Instanton calculus and chiral one-point functions in supersymmetric gauge theories

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Abstract

We compute topological one-point functions of the chiral operator $\operatorname{Tr} \varphi^k$ in the maximally confining phase of $U(N)$ supersymmetric gauge theory. These one-point functions are polynomials in the equivariant parameter $\hbar$ and the parameter of instanton expansion $q = \Lambda^{2N}$ and are of particular interest from gauge/string theory correspondence, since they are related to the Gromov–Witten theory of $\mathbf{P}^1$. Based on a combinatorial identity that gives summation formula over Young diagrams of relevant functions, we find a relation among chiral one-point functions, which recursively determines the $\hbar$ expansion of the generating function of one-point functions. Using a result from the operator formalism of the Gromov–Witten theory, we also present a vacuum expectation value of the loop operator $\operatorname{Tr} e^{it\varphi}$.

1   Introduction

Recently there is a substantial progress in the instanton calculus of four-dimensional gauge theories [1–8]. In particular, Nekrasov proposed a partition function $Z_{\text{Nek}}(\epsilon_i, a_\ell, \Lambda)$ that encodes the information of the instanton counting in four-dimensional gauge theory. In [1], integrations over the instanton moduli space are evaluated by equivariant localization principle, where the equivariant parameters (ϵ_1, ϵ_2) of the toric action on $\mathbb{C}^2 \simeq \mathbb{R}^4$ can be identified as those of the spacetime non-commutativity, or physically the graviphoton background. The fixed points of the toric action are labeled by the partitions or, in other words, the Young diagrams. Consequently the non-perturbative partition function and correlation functions are expressed as summations of the functions on the set of Young diagrams. We can show that a five-dimensional lift (or "trigonometric" lift) of Nekrasov's partition function $Z_{\text{Nek}}^{\text{5D}}$ is nothing but the partition function of topological string (the generating function of Gromov–Witten invariants) $Z_{\text{top str}}^{(K_s)}$ on a local toric Calabi–Yau 3-fold K_S, where S is an appropriate toric surface [9–13]. The correspondence of Nekrasov's partition function and the generating function of the Gromov–Witten invariants of local Calabi–Yau manifold, $Z_{\text{Nek}}^{\text{5D}} \equiv Z_{\text{top str}}^{(K_s)}$, is one example of gauge/string correspondence in topological theory [14], which is expected from the idea of geometric engineering [15].

In this paper, we explore another example of gauge/string correspondence which involves the topological one-point functions. In $U(N)$ supersymmetric gauge theory in four dimensions, there are chiral observables $\text{Tr}\,\varphi^k$, where φ is the (Higgs) scalar field in the adjoint representation. We will present a result on the computation of the vacuum expectation value of $\text{Tr}\,\varphi^k$ in the $U(1)$ gauge theory or in the maximally confining phase of the $U(N)$ theory, where the effective low-energy symmetry is reduced to $U(1) \subset U(N)$. In [4], Losev *et al.* claimed that there is a gauge/string correspondence $\text{Tr}\,\varphi^{2j} \Longleftrightarrow \tau_p(\omega)$, where $\text{Tr}\,\varphi^{2j}$ are generators of the chiral ring and $\tau_p(\omega)$ is the p-th gravitational descendant of the Kähler class ω of $\mathbf{P}^1$. Thus it is expected that the correlation functions of chiral ring elements are related to the Gromov–Witten invariants of $\mathbf{P}^1$ developed by Okounkov and Pandharipande [16,17].

The one-point functions $\langle \text{Tr}\,\varphi^{2j} \rangle$ are polynomials in the parameter of instanton expansion $q := \Lambda^{2N}$ and the equivariant parameter of the toric action $\hbar = \epsilon_1 = -\epsilon_2$. In the gauge/string correspondence, these parameters play complementary roles. For example, Nekrasov's partition function allows the following two kinds of expansions:

$$Z_{\text{Nek}}(\hbar, a_\ell, \Lambda) = \sum_{k=0}^{\infty} \Lambda^{2N \cdot k} Z_k(\hbar, a_\ell) = \exp\left(-\sum_{r=0}^{\infty} \hbar^{2r-2} F_r(a_\ell, \Lambda) \right). \quad (1.1)$$

In gauge theory, we primarily want to sum up the instanton expansion in Λ, which is achieved for example in the Seiberg-Witten theory. On the other hand the expansion in $\hbar$ is identified with the genus expansion in the corresponding (topological) string theory. The genus zero part $F_0(a_\ell, \Lambda)$ gives the prepotential of the Seiberg–Witten theory and higher order terms are expected to represent gravitational corrections [1, 11, 18–20].

In this paper, we first take the viewpoint of gauge theory and the one-point functions $\langle \text{Tr}\, \varphi^{2j} \rangle$ are defined not for each fixed instanton number but by summing up all the instanton numbers. We emphasize this point, since the partition function $Z_{U(1)}$ that appears in the definition of one-point functions contains the contributions from all the instanton numbers. We show that the chiral one-point functions $\langle \text{Tr}\, \varphi^{2j} \rangle$ satisfy the relation

$$\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \text{Tr}\, \varphi^{2j} \rangle = \frac{(2r)!}{(r!)^2} q^r, \tag{1.2}$$

which is one of the main results in the paper. The coefficients c_j^r are defined by $\prod_{j=0}^{r-1}(x^2 - j^2) = \sum_{j=1}^{r} c_j^r x^{2j}$ or a specialization of the elementary symmetric functions $e_n(x)$; $c_j^r = (-1)^{r-j} e_{r-j}(1^2, 2^2, \ldots, (r-1)^2)$. From the above linear relations (1.2) among one-point functions, we can compute the expansion in $\hbar^2$ of the generating function $T(z)$ of one-point functions $\langle \text{Tr}\, \varphi^{2j} \rangle$, order by order. Technically, our proof of the relation (1.2) is based on combinatorial identities, which we obtain by considering the power sums of Jucys–Murphy elements in the class algebras of symmetric groups.

Complementary to the above computation is the computation of the Gromov–Witten invariants in [16, 17] by operator method, which gives all genus results for each fixed instanton number on the gauge theory side. In this sense, the operator formalism naturally provides the generating function of the identities (1.2) for each instanton sector. Summing up all the instanton numbers, we can calculate the vacuum expectation value of the loop operator without difficulty. Our final result is

$$\langle \text{Tr}\, e^{it\varphi} \rangle = I_0(2\sqrt{q}\,\text{sh}(it\hbar)/\hbar), \tag{1.3}$$

with $\text{sh}(z) = e^{z/2} - e^{-(z/2)}$ and $I_n(x)$ being the modified Bessel functions. It is remarkable that the modified Bessel functions appear frequently in the computation of the correlation functions of the loop operator [21–24]. In our case $\langle \text{Tr}\, e^{it\varphi} \rangle = I_0(2i\sqrt{q}\,t)$ when $\hbar \to 0$ and the effect of the equivariant deformation by $\hbar$ is taken care of simply by renormalizing the parameter t as $it \to \text{sh}(it\hbar)/\hbar$.

The paper is organized as follows. In Section 2, we review the basic tools in instanton calculus; the ADHM construction of the instanton moduli space

and localization formula concerning the toric action on the moduli space. In Section 3, we consider the one-point function $\langle \operatorname{Tr} \varphi^{2j} \rangle$ and derive (1.2). The genus expansion of the generating function $T(z)$ is worked out in Section 4. Computation in the operator formalism and comparison with the Gromov–Witten theory are made in Section 5. The generating function of the relation (1.2) is naturally related to the loop operator and we calculate its vacuum expectation value in Section 6. Finally we prove a crucial combinatorial formula in Appendix.

It has been argued that the generating function of the Gromov–Witten invariants of $\mathbf{P}^1$ is a tau-function of Toda lattice hierarchy [16, 17]. In this paper, we have obtained the $\hbar$ expansion of the generating function $T(z)$ of chiral one-point functions. It is interesting to clarify a relation of this genus expansion to integrable hierarchy and matrix models. For a recent paper in this direction, see [25].

2   ADHM construction and localization formula

For describing the moduli space of instantons, there is a strong tool called ADHM construction. The instanton moduli space $\mathcal{M}_{N,k}$ of $U(N)$ gauge theory on $\mathbb{C}^2$ with instanton number k is constructed by introducing matrices[1] $B_1, B_2 \in M_{\mathbb{C}}(k,k)$, $J \in M_{\mathbb{C}}(N,k)$ and $I \in M_{\mathbb{C}}(k,N)$ on $\mathbb{C}$. Combining these matrices and coordinates (z_1, z_2) of $\mathbb{C}^2$, we define an $(N + 2k) \times 2k$ matrix

$$\Delta := \begin{pmatrix} J & I^\dagger \\ B_1 - z_1 & -B_2^\dagger + \overline{z_2} \\ B_2 - z_2 & B_1^\dagger - \overline{z_1} \end{pmatrix}. \tag{2.1}$$

We construct an $(N + 2k) \times N$ matrix U, whose column vectors consist of a basis of the kernel of Δ, i.e., a matrix U that satisfies $\Delta^\dagger U = 0$. A $U(N)$ connection A is defined by $A := U^\dagger(z) d_{\mathbb{C}^2} U(z)$. Then from the self-duality of A and the normalization condition on U, we obtain ADHM equations,

$$\begin{cases} \mu_{\mathbb{C}} := [B_1, B_2] + IJ = 0, \\ \mu_{\mathbb{R}} := [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0. \end{cases} \tag{2.2}$$

Elements in $\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0)$ give k-instantons of the $U(N)$ gauge theory on $\mathbb{C}^2$.

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[1] $M_{\mathbb{C}}(m,n)$ denotes the set of $m \times n$ complex matrices.

Since ADHM equations (2.2) are invariant under the action

$$(B_1, B_2, J, I) \mapsto (T_\phi^{-1} B_1 T_\phi, T_\phi^{-1} B_2 T_\phi, J T_\phi, T_\phi^{-1} I), \qquad (2.3)$$

of $T_\phi = \exp(i\phi) \in U(k)$, we may consider the quotient

$$\mathcal{M}_{N,k}^0 := \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0)/U(k), \qquad (2.4)$$

which is isomorphic to the moduli space of k-instantons of the $U(N)$ gauge theory. However, in general, $\mathcal{M}_{N,k}^0$ is singular and we consider a smooth manifold

$$\mathcal{M}_{N,k}^\zeta := \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\zeta)/U(k),$$

instead, as a resolution of $\mathcal{M}_{N,k}^0$. We note that there are several viewpoints on the manifold $\mathcal{M}_{N,k}^\zeta$. Each viewpoint has its own advantages. Firstly, it can be regarded as the moduli space of instantons on the non-commutative $\mathbb{C}^2$, where ζ corresponds to the non-commutative parameter, $[z_1, \overline{z_1}] = -\zeta/2, [z_2, \overline{z_2}] = -\zeta/2$. One can also regard $\mathcal{M}_{N,k}^\zeta$ as the moduli space of framed torsion-free sheaves (E, Φ) on $\mathbf{P}^2$, where E is a torsion-free sheaf of rank N with $\langle c_2(E), [\mathbf{P}^2] \rangle = k$ and locally free at a neighborhood of l_∞ (the line at infinity) and Φ is an isomorphism $\Phi : E|_{l_\infty} \longrightarrow \mathcal{O}_{l_\infty}^{\oplus N}$, called framing operator. Finally, if we set the rank N of gauge group to 1, $\mathcal{M}_{1,k}^0$ and $\mathcal{M}_{1,k}^\zeta$ are isomorphic to the symmetric product $S^k(\mathbb{C}^2)$ of $\mathbb{C}^2$ and the Hilbert schemes of points $(\mathbb{C}^2)^{[k]}$ on $\mathbb{C}^2$, respectively.

In four-dimensional gauge theory with the gauge group $U(N)$, we can consider the two kinds of toric action:

- $\xi_{\mathbb{C}^2}$ action on $(z_1, z_2) \in \mathbb{C}^2$ defined by $(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$. Physically, this introduces a constant (electro-magnetic) flux on $\mathbb{C}^2$ and makes it *non-commutative*. This is called "Ω background" of Nekrasov. In the following, we often put $\hbar = \epsilon_1 = -\epsilon_2$ (this is the self-duality or "Calabi–Yau" condition).
- The action of the maximal torus $(e^{ia_1}, \ldots, e^{ia_N}) \in U(1)^N$ on $U(N)$. In $\mathcal{N} = 2$ supersymmetric gauge theory, the corresponding equivariant parameters a_ℓ are identified as vacuum expectation values of the Higgs scalar in the vector multiplet.

These toric actions induce the following action of $T = U(1)^2 \times U(1)^N$ on the instanton moduli space $\mathcal{M}_{N,k}^\zeta$, which allows us to employ a powerful tool of localization formula:

$$\xi : (B_1, B_2, I, J) \mapsto (T_{\epsilon_1} B_1, T_{\epsilon_2} B_2, I T_a^{-1}, T_{\epsilon_1} T_{\epsilon_2} T_a J), \qquad (2.5)$$

where $T_{\epsilon_k} := e^{i\epsilon_k} \in U(1)$ and $T_a := \mathrm{diag}\,(e^{ia_1}, \ldots, e^{ia_N}) \in U(1)^N$. Let ξ denote the vector field associated with the toric action. The equivariant

differential operator $d_\xi := d_{\mathbb{C}^2} + d_\mathcal{M} - \iota_\xi$ on $\Omega^\bullet(\mathbb{C}^2 \times \mathcal{M}_{N,k}^\zeta) \otimes \mathbb{C}[\mathfrak{g}]$ satisfies $d_\xi^2 = -\mathcal{L}_\xi$, where $\mathcal{L}_\xi$ is the Lie derivative associated to the action ξ. We define $\mathcal{A} := U^\dagger d_\xi U$ and $\mathcal{F} := d_\xi(U^\dagger d_\xi U)$. Mathematically, $\mathcal{A}$ defines a connection on a rank N vector bundle $\mathcal{E}$ on $\mathbb{C}^2 \times \mathcal{M}_{N,k}^\zeta$, called universal bundle, and $\mathcal{F}$ is the curvature of $\mathcal{A}$. This identification was first provided in topological gauge theory [26, 27]. Since $\iota_\xi U = 0$, we obtain the following decomposition concerning the direct product $\mathbb{C}^2 \times \mathcal{M}_{N,k}^\zeta$:

$$
\begin{aligned}
\mathcal{A} &= U^\dagger d_{\mathbb{C}^2} U + U^\dagger d_\mathcal{M} U =: A + C, \\
\mathcal{F} &= d_{\mathbb{C}^2}(U^\dagger d_{\mathbb{C}^2} U) + d_{\mathbb{C}^2}(U^\dagger d_\mathcal{M} U) + d_\mathcal{M}(U^\dagger d_{\mathbb{C}^2} U) \\
&\quad + (d_\mathcal{M} U^\dagger)(d_\mathcal{M} U) - U^\dagger \mathcal{L}_\xi U \\
&= F_{\mu\nu} dx^\mu dx^\nu + \{\lambda_m dz^m + \psi_{\overline{m}} d\overline{z}^{\overline{m}}\} + \{(d_\mathcal{M} U^\dagger)(d_\mathcal{M} U) - U^\dagger \mathcal{L}_\xi U\} \\
&=: F + \Psi + \varphi.
\end{aligned}
\tag{2.6}
$$

In $\mathcal{N} = 1$ supersymmetric Yang–Mills theory, the components A, F, $\lambda_m dz^m$, $\psi_{\overline{m}} d\overline{z}^{\overline{m}}$ and φ are identified with the gauge connection, the field strength (curvature), gaugino, chiral matter field and scalar field, respectively [8]. We can see easily that it is only the scalar φ that depends on ξ.

The topological partition function and correlation functions are defined by the equivariant integration on $\mathcal{M}_{N,k}^\zeta$ and they are Laurent series in the equivariant parameters ϵ_i and a_ℓ. Namely, the correlator $\langle \mathcal{O} \rangle$ of an operator $\mathcal{O}$ is defined by

$$
\langle \mathcal{O} \rangle := \frac{1}{V\mathcal{Z}} \int_\mathcal{M} \left\{ \int_{\mathbb{C}^2} \mathcal{O} \right\} \exp(-\mathcal{S}_{\mathcal{N}=1}),
\tag{2.7}
$$

where the action for $\mathcal{N} = 1$ supersymmetric gauge theory $\mathcal{S}_{\mathcal{N}=1}$ is defined by that of $\mathcal{N} = 2$ theory $\mathcal{S}_{\mathcal{N}=2}$ perturbed by a superpotential $W(\Phi)$. The correlator $\langle \mathcal{O} \rangle$ is normalized by the volume V of the non-commutative $\mathbb{C}^2$ and the partition function $\mathcal{Z} := \int_\mathcal{M} \exp(-\mathcal{S}_{\mathcal{N}=1})$. The integral is over $\mathcal{M} := \sqcup_k \mathcal{M}_{N,k}^\zeta$, that is, when we compute the correlation function, we take a sum over the instanton number k. Since they are computed by the instanton calculus that employs the equivariant cohomology and the localization formula for the equivariant integral, let us first review the localization formula briefly.

Let M be a smooth manifold of dimension $2l$ acted by a compact Lie group G. The vector field associated to the G-action is denoted by ξ. For the G-fixed point set $\Omega^p(M)^\xi := \{\sigma \in \Omega^p(M) | \mathcal{L}_\xi \sigma = 0\}$ of p-forms on M, an element of $\Omega^\bullet(M)^\xi \otimes \mathbb{C}[\mathfrak{g}]$ is called equivariant differential form associated to the vector field ξ, where $\mathfrak{g}$ is the Lie algebra of G. Then we can define the cohomology $H_\xi^p(M)$, which is called equivariant de Rham cohomology,

for equivariant differential forms using the differential operator $d_\xi := d - \iota_\xi$. An equivariant differential form $\mu \in \Omega^p(M)^\xi \otimes \mathbb{C}[\mathfrak{g}]$ is called equivariantly exact (resp. closed), if μ is written as $d_\xi \nu$ using an equivariant form $\nu \in \Omega^{p-1}(M)^\xi \otimes \mathbb{C}[\mathfrak{g}]$ (resp. $d_\xi \mu = 0$). The integral of equivariant forms on M

$$\int_M : \Omega^\bullet(M)^\xi \otimes \mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{g}] \tag{2.8}$$

defines a homomorphism and is called equivariant integral. For calculating the equivariant integral, we can use a very powerful formula of localization [28, 29];

Theorem (**Localization formula**). *If all fixed points of the G-action on M are isolated, the integral of an equivariantly closed form μ is given by*

$$\int_M \mu = (-2\pi)^l \sum_{s \in M^G} \frac{\mu_0(s)}{\det^{1/2} \mathcal{L}_\xi(s)}, \tag{2.9}$$

where $\mathcal{L}_\xi$ is the homomorphism $\mathcal{L}_{\xi_i}^j := \partial \xi_i / \partial x^j : T_s M \longrightarrow T_s M$, μ_0 is the zero-form part of μ and M^G is the G-fixed points set on M.

When the group G is $U(1)^r$, $\det^{1/2} \mathcal{L}_\xi(s) = \prod_i (k_i(s) \cdot \epsilon)$, where $(k_1(s), \dots, k_l(s)) \in (\mathbb{Z}^r)^l$ are the weights of the representation of $U(1)^r$ at $s \in M^G$ and ϵ is the generator of $\mathfrak{g}$. Instanton part $\mathcal{Z}^{\text{inst}} := \sum_{k=0}^\infty q^k \int_{\mathcal{M}_{k,N}^\zeta} 1$ of Nekrasov's partition function Z_{Nek} for $\mathcal{N} = 2$ super Yang–Mills theory can be obtained by the equivariant integration of "1" on the moduli space $\mathcal{M}$ of instantons on $\mathbb{C}^2$. By the work of Nakajima [30], the fixed points of $U(1)^2 \times U(1)^N$ action on $\mathcal{M}_{N,k}^\zeta$ are in one-to-one correspondence with N-tuples of Young diagrams whose total number of boxes is equal to k. Let $\mathcal{P}_N(k)$ be the set of such N-tuples of Young diagrams. Using localization formula, we have the explicit form of Nekrasov's partition functions as follows:

$$\mathcal{Z}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q) = \sum_{k=0}^\infty \sum_{\underline{Y} \in \mathcal{P}_N(k)} \frac{q^k}{\prod_{\alpha, \beta=1}^N n_{\alpha, \beta}^{\underline{Y}}(\epsilon_1, \epsilon_2, \vec{a})}, \tag{2.10}$$

where

$$n_{\alpha, \beta}^{\underline{Y}}(\epsilon_1, \epsilon_2, \vec{a}) := \prod_{s \in Y_\alpha} (-l_{Y_\beta}(s)\epsilon_1 + (a_{Y_\alpha}(s) + 1)\epsilon_2 + a_\beta - a_\alpha)$$

$$\times \prod_{t \in Y_\beta} ((l_{Y_\alpha}(t) + 1)\epsilon_1 - a_{Y_\beta}(t)\epsilon_2 + a_\beta - a_\alpha), \tag{2.11}$$

$l_Y(s) := \nu_j - i$, $a_Y(s) := \mu_i - j$ for $s = (i, j) \in Y = (\mu_1 \geq \mu_2 \geq \cdots)$ and $Y^\vee = (\nu_1 \geq \nu_2 \geq \cdots)$ is the transpose of the Young diagram.

We can compute the correlation function $\langle \mathcal{O} \rangle$ using the localization formula, if we can find an extension of the operator $\mathcal{O}$ to a ξ-equivariantly closed form. In the following, we set $\epsilon_1 = -\epsilon_2 = \hbar$ for simplicity. The action of $U(1)$ on $\mathbb{C}^2$ is defined by $\xi_{\mathbb{C}^2} := i\hbar(z^1 \partial_{z^1} - z^2 \partial_{z^2} - h.c.)$. We find the following forms are invariant under $\xi_{\mathbb{C}^2}$-action and closed with respect to $d_{\xi_{\mathbb{C}^2}} := d - \iota_{\xi_{\mathbb{C}^2}}$, namely they are $\xi_{\mathbb{C}^2}$-equivariantly closed forms [8],

$$
\begin{aligned}
\alpha_{(0,0)} &:= 1, \\
\alpha_{(2,0)} &:= dz^1 \wedge dz^2 + i\hbar z^1 z^2, \\
\alpha_{(0,2)} &:= d\overline{z}^1 \wedge d\overline{z}^2 - i\hbar \overline{z}^1 \overline{z}^2, \\
\alpha_{(2,2)} &:= \alpha_{(2,0)} \wedge \alpha_{(0,2)}.
\end{aligned}
\tag{2.12}
$$

In terms of the curvature $\mathcal{F}$ on the universal sheaf $\mathcal{E}$ over $\mathbb{C}^2 \times \mathcal{M}_{N,k}^{\zeta}$, we have the ξ-equivariant extension $\mathrm{Tr}\, \varphi^J \mapsto \alpha_{(2,2)} \wedge \mathrm{Tr}\, \mathcal{F}^J$. It is equivariantly closed, since $\mathcal{F} = d_{\xi}(\overline{U} d_{\xi} U)$ is exact. Hence, the equivariant extension of scalar correlator is given by

$$
\langle \mathrm{Tr}\, \varphi^J \rangle = \frac{1}{V\mathcal{Z}} \int_{\mathcal{M}} \int_{\mathbb{C}^2} \alpha_{(2,2)} \wedge \mathrm{Tr}\, \mathcal{F}^J \exp[-\mathcal{S}_{\mathcal{N}=1}].
\tag{2.13}
$$

These correlation functions should be regarded as equivariant integral (2.8), which one can compute by the localization formula. These equivariant integrals are Laurent series in $\hbar$ and a_ℓ from which we can obtain the original correlator (2.7) in the limit $\hbar \to 0$. We note the scalar correlators $\langle \mathrm{Tr}\, \varphi^J \rangle$ are independent of the superpotential $W(\varphi)$ and the same as $\mathcal{N} = 2$ calculation [8].

3 One-point function in maximally confining phase

Chiral operators $\mathcal{O}$ in supersymmetric field theories are, by definition [31], annihilated by the fermionic charges $\overline{Q}_{\dot{\alpha}}$ of one chirality; $[\overline{Q}_{\dot{\alpha}}, \mathcal{O}]_{\pm} = 0$, considered modulo $\overline{Q}_{\dot{\alpha}}$-exact operators; $\mathcal{O} \simeq \mathcal{O} + [\overline{Q}_{\dot{\alpha}}, \Lambda]_{\pm}$. From the supersymmetry algebra in four dimensions, $[Q_\alpha, \overline{Q}_{\dot{\alpha}}]_+ = \sigma^\mu_{\alpha\dot{\alpha}} P_\mu$, we can see that the correlation functions of chiral operators are "topological" in the sense that they are independent of the positions of operators. Especially, topological one-point functions characterize the vacuum structure (phase) of the theory.

As we have seen in Section 2, the computation of one-point function $\langle \mathrm{Tr}\, \varphi^{2n} \rangle$ involves the Chern class $\mathrm{Tr}\, \mathcal{F}^{2n}$ of a universal sheaf $\mathcal{E}$ on $\mathbb{C}^2 \times \mathcal{M}$. Over the instanton moduli space $\mathcal{M}_{N,k}^{\zeta}$ we have two vector bundles W and V of rank N and k, which naturally arise in the ADHM construction.

The ADHM data are identified as $B_1, B_2 \in \mathrm{Hom}(V, V)$ and $J, I^\dagger \in \mathrm{Hom}\,(W, V)$.[2] Roughly speaking, the vector bundle W comes from a local trivialization of the instanton at infinity,[3] while V is the bundle of Dirac zero modes. The fiber of V is the space of (normalizable) solutions to the Dirac equation in the instanton background. The Riemann–Roch theorem tells us that the number of Dirac zero modes is just the instanton number k. From vector bundles E_1 on $\mathbb{C}^2$ and E_2 on $\mathcal{M}_{N,k}^\zeta$, we can construct an (external) tensor product bundle $E_1 \boxtimes E_2 := p_1^* E_1 \otimes p_2^* E_2$ on $\mathbb{C}^2 \times \mathcal{M}_{N,k}^\zeta$, where p_i denotes the projection to the i-th component. Then as an element of the equivariant K-cohomology group, the universal sheaf is isomorphic to the virtual vector bundle [4, 32]:

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{C}^2} \boxtimes W \oplus (S^- - S^+) \boxtimes V, \tag{3.1}$$

where $S^\pm$ are positive and negative spinor bundles on $\mathbb{C}^2$. Their characters are

$$\mathrm{Ch}(S^+)(t) = 1 + e^{it(\epsilon_1 + \epsilon_2)}, \quad \mathrm{Ch}(S^-)(t) = e^{it\epsilon_1} + e^{it\epsilon_2}. \tag{3.2}$$

According to [4, 32], at a fixed point of the toric action labeled by N-tuples of Young diagrams Y_α, the Chern character of $\mathcal{E}$ is given by

$$\mathrm{Ch}(\mathcal{E})_{\underline{Y}}(t) = \sum_{\alpha=1}^N e^{it a_\alpha} - (1 - e^{it\epsilon_1})(1 - e^{it\epsilon_2})$$
$$\times \sum_{\alpha=1}^N \sum_{(k,\ell) \in Y_\alpha} e^{it a_\alpha + it\epsilon_1(k-1) + it\epsilon_2(\ell-1)}, \tag{3.3}$$

and we have

$$\mathrm{Ch}(\mathcal{E})(t) = \sum_{\underline{Y} \in \mathcal{P}_N(k)} \mathrm{Ch}(\mathcal{E})_{\underline{Y}}(t). \tag{3.4}$$

The n-th Chern class $c_n(\mathcal{E})$ is defined by the expansion

$$\mathrm{Ch}(\mathcal{E})_{\underline{Y}}(t) = \sum_{n=0}^\infty \frac{(it)^n}{n!} c_n(\mathcal{E})_{\underline{Y}}. \tag{3.5}$$

Since we identify $\mathcal{F}$ as a curvature on the universal bundle $\mathcal{E}$, we have $\mathrm{Tr}\,\mathcal{F}^n = c_n(\mathcal{E})$.

In the following, we consider $U(1)$ theory ($N = 1$) and we put $\varphi_{cl} = a = 0$ for simplicity. The fixed points are labeled by a single Young diagram Y

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[2]In the description of ADHM construction in terms of D-branes in type IIB theory, V and W are the Chan–Paton bundles for D(-1)-branes and D3-branes, respectively.

[3]The moduli space $\mathcal{M}_{N,k}^\zeta$ is defined by the quotient by the gauge transformations that fix the "framing" at infinity.

and the contribution to the character is

$$\mathrm{Ch}(\mathcal{E})_Y = 1 - (1 - e^{it\epsilon_1})(1 - e^{it\epsilon_2}) \sum_{(m,\ell)\in Y} e^{it\epsilon_1(m-1)+it\epsilon_2(\ell-1)}. \qquad (3.6)$$

It is known that the moduli space of instantons $\mathcal{M}_{1,k}$ for $U(1)$ case is nothing but the Hilbert scheme of k-points on $\mathbb{C}^2$, $(\mathbb{C}^2)^{[k]}$ and that there is a natural vector bundle $\mathcal{V}$ on $(\mathbb{C}^2)^{[k]}$ of rank k, called tautological vector bundle. One can show that the (equivariant) character of $\mathcal{V}$ is the same as the vector bundle V [30, 33]:

$$\mathrm{Ch}(\mathcal{V}) = \sum_{|Y|=k} \sum_{(m,\ell)\in Y} e^{it\epsilon_1(m-1)+it\epsilon_2(\ell-1)}. \qquad (3.7)$$

Putting $\epsilon_1 = -\epsilon_2 = \hbar$ and comparing (3.5) and (3.6), we find $\mathrm{Tr}\,\varphi_Y^{2n} = c_{2n}(\mathcal{E})_Y$, $(n > 0)$, is given by:

$$\begin{aligned}
\mathrm{Tr}\,\varphi_Y^{2n} &= \hbar^{2n} \sum_{(k,\ell)\in Y} [(k-\ell+1)^{2n} + (k-\ell-1)^{2n} - 2(k-\ell)^{2n}] \\
&= \hbar^{2n} \sum_{m=0}^{n-1} 2\binom{2n}{2m} \sum_{\square \in Y} c(\square)^{2m}, \qquad (3.8)
\end{aligned}$$

where $c(\square) := (\ell - k)$ is the content at $\square = (k,\ell)$. On the other hand, computing geometric series, we have

$$\begin{aligned}
\sum_{(k,\ell)\in Y} &(1 - e^{i\epsilon_1})(1 - e^{i\epsilon_2})e^{i(k-1)\epsilon_1 + i(\ell-1)\epsilon_2} \\
&= \sum_{\ell=1}^{d} \sum_{k=1}^{\mu_\ell} (1 - e^{i\epsilon_1})(1 - e^{i\epsilon_2})e^{i(k-1)\epsilon_1 + i(\ell-1)\epsilon_2} \qquad (3.9) \\
&= \sum_{\ell=1}^{d} (e^{i(\epsilon_1\mu_\ell + \epsilon_2\ell)} - e^{i(\epsilon_1\mu_\ell + \epsilon_2(\ell-1))} - e^{i\epsilon_2\ell} + e^{\epsilon_2(\ell-1)}),
\end{aligned}$$

where d is the number of rows of Y and μ_ℓ is the number of boxes in the ℓ-th row. Hence, we obtain another expression of (3.8):

$$\mathrm{Tr}\,\varphi_Y^{2n} = \hbar^{2n} \sum_{\ell=1}^{d} [(\mu_\ell - (\ell-1))^{2n} - (\mu_\ell - \ell)^{2n} - (\ell-1)^{2n} + \ell^{2n}], \qquad (3.10)$$

which we often find in the literature.

The partition function of $U(1)$ gauge theory is

$$Z_{U(1)} = \sum_{k=0}^{\infty} \sum_{|Y|=k} \frac{1}{\prod_{\square \in Y} (\hbar h(\square))^2} q^k, \tag{3.11}$$

where $h(\square)$ is the hook length at $\square$ and $q = \Lambda^2$ is the parameter of instanton expansion. The weight $\mu(Y)^2 = \prod_{\square \in Y} (h(\square))^{-2}$ defining $Z_{U(1)}$ is called the Plancherel measure on the space of (random) partitions. The Plancherel measure is regarded as a discretization of the Vandermonde measure on random matrix. It is a classical result in representation theory that

$$\prod_{\square \in Y} \frac{1}{h(\square)} = \frac{\dim S^Y}{k!}, \tag{3.12}$$

where S^Y is the irreducible representation of the symmetric group labeled by a Young diagram Y. By the Plancherel formula $\sum_{|Y|=k} (\dim S^Y)^2 = k!$, we obtain

$$\sum_{|Y|=k} \prod_{\square \in Y} h(\square)^{-2} = \frac{1}{k!}. \tag{3.13}$$

Hence, we find that the summation over the instanton number k in (3.11) is organized into a simple form [4]:

$$Z_{U(1)} = \exp\left(\frac{q}{\hbar^2}\right). \tag{3.14}$$

The correlation functions of our interest are

$$\langle \mathrm{Tr}\, \varphi^{2n} \rangle = \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \sum_{|Y|=k} \frac{\mathrm{Tr}\, \varphi_Y^{2n}}{\hbar^{2k} \prod_{\square \in Y} h(\square)^2} q^k. \tag{3.15}$$

Substituting formula (3.8), we have

$$\langle \mathrm{Tr}\, \varphi^{2n} \rangle \exp\left(\frac{q}{\hbar^2}\right) = 2 \sum_{m=0}^{n-1} \binom{2n}{2m} \sum_{k=1}^{\infty} S_m(k) \hbar^{2(n-k)} q^k, \tag{3.16}$$

where we have introduced

$$S_n(k) := \sum_{|Y|=k} \frac{\sum_{\square \in Y} c(\square)^{2n}}{\prod_{\square \in Y} h(\square)^2}. \tag{3.17}$$

Thus the computation of $\langle \mathrm{Tr}\, \varphi^{2n} \rangle$ is equivalent to giving summation formula for $S_n(k)$ over Young diagrams. For example, a "trivial" formula $S_0(k) =$

$1/(k-1)!$ implies $\langle \operatorname{Tr} \varphi^2 \rangle = 2q$. Looking at the instanton expansion of lower degree explicitly, we find

$$(6q^2 + 2\hbar^2 q)e^{\frac{q}{\hbar^2}} = \sum_{k=1}^{\infty}(2S_0(k) + 12S_2(k))\hbar^{4-2k}q^k,$$

$$(20q^3 + 30\hbar^2 q^2 + 2\hbar^4 q)e^{\frac{q}{\hbar^2}} = \sum_{k=1}^{\infty}(2S_0(k) + 30S_2(k)$$
$$+ 30S_4(k))\hbar^{6-2k}q^k, \qquad (3.18)$$

$$(70q^4 + 280\hbar^2 q^3 + 126\hbar^4 q^2 + 2\hbar^6 q)e^{\frac{q}{\hbar^2}} = \sum_{k=1}^{\infty}(2S_0(k) + 56S_2(k)$$
$$+ 140S_4(k) + 56S_6(k))\hbar^{8-2k}q^k.$$

In Appendix, we prove the following formula:

$$\sum_{j=1}^{n} c_j^n S_j(k) = \frac{(2n)!}{((n+1)!)^2} \frac{1}{(k-n-1)!}, \qquad (3.19)$$

where c_j^n are defined by[4]

$$\mathcal{P}_{2n}(x) := x^{\overline{n}} \cdot x^{\underline{n}} = \prod_{j=0}^{n-1}(x^2 - j^2) = \sum_{j=1}^{n} c_j^n x^{2j}. \qquad (3.20)$$

We note that in terms of a specialization of the elementary symmetric functions $e_r(x)$, the coefficient c_j^n is given by

$$c_j^n = (-1)^{n-j}e_{n-j}(1^2, 2^2, \cdots, (n-1)^2). \qquad (3.21)$$

The formula implies, for example,

$$S_2(k) = \frac{1}{2}\frac{1}{(k-2)!}, \quad S_4(k) = \frac{1}{2}\frac{1}{(k-2)!} + \frac{2}{3}\frac{1}{(k-3)!},$$
$$S_6(k) = \frac{1}{2}\frac{1}{(k-2)!} + \frac{10}{3}\frac{1}{(k-3)!} + \frac{5}{4}\frac{1}{(k-4)!}, \qquad (3.22)$$

and we find an agreement with (3.18).

---

[4]The functions $x^{\overline{n}}$ and $x^{\underline{n}}$ are natural power functions in the calculus of difference.

In general, based on the combinatorial formula (3.19), we can derive the following relation among topological one-point functions valid in the maximally confining phase:

$$\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \operatorname{Tr} \varphi^{2j} \rangle = \frac{(2r)!}{(r!)^2} q^r. \tag{3.23}$$

In other words, the linear combination on the left hand side with extra terms of the coefficients c_j^r is a "good" combination without quantum corrections. Before embarking a proof of (3.23), let us take a look at some examples first. From (3.18) it is easy to find that[5]

$$\langle \operatorname{Tr} \varphi^2 \rangle = 2q,$$
$$\langle \operatorname{Tr} (\varphi^4 - \hbar^2 \varphi^2) \rangle = 6q^2,$$
$$\langle \operatorname{Tr} (\varphi^6 - 5\hbar^2 \varphi^4 + 4\hbar^4 \varphi^2) \rangle = 20q^3, \tag{3.24}$$
$$\langle \operatorname{Tr} (\varphi^8 - 14\hbar^2 \varphi^6 + 49\hbar^4 \varphi^4 - 36\hbar^6 \varphi^2) \rangle = 70q^4,$$

and we recognize the coefficients c_j^r in the linear combinations of $\langle \operatorname{Tr} \varphi^{2n} \rangle$.

For the proof of (3.23), we first plug formula (3.8) into the definition (3.15) of $\langle \operatorname{Tr} \varphi^{2j} \rangle$ to obtain

$$\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \operatorname{Tr} \varphi^{2j} \rangle Z_{U(1)}$$

$$= \hbar^{2r} \sum_{k=1}^{\infty} \sum_{|Y|=k} \left(\frac{q}{\hbar^2} \right)^k \sum_{j=1}^{r} c_j^r \frac{\sum_{\square \in Y} (c(\square)+1)^{2j} - 2(c(\square))^{2j} + (c(\square)-1)^{2j}}{\prod_{\square \in Y} h(\square)^2}$$

$$= \hbar^{2r} \sum_{k=1}^{\infty} \sum_{|Y|=k} \left(\frac{q}{\hbar^2} \right)^k \frac{\sum_{\square \in Y} [\mathcal{P}_{2r}(c(\square)+1) - 2\mathcal{P}_{2r}(c(\square)) + \mathcal{P}_{2r}(c(\square)-1)]}{\prod_{\square \in Y} h(\square)^2}$$

$$= 2r(2r-1)\hbar^{2r} \sum_{k=1}^{\infty} \sum_{|Y|=k} \left(\frac{q}{\hbar^2} \right)^k \frac{\sum_{\square \in Y} \prod_{j=0}^{r-2} (c(\square)^2 - j^2)}{\prod_{\square \in Y} h(\square)^2},$$
$$\tag{3.25}$$

where in the last line we have used the following relation satisfied by $\mathcal{P}_{2n}(x)$:[6]

$$\Delta^2 \mathcal{P}_{2n}(x) := \mathcal{P}_{2n}(x+1) - 2\mathcal{P}_{2n}(x) + \mathcal{P}_{2n}(x-1) = 2n(2n-1)\mathcal{P}_{2n-2}(x). \tag{3.26}$$

---

[5] These one-point functions in the limit $\hbar \to 0$ were computed in [8].

[6] This formula is a discrete version of $\frac{d^2}{dx^2} x^{2n} = 2n(2n-1)x^{2n-2}$.

Finally as is shown in Appendix, formula (3.19) is equivalent to

$$\sum_{|Y|=k} \frac{\sum_{\square \in Y} \prod_{j=0}^{r-2}(c(\square)^2 - j^2)}{\prod_{\square \in Y} h(\square)^2} = \frac{(2(r-1))!}{(r!)^2} \frac{1}{(k-r)!}, \qquad (3.27)$$

which allows us to factorize the partition function $Z_{U(1)}$ as follows:

$$\sum_{j=1}^{r} c_j^r \hbar^{2(r-j)} \langle \mathrm{Tr}\, \varphi^{2j} \rangle Z_{U(1)} = \frac{(2r)!}{(r!)^2} q^r \sum_{k=1}^{\infty} \frac{1}{(k-r)!} \left(\frac{q}{\hbar^2}\right)^{k-r}$$

$$= \frac{(2r)!}{(r!)^2} q^r \exp\left(\frac{q}{\hbar^2}\right). \qquad (3.28)$$

Dividing both sides by the partition function $Z_{U(1)}$, we obtain (3.23).

4   Genus expansion

From the relation (3.23) derived in Section 3, we can compute the expansion of the generating function

$$T(z) := \left\langle \mathrm{Tr}\, \frac{1}{z-\varphi} \right\rangle = \sum_{n=0}^{\infty} z^{-n-1} \langle \mathrm{Tr}\, \varphi^n \rangle, \qquad (4.1)$$

in $\hbar^2$ iteratively. The expansion should be compared with the genus expansion of topological strings and/or matrix models. Recall that the coefficients c_j^r are defined by

$$\prod_{j=0}^{r-1}(x^2 - j^2) = \sum_{j=1}^{r} c_j^r x^{2j}. \qquad (4.2)$$

We find

$$c_r^r = 1, \quad c_{r-1}^r = -\sum_{j=0}^{r-1} j^2 = -\frac{1}{6}r(r-1)(2r-1). \qquad (4.3)$$

Substituting these to the relation (3.23), we obtain

$$\langle \mathrm{Tr}\, \varphi^{2r} \rangle = \frac{(2r)!}{(r!)^2} q^r + \frac{\hbar^2}{6} r(r-1)(2r-1) \frac{(2r-2)!}{((r-1)!)^2} q^{r-1} + O(\hbar^4)$$

$$= \frac{(2r)!}{(r!)^2} q^r + \frac{\hbar^2}{12} \frac{(2r)!}{(r-1)!(r-2)!} q^{r-1} + O(\hbar^4). \qquad (4.4)$$

Hence

$$T(z) = \sum_{n=0}^{\infty} z^{-2n-1} \frac{(2n)!}{(n!)^2} q^n + \sum_{n=2}^{\infty} z^{-2n-1} \frac{\hbar^2}{12} \frac{(2n)!}{(n-1)!(n-2)!} q^{n-1} + O(\hbar^4). \qquad (4.5)$$

We note that the Taylor expansion

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n, \quad |x| < \frac{1}{4}, \tag{4.6}$$

implies

$$T(z) = \frac{1}{z\sqrt{1-(4q/z^2)}} = \frac{1}{\sqrt{z^2-4q}}, \quad \hbar \to 0. \tag{4.7}$$

Combining the expansion (4.6) and its derivatives, we find

$$\frac{24x(x+1)}{(1-4x)^{7/2}} = \sum_{n=2}^{\infty} \frac{(2n)!}{(n-1)!(n-2)!} x^{n-1}, \tag{4.8}$$

which implies the generating function $T(z)$ up to genus one:

$$T(z) = \frac{1}{\sqrt{z^2-4q}} + \hbar^2 \frac{2q(q+z^2)}{(z^2-4q)^{7/2}} + O(\hbar^4)$$

$$= \frac{1}{\sqrt{z^2-4q}} \left(1 + \hbar^2 \frac{2q(q+z^2)}{(z^2-4q)^3} + O(\hbar^4)\right). \tag{4.9}$$

Similarly, the relation (3.23) implies the genus two part of $\langle \mathrm{Tr}\,\varphi^{2r} \rangle$ is

$$\hbar^4 (c_{r-1}^r c_{r-2}^{r-1} - c_{r-2}^r) \frac{(2r-4)!}{((r-2)!)^2} q^{r-2}. \tag{4.10}$$

From the definition of c_r^j, we find

$$c_{r-1}^r c_{r-2}^{r-1} = \frac{1}{36} r(r-1)(2r-1)(r-2)(r-1)(2r-3), \tag{4.11}$$

and

$$c_{r-2}^r = \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} i^2 j^2 = c_{r-1}^r c_{r-2}^{r-1} - \frac{1}{6} \sum_{i=1}^{r-2} i^3 (i+1)(2i+1)$$

$$= c_{r-1}^r c_{r-2}^{r-1} - \frac{1}{360} r(5r-11)(r-1)(r-2)(2r-1)(2r-3). \tag{4.12}$$

Hence, the genus two part of the generating function $T(z)$ is given by

$$\frac{\hbar^4}{1440} z^{-5} \sum_{n=3}^{\infty} (5n-11) \frac{(2n)!}{(n-2)!(n-3)!} \left(\frac{q}{z^2}\right)^{n-2}. \tag{4.13}$$

From the Taylor expansion (4.8) which we have used for genus one part, we further obtain

$$\frac{2880x(27x^3 + 118x^2 + 37x + 1)}{(1-4x)^{13/2}} = \sum_{n=3}^{\infty} (5n-11) \frac{(2n)!}{(n-2)!(n-3)!} x^{n-2}. \tag{4.14}$$

In summary, the generating function up to genus two is

$$T(z) = \frac{1}{\sqrt{z^2 - 4q}} \left(1 + \hbar^2 \frac{2q(q + z^2)}{(z^2 - 4q)^3} \right.$$
$$\left. + \hbar^4 \frac{2q(27q^3 + 118q^2 z^2 + 37q z^4 + z^6)}{(z^2 - 4q)^6} + O(\hbar^6) \right). \qquad (4.15)$$

5   Computations in operator formalism

Due to the correspondence of Young diagrams (or Maya diagrams) and fermion Fock states with neutral charge, operator formalism is very powerful for computations of summations over functions on the set of Young diagrams. Let us introduce a pair of charged (NS) free fermions

$$\psi(z) = \sum_{r \in \mathbb{Z}+(1/2)} \psi_r z^{-r-(1/2)}, \quad \psi^*(z) = \sum_{s \in \mathbb{Z}+(1/2)} \psi_s^* z^{-s-(1/2)}, \qquad (5.1)$$

with the anti-commutation relation

$$\{\psi_r, \psi_s^*\} = \delta_{r+s,0}, \quad r, s \in \mathbb{Z} + \frac{1}{2}. \qquad (5.2)$$

The Fock vacuum $|0\rangle$ is defined by

$$\psi_r |0\rangle = \psi_s^* |0\rangle = 0, \quad r, s > 0. \qquad (5.3)$$

Using the Young/Maya diagram correspondence, for each partition λ, we have a state $|\lambda\rangle$ in the charge zero sector of the fermion Fock space, which is given by

$$|\lambda\rangle = \prod_{i=1}^{\infty} \psi_{i-\lambda_i-(1/2)} \|0\rangle\rangle, \qquad (5.4)$$

with

$$\psi_s^* \|0\rangle\rangle = 0, \quad \forall s. \qquad (5.5)$$

Recall the standard bosonization rule

$$J(z) =: \psi(z)\psi^*(z) := \sum_{n \in \mathbb{Z}} J_n z^{-n-1}, \quad J_n = \sum_{r \in \mathbb{Z}+\frac{1}{2}} : \psi_r \psi_{n-r}^* :,$$

$$J(z) = i\partial\phi(z), \quad \psi(z) =: e^{i\phi(z)} :, \quad \psi^*(z) =: e^{-i\phi(z)} :, \qquad (5.6)$$

where : : means the normal ordering. Now a crucial point is the following formula:

$$\exp\left(\frac{J_{-1}}{\hbar}\right) |0\rangle = \sum_{k=0}^{\infty} \frac{1}{\hbar^k} \sum_{|\lambda|=k} \frac{1}{\prod_{\square \in \lambda} h(\square)} |\lambda\rangle, \qquad (5.7)$$

which is equation (5.29) of [5]. In the language of symmetric functions, the corresponding formula is given in [34].

It is instructive to compute $J_{-1}|0\rangle$, $J_{-1}^2|0\rangle$, $J_{-1}^3|0\rangle$, $J_{-1}^4|0\rangle$, ..., iteratively. One can recognize that the action of

$$J_{-1} = \psi_{-1/2}\psi_{-1/2}^* + \psi_{-3/2}\psi_{1/2}^* + \psi_{-5/2}\psi_{3/2}^* + \psi_{-7/2}\psi_{5/2}^* + \cdots$$
$$- \psi_{-3/2}^*\psi_{1/2} - \psi_{-5/2}^*\psi_{3/2} - \psi_{-7/2}^*\psi_{5/2} + \cdots, \tag{5.8}$$

on the fermion Fock states is to move "black ball" to the right by one unit whenever possible, if the vacuum is identified as the Maya diagram whose negative positions are completely filled with "black balls". The combinatorics of this procedure gives

$$J_{-1}^k|0\rangle = \sum_{|\lambda|=k} \frac{k!}{\prod_{\square\in\lambda} h(\square)}|\lambda\rangle = \sum_{|\lambda|=k}(\dim S^\lambda)|\lambda\rangle. \tag{5.9}$$

It is easy to compute

$$\langle 0|e^{J_1/\hbar}e^{J_{-1}/\hbar}|0\rangle = \langle 0|e^{[J_1,J_{-1}]/\hbar^2}|0\rangle = e^{1/\hbar^2} = \sum_{k=0}^{\infty}\frac{1}{\hbar^{2k}k!}. \tag{5.10}$$

On the other hand,

$$\langle 0|e^{J_1/\hbar}e^{J_{-1}/\hbar}|0\rangle = \sum_{\mu,\lambda}\langle\mu|\frac{1}{\hbar^{|\mu|}\prod_{\square\in\mu}h(\square)}\cdot\frac{1}{\hbar^{|\lambda|}\prod_{\square\in\lambda}h(\square)}|\lambda\rangle$$
$$= \sum_{\lambda}\frac{1}{\hbar^{2|\lambda|}\prod_{\square\in\lambda}h(\square)^2}, \tag{5.11}$$

where we have used $\langle\mu|\lambda\rangle = \delta_{\mu,\lambda}$. Comparing the coefficients of $\hbar^{-2k}$ of both sides, we recover the identity (3.13). Recall that what we want to compute is $S_n(k)$ defined by (3.17). Let us introduce the generating function of $S_n(k)$:

$$\mathrm{Ch}[k](z) := \sum_{|\lambda|=k}\frac{\sum_{\square\in\lambda}\exp(zc(\square))}{\prod_{\square\in\lambda}h(\square)^2} = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}S_n(k), \tag{5.12}$$

where we have used the fact that $c(\square)$ is odd under the transpose of the Young diagram. This generating function gives the Chern character of the tautological vector bundle over $(\mathbb{C}^2)^{[k]}$ considered in [33]. The sum in the numerator is

$$\sum_{\square\in\lambda}\exp(zc(\square)) = \sum_{j=1}^{d(\lambda)}\sum_{i=1}^{\lambda_j}e^{z(i-j)} = \sum_{j=1}^{d(\lambda)}\frac{e^{z(\lambda_j-j+(1/2))} - e^{z(-j+(1/2))}}{e^{z/2} - e^{-z/2}}. \tag{5.13}$$

Following Okounkov and Pandharipande [16,17], we consider the operator

$$\mathcal{E}_n(z) := \sum_{r\in\mathbb{Z}+(1/2)}e^{z(r-(n/2))}E_{n-r,r}, \tag{5.14}$$

where $E_{r,s} :=: \psi_r \psi_s^* :$ is the standard basis of $\mathfrak{gl}(\infty)$ acting on the fermion Fock space.[7] We can see that $\mathcal{E}_n(z)$ satisfies the commutation relation

$$[\mathcal{E}_n(z), \mathcal{E}_m(w)] = \mathrm{sh}(nw - mz)\mathcal{E}_{n+m}(z + w) + \delta_{n+m,0}\frac{\mathrm{sh}\, n(z + w)}{\mathrm{sh}\,(z + w)}, \quad (5.15)$$

where $\mathrm{sh}(z) := e^{z/2} - e^{-z/2}$. We have $\mathcal{E}_n(0) = J_n$, with J_n being the modes of the standard $U(1)$ current of fermions. We also find an important relation

$$\begin{aligned}
\mathcal{E}_0(z)|\lambda\rangle &= \sum_{i=1}^{\infty}(e^{z(\lambda_i - i + (1/2))} - e^{z((1/2)-i)})|\lambda\rangle \\
&= \sum_{i=1}^{d(\lambda)}(e^{z(\lambda_i - i + (1/2))} - e^{z((1/2)-i)})|\lambda\rangle \\
&= \mathrm{sh}(z)\sum_{\square\in\lambda}\exp(zc(\square))|\lambda\rangle. \quad (5.16)
\end{aligned}$$

The second term comes from $\mathcal{E}_0(z)\|0\rangle\rangle = -(\mathrm{sh}(z))^{-1}\|0\rangle\rangle$, which can be calculated directly from the definition of $\|0\rangle\rangle$ or the consistency $\mathcal{E}_0(z)|0\rangle = 0$. By formula (5.9), the generating function of $S_n(k)$ is expressed in operator formalism as follows:

$$(k!)^2\mathrm{sh}(z)\mathrm{Ch}[k](z) = \langle 0|J_1^k\mathcal{E}_0(z)J_{-1}^k|0\rangle. \quad (5.17)$$

The right hand side can be computed by the commutation relation (5.15), which implies

$$J_1^k\mathcal{E}_0(z) = \sum_{\ell=0}^{k}\binom{k}{\ell}\mathrm{sh}^\ell(z)\mathcal{E}_\ell(z)J_1^{k-\ell}. \quad (5.18)$$

We also use

$$J_1^n J_{-1}^k|0\rangle = \frac{k!}{(k-n)!}J_{-1}^{k-n}|0\rangle \quad (n \le k), \quad (5.19)$$

which is derived from $e^{zJ_1}e^{wJ_{-1}}|0\rangle = e^{[zJ_1, wJ_{-1}]}e^{wJ_{-1}}e^{zJ_1}|0\rangle = e^{zw}e^{wJ_{-1}}|0\rangle$. By these formulae, we obtain

$$\langle 0|J_1^k\mathcal{E}_0(z)J_{-1}^k|0\rangle = \sum_{\ell=0}^{k}\binom{k}{\ell}\mathrm{sh}^\ell(z)\langle 0|\mathcal{E}_\ell(z)\frac{k!}{\ell!}J_{-1}^\ell|0\rangle = (k!)^2\sum_{\ell=1}^{k}\frac{\mathrm{sh}(z)^{2\ell-1}}{(\ell!)^2(k-\ell)!}. \quad (5.20)$$

---

[7]Originally the definition in [16, 17] has the constant term $\frac{\delta_{n,0}}{\mathrm{sh}(z)}$, which eliminates the central extension term in the commutation relation (5.15). Note also that our convention of the anti-commutation relation (5.2) is different from the original one in [16, 17], where the right hand side is $\delta_{r,s}$.

The contribution from $\ell = 0$ is simply zero because $\langle 0|\mathcal{E}_0(z)|0\rangle = 0$. Thus we find[8]

$$\text{Ch}[k](z) = \sum_{\ell=1}^{k} \frac{\text{sh}(z)^{2\ell-2}}{(\ell!)^2(k-\ell)!}. \tag{5.21}$$

We note that the constant term of (5.21) gives

$$\text{Ch}[k](0) = \frac{1}{(k-1)!}, \tag{5.22}$$

which is consistent with

$$S_0(k) = \frac{k}{k!}, \tag{5.23}$$

derived from (3.13). We have computed the Taylor expansion of $\text{Ch}[k](z)$ for each fixed k and found exact agreements with the results of formula (3.19) proved in Appendix. Note that formula (3.19) rather gives $S_n(k)$ as a function of k for each fixed n.

6 Loop operator

In this section,[9] let us consider the vacuum expectation value of the loop operator

$$\langle \text{Tr}\, e^{it\varphi} \rangle = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \langle \text{Tr}\, \varphi^n \rangle. \tag{6.1}$$

Note that the loop operator is related to the resolvent operator simply by the Laplace transformation:

$$T(z) = \left\langle \text{Tr}\, \frac{1}{z-\varphi} \right\rangle = \int_0^{\infty} dl\, e^{-lz} \langle \text{Tr}\, e^{l\varphi} \rangle. \tag{6.2}$$

Plugging (3.15) with (3.8) into (6.1), we find the loop operator $\langle \text{Tr}\, e^{it\varphi} \rangle$ can be expressed as ($z = it\hbar$)

$$\langle \text{Tr}\, e^{it\varphi} \rangle - 1 = \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \left[\frac{q}{\hbar^2}\right]^k$$
$$\times \sum_{|Y|=k} \frac{\sum_{\square \in Y}(e^{z(c(\square)+1)} + e^{z(c(\square)-1)} - 2e^{z(c(\square))})}{\prod_{\square \in Y}(h(\square))^2}, \tag{6.3}$$

---

[8]From the Lascoux–Thibon formula used in Appendix, we see that $\text{Ch}[k](z) = \sum_{m=1}^{k} \phi_{(1^m)}(z) \frac{1}{(k-m)!}$.

[9]We thank Amihay Hanany and Hiroyuki Ochiai for sharing the ideas that led us to make the following computations.

which can further be put into

$$\langle \mathrm{Tr}\, e^{it\varphi} \rangle - 1 = \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \left[\frac{q}{\hbar^2}\right]^k \mathrm{sh}^2(z)\mathrm{Ch}[k](z), \qquad (6.4)$$

using the function $\mathrm{Ch}[k](z)$ defined in (5.12). Since we have already evaluated $\mathrm{Ch}[k](z)$, let us plug the final expression of $\mathrm{Ch}[k](z)$ in (5.21) into (6.4):

$$\langle \mathrm{Tr}\, e^{it\varphi} \rangle - 1 = \frac{1}{Z_{U(1)}} \sum_{k=1}^{\infty} \sum_{\ell=1}^{k} \left[\frac{q}{\hbar^2}\right]^k \frac{\mathrm{sh}(z)^{2\ell}}{(\ell!)^2 (k-\ell)!}$$

$$= \frac{1}{Z_{U(1)}} \sum_{\ell=1}^{\infty} \sum_{k=\ell}^{\infty} \left[\frac{q}{\hbar^2}\right]^k \frac{\mathrm{sh}(z)^{2\ell}}{(\ell!)^2 (k-\ell)!}, \qquad (6.5)$$

where in the last equation we have exchanged the k summation and the ℓ summation. If we perform the k summation first,

$$\sum_{k=\ell}^{\infty} \frac{1}{(k-\ell)!} \left[\frac{q}{\hbar^2}\right]^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{q}{\hbar^2}\right]^{k+\ell} = Z_{U(1)} \left[\frac{q}{\hbar^2}\right]^{\ell}, \qquad (6.6)$$

we find finally the loop operator is given as

$$\langle \mathrm{Tr}\, e^{it\varphi} \rangle = 1 + \sum_{\ell=1}^{\infty} \frac{1}{(\ell!)^2} \left[\frac{q}{\hbar^2}\mathrm{sh}^2(z)\right]^{\ell} = I_0(2\sqrt{q}\,\mathrm{sh}(it\hbar)/\hbar), \qquad (6.7)$$

with $I_n(x)$ being the modified Bessel functions. It is interesting that we can perform the instanton sum of the loop operator in a closed form and obtain the exact result (6.7) in the parameter $\hbar$.

Finally we can also obtain an exact result on $T(z)$ from (6.7). The Laplace transformation (6.2) implies

$$T(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{q}{\hbar^2}\right)^n \int_0^{\infty} dl\, e^{-lz}\, \mathrm{sh}^{2n}(l\hbar)$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{q}{\hbar^2}\right)^n \sum_{m=-n}^{n} \frac{(-1)^{n-m}}{(n-m)!(n+m)!} \frac{1}{z-m\hbar}. \qquad (6.8)$$

By computing the residue at $z = m\hbar(-n \le m \le n)$, we have the partial fraction expansion[10]

$$\prod_{m=-n}^{n} \frac{1}{z-m\hbar} = \hbar^{-2n} \sum_{m=-n}^{n} \frac{(-1)^{n-m}}{(n-m)!(n+m)!} \frac{1}{z-m\hbar}, \qquad (6.9)$$

---

[10]We note that the same formula has been used in the proof of (A.3).

which gives

$$T(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} q^n \prod_{m=-n}^{n} \frac{1}{z - m\hbar} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} q^n z^{-2n-1} \prod_{m=1}^{n} \sum_{k=0}^{\infty} \left(\frac{m\hbar}{z} \right)^{2k}.$$

(6.10)

This is a rather simple answer to the $\hbar$ dependent of $T(z)$, which is consistent with the $\hbar$ expansion (up to genus two) presented in Section 4.

Acknowledgments

Appendix A   Proof of the combinatorial identity

The aim of this appendix is to give a proof to the following theorem.

Theorem A.1.

$$\sum_{\lambda \vdash k} \frac{\sum_{x \in \lambda} \prod_{i=0}^{r-1} (c(x)^2 - i^2)}{\prod_{x \in \lambda} h(x)^2} = \frac{(2r)!}{((r+1)!)^2} \cdot \frac{\prod_{i=0}^{r} (k-i)}{k!},$$

(A.1)

where λ runs over all partitions of k, i.e., the Young diagrams with k boxes.

We put

$$S_r(k) = \sum_{\lambda \vdash k} \frac{\sum_{x \in \lambda} c(x)^{2r}}{\prod_{x \in \lambda} h(x)^2}, \quad T_r(k) = \sum_{\lambda \vdash k} \frac{\sum_{x \in \lambda} \prod_{i=0}^{r-1} (c(x)^2 - i^2)}{\prod_{x \in \lambda} h(x)^2},$$

and denote by e_i and h_i the i-th elementary and complete symmetric polynomial, respectively. Then we have

$$T_r(k) = \sum_{p=1}^{r} (-1)^{r-p} e_{r-p}(1^2, 2^2, \ldots, (r-1)^2) S_p(k).$$

Since the matrices

$$((-1)^{j-i}e_{j-i}(x_1,\ldots,x_j))_{0\le i,j\le N} \quad \text{and} \quad (h_{j-i}(x_1,\ldots,x_{i+1}))_{0\le i,j\le N}$$

are inverses to each other, we see that

$$S_r(k) = \sum_{p=1}^{r} h_{r-p}(1^2,2^2,\ldots,p^2)T_p(k).$$

Thus the identity (A.1) is equivalent to

$$S_r(k) = \sum_{p=1}^{r} h_{r-p}(1^2,2^2,\ldots,p^2)\frac{(2p)!}{((p+1)!)^2}\frac{k^{\underline{p+1}}}{k!}, \qquad (A.2)$$

where $k^{\underline{p+1}}$ denotes the falling factorial

$$k^{\underline{p+1}} = k(k-1)\cdots(k-p).$$

We prove the identity (A.2) by using the Jucys–Murphy elements L_i ($1 \le i \le k$) in the group ring $\mathbb{C}[\mathfrak{S}_k]$ of the symmetric group $\mathfrak{S}_k$ and the Lascoux–Thibon formula for them. The Jucys–Murphy elements [36, 38] are defined to be the sum of transpositions

$$L_i = (1,i) + (2,i) + \cdots + (i-1,i).$$

Note that $L_1 = 0$, but it is convenient to include this case. A key property of Jucys–Murphy elements is the following.

Proposition A.2. 1. *The Jucys–Murphy elements $L_1,\ldots,L_k$ are commutative.*
  2. *On the irreducible representation S^λ of $\mathfrak{S}_k$ corresponding to a partition λ, the operators $L_1,\ldots,L_k$ are simultaneously diagonalizable and the eigenvalues of L_i are the contents $\{c(x) : x \in \lambda\}$ of λ.*

Proposition A.3. *Let $f(z)$ be a polynomial. Then the quantity*

$$k!\sum_{\lambda\vdash k}\frac{\sum_{x\in\lambda}f(c(x))}{\prod_{x\in\lambda}h(x)^2}$$

is equal to the coefficient of the identity element in $f(L_1) + \cdots + f(L_k)$.

Proof. Since $f(L_1) + \cdots + f(L_k)$ is symmetric in $L_1,\ldots,L_k$, we see that $f(L_1) + \cdots + f(L_k)$ acts on S^λ as the scalar multiplication by $\sum_{x\in\lambda}f(c(x))$. Hence, the trace of the operator $f(L_1) + \cdots + f(L_k)$ on S^λ is equal to

$f^\lambda \sum_{x\in\lambda} f(c(x))$, where f^λ is the dimension of S^λ. Since the left regular representation of $\mathfrak{S}_k$ on $\mathbb{C}[\mathfrak{S}_k]$ is decomposed as

$$\mathbb{C}[\mathfrak{S}_k] \cong \bigoplus_{\lambda\vdash k} (S^\lambda)^{\oplus f^\lambda},$$

the trace of $f(L_1) + \cdots + f(L_k)$ on $\mathbb{C}[\mathfrak{S}_k]$ is given by

$$\sum_{\lambda\vdash k} (f^\lambda)^2 \sum_{x\in\lambda} f(c(x)) = (k!)^2 \sum_{\lambda\vdash k} \frac{\sum_{x\in\lambda} f(c(x))}{\prod_{x\in\lambda} h(x)^2},$$

because $f^\lambda = k! / \prod_{x\in\lambda} h(x)$.

On the other hand, the trace of the operator $g \in \mathfrak{S}_k$ on $\mathbb{C}[\mathfrak{S}_k]$ is equal to $k!$ if g is the identity element and 0 otherwise. Hence, we see that

$$k! \sum_{\lambda\vdash k} \frac{\sum_{x\in\lambda} f(c(x))}{\prod_{x\in\lambda} h(x)^2}$$

is the coefficient of the identity element in $f(L_1) + \cdots + f(L_k)$. $\qquad\square$

Now we recall the Lascoux–Thibon formula, which expresses the power-sums of Jucys–Murphy elements as linear combinations of the class sums C_μ. For a partition μ, we denote by C_μ the sum of all permutations with cycle type μ and put $z_\mu = \prod_{i\geq 1} i^{m_i} m_i!$, where m_i is the multiplicity of i in μ.

Theorem A.4 (Lascoux–Thibon [37, §4])**.** *Given a partition κ of m, we define a formal power series $\phi_\kappa(t) = \sum_{r\geq 0} \phi_{\kappa,r} t^r / r!$ by substituting $q = e^t$ in*

$$\frac{(1-q^{-1})^{m-1}}{m! z_\kappa} \frac{\prod_i (q^{\kappa_i} - 1)}{q - 1}.$$

Then we have

$$L_1^r + \cdots + L_k^r = \sum_{m=1}^{r+1} \sum_{\substack{\kappa\vdash m \\ l(\kappa)\leq r-m+2}} \phi_{\kappa,r} \frac{z_{\kappa\cup(1^{k-m})}}{(k-m)!} C_{\kappa\cup(1^{k-m})}.$$

Corollary A.5. *If $r \geq 1$, then the coefficient of the identity element in $L_1^{2r} + \cdots + L_k^{2r}$ is given by*

$$\sum_{p=1}^{r} \frac{2^p}{((p+1)!)^2} k^{p+1} \sum_{\substack{r_1+\cdots+r_p=r \\ r_1,\ldots,r_p>0}} \binom{2r}{2r_1, 2r_2, \ldots, 2r_p},$$

where the inner sum is taken over all p-tuples of positive integers $(r_1, \ldots, r_p)$ with $r_1 + \cdots + r_p = r$, and $\binom{2r}{2r_1, 2r_2, \ldots, 2r_p}$ is the multinomial coefficient.

Proof. We consider the coefficient of $C_{(1^k)}$ in the Lascoux–Thibon formula. If $\kappa = (1)$, then $\phi_{(1)} = 1$ and $\phi_{(1),r} = 0$ for $r \geq 1$. If $\kappa = (1^m)$ with $m \geq 2$, then

$$\phi_{(1^m)}(t) = \frac{1}{(m!)^2}(e^t - 2 + e^{-t})^{m-1},$$

and

$$\phi_{(1^m),2r} = \frac{2^{m-1}}{(m!)^2} \sum_{\substack{r_1+\cdots+r_{m-1}=r \\ r_1,\ldots,r_{m-1}>0}} \binom{2r}{2r_1, 2r_2, \ldots, 2r_{m-1}},$$

where the sum is taken over all $(m-1)$-tuples of positive integers $(r_1, \ldots, r_{m-1})$ with $r_1 + \cdots + r_{m-1} = r$. □

Now the proof of (A.2) is completed by showing the following lemma.

Lemma A.6. *If $r \geq p$, then we have*

$$\frac{2^p}{(2p)!} \sum_{\substack{r_1+\cdots+r_p=r \\ r_1,\ldots,r_p>0}} \binom{2r}{2r_1, 2r_2, \ldots, 2r_p} = h_{r-p}(1^2, 2^2, \ldots, p^2). \tag{A.3}$$

Proof. First we simplify the summation on the left hand side of (A.3). We put

$$M_p(r) = \sum_{\substack{r_1+\cdots+r_p=r \\ r_1,\ldots,r_p\geq 0}} \binom{2r}{2r_1, \ldots, 2r_p},$$

$$N_p(r) = \sum_{\substack{r_1+\cdots+r_p=r \\ r_1,\ldots,r_p>0}} \binom{2r}{2r_1, \ldots, 2r_p}.$$

(We define $M_0(r) = N_0(r) = 0$.) It follows from the multinomial theorem that

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_p)\in\{1,-1\}^p} (\varepsilon_1 x_1 + \cdots + \varepsilon x_p)^{2r}$$

$$= \sum_{r_1,\ldots,r_p} \binom{2r}{r_1, \ldots, r_p}(x_1^{r_1} + (-x_1)^{r_1}) \cdots (x_p^{r_p} + (-x_1)^{r_p}),$$

where the sum is taken over all p-tuples $(r_1, \ldots, r_p)$ of non-negative integers with $r_1 + \cdots + r_p = r$. Substituting $x_1 = \cdots = x_p = 1$, we obtain

$$M_p(r) = \frac{1}{2^p} \sum_{i=0}^{p} \binom{p}{i}(p - 2i)^{2r}.$$

By applying the principle of inclusion–exclusion, we have

$$
\begin{aligned}
N_p(r) &= \sum_{j=0}^{p}(-1)^j\binom{p}{j}M_{p-j}(r) \\
&= \frac{1}{2^p}\left(\sum_{k=0}^{\lfloor p/2\rfloor}\left[\sum_{i=0}^{k}2^{2i}\binom{p}{2i}\binom{p-2i}{k-i}\right]((p-2k)^{2r}+(-p+2k)^{2r})\right. \\
&\qquad\qquad -\sum_{k=0}^{\lfloor (p-1)/2\rfloor}\left[\sum_{i=0}^{k}2^{2i+1}\binom{p}{2i+1}\binom{p-2i-1}{k-i}\right] \\
&\qquad\qquad\left. \times((p-2k-1)^{2r}+(-p+2k+1)^{2r})\right).
\end{aligned}
$$

By using the Chu–Vandermonde formula (see e.g., [35, Cor. 2.2.3]), we see that

if $0\le 2k\le 2p$,

$$
\sum_{i=0}^{k}2^{2i}\binom{p}{2i}\binom{p-2i}{k-i}=\frac{p^{\underline{k}}}{k!}\sum_{i=0}^{k}\frac{(p-k)^{\underline{i}}k^{\underline{i}}}{i!(i-1/2)^{\underline{i}}}=\binom{2p}{2k},
$$

if $0\le 2k+1\le 2p$,

$$
\begin{aligned}
\sum_{i=0}^{k}2^{2i+1}\binom{p}{2i+1}\binom{p-2i-1}{k-i}&=\frac{2p^{\underline{k+1}}}{k!}\sum_{i=0}^{k}\frac{(p-k-1)^{\underline{i}}k^{\underline{i}}}{i!(i+1/2)^{\underline{i}}} \\
&=\binom{2p}{2k+1}.
\end{aligned}
$$

Therefore, we conclude that

$$
N_p(r)=\frac{2}{2^p}\sum_{i=1}^{p}(-1)^{p-i}\binom{2p}{p-i}i^{2r}.
$$

Now we are in position to complete the proof of (A.3) by using generating functions. The generating function of the right hand sides is

$$
\begin{aligned}
\sum_{r=p}^{\infty}\frac{2^p}{(2p)!}N_p(r)z^{r-p}&=\sum_{r=p}^{\infty}\frac{2}{(2p)!}\left(\sum_{i=1}^{p}(-1)^{p-i}\binom{2p}{p-i}i^{2r}\right)z^{r-p} \\
&=\sum_{i=1}^{p}(-1)^{p-i}\frac{2i^{2p}}{(p-i)!(p+i)!}\frac{1}{1-i^2z}.
\end{aligned}
$$

By considering the partial fraction expansion, we see that

$$\sum_{i=1}^{p}(-1)^{p-i}\frac{2i^{2p}}{(p-i)!(p+i)!}\frac{1}{1-i^2z}=\prod_{i=1}^{p}\frac{1}{1-i^2z},$$

which is the generating function $\sum_{r=p}^{\infty}h_{r-p}(1^2,\ldots,p^2)z^{r-p}$. This completes the proof of (A.3). $\qquad\square$

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