounded below as a function of β for a given boundary system. Some partial results were found in [6]. It does not seem that supersymmetry helps to get any stronger results on a lower bound for s. The methods of [6] can be easily generalized to study the rate of change of the boundary free energy at low temperature in the supersymmetric case, but again we find nothing conclusive. New methods are needed to put a definite lower bound either on s or on $\ln z$. The second law of boundary thermodynamics, which holds in general, and the positivity of the boundary energy for supersymmetric systems both suggest that boundaries of systems critical in the bulk behave in some respects like isolated thermodynamic systems. The absence of lower bounds on s and $\ln z$ would weaken this analogy. The absence of lower bounds also prevents

[5] Note that the limits $L \to \infty$ and $\beta \to \infty$ do not commute.

the gradient formula from definitively controlling the infrared limits of the boundary RG.

Finally, it would be desirable to have some physical insight into the crucial roles of bulk conformal invariance and canonical UV boundary behavior in the picture of boundary physics that is provided by the two gradient formulas.

2 Supersymmetry in the presence of a boundary in 2d and 1+1d

A near critical one-dimensional quantum system with boundary, at temperature $T = 1/\beta$, can be described by a two-dimensional Euclidean quantum field theory on a semi-infinite cylinder with coordinates (x, τ), as defined in the introduction. Space is the half-line $0 \leq x < \infty$. Correlation functions of bosonic fields are periodic in euclidean time τ, with period β, while correlation functions of fermionic fields are anti-periodic. The Wick rotation to real time is given by $\tau = \mathrm{i}t$.[6] It is convenient to introduce a complex coordinate $w = x + \mathrm{i}\tau = x - t$, and its complex conjugate $\bar{w} = x - \mathrm{i}\tau = x + t$. We set the RG scale μ to 1, since variation of the RG scale is equivalent to variation of β.

2.1 Spinor conventions

A Dirac spinor $\hat{\epsilon}$ in two dimensions has two complex components

$$\hat{\epsilon} = \begin{pmatrix} \hat{\epsilon}_+ \\ \hat{\epsilon}_- \end{pmatrix}, \tag{2.1}$$

where $\hat{\epsilon}_+$ and $\hat{\epsilon}_-$ are the positive and negative chirality components. The euclidean reality condition is $(\hat{\epsilon}_+)^* = \hat{\epsilon}_-$. We use $\mu, \nu, \ldots$ for vector indices and $r, s, \ldots$ for spinor indices. Spinor indices are raised and lowered according to the rule $\hat{\epsilon}^+ = 2\hat{\epsilon}_-$, $\hat{\epsilon}^- = 2\hat{\epsilon}_+$. Our Dirac matrices γ^μ are

$$\gamma^w = \gamma^x - \gamma^t = \begin{pmatrix} 0 & 2\mathrm{i} \\ 0 & 0 \end{pmatrix}, \quad (\gamma^w)^-_+ = 2\mathrm{i}, \quad \gamma^w_{++} = \mathrm{i}, \tag{2.2}$$

$$\gamma^{\bar{w}} = \gamma^x + \gamma^t = \begin{pmatrix} 0 & 0 \\ -2\mathrm{i} & 0 \end{pmatrix}, \quad (\gamma^{\bar{w}})^+_- = -2\mathrm{i}, \quad \gamma^{\bar{w}}_{--} = -\mathrm{i}. \tag{2.3}$$

[6] Again, we will abuse notation by writing fields and operators as functions of τ working in euclidean time, and as functions of t when working in real time.

2.2 Supersymmetry transformations

We now assume that the system at hand is endowed with an action of local supersymmetry transformations $\delta_{\hat\epsilon}$ labeled by fermionic real spinor fields $\hat\epsilon^r(x,\tau)$, antiperiodic in τ. These are the superpartners of the ordinary deformations of space–time. The transformations satisfy the algebra

$$[\delta_{\hat\epsilon_1}, \delta_{\hat\epsilon_2}] = 2\hat\epsilon_1^r \hat\epsilon_2^s \gamma^\mu_{rs} \partial_\mu. \tag{2.4}$$

The vector fields on the right-hand side of (2.4) must preserve the boundary, which requires a condition $\hat\epsilon^+ = \pm\hat\epsilon^-$ on the boundary. The choice of sign is conventional. We adopt

$$\hat\epsilon^+(0,\tau) = \hat\epsilon^-(0,\tau) \equiv \hat\epsilon(\tau). \tag{2.5}$$

The supersymmetry transformations are generated by a local fermionic current $G_{\mu r}$ whose Ward identities are

$$\langle \delta_{\hat\epsilon}\mathcal{O}\rangle = \iint dx d\tau\; \partial^\mu \hat\epsilon^r \langle\, G_{\mu r}(x,\tau)\mathcal{O}\,\rangle_{\rm c}, \tag{2.6}$$

where $\mathcal{O}$ stands for an arbitrary insertion of local operators and the spinor field $\hat\epsilon^r(x,\tau)$ vanishes at large x. The operator $G_{\mu r}(x,\tau)$ in the above expression is understood as a distribution on the half-cylinder that can have singularities on the boundary and at the points of insertion of other local operators. Choosing $\hat\epsilon^r$ to vanish near the insertions we obtain the conservation equation

$$\partial^\mu G_{\mu r}(x,\tau) = 0, \tag{2.7}$$

where the derivative is taken in the distributional sense.

The Ward identity (2.6) implies that the system with boundary is invariant under a single global supersymmetry transformation $\mathcal{O} \to \mathcal{O} + \hat\epsilon\delta\mathcal{O}$ that is generated by a conserved fermionic supercharge

$$\hat\epsilon\delta\mathcal{O} = [\mathrm{i}\hat\epsilon\hat Q,\, \mathcal{O}], \tag{2.8}$$

where

$$\hat Q = \int dx\; \hat\rho(x,t), \tag{2.9}$$

$$\partial_t\hat\rho(x,t) + \partial_x\hat\jmath(x,t) = 0, \tag{2.10}$$

$$\hat\rho(x,t) = G_{t+}(x,t) + G_{t+}(x,t), \tag{2.11}$$

$$\hat\jmath(x,t) = -G_{x+}(x,t) - G_{x+}(x,t). \tag{2.12}$$

The supercharge density $\hat{\rho}(x,t)$, the supercurrent $\hat{\jmath}(x,t)$, and the supercharge $\hat{Q}$ are all self-adjoint operators. To derive explicitly the conservation of $\hat{Q}$ and the global supersymmetry transformation it generates, substitute in the Ward identity a general spinor field $\hat{\epsilon}^r(x,\tau)$ that is constant in x and obeys the boundary condition (2.5). This yields, in particular, the result

$$\langle\, \mathrm{i}\hat{Q}(\tau)\,\hat{\phi}_a(0)\,\rangle = \frac{1}{2}\mathrm{sign}(\tau)\,\delta\hat{\phi}_a(0) = \frac{1}{2}\mathrm{sign}(\tau)\,\{\mathrm{i}\hat{Q},\,\hat{\phi}_a(0)\} \tag{2.13}$$

for $\hat{\phi}_a(\tau)$ a fermionic operator localized on the boundary. The right-hand side is the unique solution of the Ward identity anti-periodic in $-\beta/2 \le \tau \le \beta/2$.

The bosonic stress-energy tensor satisfies the Ward identity

$$\langle\, v^\mu \partial_\mu \mathcal{O}\,\rangle = \iint dx d\tau\; \partial^\mu v^\nu \langle\, T_{\mu\nu}(x,\tau)\,\mathcal{O}\,\rangle_{\mathrm{c}} \tag{2.14}$$

from which we get

$$\partial_t \mathcal{O} = [\mathrm{i}H,\,\mathcal{O}] \tag{2.15}$$

with hamiltonian

$$H = \int dx\; T_{tt}(x,t). \tag{2.16}$$

Consistency of the supersymmetry algebra (2.4) and the two Ward identities requires $G_{\mu r}$ and $T_{\mu\nu}$ to be superpartners:

$$\delta_{\hat{\epsilon}} G_{\mu r}(x,\tau) = -2\hat{\epsilon}^s \gamma^\nu_{rs} T_{\mu\nu}(x,\tau). \tag{2.17}$$

The global variations are

$$\{\hat{Q},\,G_{\mu+}\} = -2T_{\mu w}, \quad \{\hat{Q},\,G_{\mu-}\} = 2T_{\mu\bar{w}}. \tag{2.18}$$

In particular, the global variation of the supercharge density gives the energy density,

$$\{\hat{Q},\,\hat{\rho}(x,t)\} = 2T_{tt}(x,t) \tag{2.19}$$

implying the supersymmetry operator algebra

$$\hat{Q}^2 = H, \tag{2.20}$$

which is consistent with the global transformation algebra $\delta^2\mathcal{O} = \mathrm{i}\partial_t\mathcal{O}$ that follows from (2.4).

2.3 Bulk superconformal invariance

A theory that is superconformal in the bulk satisfies the operator equation

$$(\gamma^{\mu})^{r}_{s} G_{\mu r}(x,\tau) = 0, \quad x > 0. \tag{2.21}$$

We write, in the bulk,

$$G_{\mu r}(x,\tau) = G^{\text{bulk}}_{\mu r}(x,\tau), \quad x > 0. \tag{2.22}$$

The bulk superconformal equation reads, in complex coordinates,

$$G^{\text{bulk}}_{\bar{w}+}(x,\tau) = G^{\text{bulk}}_{w-}(x,\tau) = 0. \tag{2.23}$$

By (2.17), the bulk superconformal condition implies the ordinary conformal invariance condition for the bulk stress-energy tensor, $T^{\mu}_{\mu}(x,\tau) = 0$, $x > 0$. The conservation law for the nonvanishing bulk currents is

$$\partial_{\bar{w}} G^{\text{bulk}}_{w+} = \partial_{w} G^{\text{bulk}}_{\bar{w}-} = 0 \tag{2.24}$$

so they are holomorphic and antiholomorphic, respectively. They are related to the conventional superconformal currents by

$$G^{\text{bulk}}_{w+}(w) = \frac{\mathrm{e}^{\pi \mathrm{i}/4}}{2\pi} G(-\mathrm{i}w), \quad G^{\text{bulk}}_{\bar{w}-}(\bar{w}) = \frac{\mathrm{e}^{-\pi \mathrm{i}/4}}{2\pi} \bar{G}(\mathrm{i}\bar{w}). \tag{2.25}$$

The conventional superconformal currents are adapted to the alternate quantization, called the *bulk quantization*, in which $-x$ is the euclidean time coordinate, τ is the spatial coordinate, and $-\mathrm{i}w = \tau - \mathrm{i}x$ is the complex coordinate. This rotation by $\pi/2$ is responsible for the factors of $(-\mathrm{i})^{\pm 3/2}$ in the relation between the spin-3/2 superconformal currents.

Bulk superconformal invariance implies in addition that the currents decay at spatial infinity as

$$G^{\text{bulk}}_{\mu r}(x,\tau) \sim \exp(-3\pi x/\beta), \quad x \to \infty. \tag{2.26}$$

This is equivalent to the superconformal condition $G_{-1/2}|0\rangle = \bar{G}_{-1/2}|0\rangle = 0$ on the bulk ground state $|0\rangle$ at $x = \infty$ in the bulk quantization. The operators $G_{-1/2}$, $\bar{G}_{-1/2}$ are the usual Fourier modes of $G(-\mathrm{i}w)$ and $\bar{G}(\mathrm{i}\bar{w})$, respectively. The bulk ground state is the only state in the bulk quantization that contributes at large x in the limit where the bulk system is infinitely long, $L/\beta \to \infty$.

2.4 The boundary supercharge

When the bulk system is superconformally invariant, the chirality of the bulk currents $G^{\rm bulk}_{w+}$, $G^{\rm bulk}_{\bar{w}-}$ ensures that they stay finite on the boundary.[7] The total current can be written as

$$G_{\mu r}(x,\tau) = G^{\rm bulk}_{\mu r}(x,\tau) - \hat{\theta}_{\mu r}(\tau)\delta(x). \tag{2.27}$$

Boundary terms proportional to derivatives of $\delta(x)$ are excluded by our assumption that the system has no boundary operators of negative ultraviolet scaling dimension.

Substituting the expansion (2.27) into the Ward identity (2.6) and integrating by parts, we derive the boundary conservation equations

$$\hat{\theta}_{xr}(\tau) = 0, \tag{2.28}$$

$$\partial_\tau[\hat{\theta}_{\tau+}(\tau) + \hat{\theta}_{\tau-}(\tau)] = G^{\rm bulk}_{x+}(0,\tau) + G^{\rm bulk}_{x-}(0,\tau). \tag{2.29}$$

It is convenient to introduce the operators

$$\hat{\theta} = \frac{\mathrm{i}}{2}(\hat{\theta}_{\tau+} + \hat{\theta}_{\tau-}) = \frac{1}{2}(\hat{\theta}_{t+} + \hat{\theta}_{t-}), \tag{2.30}$$

$$\hat{q} = -2\hat{\theta}. \tag{2.31}$$

The boundary conservation equation now reads

$$-2\mathrm{i}\partial_\tau\hat{\theta}(\tau) = G^{\rm bulk}_{x+}(0,\tau) + G^{\rm bulk}_{x-}(0,\tau) \tag{2.32}$$

or, switching to real time,

$$\partial_t\hat{q}(t) + \hat{\jmath}^{\rm bulk}(0,t) = 0, \tag{2.33}$$

where $\hat{q}(t) = -2\hat{\theta}(t)$ is the boundary supercharge. The supercharge density and supercurrent are separated into bulk and boundary parts:

$$\hat{\rho}(x,t) = \hat{q}(t)\delta(x) + \hat{\rho}^{\rm bulk}(x,t),$$
$$\hat{\jmath}(x,t) = \hat{\jmath}^{\rm bulk}(x,t) \tag{2.34}$$

[7] For a nonconformal bulk theory, a blow-up in the bulk supercurrent $G^{\rm bulk}_{\mu r}$ at the boundary would be compensated by subtractions in the construction of the total distributional current $G_{\mu r}$.

and the bulk parts are written in terms of the chiral currents

$$\hat{\rho}^{\text{bulk}}(x,t) = G_{t+}^{\text{bulk}}(x,t) + G_{t-}^{\text{bulk}}(x,t),$$
$$\hat{\jmath}^{\text{bulk}}(x,t) = -G_{x+}^{\text{bulk}}(x,t) - G_{x-}^{\text{bulk}}(x,t). \tag{2.35}$$

The stress-energy tensor is obtained by varying the supercurrent, equation (2.17), so it takes the form

$$T_{\mu\nu}(x,\tau) = T_{\mu\nu}^{\text{bulk}}(x,\tau) - \theta_{\mu\nu}(\tau)\delta(x), \tag{2.36}$$

where the only nonvanishing boundary component is $\theta_{\tau\tau}$. Again, it is convenient to introduce

$$\theta(\tau) = -\theta_{\tau\tau}(\tau) = \theta_{tt}(\tau) \tag{2.37}$$

so the boundary energy is $-\theta(t)$.

Because of the bulk conformal invariance, the trace of the stress-energy tensor lives entirely in the boundary

$$T_{\mu}^{\mu}(x,\tau) = \theta(\tau)\delta(x) \tag{2.38}$$

so $\theta(\tau)$ expresses the departure from conformal invariance in the system with boundary.[8] From (2.18) we see that the operators $\hat{\theta}(\tau)$ and $\theta(\tau)$ are superpartners:

$$\delta\hat{\theta}(\tau) = \mathrm{i}\theta(\tau), \qquad \{\hat{Q},\, \hat{\theta}(t)\} = \theta(t). \tag{2.39}$$

We choose a complete set $\{\hat{\phi}_a(\tau)\}$ of self-adjoint fermionic boundary operators. Their self-adjoint superpartners are the bosonic boundary operators $\phi_a(\tau)$,

$$\delta\hat{\phi}_a(\tau) = \mathrm{i}\phi_a(\tau), \qquad \{\hat{Q},\, \hat{\phi}_a(\tau)\} = \phi_a(\tau). \tag{2.40}$$

The space of supersymmetric boundary conditions is parameterized by the boundary coupling constants λ^a coupled to the $\phi_a(\tau)$ as in equation (1.9). These couplings preserve supersymmetry because

$$\delta\phi_a(\tau) = \mathrm{i}\partial_\tau\hat{\phi}_a(\tau) \tag{2.41}$$

so the variation of the lagrangian is a total derivative in time.

[8]This formula motivates the choice of sign in equation (2.36) defining $\theta(\tau)$.

Expanding $\hat{\theta}(\tau)$ in the complete set of fermionic boundary operators,

$$\hat{\theta}(\tau) = \beta^a \hat{\phi}_a(\tau) \tag{2.42}$$

so

$$\theta(\tau) = \beta^a \phi_a(\tau) \tag{2.43}$$

so the coefficients β^a are the boundary beta functions. The entire system becomes superconformally invariant when $\hat{\theta}(\tau)$ vanishes, given the bulk superconformal invariance. Then, from (2.29), the boundary conservation equation becomes $\mathrm{e}^{3\pi\mathrm{i}/4} G = \mathrm{e}^{-3\pi\mathrm{i}/4} \bar{G}$, in terms of the conventional superconformal currents, which is the standard superconformal gluing condition on the cylinder.

In proving the gradient formula, we will use correlation functions and response functions of the boundary supercharge $\hat{q}(\tau)$ and the bulk currents $G^{\mathrm{bulk}}_{\mu r}(x,\tau)$. We suppose that the correlation functions of the physical currents $G_{\mu r}(x,\tau)$ are given. We can define the correlation functions of $\hat{q}(\tau)$ by an approximation such as

$$\hat{q}_\epsilon(\tau) = \int_0^\epsilon dx\ \hat{\rho}(x,\tau), \tag{2.44}$$

$$\langle\, \hat{q}(\tau)\, \mathcal{O}\,\rangle = \lim_{\epsilon\to 0} \langle\, \hat{q}_\epsilon(\tau)\, \mathcal{O}\,\rangle. \tag{2.45}$$

The approximation can be controlled by virtue of the bulk superconformal invariance and the consequent chirality of the bulk currents,

$$\langle\, \hat{q}_{\epsilon'}(\tau)\, \mathcal{O}\,\rangle - \langle\, \hat{q}_\epsilon(\tau)\, \mathcal{O}\,\rangle = \int_\epsilon^{\epsilon'} dx\ \langle\, [G^{\mathrm{bulk}}_{w+}(x,\tau) + G^{\mathrm{bulk}}_{\bar{w}-}(x,\tau)]\, \mathcal{O}\,\rangle. \tag{2.46}$$

Canonical UV behavior at the boundary ensures that the correlation functions of $\hat{q}_\epsilon(\tau)$ exist in the limit and are independent of the method of approximation, up to a limited set of possible contact terms in τ. The boundary supercharge $\hat{q}(\tau)$, so defined, then differs within correlation functions from the linear combination $\beta^a \hat{\phi}_a(\tau)$ of physical boundary operators by a similarly limited set of possible contact terms in τ. We need only ensure that our calculations are insensitive to these limited sets of possible contact terms.

3 Proof of the gradient formula using Euclidean field theory

We assume that our supersymmetric 1-D system with boundary is unitary and is superconformally invariant in the bulk. We also assume some regularity in the short distance behavior at and near the boundary. We require the following limits to exist (in the distributional sense):

$$\lim_{\epsilon\to 0} \epsilon\langle\hat{\phi}_a(\epsilon\tau)\hat{\phi}_b(0)\rangle, \quad \lim_{\epsilon\to 0} \epsilon\langle\hat{\phi}_a(\epsilon\tau)\hat{\theta}(0)\rangle, \quad \lim_{\epsilon\to 0} \epsilon^2\langle\hat{G}^{\mathrm{bulk}}_{\mu r}(\epsilon x, \epsilon\tau)\hat{\phi}_a(0)\rangle, \tag{3.1}$$

and we require that there be no operators whose UV scaling dimension is negative. These requirements on the short distance behavior are satisfied if there is a supersymmetric short distance fixed point of the RG (thereby permitting canonical scaling analysis), and if the UV fixed point theory satisfies a weak cluster decomposition principle, that correlation functions should not grow at large separation (thereby forbidding negative dimension operators). Our short distance assumptions imply constraints on the contact terms that can occur in boundary correlation functions:

$$\lim_{\epsilon\to 0} \int\limits_{|\tau|<\epsilon} d\tau\, \tau^k \langle\hat{\phi}_a(\tau)\hat{\phi}_b(0)\rangle = \lim_{\epsilon\to 0} \int\limits_{|\tau|<\epsilon} d\tau\, \tau^k \langle\hat{\phi}_a(\tau)\hat{\theta}(0)\rangle = 0, \quad \text{for } k \geq 1. \tag{3.2}$$

The Ward identities for conformal Killing spinor fields are of particular interest, given bulk superconformal invariance. A spinor field $\hat{\epsilon}^r(x,\tau)$ is a conformal Killing spinor field if there exists a spinor field $\eta_s(x,\tau)$ such that

$$\partial^\mu \hat{\epsilon}^r = (\gamma^\mu)^r_s \hat{\eta}^s \tag{3.3}$$

(which means that the local supersymmetry transformation generated by $\hat{\epsilon}^r$ is compensated by the super-Weyl transformation generated by $\hat{\eta}^s$). In complex coordinates equation (3.3) reads

$$\partial^w \hat{\epsilon}^+ = 2\partial_{\bar{w}}\hat{\epsilon}^+ = 0, \quad \partial^{\bar{w}} \hat{\epsilon}^- = 2\partial_w\hat{\epsilon}^- = 0, \quad \partial^{\bar{w}} \hat{\epsilon}^+ = -4\mathrm{i}\hat{\eta}_+, \quad \partial^w \hat{\epsilon}^- = 4\mathrm{i}\hat{\eta}_- \tag{3.4}$$

So the conformal Killing condition is the condition that the components $\hat{\epsilon}^+$ and $\hat{\epsilon}^-$ be holomorphic and antiholomorphic, respectively (and complex conjugate to each other, to satisfy the euclidean reality condition).

We choose a certain special conformal Killing spinor field for each point τ' on the boundary:

$$\hat{\epsilon}^+(w) = \hat{\epsilon}_0 \cosh\left[\frac{\pi(w - \mathrm{i}\tau')}{\beta}\right], \quad \hat{\epsilon}^-(\bar{w}) = \hat{\epsilon}_0 \cosh\left[\frac{\pi(\bar{w} + \mathrm{i}\tau')}{\beta}\right], \tag{3.5}$$

where $\hat{\epsilon}_0$ is an arbitrary real fermionic constant. This special spinor field $\hat{\epsilon}^r(x,\tau)$ is antiperiodic in τ, satisfies the conformal Killing constraints (3.4) with

$$\hat{\eta}_+ = \hat{\epsilon}_0 \eta(w - \mathrm{i}\tau'), \quad \hat{\eta}_- = \hat{\epsilon}_0 \bar{\eta}(\bar{w} + \mathrm{i}\tau') \quad \eta(w) = \frac{\mathrm{i}\pi}{2\beta} \sinh\left(\frac{\pi w}{\beta}\right), \tag{3.6}$$

and satisfies the boundary condition (2.5) with boundary spinor field

$$\hat{\epsilon}(\tau) = \hat{\epsilon}_0 \cos\left[\frac{\pi(\tau - \tau')}{\beta}\right]. \tag{3.7}$$

Let us consider the Ward identity (2.6) corresponding to this special conformal spinor field, with the insertion of a single boundary fermion field $\hat{\phi}_a(\tau')$,

$$\langle \delta_{\hat{\epsilon}} \hat{\phi}_a(\tau') \rangle = \iint dx d\tau \; \partial^\mu \hat{\epsilon}^r(x,\tau) \langle G_{\mu r}(x,\tau) \hat{\phi}_a(\tau') \rangle. \tag{3.8}$$

Even though the special spinor field $\hat{\epsilon}^r$ blows up at large x, it can be used in the Ward identity because of the asymptotic condition (2.26) that follows from superconformal invariance of the bulk ground state. We can substitute on the left-hand side the global variation

$$\langle \delta_{\hat{\epsilon}} \hat{\phi}_a(\tau') \rangle = \hat{\epsilon}(\tau') \langle \delta \hat{\phi}_a(\tau') \rangle = \mathrm{i}\hat{\epsilon}_0 \langle \phi_a \rangle \tag{3.9}$$

because the first derivatives $\partial_\mu \hat{\epsilon}^r$ of our special spinor field vanish at the insertion point, and because any higher derivative contributing to $\delta_{\hat{\epsilon}} \hat{\phi}_a$ would have a negative dimension boundary operator as coefficient. By translation invariance in τ we can choose $\tau' = 0$ in the Ward identity (3.8) without loss of generality. Substituting (2.27) into (3.8), using the conformal Killing property (3.3) and dropping the common factor $\mathrm{i}\hat{\epsilon}_0$ we obtain

$$\begin{aligned} \langle \phi_a \rangle = & \iint dx d\tau \; \left[4\eta(\bar{w}) \langle G^{\mathrm{bulk}}_{w-}(x,\tau)\, \hat{\phi}_a(0) \rangle - 4\eta(w) \langle G^{\mathrm{bulk}}_{\bar{w}+}(x,\tau)\, \hat{\phi}_a(0) \rangle \right] \\ & + \int d\tau \; 4\eta(\mathrm{i}\tau) \left[\langle \hat{\theta}(\tau)\, \hat{\phi}_a(0) \rangle + \langle \frac{1}{2}(\hat{\theta}_{x-}(\tau) - \hat{\theta}_{x+}(\tau))\, \hat{\phi}_a(0) \rangle \right]. \end{aligned} \tag{3.10}$$

Taking into account the explicit form (3.6) of $\eta(w)$ we get

$$\langle \phi_a \rangle = \frac{1}{\beta} \frac{\partial \ln z}{\partial \lambda^a} = E - \frac{2\pi}{\beta} \int_0^\beta d\tau \; \sin(\pi\tau/\beta) \langle \hat{\theta}(\tau) \hat{\phi}_a(0) \rangle, \tag{3.11}$$

where

$$E = \iint dx d\tau \ \left[4\bar\eta(\bar w)\langle G^{\rm bulk}_{w-}(x,\tau)\hat\phi_a(0)\rangle - 4\eta(w)\langle G^{\rm bulk}_{\bar w+}(x,\tau)\hat\phi_a(0)\rangle\right] + \int d\tau \ 2\eta({\rm i}\tau)\left[\langle\hat\theta_{x-}(\tau)\hat\phi_a(0)\rangle - \langle\hat\theta_{x+}(\tau)\hat\phi_a(0)\rangle\right]. \quad (3.12)$$

We now argue that the quantity E vanishes under the assumptions on UV behavior. The correlation functions of the bulk currents $G^{\rm bulk}_{\bar w+}(x,\tau)$, $G^{\rm bulk}_{w-}(x,\tau)$ vanish up to contact terms, because of the bulk conformal invariance (2.23). Thus the two-point functions in the first line of (3.12) are linear combinations of $\delta(x)\delta(\tau)$ and its derivatives.[9] The assumptions on UV behavior then imply that the correlators $\langle G^{\rm bulk}_{\bar w+}(x,\tau)\hat\phi_a(0)\rangle$, $\langle G^{\rm bulk}_{w-}(x,\tau)$ $\hat\phi_a(0)\rangle$ are each proportional to $\delta(x)\delta(\tau)$. There are no higher order contact terms. Such terms however vanish upon integration in (3.12) because the functions $\eta(w)$, $\bar\eta(\bar w)$ vanish at the insertion point $x=0$, $\tau=0$. Therefore the term in the first line in (3.12) vanishes. The terms in the second line contain the operators $\hat\theta_{x\pm}$ that vanish by the equations of motion (2.29), so their correlators are pure contact terms. It follows from (3.2) that the contact terms in the correlators in the second line of E can be no more singular than $\delta(\tau)$, and hence vanish upon integration with $\eta({\rm i}\tau)$, which vanishes at $\tau=0$. Therefore $E=0$.

Next, we substitute $\beta^a\hat\phi_a$ for $\hat\theta$ in (3.11). The canonical UV behavior (3.2) makes this possible. The correlation function might be changed by a contact term, but nothing more singular than $\delta(\tau)$. The smearing function $\sin(\pi\tau/\beta)$ vanishes at $\tau=0$ so such a contact term would have no effect.[10] We obtain the gradient formula

$$\frac{\partial \ln z}{\partial\lambda^a} = -g^S_{ab}\beta^b \quad (3.13)$$

with

$$g^S_{ab} = 2\pi\int_0^\beta d\tau \ \sin(\pi\tau/\beta)\langle\hat\phi_a(\tau)\hat\phi_b(0)\rangle. \quad (3.14)$$

[9]There are no terms of the form $f(\tau)\delta(x)$ where f is a smooth function because the supercurrent has been split into bulk and boundary parts so that such terms are all contained in the $\langle\hat\theta(\tau)\hat\phi_i(0)\rangle$ correlators.

[10]A similar step is implicitly present in the proof of bosonic gradient formula given in [3].

To see that the metric g^S_{ab} is positive definite, we rewrite it

$$\int_0^\beta d\tau\ \sin(\pi\tau/\beta)\langle\hat{\phi}_a(\tau)\hat{\phi}_b(0)\rangle = \lim_{\epsilon\to 0}\int_\epsilon^{\beta-\epsilon} d\tau\ \sin(\pi\tau/\beta)\langle\hat{\phi}_a(\tau)\hat{\phi}_b(0)\rangle, \tag{3.15}$$

again making use of the canonical UV behavior (3.2). The operators $\hat{\phi}_a$ are self-adjoint, so the two-point function at finite separation is positive by reflection positivity. Therefore the right-hand side of (3.15) is positive.

The proof depends on the canonical UV behavior at three points: the vanishing of the term E in (3.11), the substitution of $\beta^a\hat{\phi}_a$ for $\hat{\theta}$, and the positivity of the metric. The issue in all three cases is that operator identities apply in correlation functions only up to contact terms. The technique of the present proof is a subtle improvement on the proof for the general gradient formula [3]. There we used the bulk and boundary conservation equations separately. Here we use the single Ward identity (2.6). This is more economic and also more transparent as we do not need to worry about the contact terms associated with the separate conservation equations. In essence the above euclidean proof hinges on the special Ward identity plus the assumptions about canonical UV behavior.

4 Proof of the gradient formula using real-time field theory

Here we give a second proof of the gradient formula (1.11), using real-time methods to evaluate

$$\frac{\partial \ln z}{\partial \lambda^a} = \beta\langle\, \phi_a\,\rangle = \beta\langle\,\{\hat{Q},\,\hat{\phi}_a\}\,\rangle. \tag{4.1}$$

First, we separate the supercharge into the contribution $\hat{q}_\epsilon(t)$ from a neighborhood of the boundary and the contribution $\hat{Q}_\epsilon(t)$ from the rest of the system:

$$\hat{q}_\epsilon(t) = \int_0^\epsilon dx\ \hat{\rho}(x,t) \quad \hat{Q}_\epsilon(t) = \hat{Q} - \hat{q}_\epsilon(t). \tag{4.2}$$

Let

$$f_{a,\epsilon}(\omega) = \int_{-\infty}^{\infty} dt\ \mathrm{e}^{\mathrm{i}\omega t}\langle\,\{\hat{q}_\epsilon(t),\,\hat{\phi}_a(0)\}\,\rangle, \tag{4.3}$$

$$F_{a,\epsilon}(\omega) = \int_{-\infty}^{\infty} dt\ \mathrm{e}^{\mathrm{i}\omega t}\langle\,\{\hat{Q}_\epsilon(t),\,\hat{\phi}_a(0)\}\,\rangle, \tag{4.4}$$

so that

$$2\pi\delta(\omega)\langle\, \phi_a \,\rangle = f_{a,\epsilon}(\omega) + F_{a,\epsilon}(\omega). \tag{4.5}$$

It is convenient to introduce an IR regulator $\delta > 0$ into equation (4.4),

$$F_{a,\epsilon}(\omega) = \lim_{\delta\to 0} \int_{-\infty}^{\infty} dt\; \mathrm{e}^{\mathrm{i}\omega t - \delta|t|} \langle\, \{\hat{Q}_\epsilon(t),\, \hat{\phi}_a(0)\}\,\rangle, \tag{4.6}$$

in order to regularize the singularity at $\omega = 0$ in intermediate stages of our calculation.

Locality tells us that, for t sufficiently near 0,

$$\{\hat{Q}_\epsilon(t),\, \hat{\phi}_a(0)\} = 0. \tag{4.7}$$

We combine this with charge conservation at $x = \epsilon$,

$$\partial_t \hat{Q}_\epsilon(t) = \hat{\jmath}^{\mathrm{bulk}}(\epsilon, t), \tag{4.8}$$

to get the identity

$$\{\hat{Q}_\epsilon(t),\, \hat{\phi}_a(0)\} = \int_0^t dt'\; \{\hat{\jmath}^{\mathrm{bulk}}(\epsilon, t'),\, \hat{\phi}_a(0)\}\;. \tag{4.9}$$

We use this identity in (4.6) to derive

$$\begin{aligned} F_{a,\epsilon}(\omega) &= \lim_{\delta\to 0} \left[\frac{R^+_{a,\epsilon}(\omega)}{\omega + \mathrm{i}\delta} + \frac{R^-_{a,\epsilon}(\omega)}{\omega - \mathrm{i}\delta} \right] \\ &= \mathrm{i}\pi\delta(\omega) \left[R^-_{a,\epsilon}(0) - R^+_{a,\epsilon}(0) \right] + \mathcal{P}(1/\omega) \left[R^+_{a,\epsilon}(\omega) + R^-_{a,\epsilon}(\omega) \right], \end{aligned} \tag{4.10}$$

where $R^\pm_{a,\epsilon}(\omega)$ are the response functions

$$R^\pm_{a,\epsilon}(\omega) = \pm \int_0^{\pm\infty} dt\; \mathrm{e}^{\mathrm{i}\omega t} \langle\, \{\mathrm{i}\hat{\jmath}^{\mathrm{bulk}}(\epsilon, t),\, \hat{\phi}_a(0)\}\,\rangle. \tag{4.11}$$

We do without the IR regulator δ in the construction of the response functions, because they are regular at $\omega = 0$, otherwise the correlation functions $F_{a,\epsilon}(\omega)$ would be more singular than $\delta(\omega)$, meaning that the real-time correlators would grow with time. $R^+_{a,\epsilon}(\omega)$ is analytic in the upper half-plane, and $R^-_{a,\epsilon}(\omega)$ is analytic in the lower half-plane.

The bulk supercurrent separates into the two chiral superconformal currents,

$$\hat{\jmath}^{\text{bulk}}(x,t) = -G^{\text{bulk}}_{w+}(x,t) - G^{\text{bulk}}_{\bar{w}-}(x,t). \tag{4.12}$$

Chirality implies that

$$\begin{aligned} G^{\text{bulk}}_{w+}(\epsilon,t) &= G^{\text{bulk}}_{w+}(\epsilon - t, 0), \qquad t < +\epsilon, \\ G^{\text{bulk}}_{\bar{w}-}(\epsilon,t) &= G^{\text{bulk}}_{\bar{w}-}(\epsilon + t, 0), \qquad t > -\epsilon \end{aligned} \tag{4.13}$$

so, by locality of the equal-time anti-commutators,

$$\begin{aligned} \{-\mathrm{i}G^{\text{bulk}}_{w+}(\epsilon,t),\, \hat{\phi}_a(0)\} &= 0, \qquad t < +\epsilon, \\ \{-\mathrm{i}G^{\text{bulk}}_{\bar{w}-}(\epsilon,t),\, \hat{\phi}_a(0)\} &= 0, \qquad t > -\epsilon \end{aligned} \tag{4.14}$$

so

$$\begin{aligned} \{\mathrm{i}\hat{\jmath}^{\text{bulk}}(\epsilon,t),\, \hat{\phi}_a(0)\} &= \{-\mathrm{i}G^{\text{bulk}}_{\bar{w}-}(\epsilon,t),\, \hat{\phi}_a(0)\}, \qquad t < +\epsilon, \\ \{\mathrm{i}\hat{\jmath}^{\text{bulk}}(\epsilon,t),\, \hat{\phi}_a(0)\} &= \{-\mathrm{i}G^{\text{bulk}}_{w+}(\epsilon,t),\, \hat{\phi}_a(0)\}, \qquad t > -\epsilon \end{aligned} \tag{4.15}$$

so we can write

$$\begin{aligned} R^{+}_{a,\epsilon}(\omega) &= \int_0^\infty \mathrm{d}t\ \mathrm{e}^{\mathrm{i}\omega t} \langle\, \{-\mathrm{i}G^{\text{bulk}}_{w+}(\epsilon,t),\, \hat{\phi}_a(0)\}\,\rangle \\ &= \int_{-\infty}^\infty dt\ \mathrm{e}^{\mathrm{i}\omega t} \langle\, \{-\mathrm{i}G^{\text{bulk}}_{w+}(\epsilon,t),\, \hat{\phi}_a(0)\}\,\rangle, \end{aligned} \tag{4.16}$$

$$\begin{aligned} R^{-}_{a,\epsilon}(\omega) &= \int_{-\infty}^0 dt\ \mathrm{e}^{\mathrm{i}\omega t} \langle\, \{-\mathrm{i}G^{\text{bulk}}_{\bar{w}-}(\epsilon,t),\, \hat{\phi}_a(0)\}\,\rangle \\ &= \int_{-\infty}^\infty dt\ \mathrm{e}^{\mathrm{i}\omega t} \langle\, \{-\mathrm{i}G^{\text{bulk}}_{\bar{w}-}(\epsilon,t),\, \hat{\phi}_a(0)\}\,\rangle. \end{aligned} \tag{4.17}$$

The dependence on ϵ is trivial because of the chirality, now in the form

$$G^{\text{bulk}}_{w+}(\epsilon,t) = G^{\text{bulk}}_{w+}(0, t-\epsilon), \qquad G^{\text{bulk}}_{\bar{w}-}(\epsilon,t) = G^{\text{bulk}}_{\bar{w}-}(0, t+\epsilon). \tag{4.18}$$

We have

$$R^{+}_{a,\epsilon}(\omega) = \mathrm{e}^{+\mathrm{i}\omega\epsilon} R^{+}_{a}(\omega), \qquad R^{-}_{a,\epsilon}(\omega) = \mathrm{e}^{-\mathrm{i}\omega\epsilon} R^{-}_{a}(\omega), \tag{4.19}$$

with

$$R_a^+(\omega) = \int_{-\infty}^{\infty} dt\; \mathrm{e}^{\mathrm{i}\omega t} \langle \{ -\mathrm{i} G^{\mathrm{bulk}}_{w+}(0,t),\, \hat\phi_a(0) \} \rangle,$$
$$R_a^-(\omega) = \int_{-\infty}^{\infty} dt\; \mathrm{e}^{\mathrm{i}\omega t} \langle \{ -\mathrm{i} G^{\mathrm{bulk}}_{\bar w-}(0,t),\, \hat\phi_a(0) \} \rangle. \tag{4.20}$$

The limit $\epsilon \to 0$ of equation (4.10) is now taken easily,

$$\begin{aligned} F_a(\omega) &\equiv \lim_{\epsilon\to 0} F_{a,\epsilon}(\omega) = \lim_{\delta\to 0} \left[\frac{R_a^+(\omega)}{\omega + \mathrm{i}\delta} + \frac{R_a^-(\omega)}{\omega - \mathrm{i}\delta} \right] \\ &= \mathrm{i}\pi\delta(\omega) \left[R_a^-(0) - R_a^+(0) \right] + \mathcal{P}(1/\omega) \left[R_a^+(\omega) + R_a^-(\omega) \right]. \end{aligned} \tag{4.21}$$

Then, from equation (4.5), we get the limit

$$f_a(\omega) \equiv \lim_{\epsilon\to 0} f_{a,\epsilon}(\omega) = 2\pi\delta(\omega) \langle \phi_a \rangle - F_a(\omega). \tag{4.22}$$

We have thus used the chirality of the bulk superconformal currents to construct the correlation functions of $\hat q(t) = \lim_{\epsilon\to 0} \hat q_\epsilon(t)$,

$$f_a(\omega) = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dt\; \mathrm{e}^{\mathrm{i}\omega t} \langle \{ \hat q_\epsilon(t),\, \hat\phi_a(0) \} \rangle = \int_{-\infty}^{\infty} dt\; \mathrm{e}^{\mathrm{i}\omega t} \langle \{ \hat q(t),\, \hat\phi_a(0) \} \rangle. \tag{4.23}$$

At this point, we could assume that $f_a(\omega)$ has no delta-function contribution at $\omega = 0$, and conclude from (4.22) that

$$\langle \phi_a \rangle = \frac{\mathrm{i}}{2} R_a^-(0) - \frac{\mathrm{i}}{2} R_a^+(0). \tag{4.24}$$

This is the assumption that the boundary correlators decay in time, that all boundary degrees of freedom return to equilibrium after any perturbation in the boundary. This is essentially the assumption that all boundary degrees of freedom couple to the bulk, thereby thermalizing. This tack was taken in [6]. In fact, we will not need to make this thermalization assumption to prove the gradient formula.

Our next step is to show that the global bulk superconformal invariance expresses itself by vanishing formulas

$$R_a^+(\mathrm{i}\pi/\beta) = 0, \qquad R_a^-(-\mathrm{i}\pi/\beta) = 0. \tag{4.25}$$

First, we use the usual relation between thermal correlation functions and expectation values of anti-commutators:

$$\langle\, \{G_{w+}^{\mathrm{bulk}}(0,t),\, \hat{\phi}_a(0)\}\,\rangle = \langle\, G_{w+}^{\mathrm{bulk}}(0,t)\,\hat{\phi}_a(0)\,\rangle + \langle\, G_{w+}^{\mathrm{bulk}}(0,t-i\beta)\,\hat{\phi}_a(0)\,\rangle \tag{4.26}$$

to obtain

$$\langle\, G_{w+}^{\mathrm{bulk}}(x,t)\,\hat{\phi}_a(0)\,\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\mathrm{e}^{\mathrm{i}\omega(x-t)}}{1+\mathrm{e}^{-\omega\beta}} R_a^+(\omega). \tag{4.27}$$

This expression analytically continues to euclidean time $\tau = \mathrm{i}t$ for $0 < \tau < \beta$,

$$\langle\, G_{w+}^{\mathrm{bulk}}(x,t)\,\hat{\phi}_a(0)\,\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\mathrm{e}^{\mathrm{i}\omega x-\omega\tau}}{1+\mathrm{e}^{-\omega\beta}} R_a^+(\omega). \tag{4.28}$$

If we take $x > 0$, we can deform the contour of integration into the upper half-plane, where the response function $R_a^+(\omega)$ is analytic. The euclidean correlation function is then expressed as a sum of the residues at the thermal poles

$$\langle\, G_{w+}^{\mathrm{bulk}}(x,\tau)\,\hat{\phi}_a(0)\,\rangle = \sum_{k=1}^{\infty} \mathrm{e}^{-\omega_k(x+i\tau)}\,\mathrm{i}\beta^{-1} R_a^+(\mathrm{i}\omega_k), \qquad \omega_k = \frac{2\pi}{\beta}\left(k-\frac{1}{2}\right). \tag{4.29}$$

The same thermal correlation function is given in the bulk quantization, where $-x$ is the euclidean time, as the matrix element

$$\langle\, G_{w+}^{\mathrm{bulk}}(x,\tau)\,\hat{\phi}_a(0)\,\rangle = \langle B|\hat{\phi}_a(0)\, G_{w+}^{\mathrm{bulk}}(x,\tau)|0\rangle, \tag{4.30}$$

where $|0\rangle$ is the superconformal bulk ground state and $\langle B|$ is the bulk state representing the boundary condition at $x = 0$. The global superconformal invariance condition in the bulk, $G_{-1/2}|0\rangle = 0$, implies that the $k = 1$ term vanishes in the sum (4.29) over the thermal poles. Therefore $R_a^+(\mathrm{i}\pi/\beta) = 0$. Similarly, using the analyticity of $R_a^-(\omega)$ in the lower half-plane and the global bulk superconformal condition $\bar{G}_{-1/2}|0\rangle = 0$, we derive the other vanishing formula $R_a^-(-\mathrm{i}\pi/\beta) = 0$. The error in these vanishing formulas is exponentially small in L/β, the exponent given by the scaling dimension of the most relevant operator in the bulk superconformal field theory.[11]

We can now derive a sum rule

$$\begin{aligned}\int \frac{d\omega}{2\pi}\,\frac{\pi^2/\beta^2}{\omega^2+\pi^2/\beta^2}\,F_a(\omega) \\ = \lim_{\delta\to 0}\int \frac{d\omega}{2\pi}\,\frac{\pi^2/\beta^2}{\omega^2+\pi^2/\beta^2}\,\frac{R_a^+(\omega)}{\omega+\mathrm{i}\delta} + \lim_{\delta\to 0}\int \frac{d\omega}{2\pi}\,\frac{\pi^2/\beta^2}{\omega^2+\pi^2/\beta^2}\frac{R_a^-(\omega)}{\omega-\mathrm{i}\delta} \\ = -\frac{\mathrm{i}}{2}R_a^+(\mathrm{i}\pi/\beta)+\frac{\mathrm{i}}{2}R_a^-(-\mathrm{i}\pi/\beta)=0. \qquad (4.31)\end{aligned}$$

The calculation starts from equation (4.21) for $F_a(\omega)$. In the first step, we can exchange the integral over ω with the removal of the IR regulator and separate the two integrals, as long as $R_a^\pm(\omega)/\omega^3$ is integrable at infinity. Then the contours of integration are deformed into the upper and lower half-planes, respectively. The growth condition on $R_a^\pm(\omega)$ justifies discarding the contours at infinity. The last step uses the vanishing formulas (4.25).

Canonical UV behavior at the boundary guarantees that $R_a^\pm(\omega)$ grows at most as ω, which more than satisfies the growth condition. The conformal supercurrents $G_{\mu r}^{\mathrm{bulk}}(x,t)$ have canonical dimension 3/2, whereas the boundary fields $\hat\phi_a(t)$ have canonical UV dimension 1/2. We assume, as an aspect of the canonical UV behavior, that there are no negative dimension boundary operators, so no such operators can occur in operator products of the bulk currents and the boundary fields. Therefore the response functions $R_a^\pm(\omega)$ defined by equations (4.20) and (4.20) have canonical UV dimension 1, and can grow no faster than ω at large ω. The leeway between the canonical growth rate ω and the growth rate ω^2 where the proof breaks down allows for the possibility of fermionic boundary fields with UV scaling dimensions slightly larger than 1/2, as in the $\alpha'\to 0$ limit of string theory.

Combining the sum rule (4.31) with equation (4.22), we get

$$\frac{\partial \ln z}{\partial \lambda^a} = \beta\langle\,\phi_a\,\rangle = \beta\int\frac{d\omega}{2\pi}\,\frac{\pi^2/\beta^2}{\omega^2+\pi^2/\beta^2}\,f_a(\omega). \qquad (4.32)$$

We substitute $\hat q(t) = -2\beta^a\hat\phi_a(t)$ in (4.23) to obtain

$$f_a(\omega) = \int_{-\infty}^{\infty} dt\;\mathrm{e}^{\mathrm{i}\omega t}\langle\,\{-2\beta^b\hat\phi_b(t),\,\hat\phi_a(0)\}\,\rangle = -2\beta^b f_{ab}(\omega) \qquad (4.33)$$

[11] A purely real-time proof of the gradient formula would require a real-time proof of the vanishing formulas from bulk superconformal invariance, without appealing to the euclidean field theory.

with

$$f_{ab}(\omega) = \int_{-\infty}^{\infty} dt\ \mathrm{e}^{\mathrm{i}\omega t} \langle\, \{ \hat{\phi}_b(t),\, \hat{\phi}_a(0) \}\, \rangle. \tag{4.34}$$

Now we have

$$\frac{\partial \ln z}{\partial \lambda^a} = \beta \langle\, \phi_a \,\rangle = -2\beta \int \frac{d\omega}{2\pi}\, \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\, f_{ab}(\omega)\beta^b, \tag{4.35}$$

which is the gradient formula

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}\beta^b \tag{4.36}$$

with metric

$$g^S_{ab} = \int d\omega\ \frac{\pi/\beta}{\omega^2 + \pi^2/\beta^2}\, f_{ab}(\omega). \tag{4.37}$$

We can integrate out ω to get the equivalent formula

$$g^S_{ab} = \int_{-\infty}^{\infty} dt\ \pi \mathrm{e}^{-\pi|t|/\beta} \langle\, \{ \hat{\phi}_b(t),\, \hat{\phi}_a(0) \}\, \rangle. \tag{4.38}$$

The assumption of canonical UV behavior implies that the correlation functions $f_{ab}(\omega)$ grow no faster than $|\omega|^0$, so the metric is well defined. By unitarity, the $f_{ab}(\omega)$, for each ω, form a nonnegative hermitian matrix, and $f_{ab}(-\omega) = f_{ba}(\omega)$, so the metric g^S_{ab} is symmetric and nonnegative. Any null vector for the metric, $g^S_{ab}v^a v^b = 0$, would be a null vector for $f_{ab}(\omega)$ for all ω, which would imply $v^a \hat{\phi}_a(t) = 0$, so $v^a = 0$, since the $\hat{\phi}_a$ are linearly independent. Therefore g^S_{ab} is a positive-definite metric on the space of boundary conditions.

Note that we have made no assumptions on the IR behavior of $f_a(\omega)$. Equation (4.22) allows for the possibility that $f_a(\omega)$ contains a long-time contribution proportional to $\delta(\omega)$, which is to say that the boundary energy could fail to thermalize after a local perturbation, as when the boundary contains a decoupled sub-system.

The assumption of canonical UV behavior at the boundary enters the real-time proof at several points. We defined the correlation functions of the boundary supercharge $\hat{q}(t)$ through the regularization procedure $\hat{q}(t) = \lim_{\epsilon\to 0} \hat{q}_\epsilon(t)$. We could have used some other regularized separation of the boundary from the rest of the system. This could have modified $\hat{q}(t)$ by some boundary operator, but that operator would have negative UV scaling dimension, which is excluded by the assumption of canonical UV behavior. We assumed canonical UV scaling of the correlation functions of the boundary fields with the bulk superconformal currents when we derived the superconformal sum rule (4.31). This requires an upper bound on the UV scaling dimensions of the boundary fields, and also the absence of negative dimension operators, which could have nonzero expectation values at finite temperature. Finally, we replaced $\hat{\theta}(t)$ by $\beta^a \hat{\phi}_a(t)$ in correlation functions.

The key step in the proof is the separation of the boundary from the rest of the system by means of the sum rule (4.31). Both bulk superconformal invariance and canonical UV behavior at the boundary are needed to derive the sum rule. The UV regularity makes it possible to write a sum rule if just one subtraction can be taken. The bulk superconformal invariance expressed in the vanishing formulas (4.25) allows us to make that subtraction (at a low thermal energy). The bulk superconformal invariance also enters at short distance when the chirality of the superconformal currents is used to construct the boundary supercharge. It would be good to have a physical understanding of the need for this combination of ultraviolet and infrared technical conditions.

5 Comparison of the two proofs

We should check that the two proofs yield the same gradient formula. The euclidean proof produces formula (3.14) for the metric in terms of the euclidean two-point functions of the boundary fields. The euclidean two-point functions can be written in terms of the real-time response functions

$$\langle \hat{\phi}_b(\tau)\hat{\phi}_a(0) \rangle_{eq} \frac{1}{2\pi} \int \mathrm{d}\omega \; \frac{\mathrm{e}^{-\omega\tau}}{1+\mathrm{e}^{-\beta\omega}} f_{ab}(\omega) \tag{5.1}$$

for $0 < \tau < \beta$. Substituting in the euclidean formula (3.14) and carrying out the integral over τ, we get the real-time formula

$$\begin{aligned} g^S_{ab} &= 2\pi \int_0^\beta d\tau \; \sin\left(\frac{\pi\tau}{\beta}\right) \frac{1}{2\pi} \int \mathrm{d}\omega \; \frac{\mathrm{e}^{-\omega\tau}}{1+\mathrm{e}^{-\beta\omega}} f_{ab}(\omega) \\ &= \int d\omega \; \frac{\pi/\beta}{\omega^2+\pi^2/\beta^2} \; f_{ab}(\omega), \end{aligned} \tag{5.2}$$

so the gradient formulas are the same.

In the euclidean proof, the choice of the special spinor field in the Ward identity is actually not unique. In particular, the real-time proof can be translated into a euclidean proof that uses a somewhat different special spinor field than (3.5), namely

$$\hat{\epsilon}^{+}(x,\tau) = \begin{cases} \hat{\epsilon}_0 \cos\left[\dfrac{\pi(\tau-\tau')}{\beta}\right], & 0 \le x \le \epsilon, \\ \hat{\epsilon}_0 \cosh\left[\dfrac{\pi(w-\epsilon-i\tau')}{\beta}\right], & \epsilon \le x, \end{cases} \tag{5.3}$$

$$\hat{\epsilon}^{-}(x,\tau) = \begin{cases} \hat{\epsilon}_0 \cos\left[\dfrac{\pi(\tau-\tau')}{\beta}\right], & 0 \le x \le \epsilon, \\ \hat{\epsilon}_0 \cosh\left[\dfrac{\pi(\bar{w}-\epsilon+i\tau')}{\beta}\right], & \epsilon \le x. \end{cases} \tag{5.4}$$

This special spinor field is constant in x within a collar $0 \le x < \epsilon$ around the boundary, and conformally Killing outside the collar.[12] This version of the proof perhaps has a slight advantage, since it uses directly an explicit construction of the correlation functions of $\hat{\theta}(\tau)$ from the physical correlation functions of $G_{\mu r}(x,\tau)$, by taking the limit $\epsilon \to 0$. The dependence on canonical UV behavior is somewhat rearranged between the two proofs, though not in any way that seems significant.

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[12]Strictly speaking, we should smooth over a small interval in $\epsilon > 0$ so that the special spinor field becomes smooth in x and τ. The proof is not affected by smoothing in ϵ, as can be seen, for example, in the real-time equation (4.19).

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