

Positive Lyapunov exponents and localization bounds for strongly mixing potentials

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Abstract

For a one-dimensional discrete Schrödinger operator with a weakly coupled potential given by a strongly mixing dynamical system with power law decay of correlations, we derive for all energies including the band edges and the band center a perturbative formula for the Lyapunov exponent. Under adequate hypothesis, this shows that the Lyapunov exponent is positive on the whole spectrum. This in turn implies that the Hausdorff dimension of the spectral measure is zero and that the associated quantum dynamics grows at most logarithmically in time.

1 Introduction

Let Σ be a topological space and $\Omega = \Sigma^{\mathbb{Z}}$ the associated Tychonov product space. Furthermore let \mathbf{P} be a probability measure on Ω which is invariant and ergodic w.r.t. the left shift $S : \Omega \rightarrow \Omega$. Now given a measurable

real-valued function V on Ω and a coupling constant $\lambda > 0$, one can associate an ergodic family of Jacobi matrices $(H_{\lambda,\omega})_{\omega \in \Omega}$ (also called discrete Schrödinger operators) each acting on $\ell^2(\mathbb{Z})$:

$$H_{\lambda,\omega}|n\rangle = |n+1\rangle + \lambda V(S^n\omega)|n\rangle + |n-1\rangle, \quad (1.1)$$

where $|n\rangle$ is the Dirac notation for the state in $\ell^2(\mathbb{Z})$ localized at site $n \in \mathbb{Z}$. If $\mathbf{P} = \mathbf{p}^{\otimes \mathbb{Z}}$ is a product measure of a compactly supported probability measure \mathbf{p} on Σ so that the random variables of the sequence $(V(S^n\omega))_{n \in \mathbb{Z}}$ of potential values are independent, the model exhibits the so-called Anderson localization, namely the spectrum of $H_{\lambda,\omega}$ is \mathbf{P} -almost surely pure-point with exponentially localized eigenstates [14], and the induced quantum dynamics is bounded in time (in the precise sense given below). The question considered in this work (and many others, see the reviews [5, 9] and references therein) concerns the spectral properties as well as the quantum dynamics in situations where \mathbf{P} is not a product measure so that the random variables $(V(S^n\omega))_{n \in \mathbb{Z}}$ are correlated. This situation typically arises when the dynamical system (Ω, S, \mathbf{P}) is the symbolic dynamics associated to a (possibly weakly) hyperbolic discrete time dynamics; then Σ is the Markov partition. If now the correlations of the potential decay sufficiently fast, then one expects that the model is still in the regime of Anderson localization. Here we complement on the prior work [1, 4] and prove that this holds at least in a weak sense when the correlations satisfy a power law decay.

The proof of localization for these models is based on the positivity of the Lyapunov exponent. This positivity can either be established by Kotani theory [5], a version of Furstenberg's theorem for correlated random matrices (work by Avila and Damanik cited in [5]) or by a perturbative calculation (for small λ) of the Lyapunov exponent. This latter calculation was first done by Chulaevsky and Spencer [4] by carrying over the argument of Thouless [18], in a version given by Pastur and Figotin [14], to the case of correlated potential values. The resulting formula is recalled in Section 2. Based on this result, Bourgain and Schlag then proved localization [1]. The only flaw left is that in [4] (and actually already in [14]) not all energies could be dealt with, but the band center and the band edges were spared out. Here we show how the techniques of our prior works [16, 17] on anomalies and band edges [6, 12] combine with those of [4] to rigorously control the perturbation theory for the Lyapunov exponent also at these energies. Instead of repeating the rather complicated proofs of [1], we then adapt to the case of correlated potentials the elementary and short argument of [10], showing that positivity of the Lyapunov exponents implies at most logarithmic growth of quantum dynamics and hence, by Guarneri's inequality [8], zero Hausdorff dimension of the spectral measures. Even though

this is a weaker localization result than pure-point spectrum with exponential localized eigenfunctions, it proves the behavior which is stable under perturbation and we hence consider, as argued in [10], that it already captures the physically relevant effect. In the next section, the results and the precise hypothesis are described and discussed in detail. The other sections contain the proofs.

2 Set-up and results

In order to fix terms and notations, we have to begin by reviewing some basic definitions of symbolic dynamics and strong mixing [3, 13]. Let Σ be a countable set furnished with the discrete topology. We designate a reference element $0 \in \Sigma$. For any subset $I \subset \mathbb{Z}$ and $\omega = (\sigma_n)_{n \in \mathbb{Z}} \in \Omega$, let us define

$$\pi_I(\omega) = (\hat{\sigma}_n)_{n \in \mathbb{Z}}, \quad \hat{\sigma}_n = 0 \text{ for } n \notin I, \quad \hat{\sigma}_n = \sigma_n \text{ for } n \in I.$$

For a bounded, measurable function $g : \Omega \rightarrow V$ into a real, normed vector space $(V, \|\cdot\|)$, the variation on I is defined by

$$\text{Var}_I(g) = \sup_{\pi_I(\omega) = \pi_I(\omega')} \|g(\omega) - g(\omega')\|.$$

Then g is called quasi-local with rate $0 < r < 1$ if and only if there exists a constant $C = C(g)$ such that for any $m, n \geq 1$,

$$\text{Var}_{[-m,n]}(g) \leq C(g)r^{m \wedge n}, \quad m \wedge n = \min\{m, n\}. \tag{2.1}$$

The set of all quasi-local functions with rate r is denoted by $\mathcal{Q}_r(V)$.

Next let us state precisely the strong mixing hypothesis used in this work. For $m < n$ and $a_k \in \Sigma$ with $m \leq k \leq n$, the associated cylinder set is denoted by $A_{m,n} = A_{m,n}(a_m, \dots, a_n) = \{\omega = (\sigma_k)_{k \in \mathbb{Z}} \mid \sigma_k = a_k, m \leq k \leq n\}$. Then the invariant measure \mathbf{P} on the shift space (Ω, \mathbb{Z}) is said to satisfy a power law ψ -mixing [2] with exponent $\alpha > 0$ if there is a constant $C > 0$ such that for all $k < l < m < n$ and all $A_{k,l}, A_{m,n}$, one has

$$|\mathbf{P}(A_{k,l} \cap A_{m,n}) - \mathbf{P}(A_{k,l})\mathbf{P}(A_{m,n})| \leq C\mathbf{P}(A_{k,l})\mathbf{P}(A_{m,n})|m - l|^{-\alpha}. \tag{2.2}$$

Equivalently, for any $\pi_{[k,l]}$ -measurable function g_1 and $\pi_{[m,n]}$ -measurable function g_2 , there holds

$$|\mathbf{E}(g_1 g_2) - \mathbf{E}(g_1)\mathbf{E}(g_2)| \leq C\mathbf{E}(|g_1|)\mathbf{E}(|g_2|)|m - l|^{-\alpha}, \tag{2.3}$$

where $k < l < m < n$ and C are as above. This also implies ergodicity. Examples when (2.3) holds are given in Remarks 2.3 and 2.4 below, after the main results are stated. Averages over ω w.r.t. \mathbf{P} are denoted by \mathbf{E} , or also by \mathbf{E}_ω if the dependence on ω is retained in the integrand. Furthermore, the set of centered quasi-local functions will be denoted by $\mathcal{Q}_r^0(V) = \{g \in \mathcal{Q}_r(V) \mid \mathbf{E}(g) = 0\}$.

Throughout we suppose that the potential in (1.1) is given by a centered real-valued quasi-local function $V \in \mathcal{Q}_r^0(\mathbb{R})$. It is well known and verified in Lemmas 5.4 that (2.3) implies the decay of correlations $|\mathbf{E}(V(\omega)V(S^n\omega))| \leq C|n|^{-\alpha}$ for some constant C . For $\alpha > 1$, one can hence define its (positive) spectral density $D_V(k)$ at $k \in [0, 2\pi)$:

$$D_V(k) = \sum_{n \in \mathbb{Z}} e^{ikn} \mathbf{E}_\omega(V(\omega)V(S^n\omega)) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\omega \left(\left| \sum_{n=0}^{N-1} e^{ikn} V(S^n\omega) \right|^2 \right).$$

As final preparation let us recall the definition of the Lyapunov exponent $\gamma_\lambda(E)$ at energy $E \in \mathbb{C}$ associated to (1.1). If the transfer matrices are defined by

$$\mathcal{T}_{\lambda,\omega}^E = \begin{pmatrix} E - \lambda V(\omega) & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{Q}_r(\mathrm{SL}(2, \mathbb{R})), \tag{2.4}$$

then

$$\gamma_\lambda(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_\omega \log \left(\left\| \prod_{n=1}^N \mathcal{T}_{\lambda,S^n\omega}^E \right\| \right).$$

The main result of Chulaevsky and Spencer [4] is that for $\alpha > 2$ and at an energy $E = 2 \cos(k)$ in the spectrum $[-2, 2]$ of the discrete Laplacian away from the band edges $E = -2, 2$ and the band center $E = 0$, one has

$$\gamma_\lambda(E) = \lambda^2 \frac{D_V(k)}{8 \sin^2(k)} + \mathcal{O} \left(\frac{\lambda^{(3\alpha+2)/(\alpha+2)}}{d(k)} \right). \tag{2.5}$$

where $d(k)$ denotes the distance of k from $0 \pmod{\frac{\pi}{2}}$. As we need to build up the whole formalism anyway, the main element of the proof of (2.5) is reproduced in Section 6. As indicated, the control of the error terms breaks down at the band edges and the band center. Our first result provides perturbative formulas for the Lyapunov exponent at these energies, generalizing, respectively, our prior results for independent potential values [16, 17].

Theorem 2.1. *Assume $\alpha > 2$, $D_V(0) > 0$ and $D_V(\pi) > 0$ (the latter is only needed for (i)).*

(i) *The Lyapunov exponent near the band center $E = 0$ is given by*

$$\gamma_\lambda(\epsilon\lambda^2) = \lambda^2 \frac{D_V(\pi)}{8} \int_0^\pi d\theta \rho_\epsilon(\theta) (1 + \cos(4\theta)) + \mathcal{O}(\lambda^{(3\alpha+2)/(\alpha+2)}), \tag{2.6}$$

where ρ_ϵ is a π -periodic smooth probability density.

(ii) Up to errors of order $\mathcal{O}(\lambda^{(3\alpha+2)/(3\alpha+6)})$, the Lyapunov exponent near the upper band edge $E = 2$ is given by

$$\gamma_\lambda(2 + \epsilon\lambda^{4/3}) = \lambda^{2/3} \left(\frac{1 - \epsilon}{2} \int d\theta \rho_\epsilon(\theta) \sin(2\theta) + \frac{D_V(0)}{8} \int d\theta \rho_\epsilon(\theta) (1 + 2 \cos(2\theta) + \cos(4\theta)) \right), \quad (2.7)$$

where ρ_ϵ is a π -periodic smooth probability density written out explicitly in Section 8. The same formula holds at the lower band edge $E = -2$.

The formulas (2.5), (2.6) and (2.7) combined allow to study the Lyapunov exponents at all energies $[-2, 2]$. In order to assure positivity for $\lambda > 0$, one first has to check that the spectral density is positive (cf. Remark 2.5 below) and then prove that the integrals appearing in (2.6) and (2.7) are positive. This is immediate for (2.6). For (2.7) we could not produce an analytic proof, but, given the explicit formula (8.1) for ρ_ϵ , one can readily do a numerical evaluation.

Nevertheless, the three formulas are not yet sufficient to prove uniform positivity of the Lyapunov exponent on the whole spectrum for some fixed small, but positive value of λ . Indeed, for once the non-random spectrum $\sigma(H_{\lambda,\omega})$ may (and typically will) fill the whole interval $[-2 - \lambda\|V\|_\infty, 2 + \lambda\|V\|_\infty]$, where $\|V\|_\infty = \mathbf{P} - \text{ess sup } |V(\omega)|$ (use approximate eigenfunctions as Weyl sequences in order to show this). For an energy $2 + \epsilon\lambda$, $\epsilon > 0$, the asymptotics (2.7) then says nothing. However, one can combine the techniques of this work with those of [17] in order to prove, as in the case of independent potential values [17, Section 8],

$$\gamma_\lambda(2 + \epsilon\lambda^\eta) = \sqrt{\epsilon\lambda^\eta} + \mathcal{O}(\lambda^{1-\eta/4}, \lambda^{7\eta/4-1}, \lambda^{(\eta/4)(3\alpha+2)/(\alpha+2)}), \quad \epsilon > 0, \quad (2.8)$$

where $\frac{4}{5} < \eta < \frac{4}{3}$ is such that the error terms are of lower order than $\lambda^{\eta/2}$ (in particular, $\eta = 1$ is allowed for α sufficiently large). Moreover, the formulas (2.5) and (2.7) do not imply positivity of the Lyapunov exponent at a fixed λ for all energies in $[2 - \lambda, 2)$ because the error term in (2.5) explodes as one approaches the band edge. However, once again one can transpose [17, Section 8] to the case of a strongly mixing potential:

$$\gamma_\lambda(2 - \epsilon\lambda^\eta) = \lambda^{2-\eta} \frac{D_V(0)}{8\epsilon} + \mathcal{O}(\lambda^{4-(5\eta/2)}, \lambda^{3\eta/2}, \lambda^{(1-(\eta/2))(3\alpha+2)/(\alpha+2)}), \quad \epsilon > 0, \quad (2.9)$$

where again $\frac{4}{5} < \eta < \frac{4}{3}$ has to assure that the error terms are subdominant. A careful analysis now allows to show (modulo the issues discussed above) that for λ sufficiently small the Lyapunov exponent is positive on $[2 - c, 2 + c]$ for $c > 0$. We do not provide the detailed argument here, but do claim to have presented all the essential ingredients in order to complete it. Similarly, by

analyzing the Lyapunov exponent $\gamma_\lambda(\epsilon\lambda^\eta)$, $1 \leq \eta \leq 2$, using the techniques of [16, Section 5.1] or [17, Section 5], one can show that the Lyapunov exponent is positive near the band center for λ sufficiently small.

Let us now assume that uniform positivity of the Lyapunov exponent has been verified for all energies in the spectrum, either by the above or some other argument, and then deduce localization estimates from this. One standard way to quantify the spreading (delocalization) of an initially localized wave packet $|0\rangle$ under the quantum mechanical time evolution $e^{-itH_{\lambda,\omega}}$ is to consider the growth of (time and disorder averaged) moments of the position operator X on $\ell^2(\mathbb{Z})$:

$$M_T^q = \int_0^\infty \frac{dt}{T} e^{-(t/T)} \mathbf{E}_\omega \langle 0 | e^{iH_{\lambda,\omega}t} |X|^q e^{-iH_{\lambda,\omega}t} |0\rangle, \quad q > 0. \quad (2.10)$$

Boundedness of M_T^q uniformly in time is called dynamical localization. Logarithmic growth in time as obtained in the following theorem is quite close to that.

Theorem 2.2. *Consider an ergodic family of Jacobi matrices $(H_{\lambda,\omega})_{\omega \in \Omega}$ of the form (1.1) with a quasi-local potential V and an invariant measure \mathbf{P} satisfying (2.3) with $\alpha > 0$. Suppose that the spectrum is included in an open interval (E_0, E_1) on which the Lyapunov exponent is uniformly positive:*

$$\gamma_\lambda(E) \geq \gamma_0 > 0, \quad E \in (E_0, E_1). \quad (2.11)$$

Then for any $\beta > 2$ there exists a constant $C(\beta, q)$ such that

$$M_T^q \leq (\log T)^{q\beta} + C(\beta, q). \quad (2.12)$$

Furthermore, the Hausdorff dimension of the spectral measure of $H_{\lambda,\omega}$ vanishes \mathbf{P} -almost surely.

The elementary proof (fitting on 4–5 pages) of (2.12) is almost completely contained in [10]. It is therefore not reproduced here, but we discuss in detail in Section 9 the only step that has to be modified. As already indicated in the introduction, the last statement then follows directly from Guarneri's inequality [8].

Now follow remarks on when the hypotheses of the above theorems are satisfied.

Remark 2.3. The strong mixing condition (2.3) clearly holds if \mathbf{P} is the product measure of some probability measure on Σ , because the functions g_1 and g_2 are then independent. The mixing condition also holds if \mathbf{P} stems from a Markov process given by a stochastic kernel having only one invariant measure on a countable set Σ . Then the decay on the r.h.s. of (2.3) is actually exponential, with rate given by the Perron–Frobenius gap of the

stochastic kernel. Yet more general, let us consider a hyperbolic dynamical system (X, T) (Axiom A) given by a map $T : X \rightarrow X$. Then one has a finite Markov partition Σ , with associated symbolic dynamics (Ω, S) [3], and there is a wealth of the so-called Gibbs measures associated to Hölder continuous (i.e., quasi-local) functions which all satisfy (2.3) with an exponential mixing rate [3, Proposition 2.4]. Two standard examples of this type already cited in [4] are the period doubling map and the Arnold cat maps. Moreover, if the phase space X is a manifold, then any differentiable real function on this manifold gives rise to a quasi-local potential under the coding map. For all these examples with exponential ψ -mixing, the error bounds in (2.5), (2.6) and (2.7) are given by the error bounds of the independent case [14, 16, 17] multiplied by $\log^2(\lambda)$. The error bounds in the independent case are recovered by sending $\alpha \rightarrow \infty$ in (2.5), (2.6), (2.7), (2.8) and (2.9).

Remark 2.4. Concrete examples of dynamical systems (X, T) having not an exponential, but only a power law decay in (2.3) have only be analyzed more recently. Necessarily T is then not uniformly hyperbolic, but it is supposed to have only a few parabolic points. Such examples can be constructed even if X is an interval, but the invariant measure then has a non-normalizable density w.r.t. the Lebesgue measure. It is, however, possible to construct a symbolic dynamics over a countable alphabet Σ which then has a shift-invariant probability measure \mathbf{P} satisfying the strong mixing estimates (2.2) and (2.3). Instead of producing a long citation list, we refer to the references in [7] which contains a proof of (2.3) for several concrete examples. It is precisely in order to deal with these cases at the verge that we bothered to work with (2.3) instead of exponential mixing.

Remark 2.5. The positivity of the spectral density $D_V(k)$ can for some examples be checked by an explicit calculation, but there are also further techniques available in order to verify this [2]. The case of $D_V(0)$ is particularly well studied because of its importance for central limit theorems [13, 7]. For the Gibbs measures of Remark 2.3 and the examples of Remark 2.4, $D_V(0) = 0$ holds if and only if $V = v \circ S - v$ is a cocycle given by another quasi-local function v . By suspension, one can deal similarly with $k = \pi$ and actually any rational $\frac{k}{2\pi}$.

Remark 2.6. The above results transpose if \mathbb{Z} is replaced by \mathbb{N} , namely for $\Omega = \Sigma^{\mathbb{N}}$ furnished with the left shift and $H_{\lambda, \omega}$ acts on $\ell^2(\mathbb{N})$. As the inverse S^{-1} of the left shift operator is not defined in that case, one needs to replace in all proofs functions like $g \circ S^{-n}$ for $n > 0$ by $(U^*)^n g$, where U^* is the $L^2(\Sigma^{\mathbb{N}}, \mathbf{P})$ -adjoint operator of $U : g \mapsto g \circ S$.

3 Anomalies at band center and band edge

Let us begin by recalling that the transfer matrix $\mathcal{T}_{\lambda,\omega}^E \in \text{SL}(2, \mathbb{R})$ given in (2.4) is elliptic for an energy $E = 2 \cos(k) \in (-2, 2)$ and $\lambda = 0$, and it can hence, to zeroth order in λ , be transformed into a rotation. More explicitly,

$$M\mathcal{T}_{\lambda,\omega}^E M^{-1} = R_k \left(\mathbf{1} + \lambda \frac{V(\omega)}{\sin(k)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \tag{3.1}$$

where

$$R_k = \begin{pmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{pmatrix}, \quad M = \frac{1}{\sqrt{\sin(k)}} \begin{pmatrix} \sin(k) & 0 \\ -\cos(k) & 1 \end{pmatrix}.$$

In the next section we will consider the action of the matrix (3.1) on the real projective line, which is identified with a circle. To lowest order λ^0 , this action induced by (3.1) is then a rotation on the circle. For irrational $\frac{k}{2\pi}$, there is a unique invariant measure given by the Lebesgue measure. For rational $\frac{k}{2\pi} = \frac{p}{q}$ at least Birkhoff sums of harmonics of order lower than q vanish.

At the band center $k = \frac{\pi}{2}$, the square of the transfer matrix (3.1) (note that $M = \mathbf{1}$ here) is the unit matrix and one can only control the lowest order harmonic, which turns out not to be sufficient for the calculation of the Lyapunov exponent. It is then more convenient to consider directly the square of the transfer matrix

$$\begin{aligned} \mathcal{T}_{\lambda,S\omega}^{\epsilon\lambda^2} \mathcal{T}_{\lambda,\omega}^{\epsilon\lambda^2} &= - \begin{pmatrix} 1 - \lambda^2 V(\omega)V(S\omega) & \epsilon\lambda^2 - \lambda V(S\omega) \\ -\epsilon\lambda^2 + \lambda V(\omega) & 1 \end{pmatrix} + \mathcal{O}(\lambda^3) \\ &= - \exp \left(\lambda \begin{pmatrix} 0 & -V(S^n\omega) \\ V(\omega) & 0 \end{pmatrix} \right) \\ &\quad + \frac{\lambda^2}{2} \begin{pmatrix} -V(\omega)V(S\omega) & 2\epsilon \\ -2\epsilon & V_\omega V(S\omega) \end{pmatrix} + \mathcal{O}(\lambda^3), \end{aligned} \tag{3.2}$$

and to group the coordinates of ω in pairs and consider $\tilde{\Omega} = \tilde{\Sigma}^{\mathbb{Z}}$, where $\tilde{\Sigma} = \Sigma \times \Sigma$, and furnish it with a probability $\tilde{\mathbf{P}}$ naturally induced by \mathbf{P} . Again the suspension $(\tilde{\Omega}, \tilde{\mathbf{P}})$ is a shift space with power law mixing. However, the matrix (3.2) is now in the form of an anomaly as discussed at the end of this section.

At a band edge, e.g., $E = -2$ and $k = \pi$, the basis change in (3.1) becomes singular and one has a Krein collision. Nevertheless, the transfer matrix at

$\lambda = 0$ can be transformed into a non-diagonalizable Jordan normal form:

$$N\mathcal{T}_{\lambda,\omega}^{-2+\epsilon\lambda^{4/3}}N^{-1} = - \begin{pmatrix} 1 + \lambda V(\omega) - \epsilon\lambda^{4/3} & 1 \\ \lambda V(\omega) - \epsilon\lambda^{4/3} & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Let us further conjugate this matrix by $N_\lambda = \begin{pmatrix} \lambda^{2/3} & 0 \\ 0 & 1 \end{pmatrix}$ in order to get again an anomaly (cf. [17] for a motivation of this conjugation):

$$N_\lambda N\mathcal{T}_{\lambda,\omega}^{-2+\epsilon\lambda^{4/3}}N^{-1}N_\lambda^{-1} = - \exp \left(\lambda^{1/3} \begin{pmatrix} 0 & 0 \\ V(\omega) & 0 \end{pmatrix} + \lambda^{2/3} \begin{pmatrix} 0 & 1 \\ -\epsilon & 0 \end{pmatrix} + \mathcal{O}(\lambda) \right). \tag{3.3}$$

Resuming, after adequate basis change and possibly regrouping of terms, one has to study in each of the three situations (3.1), (3.2) and (3.3) families of random matrices $(T_{\lambda,\omega})_{\lambda \geq 0, \omega \in \Omega} \in \mathcal{Q}_r(\text{SL}(2, \mathbb{R}))$ of the following form:

$$T_{\lambda,\omega} = \pm R_k \exp(\lambda^\eta P_{1,\omega} + \lambda^{2\eta} P_{2,\omega} + \mathcal{O}(\lambda^{3\eta})), \tag{3.4}$$

where $\eta > 0$, $P_{j,\omega} \in \mathcal{Q}_r(\text{sl}(2, \mathbb{R}))$ for $j = 1, 2$, $\mathbf{E}(P_{1,\omega}) = 0$ and the error term $\mathcal{O}(\lambda^{3\eta})$ is uniformly bounded (i.e., the bound is ω -independent). If $k = 0, \pi$, namely at a band center (3.2) and a band edge (3.3), such a family is said to have an anomaly of second order [16, 17]. In the following sections, we treat general families of the form (3.4), and then go back to the explicit cases in Section 8 in order to complete the proof of Theorem 2.1.

4 Phase shift dynamics

The bijective action \mathcal{S}_T of a matrix $T \in \text{SL}(2, \mathbb{R})$ on $S_\pi^1 = \mathbb{R}/\pi\mathbb{Z} = [0, \pi)$ is given by

$$e_{\mathcal{S}_T(\theta)} = \pm \frac{Te_\theta}{\|Te_\theta\|}, \quad e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad \theta \in S_\pi^1, \tag{4.1}$$

with an adequate choice of the sign. This defines a group action, namely $\mathcal{S}_{TT'} = \mathcal{S}_T\mathcal{S}_{T'}$. In order to shorten notations, we write $\mathcal{S}_{\lambda,\omega} = \mathcal{S}_{T_{\lambda,\omega}}$ where $T_{\lambda,\omega}$ is of the form (3.4). One thus has $\mathcal{S}_{\lambda,\omega}(\theta) = \theta + k + \mathcal{O}(\lambda)$.

Given an initial angle θ_0 , iterating this dynamics by the left shift on Ω defines a stochastic process $\theta_n(\omega)$, also simply denoted by θ_n below:

$$\theta_0(\omega) = \theta_0, \quad \theta_{n+1}(\omega) = \mathcal{S}_{\lambda,S^n\omega}(\theta_n(\omega)). \tag{4.2}$$

In order to analyze the dynamics in more detail, let us introduce for $j = 1, 2$ the trigonometric polynomials

$$p_{j,\omega}(\theta) = \Im m \left(\frac{\langle v|P_{j,\omega}|e_\theta \rangle}{\langle v|e_\theta \rangle} \right), \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{4.3}$$

One then has (cf. [17] for details)

$$\mathcal{S}_{\lambda,\omega}(\theta) = \theta + k + \sum_{j=1}^2 \lambda^{j\eta} p_{j,\omega}(\theta) + \frac{1}{2} \lambda^{2\eta} p_{1,\omega} \partial_{\theta} p_{1,\omega}(\theta) + \mathcal{O}(\lambda^{3\eta}). \quad (4.4)$$

Due to [11], Lemma 3.1 and a telescoping argument, the Lyapunov exponent $\gamma(\lambda)$ characterizing the exponential growth of the products of matrices in the ergodic family $(T_{\lambda,S^n\omega})_{n \geq 0}$ is given by

$$\gamma(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_{\theta_0} \mathbf{E}_{\omega} \sum_{n=0}^{N-1} \log(\|T_{\lambda,S^n\omega} e_{\theta_n(\omega)}\|), \quad (4.5)$$

where \mathbf{E}_{θ_0} denotes an average over the initial condition θ_0 w.r.t. an arbitrary continuous probability measure on S^1_{π} . As our interest is perturbation theory for $\gamma(\lambda)$ w.r.t. λ , we shall need the following expansions for the summands of (4.5) (e.g., [17] contains the algebraic proof):

Lemma 4.1. *Set*

$$\alpha_{j,\omega} = \langle v | P_{j,\omega} | v \rangle, \quad \beta_{j,\omega} = \langle \bar{v} | P_{j,\omega} | v \rangle, \quad \gamma_{j,\omega} = \langle \bar{v} | |P_{j,\omega}|^2 | v \rangle.$$

Then $p_{j,\omega}(\theta) = \Im m(\alpha_{j,\omega} - \beta_{j,\omega} e^{2i\theta})$. Furthermore,

$$\begin{aligned} \log(\|T_{\lambda,\omega} e_{\theta}\|) &= \Re e \left(\sum_{j=1}^2 \lambda^{j\eta} \beta_{j,\omega} e^{2i\theta} + \frac{\lambda^{2\eta}}{2} \left(|\beta_{1,\omega}|^2 + \gamma_{1,\omega} e^{2i\theta} - \beta_{1,\omega}^2 e^{4i\theta} \right) \right) \\ &+ \mathcal{O}(\lambda^{3\eta}). \end{aligned} \quad (4.6)$$

Formula (4.5) and also its perturbative evaluation based on (4.6) hence leads us to consider sums of the type

$$\hat{I}_N(\mathcal{G}) = \frac{1}{N} \mathbf{E}_{\omega} \sum_{n=0}^{N-1} \mathcal{G}(S^n\omega, \theta_n(\omega)), \quad \hat{I}(\mathcal{G}) = \lim_{N \rightarrow \infty} \hat{I}_N(\mathcal{G}), \quad (4.7)$$

for functions \mathcal{G} on $\Omega \times S^1_{\pi}$ of the type $\mathcal{G}(\omega, \theta) = \sum_j g_j(\omega) f(\theta)$. More explicitly, the above lemma shows that one only needs functions of the form $g(\omega) e^{2i\theta}$ and $g(\omega) e^{4i\theta}$ with $g \in \mathcal{Q}_r(\mathbb{C})$. For a π -periodic function $f \in C(S^1_{\pi})$, we also introduce

$$I_N(f) = \frac{1}{N} \mathbf{E} \sum_{n=0}^{N-1} f(\theta_n), \quad I(f) = \lim_{N \rightarrow \infty} I_N(f),$$

This is a Birkhoff sum of the process $\theta_n = \theta_n(\omega)$. In the sum (4.7) there is, moreover, an explicit dependence of \mathcal{G} on ω , hence let us use the term Birkhoff-like sums for the sums $\hat{I}(\mathcal{G})$. The defined limits may not exist but one can work with lim sup and lim inf and all estimates in the next sections are valid for both.

5 From Birkhoff-like sums to Birkhoff sums

The aim of this section is, as indicated in the title, to reduce the perturbative evaluation of the Birkhoff-like sums (4.7) to the evaluation of Birkhoff sums by invoking the correlation decay (2.3).

Proposition 5.1. *Suppose $\alpha > 2$ and $k = 0$. Let $g \in \mathcal{Q}_r(\mathbb{C})$ and $f \in C^2(S^1_\pi)$. Define $\mathcal{G}(\omega, \theta) = g(\omega)f(\theta)$. Then*

$$\hat{I}(\mathcal{G}) = \mathbf{E}(g)I(f) + \mathcal{O}(\lambda^{\eta(\alpha/1+\alpha)}). \tag{5.1}$$

If $\mathbf{E}(g) = 0$, one has the following convergent expression for the next higher order contribution:

$$\hat{I}(\mathcal{G}) = \lambda^\eta \sum_{j=1}^\infty I(f_j) + \mathcal{O}(\lambda^{\eta(2\alpha/2+\alpha)}), \quad f_j(\theta) = \mathbf{E}_\omega(g(S^j\omega)p_{1,\omega}(\theta))f'(\theta). \tag{5.2}$$

Remark. For $k \neq 0$ equations (5.1) and (5.2) hold as well if on their r.h.s. one replaces $f(\theta)$ and $f_j(\theta)$ by $\hat{f}(\theta) = f(\theta + 6\lambda^{-\eta/(1+\alpha)}k)$ and $\hat{f}_j(\theta) = f_j(\theta + (6\lambda^{-\eta/(1+\alpha)} - j)k)$ respectively. For the proof, none of the lemmata have to be changed. Only the final argument following them can easily be modified.

The first lemma needed for the proof is mainly contained in [4]. We provide a few more details of the proof and use the notations of this work.

Lemma 5.2. *One has for $m, n \geq 1$*

$$\text{Var}_{[-m-n, n+m]}(\theta_n) \leq \mathcal{O}(r^m \lambda^\eta).$$

Proof. Using equation (4.2),

$$\begin{aligned} |\theta_{n+1}(\omega) - \theta_{n+1}(\omega')| &\leq |\mathcal{S}_{\lambda, S^n \omega}(\theta_n(\omega)) - \mathcal{S}_{\lambda, S^n \omega}(\theta_n(\omega'))| \\ &\quad + |\mathcal{S}_{\lambda, S^n \omega}(\theta_n(\omega')) - \mathcal{S}_{\lambda, S^n \omega'}(\theta_n(\omega'))|, \end{aligned}$$

one deduces

$$\begin{aligned} |\theta_{n+1}(\omega) - \theta_{n+1}(\omega')| &\leq \left(\sup_{\omega, \theta} |\mathcal{S}'_{\lambda, \omega}(\theta)| \right) |\theta_n(\omega) - \theta_n(\omega')| \\ &\quad + \sup_{\theta} |\mathcal{S}_{\lambda, S^n \omega}(\theta) - \mathcal{S}_{\lambda, S^n \omega'}(\theta)|. \end{aligned} \tag{5.3}$$

Using the estimate

$$\left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\| = \left\| \frac{x - x'}{\|x\|} + x' \left(\frac{\|x'\| - \|x\|}{\|x\|\|x'\|} \right) \right\| \leq \frac{2}{\|x\|} \|x - x'\|$$

and the definition of $\mathcal{S}_{\lambda,\omega}$, it follows

$$\begin{aligned} \|e_{\mathcal{S}_{\lambda,\omega}(\theta)} - e_{\mathcal{S}_{\lambda,\omega'}(\theta)}\| &\leq \frac{2}{\|T_{\lambda,\omega}e_\theta\|} \|T_{\lambda,\omega} - T_{\lambda,\omega'}\| \\ &\leq 2 \left(\sup_{\omega} \|T_{\lambda,\omega}^{-1}\| \right) \|T_{\lambda,\omega} - T_{\lambda,\omega'}\|. \end{aligned}$$

This implies

$$\sup_{\theta} |\mathcal{S}_{\lambda,S^n\omega}(\theta) - \mathcal{S}_{\lambda,S^n\omega'}(\theta)| \leq C_1\lambda^\eta \|Q_{\lambda,S^n\omega} - Q_{\lambda,S^n\omega'}\|, \tag{5.4}$$

where C_1 is a constant and $T_{\lambda,\omega} = \mathbf{1} + \lambda^\eta Q_{\lambda,\omega}$ for a matrix-valued function $Q_{\lambda,\omega}$ that is analytic in λ^η and uniformly quasi-local for small λ (i.e., the constant and rate is λ -independent). Furthermore, one has

$$\sup_{\omega,\theta} |\mathcal{S}'_{\lambda,\omega}(\theta)| \leq 1 + C_2\lambda^\eta$$

for λ sufficiently small and some constant C_2 . Applying this and (5.4) to (5.3) one gets

$$\text{Var}_I(\theta_{n+1}) \leq (1 + C_2\lambda^\eta)\text{Var}_I(\theta_n) + C_1\lambda^\eta\text{Var}_I(Q_{\lambda,S^n\omega}).$$

Iterating this estimate and using $\text{Var}_I(\theta_0) = 0$, it follows that

$$\begin{aligned} \text{Var}_{[-m-n,n+m]}(\theta_n) &\leq \sum_{j=1}^n (1 + C_2\lambda^\eta)^{j-1} C_1\lambda^\eta \text{Var}_{[-m-n,n+m]}(Q_{\lambda,S^{n-j}\omega}) \\ &\leq C_1\lambda^\eta C_3 r^{m+1} \sum_{j=0}^\infty (1 + C_2\lambda^\eta)^j r^j = \mathcal{O}(\lambda^\eta r^m) \end{aligned}$$

for λ sufficiently small. □

In order to state the next two lemmata, we introduce the following notation extending (4.7):

$$\hat{I}_N^m(\mathcal{G}) = \frac{1}{N} \mathbf{E} \sum_{n=0}^{N-1} \mathcal{G}(S^{m+n}\omega, \theta_n(\omega)), \quad \hat{I}^m(\mathcal{G}) = \lim_{N \rightarrow \infty} \hat{I}_N^m(\mathcal{G}).$$

Lemma 5.3. *Let $g_1, g_2 \in \mathcal{Q}_r(\mathbb{C})$ and $f \in C^1(S_\pi^1)$. Furthermore let $k \geq l \geq 0$ and $m \geq 1$. Then*

$$\begin{aligned} \mathbf{E}_\omega(g_1(S^{3m+k+n}\omega)g_2(S^{3m+l+n}\omega)f(\theta_n(\omega))) \\ = \mathbf{E}(f(\theta_n))\mathbf{E}(g_1 \circ S^{k-l}g_2) + \mathcal{O}(m^{-\alpha}), \end{aligned} \tag{5.5}$$

uniformly in k, l and n . This implies, for $\mathcal{G}(\omega, \theta) = g_1(S^k\omega)g_2(S^l\omega)f(\theta)$,

$$\hat{I}^{3m}(\mathcal{G}) = \mathbf{E}((g_1 \circ S^{k-l})g_2)I(f) + \mathcal{O}(m^{-\alpha}). \tag{5.6}$$

Proof. By Lemma 5.2 and because f is Lipschitz-continuous, one has uniformly in n

$$|f(\theta_n(\omega)) - f(\theta_n(\pi_{[-m-n, n+m]}(\omega)))| \leq \mathcal{O}(\lambda^\eta r^m).$$

As g_1 and g_2 are quasi-local and therefore bounded, one also deduces uniformly in k, n and l

$$|g_1(S^{k+n+3m}\omega)g_2(S^{l+n+3m}\omega) - ((g_1 \circ S^k)(g_2 \circ S^l)) \circ S^{m+3m} \circ \pi_{[n+2m, n+k+4m]}(\omega)| \leq \mathcal{O}(r^m).$$

Let us denote the two functions inside the modulus by g and \hat{g} , respectively. Similarly denote $f \circ \theta_n \circ \pi_{[-n-m, n+m]}$ by \hat{f} . Now consider $\mathbf{E}(g(f \circ \theta_n))$. As the functions f and g are bounded, it follows from the estimates above and (2.3) that with errors of order $\mathcal{O} = \mathcal{O}(m^{-\alpha}) \geq \mathcal{O}(r^m) \geq \mathcal{O}(\lambda^\eta r^m)$ (for big m and small λ) in each step we get

$$\begin{aligned} \mathbf{E}(gf(\theta_n)) &= \mathbf{E}(\hat{g}\hat{f}(\theta_n)) + \mathcal{O} = \mathbf{E}(\hat{g}\hat{f}) + \mathcal{O} = \mathbf{E}(\hat{g})\mathbf{E}(\hat{f}) + \mathcal{O} \\ &= \mathbf{E}(g)\mathbf{E}(f(\theta_n)) + \mathcal{O}. \end{aligned}$$

This finishes the proof. □

Replacing $g_2(S^{3m+n+l}\omega)$ by $g_2(S^{l+n}\omega)$ for $0 \leq l \leq k$, one can modify the argument by grouping g_2 and f together. This gives the following

Lemma 5.4. *Let $g_1, g_2 \in \mathcal{Q}_r(\mathbb{C})$ and let $f \in C^1(S_\pi^1)$. Then one has for $0 \leq l \leq k$ and $m \geq 1$*

$$\begin{aligned} \mathbf{E}_\omega \left(g_1(S^{3m+n+k}\omega)g_2(S^{l+n}\omega)f(\theta_n(\omega)) \right) &= \mathbf{E}(g_1)\mathbf{E}_\omega \left(g_2(S^{l+n}(\omega))f(\theta_n(\omega)) \right) \\ &\quad + \mathcal{O}(m^{-\alpha}), \end{aligned}$$

uniformly in l, k and n . This implies for $\mathcal{G}(\omega, \theta) = g_1(S^{3m+k}\omega)g_2(S^l\omega)f(\theta)$

$$\hat{I}(\mathcal{G}) = \mathbf{E}(g_1)\hat{I}(g_2(S^l\omega)f(\theta)) + \mathcal{O}(m^{-\alpha}), \tag{5.7}$$

and leads, for $f = 1$ and $l = 0$, to

$$\mathbf{E}(g_1(S^{3m+k}(\omega))g_2(\omega)) = \mathbf{E}(g_1)\mathbf{E}(g_2) + \mathcal{O}(m^{-\alpha}). \tag{5.8}$$

Proof of Proposition 5.1. By Taylor expansions and $p_{1,S^{n+j}\omega}(\theta_{n+j}) = p_{1,S^{n+j}\omega}(\theta_n) + \mathcal{O}(j\lambda^\eta)$, one finds

$$f(\theta_{n+6m}) = f(\theta_n) + \lambda^\eta \sum_{j=0}^{6m-1} p_{1,S^{n+j}\omega}(\theta_n) f'(\theta_n) + \mathcal{O}(m^2\lambda^{2\eta}).$$

Therefore multiplying with $g \circ S^{6m+n}$ and averaging over ω and n gives

$$\hat{I}(\mathcal{G}) = \hat{I}^{6m}(\mathcal{G}) + \lambda^\eta \sum_{j=0}^{6m-1} \hat{I}(\mathcal{G}_j) + \mathcal{O}(m^2\lambda^{2\eta}),$$

where $\mathcal{G}_j(\omega, \theta) = g(S^{6m}\omega)p_{1,S^j\omega}(\theta)f'(\theta)$. As $p_{1,\omega}(\theta)$ is a trigonometric polynomial in θ , Lemma 5.3 can be applied to each summand in order to obtain

$$\hat{I}^{6m}(\mathcal{G}) = \mathbf{E}(g)I(f) + \mathcal{O}(m^{-\alpha}).$$

Because the functions \mathcal{G}_j are uniformly bounded, one has $\lambda^\eta \sum_{j=0}^{6m-1} \hat{I}(\mathcal{G}_j) = \mathcal{O}(m\lambda^\eta)$. Using $m = \lambda^{-\eta\frac{1}{1+\alpha}}$ now proves the first part.

Now let $\mathbf{E}(g) = 0$. Again because $p_{1,\omega}$ is a trigonometric polynomial, Lemma 5.3 gives, for $j \geq 3m$ and f_j as defined in (5.2),

$$\hat{I}(\mathcal{G}_j) = I(\mathbf{E}_\omega(g(S^{6m-j}\omega)p_{1,\omega})f') + \mathcal{O}(m^{-\alpha}) = I(f_{6m-j}) + \mathcal{O}(m^{-\alpha}).$$

Using Lemma 5.4, one obtains for $j < 3m$

$$\hat{I}(\mathcal{G}_j) = \mathbf{E}(g)\hat{I}^j(p_{1,\omega}f'(\theta)) + \mathcal{O}(m^{-\alpha}) = \mathcal{O}(m^{-\alpha}).$$

All together, one has

$$\hat{I}(\mathcal{G}) = \lambda^\eta \sum_{j=3m}^{6m-1} I(f_{6m-j}) + \mathcal{O}(m^2\lambda^{2\eta}, \lambda^\eta m^{1-\alpha}, m^{-\alpha}).$$

Because (5.8) gives

$$\begin{aligned} \sum_{j=3m+1}^\infty |f_j(\theta)| &= \sum_{j=3m+1}^\infty |\mathbf{E}_\omega(g(S^j\omega)p_{1,\omega}(\theta))f'(\theta)| \leq C \sum_{j=3m+1}^\infty j^{-\alpha} \\ &= \mathcal{O}(m^{1-\alpha}), \end{aligned}$$

one therefore deduces

$$\hat{I}(\mathcal{G}) = \lambda^\eta \sum_{j=1}^\infty I(f_j) + \mathcal{O}(m^2\lambda^{2\eta}, \lambda^\eta m^{1-\alpha}, m^{-\alpha}).$$

Finally choosing $m = \lambda^{-\frac{2\eta}{\alpha+2}}$ concludes the proof. □

6 Oscillatory sums away from band center and edges

As already explained in Section 4, for the calculation of the Lyapunov exponent one needs to evaluate the Birkhoff-like sums of functions of the type $\mathcal{G}(\omega, \theta) = g(\omega)e^{2ij\theta}$, $j = 1, 2$. This is done in Proposition 6.1 below for energies away from the band center and band edge. By applying it to the terms appearing when (4.6) is replaced in (4.5), this result allows to complete the proof of formula (2.5). As the straight-forward algebraic calculations are carried out in detail e.g., in [4, 11] and we present a similar calculation for the band edge in Section 8, we skip the details.

Proposition 6.1. *Let $\alpha > 2$. Suppose that the lowest order rotation phase k in the dynamics (4.4) satisfies $d(k) = \text{dist}(k \bmod \frac{\pi}{2}, 0) > 0$. Consider $\mathcal{G}_j(\omega, \theta) = g(\omega)e^{2ij\theta}$ with $j = 1, 2$ and $g \in \mathcal{Q}_r(\mathbb{R})$. Then*

$$\hat{I}(\mathcal{G}_j) = \mathcal{O}\left(\frac{\lambda^{\eta(\alpha/1+\alpha)}}{d(k)}\right).$$

If, moreover, $\mathbf{E}(g) = 0$,

$$\hat{I}(\mathcal{G}_1) = \lambda^\eta \sum_{j=1}^\infty \mathbf{E}_\omega(g(S^j\omega)\overline{\beta_{1,\omega}}) + \mathcal{O}\left(\frac{\lambda^{\eta(2\alpha/2+\alpha)}}{d(k)}\right). \tag{6.1}$$

Proof. ([4, 11, 14]). The dynamics and the definition of the Birkhoff sums imply $I_N(e^{2ij\theta}) = e^{2ijk}I_N(e^{2ij\theta}) + \mathcal{O}(N^{-1}, \lambda^\eta)$. This implies $I(e^{2ij\theta}) = \mathcal{O}(d(k)^{-1}\lambda^\eta)$. Therefore the modifications of (5.1) and (5.2) in Proposition 5.1 mentioned in the remark are irrelevant. The bound (5.1) thus implies the first statement. The formula (6.1) now follows after a short calculation from (5.2), the identity $p_{1,\omega}(\theta) = \Im m(\alpha_{1,\omega} - \beta_{1,\omega}e^{2i\theta})$ and the first statement. \square

7 Fokker–Planck operator for drift-diffusion

We now focus on energies for which the rotation angle k in (4.4) satisfies $k \bmod \frac{\pi}{2} = 0$ so that the argument of Proposition 6.1 does not apply in order to calculate the Birkhoff sum $I(e^{2i\theta})$. For this purpose, let us introduce the bilinear form

$$\langle g_1, g_2 \rangle_\Omega = \mathbf{E}_\omega(g_1(\omega)g_2(\omega)) + 2 \sum_{m=1}^\infty \mathbf{E}_\omega(g_1(\omega)g_2(S^m\omega)), \quad g_1, g_2 \in \mathcal{Q}_r^0(\mathbb{R}),$$

which by (5.8) is well-defined. Note that $D_V(0) = \langle V, V \rangle_\Omega$. Let us use the notation $p_j(\omega, \theta) = p_{j,\omega}(\theta)$ and $p'_j = \partial_\theta p_j$. Then expressions like $\langle p_1, p'_1 \rangle_\Omega$ are functions of θ on S^1_π .

Proposition 7.1. *Let the family $T_{\lambda,\omega}$ be as in (3.4) with $k = 0$, and $F \in C^3(S_\pi^1)$. For $f \in C^1(S_\pi^1)$ given by*

$$f = \langle p_1, p_1 \rangle_\Omega F'' + (\langle p_1, p'_1 \rangle_\Omega + 2\mathbf{E}(p_{2,\omega})) F', \tag{7.1}$$

one then has for $\alpha > 2$

$$I(f) = \mathcal{O}(\lambda^{\eta(\alpha-2/\alpha+2)}).$$

Proof. By a Taylor expansion, one has with errors of order $\mathcal{O} = \mathcal{O}(\lambda^{3\eta})$

$$\begin{aligned} F(\mathcal{S}_{\lambda,\omega}(\theta)) &= F(\theta) + \sum_{k=1}^2 \lambda^{k\eta} p_{k,\omega}(\theta) F'(\theta) \\ &\quad + \lambda^{2\eta} \frac{1}{2} [F'(\theta) p_{1,\omega}(\theta) p'_{1,\omega}(\theta) + p_{1,\omega}^2(\theta) F''(\theta)] + \mathcal{O}. \end{aligned}$$

We now use this for $\theta = \theta_n$ and average over n . Because $p_{1,\omega}$ is centered and a polynomial, one can apply equation (5.2) of Proposition 5.1 to the term with power λ^η and (5.1) to the other terms. This gives

$$\begin{aligned} I(F) &= I(F) + \frac{1}{2} \lambda^{2\eta} (I(\langle p_1, p'_1 \rangle_\Omega F') + I(\langle p_1, p_1 \rangle_\Omega F'')) \\ &\quad + 2I(\mathbf{E}_\omega(p_{2,\omega}) F') + \mathcal{O} \end{aligned}$$

with errors of order $\mathcal{O} = \mathcal{O}(\lambda^{\eta(3\alpha+2/\alpha+2)})$. As the functional I is linear, resolving this equation for $I(f)$ gives the desired result. \square

This proposition shows that we can control error terms on Birkhoff sums for a function f , if f is in the image of the operator \mathcal{L} on functions on S_π^1 given by

$$\mathcal{L} = (p\partial_\theta + q)\partial_\theta, \quad p = \langle p_1, p_1 \rangle_\Omega, \quad q = \langle p_1, p'_1 \rangle_\Omega + 2\mathbf{E}(p_{2,\omega}). \tag{7.2}$$

As one needs to calculate Birkhoff sums $I(f)$ perturbatively, we are looking for some class of functions where $\lim_{\lambda \rightarrow 0} I(f)$ exists. For f in the image under \mathcal{L} of $C^3(S_\pi^1)$, this limit is 0. Thus, if this map is given by the scalar product with some L^2 -function ρ , one has $\rho \in \text{Ran}(\mathcal{L})^\perp = \text{Ker}(\mathcal{L}^*)$, where the formal adjoint is given by

$$\mathcal{L}^* = \partial_\theta(\partial_\theta p - q).$$

\mathcal{L}^* is a forward Kolmogorov or Fokker–Planck operator describing the drift-diffusion dynamics of the process θ_n on S_π^1 , and \mathcal{L} is the associated backward Kolmogorov operator [15]. It will be shown that in the situations considered here, $\text{Ker}(\mathcal{L}^*)$ is spanned by a smooth, L^1 -normalized function ρ . Furthermore, the following theorem shows that $f \in \text{Ker}(\mathcal{L}^*)^\perp \cap C^2(S_\pi^1)$ turns out to be sufficient for finding a solution $F \in C^3(S_\pi^1)$ of the differential equation

(7.1) so that Proposition 7.1 actually applies. Even though contained in [17], let us give the proof for the sake of completeness.

Theorem 7.2. *Suppose that $p(\hat{\theta}) = 0$ for at most one angle $\hat{\theta} \in S^1_\pi$. Furthermore suppose $q(\hat{\theta}) \neq 0$ in that case. Then the Fokker–Planck operator \mathcal{L}^* has a unique groundstate $\rho \in C^\infty(S^1_\pi)$, which is non-negative and normalized. Furthermore, for $f \in C^2(S^1_\pi)$, one has*

$$I(f) = \int_0^\pi d\theta \rho(\theta) f(\theta) + \mathcal{O}(\lambda^{\eta(\alpha-2/\alpha+2)}).$$

Proof. Integrating the equation $\mathcal{L}^* \rho = 0$ once gives

$$(p\partial_\theta + (\partial_\theta p) - q)\rho = C, \tag{7.3}$$

where C is some real constant. As $I(f + c) = c + I(f)$ for $c = \langle \rho, f \rangle$, we may assume $\int_0^\pi d\theta \rho(\theta) f(\theta) = 0$ once we found the normalized solution of (7.3). Proposition 7.1 then gives the bound on $I(f)$ if one finds a solution $G \in C^2(S^1_\pi)$ of

$$(p\partial_\theta + q)G = f, \quad \int_0^\pi d\theta G(\theta) = 0. \tag{7.4}$$

First let us consider the case $p > 0$. Then there is no singularity and \mathcal{L}^* is elliptic. The groundstate ρ and the function G can be calculated. For some $\tilde{\theta}$ set

$$w(\theta) = \int_{\tilde{\theta}}^\theta d\xi \frac{q(\xi)}{p(\xi)}, \quad W(\theta) = \int_{\tilde{\theta}}^\theta d\xi \frac{e^{w(\xi)}}{p(\xi)} f(\xi), \quad \tilde{W}(\theta) = \int_{\tilde{\theta}}^\theta d\xi e^{-w(\xi)}. \tag{7.5}$$

Then

$$\rho = C_1 \frac{e^w}{p} (C_2 \tilde{W} + 1), \quad G = e^{-w} (W + C_3), \tag{7.6}$$

where C_2 is fixed by the condition that ρ is π -periodic and $C_1 > 0$ is a normalization constant. This fixes $C = C_1 C_2$ in (7.3). G is a solution of the first equation of (7.4) and for $C \neq 0$ the constant C_3 is fixed by the condition that G is π -periodic. Furthermore one has

$$0 = \int \rho f = \int \rho (p\partial_\theta + q)G = - \int G(\partial_\theta p - q)\rho = -C \int G(\theta). \tag{7.7}$$

Thus G is a solution of (7.4). If $C = 0 \Leftrightarrow C_2 = 0$, then w is π -periodic as well as W which follows from $\int \rho f = 0$. Therefore G is π -periodic and C_3 is chosen such that the integral in (7.4) vanishes.

Now let $p(\hat{\theta}) = 0$ for exactly one $\hat{\theta} \in S^1_\pi$ and for sake of concreteness let $q(\hat{\theta}) > 0$ which implies $\tilde{q}(\hat{\theta}) > 0$. Then choose $\tilde{\theta} \in (\hat{\theta}, \hat{\theta} + \pi)$ in the first equation of (7.5), $\tilde{\theta} = \hat{\theta}$ in the second one and $\tilde{\theta} = \hat{\theta} + \pi$ in the third one. As $\lim_{\theta \downarrow \hat{\theta}} e^{w(\theta)} = 0$ and $\lim_{\theta \uparrow \hat{\theta} + \pi} e^{w(\theta)} = \infty$ in this case, w , W and \tilde{W} are well

defined for $\theta \in (\hat{\theta}, \hat{\theta} + \pi)$. Using de l'Hospital's rule, one can prove by induction (see [17] for details) that

$$\rho = C \frac{e^w}{p} \tilde{W}, \quad G = e^{-w} W$$

can both be continued to a smooth (even at $\hat{\theta}$) and π -periodic function. $C > 0$ is again a normalization constant and hence equation (7.7) shows that G solves (7.4). □

Before applying this result in order to prove Theorem 2.1, let us present another derivation of the equation $\mathcal{L}^* \rho = 0$, albeit a formal one, which shows that ρ is the lowest order approximation for the asymptotic invariant measure of the process θ_n . Expanding the function $\mathcal{S}_{\lambda, \omega}^N = \mathcal{S}_{\lambda, S^{N-1}\omega} \circ \dots \circ \mathcal{S}_{\lambda, S\omega} \circ \mathcal{S}_{\lambda, \omega}$ shows that the coefficients of

$$\mathcal{S}_{\lambda, \omega}^N(\theta) = \theta + \lambda^\eta \hat{p}_\omega^N(\theta) + \frac{1}{2} \lambda^{2\eta} \hat{q}_\omega^N(\theta) + \mathcal{O}(\lambda^{3\eta})$$

are

$$\hat{p}_\omega^N = \sum_{n=0}^{N-1} p_{1, S^n \omega}, \quad \hat{q}_\omega^N = \sum_{n=0}^{N-1} \left(p_{1, S^n \omega} + \sum_{j=0}^{n-1} p_{1, S^j \omega} \right) p'_{1, S^n \omega} + 2 \sum_{n=0}^{N-1} p_{2, S^n \omega}.$$

An invariant measure $\nu_{\lambda, N}$ for N steps of the dynamics θ_n on S_π^1 satisfies

$$\int_0^\pi \nu_{\lambda, N}(d\theta) f(\theta) = \mathbf{E} \int_0^\pi \nu_{\lambda, N}(d\theta) f(\mathcal{S}_{\lambda, S^{N-1}\omega}^N(\theta)), \quad f \in C(S_\pi^1). \quad (7.8)$$

Supposing $\nu_{\lambda, N}(d\theta) = \rho_{\lambda, N}(\theta) d\theta = \rho_N(\theta) d\theta + o(\lambda^0)$, (7.8) leads to

$$\mathcal{L}_N^* \rho_N = 0, \quad \mathcal{L}_N^* = \partial_\theta (\partial_\theta \mathbf{E}((\hat{p}_{1, \omega}^N)^2) - \mathbf{E}(\hat{q}_\omega^N)).$$

Using the stationarity of \mathbf{P} and the definitions of \hat{p}_ω^N and \hat{q}_ω^N , one deduces

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}((\hat{p}_{1, \omega}^N)^2) = p, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}(\hat{q}_\omega^N) = q,$$

where the convergences are uniform in θ . This shows that $\frac{1}{N} \mathcal{L}_N^* \rightarrow \mathcal{L}^*$ weakly for $N \rightarrow \infty$.

8 Application to the band center and band edge

This section contains the proof of Theorem 2.1. Let us first consider item (i), that is the band center. As described in Section 3 we have to work with the probability space $\tilde{\Omega} = (\Sigma \times \Sigma)^\mathbb{Z}$ which is isomorphic to Ω by the

pairing isomorphism \mathcal{P} . Using this isomorphism and the potential V , which is defined on Ω , let us define the two random variables on $\tilde{\Omega}$

$$v_{\tilde{\omega}} = V(\mathcal{P}^{-1}(\tilde{\omega})) = V(\omega), \quad u_{\tilde{\omega}} = V(S\mathcal{P}^{-1}(\tilde{\omega})) = V(S\omega).$$

Then according to equation (3.2) the family of matrices we have to consider is given by

$$T_{\lambda, \tilde{\omega}} = -\exp \left[\lambda \begin{pmatrix} 0 & -u_{\tilde{\omega}} \\ v_{\tilde{\omega}} & 0 \end{pmatrix} + \frac{\lambda^2}{2} \begin{pmatrix} -u_{\tilde{\omega}}v_{\tilde{\omega}} & 2\epsilon \\ -2\epsilon & u_{\tilde{\omega}}v_{\tilde{\omega}} \end{pmatrix} + \mathcal{O}(\lambda^3) \right].$$

In this situation one has $\alpha_{1, \tilde{\omega}} = i(v_{\tilde{\omega}} + u_{\tilde{\omega}})/2$, $\beta_{1, \tilde{\omega}} = i(u_{\tilde{\omega}} - v_{\tilde{\omega}})/2$, $\alpha_{2, \tilde{\omega}} = -i\epsilon$ and $\beta_{2, \tilde{\omega}} = -\frac{1}{2}u_{\tilde{\omega}}v_{\tilde{\omega}}$. Using Lemma 4.1 and $\langle v - u, v - u \rangle_{\tilde{\Omega}} = 2D_V(\pi)$ and $\langle v + u, v + u \rangle_{\tilde{\Omega}} = 2D_V(0)$, one obtains that the polynomials (7.2) are explicitly given by

$$p(\theta) = \frac{1}{2}D_V(0) + \frac{1}{2}D_V(\pi) \cos^2(2\theta), \quad q(\theta) = -\frac{1}{2}D_V(\pi) \sin(4\theta) - \epsilon.$$

By assumption on V , one has $p > 0$ uniformly on S^1_π . By Theorem 7.2 there is thus a smooth, positive and L^1 -normalized groundstate ρ_ϵ for the operator \mathcal{L}^* (which can readily be written out). Furthermore, one checks $\gamma_{1, \tilde{\omega}} = (v_{\tilde{\omega}}^2 - u_{\tilde{\omega}}^2)/2$. Then equation (4.6), Theorem 4.2 and Proposition 5.1 combined with some algebra leads to (2.6) for $\gamma_\lambda(\epsilon\lambda^2) = \frac{1}{2}\gamma(\lambda)$.

Now let us prove Theorem 2.1(ii). Hence let $T_{\lambda, \omega} = N_\lambda N \mathcal{T}_{\lambda, \omega}^{-2+\epsilon\lambda^2} N^{-1} N_\lambda^{-1}$ be the anomaly given in (3.3). As $\alpha_{1, \omega} = iV(\omega)/2$, $\beta_{1, \omega} = -iV(\omega)/2$, $\alpha_{2, \omega} = -i(\epsilon + 1)/2$ and $\beta_{2, \omega} = i(\epsilon - 1)/2$, one deduces, using $\langle V, V \rangle_\Omega = D_V(0)$,

$$p(\theta) = D_V(0) \cos^4(\theta), \quad q(\theta) = -\epsilon - 1 + (1 - \epsilon) \cos(2\theta) - 2D_V(0) \cos^3(\theta) \sin(\theta).$$

By assumption on V one has $p(\theta) > 0$ for $\theta \notin \frac{\pi}{2}$, and as $q(\frac{\pi}{2}) = -2 \neq 0$, there is a unique groundstate $\rho_\epsilon \in C^\infty(S^1_\pi)$ by Theorem 7.2. Explicitly, one obtains

$$\begin{aligned} \rho_\epsilon(\theta) &= C \int_{-(\pi/2)}^\theta d\xi \frac{\cos^2(\xi)}{\cos^6(\theta)} \\ &\times \exp \left(\frac{2}{3D_V(0)} (\tan^3(\xi) - \tan^3(\theta) + 3\epsilon \tan(\xi) - 3\epsilon \tan(\theta)) \right), \end{aligned} \tag{8.1}$$

where C is some normalization constant. Furthermore, one checks $\gamma_{1, \omega} = V(\omega)^2/2$ and hence (4.6), Proposition 5.1 and Theorem 7.2 imply (2.7).

9 Bound on the quantum dynamics

As already said above, the proof of Theorem 2.2 follows exactly the proof of [10], Theorem 1 given in Sections 3 and 4 therein, except that the proof of [10], Lemma 4 has to be refined in order to deal with strong mixing (2.3) instead of independent potential values $V(S^n\omega)$. The conclusion of the following lemma is hence exactly the same as of [10], Lemma 4 and we thereby consider the proof of Theorem 2.2 to be complete.

Let us set $U = \{E \in \mathbb{C} \mid E_0 \leq \Re e(E) \leq E_1, |\Im m(E)| \leq 1\}$. Furthermore introduce the transfer matrices over several sites:

$$\mathcal{T}_{\lambda,\omega}^E(k, m) = \prod_{n=m}^{k-1} \mathcal{T}_{\lambda,S^n\omega}^E, \quad k > m,$$

Furthermore, $\mathcal{T}_{\lambda,\omega}^E(k, m) = (\mathcal{T}_{\lambda,\omega}^E(m, k))^{-1}$ for $k < m$ and $\mathcal{T}_{\lambda,\omega}^E(m, m) = \mathbf{1}$.

Lemma 9.1. *Let $E \in U$ and $N \in \mathbb{N}$. Then there is a constant \hat{C} such that the set*

$$\hat{\Omega}_N(E) = \left\{ \omega \in \Omega \mid \max_{0 \leq n \leq N} \|\mathcal{T}_{\lambda,\omega}^E(n, 1)\|^2 \geq e^{\hat{C}N^{1/2}} \right\}$$

satisfies

$$\mathbf{P}(\hat{\Omega}_N(E)) \geq 1 - e^{-\hat{C}N^{1/2}}.$$

Proof. For sake of notational simplicity, we will drop the index λ on the transfer matrices $\mathcal{T}_{\lambda,\omega}^E$. Let us fix $E \in U$ and $N \in \mathbb{N}$ and then split N into $\frac{N}{N_3}$ pieces of length $N_3 = N_0 + N_1 + 2N_2$. For $j = 0, \dots, \frac{N}{N_3}$, we consider the following events:

$$\begin{aligned} \Omega_j^0 &= \left\{ \omega \in \Omega \mid \|\mathcal{T}_\omega^E(jN_3 + N_0, jN_3)\| \leq e^{1/2\gamma_0 N_0} \right\}, \\ \Omega_j^1 &= \left\{ \omega \in \Omega \mid \|\mathcal{T}_{\pi_{[jN_3 - N_2, N_3j + N_0 + N_2]}(\omega)}^E(jN_3 + N_0, jN_3)\| \leq e^{2/3\gamma_0 N_0} \right\}, \\ \Omega_j^2 &= \left\{ \omega \in \Omega \mid \|\mathcal{T}_\omega^E(jN_3 + N_0, jN_3)\| \leq e^{3/4\gamma_0 N_0} \right\}. \end{aligned}$$

First we note that uniformly in ω and for some $\gamma_1 > 0$

$$\|\mathcal{T}_\omega^E(n, m)\| \leq e^{\gamma_1|n-m|}.$$

Therefore the hypothesis (2.11) implies as in the proof of [10], Lemma 3 that, for $E \in U$ and $N_0 \in \mathbb{N}$, we have

$$\mathbf{P}(\Omega_j^2) \leq 1 - p_0 < 1, \quad p_0 > 0. \tag{9.1}$$

To shorten notations let us define $\pi_j = \pi_{[jN_3 - N_2, jN_3 + N_0 + N_2]}$ and $\mathcal{T}_{\omega, j}^E = \mathcal{T}_{\omega}^E(N_3j + N_0, N_3j)$. Using the quasi-locality of $g(\omega) = \mathcal{T}_{\omega}^E$ we get

$$\begin{aligned} & \left\| \mathcal{T}_{\omega, j}^E - \mathcal{T}_{\pi_j(\omega), j}^E \right\| \\ &= \left\| \sum_{l=jN_3}^{jN_3 + N_0 - 1} \left(\prod_{k=jN_3}^{l-1} \mathcal{T}_{S^k \omega}^E \right) \left[\mathcal{T}_{S^l \omega}^E - \mathcal{T}_{S^l \pi_j(\omega)}^E \right] \left(\prod_{k=l+1}^{jN_3 + N_0 - 1} \mathcal{T}_{S^k \pi_j(\omega)}^E \right) \right\| \\ &\leq N_0 \left(\sup_{\omega} (\mathcal{T}_{\omega}^E) \right)^{N_0 - 1} Cr^{N_2}, \end{aligned}$$

where $C = C(g)$ as in (2.1). Now choosing $N_2 = cN_0$ for an adequate constant c , it follows that

$$\left\| \mathcal{T}_{\omega, j}^E - \mathcal{T}_{\pi_j(\omega), j}^E \right\| \leq e^{1/2\gamma_0 N_0}.$$

Therefore for $\omega \in \Omega_j^0$

$$\left\| \mathcal{T}_{\pi_j(\omega), j}^E \right\| \leq \left\| \mathcal{T}_{\omega, j}^E \right\| + e^{1/2\gamma_0 N_0} \leq 2e^{1/2\gamma_0 N_0} \leq e^{2/3\gamma_0 N_0}$$

for N_0 large enough, implying $\Omega_j^0 \subset \Omega_j^1$. By a similar calculation, one obtains the second inclusion of

$$\Omega_j^0 \subset \Omega_j^1 \subset \Omega_j^2. \tag{9.2}$$

By (9.1) this implies

$$\mathbf{P}(\Omega_j^1) \leq \mathbf{P}(\Omega_j^2) \leq 1 - p_0.$$

Now clearly Ω_j^1 is $\pi_j = \pi_{[jN_3 - N_2, jN_3 + N_0 + N_2]}$ -measurable. Therefore the strong mixing condition (2.2) implies that $\mathbf{P}(\Omega_0^1 \cap \Omega_1^1) \leq \mathbf{P}(\Omega_0^1)\mathbf{P}(\Omega_1^1)(1 + CN_1^{-\alpha}) \leq (1 - p_0)^2(1 + CN_1^{-\alpha})$. At the next step, one obtains $\mathbf{P}(\Omega_0^1 \cap \Omega_1^1 \cap \Omega_2^1) \leq (1 - p_0)^3(1 + CN_1^{-\alpha})^2$. Iteration and (9.2) therefore give

$$\mathbf{P} \left(\bigcap_{j=0, \dots, N/N_3} \Omega_j^0 \right) \leq \mathbf{P} \left(\bigcap_{j=0, \dots, N/N_3} \Omega_j^1 \right) \leq ((1 - p_0)(1 + CN_1^{-\alpha}))^{N/N_3}.$$

Now let us choose N_1 sufficiently large such that $1 - p_1 = (1 - p_0)(1 + CN_1^{-\alpha}) < 1$. Then

$$\mathbf{P} \left(\left\{ \omega \in \Omega \mid \max_{0 \leq j \leq N/N_3} \left\| \mathcal{T}_{\omega}^E(jN_3 + N_0, jN_3) \right\|^2 \leq e^{\gamma_0 N_0} \right\} \right) \leq (1 - p_1)^{N/N_3}.$$

Furthermore $\mathcal{T}_{\omega}^E(jN_3 + N_0, jN_3) = \mathcal{T}_{\omega}^E(jN_3 + N_0, 1)\mathcal{T}_{\omega}^E(jN_3, 1)^{-1}$. As $A = BC$ implies either $\|B\| \geq \|A\|^{1/2}$ or $\|C\| \geq \|A\|^{1/2}$ for arbitrary matrices,

and $\|A^{-1}\| = \|A\|$ for $A \in \mathrm{SL}(2, \mathbb{C})$, it therefore follows that

$$\mathbf{P} \left(\left\{ \omega \in \Omega \left| \max_{0 \leq j \leq N/N_3} \max \left\{ \|\mathcal{T}_\omega^E(jN_3, 1)\|^2, \|\mathcal{T}_\omega^E(jN_3 + N_0, 1)\|^2 \right\} \geq e^{1/2\gamma_0 N_0} \right. \right\} \right)$$

is greater or equal than $1 - (1 - p_1)^{N/N_3}$. Choosing $N_0 = cN^{1/2}$ with adequate c concludes the proof. \square

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