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Topological gauge theories on local spaces and black hole entropy countings

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Abstract

We study cohomological gauge theories on total spaces of holomorphic line bundles over complex manifolds and obtain their reduction to the base manifold by $U(1)$-equivariant localization of the path integral. We exemplify this general mechanism by proving via exact path integral localization a reduction for local curves conjectured in hep-th/0411280, relevant to the calculation of black hole entropy/Gromov–Witten invariants. Agreement with the four-dimensional gauge theory is recovered by taking into account in the latter non-trivial contributions coming from one-loop fluctuation determinants at the boundary of the total space. We also study a class of abelian gauge theories on Calabi–Yau local surfaces, describing the quantum foam for the A-model, relevant to the calculation of Donaldson–Thomas invariants.

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1 Introduction

Topological theories are a natural and amusing instrument to quantify some non-perturbative aspects of superstring theory. Actually, BPS protection reduces the evaluation of F-terms to a restricted configuration space and this, in favorable conditions, allows their exact calculation. The language of topological theories renders manifest the non-renormalization properties of such terms and clarifies the above configuration space reduction [4, 7]. In particular, this applies both to the world-sheet and the gauge theory approaches to the counting of the entropy of BPS black holes in superstring theories [28]. On the world-sheet side one finds that few, but remarkable, all-loop calculations of the effective theory of a Calabi–Yau compactification can be recasted in terms of amplitudes of a topological theory of strings counting the number of inequivalent world-sheet instantons in the Calabi–Yau, $Z_{\text{top}} = \sum_g \lambda_{\text{top}}^{2g-2} F_g$. The counting of D-brane bound states has been advocated in [28] to provide a non-perturbative completion of these amplitudes via a conjectured S-duality in topological strings [27]. A natural setting to describe this duality is topological M-theory [3, 5, 9, 11, 14, 18, 26]. More precisely, in [28], the suggestive relation $Z_{\text{BH}} \sim |Z_{\text{top}}|^2$ was proposed to hold in the limit of large black holes charges up to $O(e^{-N})$ corrections. In the dual D-brane language, the black hole BPS multiplicities gets calculated by the supersymmetric partition function of the twisted gauge theory living on the D-brane system. It is therefore interesting to find exact calculational methods in such a framework. In this paper, we study in particular cohomological gauge theories on local spaces, that is, on spaces whose geometry can be obtained by zooming in the vicinity of a given non-trivial cycle. Namely, in the case of complex codimension one, one has the total space over the cycle of an appropriate line bundle. On this space, there is a natural $U(1)$-symmetry acting on the fiber which can be used to simplify the relevant path integral calculation. Actually, we will show how to realize the dimensional reduction on the base by using such a symmetry. This will be described in detail for the D4/D2/D0 system considered in [2, 29]. We will also present a more general mechanism for the D6/D2/D0 system [22].

In [2], it has been proposed that the partition function of the relevant twisted $\mathcal{N} = 4$ gauge theory on the total space $\mathcal{L} \to \Sigma$, where Σ is a Riemann surface and $\mathcal{L}$ a line bundle over it, can be evaluated by reducing to a q-deformed YM_2 on Σ [20]. This was verified by comparing the factorized structure of the partition function in the large N limit with topological string amplitudes on the relevant Calabi–Yau local curve $\mathcal{L} \oplus \mathcal{K}\mathcal{L}^{-1} \to \Sigma$, where $\mathcal{K}$ is the canonical line bundle on the base Σ and the D4-branes wrap the cycle $\mathcal{L} \to \Sigma$ [12].

On the other hand, a comparison with the four-dimensional gauge theory has been done in the genus zero case when the instanton counting can be performed [17, 19]. In this case, agreement was found up to some perturbative contributions which can be recasted as one-loop fluctuations of the Chern–Simons theory living at the boundary of the total space [19]. In this paper, we suggest that these contributions have actually a very simple interpretation in the four-dimensional gauge theory. The crucial point is that we are quantizing the theory on a non-compact space. Henceforth, the one-loop determinants do not cancel completely as it happens on compact manifolds, but they receive a non-trivial contribution precisely from the boundary, where the four-dimensional action reduces to a Chern–Simons term.[1] By including these contributions in the four-dimensional instanton counting one can recover agreement with the q-deformed YM_2 results.

Let us briefly outline the content of the paper. In Section 2, we start describing the geometric set-up, then we obtain the reduction of the topological action on the base as well as the natural implementation of the $U(1)$-equivariant BRST symmetry in our problem. In Section 3, we analyze in close detail the D4/D2/D0 system and we discuss its reduction to the q-deformed YM_2 by using the relevant localization of the path integral. In Section 4, we extend our procedure to the case of a topological gauge theory on a local Calabi–Yau surface and we propose an analogous reduced approach to the counting of D6/D2/D0 BPS black hole entropy. In Section 5, we discuss few open issues.

2 Reduction of cohomological gauge theories on local spaces

2.1 Cohomological gauge theories on local spaces

Let Σ be a complex manifold with $\dim_{\mathbf{C}} \Sigma = n$ and $\mathcal{L}$ be an holomorphic line bundle over it. Let $M = \mathcal{L} \to \Sigma$ be the total space of $\mathcal{L}$. Let E be a gauge bundle over Σ with structure group G. It extends canonically to a gauge bundle on M that we will still call E for notational simplicity. Let $A \in \mathrm{Conn}(E, M)$ be a connection of E on M and let us consider a topological invariant functional $S_{\mathrm{top}}(A)$ that is invariant under continuous deformations of the connection A along $\mathrm{Conn}(E, M)$. Typically

$$S_{\mathrm{top}}(A) = \int_M [P(F) \wedge K], \tag{2.1}$$

[1] In the case of flat $\mathbf{R}^4$ space, the boundary contribution is trivial.

where $P(F)$ is an Ad-invariant polynomial in the curvature $F = dA + A \wedge A$ and $K \in H^\bullet(\Sigma)$ is an element in the even cohomology of Σ. In (2.1), the top component of the integrand is understood.

This kind of theories naturally arise in D-brane/black holes entropy countings. These countings reduce to the evaluation of the partition function of the cohomological gauge theories obtained by quantizing (2.1). Actually, since M is non compact, we wish to sum over all the gauge field configurations at the boundary. This can be achieved either directly in the path integral by implementing the summation over the boundary fluctuations via the associated Chern–Simons-like theory, or by analyzing the sum over the boundary values in terms of the holonomy of gauge field at infinity. The first approach leads then to work on the total space M giving the result in terms of bulk and boundary contributions. The second, as we will show in detail later, leads to a reduction of the calculation of the path integral via an associated topological theory defined on the base Σ.

By construction ∂M is the total space of the circle bundle $\arg(\mathcal{L})$, namely $\partial M = \arg \mathcal{L} \to \Sigma$. Therefore, to parametrize the boundary conditions, we can specify the holonomy of A along each S^1 fiber on Σ, that is

$$e^{i\Phi} = P \exp\left\{ i \int_{S^1 \times \mathrm{Pt}} A \right\} \tag{2.2}$$

at any point $\mathrm{Pt} \in \Sigma$. Summing over all the boundary conditions, then will mean path integrate over the holonomy field Φ.

On M it is natural to distinguish horizontal and vertical directions. Let $\gamma \in \mathrm{Conn}(\mathcal{L}, \Sigma)$ be a reference connection for $\mathcal{L}$ and let w be a coordinate on the fiber. Then, the differential element $Dw = dw + \gamma w$ is covariant, namely Dw transform like a section of $\mathcal{L}$. This defines a decomposition of the Dolbeault differential $\partial = \partial^h + \partial^v$, where

$$\begin{aligned} \partial^h &= \partial_\Sigma - \gamma w \partial_w, \\ \partial^v &= Dw \partial_w, \end{aligned}$$

and ∂_Σ is the Dolbeaux differential on the base. The de Rham differential gets decomposed too. Similarly, the gauge connection splits as

$$A = A^h + A^v = A^h + \varphi Dw + \varphi^\dagger D\bar{w}, \tag{2.3}$$

where both $A^h \in \mathrm{Conn}(E, \Sigma)$ and $\varphi \in \Gamma(\mathrm{Ad}\, E, \Sigma)$ depend parametrically on the fiber coordinate w.

The above holonomy assignment (2.2) corresponds, up to a gauge transformation, to the boundary conditions

$$i_\theta A = i(w\varphi - \bar{w}\varphi^\dagger) \sim \frac{\Phi}{2\pi} \quad \text{as} \quad w \sim \infty, \tag{2.4}$$

where $\theta = i(w\partial_w - \bar{w}\partial_{\bar{w}})$ is the fundamental vector field generating the $U(1)$-action on the fiber $w \to e^{i\theta}w$.

We will now show how the topological theory (2.1) can be calculated in terms of reduced field configurations on the base manifold. We will first analyse (2.1) at the classical level, and then show how the equivariant localization with respect to the $U(1)$-action $w \to e^{i\theta}w$ on the fiber allows us to dimensionally reduce the theory at the path integral level.

2.2 Covariant dimensional reduction of the classical action

Since the classical topological action is independent upon the specific connection of the given vector bundle E and at given boundary conditions we use to calculate it, we can choose to evaluate it by using some particular class of connections, namely $U(1)$-invariant ones. On top of it, to simplify our life, we can perform the calculation in a given gauge with respect to the G bundle. We choose to work in radial gauge $i_R A = 0$, where $R = w\partial_w + \bar{w}\partial_{\bar{w}}$ is the invariant vector field generating the fiber dilatations. This means that we consider gauge connections of the form

$$\hat{A} = \alpha + \frac{\rho}{2\pi}\mathrm{Im}\left(\frac{Dw}{w}\right), \tag{2.5}$$

where $\alpha \in \mathrm{Conn}(E, \Sigma)$. α and ρ do not depend on $\arg(w)$ and $\rho \in \mathrm{Adj}(E, \Sigma)$ satisfies the boundary condition $\rho \sim \Phi$ as $w \sim \infty$.

Although the calculation can be performed in arbitrary dimensions, let us do it explicitly in the case $n = 1$ first. The relevant possible terms in (2.1) are $\int_M \mathrm{Tr}(F \wedge F)$ and $\int_M K \wedge \mathrm{Tr}(F)$. By simply substituting (2.5), using the Stokes theorem and the boundary condition for ρ, we get

$$\int_M \mathrm{tr}\, \hat{F} \wedge K = \int_\Sigma \mathrm{tr}\, \Phi K \tag{2.6}$$

and

$$\int_M \mathrm{tr}(\hat{F} \wedge \hat{F}) = \int_\Sigma \left[2\,\mathrm{tr}\,(\Phi f) + \frac{1}{2\pi}\mathrm{tr}\,\Phi^2 R_{\mathcal{L}}\right], \tag{2.7}$$

where $f = d_\Sigma a + a \wedge a,\ a = \alpha|_{w\sim\infty}$ and $R_{\mathcal{L}} = d_\Sigma\,\mathrm{Im}\,\gamma$ is the curvature of $\mathcal{L}$.

In the general case,[2] we can proceed by using equivariant localization under the $U(1)$-invariance in the evaluation of the topological invariant (2.1). The equivariant extension of the gauge curvature at the fixed point (2.5) is given by

$$\tilde{F} = \hat{F} + \mu, \tag{2.8}$$

where μ is the moment map of the $U(1)$-action, $d_{\hat{A}}\mu = -i_\theta \hat{F}$. By inspection, one gets $\mu = \frac{\rho}{2\pi}$. The localization formula [6] then gives

$$\int_M [P(F) \wedge K] = \int_M [P(\tilde{F}) \wedge K] = 2\pi \int_\Sigma \left[\frac{P\left(f + \frac{\Phi}{2\pi}(1 + R_{\mathcal{L}})\right) \wedge K}{(1 + R_{\mathcal{L}})} \right], \tag{2.9}$$

where the factor $\frac{2\pi}{1+R_{\mathcal{L}}}$ is the inverse Euler class of the normal bundle, that is of the line bundle $\mathcal{L}$ itself. Notice that since $K \in H^\bullet(\Sigma)$, then $i_\theta K = 0$ and K is trivially equivariantly closed. In particular, by specifying $n = 1$ in (2.9), we generate as possible terms exactly (2.6) and (2.7).

Therefore, we see that if we constrain the connection to be $U(1)$-invariant, then the topological action (2.1) gets reduced to (2.9), which depends on a connection for E on the base and the holonomy field. Our next aim will be to implement the above reduction in the full gauge-fixed topological theory.

2.3 Reduction via $U(1)$-equivariant localization: the four-dimensional case

To implement the reduction of the previous section at the path integral level, we use equivariant localization with respect to the lifting on the field space of the $U(1)$-action on the fiber $w \to e^{i\theta} w$, $\theta \in S^1$.

Before doing it, let us discuss some aspects of the topological gauge symmetry at hand. We restrict for simplicity to the four-dimensional case ($n = 1$), although the following strategy can be generalized to higher dimensions. This will be exemplified later for $n = 2$.

Let us consider first the topological action $\frac{1}{2}\int_M \mathrm{tr}\, F \wedge F$. This is invariant under the BRST action on the space of connections with arbitrary boundary conditions $sA = \Psi$ if $\Psi \sim d_A c$ at the boundary. Actually, this is the usual scheme to quantize the four-dimensional topological theory and to show that it localizes on instanton solutions. As it is well known, the semiclassical limit

[2] Actually one could proceed also in the direct way. This implies the use of the standard integral transgression formula to solve $P(F) = dQ(A, F)$, passing to the boundary via Stoke's theorem, calculating the integral along the circle at infinity and finally integrating back the transgression formula.

is exact in these cohomological theories and the path integral is reduced to the integration of the relevant observables over the instanton moduli space times the contribution of the determinants of the one-loop fluctuations around the instanton vacua. These determinants usually cancel to one due to supersymmetry. Notice however that there is a subtlety in this case due to the non-compactness of the manifold on which we are quantizing the theory. The functional integration of the fields at the boundary produces a non-trivial one-loop determinant which has to be taken into account. Then schematically one gets

$$Z_M = Z_{\partial M}^{\mathrm{CS(1-loop)}} Z_M^{\mathrm{instantons}}. \tag{2.10}$$

The one-loop Chern–Simons partition function comes from the integration along the field fluctuations at the boundary while the second factor comes from the bulk instantons.

The same calculation can be performed in a different way by parametrizing the boundary values of the connection via its holonomy. So we calculate

$$Z_M = \int D[\Phi] Z_M(\Phi), \tag{2.11}$$

where $Z_M(\Phi)$ is the partition function calculated at fixed holonomy Φ. As we will show in detail in the next section, by applying this procedure to the gauge theory relevant for the D4/D2/D0 system, one gets the partition function of the q-deformed YM_2. The quantum measure $D[\Phi]$ has been discussed in [2] and we will come back to it later on. Notice that the explicit calculations of the q-deformed YM_2 on the sphere precisely reproduce the structure in (2.10) [19].

Let us put again the question about the boundary conditions on gauge parameters. Now, since the holonomy is fixed, we can assign them differently. Actually, at fixed holonomy, we require the natural boundary conditions

$$d_\theta A^h \sim 0, \quad i_\theta A \sim \Phi/2\pi, \tag{2.12}$$

$$d_\theta \Psi^h \sim 0, \quad i_\theta \Psi \sim 0. \tag{2.13}$$

up to gauge transformations, which we want to insist on. Actually the topological action is not shift symmetric under the above boundary conditions. In fact, we have

$$\begin{aligned} s\frac{1}{2}\int_M \mathrm{tr}\, F\wedge F &= \int_M \mathrm{tr}\, F\wedge d_A\Psi = \int_{\partial M} \mathrm{tr}\,(F^{hh}\wedge \Psi_\theta + F_\theta^h \wedge \Psi^h) \\ &= \int_{\partial M} \mathrm{tr}\, d_{A^h} A_\theta \wedge \Psi^h = \int_\Sigma \mathrm{tr}\, d_a\Phi\wedge\psi, \end{aligned} \tag{2.14}$$

where $A^h \sim a$ and $\Psi^h \sim \psi$ as $w \sim \infty$. Notice that, if K is a 2-form on Σ, we have

$$s \int_M K \wedge \operatorname{tr} F = \int_{\partial M} K \wedge \Psi = 0$$

since $\Psi_\theta \sim 0$.

In order to cure the above lack of symmetry, one can improve the topological action by adding the following topological observable localized on Σ

$$\mathcal{O}^{(2)}_\Sigma = \int_\Sigma \operatorname{tr}\left(\phi F + \frac{1}{2}\Psi \wedge \Psi\right). \tag{2.15}$$

Let us read now the above deformation from the point of view of the calculation (2.11) with fixed holonomy at infinity. We just noticed in (2.14) that we need to improve our action. In fact, we have to improve the BRST symmetry too. Let us consider the quantity $I = \frac{1}{2}\int_M \operatorname{Tr} F \wedge F + \mathcal{O}^{(2)}_{\Sigma_\infty}$ where Σ_∞ is the copy of the base at infinity, and calculate its BRST transform under a BRST operator s_θ to be determined up to its fixed shift action on the connection $s_\theta A = \Psi$. We have

$$s_\theta I = \int_{\Sigma_\infty} \operatorname{tr}(s_\theta \phi F) + \operatorname{tr}(s_\theta \Psi - d_A \phi - d_A \Phi)\Psi, \tag{2.16}$$

which is zero for

$$s_\theta A = \Psi, \quad s_\theta \Psi = d_A(\phi + 2\pi i_\theta A) + 2\pi i_\theta F \quad \text{and} \quad s_\theta \phi = 0 \tag{2.17}$$

due to the boundary condition $i_\theta \Psi \sim 0$ at infinity. Actually, we can rewrite (2.17) as

$$s_\theta A = \Psi, \quad s_\theta \Psi = d_A \phi + 2\pi L_\theta A \quad \text{and} \quad s_\theta \phi = 0 \tag{2.18}$$

which satisfies $s_\theta^2 = \delta_\phi + 2\pi L_\theta$, where $L_\theta = d i_\theta + i_\theta d$ is the Lie derivative along the vector field θ. This reveals s_θ to be nothing but the $U(1)$-equivariant extension of the original BRST symmetry.

Alternatively, by redefining

$$\phi' = \phi + 2\pi i_\theta A \tag{2.19}$$

we can rewrite (2.18) as

$$s_\theta A = \Psi, \quad s_\theta \Psi = d_A \phi' + 2\pi i_\theta F \quad \text{and} \quad s_\theta \phi' = 2\pi i_\theta \Psi, \tag{2.20}$$

and we have $s_\theta^2 = \delta_{\phi'} + 2\pi \mathcal{L}_\theta$, where $\mathcal{L}_\theta = d_A i_\theta + i_\theta d_A$ is the *covariant* Lie derivative. Notice that the field redefinition (2.19) changes the boundary conditions of the field ϕ due to (2.4). Actually these are no longer vanishing, but $\phi' \sim \Phi$ as $w \sim \infty$.

For later use, let us notice also that, due to (2.13), we have $s_\theta \int_M K \wedge \operatorname{tr} F = 0$.

3 D4/D2/D0-branes on local Calabi–Yau's

Let us here consider the topological gauge theory studied in [2, 29]. Consider a Riemann surface Σ, a line bundle $\mathcal{L}$ over it and the CY local curve $X = \mathcal{L} \oplus \mathcal{L}^{-1}\mathcal{K} \to \Sigma$, where $\mathcal{K}$ is the canonical line bundle over Σ. Let us place N D4-branes on $M = \mathcal{L} \to \Sigma$ and any number of D0 and D2 wrapping Σ.

Actually, the entropy of such a D-brane system is calculated by the Vafa–Witten twisted $\mathcal{N} = 4$ SYM with gauge group $U(N)$ and classical action

$$S_{\text{top}} = \frac{1}{2g_s}\int_M \mathrm{tr}(F \wedge F) + \frac{\theta}{g_s}\int_M \mathrm{tr}\, F \wedge K \tag{3.1}$$

with K the unit volume form on Σ.

In the previous section, we have been proving that the path integral for such a theory gets localized on $U(1)$-invariant configurations. In particular, the reduction of the various terms in (3.1) reads

$$S_{\text{red}} = \frac{1}{g_s}\int_\Sigma \mathrm{tr}(\Phi f) + \frac{\theta}{g_s}\int_\Sigma \mathrm{tr}\, \Phi K + \frac{1}{2g_s}\int_\Sigma \mathrm{tr}\, \Phi^2 \frac{R_{\mathcal{L}}}{2\pi}. \tag{3.2}$$

Let us notice the appearance of the last term quadratic in Φ which we obtained just by insisting on a covariant reduction scheme. The curvature of the line bundle is $R_{\mathcal{L}} = -2\pi p K$, where $\deg(\mathcal{L}) = -p$. In this way, one obtains precisely the topological action for the q-deformed YM_2 considered in [2].

Let us now discuss the evaluation of the partition function for the D4/D2/D0 system

$$Z_{\text{D4/D2/D0}} = \int \mathcal{D}[A, \Psi, \ldots]\exp\left[-S_{\text{top}} - S_{g\cdot f.} - \frac{1}{g_s}(\mathcal{O}_2 + \theta\mathcal{O}_2^{[K]} + \mathcal{O}_4)\right], \tag{3.3}$$

where

$$\mathcal{O}_2 = \int_\Sigma \mathrm{tr}\left(F\phi + \frac{1}{2}\Psi \wedge \Psi\right), \tag{3.4}$$

$$\mathcal{O}_2^{[K]} = \int_\Sigma \mathrm{tr}\, \phi K, \tag{3.5}$$

$$\mathcal{O}_4 = \frac{1}{2}\int_\Sigma \mathrm{tr}\, \phi^2 R_{\mathcal{L}}. \tag{3.6}$$

The subscript index in (3.6) is the ghost number.

Notice that, according to the prescription described in Section 2.3, in (3.3) we inserted the topological observables (3.6) with positive ghost numbers. Due to the absence of a ghost number anomaly in the twisted $\mathcal{N} = 4$

SYM theory, the v.e.v. of these observables is actually vanishing and the path integral is insensitive to their presence. Nonetheless, their insertion simplifies the task of dimensional reduction. This procedure is completely analogous to the mass deformation of the Vafa–Witten theory which allows to compute the twisted $\mathcal{N}=4$ partition function in terms of $\mathcal{N}=1$ vacua countings. In our case, the observables (3.6) break the anti-BRST of the $N_T=2$ balanced topological field theory [8, 15], thus reducing the twisted supersymmetry to $N_T=1$. Indeed, the deformation (3.3) suggests an inspiring relation with the Donaldson intersection theory on the instanton moduli space. It would be very interesting to study in deeper detail this connection by analyzing the $U(1)$-localization procedure for Witten's TYM theory [30]. For Riemann surfaces of genus zero, this could provide a direct derivation of the blow-up formulae for Donaldson polynomials [16] that were obtained in [23, 24] by using the low-energy Seiberg–Witten effective theory.

At the fixed locus under the $U(1)$-action, the observables (3.6) give

$$\mathcal{O}_2^{\mathrm{red}} = \int_\Sigma \mathrm{tr}\left((f+R_{\mathcal{L}}\Phi)\phi + \frac{1}{2}\psi\wedge\psi\right), \tag{3.7}$$

$$\mathcal{O}_2^{[K]\,\mathrm{red}} = \int_\Sigma \mathrm{tr}\,\phi K, \tag{3.8}$$

$$\mathcal{O}_4^{\mathrm{red}} = \frac{1}{2}\int_\Sigma \mathrm{tr}\,\phi^2 R_{\mathcal{L}}, \tag{3.9}$$

where all the fields depend on the base Σ only. We calculate the total classical action first by reducing $S_{\mathrm{top}} + \frac{1}{g_s}[\mathcal{O}_2 + \theta\mathcal{O}_2^{[K]} + \mathcal{O}_4]$ and obtain

$$S_{\mathrm{red}} = \frac{1}{g_s}\int_\Sigma \left[\theta\,\mathrm{tr}(\phi')K + \mathrm{tr}(\phi' f + \psi^2) + \frac{1}{2}\mathrm{tr}(\phi'^2)R_{\mathcal{L}}\right], \tag{3.10}$$

where $\phi' = \phi + \Phi$. Notice that this is *precisely* the shift (2.19) so that we can interpret the reduced theory as a two-dimensional topological gauge theory on Σ.

Notice however that since we choose $\Sigma = \Sigma_\infty$, the vanishing boundary conditions on ϕ imply that on the base image at infinity $\phi' = \Phi$. Therefore, we get the reduced theory with action

$$S_{\mathrm{YM}_2} = \frac{1}{g_s}\int_\Sigma \left[\theta\,\mathrm{tr}(\Phi)K + \mathrm{tr}(\Phi f + \psi^2) + \frac{1}{2}\mathrm{tr}(\Phi^2)R_{\mathcal{L}}\right] \tag{3.11}$$

that is the topological action of the 2d Yang–Mills. The reduced BRST operator is then

$$sa = \psi, \quad s\psi = d_a\Phi \quad \text{and} \quad s\Phi = 0. \tag{3.12}$$

3.1 Equivariant localization of the path integral

In order to show the dimensional reduction at the path integral level, we have to face the full gauge-fixed theory. The full field content of the Vafa–Witten twisted $\mathcal{N}=4$ theory is given by the $N_T=2$ connection multiplet $(A,\Psi_\pm,H)$, a selfdual tensor multiplet $(B^+,\chi^+_\pm,H^+)$ and the Cartan multiplet [15]. The relevant $U(1)$-equivariant BRST transformations on the selfdual multiplet are

$$s_\theta B^+ = \chi^+, \quad s_\theta \chi^+ = [\phi', B^+] + \mathcal{L}_\theta B^+. \tag{3.13}$$

The gauge-fixed action of the theory is given in terms of the action potential [15]

$$\begin{aligned}
\mathcal{F} &= \mathcal{F}_1 + \mathcal{F}_2,\\
\mathcal{F}_1 &= \int_M \mathrm{Tr}[B^+_{\mu\nu}(F^{\mu\nu} + [B^\mu_\rho, B^{\rho\nu}])],\\
\mathcal{F}_2 &= \int_M \mathrm{Tr}[\chi^+_+ \wedge \chi^+_- + \Psi_+ \wedge *\Psi_-],
\end{aligned} \tag{3.14}$$

where we do not consider the gauge fixing of the Cartan multiplet since it can be universally re-absorbed [15].

Let us now show that the modes with non-zero transverse momentum can be integrated out and that they do not contribute to the topological partition function. In order to show this, we split all field configurations in terms of zero and non-zero modes with respect to the operator which defines the transverse momentum. Because of the gauge symmetry of the problem, we have to consider the operator $d^v_{A_\infty}$, where A_∞ is the connection at the boundary $w \sim \infty$. The integrability of F^{vv} along the fibers requires that $F^{vv}_\infty = (d^v_{A_\infty})^2 = 0$. To choose the appropriate gauge-fixing, let us rescale the fields as

$$\begin{aligned}
A &= A_\infty + x\delta A^h + x^{-1}\delta A^v,\\
B^+ &= x^{-1}\delta B^{(2,0)} + c.c. + (b_\infty + \delta b)\omega_x.
\end{aligned} \tag{3.15}$$

where δ[fields] denote the non-zero modes with respect to $d^v_{A_\infty}$. We will consider the scaling gauge $x \to \infty$. In (3.15) we used the rescaled Kähler form $\omega_x = x\omega^{hh} + x^{-\alpha}\omega^{vv}$ obtained from the block diagonal $\omega = \omega^{hh} + \omega^{vv} = i\partial\bar{\partial} f$, where f is the Kähler potential. Notice that in (3.15) we retained only the zero modes for the connection multiplet and the Kähler component of the selfdual multiplet, while we did not for the $(2,0)$ part of the selfdual multiplet. Actually $B^{(2,0)}$ would have a zero mode structure

$$B^{(2,0)}_{\infty} = \beta \frac{Dw}{w} \tag{3.16}$$

up to gauge conjugation by $e^{i\theta\Phi}$ and where $\beta \in T^{(1,0)}(\Sigma, \mathrm{Adj}\, E)$. This field configuration is singular as $w \sim 0$ and we do not consider it to belong to the allowed field space.[3]

In (3.15) we just gave the rescaling of each top member of the $N_T = 2$ multiplets, the others being equal because of BRST invariance. This guarantees that such a rescaling does not generate any non-trivial Jacobian from the quantum measure in the path integral. The configuration A_∞ is taken to satisfy the holonomy assignment and is given, up to gauge conjugation, by

$$A_\infty = a + \frac{\Phi}{2\pi} \mathrm{Im} \left(\frac{Dw}{w} \right).$$

To calculate the rescaled gauge fixing action, we apply the rescalings (3.15) to (3.14) and obtain that if $\alpha > 2$, then

$$\mathcal{F}_1 = \int_M \delta B^{(2,0)} \bar{\partial}^v_{A_\infty} (\delta A^h)^{(0,1)} + c.c. + \delta b \omega^{hh} d^v_{A_\infty} \delta A^v + O(1/x) \tag{3.17}$$

while $\mathcal{F}_2 = O(1/x)$ and does not contribute in the scaling limit.

In order to integrate the leftover transverse momentum zero modes for the BRST doublets $(\bar{\psi}, H)_\infty$ and $(b, \chi_+)_\infty (\chi_-, h)_\infty$, we add to the gauge fixing action the exact term

$$s_\theta \int_\Sigma \mathrm{tr}(b[\Phi, \chi_-] + H \wedge *[\Phi, \bar{\psi}])$$

which path integrates to $\det(\mathrm{Adj}_\Phi)^2$ in the fermionic sector and to det $(\mathrm{Adj}_\Phi)^{-2}$ in the bosonic one, so giving no contribution to the partition function.

Therefore, in the scaling gauge $x \to \infty$, the leftover term (3.17) gauge-fixes all the fluctuation in the connection and in the selfdual $N_T = 2$ multiplets.

Summarizing, in the scaled gauge we then find that all the modes with non-vanishing transverse momentum get gauge-fixed to zero and the path integral evaluation reduces to the theory dimensionally reduced on the basis

[3]Notice that, if one would keep it, this sector would extend the YM_2 theory on Σ to a deformed Hitchin system with equations

$$f + \Phi R_{\mathcal{L}} = [\beta, \bar{\beta}], \quad \bar{\partial}_a \beta = 0 \quad \text{and} \quad d_a \Phi = 0,$$
$$[\Phi, \beta] = 0, \quad d_a b = 0.$$

with action (3.11). Notice that, due to the compactness of the holonomy field Φ, its path integral measure has to be suitably defined as in [2], therefore, giving the q-deformation of YM_2 on Σ.

4 Quantum foam on the local CY surface

Let us consider in this section a topological abelian $U(1)$ gauge theory on a generic CY local surface $M = \mathcal{L} \to \Sigma$, where $\mathcal{L} = \mathcal{K}$ is the canonical line bundle of the complex surface Σ itself. This theory describes a system of D2/D0 dissolved in a D6-brane wrapping M. The topological partition function of this theory computes the Donaldson–Thomas invariants of the CY it is defined on. Moreover, it has been proposed in [22] as weak coupling expansion in g_s for the topological A-model. Our reduction formula works also if the line bundle $\mathcal{L}$ is not constrained to be $\mathcal{K}$ by the Calabi–Yau condition.

The quantum foam topological action is

$$S_{qf}(A) = \frac{g_s}{3} \int_M F \wedge F \wedge F + \int_M k_0 \wedge F \wedge F, \tag{4.1}$$

where $F = dA$ is the abelian curvature and $k_0 \in H^2(\Sigma, Z)$. By definition, the topological version to consider is the twisted maximally supersymmetric one.

As in the four-dimensional case, we can consider the quantization of the theory from two equivalent points of view. Its partition function can be evaluated directly, by giving

$$Z_{qf} = Z^{(5d-\mathrm{oneloop})} Z_{\mathrm{bulk}}, \tag{4.2}$$

where $Z^{(5d-\mathrm{oneloop})}$ is the one-loop path integral of the topological boundary theory with action

$$S_{5d}(A) = \frac{g_s}{3} \int_{\partial M} A \wedge dA \wedge dA + \int_M k_0 \wedge A \wedge dA. \tag{4.3}$$

The bulk contribution Z_{bulk} is the partition function of the topological theory on M with vanishing boundary conditions.

A second equivalent calculational scheme is again obtained by specifying the (abelian) holonomy $e^{i\phi} = e^{i\int_{S^1_\infty} A}$. Therefore, we have also

$$Z_{qf} = \int \mathcal{D}[\Phi] Z_{qf}(\Phi). \tag{4.4}$$

By applying the same procedure we developed in the previous sections, we can now reduce the evaluation of the partition function to that of an abelian topological gauge theory on the four-dimensional base manifold.

The only peculiar ingredient to be fixed is the observable of the initial theory to be added in order to produce the equivariant extension of the action functional. This is given by a suitable linear combination of

$$\mathcal{O}_1 = \int_\Sigma \Psi \wedge \Psi \wedge F + \phi F \wedge F \quad \text{and} \quad \mathcal{O}_2 = \int_\Sigma k_0 \wedge \left(\frac{1}{2} \Psi \wedge \Psi + \phi F \right). \tag{4.5}$$

We can calculate then, by using (2.9)

$$\int_M \tilde{F} \wedge \tilde{F} \wedge \tilde{F} = \int_\Sigma \left[\Phi^3 \frac{R_\mathcal{K}}{2\pi} \wedge \frac{R_\mathcal{K}}{2\pi} + 3\Phi^2 f \wedge \frac{R_\mathcal{K}}{2\pi} + 3\Phi f \wedge f \right]$$

and

$$\int_M k_0 \wedge \tilde{F} \wedge \tilde{F} = \int_\Sigma k_0 \wedge \left[2f\Phi + \frac{R_\mathcal{K}}{2\pi} \Phi^2 \right],$$

and we get that the quantum foam reduces to the topological action on the base Σ

$$\begin{aligned} S_{qf-4d} = g_s \int_\Sigma & \left[\frac{1}{3} \Phi^3 \frac{R_\mathcal{K}}{2\pi} \wedge \frac{R_\mathcal{K}}{2\pi} + \Phi^2 f \wedge \frac{R_\mathcal{K}}{2\pi} + \Phi f \wedge f + \left(f + \Phi \frac{R_\mathcal{K}}{2\pi} \right) \wedge \psi \wedge \psi \right] \\ & + \int_\Sigma k_0 \wedge \left(\Phi f + \Phi^2 \frac{R_\mathcal{K}}{2\pi} + \frac{1}{2} \psi \wedge \psi \right) \end{aligned} \tag{4.6}$$

that is closed under the reduced BRST action $sa = \psi$, $s\psi = d\Phi$ and $s\Phi = 0$. The maximally SUSY twisted theory that we are considering is the one studied in [21] on Kähler threefolds. One can extend in full analogy to this case too the off-shell localization of the path integral that we discussed for the Vafa–Witten theory in the previous section. Due to the abelian nature of the theory, the quantum measure over Φ is the usual translational invariant one for a single S^1 valued field. We would get an analog of the q-deformation in a non-abelian version of the topological theory at hand. Therefore, we find that the calculation scheme at fixed holonomy reduces to the cohomological gauge theory (4.6)

$$Z_{qf} = \int \mathcal{D}[a, \psi, \Phi] e^{-S_{qf-4d}}.$$

It is tempting to treat the theory defined by (4.6) in a perturbative expansion in g_s. Indeed, the propagators for this theory would be given by the terms in the second line of (4.6) and as such would be localized on two-dimensional cycles. Moreover, the interaction terms in the first line give rise to cubic vertices. This structure is similar to the one found for the topological vertex computing Gromov–Witten invariants on toric CYs [1] and their generalization to degenerated torus actions studied in [13]. It would be nice to try to make a closer comparison with these formalisms.

5 Conclusions and open issues

In this paper, we have been developing a general reduction scheme for cohomological gauge theories on local spaces. Actually we have been studying in detail the case of twisted maximally supersymmetric gauge theories, but we believe that a more general reduction scheme holds also for other cases. In particular, a tempting relation among Donaldson invariants and two-dimensional Yang–Mills theories would appear in the reduction of the twisted $\mathcal{N} = 2$ theory. Let us here discuss few open issues raised by the analysis we performed in this paper.

As far as the comparison among the calculation of the D4/D2/D0 partition function in the scheme in four and in two dimensions is concerned, it remains to fully recognize the precise encoding by the $q\mathrm{YM}_2$ of the bulk point-like instantons [17, 19]. Actually our reduction mechanism gives non-trivial results only for gauge fields with non-trivial holonomy at infinity, suggesting the possible existence of a further branch. This could be related to the fact that in the vicinity of the base manifold we assumed some regularity conditions for the fields, see Section 3.1. It is possible that our criteria are too strict and that we should include more general field configurations. In particular, the inclusion of singular fields via a suitable compactification of the space of $U(1)$-invariant configurations could provide the missing terms. From the analysis of [25], one could expect that at least in the rank one case this compactification contains the symmetric product $\Sigma^{[k]}$ of k copies of the base manifold Σ, k being the instanton number. This would properly take into account the contribution of point-like instantons.

In Section **4**, we generalized the dimensional reduction of the path integral to the six-dimensional abelian gauge theory on the local CY surface $\mathcal{K} \to \Sigma$. One should be able to compare the calculation of the partition function of the reduced gauge theory on the four manifold Σ with the topological string partition function on the local Calabi–Yau, for example in the case $\Sigma = \mathbb{C}P^2$ and $\mathcal{K} = \mathcal{O}(-3)$.

The ability to show that the fields with non-zero vertical momentum do not really contribute to the topological theory could be extended in other cases. The mechanism we have shown could generalize to Schwarz-type theories, like Chern–Simons and holomorphic Chern–Simons, giving an off-shell reduction for the constructions made in [10]. Moreover, it could extend to cohomological gauge theories on more general local spaces with higher number of non-compact directions.

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