Adv. Theor. Math. Phys. **13** (2009) 217–257

Kähler–Ricci flow, Morse theory, and vacuum structure deformation of $N = 1$ supersymmetry in four dimensions

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Abstract

We address some aspects of four-dimensional chiral $N = 1$ supersymmetric theories on which the scalar manifold is described by Kähler geometry and can further be viewed as Kähler–Ricci soliton generating a one-parameter family of Kähler geometries. All couplings and solutions, namely the BPS domain walls and their supersymmetric Lorentz invariant vacua turn out to be evolved with respect to the flow parameter related to the soliton. Two models are discussed, namely $N = 1$ theory on Kähler–Einstein manifold and $U(n)$ symmetric Kähler–Ricci soliton with positive definite metric. In the first case, we find that the evolution of the soliton causes topological change and correspondingly, modifies the

e-print archive: http://lanl.arXiv.org/abs/hep-th/0708.1036

Morse index of the nondegenerate vacua realized in the parity transformation of the Hessian matrix of the scalar potential after hitting singularity, which is natural in the global theory and for nondegenerate Minkowskian vacua of the local theory. However, such situation is not trivial in anti de Sitter vacua. In an explicit model, we find that this geometric (Kähler–Ricci) flow can also change the index of the vacuum before and after singularity. Finally in the second case, since around the origin the metric is diffeomorphic to $\mathbb{C}\mathrm{P}^{n-1}$, we have to consider it in the asymptotic region. Our analysis shows that no index modification of vacua is present in both global and local theories.

1 Introduction

Geometric evolution equation which is called Ricci flow equation introduced by Hamilton in [1] has been prominently studied by mathematicians due to its achievement in solving famous three-dimensional puzzles, namely Poincaré and Thurston's geometrization conjectures [2].[1] Moreover, in higher dimensional manifold, particularly in compact Kähler manifold with the first Chern class $c_1 = 0$ or $c_1 < 0$, one can then reprove the well-known Calabi conjecture using the so called Kähler–Ricci flow equation [6]. Several authors have studied to construct a soliton of the Kähler–Ricci flow with $U(n)$ symmetry [7–9]. Such a solution is one of our interest and related to the subject of this paper.

In the physical context, this Kähler–Ricci flow can be regarded as a one-loop approximation to the physical renormalization group equation of two-dimensional theories at quantum level. For example, some authors have discussed it in the context of Wilsonian renormalization group coming from $N = 2$ supersymmetry in two and three dimensions [10, 11].[2] However this is not the case in higher dimension, particularly in four dimensions.

The purpose of this paper is to study the nature of chiral $N = 1$ supersymmetric theories in four dimensions together with its solutions, when the scalar manifold described by the Kähler manifold is no longer static. In the sense that it is a Kähler–Ricci soliton generating a one-parameter family of Kähler manifolds which evolves with respect to a parameter, say τ. This further imply that all couplings such as scalar potential, fermion mass matrix, and gravitino mass do have evolution with respect to the flow parameter τ.

[1] For a detailed and comprehensive proof of these conjectures is given in [3–5].
[2] We thank M. Nitta for informing us these papers.

Particularly in local theory such deformation happens in its solitonic solution, like BPS domain walls that have residual supersymmetry. This can be easily seen because all quantities such as warped factor, BPS equations, and the beta function of field theory in the context of anti de Sitter (AdS)/conformal field theory (CFT) correspondence (for a review, see for example [12]) do depend on τ. In this paper, we carry out the analysis on critical points of the scalar potential describing Lorentz invariant vacua on which the BPS equations and the beta function vanish. The first step is to consider the second-order analysis using Hessian matrix of the scalar potential which gives the index of the vacua. The next step is to perform a first-order analysis of the beta function for verifying the existence of an AdS vacuum which corresponds to a CFT in three dimensions. We realize our analysis into two models as follows.

In the first model we have a global and local $N = 1$ supersymmetric theories on the soliton where the initial manifold is Kähler–Einstein whose "cosmological constant" is chosen, for example, to be positive, namely $\Lambda > 0$. By taking $\tau \geq 0$ the flow turns out to be collapsed to a point at $\tau = 1/2\Lambda$ and correspondingly, both global and local $N = 1$ theories blow up. Furthermore, this geometric flow interpolates between two different theories which are disconnected by the singularity at $\tau = 1/2\Lambda$, namely $N = 1$ theories on Kähler–Einstein manifold with $\Lambda > 0$ in the interval $0 \leq \tau < 1/2\Lambda$, while for $\tau > 1/2\Lambda$ it becomes $N = 1$ theories on Kähler–Einstein manifold with $\Lambda < 0$ and opposite metric signature which yields a theory with wrong-sign kinetic terms and therefore, might be considered as "ghosts" at the quantum level.[3]

Consequently, in global $N = 1$ chiral theory such evolution of Kähler–Einstein soliton results in the deformation of the supersymmetric Minkowskian vacua which can be immediately observed from the parity transformation of the Hessian matrix of the scalar potential that changes the index of the vacua if they are nondegenerate. In local theory, we get the same situation for nondegenerate Minkowskian vacua. These happen because only the geometric (Kähler–Ricci) flow affects the index of the vacua in the second-order analysis.

In AdS ground states we have however a different situation. In this case both the geometric flow and the coupling quantities such as the holomorphic superpotential and the fermionic mass do play a role in determining the index of the vacua. Therefore, we have to enforce certain conditions on the Hessian matrix of the scalar potential in order to obtain the parity transformation. In addition, there is a possibility of having a condition

[3]This aspect will be addressed elsewhere.

where the degeneracy of the vacua can also be modified by the Kähler–Ricci flow. For example, in $\mathbb{C}\mathrm{P}^1$ model with linear superpotential, we find that this geometric flow transforms a nondegenerate vacuum to a degenerate vacuum and vice versa with respect to τ before and after singular point at $\tau = 1/4$. In this model, degenerate vacuum possibly could not exist particularly in the infrared (IR) (low energy scale) region.

Now we turn to consider the second model, namely $N = 1$ theories on $U(n)$ invariant Kähler–Ricci soliton constructed over line bundle of $\mathbb{C}\mathrm{P}^{n-1}$ with positive definite metric for $\tau \geq 0$ [9]. Near the origin, the soliton turns to the Kähler–Einstein manifold, namely $\mathbb{C}\mathrm{P}^{n-1}$. Hence, we have to analyze its behavior in the asymptotic region. By taking $\Lambda > 0$ case, we find that the soliton collapses into a Kähler cone at $\tau = 1/2\Lambda$ which is singular. Moreover, the soliton is asymptotically dominated by the cone metric which is positive definite for all $\tau \geq 0$. As a result the vacuum is static, or in other words it does not depend on τ. Therefore, no parity transformation of the Hessian matrix and no other vacuum modification caused by the geometric flow exist.

The organization of this paper is as follows. In Section 2 we introduce the notion of Kähler–Ricci flow equation with its solutions. Section 3 consists of two part discussions about the global and local $N = 1$ supersymmetry on general Kähler–Ricci soliton. Then, we construct BPS domain wall solutions in Section 4. In Section 5 we discuss the behavior of the supersymmetric Lorentz invariant vacua on Kähler–Einstein manifold and gives an explicit model. Section 6 provides a review of $U(n)$ symmetric Kähler–Ricci soliton together with our analysis on its vacua. We finally conclude our results in Section 7.

2 Evolution equation: Kähler–Ricci flow

This section is devoted to give a background about Ricci flow defined on a complex manifold endowed with Kähler metric which is called Kähler–Ricci flow. A remarkable fact is that the solution of this Kähler–Ricci flow remains Kähler as demanded by $N = 1$ supersymmetry. There are some excellent references on this subject for interested reader, for example, in [7–9].[4] We collect some evolution equations of Riemann curvature, Ricci tensor, and Ricci scalar in Appendix C.

A Kähler manifold $(\mathbf{M}, g(\tau))$ is a Kähler–Ricci soliton if it satisfies

$$\frac{\partial g_{i\bar{j}}}{\partial \tau}(z, \bar{z}; \tau) = -2R_{i\bar{j}}(z, \bar{z}; \tau), \quad 0 \leq \tau < T, \tag{2.1}$$

[4] We also recommend [13] for a detailed look on Ricci flow.

where $(z, \bar{z}) \in \mathbf{M}$. Furthermore, the flow $g(\tau)$ has the form

$$g(\tau) = \sigma(\tau)\psi_\tau^*(g(0)), \quad 0 \leq \tau < T, \tag{2.2}$$

with $\sigma(\tau) \equiv (1 - 2\Lambda\tau)$, the map ψ_τ is a diffeomorphism, and $g(0)$ is the initial metric at $\tau = 0$ which fulfills the following identity

$$-2R_{i\bar{j}}(0) = \nabla_i Y_{\bar{j}}(0) + \bar{\nabla}_{\bar{j}} Y_i(0) - 2\Lambda g_{i\bar{j}}(0), \tag{2.3}$$

for some real constant Λ and some holomorphic vector fields[5]

$$Y(0) = Y^i(z,0)\partial_i + Y^{\bar{i}}(\bar{z},0)\bar{\partial}_{\bar{i}}, \tag{2.4}$$

on $\mathbf{M}$ where $i, j = 1, \ldots, \dim_{\mathbb{C}}(\mathbf{M})$. The vector field $Y(0)$ is related to a τ-dependent vector field $X(\tau)$ by

$$X(\tau) = \frac{1}{\sigma(\tau)} Y(0), \tag{2.5}$$

which generates a family of diffeomorphisms ψ_τ and satisfies the following equation

$$\begin{aligned} \frac{\partial \hat{z}^i}{\partial \tau} &= X^i(\hat{z}, \tau), \\ \frac{\partial \hat{\bar{z}}^{\bar{i}}}{\partial \tau} &= X^{\bar{i}}(\hat{\bar{z}}, \tau), \end{aligned} \tag{2.6}$$

with

$$\hat{z} \equiv \psi_\tau(z). \tag{2.7}$$

2.1 Kähler–Einstein manifold

We turn to consider the case where $Y(0)$ vanishes. In this case, the initial metric is Kähler–Einstein

$$R_{i\bar{j}}(0) = \Lambda\, g_{i\bar{j}}(0), \tag{2.8}$$

for some constants $\Lambda \in \mathbb{R}$. Taking

$$g_{i\bar{j}}(\tau) = \rho(\tau)\, g_{i\bar{j}}(0), \tag{2.9}$$

and then from the definition of Ricci tensor, one finds that

$$R_{i\bar{j}}(\tau) = R_{i\bar{j}}(0) = \Lambda\, g_{i\bar{j}}(0). \tag{2.10}$$

So the solution of (2.1) can be obtained as

$$g_{i\bar{j}}(\tau) = (1 - 2\Lambda\tau)\, g_{i\bar{j}}(0), \tag{2.11}$$

with $\rho(\tau) = \sigma(\tau) = (1 - 2\Lambda\tau)$. Moreover, the Kähler potential has the form

$$K(\tau) = (1 - 2\Lambda\tau)\, K(0). \tag{2.12}$$

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[5]Holomorphicity of $Y(0)$ follows from the fact that the complex structure J on $\mathbf{M}$ satisfies $\psi_\tau^*(J) = J$.

If we take the Kähler–Ricci flow (2.1) to evolve for $\tau \geq 0$, then for the case $\Lambda < 0$ the metric (2.11) is smoothly expanded and no collapsing flow exists. On the other hand, in the case $\Lambda > 0$ it collapses to a point at $\tau = 1/2\Lambda$, but reappear for $\tau > 1/2\Lambda$ which is diffeomorphic to a manifold with $\Lambda < 0$ and its metric signature is opposite with the initial metric defined in (2.8).[6]

Note that in both cases the flow also becomes singular as $\tau \to +\infty$. In addition, if the initial metric is Ricci flat, namely $\Lambda = 0$, then it is clear that the metric does not change under (2.1).

2.2 Gradient Kähler–Ricci soliton

Now let us discuss the $Y(0) \neq 0$ case. Our particular interest is the case where the holomorphic vector field (2.5) can also be written as

$$Y^i(0) = g^{i\bar{j}} \bar{\partial}_{\bar{j}} P(z, \bar{z}) \tag{2.13}$$

for some real functions $P(z, \bar{z})$ on $\mathbf{M}$. Thus, we say that $g(\tau)$ is a gradient Kähler–Ricci soliton [7, 8]. In this case, in general, there are three possible solutions. For $\Lambda > 0$ we have a shrinking gradient Kähler–Ricci soliton, whereas the soliton is an expanding gradient Kähler–Ricci soliton for $\Lambda < 0$. The case where $\Lambda = 0$ is a steady gradient Kähler–Ricci soliton. In fact, on any compact Kähler manifold, a gradient steady or expanding Kähler–Ricci soliton is necessarily a Kähler–Einstein manifold because the real function $P(z, \bar{z})$ is just a constant [3]. We give an explicit construction of these expanding and shrinking Kähler–Ricci solitons in Section 6.

3 $N = 1$ supersymmetric theory on Kähler–Ricci soliton

In this section we discuss $N = 1$ supersymmetric theory in four dimensions whose nonlinear σ-model describing a Kähler geometry[7] $(\mathbf{M}, g(\tau))$ satisfies one-parameter Kähler–Ricci flow equation (2.1). This equation tells us that the dynamics of the metric $g_{i\bar{j}}$ with respect to τ is determined by the Ricci tensor $R_{i\bar{j}}$. Thus, it follows that the $N = 1$ theory is deformed with respect to τ as discussed in the following. We divide the discussion into two parts. In the first part we construct a global $N = 1$ chiral supersymmetry as a toy model. Then, in the second part we extend the construction to a local

[6] See discussion in Sections 5.1 and 5.2.

[7] Note that in four dimensional global $N = 1$ supersymmetry the scalar manifold $\mathbf{M}$ is Kähler, whereas local $N = 1$ supersymmetry demands that the scalar manifold $\mathbf{M}$ has to be Hodge–Kähler [14, 15], see the next subsection.

supersymmetry, namely $N = 1$ chiral supergravity which corresponds to our analysis for the most part of this paper.

3.1 Global $N = 1$ chiral supersymmetry

First of all, let us focus on the properties of the deformed global $N = 1$ chiral supersymmetric theory in four dimensions. The spectrum of the theory consists of a spin-$\frac{1}{2}$ fermion χ^i and a complex scalar z^i with $i, j = 1, \ldots, n_c$. As mentioned above, the complex scalars $(\bar{z}^{\bar{i}}, z^i)$ parameterize a Kähler geometry $(\mathbf{M}, g(\tau))$. The construction of the global $N = 1$ theory on Kähler–Ricci soliton is as follows. First, we consider the chiral Lagrangian in (for a pedagogical review of $N = 1$ supersymmetry, see for example [14]), say $\mathcal{L}^0_{\text{global}}$, where the metric of the scalar manifold is static. Then, replacing all geometric quantities such as the metric $g_{i\bar{j}}(0)$ by the soliton $g_{i\bar{j}}(\tau)$, the on-shell $N = 1$ chiral Lagrangian can be written down up to 4-fermion term as

$$\begin{aligned} \mathcal{L}_{\text{global}} &= g_{i\bar{j}}(\tau)\, \partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} - \frac{\mathrm{i}}{2} g_{i\bar{j}}(\tau) \left(\bar{\chi}^i \gamma^\mu \partial_\mu \chi^{\bar{j}} + \bar{\chi}^{\bar{j}} \gamma^\mu \partial_\mu \chi^i \right) \\ &\quad + M_{ij}(\tau) \bar{\chi}^i \chi^j + \bar{M}_{\bar{i}\bar{j}}(\tau)\ \bar{\chi}^{\bar{i}} \chi^{\bar{j}} - V(\tau), \end{aligned} \tag{3.1}$$

where the scalar potential has the form

$$V(z, \bar{z}; \tau) = g^{i\bar{j}}(\tau)\, \partial_i W\, \bar{\partial}_{\bar{j}} \bar{W}, \tag{3.2}$$

with $W \equiv W(z)$. The metric $g_{i\bar{j}}(\tau) \equiv \partial_i \bar{\partial}_{\bar{j}} K(\tau)$ is the solution of (2.1), where the real function $K(\tau)$ is a Kähler potential. Fermionic mass-like quantity $M_{ij}(\tau)$ is then given by

$$M_{ij}(\tau) \equiv \nabla_i \partial_j W = \partial_i \partial_j W - \Gamma^k_{ij}(\tau)\, \partial_k W. \tag{3.3}$$

and it is related to the first derivative of the scalar potential (3.2) with respect to $(z, \bar{z})$

$$\begin{aligned} \partial_i V &= g^{j\bar{l}}(\tau) M_{ij}(\tau)\, \bar{\partial}_{\bar{l}} \bar{W}, \\ \partial_{\bar{i}} V &= g^{\bar{j}l}(\tau) \bar{M}_{\bar{i}\bar{j}}(\tau) \partial_l W. \end{aligned} \tag{3.4}$$

The supersymmetry transformation of the fields up to 3-fermion terms leaving the Lagrangian invariant (3.1) are

$$\begin{aligned} \delta z^i &= \bar{\chi}^i \epsilon_1, \\ \delta \chi^i &= \mathrm{i} \partial_\mu z^i \gamma^\mu \epsilon^1 + g^{i\bar{j}}(\tau) \bar{\partial}_{\bar{j}} \bar{W} \epsilon_1. \end{aligned} \tag{3.5}$$

Since the evolution of the metric $g_{i\bar{j}}(\tau)$ is determined by the Ricci tensor $R_{i\bar{j}}(\tau)$, one can directly see that the dynamics of the scalar potential (3.2) and the fermionic mass (3.3) has a similar behavior with respect to τ, namely

$$\begin{aligned}\frac{\partial V(\tau)}{\partial\tau} &= 2R^{i\bar{j}}(\tau)\,\partial_i W\,\bar{\partial}_{\bar{j}}\bar{W},\\ \frac{\partial M_{ij}(\tau)}{\partial\tau} &= 2g^{k\bar{l}}(\tau)\nabla_i R_{j\bar{l}}(\tau)\,\partial_k W,\end{aligned} \tag{3.6}$$

where $R^{i\bar{j}} \equiv g^{i\bar{l}}g^{k\bar{j}}R_{k\bar{l}}$.

To make the above formula clearer, we now turn to discuss an example where the initial geometry (at $\tau = 0$) is Kähler–Einstein manifold, as discussed in section 2. This follows that the equations in (3.6) are simplified into

$$\begin{aligned}\frac{\partial V(\tau)}{\partial\tau} &= 2\Lambda\,\sigma(\tau)^{-2}\,g^{i\bar{j}}(0)\,\partial_i W\,\bar{\partial}_{\bar{j}}\bar{W} = 2\Lambda\,\sigma(\tau)^{-2}\,V(0),\\ \frac{\partial M_{ij}(\tau)}{\partial\tau} &= 0.\end{aligned} \tag{3.7}$$

The first equation in (3.7) has the solution

$$V(\tau) = \sigma(\tau)^{-1}\,V(0), \tag{3.8}$$

whereas the second equation tells us that the fermion mass matrix $M_{ij}(\tau)$ does not depend on τ. As the flow (2.11) deforms for $\tau \geq 0$, the Lagrangian (3.1) can be smoothly defined and there is no singularity for the case $\Lambda < 0$. This follows that in this case we have a well-defined theory as the Kähler geometry changed with respect to τ. However, in the case $\Lambda > 0$ since the flow (2.11) shrinks to a point at $\tau = 1/2\Lambda$ the scalar potential (3.8) becomes singular and the Lagrangian (3.1) diverges. Moreover, as mentioned above for $\tau > 1/2\Lambda$, the flow evolves again and it is related to another $N = 1$ global theory on a Kähler–Einstein manifold with $\Lambda < 0$ and different metric signature.

To be precise, let $\mathbf{M}_0$ be the initial geometry which is Kähler–Einstein manifold with $\Lambda > 0$ whose metric $g_{i\bar{j}}(0)$ is positive definite. Next, we cast (2.10) to the following form

$$R_{i\bar{j}}(\tau) = \widehat{\Lambda}(\tau)\,g_{i\bar{j}}(\tau), \tag{3.9}$$

where

$$\begin{aligned}\widehat{\Lambda}(\tau) &\equiv \sigma(\tau)^{-1}\Lambda,\\ g_{i\bar{j}}(\tau) &\equiv \sigma(\tau)\,g_{i\bar{j}}(0).\end{aligned} \tag{3.10}$$

In the interval $0 \leq \tau < 1/2\Lambda$, the solution is diffeomorphic to the initial geometry. However, for $\tau > 1/2\Lambda$ we have a Kähler–Einstein geometry $\widehat{\mathbf{M}_0}$ with a negative definite metric $g_{i\bar{j}}(\tau)$ and $\widehat{\Lambda}(\tau) < 0$. As mentioned in the introduction, such a metric produces a theory with wrong-sign kinetic terms and could be interpreted as ghosts.

Similar result can also be achieved for the case where the initial geometry is a Kähler–Einstein geometry with indefinite metric where $\Lambda > 0$ [16]. Then, we have also an indefinite Kähler–Einstein metric but with $\widehat{\Lambda} < 0$ for $\tau > 1/2\Lambda$.

3.2 $N = 1$ chiral supergravity

In this subsection, we generalize the previous result to a local $N = 1$ supersymmetry. In four dimensions the spectrum of a generic chiral $N = 1$ supergravity theory consists of a gravitational multiplet and n_c chiral multiplets. These multiplets are decomposed of the following component fields:

- A gravitational multiplet

$$(g_{\mu\nu}, \psi^1_\mu), \quad \mu = 0, \ldots, 3. \tag{3.11}$$

  This multiplet consist of the graviton $g_{\mu\nu}$ and a gravitino ψ^1_μ. For the gravitino $\psi^1_\mu, \psi_{1\mu}$ and the upper or lower index denotes left or right chirality, respectively.
- n_c chiral multiplets

$$(z^i, \chi^i), \quad i = 1, \ldots, n_c. \tag{3.12}$$

  Each chiral multiplet consist of a spin-$\frac{1}{2}$ fermion χ^i and a complex scalar z^i.

Local supersymmetry further requires the scalar manifold $\mathbf{M}$ spanned by the complex scalars $(\bar{z}^{\bar{i}}, z^i)$ to be a Hodge–Kähler manifold with an additional $U(1)$-connection

$$Q \equiv -\frac{1}{M_P^2}\left(K_i\, dz^i - K_{\bar{i}}\, d\bar{z}^{\bar{i}}\right), \tag{3.13}$$

where $K_i \equiv \partial_i K$ and the metric $g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$ where the Kähler potential K is an arbitrary real function [14, 15]. In our case since $g_{i\bar{j}}$ is the solution of (2.1), then K and Q must be τ-dependent.

Similar as in the global case, we first consider the chiral theory studied in [14, 15]. Then, by employing the same procedure the $N = 1$ supergravity

Lagrangian up to four-fermions terms on Kähler–Ricci soliton has the following form

$$\begin{aligned}
\mathcal{L}_{\rm local} = & -\frac{M_P^2}{2}R + g_{i\bar{j}}(\tau)\partial_\mu z^i \partial^\mu \bar{z}^{\bar{i}} + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-h}}\left(\bar{\psi}^1_\mu \gamma_\sigma \widetilde{\nabla}_\nu \psi_{1\lambda} - \bar{\psi}_{1\mu}\gamma_\sigma \widetilde{\nabla}_\nu \psi^1_\lambda\right) \\
& - \frac{\rm i}{2} g_{i\bar{j}}(\tau)\left(\bar{\chi}^i \gamma^\mu \nabla_\mu \chi^{\bar{j}} + \bar{\chi}^{\bar{j}}\gamma^\mu \nabla_\mu \chi^i\right) \\
& - g_{i\bar{j}}(\tau)\left(\bar{\psi}_{1\nu}\gamma^\mu\gamma^\nu \chi^i \partial_\mu \bar{z}^{\bar{j}} + \bar{\psi}^1_\nu \gamma^\mu \gamma^\nu \chi^{\bar{j}} \partial_\mu z^i\right) \\
& + L(\tau)\,\bar{\psi}^1_\mu \gamma^{\mu\nu}\psi^1_\nu + \bar{L}(\tau)\,\bar{\psi}_{1\mu}\gamma^{\mu\nu}\psi_{1\nu} + {\rm i} g_{i\bar{j}}(\tau) \\
& \times \left(\bar{N}^{\bar{j}}(\tau)\bar{\chi}^i \gamma^\mu \psi^1_\mu + N^i(\tau)\bar{\chi}^{\bar{j}}\gamma^\mu \psi_{1\mu}\right) \\
& + \mathcal{M}_{ij}(\tau)\bar{\chi}^i \chi^j + \bar{\mathcal{M}}_{\bar{i}\bar{j}}(\tau)\ \bar{\chi}^{\bar{i}}\chi^{\bar{j}} - \mathcal{V}(\tau),
\end{aligned} \tag{3.14}$$

where $h \equiv \det(g_{\mu\nu})$ and $L(\tau)(\bar{L}(\tau))$ can be written in terms of an (anti)-holomorphic superpotential function of $W(z)(\bar{W}(\bar{z}))$,

$$\begin{aligned}
L(\tau) &= e^{K(\tau)/2M_P^2} W(z), \\
\bar{L}(\tau) &= e^{K(\tau)/2M_P^2} \bar{W}(\bar{z}),
\end{aligned} \tag{3.15}$$

with $K(\tau)$ is a Kähler potential of the chiral multiplets, and the quantities $N^i(\tau), \mathcal{M}_{ij}(\tau)$ are given by:

$$\begin{aligned}
N^i(\tau) &= g^{i\bar{j}}(\tau)\bar{\nabla}_{\bar{j}}\bar{L}(\tau), \\
\mathcal{M}_{ij}(\tau) &= \frac{1}{2}\nabla_i \nabla_j L(\tau).
\end{aligned} \tag{3.16}$$

On the other hand, the $N = 1$ scalar potential can be expressed in terms of $L(\bar{L})$ as

$$\mathcal{V}(\tau) = g^{i\bar{j}}(\tau)\nabla_i L(\tau)\,\bar{\nabla}_{\bar{j}}\bar{L}(\tau) - \frac{3}{M_P^2}L(\tau)\bar{L}(\tau), \tag{3.17}$$

where $\nabla_i L(\tau) = \partial_i L(\tau) + \frac{1}{2M_P^2}K_i(\tau)\,L(\tau)$. Then the first derivative of $\mathcal{V}(\tau)$ with respect to $(\bar{z}^{\bar{i}}, z^i)$ are of the form

$$\begin{aligned}
\partial_{\bar{i}}\mathcal{V} &= \frac{1}{2}\mathcal{M}_{kj}(\tau)N^j(\tau) - \frac{1}{M_P^2}N_k(\tau)\bar{L}(\tau), \\
\partial_{\bar{i}}\mathcal{V} &= \frac{1}{2}\bar{\mathcal{M}}_{\bar{k}\bar{j}}(\tau)\bar{N}^{\bar{j}}(\tau) - \frac{1}{M_P^2}\bar{N}_{\bar{k}}(\tau)L(\tau).
\end{aligned} \tag{3.18}$$

The supersymmetry transformation laws up to 3-fermion terms leaving invariant (3.14) are:

$$\begin{aligned}
\delta\psi_{1\mu} &= M_P\left(\widetilde{\mathcal{D}}_\mu\epsilon_1 + \frac{\mathrm{i}}{2}L(\tau)\,\gamma_\mu\,\epsilon^1\right),\\
\delta\chi^i &= \mathrm{i}\partial_\mu z^i\gamma^\mu\epsilon^1 + N^i(\tau)\epsilon_1,\\
\delta e^a_\mu &= -\frac{1}{M_P}(\mathrm{i}\bar{\psi}_{1\mu}\gamma^a\epsilon^1 + \mathrm{i}\bar{\psi}^{1\mu}\gamma^a\epsilon_1),\\
\delta z^i &= \bar{\chi}^i\epsilon_1,
\end{aligned} \tag{3.19}$$

where $\widetilde{\mathcal{D}}_\mu\epsilon_1 = \partial_\mu\epsilon_1 - \frac{1}{4}\gamma_{ab}\,\omega^{ab}_\mu\epsilon_1 + \frac{\mathrm{i}}{2}Q_\mu(\tau)\epsilon_1$, and $Q_\mu(\tau)$ is a $U(1)$-connection which is the first derivative of the Kähler potential $K(\tau)$ with respect to x^μ. Here, we have defined $\epsilon_1 \equiv \epsilon_1(x,\tau)$. Furthermore the evolution of the coupling quantities beside the metric $g_{i\bar{j}}(\tau)$ in (3.14) are given by

$$\begin{aligned}
\frac{\partial L(\tau)}{\partial\tau} &= \frac{K_\tau(\tau)}{2M_P^2}L(\tau),\\
\frac{\partial N^i(\tau)}{\partial\tau} &= 2R^i_j(\tau)N^j(\tau) + \frac{K_\tau(\tau)}{2M_P^2}N^i(\tau) + g^{i\bar{j}}(\tau)\frac{K_{\bar{j}\tau}(\tau)}{M_P^2}\bar{L}(\tau),\\
\frac{\partial\mathcal{V}(\tau)}{\partial\tau} &= \frac{\partial N^i(\tau)}{\partial\tau}N_i(\tau) + \frac{\partial N_i(\tau)}{\partial\tau}N^i(\tau) - \frac{3K_\tau(\tau)}{M_P^2}|L(\tau)|^2,\\
\frac{\partial\mathcal{M}_{ij}(\tau)}{\partial\tau} &= -R_{j\bar{l}}(\tau)\nabla_i\bar{N}^{\bar{l}}(\tau)\\
&\quad + \frac{1}{2}g_{j\bar{l}}(\tau)\left(\partial_i\frac{\partial\bar{N}^{\bar{l}}(\tau)}{\partial\tau} + \frac{K_{i\tau}(\tau)}{2M_P^2}\bar{N}^{\bar{l}}(\tau) + \frac{K_i(\tau)}{2M_P^2}\frac{\partial\bar{N}^{\bar{l}}(\tau)}{\partial\tau}\right).
\end{aligned} \tag{3.20}$$

Similar as in the global case, we finally take an example where the initial metric is Kähler–Einstein. In this case the metric evolves with respect to τ as in (2.11) and the $U(1)$-connection takes the form

$$Q(\tau) \equiv -\frac{\sigma(\tau)}{M_P^2}\left(\partial_i K(0)dz^i - \bar{\partial}_{\bar{i}}K(0)d\bar{z}^{\bar{i}}\right), \tag{3.21}$$

since the Kähler potential is given by (2.12). Taking $\tau \geq 0$, the theory is well defined for $\Lambda < 0$, while for $\Lambda > 0$ it becomes singular at $\tau = 1/2\Lambda$. In the latter case, namely $\Lambda > 0$ case, the $N = 1$ local theory is changed and the shape of the scalar manifold is a Kähler–Einstein manifold with $\Lambda < 0$ and moreover, the metric signature has opposite sign. Thus, we have a gravitational multiplet coupled to chiral multiplets which might be considered as ghosts.

4 Deformation of $N = 1$ supergravity BPS domain walls

Here, we turn our attention to discuss a ground state which breaks Lorentz invariance partially and preserves half of the supersymmetry of the parental theory, namely the BPS domain wall in $N = 1$ supergravity. This type of solution was firstly discovered in [17]. Interested reader can further consult [18] for an excellent review of this subject. In this section some conventions follow rather closely to [19, 20].

The first step is to take the ansatz metric as

$$ds^2 = a^2(u,\tau)\,\eta_{\underline{\nu}\,\underline{\lambda}}\,dx^{\underline{\nu}}\,dx^{\underline{\lambda}} - du^2, \tag{4.1}$$

where $\underline{\nu}, \underline{\lambda} = 0, 1, 2,\ \eta_{\underline{\nu}\,\underline{\lambda}}$ is a three-dimensional Minkowskian metric, and then the corresponding Ricci scalar has the form

$$R = 6\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right], \tag{4.2}$$

where $a' \equiv \partial a/\partial u$. Here, $a(u,\tau)$ is the warped factor assumed to be τ dependent. For the shake of simplicity we set $\psi_{1\mu} = \chi^i = 0$ on the background (4.1) and then, the supersymmetry transformation (3.19) becomes

$$\begin{aligned}
\frac{1}{M_P}\delta\psi_{1u} &= D_u\,\epsilon_1 + L(\tau)\gamma_u\epsilon^1 + \frac{\mathrm{i}}{2}Q_u(\tau)\,\epsilon_1,\\
\frac{1}{M_P}\delta\psi_{1\underline{\nu}} &= \partial_{\underline{\nu}}\,\epsilon_1 + \frac{1}{2}\gamma_{\underline{\nu}}\left(-\frac{a'}{a}\gamma_3\epsilon_1 + \mathrm{i}e^{K(\tau)/2M_P^2}\,W\epsilon^1\right) + \frac{\mathrm{i}}{2}Q_{\underline{\nu}}(\tau)\,\epsilon_1,\\
\delta\chi^i &= \mathrm{i}\partial_\mu z^i\,\gamma^\mu\epsilon^1 + N^i(\tau)\epsilon_1.
\end{aligned} \tag{4.3}$$

Furthermore, all supersymmetric variations (4.3) vanish in order to have residual supersymmetry on the ground states. We then simply assign that ϵ_1 and z^i depend on both u and τ. Thus, the first equation in (4.3) shows that ϵ_1 indeed depends on u, while the second equation gives a projection equation

$$\frac{a'}{a}\gamma_3\epsilon_1 = \mathrm{i}L(\tau)\epsilon^1, \tag{4.4}$$

which further gives

$$\frac{a'}{a} = \pm\,|L(\tau)|. \tag{4.5}$$

Thus, equation (4.5) shows that the warped factor a is indeed τ-dependent, which is consistent with our ansatz (4.1). Introducing a real function $\mathcal{W}(\tau) \equiv$

$|L(\tau)|$, the third equation in (4.3) results in a set of BPS equations

$$z^{i\prime} = \mp 2 g^{i\bar{j}}(\tau) \bar{\partial}_{\bar{j}} \mathcal{W}(\tau),$$
$$\bar{z}^{\bar{i}\prime} = \mp 2 g^{j\bar{i}}(\tau) \partial_j \mathcal{W}(\tau), \tag{4.6}$$

describing gradient flows. It is important to note that using (4.6) it can be shown that (4.5) is a monotonic decreasing function and corresponds to the c -function in the holographic correspondence [21]. Moreover, in the context of CFT the supersymmetric flows which is relevant for our analysis are described by a beta function

$$\beta^i \equiv a \frac{dz^i}{da} = -2 g^{i\bar{j}}(\tau) \frac{\bar{\partial}_{\bar{j}} \mathcal{W}}{\mathcal{W}}, \tag{4.7}$$

together with its complex conjugate, after employing (4.5) and (4.6). In this description the scalar fields play a role as coupling constants and the warp factor a can be viewed as an energy scale [21, 22].

In the analysis it is convenient that the potential (3.17) can be shaped into

$$\mathcal{V}(z, \bar{z}; \tau) = 4\, g^{i\bar{j}}\, \partial_i \mathcal{W}\, \bar{\partial}_{\bar{j}} \mathcal{W} - \frac{3}{M_P^2}\, \mathcal{W}^2, \tag{4.8}$$

whose first derivative with respect to $(z, \bar{z})$ is given by a set of the following equation

$$\partial_i \mathcal{V} = 4\, g^{j\bar{k}}\, \nabla_i \partial_j \mathcal{W}\, \bar{\partial}_{\bar{k}} \mathcal{W} + 4\, g^{j\bar{k}}\, \partial_j \mathcal{W}\, \partial_i \bar{\partial}_{\bar{k}} \mathcal{W} - \frac{6}{M_P^2}\, \mathcal{W}\, \partial_i \mathcal{W},$$
$$\bar{\partial}_{\bar{i}} \mathcal{V} = 4\, g^{j\bar{k}}\, \bar{\nabla}_{\bar{i}} \bar{\partial}_{\bar{k}} \mathcal{W}\, \partial_j \mathcal{W} + 4\, g^{j\bar{k}}\, \bar{\partial}_{\bar{k}} \mathcal{W}\, \bar{\partial}_{\bar{i}} \partial_j \mathcal{W} - \frac{6}{M_P^2}\, \mathcal{W}\, \bar{\partial}_{\bar{i}} \mathcal{W}, \tag{4.9}$$

where $\nabla_i \partial_j \mathcal{W} = \partial_i \partial_j \mathcal{W} - \Gamma^k_{ij} \partial_k \mathcal{W}$. Note that as mentioned in the preceding section, it is possible that the theory becomes unphysical in the sense that it has wrong-sign kinetic term for finite τ. For example, it happens when the flow described by (2.11) for $\Lambda > 0$ and $\tau > 1/2\Lambda$. So, we have BPS domain walls whose dynamics described by the warped factor $a(u, \tau)$ are controlled by ghosts.

Now let us turn to consider the gradient flow equation (4.6) and the first derivative of the scalar potential (4.9). Critical points of (4.6) are determined by

$$\partial_i \mathcal{W}(\tau) = \bar{\partial}_{\bar{i}} \mathcal{W}(\tau) = 0, \tag{4.10}$$

which leads to

$$\partial_i \mathcal{V} = \bar{\partial}_{\bar{i}} \mathcal{V} = 0. \tag{4.11}$$

It turns out that the critical points of $\mathcal{W}(\tau)$ are related to the Lorentz invariant vacua described by the $N = 1$ scalar potential $V(z, \bar{z}; \tau)$. Moreover, the existence of such points can be checked by the beta function (4.7), which means that these could be in the ultraviolet (UV) region if $a \to \infty$ or in the infrared (IR) region if $a \to 0$. These aspects will be addressed in Section 5 and Section 6.

5 Supersymmetric vacua on Kähler–Einstein manifold

The aim of this section is to present analysis of deformation of supersymmetric Lorentz invariant ground states on the Kähler–Ricci flow when the initial geometry is Kähler–Einstein geometry. As mentioned above, these vacua correspond to critical points of (4.6). In particular, our interest is the AdS vacua which correspond to the CFT. Our analysis here is in the context of the Morse theory [23][8] and the Morse–Bott theory [25,26], and in addition, applying the RG flow analysis to check the existence of such vacua. We organize this section into two parts. First, we perform the analysis to the vacua of the global $N = 1$ theory. Then we generalize it to the local $N = 1$ theory which are related to the BPS domain wall solutions. We leave the proof of some theorems here in Appendix B.

5.1 Global supersymmetric case

Let us first investigate some properties of supersymmetric Lorentz invariant ground states of the theory where all fermions and $\partial_\mu z^i$ are set to be zero.[9]

In a smooth region, *i.e.* there is no collapsing flow, at the ground state $p_0 \equiv (z_0, \bar{z}_0)$ we find that supersymmetric conditions

$$\partial_i W(z_0) = \bar{\partial}_{\bar{i}} \bar{W}(\bar{z}_0) = 0, \tag{5.1}$$

imply the set of equation in (3.4) and supersymmetry transformation (3.5) to be vanished. In other words, critical point of holomorphic superpotential $W(z)$ defines a vacuum of the theory and z_0 does not depend on τ. Thus, the vacuum is fixed in this case. Moreover, it is easy to see from (3.6) that at the vacua the scalar potential and the mass matrix of fermions do not change with respect to τ. So in order to characterize the ground states, we have to consider the second-order derivative of the scalar potential (3.2)

[8] An excellent background of Morse theory is also given, for example, in [24].

[9] For the rest of the paper we refer Lorentz invariant vacuum as vacuum or ground state.

with respect to $(z, \bar{z})$ called Hessian matrix evaluated at p_0, whose nonzero component has the form

$$\partial_i \bar{\partial}_{\bar{j}} V(p_0, \tau) = \sigma(\tau)^{-1} g^{l\bar{k}}(p_0; 0) \partial_i \partial_j W(z_0)\, \bar{\partial}_{\bar{k}} \bar{\partial}_{\bar{j}} \bar{W}(\bar{z}_0), \qquad (5.2)$$

where $g_{l\bar{k}}(0)$ is a Kähler–Einstein metric and we have assumed here that $g_{l\bar{k}}(p_0, 0)$ is invertible. Then (5.2) leads to the following statements:

Lemma 5.1. *The scalar potential $V(\tau)$ is a Morse function if and only if the superpotential $W(z)$ has no degenerate critical points.*

We want to mention here that the above lemma implies that the superpotential is at least general quadratic function, namely, $W(z) = a_{ij}(z - z_0)^i (z - z_0)^j$, where the matrix coupling (a_{ij}) is invertible.[10] Correspondingly, all eigenvalues of the fermion mass matrix (3.3) evaluated at p_0 are nonzero. For the rest of the paper, if $V(\tau)$ is a Morse function, then we name it the Morse potential. Furthermore, we have the following consequence.

Corollary 5.2. *All supersymmetric ground states of the Morse potential $V(\tau)$ are nondegenerate* (*isolated*).

Therefore, since $V(\tau)$ is a Morse function we can assign any supersymmetric vacuum by Morse index describing the number of negative eigenvalue of the Hessian matrix (5.2).[11] Looking at (5.2), we find that for $\tau \geq 0$ and $\Lambda > 0$

$$\partial_i \bar{\partial}_{\bar{j}} V(p_0, \tau > 1/2\Lambda) = -\partial_i \bar{\partial}_{\bar{j}} V(p_0, \tau < 1/2\Lambda), \qquad (5.3)$$

which shows the existence of parity transformation of the Hessian matrix at p_0 caused by the metric (2.11) mapping the Morse index from, say λ, to another Morse index $2n_c - \lambda$. So, we can write our main result of global supersymmetric vacua on Kähler–Einstein manifold as follows.

Theorem 5.3. *If p_0 is an isolated supersymmetric vacuum of the Morse index λ, then there exist a parity transformation of the matrix* (5.2) *that changes the index λ to another Morse index* $(2n_c - \lambda)$ *due to the deformation of Kähler–Einstein manifold. Furthermore, around p_0 we can introduce real*

[10] For constant and linear superpotential, we have degenerate supersymmetric vacua of the global theory in the sense that they are trivial in the second-order analysis. On the other side, all nonsupersymmetric vacua are degenerate in this global case.

[11] These negative eigenvalues describe the unstable directions of a vacuum.

local coordinates $X_p(\tau) = |\sigma(\tau)|^{-1/2} X_p(0)$ *with* $p = 1, \ldots, 2n_c$ *such that*

$$V(\tau) = \varepsilon(\sigma)\Big(- X_1^2(\tau) - \cdots - X_\lambda^2(\tau) + X_{\lambda+1}^2(\tau) + \cdots + X_{2n_c}^2(\tau)\Big), \quad (5.4)$$

where

$$\varepsilon(\sigma) = \begin{cases} 1 & \textit{if} \;\; 0 \le \tau < 1/2\Lambda, \\ -1 & \textit{if} \;\; \tau > 1/2\Lambda, \end{cases} \quad (5.5)$$

with $\Lambda > 0$ *for* $\tau \ge 0$.

Note that for $\Lambda < 0$, the factor $\sigma(\tau) \equiv (1 - 2\Lambda\tau) > 0$, and in this case we have only $\varepsilon(\sigma) = 1$ for $\tau \ge 0$. Similar result can also be obtained for static case, namely, for $\Lambda = 0$ or the Calabi-Yau manifold.

Let us start to discuss Theorem 5.3 by taking the example discussed in the previous section where the initial geometry is a Kähler–Einstein geometry $\mathbf{M}_0$ with positive definite metric and $\Lambda > 0$. Looking at (3.9), we find that the flow gives the same shape as the initial geometry $\mathbf{M}_0$ for $0 \le \tau < 1/2\Lambda$, while for $\tau > 1/2\Lambda$ the flow turns into a Kähler–Einstein geometry $\widehat{\mathbf{M}}_0$ with negative definite metric and $\widehat{\Lambda} < 0$. Therefore, this geometrical deformation causes the parity transformation of the matrix (5.2) that changes the index λ for $\tau < 1/2\Lambda$ to $(2n_c - \lambda)$ for $\tau > 1/2\Lambda$ of the same ground state.

5.2 Local supersymmetric case

Similar as preceding subsection we consider here the smooth region where the geometric flow does not vanish. Our interest here is to study supersymmetric vacuum $p_0 \equiv (z_0, \bar{z}_0)$ in local case which demands that

$$\partial_i \mathcal{W}(p_0;\tau) = 0 \Rightarrow \partial_i W(z_0) + \sigma(\tau) \frac{K_i(p_0;0)}{M_P^2} W(z_0) = 0, \quad (5.6)$$

has to hold and correspondingly, both BPS equations in (4.6) and the beta function (4.7) vanish. Then the τ-dependent scalar potential (3.17) becomes

$$\mathcal{V}(p_0;\tau) = -\frac{3}{M_P^2} \mathrm{e}^{\sigma(\tau)K(p_0;0)/M_P^2} |W(z_0)|^2 = -\frac{3}{M_P^2} \mathcal{W}^2(p_0;\tau), \quad (5.7)$$

which is zero or negative definite standing for cosmological constant of the spacetime on which the Ricci scalar (4.2) has the form

$$R = 12\, \mathcal{W}^2(p_0;\tau), \quad (5.8)$$

since $(a'/a)'(p_0) = 0$. Hence, the warped factor $a(u,\tau)$ is simply

$$a(u,\tau) = a_0(\tau) \exp\Big[\pm \mathcal{W}(p_0;\tau) u\Big]. \quad (5.9)$$

Furthermore, the Hessian matrix of the scalar potential (3.17) in this case is given by

$$\begin{aligned}
\partial_i \partial_j \mathcal{V}(p_0;\tau) &= -\frac{1}{M_P^2}\mathcal{M}_{ij}(p_0;\tau)\bar{L}(p_0;\tau),\\
\bar{\partial}_{\bar{i}} \bar{\partial}_{\bar{j}} \mathcal{V}(p_0;\tau) &= -\frac{1}{M_P^2}\bar{\mathcal{M}}_{\bar{i}\bar{j}}(p_0;\tau)L(p_0;\tau),\\
\partial_i \bar{\partial}_{\bar{j}} \mathcal{V}(p_0;\tau) &= \sigma(\tau)^{-1} g^{k\bar{l}}(p_0;0)\mathcal{M}_{ik}(p_0;\tau)\bar{\mathcal{M}}_{\bar{l}\bar{j}}(p_0;\tau)\\
&\quad - \frac{2\sigma(\tau)}{M_P^4} g_{i\bar{j}}(p_0;0)\mathcal{W}^2(p_0;\tau),
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
\mathcal{M}_{ij}(p_0;\tau) &= e^{\sigma(\tau)K(p_0;0)/2M_P^2}\\
&\quad \times \left(\partial_i\partial_j W(z_0) + \frac{\sigma(\tau)}{M_P^2}K_{ij}(p_0;0)W(z_0) + \frac{\sigma(\tau)}{M_P^2}K_j(p_0;0)\partial_i W(z_0)\right),
\end{aligned} \tag{5.11}$$

and we have assumed that $g_{l\bar{k}}(p_0, 0)$ is invertible. As mentioned in the previous subsection, the Morse index of a vacua is given by the negative eigenvalues of (5.10) describing unstable directions. Along these directions, the gradient flows provided by (4.6) are unstable and therefore, we have unstable walls in the context of dynamical system. On the other side, we obtain a stable solution along stable direction described by the positive eigenvalues of (5.10). Finally, to check the existence of such vacua in the IR and UV regions, we need to consider a first-order expansion of the beta function (4.7) at p_0 provided by

$$\mathcal{U} \equiv -\begin{pmatrix} \partial_j \beta^i & \partial_j \bar{\beta}^{\bar{i}} \\ \bar{\partial}_{\bar{j}} \beta^i & \bar{\partial}_{\bar{j}} \bar{\beta}^{\bar{i}} \end{pmatrix}(p_0;\tau), \tag{5.12}$$

where

$$\begin{aligned}
\partial_j \beta^i(p_0;\tau) &= -2\sigma(\tau)^{-1} g^{i\bar{k}}(p_0;0)\frac{\partial_j \bar{\partial}_{\bar{k}}\mathcal{W}(p_0;\tau)}{\mathcal{W}(p_0;\tau)} = -\frac{1}{M_P^2}\,\delta^i_j,\\
\bar{\partial}_{\bar{j}} \beta^i(p_0;\tau) &= -2\sigma(\tau)^{-1} g^{i\bar{k}}(p_0;0)\frac{\bar{\partial}_{\bar{j}} \bar{\partial}_{\bar{k}}\mathcal{W}(p_0;\tau)}{\mathcal{W}(p_0;\tau)},
\end{aligned} \tag{5.13}$$

and the rests are their complex conjugate. In order to have a vacuum in the UV region, at least one eigenvalue of the matrix (5.12) must be positive, because the RG flow departs the region in this direction. On the other side, in the IR region the RG flow approaches a vacuum in the direction of negative eigenvalue of (5.12). In the following, we consider two cases, namely the Morse theory analysis for nondegenerate vacua and then, degenerate vacua using Morse–Bott theory.

Let us first discuss the case in which the scalar potential (3.17) is Morse potential. For $W(z_0) = 0$, the ground states are flat Minkowskian spacetime and we regain the condition (5.1), which means that p_0 is fixed and does not evolve with respect to τ. Additionally, the warped factor equals $a_0(\tau)$ and can be rescaled by coordinate redefinition. In this case, the only nonzero coupling is the fermionic mass matrix defined in (3.16) which deforms with respect to τ as one can directly see from (5.11). This follows that the non zero components of the Hessian (5.10) are

$$\partial_i \bar{\partial}_{\bar{j}} \mathcal{V}(p_0;\tau) = \mathrm{e}^{\sigma(\tau) K(p_0;0)/M_P^2} \sigma(\tau)^{-1} g^{l\bar{k}}(p_0;0)\, \partial_i \partial_l W(z_0)\, \bar{\partial}_{\bar{k}} \bar{\partial}_{\bar{j}} \bar{W}(\bar{z}_0), \tag{5.14}$$

which means that the nondegeneracy of the vacua is determined by the holomorphic superpotential $W(z)$. Moreover, the property of the vacua does not change if we set $M_P \to +\infty$, so the analysis is similar to the global case. In other words, they have the same properties as in the second-order analysis for $W(z_0) = 0$. Then, in this case the holomorphic superpotential $W(z)$ has at least general quadratic form discussed in the preceding subsection. Note that in this case the matrix (5.12) is meaningless because these are not related to the CFT in three dimensions and, additionally, it diverges.

The other case, namely $W(z_0) \neq 0$, is curved symmetric spacetime with negative cosmological constant, called AdS. In this case, the vacuum p_0 depends on τ if $K_j(p_0;0) \neq 0$, which means that our ground state $p_0(\tau)$ and, moreover, all couplings (3.15) to (3.17) do deform with respect to τ.[12] In addition, we assume that the sign of the initial metric evaluated at $p_0(\tau)$, namely $g_{i\bar{j}}(p_0(\tau);0)$, is unchanged and well defined for $\tau \geq 0$ but not at the singular flow. Then from (5.9) we can choose, for example, a model with positive sign where for finite τ, the UV region can be defined around $u \to +\infty$ and $a \to +\infty$. Additionally, the IR region in this case is around $u \to -\infty$ and $a \to 0$.

Unlike Minkowskian cases, in AdS vacua we cannot trivially observe geometrical change of the vacua after the metric (2.11) hits the singularity at $\tau = 1/2\Lambda$ with $\Lambda > 0$ indicated by the parity transformation of the Hessian matrix (5.10). Therefore, one has to enforce additional conditions in order to

[12] Since the vacua depend on τ, so the metric (2.11) may change the nature of the vacua. Therefore, such vacua are dynamical with respect to τ.

achieve such particular situation. The next step is to analyze the eigenvalues of the matrix (5.12) whether such vacua exist in the UV and IR regions.

Before coming to the result described by the theorem below, let us define some quantities related to the Hessian matrix (5.10) in the following:

$$\begin{aligned} \mathcal{V}_{ij}(p_0(\tau);\tau) &\equiv -\frac{\varepsilon(\sigma)}{M_P^2}\,\mathcal{M}_{ij}(p_0(\tau);\tau)\bar{L}(p_0(\tau);\tau), \qquad (5.15)\\ \mathcal{V}_{i\bar{j}}(p_0(\tau);\tau) &\equiv |\sigma(\tau)|^{-1}g^{k\bar{l}}(p_0(\tau);0)\mathcal{M}_{ik}(p_0(\tau);\tau)\bar{\mathcal{M}}_{\bar{l}\bar{j}}(p_0(\tau);\tau)\\ &\quad -\frac{2|\sigma(\tau)|}{M_P^4}g_{i\bar{j}}(p_0(\tau);0)\mathcal{W}^2(p_0(\tau);\tau), \end{aligned}$$

where $\varepsilon(\sigma)$ is given in (5.5). Particular result in a nondegenerate AdS case where the parity transformation of the Hessian matrix (5.10) is manifest can then be written as follows.

Theorem 5.4. *Let $\mathcal{V}(\tau)$ be a Morse potential and $p_0(\tau) = q_0$ be an isolated AdS ground state of the Morse index λ for $0 \le \tau < 1/2\Lambda$ with $\Lambda > 0$. Then, there are real local coordinates $Y_p(\tau)$ around q_0 with $p = 1, \ldots, 2n_c$ such that*

$$\mathcal{V}(\tau) = \mathcal{V}(q_0;\tau) - Y_1^2(\tau) - \cdots - Y_\lambda^2(\tau) + Y_{\lambda+1}^2(\tau) + \cdots + Y_{2n_c}^2(\tau). \quad (5.16)$$

Suppose that for all $i, j = 1, \ldots, n_c$ the following inequalities

$$\begin{aligned} &\mathrm{Re}\big(\mathcal{V}_{ij}(p_0(\tau);\tau)\big) > 0, \quad \mathrm{Im}\big(\mathcal{V}_{ij}(p_0(\tau);\tau)\big) > 0,\\ &\mathrm{Re}\big(\mathcal{V}_{i\bar{j}}(p_0(\tau);\tau)\big) > 0, \quad \mathrm{Im}\big(\mathcal{V}_{i\bar{j}}(p_0(\tau);\tau)\big) > 0, \end{aligned} \qquad (5.17)$$

hold for $\tau \ge 0$ and $\tau \ne 1/2\Lambda$. Let $p_0(\tau) = \hat{q}_0$ be another isolated vacuum for $\tau > 1/2\Lambda$ and both $(q_0, \hat{q}_0)$ exist in the UV or IR regions. Then, Kähler–Ricci flow causes a parity transformation of the Hessian matrix (5.10)*, that maps q_0 of the index λ to $\hat{q}_0$ of the index $(2n_c - \lambda)$ such that around $\hat{q}_0$ real local coordinates $\widehat{Y}_p(\tau) \ne Y_p(\tau)$ exist and*

$$\mathcal{V}(\tau) = \mathcal{V}(\hat{q}_0;\tau) + \widehat{Y}_1^2(\tau) + \cdots + \widehat{Y}_\lambda^2(\tau) - \widehat{Y}_{\lambda+1}^2(\tau) - \cdots - \widehat{Y}_{2n_c}^2(\tau), \quad (5.18)$$

for $\tau > 1/2\Lambda$.

Our comments of the above theorem are in order. Firstly, in general $q_0 \ne \hat{q}_0$ and they are called parity pair of the vacua in the sense that q_0 has λ negative eigenvalues of the Hessian matrix (5.10), while $2n_c - \lambda$ negative eigenvalues belong to $\hat{q}_0$ caused by the geometric flow (2.11) after passing the singularity at $\tau = 1/2\Lambda$. On the other side, if the inequalities (5.17) do not hold, then q_0 and $\hat{q}_0$ would have the same index λ.

Secondly, we can reconsider the case discussed in the global case where the initial geometry is taken to be a Kähler–Einstein geometry $\mathbf{M}_0$ with positive definite metric and $\Lambda > 0$. However, in the dynamical AdS case since $p_0(\tau)$, the holomorphic superpotential $W(z_0(\tau))$ together with the fermion mass matrix (5.11) also control the properties of the ground states in the second-order analysis described by (5.10). In other words, there are two aspects that play a role in the second-order analysis of vacua, namely the Kähler–Ricci flow (2.11) and the dynamics of coupling quantities which depend both on the form of the superpotential and the geometry. Note that $g_{i\bar{j}}(p_0(\tau);0)$ remains positive definite for $\tau \geq 0$ in this case.

Thirdly, the inequalities (5.17) mean that all components of the matrix (5.10) have at least to change the sign as the Kähler–Einstein geometry evolves. This can be easily seen in a special case where

$$\mathcal{M}_{ij}(p_0(\tau);\tau) = 0, \tag{5.19}$$

holds for $\tau \geq 0$ and $\tau \neq 1/2\Lambda$. Consequently, we have a theory in which all spin-$\frac{1}{2}$ fermionic are massless. Then the Hessian matrix (5.10) has simple diagonal form

$$\partial_i \bar{\partial}_{\bar{j}} \mathcal{V}(p_0(\tau);\tau) = -\frac{2\sigma(\tau)}{M_P^4}\, g_{i\bar{j}}(p_0(\tau);0)\, \mathcal{W}^2(p_0(\tau);\tau). \tag{5.20}$$

Hence, in this case, the condition (5.17) can be simplified by stating that the sign of $g_{i\bar{j}}(p_0(\tau);0)$ is fixed with respect to τ in order the parity to be revealed. In $\Lambda > 0$ case, we have Morse index $2n_c$ for $\tau < 1/2\Lambda$ and then, it changes to 0 for $\tau > 1/2\Lambda$. In other words, we have unstable walls for $\tau < 1/2\Lambda$, which become stable after hitting the singularity at $\tau = 1/2\Lambda$. Such situation, however, does not occur in $\Lambda < 0$ case and the index is still $2n_c$ for $\tau \geq 0$.

Furthermore, the condition (5.19) also leads to

$$\partial_i \partial_j \mathcal{W}(p_0(\tau);\tau) = 0. \tag{5.21}$$

Then, the matrix (5.12) becomes simply

$$\mathcal{U} = \frac{1}{M_P^2}\begin{pmatrix} \delta^i_j & 0 \\ 0 & \delta^{\bar{i}}_{\bar{j}} \end{pmatrix}, \tag{5.22}$$

which shows that these vacua only live in the UV region.

On the other hand, it is also of interest to consider a case where the ground states are determined by the critical points of the holomorphic superpotential $W(z)$, i.e., equation (5.1) is fulfilled. Then the $U(1)$-connection (3.21) evaluated at p_0 vanishes in all order analysis.[13] In other words, the supersymmetric condition (5.6) reduces to

$$\partial_i W(z_0) = 0, \quad W(z_0) \neq 0, \tag{5.23}$$

In this case, the ground state p_0 is coming from the critical points of $W(z)$ fixed with respect to τ. So, the second-order properties of vacua described by (5.10) are fully controlled by the geometric flow and the parity transformation appears if the condition (5.17) is also fulfilled.

We finally put some remarks for the case of Morse potential. In dynamical AdS vacua if the condition (5.17) is not satisfied, then the parity does not emerge but the deformation of the vacua still exists due to the Kähler–Ricci flow. These vacua admit the same Morse index λ, see the above example for $\Lambda < 0$ and $\tau \geq 0$. Similar things happen for the static case. In addition, since the scalar potential (3.17) is a Morse potential, then it is impossible to have a degenerate vacuum. Therefore, any vacuum does not deform to another vacuum with different Morse index in $0 \leq \tau < 1/2\Lambda$ with $\Lambda > 0$. Thus, no index modification of the vacua exists in the interval.

Now we turn to consider when the ground states are degenerate. This can occur if the Hessian matrix (5.10) has m zero eigenvalues with $m \leq 2n_c$. Therefore, the vacua can be viewed as m-dimensional submanifold of the Kähler–Einstein manifold. For the case at hand, the scalar potential (3.17) is a Morse–Bott function (or Morse–Bott potential), which is a generalization of a Morse function. If the vacua are Minkowskian, then the superpotential $W(z)$ must be constant or take a linear form as discussed in Section 5.1 since the other forms are excluded. Thus, we have $m = 2n_c$ and the vacuum manifold could then be the Kähler–Einstein manifold $\mathbf{M}_0$ or $\widehat{\mathbf{M}}_0$ depending on τ and no parity transformation of the matrix (5.10) caused by the flow (2.11) appears in this model. Note that in the model at hand we also have singular (5.12) because it is undefined for $\partial_i\partial_j\mathcal{W}(p_0(\tau);\tau) = \mathcal{W}(p_0(\tau);\tau) = 0$. Hence, this means that the three-dimensional CFT does not exist and we do not have the correspondence in the Minkowskian ground states.

In AdS vacua our construction is as follows. Let S be an m-dimensional vacuum submanifold of a Kähler–Einstein geometry $\mathbf{M}_0$.[14] Then at any $p_0 \in S$ we can split the tangent space $T_{p_0}\mathbf{M}_0$ as

$$T_{p_0}\mathbf{M}_0 = T_{p_0}S \oplus N_{p_0}S, \tag{5.24}$$

[13] This case has been considered in reference [20] for model with a chiral multiplet.

[14] We could also consider $\mathbf{M}_0$ as a set of a disjoint union of some connected smooth manifolds with finite dimension, see, for example, [25, 26].

where $T_{p_0}S$ is the tangent space of S and $N_{p_0}S$ is the normal space of S. Moreover, the Hessian matrix (5.10) is nondegenerate in the normal direction to S.

Unlike the previous case, since the scalar potential is Morse–Bott potential, thus we have a rich and complicated structure of vacua, particularly in the dynamical case. In the following we list some possibilities. The first possibility is a similar situation as in Theorem 5.4 which can be written down in the following statements.

Theorem 5.5. *Let $\mathcal{V}(\tau)$ be Morse–Bott potential. At any degenerate AdS vacuum $q_0 \in S$ where S is an m-dimensional vacuum submanifold of a Kähler–Einstein geometry $\mathbf{M}_0$, then, in the direction of $N_{q_0}S$ we can write the scalar potential* (3.17) *as*

$$\mathcal{V}(\tau) = \mathcal{V}(S;\tau) - Y_1^2(\tau) - \cdots - Y_{\underline{\lambda}}^2(\tau) + Y_{\underline{\lambda}+1}^2(\tau) + \cdots + Y_{2n_c-m}^2(\tau), \tag{5.25}$$

for $0 \le \tau < 1/2\Lambda$ with $\Lambda > 0$. Suppose that there exists another m-dimensional vacuum submanifold $\widehat{S}$ of a Kähler–Einstein geometry $\widehat{\mathbf{M}}_0$ for $\tau > 1/2\Lambda$, the inequalities (5.17) *are satisfied, and both submanifolds $(S, \widehat{S})$ live in the UV or IR regions. Therefore, the deformation of $\mathbf{M}_0$ via Kähler–Ricci flow causes a parity transformation of the Hessian matrix* (5.10) *that changes $q_0 \in S$ to $\hat{q}_0 \in \widehat{S}$ such that in the direction of $N_{\hat{q}_0}\widehat{S}$ the scalar potential* (3.17) *has the form*

$$\mathcal{V}(\tau) = \mathcal{V}(\widehat{S};\tau) + \widehat{Y}_1^2(\tau) + \cdots + \widehat{Y}_{\underline{\lambda}}^2(\tau) - \widehat{Y}_{\underline{\lambda}+1}^2(\tau) - \cdots - \widehat{Y}_{2n_c-m}^2(\tau), \tag{5.26}$$

for $\tau > 1/2\Lambda$ with $\widehat{Y}_{\underline{p}}(\tau) \neq Y_{\underline{p}}(\tau)$, where $\underline{p} = 1, \ldots, 2n_c - m$.

This theorem implies that each $q_0 \in S \subseteq \mathbf{M}_0$ has the same index $\underline{\lambda}$ for $0 \le \tau < 1/2\Lambda$. In other words, S is a vacuum submanifold of the index $\underline{\lambda}$. Moreover, since the ground states are τ-dependent, i.e., $p_0(\tau)$, and if the Hessian matrix (5.10) is changed in sign after passing the singular point, then the submanifold S deforms to the submanifold $\widehat{S} \subseteq \widehat{\mathbf{M}}_0$ of the index $(2n_c - \underline{\lambda})$ for $\tau > 1/2\Lambda$. Both S and $\widehat{S}$ are called parity pair for similar reason as in the nondegenerate case, namely $\widehat{S}$ of the index $(2n_c - \underline{\lambda})$ has the same dimension as S and exists after the geometric flow (2.11) hits the singularity which splits $\mathbf{M}_0$ and $\widehat{\mathbf{M}}_0$.

On the other side, in the static case where the $U(1)$-connection evaluated at p_0 is zero, the vacuum submanifold S stays fixed, which means that it does not change with respect to τ. In other words, we have $S = \widehat{S}$, but its index could change due to Kähler–Ricci flow.

The last possible situation is as follows.

Theorem 5.6. *Suppose $\mathcal{V}(\tau)$ is a Morse–Bott potential and the conditions* (5.17) *do not hold. Let S be an m-dimensional vacuum submanifold of $\mathbf{M}_0$ with index $\underline{\lambda}$ in $0 \leq \tau < \tau_0$ and $\tau_0 < 1/2\Lambda$. Then we have the following cases. S deforms to*

1. *an n-dimensional vacuum submanifold $S_1 \subseteq \mathbf{M}_0$ of the index $\underline{\lambda}_1$ in $\tau_0 \leq \tau < 1/2\Lambda$. If $n \neq m$, then the index $\underline{\lambda}_1 \in \{0, \ldots, 2n_c - n\}$. However, if $n = m$, then $\underline{\lambda}_1 \neq \underline{\lambda}$ and $\underline{\lambda}_1 \in \{0, \ldots, 2n_c - n\}$.*
2. *an n_1-dimensional vacuum submanifold $\widehat{S}_1 \subseteq \widehat{\mathbf{M}}_0$ of the index $\underline{\lambda}_2$ in $\tau > 1/2\Lambda$. If $n_1 \neq m$, then we have $\underline{\lambda}_2 \in \{0, \ldots, 2n_c - n_1\}$. But, if $n_1 = m$, then $\underline{\lambda}_2 \neq 2n_c - \underline{\lambda}$ and $\underline{\lambda}_2 \in \{0, \ldots, 2n_c - n_1\}$.*

Furthermore, S_1 and $\widehat{S}_1$ are not the parity pair. All S, S_1, and $\widehat{S}_1$ may exist in the UV or IR regions.

This leads to a result that both the dimension and the index of the vacuum manifold may also be changed before or after the singularity, which is not related to the case stated in Theorem 5.5. For example, there is a situation where a nondegenerate vacuum changes to another degenerate vacuum as τ runs before or after singularity. We give an example of this situation, namely $\mathbb{C}P^1$ model with linear superpotential, in the next subsection.

5.3 A model with linear superpotential

In this subsection, we discuss a $\mathbb{C}\mathrm{P}^{n_c}$ model with linear superpotential in order to make the statements in the previous subsection clearer. For the model at hand, the second-order derivative of the superpotential equals zero, so some difficulties have been reduced.

Let us start by considering a superpotential $W(z)$ which takes linear form as

$$W(z) = a_0 + a_i\, z^i, \tag{5.27}$$

where the constants $a_0, a_i \in \mathbb{R}$. Moreover, the scalar manifold is chosen to be $\mathbb{C}\mathrm{P}^{n_c}$ whose Kähler potential has the form

$$K(z, \bar{z}; 0) = \ln(1 + z^i \bar{z}^{\bar{i}}), \tag{5.28}$$

where z^i is the standard Fubini–Study coordinates of $\mathbb{C}\mathrm{P}^{n_c}$. In the model, the metric is positive definite with the constant $\Lambda = n_c + 1 > 0$. Then the

Kähler–Ricci flow implies that the τ-dependent Kähler potential is given by

$$K(z, \bar{z}; \tau) = \sigma(\tau) \ln(1 + z^i \bar{z}^{\bar{i}}), \tag{5.29}$$

and the modified constant is $\widehat{\Lambda}(\tau) \equiv (n_c + 1)/\sigma(\tau)$. So, in the interval $0 \leq \tau < 1/2(n_c + 1)$ the flow is diffeomorphic to $\mathbb{CP}^{n_c}$, but for $\tau > 1/2(n_c + 1)$ it becomes $\widehat{\mathbb{CP}^{n_c}}$ on which the resulting metric is negative definite with the constant $\widehat{\Lambda} < 0$.

As previously mentioned, this model has degenerate ground states in the global supersymmetric theory, but in a local supersymmetry it is still nontrivial. In local case, the supersymmetric condition (5.6) is given by

$$a_i + \frac{\sigma(\tau)\, \bar{z}^{\bar{i}}}{M_P^2 (1 + z^k \bar{z}^{\bar{k}})} (a_0 + a_j\, z^j) = 0. \tag{5.30}$$

It is easy to see that for the static case, $z^i = 0$ is a vacuum by demanding $a_i = 0$. Furthermore, the Hessian matrix (5.10) becomes simply diagonal matrix whose nonzero elements are

$$\partial_i \bar{\partial}_{\bar{j}} \mathcal{V}(p_0; \tau) = -\frac{2\sigma(\tau)}{M_P^4}\, \delta_{i\bar{j}}\, |a_0|^2. \tag{5.31}$$

Thus we have an isolated AdS spacetime for $a_0 \neq 0$ representing unstable walls whose Morse index is $2n_c$ for $0 \leq \tau < 1/2(n_c + 1)$. Then it changes to 0 for $\tau > 1/2(n_c + 1)$ describing stable walls as the geometry evolves with respect to the Kähler–Ricci equation. Such vacua exist in the UV region since we have (5.22). On the other hand, the ground states turn out to be degenerate Minkowskian spacetime if $a_0 = 0$ for any z^i. Therefore, the vacuum manifold is $\mathbb{CP}^{n_c}$ for $0 \leq \tau < 1/2(n_c + 1)$ and then becomes $\widehat{\mathbb{CP}^{n_c}}$ for $\tau > 1/2(n_c + 1)$.

In nonstatic case, it is convenient to take $n_c = 1$ case. So the ground state is chosen to be

$$z_0 = (x_0 + \mathrm{i} y_0), \tag{5.32}$$

where $x_0, y_0 \in \mathbb{R}$. In this case, the condition (5.30) results in

$$\begin{aligned} x_0(\tau) &= \frac{1}{2(\sigma(\tau) + M_P^2)} \left[-\frac{\sigma(\tau) a_0}{a_1} \pm \left(\left(\frac{\sigma(\tau) a_0}{a_1} \right)^2 - 4M_P^2(\sigma(\tau) + M_P^2) \right)^{1/2} \right], \\ y_0(\tau) &= 0, \end{aligned} \tag{5.33}$$

with $a_1 \neq 0$. Then, in order to obtain a well-defined theory, the parameter τ must fulfill the following inequalities,

$$\tau \leq \frac{1}{4} - \frac{M_P^2}{2}\left(\frac{a_1}{a_0}\right)^2\left[1 + \left(1 + \left(\frac{a_0}{a_1}\right)^2\right)^{1/2}\right],$$

$$\tau \geq \frac{1}{4} + \frac{M_P^2}{2}\left(\frac{a_1}{a_0}\right)^2\left[-1 + \left(1 + \left(\frac{a_0}{a_1}\right)^2\right)^{1/2}\right], \tag{5.34}$$

before and after singular point $\tau = 1/4$. The eigenvalues of the Hessian matrix (5.10) in this case have the form

$$\lambda^{\mathcal{V}}_{1,2}(\tau) = \frac{2|\sigma(\tau)|}{M_P^4}\left(1 + x_0^2\right)^{-2+\sigma(\tau)/M_P^2}\Bigg(-2\varepsilon(\sigma)\Big[(a_0 + a_1 x_0)^2 - \frac{1}{2}(a_1 - a_0 x_0)^2 x_0^2\Big] \pm \big|(a_0 + a_1 x_0)(a_1 - a_0 x_0)x_0\big|\Bigg), \tag{5.35}$$

where $\varepsilon(\sigma)$ is given by (5.5) with $\Lambda = 2$. Moreover, the existence of a vacuum can be checked by (5.12) whose eigenvalues are

$$\lambda^{\mathcal{U}}_{1,2}(\tau) = \frac{1}{M_P^2}\left(1 \pm \left|\frac{x_0(a_1 - a_0 x_0)}{(a_0 + a_1 x_0)}\right|\right). \tag{5.36}$$

Let us simplify the model in which both a_0 and a_1 are positive and $a_0 \gg a_1$. For $\tau \lesssim \frac{1}{4} - \frac{a_1 M_P^2}{2a_0}$ we have $\varepsilon(\sigma) = 1$, and then local minimum occurs in

$$-\frac{a_0 M_P^2}{8a_1\sqrt{2}} < \tau \lesssim \frac{1}{4} - \frac{a_1 M_P^2}{2a_0}, \tag{5.37}$$

while local maximum exists in

$$\tau < -\frac{a_0 M_P^2}{8a_1}. \tag{5.38}$$

Additionally, saddle arises between

$$-\frac{a_0 M_P^2}{8a_1} < \tau < -\frac{a_0 M_P^2}{8a_1\sqrt{2}}. \tag{5.39}$$

Also, it is possible to have intrinsic degenerate vacua[15] if

$$(a_0 + a_1 x_0)^2 = (a_1 - a_0 x_0)^2 x_0^2 \tag{5.40}$$

holds, which occurs around

$$\tau \approx -\frac{a_0 M_P^2}{8a_1}, \tag{5.41}$$

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[15] Intrinsic means that these degenerate vacua are coming from degenerate critical points of $\mathcal{W}$ [20].

whereas the other degeneracy needs

$$4(a_0 + a_1 x_0)^2 = (a_1 - a_0 x_0)^2 \, x_0^2 \tag{5.42}$$

and takes place near

$$\tau \approx -\frac{a_0 M_P^2}{8 a_1 \sqrt{2}}. \tag{5.43}$$

These mean that a nondegenerate ground state can be changed into another degenerate ground state by Kähler–Ricci flow far before singularity at $\tau = 1/4$. In other words, this situation occurs without geometrical change and further shows that the scalar potential of the model is Morse–Bott potential. Then, the analysis using (5.36) shows that all vacua may exist in the UV region because it has at least one positive eigenvalue that ensures the RG flows depart the region. However, in the IR region, local maximum and intrinsic degenerate vacua do not exist and therefore, no RG flows approach the vacua.

In $\tau \gtrsim \frac{1}{4} + \frac{a_1 M_P^2}{2a_0}$ we get $\varepsilon(\sigma) = -1$, which leads to the fact that this model admits parity transformation of the Hessian matrix (5.10). Then, the following interval

$$\frac{1}{4} + \frac{a_1 M_P^2}{2 a_0} \lesssim \tau < \frac{a_0 M_P^2}{8 a_1 \sqrt{2}} \tag{5.44}$$

shows the existence of local maximum, whereas

$$\tau > \frac{a_0 M_P^2}{8 a_1} \tag{5.45}$$

is for local minimum. Next, saddle exists for

$$\frac{a_0 M_P^2}{8 a_1 \sqrt{2}} < \tau < \frac{a_0 M_P^2}{8 a_1}, \tag{5.46}$$

while intrinsic degenerate vacua occur near

$$\tau \approx \frac{a_0 M_P^2}{8 a_1}, \tag{5.47}$$

and the other degenerate vacua arise around

$$\tau \approx \frac{a_0 M_P^2}{8 a_1 \sqrt{2}}. \tag{5.48}$$

Similar as in the previous case, in the UV region we have all possibilities of the vacua, but local minimum and intrinsic degenerate vacua are forbidden in the IR region.

6 Supersymmetric vacua on gradient Kähler–Ricci soliton

In this section we consider the deformation of the supersymmetric vacua on $U(n_c)$ symmetric gradient Kähler–Ricci soliton on line bundles over $\mathbb{CP}^{n_c-1}$. The organization of this section is the following: In the first part, we review the construction of the gradient Kähler–Ricci soliton on line bundles over $\mathbb{CP}^{n_c-1}$ studied in [9]. The second part is to provide an analysis of the supersymmetric ground states in $N=1$ theory.

6.1 Rotationally symmetric gradient Kähler–Ricci soliton

First of all, let us first construct a Kähler metric on $\mathbb{C}^{n_c}\backslash\{0\}$ as follows. We define the initial Kähler potential, i.e., at $\tau=0$,

$$K(z,\bar{z},0)=\phi(u), \tag{6.1}$$

where

$$u\equiv 2\ln(\delta_{i\bar{j}}z^i\bar{z}^{\bar{j}})=2\ln|z|^2, \tag{6.2}$$

on $\mathbb{C}^{n_c}\backslash\{0\}$. So our ansatz (6.1) has $U(n_c)$ symmetric. Secondly, we simplify the case by taking the vector field $Y^i(0)$ to be holomorphic and linear[16]

$$Y^i(0)=\mu z^i, \tag{6.3}$$

where $\mu\in\mathbb{R}$. Moreover, the Kähler potential (6.1) implies that the metric and its inverse have to be

$$g(0)=g_{i\bar{j}}(0)\,dz^i\,d\bar{z}^{\bar{j}}=\left[2e^{-u/2}\phi_u\,\delta_{i\bar{j}}+4e^{-u}\left(\phi_{uu}-\frac{1}{2}\phi_u\right)\bar{z}^{\bar{i}}\,z^j\right]dz^i\,d\bar{z}^{\bar{j}},$$
$$g^{-1}(0)=g^{\bar{j}i}(0)\,\bar{\partial}_{\bar{j}}\,\partial_i=\frac{\mathrm{e}^{u/2}}{2\phi_u}\left[\delta^{\bar{j}i}-\mathrm{e}^{-u/2}\,\frac{\phi_{uu}-(1/2)\phi_u}{\phi_{uu}}\,\bar{z}^{\bar{j}}z^i\right]\bar{\partial}_{\bar{j}}\,\partial_i, \tag{6.4}$$

where $\phi_u\equiv\frac{d\phi}{du}$ and $\phi_{uu}\equiv\frac{d^2\phi}{du^2}$. Since the metric (6.4) is positive definite in this case, we have then the inequalities

$$\phi_u>0,\quad \phi_{uu}>0. \tag{6.5}$$

Inserting (6.3) and (6.4) into (2.3) results in

$$\begin{aligned}\frac{2(n_c+1)}{\phi_u}&\left(\phi_{uu}-\frac{1}{2}\phi_u\right)+\frac{4}{\phi_u}\left(-\left(\phi_{uu}-\frac{1}{2}\phi_u\right)+\frac{d}{du}\left(\phi_{uu}-\frac{1}{2}\phi_u\right)\right)\\&-\frac{4}{\phi_u\phi_{uu}}\left(\phi_{uu}-\frac{1}{2}\phi_u\right)\frac{d}{du}\left(\phi_{uu}-\frac{1}{2}\phi_u\right)+4\Lambda\phi_u-8\mu\,\phi_{uu}\\&=A_0\mathrm{e}^{(1-n_c)u/2},\end{aligned} \tag{6.6}$$

[16]This vector field can be derived from the real function $P(z,\bar{z})$ in (2.13) [7–9].

where A_0 is an arbitrary constant. Now defining $\Phi \equiv \phi_u$ and then writing $\Phi_u = F(\Phi)$ yields

$$\frac{dF}{d\Phi} + \left(\frac{n_c - 1}{\Phi} - 4\mu\right) F - \left(\frac{n_c}{2} - 2\Lambda\Phi\right) = \frac{A_0}{\Phi} \mathrm{e}^{(1-n_c)u/2}. \tag{6.7}$$

For the sake of simplicity, we consider the case where the right hand side of (6.7) vanishes by taking $A_0 = 0$, so that the solution of (6.7) takes the form

$$\Phi_u = F(\Phi) = A_1 \frac{\mathrm{e}^{4\mu\Phi}}{\Phi^{n_c-1}} + \frac{\Lambda}{2\mu}\Phi + \frac{2(\Lambda-\mu)}{(4\mu)^{n_c+1}} \sum_{j=0}^{n_c-1} \frac{n_c!}{j!} (4\mu)^j \, \Phi^{j+1-n_c}, \tag{6.8}$$

where A_1 is also an arbitrary constant. For the case at hand, the Kähler–Einstein solution of (6.7), namely the $\mu = 0$ case, is excluded.

To construct the soliton, let us first recall the flow vector $X^i(\tau)$ defined in (2.5). From (6.3), we have

$$X^i(\tau) = \frac{\mu}{\sigma(\tau)} z^i, \tag{6.9}$$

which induces the diffeomorphisms

$$\hat{z} \equiv \psi_\tau(z) = \sigma(\tau)^{-\mu/2\Lambda} z. \tag{6.10}$$

Thus, the complete τ-dependent Kähler–Ricci soliton defined on $\mathbb{C}^{n_c}\backslash\{0\}$ is

$$g(z, \bar{z}; \tau) = \sigma(\tau)^{1-\mu/\Lambda} g_{i\bar{j}} \left(\sigma(\tau)^{-\mu/2\Lambda} z\right) dz^i \, d\bar{z}^{\bar{j}}. \tag{6.11}$$

Now we are ready to discuss the construction of the gradient Kähler–Ricci soliton on line bundles over $\mathbb{C}\mathrm{P}^{n_c-1}$. First of all, the metric (6.4) is nontrivial $U(n_c)$-invariant gradient Kähler–Ricci metric defined on $(\mathbb{C}^{n_c}\backslash\{0\})/\mathbb{Z}_\ell$, where $\mathbb{Z}_\ell$ acting on $\mathbb{C}^{n_c}\backslash\{0\}$ by $z \mapsto e^{2\pi \mathrm{i}/\ell} z$ with ℓ is a positive integer and $\ell \neq 0$. Thus, we replace the coordinates z using the above analysis by new coordinates which can be viewed as holomorphic polynomial of order ℓ, i.e.,

$$\xi \equiv z^\ell, \tag{6.12}$$

that parameterize $(\mathbb{C}^{n_c}\backslash\{0\})/\mathbb{Z}_\ell$. Then we construct a negative line bundle over $\mathbb{C}\mathrm{P}^{n_c-1}$, denoting by $L^{-\ell}$, by gluing $\mathbb{C}\mathrm{P}^{n_c-1}$ into $(\mathbb{C}^{n_c}\backslash\{0\})/\mathbb{Z}_\ell$ at $\xi = 0$.[17] This leads that one has to add $\mathbb{C}\mathrm{P}^{n_c-1}$ at $\xi = 0$ and analyze (6.8)

[17] We do not consider L^ℓ for $\ell > 0$ here because there is no complete gradient Kähler–Ricci soliton on it.

near $\xi = 0$. In this case, it is convenient to cast the initial metric (6.4) into the following form

$$g(0) = \Phi\, g_{\mathrm{FS}} + \Phi_v\, dw\, d\bar{w}, \tag{6.13}$$

where g_{FS} is the standard Fubini–Study metric of $\mathbb{C}\mathrm{P}^{n_c-1}$

$$g_{\mathrm{FS}} = \left(\frac{\delta_{a\bar{b}}}{1+\zeta^c\bar{\zeta}^{\bar{c}}} - \frac{\zeta^b\bar{\zeta}^{\bar{a}}}{(1+\zeta^c\bar{\zeta}^{\bar{c}})^2} \right) d\zeta^a\, d\bar{\zeta}^{\bar{b}}, \tag{6.14}$$

with $\zeta^a \equiv \xi^a/\xi^{n_c}$, $\zeta^{n_c} = 1$ and $a, b, c = 1, \ldots, n_c - 1$. Moreover, $w \equiv w(\xi, \bar{\xi})$ is a nonholomorphic coordinate and here $v \equiv 2\ln|\xi|^2$. So, to obtain $\mathbb{C}\mathrm{P}^{n_c-1}$ at $\xi = 0$, one has to take

$$\lim_{v\to-\infty} \Phi(v) = a > 0, \quad F(a) = 0, \quad \frac{dF}{d\Phi}(a) > 0 \tag{6.15}$$

with

$$a = \frac{1}{4\Lambda}(n_c - \ell). \tag{6.16}$$

For $\Lambda < 0$ case, we have $\ell > n_c$, whereas the $\Lambda > 0$ case demands $0 < \ell < n_c$.[18]

Let us consider the behavior of the gradient Kähler–Ricci soliton defined on $L^{-\ell}$ in the asymptotic region $|\xi| \to +\infty$. We take a case where Φ becomes larger as $v \to +\infty$ such that the exponential term in (6.8) suppresses to zero and the dominant term is the linear term $\frac{\Lambda}{2\mu}\Phi$. The inequalities (6.5) further restricts that we only have two possible solutions as follows. The first case is when $\Lambda < 0$ and $\mu < 0$. Then (6.8) may be written down in the form

$$\Phi_v = \frac{\Lambda}{2\mu}\,\Phi + G\left(\frac{1}{\Phi}\right). \tag{6.17}$$

Rewriting (6.17) in terms of $\Psi = 1/\Phi$ results in

$$\Phi = \mathrm{e}^{\Lambda v/2\mu} B\left(\mathrm{e}^{-\Lambda v/2\mu}\right), \tag{6.18}$$

[18] In [9] for $\Lambda < 0$ case the metric (6.11) becomes asymptotically cone-like expanding soliton, while in the case of $\Lambda > 0$ it turns out to be cone-like shrinking soliton in the asymptotic region.

for large v, where B is smooth and $B(0) > 0$. Thus, the Kähler–Ricci soliton (6.11) becomes

$$\begin{aligned} g(\xi,\bar{\xi};\tau) = 2\Bigg[& |\xi|^{-2+2\Lambda/\mu}\, B\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right)\delta_{i\bar{j}} \\ & + \Bigg\{ \left(\frac{\Lambda}{\mu}-1\right)\, |\xi|^{2\Lambda/\mu} B\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right) \\ & - \sigma(\tau)\,\frac{\Lambda}{\mu}\,\dot{B}\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right)\Bigg\}|\xi|^{-4}\bar{\xi}^{\bar{i}}\,\xi^{j}\Bigg] d\xi^{i}\, d\bar{\xi}^{\bar{j}}, \end{aligned} \tag{6.19}$$

with $\xi \in L^{-\ell}$. It is easy to see that the soliton (6.19) collapses to a cone metric at $\tau = \frac{1}{2\Lambda} < 0$

$$g(\xi,\bar{\xi};1/2\Lambda) = B(0)\, g_{\mathrm{cone}}(\xi,\bar{\xi}), \tag{6.20}$$

where

$$g_{\mathrm{cone}}(\xi,\bar{\xi}) = |\xi|^{-2+2\Lambda/\mu}\,\left(\delta_{i\bar{j}} + \left(\frac{\Lambda}{\mu}-1\right)|\xi|^{-2}\bar{\xi}^{\bar{i}}\,\xi^{j}\right) d\xi^{i}\, d\bar{\xi}^{\bar{j}}, \tag{6.21}$$

and then expands smoothly for $\tau \geq 0$.

Now we consider the second case where $\Lambda > 0$ and $\mu > 0$. In this case, as Φ becomes larger as $v \to +\infty$, the exponential term becomes dominant and then it is difficult to define Φ in this region. To get rid of it, one has to set $A_1 = 0$, which further implies that the dominant term is again the linear term $\frac{\Lambda}{2\mu}\Phi > 0$. Then, we may write (6.8) into (6.17), which leads to a similar equation like (6.18). After some computation, we get

$$\begin{aligned} g(\xi,\bar{\xi};\tau) = 2\Bigg[& |\xi|^{-2+2\Lambda/\mu}\, D\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right)\delta_{i\bar{j}} \\ & + \Bigg\{ \left(\frac{\Lambda}{\mu}-1\right)\, |\xi|^{2\Lambda/\mu} D\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right) \\ & - \sigma(\tau)\,\frac{\Lambda}{\mu}\,\dot{D}\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right)\Bigg\}|\xi|^{-4}\bar{\xi}^{\bar{i}}\,\xi^{j}\Bigg] d\xi^{i}\, d\bar{\xi}^{\bar{j}}, \end{aligned} \tag{6.22}$$

for large $|\xi|$, where D is a smooth function and $D(0) > 0$ with $\xi \in L^{-\ell}$. Therefore, the metric (6.22) converges to a cone metric (6.21) at $\tau = \frac{1}{2\Lambda} > 0$ and evolves smoothly for $\tau > \frac{1}{2\Lambda}$.

6.2 Vacuum structure of $N = 1$ theory

First of all, we recall some properties of Minkowskian ground states of the local theory discussed in (5.2) since the second-order analysis is similar to the global case by setting $M_P \to +\infty$. As pointed out before, a flat Minkowskian vacuum $\tilde{p}_0 \equiv (\xi_0, \bar{\xi}_0)$ requires

$$W(\xi_0) = \partial_i W(\xi_0) = 0, \tag{6.23}$$

together with their complex conjugate for all $i = 1, \ldots, n_c$. Again, we obtain here a static Minkowskian vacuum which is defined by the critical points of the holomorphic superpotential. The nonzero component of the Hessian matrix of the scalar potential (3.17) is given by

$$\partial_i \bar{\partial}_{\bar{j}} \mathcal{V}(\tilde{p}_0; \tau) = \mathrm{e}^{K(\tilde{p}_0;\tau)/M_P^2} g^{l\bar{k}}(\tilde{p}_0; \tau)\, \partial_i \partial_l W(\xi_0)\, \bar{\partial}_{\bar{k}} \bar{\partial}_{\bar{j}} \bar{W}(\bar{\xi}_0), \tag{6.24}$$

where $K(\tau)$ and $g^{l\bar{k}}(\tau)$ are the Kähler potential and the inverse of the metric (6.11), respectively, with vanishing scalar potential. We first assume that the scalar potential (3.17) is a Morse potential, so the holomorphic superpotential has at least quadratic form with invertible Hessian matrix discussed in Section 5.1. Near $\xi = 0$, the analysis is the same as in Section 5 since we have $\mathbb{C}\mathrm{P}^{n_c-1}$, which is a Kähler–Einstein manifold. Therefore, our interest is to consider in asymptotic region $|\xi| \to +\infty$.

For the case at hand, we take the case for $\tau \geq 0$. So, in $\Lambda < 0$ case no collapsing point exists and the soliton expands smoothly, described by (6.19), whose $U(1)$-connection and Kähler potential of the theory are given by

$$\begin{aligned} Q(\xi, \bar{\xi}; \tau) &= -\frac{\mathrm{i}}{M_P^2} |\xi|^{-2+2\Lambda/\mu}\, B\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right) \left[\bar{\xi}^{\bar{i}}\, d\xi^i - \xi^i\, d\bar{\xi}^{\bar{i}}\right], \\ K(\xi, \bar{\xi}; \tau) &= \int \mathrm{e}^{\Lambda v/2\mu} B\left(\sigma(\tau) e^{-\Lambda v/2\mu}\right) dv + c, \end{aligned} \tag{6.25}$$

respectively, where $v \equiv 2 \ln |\xi|^2$ and c is a real constant. Since B is smooth function and, in general, positive definite as demanded by the inequalities (6.5), then it is well defined and does not have other singular point, namely, for $\tau = \tau_0 \geq 0$ such that

$$B\left(\sigma(\tau_0)|\xi|^{-2\Lambda/\mu}\right) = \dot{B}\left(\sigma(\tau_0)|\xi|^{-2\Lambda/\mu}\right) = 0. \tag{6.26}$$

This can be restated that it is impossible to have $\tau_0 \geq 0$ at which the soliton (6.19) collapses to a point. On the other hand, in $\Lambda > 0$ case, the soliton (6.22) converges to a cone metric at $\tau = 1/2\Lambda$ which is singular manifold because the diffeomorphism (6.10) diverges. The function D should have the same behavior as B, i.e., it does not satisfy (6.26) for $\tau \geq 0$. Moreover, since $|\xi| \to +\infty$, and then, in order to achieve a consistent picture we have to take $|\xi| \gg \tau$ for any finite τ. Therefore, the solitons (6.19) and (6.22) can be expanded as

$$g(\xi, \bar{\xi}; \tau) = g_{\rm cone}(\xi, \bar{\xi}) + O(\xi, \bar{\xi}; \tau), \tag{6.27}$$

where $O(\xi, \bar{\xi}; \tau)$ is higher order terms which are very small compared to $g_{\rm cone}(\xi, \bar{\xi})$ given in (6.21). Hence, in both cases the solitons are invertible, positive definite, and correspondingly, we have well-defined theory for all $\tau \geq 0$. Further analysis confirms that there is no parity transformation of the Hessian matrix of (3.17) that changes the Morse index of Minkowskian ground states.

Now we turn our attention to discuss nondegenerate AdS vacuum. In this case, supersymmetry further requires

$$\partial_i W(\xi_0) + \frac{2}{M_P^2} |\xi_0|^{-2+2\Lambda/\mu} \, B\left(\sigma(\tau)|\xi_0|^{-2\Lambda/\mu}\right) \bar{\xi}_0^{\bar{i}} \, W(\xi_0) = 0 \tag{6.28}$$

for $\Lambda < 0$, which follows that the ground state also depends on τ, namely $\tilde{p}_0(\tau)$ with non zero scalar potential

$$\mathcal{V}(\tilde{p}_0; \tau) = -\frac{3}{M_P^2} \mathrm{e}^{K(\tilde{p}_0;\tau)/M_P^2} |W(\xi_0)|^2, \tag{6.29}$$

where the Kähler potential $K(\tau)$ is given in (6.25), while for $\Lambda > 0$ the function B is replaced by D. However, in the case at hand, since we have cone dominating metric, only static AdS vacuum leads the term in (6.28) and further,

$$\begin{aligned} B\left(\sigma(\tau)|\xi|^{-2\Lambda/\mu}\right) &\approx B(0), \\ K(\xi, \bar{\xi}; \tau) &\approx B(0) \frac{2\mu}{\Lambda} \, |\xi|^{2\Lambda/\mu} + c, \end{aligned} \tag{6.30}$$

for any ξ in this asymptotic region. In addition, it is impossible to find a vacuum $\tilde{p}_0$ defined by the critical points of $W(z)$ as in the Kähler–Einstein case in which the $U(1)$-connection in (6.25) evaluated at $\tilde{p}_0$ vanishes.

For degenerate cases, both Minkowskian and AdS vacua reach the same results, namely the parity transformation of the Hessian matrix of (3.17) caused by the soliton does not emerge because the static cone metric g_{cone} $(\xi, \bar{\xi})$ is the leading term. Hence, we obtain similar results as in the previous case.

Similar as in the preceding section, the next step is to check the existence of such AdS vacua using (5.12) describing the first-order expansion of the beta function (4.7). For example, in the model where all spin-$\frac{1}{2}$ fermions are massless, all AdS vacua live in the UV region because all the eigenvalues of (5.12) are positive. Moreover, such a model admits only unstable walls because only negative eigenvalues of the Hessian matrix of (3.17) survive.

Finally, we conclude all results by the following theorem.

Theorem 6.1. *Consider $N = 1$ chiral supersymmetry on $U(n_c)$ symmetric Kähler–Ricci soliton* (6.11) *in asymptotic region satisfying* (6.27)*. Suppose there exists an m-dimensional vacuum manifold S of the theory with the index λ in the IR or UV regions. Then, in the normal direction of S, we can introduce real local coordinates $X_p(\tau) \approx X_p$ with $p = 1, \ldots, 2n_c - m$ for all Λ and finite τ such that*

$$\mathcal{V}(\tau) = \mathcal{V}(S) - X_1^2 - \cdots - X_\lambda^2 + X_{\lambda+1}^2 + \cdots + X_{2n_c-m}^2. \tag{6.31}$$

Thus, no parity transformation of the Hessian matrix of (4.8) *emerges caused by the Kähler–Ricci soliton in the region.*

7 Conclusions

So far we have discussed the properties of four-dimensional chiral $N = 1$ supersymmetry on Kähler–Ricci flow satisfying (2.1). Both in global and local theories, all couplings deform with respect to the parameter τ which is related to the dynamics of the Kähler metric. Then, flat BPS domain walls of $N = 1$ supergravity have been constructed where the warped factor (4.5) and the supersymmetric flows are described by the BPS equations (4.6) and the beta function (4.7) is controlled by the holomorphic superpotential $W(z)$ and the geometric flow (2.1).

We have considered the case where the initial manifold is Kähler–Einstein with $\Lambda > 0$. In particular, in the interval $\tau \geq 0$, the flow collapses to a point at $\tau = 1/2\Lambda$ and both $N = 1$ theories diverge. Such singularity disjoints two different theories, namely in $0 \leq \tau < 1/2\Lambda$ we have $N = 1$ theories on a Kähler–Einstein manifold with $\Lambda > 0$, whereas for $\tau > 1/2\Lambda$ the case turns out to be $N = 1$ theories on another Kähler–Einstein manifold with $\Lambda < 0$ and opposite metric signature. In the latter model we have wrong-sign kinetic term which might be considered as ghosts.

In global theory, such geometrical deformation affects the vacua, which can be directly seen from the parity transformation of the Hessian matrix (5.2) that changes the Morse index of the vacua if they are nondegenerate. The same conclusion is achieved for nondegenerate Minkowskian vacua in the local theory. These are written in Theorem 5.3.

In AdS vacua, it is not trivial to observe directly such phenomenon since both the geometric flow (2.1) and the coupling quantities such as the holomorphic superpotential $W(z)$ and the fermionic mass in (3.16) generally determine the properties of the vacua in the second-order analysis. Therefore, we had to impose the conditions (5.17) in order for the parity transformation to be revealed in both the nondegenerate and degenerate cases described by Theorem 5.4 and Theorem 5.5, respectively. Particularly in the degenerate case we had another possibility that the Kähler–Ricci flow could also change an n-dimensional vacuum manifold to another m-dimensional vacuum manifold with $n \neq m$ as mentioned in Theorem 5.6. Finally, we have employed the RG flow analysis to verify the existence of the vacua since they correspond to the three dimensional CFT.

Then, we gave an explicit model, namely, the $\mathbb{C}\mathrm{P}^{n_c}$ model with linear superpotential $W(z) = a_0 + a_i\, z^i$. In the model, we obtain a static vacuum $z^i = 0$ with $a_i = 0$. For $a_0 \neq 0$, the vacuum is an isolated AdS spacetime of the index $2n_c$ describing unstable walls for $0 \leq \tau < 1/2(n_c + 1)$. In the interval $\tau > 1/2(n_c + 1)$, the index then changes to 0 and it corresponds to stable walls. All of them live in the UV region ensured by (5.22). On the other side, taking $a_0 = 0$ for any z^i the vacua become degenerate Minkowskian spacetime. Hence, the vacuum manifold is $\mathbb{C}P^{n_c}$ for $0 \leq \tau < 1/2(n_c + 1)$ and then turns into $\widehat{\mathbb{C}\mathrm{P}^{n_c}}$ for $\tau > 1/2(n_c + 1)$.

For dynamical AdS vacua, we easily take $n_c = 1$. Before singularity, namely $\tau < 1/4$, there are local minimum, local maximum, saddle, and degenerate vacua living in the UV region given by (5.37) to (5.43), whereas in the IR region we have only local minimum, saddle, and nonintrinsic degenerate vacua. This evidence shows that the Kähler–Ricci flow indeed plays a role in changing the dimension and the index of the vacua consistent with the first point in Theorem 5.6. Similar phenomenon happens after singularity, namely $\tau > 1/4$. The UV region allows all possibilities of vacua, but in the IR region there are only local maximum, saddle, and again, nonintrinsic degenerate vacua. These are the facts of the second point in Theorem 5.6.

The second example is the $U(n_c)$ invariant soliton. Around $\xi = 0$ the soliton becomes $\mathbb{C}\mathrm{P}^{n_c-1}$, which is a Kähler–Einstein manifold mentioned above. Therefore, our interest is in asymptotic region $|\xi| \to +\infty$. Again, for $\Lambda > 0$, the singularity occurs at $\tau = 1/2\Lambda$ and the soliton converges to a Kähler cone. Additionally, the soliton has positive definite metric for $\tau \geq 0$ demanded by the inequalities (6.5).

Moreover, in order to get a consistent picture we have to choose $|\xi| >> \tau$ for any finite τ. Then, in the region the soliton is dominated by the cone metric (6.21) and has positive definite for all $\tau \geq 0$. Therefore, all vacua are static and correspondingly, do not posses parity transformation of the Hessian matrix of the scalar potential (3.17) caused by the geometric flow (6.19) and (6.22). The same step as in the previous case, in AdS vacua the analysis using RG flow must be performed for proving the existence of such vacua. As an example, in a model consisting of all massless spin-$\frac{1}{2}$ fermions, all AdS vacua exist in the UV region.

Finally, we want to mention that the analysis here can be generalized to a case of curved BPS domain walls. This has been addressed for two dimensional model in [27].

Acknowledgment

We acknowledge H.-D. Cao and T. Mohaupt for email correspondence. It is also a pleasure to thank K. Yamamoto and the people at the theoretical astrophysics group and the elementary particle physics group of Hiroshima university for nice discussions and warmest hospitality during the authors stay in Theoretical Astrophysics Group, Hiroshima University. We also thank M. Nitta for notifying us the references [10, 11]. We would like to thank M. Satriawan for reading the previous version of the paper and correcting some grammar. This work is funded by ITB Alumni Association (HR IA-ITB) research project 2008 No. 1241a/K01.7/PL/2008, Riset Fundamental DP2M-DIKTI No.013/SP2H/PP/DP2M/III/2007, and 324/SP2H/PP/DP2M/III/2008.

A Convention and notation

The aim of this appendix is to assemble our conventions in this paper. The spacetime metric is taken to have the signature $(+,-,-,-)$ while the Riemann tensor is defined to be $R^{\mu}_{\ \nu\rho\lambda} = \partial_\rho \Gamma^{\mu}_{\ \nu\lambda} - \partial_\lambda \Gamma^{\mu}_{\ \nu\rho} + \Gamma^{\sigma}_{\ \nu\lambda}\Gamma^{\mu}_{\ \sigma\rho} -$

$\Gamma^{\sigma}{}_{\nu\rho}\Gamma^{\mu}{}_{\sigma\lambda}$. The Christoffel symbol is given by $\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})$ where $g_{\mu\nu}$ is the spacetime metric.

We supply the following indices:

$\mu, \nu = 0, \ldots, 3,$	label curved four-dimensional spacetime indices;
$i, j, k = 1, \ldots, n_c,$	label the $N = 1$ chiral multiplet;
$\bar{i}, \bar{j}, \bar{k} = 1, \ldots, n_c,$	label conjugate indices of i, j, k;
$a, b = 1, \ldots, n_c - 1,$	label Fubini–Study coordinates of $\mathbb{C}\mathrm{P}^{n_c-1}$;
$p, q = 1, \ldots, 2n_c,$	label real coordinates;
$\underline{p}, \underline{q} = 1, \ldots, 2n_c - m,$	label real coordinates in normal direction of m-dimensional submanifold.

Some quantities on a Kähler manifold are given as follows. $g_{i\bar{j}}$ denotes the metric of the Kähler manifold whose Levi–Civita connection is defined as $\Gamma^{l}{}_{ij} = g^{l\bar{k}}\partial_i g_{j\bar{k}}$ and its conjugate $\Gamma^{\bar{l}}{}_{\bar{i}\bar{j}} = g^{\bar{l}k}\partial_{\bar{i}} g_{\bar{j}k}$. Then the curvature of the Kähler manifold is defined as

$$R^{l}{}_{i\bar{j}k} \equiv \bar{\partial}_{\bar{j}}\Gamma^{l}{}_{ik}, \tag{A.1}$$

while the Ricci tensor has the form

$$R_{k\bar{j}} = R_{\bar{j}k} \equiv R^{i}{}_{i\bar{j}k} = \bar{\partial}_{\bar{j}}\Gamma^{i}{}_{ik} = \partial_k\bar{\partial}_{\bar{j}} \ln\big(\det(g_{i\bar{j}})\big). \tag{A.2}$$

Finally, the Ricci scalar can be written down as

$$R \equiv g^{i\bar{j}} R_{i\bar{j}}. \tag{A.3}$$

B Analysis of the Hessian matrix of the scalar potential

This section is set to give our convention of the Hessian matrix of the scalar potential and some proofs of the results particularly discussed in Section 5 since the other results can be extracted from these proofs. The analysis here is standard in Morse(-Bott) theory and we omit the Einstein summation convention. Interested reader can further read references [23–26].

First of all, we define the Hessian matrix of the scalar potential (3.17) in local theory as

$$H_{\mathcal{V}} \equiv \begin{pmatrix} \partial_i\bar{\partial}_{\bar{j}}\mathcal{V} & \partial_i\partial_j\mathcal{V} \\ \partial_{\bar{j}}\partial_{\bar{i}}\mathcal{V} & \partial_j\bar{\partial}_{\bar{i}}\mathcal{V} \end{pmatrix}(p_0), \tag{B.1}$$

where $p_0 = (z_0, \bar{z}_0)$ is a critical point (vacuum) and $i, j = 1, \ldots, 2n_c$, around which the scalar potential can be expanded as

$$\mathcal{V}(z, \bar{z}; \tau) = \mathcal{V}(p_0; \tau) + \sum_{p,q=1}^{2n_c} \frac{\partial^2 \mathcal{V}(p_0)}{\partial x^p \partial x^q} \, \delta x^p \, \delta x^q, \tag{B.2}$$

with $p, q = 1, \ldots, 2n_c$, and we have defined real coordinates such that $z^i \equiv x^i + \mathrm{i} x^{i+n_c}$ and $\delta x^p \equiv x^p - x_0^p$.

Let us first focus on nondegenerate case and the initial geometry is Kähler–Einstein with $\Lambda > 0$. For the case at hand, the matrix (B.1) is invertible and evolves as

$$H_{\mathcal{V}} = \varepsilon(\sigma) \begin{pmatrix} \mathcal{V}_{i\bar{j}}(p_0) & \mathcal{V}_{ij}(p_0) \\ \\ \mathcal{V}_{\bar{j}i}(p_0) & \mathcal{V}_{\bar{j}\bar{i}}(p_0) \end{pmatrix}, \tag{B.3}$$

where $\varepsilon(\sigma)$ is given in (5.5) and the quantities $\mathcal{V}_{i\bar{j}}(p_0)$, $\mathcal{V}_{ij}(p_0)$ are defined in (5.15). In order to obtain the parity transformation of (B.1), the real and imaginary parts of $\mathcal{V}_{i\bar{j}}(p_0)$ and $\mathcal{V}_{ij}(p_0)$ remain positive with respect to $\tau \geq 0$ and $\tau \neq 1/2\Lambda$, which means that the inequalities (5.17) hold. Therefore, the expansion (B.2) becomes

$$\mathcal{V}(z, \bar{z}; \tau) = \mathcal{V}(p_0(\tau); \tau) + \varepsilon(\sigma) \sum_{p,q=1}^{2n_c} \mathcal{V}_{pq}(p_0(\tau); \tau) \, \delta x^p \, \delta x^q, \tag{B.4}$$

with $x_0^p \equiv x_0^p(\tau)$. Since $\tau \neq 1/2\Lambda$, we can define a new coordinate $X_1(\tau)$ as

$$X_1(\tau) \equiv |\mathcal{V}_{11}(p_0(\tau); \tau)|^{1/2} \left(\delta x^1(\tau) + \sum_{p=2}^{2n_c} \delta x^p(\tau) \frac{\mathcal{V}_{p1}(p_0(\tau)}{\mathcal{V}_{11}(p_0(\tau)} \right), \tag{B.5}$$

and then, (B.4) can be expressed as

$$\mathcal{V}(z, \bar{z}; \tau) = \mathcal{V}(p_0(\tau); \tau) + \varepsilon(\sigma) \left(\pm X_1^2(\tau) + \sum_{p,q=2}^{2n_c} \mathcal{V}'_{pq}(p_0(\tau); \tau) \, \delta x^p \, \delta x^q \right). \tag{B.6}$$

Therefore, we inductively carry on the computation until $r < 2n_c$ by defining

$$X_r(\tau) \equiv |\widetilde{\mathcal{V}}_{rr}(p_0(\tau); \tau)|^{1/2} \left(\delta x^r(\tau) + \sum_{p=r+1}^{2n_c} \delta x^p(\tau) \frac{\widetilde{\mathcal{V}}_{pr}(p_0(\tau); \tau)}{\widetilde{\mathcal{V}}_{rr}(p_0(\tau); \tau)} \right), \tag{B.7}$$

so that (B.4) has the form

$$\mathcal{V}(z, \bar{z}; \tau) = \mathcal{V}(p_0(\tau); \tau) + \varepsilon(\sigma) \left(\pm \sum_{p=1}^{r} X_p^2(\tau) + \sum_{p,q=r+1}^{2n_c} \widetilde{\mathcal{V}}'_{pq}(p_0(\tau); \tau) \, \delta x^p \, \delta x^q \right), \tag{B.8}$$

where

$$X_p(\tau) = \begin{cases} Y_p(\tau) & \text{if} \quad 0 \le \tau < 1/2\Lambda, \\ \widehat{Y}_p(\tau) & \text{if} \qquad\quad \tau > 1/2\Lambda. \end{cases} \tag{B.9}$$

This completes the induction and the proof of Theorem 5.4. If we take the limit $M_P \to +\infty$ or $W(z_0) = 0$, then $X_p(\tau) = |\sigma(\tau)|^{-1/2} X_p(0)$ as discussed in Theorem 5.4.

Now we turn to consider the degenerate case starting with the expansion (B.4). Next, the infinitesimal dual basis dx^p are split into

$$dx^p = \sum_{\underline{p}=1}^{2n_c - m} \Big(T^p_{\underline{p}}\, dx^{\underline{p}} + N^p_{\underline{p}}\, dx^{\underline{p}} \Big). \tag{B.10}$$

where $T^p_{\underline{p}}\, dx^{\underline{p}}$ and $N^p_{\underline{p}}\, dx^{\underline{p}}$ are both dual tangent and normal vectors of an m-dimensional submanifold S, respectively. Furthermore, the Hessian matrix in the direction of the tangent vector vanishes, namely

$$\sum_{p=1}^{2n_c} T^p_{\underline{p}}\, \mathcal{V}_{pq}(p_0(\tau);\tau) \ = 0 \tag{B.11}$$

for each q and $\underline{p}$. Restricting to the normal subspace of S, (B.4) can be written as

$$\mathcal{V}(z,\bar{z};\tau) = \mathcal{V}(p_0(\tau);\tau) + \varepsilon(\sigma) \sum_{\underline{p},\underline{q}=1}^{2n_c - m} \mathcal{V}_{\underline{pq}}(p_0(\tau);\tau)\, \delta x^{\underline{p}}\, \delta x^{\underline{q}}, \tag{B.12}$$

where $\mathcal{V}_{\underline{pq}}(p_0(\tau);\tau) \equiv \sum_{p,q=1}^{2n_c} N^p_{\underline{p}} N^q_{\underline{q}} \mathcal{V}_{pq}(p_0(\tau);\tau)$. We can then do a similar computation as in the nondegenerate case to cast (B.12) in a bilinear form. Thus, we have proved Theorem 5.5.

C More on Kähler–Ricci flow

In this appendix, we provide dynamical equations of geometric quantities such as the inverse metric, Riemann curvature, Ricci tensor, and finally, Ricci scalar. Using the identity $g^{\bar{k}i} g_{j\bar{k}} = \delta^i_j$, we find that the evolution of the inverse metric is governed by

$$\frac{\partial g^{i\bar{j}}(\tau)}{\partial \tau} = 2R^{i\bar{j}}(\tau), \tag{C.1}$$

where $R^{i\bar{j}} \equiv g^{i\bar{l}} g^{k\bar{j}} R_{k\bar{l}}$, which results in

$$\frac{\partial {\Gamma^i}_{jk}(\tau)}{\partial \tau} = -2g^{i\bar{l}} \nabla_j R_{k\bar{l}}(\tau). \tag{C.2}$$

Then, we obtain the dynamical equation of the Riemann curvature

$$\frac{\partial R^{l}{}_{i\bar{j}k}(\tau)}{\partial\tau} = -2g^{l\bar{i}}\bar{\nabla}_{\bar{j}}\nabla_i R_{k\bar{i}}(\tau), \tag{C.3}$$

which consequently gives the evolution equation of the Ricci tensor

$$\frac{\partial R_{\bar{j}k}(\tau)}{\partial\tau} = -2\left(\Delta R_{\bar{j}k}(\tau) + g^{l\bar{i}}R^{\bar{k}}_{\bar{j}l\bar{i}}R_{\bar{k}k} - g^{l\bar{i}}R^{i}_{\ l\bar{j}k}R_{i\bar{i}}\right), \tag{C.4}$$

with $\Delta \equiv g^{i\bar{j}}\nabla_i\bar{\nabla}_{\bar{j}}$. Lastly, the dynamics of the Ricci scalar is controlled by

$$\frac{\partial R(\tau)}{\partial\tau} = -2\left(\Delta R(\tau) + R^{i\bar{j}}R_{i\bar{j}}\right). \tag{C.5}$$

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