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Invariant bundles on B-fibered Calabi–Yau spaces and the Standard model

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Abstract

We derive the Standard model gauge group together with chiral fermion generations from the heterotic string by turning on a Wilson line on a non-simply connected Calabi–Yau 3-fold with an SU(5) gauge group. For this we construct stable $\mathbf{Z_2}$-invariant $\mathrm{SU}(4) \times U(1)$ bundles on an elliptically fibered cover Calabi–Yau 3-fold of special fibration type (the B-fibration). The construction makes use of a modified spectral cover approach giving just invariant bundles.

1 Introduction

Attempts to get a (supersymmetric) phenomenological spectrum with gauge group G_{SM} and chiral matter content of the Standard model from the $E_8 \times$

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E_8 heterotic string on a Calabi–Yau space X started with embedding the spin connection in the gauge connection giving an unbroken E_6 (times a hidden E_8 coupling only gravitationally). More generally [1], one can instead of the tangent bundle embed a $G = \mathrm{SU}(n)$ bundle for $n = 4$ or 5, leading to unbroken $H = \mathrm{SO}(10)$ or $\mathrm{SU}(5)$ of even greater phenomenological interest.

If there is a freely acting group $\mathcal{G}$ on the usually simply-connected X, one can work on $X' = X/\mathcal{G}$ with $\pi_1(X') = \mathcal{G}$ allowing a further breaking of H by turning on Wilson lines; the enhanced structure group $G \times \mathcal{G}$ leads to a reduced commutator.

On X' one turns on a $\mathbf{Z_2}$ Wilson line of generator $\mathbf{1_3} \oplus -\mathbf{1_2}$ breaking $H = \mathrm{SU}(5)$ to G_{SM}

$$\mathrm{SU}(5) \longrightarrow G_{\mathrm{SM}} = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_{ew} \times U(1)_Y$$

(up to a $\mathbf{Z_6}$). The Wilson line W can be considered as a flat bundle on X' induced from the $\mathbf{Z}_2$-cover $\rho : X \to X'$ via the given embedding of $\mathbf{Z_2}$ in $H = \mathrm{SU}(5)$. This gives, from[1] $\bar{\mathbf{5}} = \bar{d} \oplus L$ and $\mathbf{10} = Q \oplus \bar{u} \oplus \bar{e}$, the fermionic matter content of the Standard model

$$\begin{aligned} \text{SM fermions} &= Q \oplus L \oplus \bar{u} \oplus \bar{d} \oplus \bar{e} \\ &= (\mathbf{3},\mathbf{2})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-4/3} \oplus (\bar{\mathbf{3}},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{1})_2. \end{aligned} \tag{1.1}$$

From the decompositions of ad_{E_8} under $G \times H = \mathrm{SU}(4) \times \mathrm{SO}(10)$ resp. $\mathrm{SU}(5) \times \mathrm{SU}(5)$

$$\mathbf{248} = (\mathbf{4},\mathbf{16}) \oplus (\overline{\mathbf{4}},\overline{\mathbf{16}}) \oplus (\mathbf{6},\mathbf{10}) \oplus (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{45}) \tag{1.2}$$

$$= (\mathbf{5},\mathbf{10}) \oplus (\overline{\mathbf{5}},\overline{\mathbf{10}}) \oplus (\mathbf{10},\overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}},\mathbf{5}) \oplus (\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}), \tag{1.3}$$

one finds that for unbroken SO(10) a consideration of the fundamental $V = \mathbf{4}$ is enough as all fermions, including a right-handed neutrino singlet, sit in the $\mathbf{16} \to \bar{\mathbf{5}} \oplus \mathbf{10} \oplus \mathbf{1}$. For unbroken SU(5) also $\Lambda^2 V = \mathbf{10}$ has to be considered; but as the $\mathbf{10}$ and the $\bar{\mathbf{5}}$ come in the same number of families (as also demanded by anomaly considerations), it is enough to adjust $\chi(X, V)$ (the V-related matter) to get all the Standard model fermions.

To describe explicitely the bundle we choose X elliptically fibered $\pi : X \to B$ in a specific way. For the Hirzebruch surfaces $B = \mathbf{F_m}$, $m = 0, 1, 2$, X turns out to be smooth. As $\pi_1(X) \neq 0$, one is actually working with a $G = \mathrm{SU}(5)$ bundle leading to an $H = \mathrm{SU}(5)$ gauge group on a space admitting a free involution τ_X (leaving the holomorphic 3-form Ω invariant) to get a smooth Calabi–Yau $X' = X/\mathbf{Z_2}$ over a base B'.

[1] the properly normalized hypercharge Y arises by a normalization factor 1/3.

To compute the generation number, one has to work on X as X' does not have a section but only a bi-section (left over from the two sections of X), and so one can not use the spectral cover method there directly. On X', the generation number is reduced by $|\mathcal{G}|$.

If the bundle V' over X' is a 3-generation bundle, then the bundle $V = \rho^* V'$ on X has six generations and is "moddable" by construction. Conversely, having constructed a bundle above on X with six generations, one assures that it can be modded out by τ_X (to get the searched for bundle on X') by demanding that V should be τ_X-invariant. So one has to specify a τ_X-invariant SU(5) bundle on X that, besides fulfilling some further requirements of the spectral cover construction (cf. below), leads to six generations. For important work related to this question, cf. [4–8] and literature cited there.

As it will be our goal to "mod" not just the Calabi–Yau spaces but also the geometric data describing the bundle (and this transformation of bundle data into geometric data uses in an essential way the elliptic fibration structure), we will search only for actions which preserve the fibration structure, i.e., $\tau_B \cdot \pi = \pi \cdot \tau_X$ with τ_B an action on the base

$$\begin{array}{ccc} X & \xrightarrow{\tau_X} & X \\ \pi \downarrow & & \downarrow \pi \\ B & \xrightarrow{\tau_B} & B \end{array} \tag{1.4}$$

Therefore, our elliptically fibered Calabi–Yau spaces will actually have two sections[2] σ_1 and $\sigma_2 = \tau_X \sigma_1$ (B-model spaces). Turning this around, if one wants to construct τ_X by choosing a specific X, we will look for a Calabi–Yau X with a type of elliptic fibration which has besides the usually assumed single section (A-model) a second one (B-model); this will then lead to a free involution τ_X on Z.

When investigating the invariance of V, we are led to consider a version of the spectral cover method especially adapted to the situation with two sections. Concretely, we will work with a modified spectral surface and Poincaré bundle (with $\Sigma = \sigma_1 + \sigma_2$)

$$C = \frac{n}{2}\Sigma + \eta, \tag{1.5}$$

$$P = \mathcal{O}(2\Delta - \Sigma_{\mathrm{I}} - \Sigma_{\mathrm{II}} - c_1). \tag{1.6}$$

Obviously, this assumes that V has *even* rank n. Therefore the original strategy to obtain an SU(5) gauge group (in the observable sector) from an SU(5) bundle has to be modified. Concretely, one chooses an SU(4) bundle

[2]We will use the same notation for a section, its image and its cohomology class.

(of commutator SO(10) in E_8) and twists this with a further invariant line bundle $\mathcal{L}$ (of $c_1(\mathcal{L}) = D = x\Sigma + \alpha$)

$$\mathcal{V} = V \otimes \mathcal{L}(D). \tag{1.7}$$

The structure group is $U(4) = \mathrm{SU}(4) \cdot U(1)_X$ but as the difference to $\mathrm{SU}(4) \times U(1)_X$ is only a discrete group $\mathbf{Z_4}$, and group-theoretical statements are here meant on the level of Lie algebras, we will refer to $\mathrm{SU}(4) \times U(1)_X$ as structure group giving $\mathrm{SU}(5) \times U(1)_X$ as gauge group. The anomalous $U(1)_X$ becomes massive by the Green–Schwarz mechanism.

This bundle $\mathcal{V}$ is then embedded in E_8 as a sum of stable bundles $\tilde{\mathcal{V}} = \mathcal{V} \oplus \det^{-1}(\mathcal{V}) = V \otimes \mathcal{L} \oplus \mathcal{L}^{-4}$ with $c_1(\tilde{\mathcal{V}}) = 0$. From the condition of effectivity of the 5-brane W in

$$c_2(V) - 10D^2 + W = c_2(X), \tag{1.8}$$

one realizes that[3] one is forced to allow for an additional twist by an invariant line bundle with $x \neq 0$. This however leads to a problem in the Donaldson–Uhlenbeck–Yau (DUY) condition $c_1(V)J^2 = 0$, where $J = \epsilon J_0 + H_B$ is a Kahler potential for which stability of V can be guaranteed (here H_B is a Kahler class on the base). As the concrete bound $\epsilon \leq \epsilon_*$ from which on J is appropriate is not known explicitely one has to solve the DUY equation in every oder in ϵ individually which leads for the constant term to $2xH_B^2 = 0$.

Therefore, one must go beyond tree-level here and invoke the 1-loop correction to the DUY equation [19] which in turn leads to further conditions assuring positivity of the dilaton ϕ and of the gauge kinetic function. These two inequalities taken together with the two inequalities assuring effectivity of W and the further inequality assuring irreducibility (resp. ampleness) of C turn out to be quite restrictive.

So we will construct rank 4 vector bundles V on X which fulfill the following conditions

- $\tilde{\mathcal{V}}$ is τ_X-invariant and satisfies 1-loop modified DUY condition;
- $c_1(\tilde{\mathcal{V}}) \equiv 0 (\mathrm{mod}\ 2)$, $c_2(X) - (c_2(V) - 10c_1(\mathcal{L})^2)$ is effective, $\chi(X, \tilde{\mathcal{V}}) = N_{\mathrm{gen}}$.

Having obtained the chiral fermions of the Standard Model, one would like to count also the number of Higgs multiplets and moduli. This will be considered elsewhere.

[3] Apart from a case, not treated in this paper, which has special features.

In Section 2, the B-model spaces along with their cohomological data are introduced. In Section 3, the modified spectral cover construction of bundles is introduced and the Chern classes of V and its twist are computed. In Section 4, we are writing down for $\mathbf{F_0}$ and $\mathbf{F_2}$ a free involution and the τ_X-action on V is described in detail. In Section 5, we give some group-theoretical details and describe the embedding in E_8; then we make the condition for the effectivity of the ensuing 5-brane explicit. In Section 6, the stability condition is made explicit and the relation with the 1-loop modified DUY equation is described. Finally, in Section 7, we list all numerical conditions which result from the analysis of these models.

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2 Calabi−Yau 3-fold with two sections

In this paper, we will consider a Calabi–Yau 3-fold X which is elliptically fibered over a Hirzebruch surface $\mathbf{F}_m$ and whose generic fiber is described by the so-called B-fiber $\mathbf{P}_{1,2,1}(4)$ instead of the usual A-fiber $\mathbf{P}_{2,3,1}(6)$ (the subscripts indicate the weights of x, y, z). X is described by a generalized Weierstrass equation which embeds X in a weighted projective space bundle over $\mathbf{F_m}$

$$y^2 + x^4 + a_2 x^2 z^2 + b_3 x z^3 + c_4 z^4 = 0, \tag{2.1}$$

where x, y, z and a, b, c are sections of K_B^{-i} with $i = 1, 2, 0$ and $i = 2, 3, 4$, respectively.

X admits two cohomologically inequivalent section σ_1, σ_2. For this consider equation (2.1) at the locus $z = 0$, i.e., $y^2 = x^4$ (after $y \to iy$). One finds eight solutions which constitute the two equivalence classes $(x, y, z) = (1, \pm 1, 0)$ in $\mathbf{P}_{1,2,1}$. We choose $y = +1$, corresponding to the section σ_1, as zero in the group law, while the other one can be brought, for special points in the moduli space, to a half-division point (in the group law) leading to the shift-involution, cf. Section 4.1. Let us keep on record the relation of divisors

$$(z) = \Sigma := \sigma_1 + \sigma_2, \quad \sigma_1 \cdot \sigma_2 = 0 \tag{2.2}$$

One finds for the Chern classes of X (cf. [13]; we use the notation $c_i = \pi^* c_i(B)$)

$$c_2(X) = 6\,\Sigma c_1 + c_2 + 5\, c_1^2, \quad c_3(X) = -36\, c_1^2. \tag{2.3}$$

From the weights a_2, b_3 and c_4 of the defining equation, one gets $5^2 + 7^2 + 9^2 - 3 - 3 - 1 = 148$ complex structure deformations over $\mathbf{F_0}$. This is consistent with the Euler number and the $h^{1,1}(X) = 4$ Kähler classes

$$h^{1,1}(X) = 4, \quad h^{2,1}(X) = 148, \quad e(X) = -288 \tag{2.4}$$

For later use let us also note the adjunction relations (with $\sigma_i := \sigma_i(B)$, $i = 1, 2$)

$$\sigma_i^2 = -c_1\sigma_i, \quad \Sigma^2 = -c_1\Sigma. \tag{2.5}$$

In this paper the base B of X is given by a Hirzebruch surface $\mathbf{F_m}$ (with $m = 0, 2$) with $H^2(B, \mathbf{Z})$ generated by the effective base and fiber classes b and f (with the intersection relations $b^2 = -m$, $b \cdot f = 1$ and $f^2 = 0$). $\eta = xb + yf$ effective, denoted as $\eta \geq 0$, means then $x \geq 0$ and $y \geq 0$. The Kähler cone is described in the appendix.

3 Bundles from spectral covers

For the description of vector bundles V on B-fibered Calabi–Yau 3-fold X, we will apply the spectral cover construction (equivalently a relative Fourier–Mukai transform). The spectral data are an effective divisor C of X ("the spectral surface") and a line bundle L on C. The correspondence between (C, L) and V was described for bundles on A-fibered Calabi–Yau in [2,3,11,12,14] and for the B-fibered case in [13,15].

For our applications, let us briefly recall some facts about the spectral cover construction for the A-models. One first forms the fiber product $X \times_B \hat{X}$ and denote the projections on X and $\hat{X}$ by p and $\hat{p}$, respectively. Points q on the fiber $\hat{E}_b$ of $\hat{X}$ will parametrize degree zero line bundles $\mathcal{L}_q$ on E_b for each $b \in B$; as it is usual, we will identify $X \xrightarrow{\sim} \widehat{X}$. There exists the so-called universal Poincaré line bundle $\mathcal{P}$ on $X \times_B \widehat{X} = X_{\mathrm{I}} \times_B X_{\mathrm{II}}$ whose restrictions to the $E_b \times q_b$ are just the $\mathcal{L}_q$. $\mathcal{P}$ is defined only up to a tensor product by the pullback of a line bundle on X_{II} and one can normalize it by letting $\mathcal{P}_{|\sigma_1 \times_B \widehat{X}} \simeq \mathcal{O}_X$ leading to $c_1(\mathcal{P}) = \Delta - \sigma_{1,\mathrm{I}} - \sigma_{1,\mathrm{II}} - c_1$. One then considers the spectral cover surface $C \subset X$ of class $C = n\sigma_1 + \eta$ which is an n-fold cover of B and forms the fiber product $X \times_B C$. The relative Fourier–Mukai transform constructs a vector bundle V from its spectral data (C, L) (where $\hat{p}_C \colon X \times_B C \to C$)

$$V = p_*((\hat{p}_C^* L) \otimes \mathcal{P}). \tag{3.1}$$

We introduce now a new procedure where we make the whole bundle construction symmetric (τ-invariant) from the beginning. For this we define for *even* n (with suitable pull-backs via p and $p_{\mathrm{II}} = \hat{p}$ understood in (3.3); Δ is the diagonal in $X \times_B X$)

$$C = \frac{n}{2}\Sigma + \eta, \tag{3.2}$$

$$c_1(P) = 2\Delta - \Sigma_{\mathrm{I}} - \Sigma_{\mathrm{II}} - c_1 \quad (p_* c_1(P) = 0) \tag{3.3}$$

As we will usually assume that C is effective and irreducible, let us review the conditions which we have to impose on η to assure this. C is effective exactly if η is, what we denote by $\eta \geq 0$, and C is irreducible when (cf. [10])

$$\eta \cdot b \geq 0, \quad \eta - \frac{n}{2}c_1 \geq 0. \tag{3.4}$$

Actually we will even assume that C is ample (cf. appendix) so that the classification of line bundles on C is discrete and L is determined by $c_1(L)$ up to isomorphism. For then $h^{1,0}(C) = 0$ as $h^1(C, \mathcal{O}_C) = h^2(X, \mathcal{O}_X(-C))$ by the long exact sequence associated to $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$; then $h^2(X, \mathcal{O}_X(-C)) = h^1(X, \mathcal{O}_X(C)) = 0$ by the Kodaira vanishing theorem.

Applying the Grothendieck–Riemann–Roch theorem to the covering $\pi_C \colon C \to B$ gives

$$\mathrm{ch}(\pi_{C*}L)\mathrm{Td}(B) = \pi_{C*}(\mathrm{ch}(L)\mathrm{Td}(C)) \tag{3.5}$$

From $c_1(V) = 0$, one derives thereby the general[4] expression for L (with $\rho := \mu + \frac{n}{2}\lambda$)

$$c_1(L) = \frac{C + c_1}{2} + \gamma \quad ((\pi_C)_*\gamma = 0), \tag{3.6}$$

$$\begin{aligned}\gamma &= \lambda\left(n\sigma_1 - \left(\eta - \frac{n}{2}c_1\right)\right) + \mu(\sigma_1 - \sigma_2) \\ &= \lambda\left(\frac{n}{2}\Sigma - \left(\eta - \frac{n}{2}c_1\right)\right) + \rho(\sigma_1 - \sigma_2).\end{aligned} \tag{3.7}$$

So for $n = 4$, and $\rho = 0$ as we later have to assure invariance of V (we had chosen c_1 even), $c_1(L)$ is an integral class if $\lambda \in \mathbf{Z} + \frac{1}{2}$ or if $\lambda \in \mathbf{Z}$ and η is even.

The Grothendieck–Riemann–Roch theorem for the covering $p\colon X \times_B C \to X$ gives

$$\mathrm{ch}(V)\mathrm{Td}(X) = p_*((\hat{p}_C^* \,\mathrm{ch}(L))\,\mathrm{ch}(P)\mathrm{Td}(X \times_B C)), \tag{3.8}$$

$$\begin{aligned}c_2(V) = {}& 2\left(\eta + \frac{n}{4}c_1\right)\Sigma - 2\rho\left(\eta - \frac{n}{2}c_1\right)(\sigma_1 - \sigma_2) - \rho^2 c_1\left(\eta - \frac{n}{2}c_1\right) \\ & + 2\eta c_1 + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta\left(\eta - \frac{n}{2}c_1\right) - \frac{n((n^2/4) - 1)}{24}c_1^2,\end{aligned} \tag{3.9}$$

$$\frac{1}{2}c_3(V) = 4\lambda\eta\left(\eta - \frac{n}{2}c_1\right). \tag{3.10}$$

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[4]The space of line bundles $\mathrm{Pic}(C)$ on C is generically not simply characterized by pullbacks of bundles from X; new divisor classes on C can appear.

Twisting by $\mathcal{L}(D)$

For group-theoretical reasons explained later and for reasons of effectivity of the ensuing 5-brane class, we still have to twist our $\mathrm{SU}(n)$ vector bundle by an invariant line bundle $\mathcal{L}(D)$ where $D = x\Sigma + \alpha$ to get $\mathcal{V} = V \otimes \mathcal{L}$ and actually then to consider $\tilde{\mathcal{V}} = V \otimes \mathcal{L} \oplus \mathcal{L}^{-4}$. Now let us compute the generation number. For the double covering $\rho : X \to X'$ with vector bundles $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}'$, one has $\chi(X, \tilde{\mathcal{V}}) = 2\chi(X', \tilde{\mathcal{V}}')$ (here $\tilde{\mathcal{V}}'$ with $\tilde{\mathcal{V}} = \rho^*\tilde{\mathcal{V}}'$ exists precisely because $\tilde{\mathcal{V}}$ is invariant). N_{gen} can be computed from the index $\chi(X, \tilde{\mathcal{V}})$. The net number of chiral matter multiplets (the "number of generations") is given by (note that $c_1(\tilde{\mathcal{V}}) = 0$ and $c_2 = 4, c_1^2 = 8, n = 4$)

$$
\begin{aligned}
N_{\mathrm{gen}} = \chi(X, V \otimes \mathcal{L} \oplus \mathcal{L}^{-4}) &= \int_X \mathrm{ch}_3(V \otimes \mathcal{L}) + \mathrm{ch}_3(\mathcal{L}^{-4}) \\
&= \frac{1}{2}c_3(V) - c_2(V)D - 10D^3 \\
&= 4\left[\lambda - x\left(\lambda^2 - \frac{1}{4}\right)\right]\eta(\eta - 2c_1) + 40x(1 - 4x^2) \\
&\quad - 4\alpha(\eta + c_1) - 60x\alpha(\alpha - xc_1).
\end{aligned}
$$

4 Involution and invariant bundles

4.1 Existence of a free $\mathbf{Z_2}$ operation

We give a free involution τ_X on X which leaves the holomorphic 3-form invariant; then $X' = X/\tau_X$ is a smooth Calabi–Yau. We assume τ_X compatible with the fibration, i.e., we assume the existence of an involution τ_B on the base B with $\tau_B \cdot \pi = \pi \cdot \tau_X$.

We will choose for τ_B the following operation in local (affine) coordinates

$$
b = (z_1, z_2) \xrightarrow{\tau_B} -b = \tau_B(b) = (-z_1, -z_2). \tag{4.1}
$$

The idea for the construction of τ_X is to combine τ_B with an operation on the fibers. A free involution on a smooth elliptic curve is given by translation by a half-division point. Such an object has to exist globally; this is the reason we have chosen the B-fibration where X possesses a second section. If we would tune $\sigma_2(b) \in E_b$ to be a half-division point, the condition $b_3 = 0$ would ensue and X would become singular. Therefore this idea has to be enhanced. Furthermore, even for a B-fibered X, those fibers lying over the discriminant locus in the base will be singular where the freeness of the shift might be lost. As the fixed point locus of τ_B is a finite set of (four) points, we can assume that it is disjoint from the discriminant locus (so points in the potentially dangerous singular fibers are still not fixed points of τ_X; for discussion cf. [13, 15]).

One finds [13] as τ_X over $\mathbf{F_m}$ with m even (i.e., m being 0 or 2) the free involution

$$(z_1, z_2; x, y, z) \xrightarrow{\tau_X} (-z_1, -z_2; -x, -y, z). \tag{4.2}$$

This exchanges the points $\sigma_1(b) = (b; 1, 1, 0)$ and $\sigma_2(-b) = (-b; 1, -1, 0)$ between the fibers E_b and $E_{-b} = E_{\tau_B(b)}$; in $\mathbf{P}_{1,2,1}$, the sign in the x-coordinate can be scaled away here in contrast to the sign in the y-coordinate. As indicated above, an involution like in equation (4.2) could not exist on the fiber alone, i.e., as a map $(x, y, z) \longrightarrow (-x, -y, z)$, because this would from equation (2.1) force one to the locus $b_3 = 0$ where X becomes singular (so only then is this defined on the fiber and so, being a free involution, a shift by a half-division point). But it can exist combined with the base involution τ_B on a subspace of the moduli space where the generic member is still smooth, from equation (2.1), the coefficient functions should transform under τ_X as a_2^+, b_3^-, c_4^+, i.e., over $\mathbf{F_0}$, say, only monomials $z_1^p z_2^q$ within $b_{6,6}$ with $p+q$ even are forbidden; similarly in $a_{4,4}$ and $c_{8,8}$, $p+q$ odd is forbidden. So the number of deformations drops to $h^{2,1}(X) = (5^2+1)/2 + (7^2-1)/2 + (9^2+1)/2 - 1 - 1 - 1 = 75$. The discriminant remains generic since enough terms in a, b, c survive, so Z is still smooth (cf. [13]). The Hodge numbers (4,148) and (3, 75) of X and X' show that indeed $e(X') = e(X)/2$ (X' has lost one divisor as the two sections are identified).

4.2 Invariance of the bundles

We describe conditions on the spectral data (C, L) for the τ_X-invariance of V. The surface C lies actually in the dual Calabi–Yau $\hat{X}$ where $\hat{\tau}$ operates (cf. the next subsection): $\tilde{V} = \tau^* V$ turns out to be again a spectral cover bundle with $\tilde{C} := \hat{\tau}(C)$ as spectral surface. As fiberwise semistable bundle, it is fixed up to the datum $\tilde{L}$. More precisely the argument for invariance now goes as follows. The symmetric form (in σ_1 and σ_2) of P suggests that V should be invariant if L is chosen also symmetric, i.e., $\rho = 0$. However, one has to take into account that on the part of P in $X_{\mathrm{II}} = \hat{X}$, and similarly on L which sits in X_{II}, actually $\hat{\tau}$ operates. The conclusion remains nevertheless correct. $\tilde{V} = \tau^* V$ will always be chosen to have again C as its spectral surface (cf. next subsection) and is of the same general form as V, i.e., a spectral cover bundle of $c_1(\tilde{V}) = 0$, just with different input parameters $\tilde{\lambda}, \tilde{\rho}$.

But these can be read off from $c_2(\tau^* V) = \tau^* c_2(V)$ (where the latter is the usual operation in X, resp. its cohomology, interchanging σ_1 and σ_2). This gives $\rho = 0$ as necessary and sufficient condition for invariance.

Remark on Fourier–Mukai transformation

Usually the spectral cover construction of V from (C, L) is interpreted as an equivalence of data in the framework of Fourier–Mukai transformations. Because of the difference between C and C_{eff} (cf. next subsection) occurring in our construction, we will not employ the idea of an inverse transform here. Let us nevertheless point to some facts related to the discussion above and related invariance arguments.

$\tilde{V} = \tau^* V$ is the Fourier–Mukai transform FM^0 of a *line bundle* $\tilde{L} = j^* \tilde{l}$ supported on $\tilde{C} = \hat{\tau}(C)$, where $j : \tilde{C} \hookrightarrow X$, i.e., $\tau^* \mathrm{FM}^0 i_* i^* l = \mathrm{FM}^0 j_* j^* \tilde{l}$, where $L = i^* l$ was the line bundle datum on C for V (here $i : C \hookrightarrow X$). For this, recall that for $\tilde{V}$ a semistable vector bundle of rank n and degree zero on the fibers of its inverse Fourier–Mukai transform $\mathrm{FM}^1(\tilde{V})$ is a torsion sheaf of pure dimension 2 on X and of rank 1 over its support which is a surface $j : \tilde{C} \hookrightarrow X$, finite of degree n over B. For $\tilde{C}$ smooth $\mathrm{FM}^1(\tilde{V}) = j_* \tilde{L}$ is just the extension by zero of some torsion-free rank 1 sheaf which is actually a *line bundle* $\tilde{L} \in \mathrm{Pic}(\tilde{C})$: for $\pi_{\tilde{C}*} \tilde{L} = \tilde{V}|_B$ is locally free and $\pi_{\tilde{C}} : \tilde{C} \to B$ is a finite flat surjective morphism, so $\tilde{L}$ is locally free as well.

Now for a Poincaré bundle which would be strictly invariant in the sense that $(\tau \times \hat{\tau})^* P = P$, one can indeed show that $\tau^* \mathrm{FM}^0(E) = \mathrm{FM}^0(\tau^* E)$ (here $E = i_* L$). For this, note that

$$\begin{aligned} \tau^* \mathrm{FM}^0(E) &= \tau^* \pi_* (\hat{\pi}^* E \otimes P) = \pi_* (\hat{\pi}^* (\hat{\tau}^* E) \otimes P) \\ &= \mathrm{FM}^0(\hat{\tau}^* E) \end{aligned} \tag{4.3}$$

Thus one gets invariance if $\mathrm{FM}^0(\hat{\tau}^* E) = \mathrm{FM}^0(E)$ for which it is sufficient that $\hat{\tau}^* E = E$.

4.3 Operation on the dual Calabi–Yau $\hat{X}$

Let us see how τ_X operates effectively in the dual Calabi–Yau $\hat{X}$. E_b has the equation

$$y^2 = x^4 + a_2^+(b) x^2 z^2 + b_3^-(b) x z^3 + c_4^+(b) z^4. \tag{4.4}$$

Similarly E_{-b} with $-b := \tau_B(b)$ has the equation

$$y^2 = x^4 + a_2^+(b) x^2 z^2 - b_3^-(b) x z^3 + c_4^+(b) z^4. \tag{4.5}$$

τ maps E_{-b} to E_b and one gets for the transformed bundle

$$\begin{aligned} V|_{E_b} &= \bigoplus_{i=1}^{n} \mathcal{O}_{E_b}(q_i(b) - \sigma_1(b)) \Longrightarrow (\tau^* V)|_{E_b} \\ &= \bigoplus_{i=1}^{n} \mathcal{O}_{E_b}(\tau q_i(-b) - \sigma_2(b)). \end{aligned} \tag{4.6}$$

As $\mathcal{O}_{E_b}(\tau q_i(-b) - \sigma_2(b)) \cong \mathcal{O}_{E_b}(t^{-1}_{\sigma_2(b)} \tau q_i(-b) - \sigma_1(b))$, where $t_{\sigma_2(b)}$ is the translation in the group law, one finds that $\tau^* V = V$ amounts fiberwise to the relation $\{q_i(b)\} = \{t^{-1}_{\sigma_2(b)} \tau q_i(-b)\}$, i.e., to $t^{-1}_{\sigma_2} \tau C = C$. So $\hat{\tau} := t^{-1}_{\sigma_2} \circ \tau$ is the relevant operation $\hat{\tau}$ on $\hat{X}$

$$\tau_X = t_{\sigma_2} \circ \hat{\tau}. \tag{4.7}$$

We did not assume that $\sigma_2(b)$ is a half-division point (so t_{σ_2} is not an involution); if one would do so by tuning $b_3 = 0$, the space X would become singular and only then acts t_{σ_2} as $(b; x, y, z) \to (b; -x, -y, z)$.

Actions in coordinates and in the group structure

Involutions τ_X covering τ_B (in the sense of equation (1.4)) are determined by involutions α (which turns out to be $\hat{\tau}$ in our case) covering τ_B with $\alpha\sigma_{1/2} = (-1)\sigma_{1/2}$ (the minus refers to the inversion in the group law) (cf. also [6]).

Now let us define a decomposition of τ in the coordinate involutions

$$\tau = \iota \circ \beta \tag{4.8}$$

with (ι, covering $\mathbf{1_B}$, is fiberwise the covering involution for the obvious map to $\mathbf{P^1_{x,z}}$)

$$\iota : (b; x, y, z) \longrightarrow (b; x, -y, z), \quad \beta : (b; x, y, z) \longrightarrow (-b; -x, y, z) \tag{4.9}$$

(β keeps σ_i fixed). One has the following relation between coordinate map and the structural maps (note that both sides just act on the fiber and interchange the σ_i)

$$\iota = t_{\sigma_2} \circ (-1). \tag{4.10}$$

First note here that ι is independent of the element q chosen as group zero; similarly the translation becomes in general $(t_{\sigma_2})_q = t_{\sigma_2} \circ t_q^{-1}$ and the inversion $(-1)_q = t_q \circ (-1)$, so the right-hand side is independent of the zero chosen.

Concerning the proof of equation (4.10) note that a holomorphic map on the fiber $\mathbf{C}/\Lambda$ is $t_q \circ \rho$ with ρ a group homomorphism, t_q a translation (here q must be σ_2). ρ lifts to a linear transformation $z \to az + b$ of $\mathbf{C}$ ($b \in \Lambda$), so a, keeping invariant the lattice Λ, is in $\mathbf{Z}$ ($\mathbf{C}/\Lambda$ in general has no complex multiplication); ρ is an isomorphism as ι is, so $a = \pm 1$ and $a = -1$ as $\rho\sigma_2 = (-1)\sigma_2$.

$$\tau = \iota \circ \beta = t_{\sigma_2} \circ \underline{(-1) \circ \beta} = t_{\sigma_2} \circ \hat{\underline{\tau}}. \tag{4.11}$$

Application to the spectral points

For the choice of our Poincare bundle in $V = p_*(p_C^* L \otimes P)$, one finds

$$P|_{E \times q} = \mathcal{L}((2q - \sigma_2) - \sigma_1), \tag{4.12}$$

giving effective spectral points $C^{\text{eff}} = \{2q - \sigma_2 | q \in C\}$ ("2" and "–" refer to the group).

What we actually want to achieve is that $\hat{\tau} C^{\text{eff}} = C^{\text{eff}}$ as the $\hat{\tau}$-invariance condition concerns the points q_i^{eff} corresponding to line bundle summands on the fiber. β, being holomorphic and sending σ_1 to itself, is a group homomorphism between respective fibers. With $2q - \sigma_2 = q - \iota q$ (operations in the group) from equation (4.10), one gets as condition

$$(-1)\beta(1 - \iota)q_i(b) = (1 - \iota)q_i(-b). \tag{4.13}$$

As (-1) and β are group homomorphisms, this will be fulfilled if we can assure that

$$q_i(-b) = \iota\beta q_i(b). \tag{4.14}$$

That is, the reduction achieved amounts to (note that fiberwise $C^{\text{eff}} = (1 - \iota)C$)

$$\iota\,\beta\, C = C \Longrightarrow (-1)\beta C^{\text{eff}} = C^{\text{eff}}. \tag{4.15}$$

The equation of the spectral cover surface

To show how this condition can be implemented we give the coordinate description of our spectral cover surface C. In the appendix, we recall the corresponding relations in the A-model elliptic fibration and derive the expression for our case.

From there one gets the equation of a spectral cover surface C (for $n = 4$)

$$w = \alpha_{20}x^2 + \alpha_{02}y + \alpha_{10}xz + \alpha_{00}z^2 = 0. \tag{4.16}$$

This shows that invariance of C under β would mean that the coefficient functions α_{ij} transform invariantly under τ_B except α_{10} which should transform anti-invariantly (or the other way around); similarly invariance of C under $\iota\beta$ as in equation (4.14) (to assure the necessary $\hat{\tau} = (-1)\beta$-invariance of C^{eff}) means that α_{02} and α_{10} have to transform with the other sign than α_{20} and α_{00}, a condition which can easily be imposed.

5 The E_8 embedding and the massive $U(1)$

We will explain now more precisely our strategy outlined in the introduction how to get an SU(5) GUT group and describe the embedding of the structure group into E_8.

The spectral cover construction leads in general to $U(n)$ bundles $\mathcal{V}$ (the "non-split case") with $c_1(\mathcal{V}) = \alpha$, cf. [9]. For us it will be enough to consider the "split case"

$$\mathcal{V} = V \otimes \mathcal{L}(D), \tag{5.1}$$

where V is an SU(n) bundle and $\mathcal{L}(D)$ is a line bundle of $c_1(\mathcal{L}(D)) = D = x\Sigma + \alpha$. So $c_1(\mathcal{V}) \equiv 0(n)$ as D is integral. Conversely, if $c_1(\mathcal{V}) \equiv 0(n)$, one can split off an *integral* class D of $c_1(\mathcal{V}) = nD$ and define a corresponding line bundle $\mathcal{L}(D)$ such that $V := \mathcal{V} \otimes \mathcal{L}(-D)$ is an SU(n) bundle, i.e., one can think of $\mathcal{V}$ then as $V \otimes \mathcal{L}(D)$.

Note that the structure group $U(n)$ arises in this case from SU$(n) \cdot U(1)$ (the latter factor is understood here always as embedded by multiples of the identity matrix), whereas for a bundle $V \oplus \mathcal{L}(D)$, the structure group would be the direct product SU$(n) \times U(1)$. Note that there is a morphism $f\colon \mathrm{SU}(n) \times U(1) \to U(n)$ sending $(a, b) \mapsto a \cdot b$. The image of this morphism is $U(n) = \mathrm{SU}(n) \cdot U(1)$, so $\mathrm{SU}(n) \cdot U(1) = (\mathrm{SU}(n) \times U(1))/\ker(f)$. The subgroup $\ker(f)$ is formed by all pairs $(\lambda \cdot \mathrm{Id}_n, \lambda^{-1})$ where $\lambda \in \mathbf{C}$ with $\lambda \cdot \mathrm{Id}_n \in \mathrm{SU}(n)$, i.e., $\lambda^n = 1$ and $\ker(f) = \mathbf{Z_n}$ (the group of nth roots of unity). As the difference between the direct product and the product is just a discrete group, and since all group-theoretical statements in this paper are understood on the level of Lie algebras, we will write SU$(4) \times U(1)$ instead of SU$(4) \cdot U(1)$ for our structure group G.

Let us make the embedding of G in E_8 more explicit. One embeds a $U(4)$ bundle block diagonally via $U(4) \ni A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & \det^{-1} A \end{pmatrix} \in \mathrm{SU}(5)$. In our case $\mathcal{V} = V \otimes \mathcal{L}$ with $c_1(\mathcal{V}) = 4c_1(\mathcal{L})$, so in this sense one works actually with the bundle $\tilde{\mathcal{V}} = \mathcal{V} \oplus \mathcal{L}^{-4}$.

The commutator of $G = \mathrm{SU}(4) \times U(1)_X$ in E_8 is given by $H = \mathrm{SU}(5) \times U(1)_X$, the observed gauge group. The adjoint representation of E_8

decomposes under $\mathrm{SU}(4)\times\mathrm{SU}(5)\times U(1)_X$ as follows (with $\mathrm{ad}(E_8) = \bigoplus_i U_i^{\mathrm{SU}(4)}\otimes R_i^{\mathrm{SO}(10)} = \bigoplus_i (U_i, R_i) = \bigoplus_i (U_i, S_i^{\mathrm{SU}(5)})_{t_i^{U(1)}})$

$$\begin{aligned}
\mathbf{248} &\overset{\mathrm{SU}(5)\times\mathrm{SU}(5)}{\longrightarrow} (\mathbf{5},\mathbf{10})\oplus(\overline{\mathbf{5}},\overline{\mathbf{10}})\oplus(\mathbf{10},\overline{\mathbf{5}})\oplus(\overline{\mathbf{10}},\mathbf{5})\oplus(\mathbf{24},\mathbf{1})\oplus(\mathbf{1},\mathbf{24}),\\
&\overset{\mathrm{SU}(4)\times\mathrm{SU}(5)\times U(1)_X}{\longrightarrow} ((\mathbf{4},\mathbf{1})_{-5}\oplus(\mathbf{4},\overline{\mathbf{5}})_3\oplus(\mathbf{4},\mathbf{10})_{-1})\\
&\quad\oplus((\overline{\mathbf{4}},\mathbf{1})_5\oplus(\overline{\mathbf{4}},\mathbf{5})_{-3}\oplus(\overline{\mathbf{4}},\overline{\mathbf{10}})_1)\oplus(\mathbf{6},\mathbf{5})_2\oplus(\mathbf{6},\overline{\mathbf{5}})_{-2}\\
&\quad\oplus(\mathbf{15},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{10})_4\oplus(\mathbf{1},\overline{\mathbf{10}})_{-4}\oplus(\mathbf{1},\mathbf{24})_0. \qquad (5.2)
\end{aligned}$$

The SU(5) representations are given as an auxiliary step. The full decomposition, identical to an auxiliary $\mathrm{SU}(4)\times\mathrm{SO}(10)$ step, leads to the right-handed neutrino ν_R.

The massless (charged) matter content is $\bigoplus(S_k)_{t_k} = \mathbf{1_{-5}}\oplus\bar{\mathbf{5}}_\mathbf{3}\oplus\mathbf{10_{-1}}\oplus\bar{\mathbf{5}}_\mathbf{-2}\oplus\mathbf{10_4}$; one can write conditions for the absence of net generations of the exotic matter given by the last tow summands; additionally (besides the gauge bosons $(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{24})_0$ of H), some neutral matter given by singlets (moduli) arises from $\mathrm{End}(V)$, i.e., $(\mathbf{15},\mathbf{1})_0$.

Precisely, those $U(1)$'s in H which occur already in G (a so-called $U(1)$ of type I; other $U(1)$'s in H are called to be of type II) are anomalous [16–19]. The anomalous $U(1)_X$ can gain a mass by absorbing some of the would be massless axions via the Green–Schwarz mechanism, that is, the gauge field is eliminated from the low energy spectrum by combining with an axion and so becoming massive. One has to check that the anomalies related to $U(1)_X$ do not cancel accidentally (i.e., that the mixed abelian-gravitational, the mixed abelian–non-abelian and the pure cubic abelian anomaly do not all vanish). Computing the anomaly coefficients of $U(1)_X$, we find (cf. [19])

$$\begin{aligned}
A_{U(1)-G^2_{\mu\nu}} &= \sum \mathrm{tr}_{(S_k)_{t_k}} q\cdot\chi(X, U_k\otimes t_k)\\
&= 10D\cdot(12\cdot(-c_2(V)+10D^2)+5c_2(X)), \qquad (5.3)\\
A_{U(1)-\mathrm{SU}(5)^2} &= \sum q_{t_k} C_2(S_k)\cdot\chi(X, U_k\otimes t_k)\\
&= 10D\cdot(2(-c_2(V)+10D^2)+c_2(X)), \qquad (5.4)\\
A_{U(1)^3} &= \sum \mathrm{tr}_{(S_k)_{t_k}} q^3\cdot\chi(X, U_k\otimes t_k)\\
&= 200D\cdot(6(-c_2(V)+10D^2)+40D^2+3c_2(X)). \qquad (5.5)
\end{aligned}$$

(with C_2 normalized to give $C_2(\bar{f}) = 1$, $C_2(\Lambda^2 f) = 3$ for SU(5)). For $x \neq 0$ (the case of interest for us, cf. below), the last two conditions are not proportional as $D^3 \neq 0$. In any case, the first condition is independent, so not all three coefficients will vanish.

5.1 Effectivity of the 5-brane

In the presence of magnetic 5-branes, the Bianchi identity H-field reads

$$dH = \operatorname{tr} R \wedge R - \frac{1}{30} \operatorname{Tr} F \wedge F + \sum_{\text{5-branes}} \delta_5^{(4)}, \tag{5.6}$$

where R and F are the associated curvature forms of the spin connection on X and the gauge connection on $\mathcal{V}$, and the last term is the source term contributed by the 5-branes. Here tr refers to the trace of the endomorphism of the tangent bundle of X and Tr denotes the trace in the adjoint representation of G.

This leads in our case of

$$\tilde{\mathcal{V}} = \mathcal{V} \oplus \mathcal{L}^{-4} \tag{5.7}$$

to the anomaly equation

$$-\mathrm{ch}_2(X) = -\mathrm{ch}_2(\mathcal{V}) - \mathrm{ch}_2(\mathcal{L}^{-4}) + W. \tag{5.8}$$

This gives with $c_1(\mathcal{V}) = -c_1(\mathcal{L}^{-4})$ and $c_1(\mathcal{L}) = D$

$$\begin{aligned} c_2(X) &= c_2(\mathcal{V}) - c_1^2(\mathcal{L}^{-4}) + W \\ &= c_2(V) - 10D^2 + W. \end{aligned} \tag{5.9}$$

Note that the last term in equation (5.6) is formally a current that integrates to one in the direction transverse to a 5-brane whose class we denote by W. The class W is the Poincaré dual of the sum of all sources and represents a class in $H_2(X, \mathbf{Z})$. Supersymmetry demands W to be the class of an effective curve in X. For simplicity, we will think of the curve of W as being irreducible.

We have a factorization $\mathcal{V} = V \otimes \mathcal{L}(D)$ where V is a $\mathrm{SU}(n)$ bundle and $\mathcal{L}(D)$ is an line bundle with $c_1(\mathcal{L}(D)) = D$. With the decomposition of W

$$W = W_B + W_F = w_B\Sigma + a_f F \tag{5.10}$$

(where W_B is an effective curve class and $a_f \geq 0$), one finds

$$\begin{aligned} W &= c_2(X) - c_2(V) + 10D^2 \\ &= \left(6c_1\Sigma + c_2 + 5c_1^2\right) - 2\left(\eta + \frac{n}{4}c_1\right)\Sigma - kF + 10D^2 \\ &= \left(6c_1 - 2\left(\eta + \frac{n}{4}c_1\right) + 10x(2\alpha - xc_1)\right)\Sigma + \left(c_2 + 5c_1^2 - kF + 10\alpha^2\right), \end{aligned} \tag{5.11}$$

giving the effectivity conditions (with $c_1^2 = 8$, $c_2 = 4$, $n = 4$)

$$w_B = 4c_1 - 2\eta + 10x(2\alpha - xc_1) \geq 0, \tag{5.12}$$

$$a_f = 48 - 2\left(\lambda^2 - \frac{1}{4}\right)\eta(\eta - 2c_1) - 2\eta c_1 + 10\alpha^2 \geq 0. \tag{5.13}$$

Note that (5.12) amounts to $\eta \leq 2c_1$ for $x = 0$, i.e., to $\eta = 2c_1$ where C is not ample and $h^{1,0}(C) \neq 0$.

6 The DUY constraint and its 1-loop modification

Like the Calabi–Yau condition on the underlying space X, the holomorphicity and stability of the vector bundle V are direct consequences of the required four-dimensional supersymmetry. The demand is that a connection A on V has to satisfy (at string tree-level) the DUY equation (J denotes a Kähler form on X)

$$F_A^{2,0} = F_A^{0,2} = 0, \quad F_A^{1,1} \wedge J^2 = 0. \tag{6.1}$$

The first equation implies the holomorphicity of V; the second equation is the Hermitian-Yang–Mills equation $F_A^{1,1} \wedge J^{n-1} = c \cdot I_F \cdot J^n$ (for $n = 3$ with $c \in \mathbf{C}$ vanishing) with the integrability condition (condition for the existence of a unique solution in case V is polystable, i.e., a sum of μ-stable bundles with the same slope) [22, 23]

$$\int_X c_1(V) \wedge J^2 = 0 \tag{6.2}$$

V is called μ-stable with respect to some Kähler class J if its slope $\mu(V) = \frac{1}{rk(V)} \int c_1(V) \wedge J^2$ is bigger than the slope of each subbundle V' of smaller rank.

If C is irreducible, then V will be stable for sufficiently small ϵ with respect to [3]

$$J = \epsilon J_0 + \pi^* H_B, \quad \epsilon > 0. \tag{6.3}$$

Here $J_0 = x_1\sigma_1 + x_2\sigma_2 + h$ with $x_1 + x_2 > 0$ and H_B is an ample divisor in B. So the volume of the fiber F of X is kept arbitrarily small compared to volumes of effective curves in the base. For details on the stability properties and generalizations, we refer to [3, 24]. Note that working on B-fibered Calabi–Yau 3-folds does not effect the stability proof of [3]. Thus our bundles are stable with respect to the Kähler class in equation (6.3).

The bundle $\mathcal{V} = V \otimes \mathcal{L}(D)$ is stable with respect to $J = \epsilon J_0 + H_B$ with ϵ small positive, $H_B \in \mathcal{C}_B$ and $J_0 = x_1\sigma_1 + x_2\sigma_2$ (with $\mathcal{C}_B$ the Kähler cone of B). The DUY constraint is

$$0 = DJ^2 = -\epsilon^2 c_1(\alpha - xc_1)\left(\sum x_i^2\right) + 2\epsilon\left(\sum x_i\right) H_B(\alpha - xc_1) + 2xH_B^2. \tag{6.4}$$

If one tries to argue that the Kähler moduli in J are tuned such that $DJ^2 = 0$, one encounters the difficulty that the relevant ϵ (for which stability is known) depends itself on the chosen J_0 and H_B. So one rather has to make vanish the coefficients of the three different ϵ-powers individually, so that $DJ^2 = 0$ independently of the actual, unknown value of ϵ. But the constant term gives $x = 0$ for which equation (5.12) can not be relaxed.

For a rank n bundle $\mathcal{V} = \oplus\mathcal{V}_i$ composed of $U(n_i)$ bundles of slopes $\mu_i = \frac{1}{n_i}\int J^2 c_1(\mathcal{V}_i) =: \mu$ (they must coincide for $\mathcal{V}$ to be polystable), one finds $\int J^2 c_1(\mathcal{V}_i) = 0$ for all i as $0 = \int J^2 c_1(\mathcal{V}) = \sum r_i\mu_i = \mu n$. For us, having $c_1(\mathcal{V}_i) = \pm 4D$ means $\int J^2 D = 0$.

The 1-loop modification

We discuss now an approach which leads to a rank 5 physical bundle for a split extension and *non-vanishing* x. This causes the slope to be non-zero and so one has to invoke the quantum corrected version of it. This will make the slope vanish at the 1-loop level and fix the dilaton.

From the condition of effectivity of the 5-brane W in $c_2(V) - 10D^2 + W = c_2(X)$, one realizes that[5] one is forced to allow for an additional twist by an invariant line bundle with $x \neq 0$. This however leads to a problem in the DUY condition $c_1(V)J^2 = 0$ where $J = \epsilon J_0 + H_B$ is a Kahler class for which stability of V can be guaranteed (here H_B is a Kahler class on the base). As the concrete bound $\epsilon \leq \epsilon_*$ for which J is appropriate is not known explicitely, one has to solve the DUY equation in every order in ϵ individually which leads for the constant term to $2xH_B^2 = 0$.

Therefore, one must go beyond tree-level here and invoke the 1-loop correction to the DUY equation [19] which in turn leads to further conditions assuring positivity of the dilaton ϕ and of the gauge kinetic function. These two inequalities taken together with the two inequalities assuring effectivity of W and the further inequality assuring irreducibility (resp. ampleness) of C turn out to be quite restrictive.

So we invoke the 1-loop correction of the DUY integrability constraint [19–21] (here $z \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$ refers to the position of W in the interval between

[5]Apart from a "separation case", which has special features.

the two E_8-walls)[6]

$$0 = \int_X c_1(\mathcal{L})J^2 - \phi \int_X c_1(\mathcal{L}) \left(-c_2(V) + 10c_1(\mathcal{L})^2 + \frac{1}{2}c_2(X) - \left(\frac{1}{2} - z\right)^2 W \right) \tag{6.5}$$

$$= DJ^2 - \phi D \left[\left(1 - \left(\frac{1}{2} - z\right)^2 \right) W - \frac{1}{2}c_2(X) \right] \tag{6.6}$$

(corresponding to a deformed DUY equation) with $\phi = l_s^4 g_s^2$ giving

$$\phi = \frac{DJ^2}{D((3/4)W - (1/2)c_2(X))}$$
$$= \frac{\mathcal{O}(\epsilon) + xH_B^2}{(\alpha - xc_1)((3/4)w_B - 3c_1) + x((3/4)a_f - 22)} > 0 \tag{6.7}$$

as condition. So it is enough to have (for $x > 0$; otherwise one has the reverse inequality)

$$(\alpha - xc_1)\left(\frac{3}{4}w_B - 3c_1\right) + x\left(\frac{3}{4}a_f - 22\right) > 0. \tag{6.8}$$

The positivity of the 1-loop corrected gauge kinetic function requires [19–21]

$$0 < J^3 - 3\phi \left[J \left(W - \frac{1}{2}c_2(X) - \left(\frac{1}{2} - z\right)^2 W \right) \right]$$
$$= \mathcal{O}(\epsilon) - 3\phi \left[\left(1 - \left(\frac{1}{2} - z\right)^2 \right) W - \frac{1}{2}c_2(X) \right] H_B.$$

As stability will hold for all small enough ϵ, it will be enough to show

$$\left(\frac{3}{4}[4c_1 - 2\eta + 10x(2\alpha - xc_1)] - 3c_1 \right) H_B < 0. \tag{6.9}$$

7 Model constraints and conclusions

Let us collect all conditions on the bundle. Note that λ can be integral or half-integral if η is even (actually α could be half-integral by considering genuine $U(n)$ bundles $\mathcal{V}$).

[6] We choose $z = 0$ to illustrate; to determine z_W, open membrane instanton effects have to be included.

We obtain the equation for the generation number of $\tilde{\mathcal{V}} = V \otimes \mathcal{L} \oplus \mathcal{L}^{-4}$ with $D = x\Sigma + \alpha$

$$\begin{aligned} N_{\text{gen}} = 4\left[\lambda - x\left(\lambda^2 - \frac{1}{4}\right)\right]\eta(\eta - 2c_1) + 40x(1-4x^2) \\ - 4\alpha(\eta + c_1) - 60x\alpha(\alpha - xc_1). \end{aligned} \tag{7.1}$$

One notes that N_{gen} is manifestly divisible by 4, except perhaps the first term. For $n = 4$ and $\rho = 0$ is λ either $\mathbf{Z} + \frac{1}{2}$ or $\lambda \in \mathbf{Z}$ with η even. It remains to discuss the term $\eta(\eta - 2c_1)$. If $\lambda \in \frac{1}{2} + \mathbf{Z}$, this term is divisible by 2; if $\lambda \in \mathbf{Z}$ as c_1 is even and $\eta \equiv c_1 \bmod 2$, this term is divisible by 4. In total, also the first term is divisible by 4.

Therefore, the physical generation number $N_{\text{gen}}/2$ downstairs on $X/\mathbf{Z_2}$ turns out to be even.

Furthermore, one obtains five inequalities as conditions: one for the irreducibility of C (this is on $\mathbf{F_0}$; when checking effectivity and irreducibility of C, note that $\eta \geq 0$ implies on $\mathbf{F_0}$ also $\eta \cdot b \geq 0$); actually for C ample, one gets a slightly sharper inequality; then two for the effectivity of W and two from the 1-loop considerations, concerning positivity of ϕ (where (7.4) is meant for $x > 0$; otherwise one has the reverse inequality) and of the gauge kinetic term[7]

$$\eta \geq 2c_1, \tag{7.2}$$

$$w_B = 4c_1 - 2\eta + 10x(2\alpha - xc_1) \geq 0, \tag{7.3}$$

$$a_f = 48 - 2\left(\lambda^2 - \frac{1}{4}\right)\eta(\eta - 2c_1) - 2\eta c_1 + 10\alpha^2 \geq 0, \tag{7.4}$$

$$\frac{1}{2}D\left[\frac{3}{4}W - \frac{1}{2}c_2(X)\right] = (\alpha - xc_1)\left(\frac{3}{4}w_B - 3c_1\right) + x\left(\frac{3}{4}a_f - 22\right) > 0, \tag{7.5}$$

$$\frac{1}{2}H_B\left[\frac{3}{4}W - \frac{1}{2}c_2(X)\right] = \frac{3}{4}\big(-2\eta + 10x(2\alpha - xc_1)\big)H_B < 0. \tag{7.6}$$

Therefore, we have constructed invariant bundles on B-fibered Calabi–Yau spaces. Thereby we can get downstairs on $X/\mathbf{Z_2}$ a number of $N_{\text{gen}}/2$ net generations of chiral fermions of the Standard model (plus an additional right-handed neutrino).

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[7]From (7.3) and (7.6), one may try to choose $0 \leq w_B \leq 4c_1$ (with $w_B \neq 4c_1$). But this is unnecessarily restrictive: (7.6) is satisfied *for all* ample H_B for $w_B - 4c_1 \leq 0$ (with $w_B \neq 4c_1$); but if $w_B - 4c_1 = (s,t)$ with already only one of them negative, then one can find *a certain* H_B for which (7.6) holds.

Appendix A

A.1 Equation of the spectral cover

Recall that in the A-model with elliptic curve $zy^2 = 4x^3 - g_2x - g_3$ in $\mathbf{P}^2 = \mathbf{P}_{\mathbf{1,1,1}}$ with zero point $p = (0, 1, 0)$, the $(z) = l$ in $\mathbf{P^2}$ becomes $(z)|_E = 3p$ on E. To encode n points on E, one chooses a homogeneous polynomial $w_{n/2}^{(\mathrm{hom})}(x, y, z)$ of degree $n/2$. One realizes that from its $3n/2$ zeroes on E (after Bezout's theorem) only n, say q_i, carry information as $n/2$ of them are always at p: for the rewriting $w_{n/2}^{(\mathrm{hom})}(x, y, z) = z^{n/2}w_{n/2}^{\mathrm{aff}}(x/z, y/z)$ shows manifestly $3n/2$ zeroes at p from the z-power and n poles at p and n zeroes at the q_i (note that x/z and y/z have a pole at p of order $-(1-3) = 2$ and $-(0-3) = 3$, resp., so the meromorphic function $w^{\mathrm{aff}}|_E$ has that divisor). Concretely for $n = 4$

$$w^{(\mathrm{hom})}(x, y, z) = a_4x^2 + a_3yz + a_2xz + a_0z^2 = z^2w^{\mathrm{aff}}\left(\frac{x}{z}, \frac{y}{z}\right). \tag{A.1}$$

This gives on E for the divisor of $w^{(\mathrm{hom})} = w$ resp. for the divisor of zeroes of w^{aff}

$$(w|_E) = \frac{n}{2}\sigma + \sum q_i, \quad (w^{\mathrm{aff}}|_E)_0 = \sum q_i. \tag{A.2}$$

Globally, as (x, y, z) have $\mathcal{L} = K_B^{-1}$-weights $(2, 3, 0)$, one has $a_i \in H^0(B, \mathcal{O}(\eta - ic_1))$.

Now in the B-model with the elliptic curve in $\mathbf{P}_{\mathbf{1,2,1}}$, one has the group zero $p_1 = (1, 1, 0)$ and the point $p_2 = (1, -1, 0)$; let $(z)|_E = p_1 + p_2 =: P$. The well-defined meromorphic functions x/z and y/z^2 on E have polar divisors P and $2P$, respectively. For $n = 4$

$$w_{n/2}^{\mathrm{aff}}\left(\frac{x}{z}, \frac{y}{z^2}\right) = \alpha_{20}\left(\frac{x}{z}\right)^2 + \alpha_{02}\frac{y}{z^2} + \alpha_{10}\frac{x}{z} + \alpha_{00}. \tag{A.3}$$

From the polar divisor of the meromorphic function $w^{\mathrm{aff}}|_E$, one reads off its total divisor[8] $(w^{\mathrm{aff}}|_E) = 2P - \sum^4 q_i$. So the corresponding homogeneous

[8]The divisor $2(\sigma_1 + \sigma_2) - \sum^4 q_i$ (in each fiber) of the meromorphic function w/z^2 gives the relation $\sum^4 q_i = 2(\sigma_1 + \sigma_2)$ or $\sum^4(q_i - \sigma_1) = 2(\sigma_2 - \sigma_1)$ in the divisor class group, or $\sum^4 q_i = 2\sigma_2$ in the group structure; so $\sum^4(2q_i - \sigma_2) = \sum^4 q_i^{\mathrm{eff}} = 0$, i.e., one has fiberwise an SU(n)-bundle, cf. Section 4.3.

polynomial $w^{(\mathrm{hom})} = w$

$$w_{n/2}^{(\mathrm{hom})}(x, y, z) = \alpha_{20}x^2 + \alpha_{02}y + \alpha_{10}xz + \alpha_{00}z^2 = z^2 w_{n/2}^{\mathrm{aff}} \tag{A.4}$$

has on E just the divisor given by the four zeroes q_i (as the double zeroes at the p_i of the z-power cancel with the double poles there of w^{aff}), i.e.,

$$(w|_E) = \sum q_i = (w^{\mathrm{aff}}|_E)_0. \tag{A.5}$$

(So this case is simpler than the A-model as no $n/2$ further zeroes at p are carried along.)

Concerning the $\mathcal{L} = K_B^{-1}$-weights of the α_{ij}, i.e., the transformation properties of the coefficient functions along the base of the fibration, x, y, z have $\mathcal{L}$-weights $1, 2, 0$ (do not confuse them with their indicated $\mathbf{P_{1,2,1}}$ weights), so $\alpha_{ij} \in H^0(B, \mathcal{O}(\eta - (i + j)c_1))$.

A.2 The Kähler cone

Concerning the base surface $B = \mathbf{F_m}$, note that the base and fiber classes b and f represent actual curves. The *effective cone* (non-negative linear combinations of classes of actual curves) is given by the condition $p \geq 0, q \geq 0$ on $\rho = pb + qf$; this we denote by $\rho \geq 0$. The *Kähler cone* $\mathcal{C}_B$ of B (where $\rho \in \mathcal{C}_B$ means $\rho\zeta > 0$ for all actual curves of classes ζ or equivalently $\rho b > 0, \rho f > 0$) is given by $\mathcal{C}_B = \{t_1 b^+ + t_2 f | t_i > 0\}$ (with $b^+ = b + mf$). For example on $\mathbf{F_2}$ one has $c_1 \notin \mathcal{C}_B$ as $c_1 b = 0$.

Let $J = x_1\sigma_1 + x_2\sigma_2 + H$ be an element in the Kähler cone $\mathcal{C}_X$ ($H \in \mathcal{C}_B$). Then demanding that its intersections with the curves F and $\sigma_i\alpha$ are non-negative amounts to

$$x_1 + x_2 > 0, \tag{A.6}$$

$$(H - x_i c_1)\alpha > 0. \tag{A.7}$$

Similarly, intersecting J^2 with σ_i and α and building also J^3 gives

$$(H - x_i c_1)^2 > 0, \tag{A.8}$$

$$\left(2\sum x_i H - \sum x_i^2 c_1\right)\alpha > 0, \tag{A.9}$$

$$\sum x_i (H - x_i c_1)^2 + \left(2\sum x_i H - \sum x_i^2 c_1\right) H > 0. \tag{A.10}$$

From this, one gets[9] the condition for J to be ample (positive)

$$J = x_1\sigma_1 + x_2\sigma_2 + H \in \mathcal{C}_X \Longleftrightarrow x_1 + x_2 > 0, \quad H - x_i c_1 \in \mathcal{C}_B \qquad \text{(A.11)}$$

So the condition for $C = \frac{n}{2}\Sigma + \eta$ to be *ample* (not only effective, cf. (3.4)) is

$$\eta - \frac{n}{2}c_1 \in \mathcal{C}_B. \qquad \text{(A.12)}$$

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[9] (A.10) is not independent as is clear for x_1 and x_2 individually non-negative, and follows in general with (A.6) (as the left-hand side of (A.10) is $\sum x_i(3H^2 - 3x_iHc_1 + 8x_i^2) \geq 8\sum x_i^3$ using (A.7) for $\alpha = H$). (A.9) follows from (A.7), giving $\Big(\sum x_i\, H - \sum x_i^2\, c_1\Big)\alpha \geq 0$, with (A.6), giving $\sum x_i H\alpha \geq 0$.

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