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Heterotic Kähler/non-Kähler transitions

Melanie Becker[1,2], Li-Sheng Tseng[2,3] and Shing-Tung Yau[3]

[1]George P. and Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843, USA
[2]Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA
[3]Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

Abstract

We show how two topologically distinct spaces — the Kähler $K3 \times T^2$ and the non-Kähler T^2 bundle over $K3$ — can be smoothly connected in heterotic string theory. The transition occurs when the base $K3$ is deformed to the $T^4/\mathbb{Z}_2$ orbifold limit. The orbifold theory can be mapped via duality to M-theory on $K3 \times K3$ where the transition corresponds to an exchange of the two $K3$'s.

1 Introduction

Background geometry affects strings and point particles very differently. From the many well-studied string dualities, we know that string theories on different geometrical spaces can be dual, that is, identical up to some

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identification. Moreover, string theory can also smoothly connect topologically different spaces. Some of these string transitions, for example, those of flops [1–3] and conifolds [4–7], have geometrical origins and are closely related to the mathematics of singularity resolutions.

It is conceivable that many vacua in the string theory landscape are connected by transitions and one may wonder if the more recently studied manifolds with torsion can be connected to more conventional string theory compactifications. Many years ago, it was conjectured [8] that Calabi–Yau manifolds can indeed be connected via transitions through non-Kähler manifolds [9, 10] (see also [11]). Kähler/non-Kähler transitions have also been described recently from the worldsheet conformal field theory point of view in [12] in the context of gauged linear sigma models [13].

In this paper, we take the space–time approach to explore transitions between Kähler and non-Kähler manifolds in the context of flux compactifications of heterotic string theory.[1] In the heterotic theory, there can be two types of fluxes — the gauge 2-form F_{MN} and the 3-form H_{MNP}. Preserving supersymmetry in four dimensions will constrain both the compactification geometry and the fluxes. It is the goal of this paper to relate two different types of spaces both of which are locally $K3 \times T^2$. The first is the geometry of the $K3 \times T^2$ manifold with non-zero $U(1)$ gauge fields. The second is the non-Kähler geometry of a T^2 bundle over $K3$, the FSY (Fu–Strominger–Yau) geometry.

This paper is organized as follows. In Section 2, we analyze the conditions under which a Kummer surface can be blown down to a $T^4/\mathbb{Z}_2$ orbifold in the presence of fluxes while maintaining supersymmetry throughout. In Section 3, we show that a transition between the Kähler geometry $K3 \times T^2$ and the non-Kähler FSY geometry can take place using the mapping to M-theory on $K3 \times K3$ where the transition corresponds to an exchange of the two $K3$'s. In Section 4, we present our conclusions. In an appendix, we work out the conditions for the FSY geometry to preserve $N = 2$ supersymmetry, which is necessary in order that a smooth transition to the $K3 \times T^2$ geometry can take place.

2 $N = 2$ heterotic compactifications

We are interested in supersymmetric compactifications to four dimensions in heterotic string theory. The conditions on the hermitian form J and the

[1] For non-compact geometries, non-Kähler to non-Kähler transitions in heterotic strings have been discussed in [14].

holomorphic (3,0)-form Ω are[2]

$$d(\|\Omega\|_J\, J \wedge J) = 0, \tag{2.1}$$

$$F^{(2,0)} = F^{(0,2)} = 0, \quad F_{mn}J^{mn} = 0, \tag{2.2}$$

$$2i\,\partial\bar{\partial}J = \frac{\alpha'}{4}[\mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F)]. \tag{2.3}$$

The standard background fields — the 3-form H and the dilaton ϕ — are determined from the supersymmetric constraints

$$H = i(\bar{\partial} - \partial)J, \tag{2.4}$$

$$\|\Omega\|_J = \mathrm{e}^{-2(\phi+\phi_0)}. \tag{2.5}$$

We focus on two special classes of solutions. The first is the $K3 \times T^2$ solution [16]. The hermitian metric and the holomorphic 3-form is taken to be

$$J = \mathrm{e}^{2\phi}J_{K3} + \frac{i}{2}dz \wedge d\bar{z}, \quad \Omega = \Omega_{K3} \wedge dz. \tag{2.6}$$

In general, ϕ is non-constant and has dependence on the $K3$ coordinates. By (2.4), this gives a contribution to the H-field. The second is the torus bundle over $K3$ solution [15, 17, 18] with the torus twisted with respect to the $K3$ base. The metric and the 3-form are generalized to

$$J = \mathrm{e}^{-2\phi}J_{K3} + \frac{i}{2}(dz+\alpha) \wedge (d\bar{z}+\bar{\alpha}), \quad \Omega = \Omega_{K3} \wedge \theta, \tag{2.7}$$

where $\theta = (dz + \alpha)$ is a globally defined (1, 0)-form. We shall only consider the case where the curvature of the torus bundle $\omega = \omega_1 + i\omega_2 = d\theta = d\alpha \in H^2(K3, \mathbb{Z})$ is a (1,1)-form; that is,

$$\omega^{(2,0)} = \omega^{(0,2)} = \omega_{mn}J_{K3}{}^{mn} = 0. \tag{2.8}$$

The H-field now has contributions from both the derivative of ϕ and also ω.

To fully describe both solutions, we have to specify the gauge bundle. In addition to being hermitian Yang–Mills (2.2), the gauge bundle must satisfy the anomaly equation (2.3). Integrating the anomaly equation over $K3$ leads to the topological condition

$$\frac{1}{16\pi^2}\int_{K3} \mathrm{tr}\, F \wedge F - \int_{K3} \omega \wedge \bar{\omega} = 24, \tag{2.9}$$

[2]We mostly follow the notation and conventions of [15]. In this paper, $F = F^aT^a$ is taken to be anti-hermitian with $\mathrm{tr}\, T^aT^b = -\delta^{ab}$. For convenience, we shall subsequently also set $2\pi\sqrt{\alpha'} = 1$.

where $\bar{\omega} = \omega_1 - i\omega_2$ is the complex conjugate of ω. It turns out that this condition is sufficient to guarantee that the anomaly equation, which for these geometries can be interpreted as a highly non-linear second-order partial differential equation for ϕ, can be solved [18, 15]. Note that by (2.2) and (2.8), both F and ω are anti-self-dual and therefore the left hand side of (2.9) is positive semi-definite. As a simplification, we will consider only direct sums of $U(1)$ gauge bundles. Dirac quantization requires that $\frac{i}{2\pi}F^a \in H^2(K3, \mathbb{Z})$. Considering also (2.2) and (2.8), the field strength F and ω indeed satisfy identical equations. This is suggestive that the gauge bundle and torus bundle under appropriate conditions might perhaps be interchangeable. In general, this is not the case. But we shall show in Section 3 that different values for the pair (F, ω) can be smoothly connected.

We point out that the $K3 \times T^2$ solution preserves $N = 2$ SUSY in four dimensions. Likewise, with $\omega \in H^{(1,1)}(K3, \mathbb{Z})$, the FSY geometry also preserves $N = 2$ SUSY. These two solutions must preserve the same amount of supersymmetry if we desire a smooth transition between them. It is worthwhile to emphasize that the conditions for spacetime supersymmetry do allow for the presence of a (2, 0) component for ω. However, the resulting four-dimensional supersymmetry would then be reduced down to $N = 1$. A discussion of the supersymmetry of the FSY geometry is provided in Appendix A.

2.1 Deforming to the orbifold limit of $K3$

We will take the $K3$ surface S to be a Kummer surface. This can be described as the blow-up of $T^4/\mathbb{Z}_2$ at all 16 fixed points. Here, we want to find the conditions under which a Kummer surface can be blown down to a $T^4/\mathbb{Z}_2$ orbifold in the presence of fluxes while maintaining supersymmetry throughout. Keeping the complex structure fixed, the supersymmetry variation conditions with non-zero fluxes were worked out in [19]. For ease of presentation, we describe below the equivalent reverse process of blowing up the fixed points for a given flux.

Let $C_i \in H_2(S, \mathbb{Z})$, $i = 1, \ldots, 16$, be a basis for the 16 blown up (-2)-curves of the Kummer surface.[3] We let $\beta_i \in H^2(S, \mathbb{Z})$ be the associated dual 2-forms. Since these rational curves are localized and thus disjoint, the matrix of intersection numbers is

$$C_i \cdot C_j = \int_S \beta_i \wedge \beta_j = -2\,\delta_{ij}. \tag{2.10}$$

[3]For a review of the mathematical aspects of $K3$ surfaces, see [20].

Note that this intersection matrix is different from that for the standard basis of $H_2(S, \mathbb{Z})$, which is given by

$$(-E_8) \otimes (-E_8) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.11}$$

where E_8 denotes the Cartan matrix of the Lie algebra E_8. Though the 16 C_i's together with the six 2-cycles of $T^4/\mathbb{Z}_2$ provide a natural set of elements in $H_2(S, \mathbb{Z})$ for the Kummer surface, this set as constituting a basis turns out to only generate a sublattice of the full $H_2(S, \mathbb{Z})$ lattice (see, for example, [21]). This "Kummer" basis however can be used as a basis for $H_2(S, \mathbf{Q})$.

Proceeding on, the area of the ith-rational curve is given by[4]

$$A_i = \int_{C_i} J = \int_S J \wedge \beta_i, \tag{2.12}$$

where we have used the dual relation associating $C_i \sim \beta_i$. In the orbifold limit, each rational curve shrinks to a point and thus $A_i = 0$. To deform away from the orbifold limit, we want to deform J such that $\delta A_i > 0$. That is, we need

$$\int_{C_i} J + \delta J = \int_{C_i} \delta J = \int_S \delta J \wedge \beta_i > 0. \tag{2.13}$$

Furthermore, to preserve supersymmetry in the presence of fluxes, i.e., non-zero torus bundle and/or gauge bundle curvature, we have to satisfy the additional conditions [19]

$$\int_S \delta J \wedge \omega = 0, \qquad \int_S \delta J \wedge F = 0. \tag{2.14}$$

These arise from varying the primitivity condition for the curvatures ω and F. Let us focus below on the torus bundle curvature as the conditions for gauge bundle are identical. Varying $\omega \wedge J = 0$ implies

$$\begin{aligned} 0 &= \delta\omega \wedge J + \omega \wedge J \\ &= i\partial\bar{\partial} f \wedge J + \omega \wedge \delta J, \end{aligned} \tag{2.15}$$

where $\delta\omega = i\partial\bar{\partial} f$ imposes that ω can only vary in its cohomology class. Taking the hodge star of (2.15) results in $\Delta f = *(\omega \wedge \delta J)$. This then implies the integral condition in (2.14), which is the necessary and sufficient condition that a solution for f exists.

[4]We will take $J = J_{K3}$ in this subsection since we are only interested here in deforming the $K3$.

It is not difficult to satisfy both (2.13) and (2.14). Using the β_i basis, let $\delta J = a^j \beta_j$ and $\omega = b^i \beta_i$.[5] Then, we have

$$\int_{C_i} \delta J = \int_S a^j \beta_j \wedge \beta_i = -2\, a_i = \delta A_i > 0, \tag{2.16}$$

$$\int_S \delta J \wedge \omega = \int_S a^j \beta_j \wedge b^i \beta_i = -2\, a_i b^i = \delta A_i b^i = 0, \tag{2.17}$$

where $\delta A_i = -2\, a_i$. Thus, as long as two of the b^i's are non-zero and also not of the same signs, there exist positive δA_i's as required for (2.16) that satisfy (2.17). This is the condition for ω (and similarly for F) that ensures that the singularities can be blown up.

As a simple example, let ω for the FSY model be given by

$$\omega = \omega_1 + i\omega_2 = (1+i)\beta_1 + (1+i)\beta_2 - 2(1+i)\beta_3, \tag{2.18}$$

and $F = 0$. With (2.10), this model satisfies that anomaly condition (2.9)

$$-\int_S \omega \wedge \bar{\omega} = -\int_S 2(\beta_1 \wedge \beta_1 + \beta_2 \wedge \beta_2) + 8\, \beta_3 \wedge \beta_3 = 24. \tag{2.19}$$

It can be easily checked that (2.16) and (2.17) are satisfied for $\delta A_1 = \delta A_2 = \delta A_3 = a > 0$, where a is an arbitrary positive constant. $\delta A_i > 0$ for $i = 4, \ldots, 16$ are not constrained. We shall show in the next section that this FSY model at the orbifold limit $K3 = T^4/\mathbb{Z}_2$ is smoothly connected to the $K3 \times T^2$ model with non-zero $U(1)$ gauge field strengths

$$\frac{F^1}{2\pi} = \frac{F^2}{2\pi} = -\frac{F^3}{4\pi} = (1+i)\, d\bar{z}_1 \wedge dz_2 + (1-i)\, dz_1 \wedge d\bar{z}_2, \tag{2.20}$$

where the superscript index in F^i, $i = 1, \ldots, 16$, denotes the 16 $U(1)$ gauge field strengths and (z_1, z_2) are the complex coordinates on $T^4/\mathbb{Z}_2$.

3 Duality at the orbifold point

At the $T^4/\mathbb{Z}_2$ orbifold of $K3$, we can map the heterotic solutions to those of M-theory on $Y = K3 \times K3$, where each $K3$ is a $T^4/\mathbb{Z}_2$ orbifold. Let us recall the chain of dualities [22, 23]. Starting from M-theory compactified on Y, we can treat the second $T^4/\mathbb{Z}_2$ orbifold as a torus fibration over $T^2/\mathbb{Z}_2$. Taking the area of the torus fiber to zero, we arrive at the type IIB theory on an orientifold $T^4/\mathbb{Z}_2 \times T^2/\mathbb{Z}_2$, where at each of the four fixed points of $T^2/\mathbb{Z}_2$, there are four D7 branes and one O7 brane. Such a brane configuration gives an SO(8)^4 enhanced gauge symmetry[24]. Now applying a T-duality along

[5]Here, we have suppressed the non-relevant terms associated with the six non-localized (1,1)-forms on $T^4/\mathbb{Z}_2$ which are orthogonal to the localized forms β_i.

the two directions of $T^2/\mathbb{Z}_2$ and then followed by an S-duality, we obtain the heterotic theory on $K3 \times T^2$ with $\mathrm{SO}(8)^4$ gauge group [23].

The duality can incorporate a non-zero 4-form G-flux. To preserve supersymmetry, the G-flux in M-theory is required to be a primitive $(2,2)$-form. The G-flux is also quantized so that $G \in H^4(Y, \mathbb{Z})$. Additionally, it must satisfy the constraint (assuming no M2-branes) [25, 26]

$$\frac{1}{2}\int_Y G \wedge G = \frac{\chi(Y)}{24}. \tag{3.1}$$

We shall take G to be the exterior product of two (1,1)-forms, one from each $K3$. In the orbifold limit, there are 19 primitive (1,1)-forms — 16 localized at each of the fixed points and three non-localized ones. They constitute the orthogonal Kummer basis, $\{\beta_i, \gamma_I\}$ with $i = 1, \ldots, 16$, $I = 1, 2, 3$, and normalized to -2, for the primitive forms in $H^{(1,1)}(S)$. Thus, the most general G-flux that we consider takes the form

$$G = C_{ij}\,\beta_i \wedge \beta'_j + C_{Ij}\,\gamma_I \wedge \beta'_j + D_{iJ}\,\beta_i \wedge \gamma'_J + D_{IJ}\,\gamma_I \wedge \gamma'_J, \tag{3.2}$$

where we have placed primes to denote forms from the second $K3$ and $C_{ij}, C_{Ij}, D_{iJ}, D_{IJ}$ are integer constants suitably chosen such that G is integral quantized and satisfies (3.1). The four different terms dualize to different types of fluxes in the heterotic theory. Let us fix our convention by performing the duality operations always on the second $K3$. Then, the D_{IJ} and D_{iJ} terms with non-localized (1,1)-forms γ'_J dualize to give a non-zero torus curvature ω. In contrast, C_{ij} and C_{Ij} terms with localized β'_j dualize to non-zero heterotic field strengths F^j. The case of $D_{IJ} \neq 0$ in particular was discussed in detail in [17, 22, 23, 27].

Of interest for us is the exchange of the two $K3$'s. What we call the first or second $K3$ is certainly inconsequential for the physical theory. But for the G-flux, such an exchange can interchange the different terms and result in a different heterotic dual theory when the "second" $K3$ is dualized. The set of C_{ij} terms and also that of the D_{IJ} terms map to itself under interchange. However, for the other two sets of terms, we have $D_{iJ} \to C_{Ji}$ and $C_{Ij} \to D_{jI}$, which implies that two types of heterotic fluxes are exchanged under interchange of the two $K3$'s. Hence, we shall focus below on fluxes of types C_{Ij} and D_{iJ}.

For simplicity, let us take the T^8 covering space of $Y = T^4/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$ to have standard periodicities $z_k \sim z_k + 1 \sim z_k + i$, for $k = 1, \ldots, 4$. Let us begin first with the D_{iJ} terms. We write out explicitly the non-localized part

$$G = D_{i1}\,\beta_i \wedge \frac{1}{2}(dz_3 \wedge d\bar{z}_4 + d\bar{z}_3 \wedge dz_4) + D_{i2}\,\beta_i \wedge \frac{1}{2i}(dz_3 \wedge d\bar{z}_4 - d\bar{z}_3 \wedge dz_4). \tag{3.3}$$

Note that we have not included the term with $\gamma_3 = \frac{1}{2}(dz_3 \wedge d\bar{z}_3 - dz_4 \wedge d\bar{z}_4)$ because it is not normalizable when the area of the torus fiber is taken to zero when mapping from M-theory to type IIB orientifold theory [22, 23]. Now re-arranging the two terms, we have

$$\begin{aligned} G &= \frac{1}{2}(D_{i1} + iD_{i2})\beta_i \wedge d\bar{z}_3 \wedge dz_4 + \frac{1}{2}(D_{i1} - iD_{i2})\beta_i \wedge dz_3 \wedge d\bar{z}_4 \\ &= \frac{1}{2}(D_3 \wedge dz_4 + \bar{D}_3 \wedge d\bar{z}_4) \\ &= \frac{1}{2}[D_3 \wedge (dx_{10} + idx_{11}) + \bar{D}_3 \wedge (dx_{10} - idx_{11})] \\ &= \frac{1}{2}(D_3 + \bar{D}_3) \wedge dx_{10} + \frac{1}{2}(D_3 - \bar{D}_3) \wedge i\,dx_{11} \\ &= H_3 \wedge dx_{10} + F_3 \wedge dx_{11}, \end{aligned} \tag{3.4}$$

where we have introduced $D_3 = D_i\beta_i \wedge d\bar{z}_3 = (D_{i1} + iD_{i2})\beta_i \wedge d\bar{z}_3$ and substituted $z_4 = x_{10} + ix_{11}$. In particular,

$$H_3 = dB_2 = \frac{1}{2}(D_i\beta_i \wedge d\bar{z}_3 + \bar{D}_i\beta_i \wedge dz_3) = \frac{1}{2}d(\alpha \wedge d\bar{z}_3 + \bar{\alpha} \wedge dz_3), \tag{3.5}$$

$$B_2 = \frac{1}{2}(\alpha \wedge d\bar{z}_3 + \bar{\alpha} \wedge dz_3), \tag{3.6}$$

where we have defined $D_i\beta_i = d\alpha$. Applying two T-dualities in the z_3 directions, the metric and the B-field of the type IIB theory get mixed. After a further S-duality, the resulting heterotic metric takes the form [22, 23, 27]

$$ds^2 = e^{2\phi}(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) + |dz_3 + \alpha|^2. \tag{3.7}$$

This is the metric of the FSY solution in the $\mathbb{Z}_2$ orbifold limit. Thus, we see that the M-theory solution, with a G-flux having a non-localized (1,1)-form γ' on the second $K3$ which we dualized, gets mapped to a heterotic FSY solution. Of course, one can check that the anomaly condition (2.9) is satisfied. This follows from (3.1),

$$\begin{aligned} 24 = \frac{1}{2}\int_Y G \wedge G &= \frac{1}{4}\int_S D_i\beta_i \wedge \bar{D}_j\beta_j \int_S dz_3 \wedge d\bar{z}_3 \wedge dz_4 \wedge d\bar{z}_4 \\ &= -\int_S D_i\beta_i \wedge \bar{D}_j\beta_j, \end{aligned} \tag{3.8}$$

which is (2.9) after setting $\omega = D_i\beta_i = (D_{i1} + iD_{i2})\beta_i$ and $F = 0$. Note that the torus bundle curvature here is localized at the fixed points of the base $T^4/\mathbb{Z}_2$.

Let us now return to the M-theory model and exchange the two $K3$'s. The G-flux of (3.3) maps to a G-flux with C_{Ij} terms

$$
\begin{aligned}
G &= C_{1i}\frac{1}{2}(dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2) \wedge \beta_i' + C_{2i}\frac{1}{2i}(dz_1 \wedge d\bar{z}_2 - d\bar{z}_3 \wedge dz_4) \wedge \beta_i' \\
&= \left[\frac{1}{2}(C_{1i} + iC_{2i})d\bar{z}_1 \wedge dz_2 + \frac{1}{2}(C_{1i} - iC_{2i})dz_1 \wedge d\bar{z}_2\right] \wedge \beta_i' \\
&\equiv \frac{1}{2}[C_i d\bar{z}_1 \wedge dz_2 + \bar{C}_i dz_1 \wedge d\bar{z}_2] \wedge \beta_i' \\
&\equiv \frac{F^i}{4\pi} \wedge \beta_i'.
\end{aligned}
\tag{3.9}
$$

Dualizing again in the z_3, z_4 directions, the resulting fluxes on the type IIB orientifold are now very different. With the G-flux localized at points on the second $K3$ as in (3.9), we have in type IIB non-zero gauge field strengths $F^i \sim C_{Ii}\gamma^I$, on the D7/O7 planes [17, 22, 23]. Unaffected by the two T-dualities and one S-duality, these gauge fluxes become the gauge field strengths of the heterotic theory on $K3 \times T^2$. The anomaly condition again follows from (3.1)

$$
\begin{aligned}
24 = \frac{1}{2}\int_Y G \wedge G &= \frac{1}{32\pi^2}\int_S F^i \wedge F^j \int_S \beta_i' \wedge \beta_j' \\
&= -\frac{1}{16\pi^2}\int_S F^i \wedge F^i.
\end{aligned}
\tag{3.10}
$$

As an example, let us compare the models (2.18) and (2.20) presented in the last section. In fact, it is easy to see that the FSY model of (2.18) arises from the G-flux

$$
G = (\beta_1 + \beta_2 - 2\beta_3) \wedge \gamma_1' + (\beta_1 + \beta_2 - 2\beta_3) \wedge \gamma_2', \tag{3.11}
$$

and that of (2.20) from

$$
G = (\gamma_1 + \gamma_2) \wedge \beta_1' + (\gamma_1 + \gamma_2) \wedge \beta_2' - 2(\gamma_1 + \gamma_2) \wedge \beta_3'. \tag{3.12}
$$

They differ just by a switch of the two $K3$'s.

In short, we have shown that for identical G-flux (3.2) and (3.9), which differs only in the assignment of what we called the first or second $K3$, the resulting dual heterotic models have very different background geometries. The first maps to an FSY-type solution and the second maps to a $U(1)$ gauge bundle on the Kähler manifold $K3 \times T^2$. Since the spectrum of the M-theory is invariant under the $K3$ exchange, it may seem that the distinct dual heterotic models must be physically identical.

This is however not the case in four dimensions. We point out that such a heterotic–heterotic duality can only be apparent when both heterotic models are compactified further on an additional circle so that the external spacetime theory is three dimensional. The low energy theory would then be identical to that of M-theory on $K3 \times K3$, which is also three dimensional. For in the process of "dualizing" from M-theory to heterotic theory, we shrank down the area of the elliptic fiber, $A_{T^2} \to 0$, to obtain a type II and then later heterotic theory in four dimensions. If we had not taken the zero area limit, then the exact duality would result in a seven-dimensional compact geometry, e.g., the FSY geometry times an additional S^1. More specifically, $A_{T^2} \sim 1/R_{S^1}$, and thus the $A_{T^2} \to 0$ limit corresponds to the decompactification limit of the extra circle in heterotic theory.

Taking into account of the deformation to the orbifold limit discussed in Section 2, we have thus demonstrated a connection between two smooth geometries, one Kähler and the other non-Kähler, via a path through an orbifold limit on the moduli space of the $K3$ surface.

4 Discussions and outlook

We have utilized the mapping to M-theory to connect Kähler and non-Kähler flux compactifications in heterotic theory. The M-theory model organizes the heterotic models' torus and gauge bundle curvatures into a single 4-form G-flux. Turning on the torus and gauge bundle curvatures correspond to turning on different types of forms for the G-flux. As we have shown, an exchange in the two $K3$'s with non-zero G-flux can lead to either the heterotic $K3 \times T^2$ model or the FSY model. Interestingly, M-theory on $K3 \times K3$ can also be dual to type IIA theory on $X_3 \times S^1$, where X_3 is a Calabi–Yau 3-fold. It has been pointed out in [28] that the exchange of two $K3$'s in this set up corresponds to mirror symmetry for the Calabi–Yau 3-fold. Following this observation, it is conceivable that the duality we have pointed to is related to a "generalized" mirror symmetry in the heterotic theory with fluxes.

The transition we have discussed preserves $N = 2$ supersymmetry in four dimensions. An interesting old question is whether the space of all $N = 2$ string vacua is actually connected. Prior to the current popularity of flux compactifications, many works in the mid-90s gave evidences to unforeseen connectedness between different vacua and thus hinted at such a possibility. But with the recent increases in flux models which are quantized in integral units, hopes of a single connected moduli space have now faded. But perhaps the simple well-known moduli space of $c = 1$ closed bosonic string theory [29, 30] will give a guide to the moduli space of $N = 2$ vacua. In $c = 1$, the moduli space includes the circle S^1 with radius R_c and the $\mathbb{Z}_2$ orbifold $S^1/\mathbb{Z}_2$ with radius R_t for the target space, geometry. Rather surprisingly, these two distinct geometries are connected or dual at precisely the point $R_c/2 = R_t = \sqrt{\alpha'}$. In addition, there are three points in the moduli space, which are disconnected to all other theories. Taking this example as a lead, we may think that the moduli space of $N = 2$ string vacua also smoothly link together topologically distinct manifolds, but yet there will also be regions of isolated vacua which do not have any moduli that connect to the rest. Our examples here of a connection between Kähler and non-Kähler geometries is an example of a somewhat surprising link.

Finally, we have for simplicity focused our attention on a subset of non-Kähler FSY solutions. It would be interesting to explore the connectedness of the moduli space when the gauge bundle is non-Abelian. Studying this may involve non-perturbative effects. For instance, on the M-theory side, we would need to consider M2-branes wrapping singular 2-cycles of the $K3$ in order to generate non-Abelian flux. On the heterotic side, wrapped branes of the sort discussed in [31] might also be required. FSY solutions with torus curvature ω having a (2,0) component should also be investigated for possible transitions. These $N = 1$ vacua would sit in the moduli space of all $N = 1$ heterotic compactifications, which include the conventional Calabi–Yau compactifications. This moduli space should incorporate the well-studied conifold transitions, both Kähler/Kähler and Kähler/non-Kähler types, which are $N = 1$ transitions in heterotic theory. We hope to explore some of these issues in future works.

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Appendix $N = 2$ supersymmetry conditions

We check that FSY geometry has $N = 2$ supersymmetry in four dimensions. Assuming $N = 1$ SUSY, we derive the additional conditions $N = 2$ SUSY imposes. Discussions and references on the SU(2) structure relevant for $N = 2$ SUSY can be found in [32, 33]. We start from the supersymmetry constraints

$$\nabla_M \eta + \frac{1}{8} H_{MNP} \gamma^{NP} \eta = 0, \tag{A.1}$$

$$\gamma^M \partial_M \phi \, \eta + \frac{1}{12} H_{MNP} \gamma^{MNP} \eta = 0, \tag{A.2}$$

$$\gamma^{MN} F_{MN} \eta = 0, \tag{A.3}$$

$N = 1$ SUSY implies the existence of a no-where vanishing spinor η_1 on the manifold X. This gives an SU(3) structure and in particular we can define

$$J_{mn} = -i{\eta_1}^\dagger \gamma_{mn} \eta_1, \quad \Omega_{mnp} = \mathrm{e}^{-2\phi} {\bar{\eta}_1}^\dagger \gamma_{mnp} \eta_1, \tag{A.4}$$

where the complex conjugate spinor is defined to be $\bar{\eta}_1 = B^* \eta_1^*$.[6] The manifold is required to be complex hermitian and the metric conformally balanced. Moreover, we have the relations

$$H = i(\bar{\partial} - \partial) J, \quad \Omega_{mnp} \bar{\Omega}^{mnp} = 8\mathrm{e}^{-4\phi}. \tag{A.5}$$

$N = 2$ SUSY implies the existence of a second no-where vanishing spinor η_2. Both spinors have the same chirality, which we take to be positive, and we shall assume that the two spinors are never parallel. We can therefore normalize so that

$$\eta_i^\dagger \eta_j = \delta_{ij}, \quad i, j = 1, 2. \tag{A.6}$$

With the additional spinor, there now exists a no-where vanishing 1-form

$$v_m = {\bar{\eta}_1}^\dagger \gamma_m \eta_2. \tag{A.7}$$

Alternatively, we can write $\eta_2 = \frac{1}{2} v_m \gamma^m \bar{\eta}_1$. In the holomorphic coordinates $J_a^b = i\delta_a^b$, we have $J_m^n v_n = i v_m$; that is, v has only holomorphic components. Moreover, the normalization of η_2 implies that $|v|^2 = g^{a\bar{b}} v_a \bar{v}_{\bar{b}} = 2$.

We point out that the existence of a no-where vanishing 1-form is a strong constraint on X. Specifically, from the Poincaré–Hopf theorem, the number of zeroes of a vector field (and its dual 1-form) is at least that of the Euler characteristic (i.e., $\geq |\chi|$). Thus, $N = 2$ SUSY having a non-vanishing 1-form requires $\chi(X) = 0$.

[6] The B matrix here satisfies $B\gamma^m B^* = -\gamma^*$ and we shall work in a basis where $B^t = B$.

Besides the vector, the additional spinor allows us to write down the forms,

$$\begin{aligned}
{\eta_2}^\dagger \gamma_{mn} \eta_1 &= \frac{1}{2} \bar{v}^s \Omega_{smn} \mathrm{e}^{2\phi} \equiv (K_2)_{mn} + i(K_1)_{mn}, \\
{\eta_2}^\dagger \gamma_{mn} \eta_2 &= -iJ_{mn} - 2v_{[m}\bar{v}_{n]} \equiv -2i(K_3)_{mn} + iJ_{mn}, \qquad \text{(A.8)}\\
\bar{\eta_1}^\dagger \gamma_{mnp} \eta_1 &= (v \wedge (K_2 + iK_1))_{mnp}, \\
\bar{\eta_2}^\dagger \gamma_{mnp} \eta_2 &= (v \wedge (K_2 - iK_1))_{mnp}, \\
\bar{\eta_2}^\dagger \gamma_{mnp} \eta_1 &= (v \wedge J)_{mnp}. \qquad \text{(A.9)}
\end{aligned}$$

Note that the hermitian form and holmorphic 3-form can be written as

$$\begin{aligned}
J &= K_3 + \frac{i}{2} v \wedge \bar{v}, \\
\Omega &= \mathrm{e}^{-2\phi}(K_2 + iK_1) \wedge v.
\end{aligned} \qquad \text{(A.10)}$$

The K_A's, with $A = 1, 2, 3$, reside in a four-dimensional subspace and give a hyperkähler structure

$$(K_A)_m{}^n (K_B)_n{}^k = -\delta_{AB} \left[\delta^k_m - \frac{1}{2}(v_m \bar{v}^k + \bar{v}_m v^k) \right] + \epsilon^C_{AB} (K_C)_m{}^k, \qquad \text{(A.11)}$$

where the additional terms in v's are present since the forms are defined in six dimensions and not in four. In particular, we have, for example, $(K_1)^n_m (K_1)^m_n = -4$ as typical for a four-dimensional hyperkähler space.

The first constraint equation (A.1) requires that all the non-vanishing forms are covariantly constant with respect to the H-connection. Note that $N = 1$ SUSY already ensures that J and $\Omega \mathrm{e}^{2\phi}$ are covariantly constant. Hence, the additional constraint of $N = 2$ comes from requiring the covariantly constancy of the 1-form

$$\nabla^H_m v_n = \nabla_m v_n - \frac{1}{2} H_m{}^r{}_n v_r = 0. \qquad \text{(A.12)}$$

The second equation (A.2) gives further differential constraints on the forms. For $A = \gamma_{n_1 \cdots n_p}$ antisymmetric combination of gamma matrices, we can re-express (A.2) as [34]

$$\partial_m \phi \, \eta_j^\dagger [A, \gamma^m]_\pm \eta_i + \frac{1}{12} H_{mnp} \eta_j^\dagger [A, \gamma^{mnp}]_\mp \eta_i = 0, \qquad \text{(A.13)}$$

$$\partial_m \phi \, \bar{\eta}_j^\dagger [A, \gamma^m]_\pm \eta_i + \frac{1}{12} H_{mnp} \bar{\eta}_j^\dagger [A, \gamma^{mnp}]_\mp \eta_i = 0, \qquad \text{(A.14)}$$

where the $+$ or $-$ sign for the brackets denotes symmetric or antisymmetric brackets, respectively. The condition on the 1-form v can be derived with

$A = \gamma_{n_1 n_2}$ resulting in the constraint

$$d[\mathrm{e}^{-2\phi}v] + i * (H \wedge \mathrm{e}^{-2\phi}v) = 0. \tag{A.15}$$

For the 2-forms and 3-forms in (A.8) and (A.9) which we will denote generically as χ_2 and χ_3, the conditions are

$$\begin{aligned} d[*(\mathrm{e}^{-2\phi}\chi_2)] &= 0, \\ d[*(e^{-2\phi}\chi_3)] &= 0. \end{aligned} \tag{A.16}$$

Now for the gauge field strength F, in addition to being holomorphic $F^{(2,0)} = F^{(0,2)} = 0$ and primitive $F^{mn}J_{mn} = 0$, we now have in general the condition $F^{mn}(\chi_2)_{mn} = 0$, which gives the additional requirement

$$F^{mn}(K_3)_{mn} = 0. \tag{A.17}$$

We now check that the FSY geometry with torus bundle curvature $\omega \in H^{(1,1)}(K3, \mathbb{Z})$ satisfies the $N = 2$ SUSY conditions. First, note that as required, the Euler characteristic of a T^2 bundle over $K3$ is zero. From the decomposition of (A.10), it is clear that we have the identification

$$v = \theta, \quad K_3 = \mathrm{e}^{2\phi}J_{K3}, \quad K_2 + iK_1 = \mathrm{e}^{2\phi}\Omega_{K3}. \tag{A.18}$$

The K_A's are thus the hyperkähler forms on the conformal $K3$. We now check that the differential equations are also satisfied. First, the covariantly constancy as in (A.12) can be shown using (A.5) and (A.10) to be equivalent to $\partial\theta = 0$. Therefore, the torus bundle curvature twist can not contain a (2,0) component. The condition (A.15) can be shown to reduce to

$$\bar{\partial}_{\bar{c}}\phi\, \theta^{\bar{c}} = 0, \quad H_{ab\bar{c}}\, \theta^{\bar{c}} = 0. \tag{A.19}$$

This easily holds as $v^{\bar{c}} = g^{\bar{c}a}v_a = (0, 0, 2)$. The conditions of (A.16) for various forms in (A.8) and (A.9) can also easily be checked to hold. And lastly, for the gauge field strength, we see that the additional requirement (A.17) is satisfied since all field strengths are hermitian Yang–Mills on the base $K3$.

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