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Exact renormalization of a noncommutative ϕ^3 model in six dimensions

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Abstract

The noncommutative self-dual ϕ^3 model in six dimensions is quantized and essentially solved, by mapping it to the Kontsevich model. The model is shown to be renormalizable and asymptotically free, and solvable genus by genus. It requires both wavefunction and coupling constant renormalization. The exact ("all-order") renormalization of the bare parameters is determined explicitly, which turns out to depend on the genus 0 sector only. The running coupling constant is also computed exactly, which decreases more rapidly than predicted by the 1-loop beta-function. A phase transition to an unstable phase is found.

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1 Introduction

This paper is the third part of a series of papers [1, 2] studying the quantization of a noncommutative "self-dual" ϕ^3 model, by mapping it to the Kontsevich model. The model is characterized by an additional potential term in the action, which takes care of the UV/IR mixing following [3–6]. In the previous papers, we discussed the cases of two and four dimensions and showed that the model is renormalizable and essentially solvable. These cases are in some sense simpler because they are super-renormalizable. This is no longer the case in six dimensions, where new renormalization is required at each order, and the full complexity of an interacting quantum field theory is found with both wavefunction and coupling constant renormalization. Indeed, the commutative ϕ^3 model in six dimensions is known [7] to be asymptotically free.

Generalizing [1,2], we show in this paper that the self-dual NC ϕ^3 model in six dimensions can be renormalized and essentially solved in terms of a genus expansion. This is possible using the results of [8, 9] on the Kontsevich model, which must however be properly renormalized. The model has six relevant parameters. In particular, both coupling constant and wavefunction renormalization are required in the six-dimensional case, in addition to tadpole and mass renormalization which were sufficient in two and four dimensions. Remarkably, we find again that the renormalization is determined by the genus 0 sector only, as in the case of two and four dimensions.

After renormalization, all n-point functions can be computed in principle in terms of a genus expansion, and we give explicit expressions for the 1-, 2-, and some 3-point functions. We also find a critical surface $\alpha = 0$ in moduli space, which separates two different phases. This provides a model which contains the full complexity of renormalization of a nonsuper-renormalizable asymptotically free quantum field theory, while being solvable and hence fully under control. It is very remarkable that a six-dimensional interacting NC field theory allows such a detailed analytical description. It can therefore also serve as a testing ground for various approximation methods, which is of interest also in a more general context.

In particular, we are able to determine exactly the RG flow of the bare parameters, as well as the running of the "physical" coupling constant i.e., the 1PI 3-point function. We find that the 1-loop beta-function for the coupling constant correctly predicts asymptotic freedom, but wrongly predicts a $(\log N)^{-1/2}$ dependence on the scale as opposed to the correct $(\log N)^{-2}$ dependence.

Beyond the self-dual case $\Omega = 1$, perturbative renormalizability of similar models has been established using variants of a renormalization group approach, see e.g., [3–5, 10–12]. The present approach using matrix model methods is certainly appropriate for the case $\Omega = 1$. We indicate in Section 5 how it might be extended to $\Omega \neq 1$, which however is not carried out here. For a related matrix approach to NC field theory, see also [13, 14].

Being a continuation of our previous work on the two- and four-dimensional case, we will be brief in certain issues which have already been discussed there, and which apply without change. Nevertheless, the present paper is essentially self-contained. In Section 2, we define the ϕ^3 model under consideration and rewrite it as Kontsevich model. We then briefly recall the most important facts about the Kontsevich model in Section 3. The main technical analysis is contained in Section 4.1, while the main results of this paper are collected in Section 4.2. The asymptotic behavior of the 1-, 2- and 3-point functions is determined in Section 4.3, and further aspects such as relation with string field theory are briefly discussed in Section 5. The perturbative results such as the 1-loop beta-function are given in Section 6, and we conclude with a general discussion and outlook.

2 The noncommutative self-dual ϕ^3 model

We consider the noncommutative ϕ^3 model on the six-dimensional quantum plane $\mathbb{R}^6_\theta$, which is generated by self-adjoint operators[1] x_i satisfying the canonical commutation relations

$$[x_i, x_j] = i\theta_{ij}, \tag{2.1}$$

for $i, j = 1, \ldots, 6$. We also introduce

$$\tilde{x}_i = \theta^{-1}_{ij} x_j, \quad [\tilde{x}_i, \tilde{x}_j] = i\theta^{-1}_{ji} \tag{2.2}$$

assuming that θ_{ij} is nondegenerate. The model to be studied is defined by the action

$$\tilde{S} = \int_{\mathbb{R}^6_\theta} \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{\mu^2}{2}\phi^2 + \Omega^2(\tilde{x}_i\phi)(\tilde{x}_i\phi) + \frac{i\lambda_0}{3!}\phi^3. \tag{2.3}$$

An additional oscillator-type potential $\Omega^2(\tilde{x}_i\phi)(\tilde{x}_i\phi)$ is included following [3–6], making the model covariant under Langmann–Szabo duality and taking care of the UV/IR mixing. The dynamical object is the scalar field $\phi = \phi^\dagger$, which is a self-adjoint operator acting on the representation space

[1] We ignore operator-technical subtleties, since the model will be regularized using a cutoff N.

$\mathcal{H}$ of the algebra (2.1). The action is written with imaginary coupling $i\lambda_0$, so that the quantization is well defined for real λ_0; otherwise the action would be unbounded. We will see however that after quantization, one can perform an analytic continuation to real $\lambda_0' = i\lambda_0$.

Noting that the ∂_i are inner derivatives $\partial_i f = -i[\tilde{x}_i, f]$, the action can be written as

$$\tilde{S} = \int -(\tilde{x}_i\phi\tilde{x}_i\phi - \tilde{x}_i\tilde{x}_i\phi\phi) + \Omega^2\tilde{x}_i\phi\tilde{x}_i\phi + \frac{\mu^2}{2}\phi^2 + \frac{i\lambda_0}{3!}\phi^3 \tag{2.4}$$

using the cyclic property of the integral. For the "self-dual" point $\Omega = 1$, this simplifies further to

$$\tilde{S} = \int \left(\tilde{x}_i\tilde{x}_i + \frac{\mu^2}{2}\right)\phi^2 + \frac{i\lambda_0}{3!}\phi^3. \tag{2.5}$$

In order to quantize the theory, we need to include a linear (tadpole) counterterm $-\mathrm{Tr}(i\lambda)A\phi$ to the action (the explicit factor $i\lambda$ is inserted for convenience). Replacing the integral by $\int \to (2\pi\theta)^3$ Tr and adding a constant term for convenience, we are led to consider the action

$$S = \mathrm{Tr}\left(\frac{1}{2}J\phi^2 + \frac{i\lambda}{3!}\phi^3 - (i\lambda)A\phi - \frac{1}{3(i\lambda)^2}J^3 - JA\right). \tag{2.6}$$

Here $J = 2Z(2\pi\theta)^3(\sum_i \tilde{x}_i\tilde{x}_i + \frac{\mu^2}{2})$ is essentially the Hamiltonian of a three-dimensional quantum mechanical harmonic oscillator. A wavefunction renormalization Z has also been introduced, which for the cubic term is absorbed in the redefined coupling constant λ, in order to simplify the notation. The field ϕ will be the renormalized, finite physical field. In the usual basis of eigenstates, J then diagonalizes as

$$J|n_1, n_2, n_3\rangle = 16\pi^3\theta^2 Z\left(n_1 + n_2 + n_3 + \frac{\mu^2\theta + 3}{2}\right)|n_1, n_2, n_3\rangle, \quad n_i \in \{0, 1, 2, \ldots\} \tag{2.7}$$

assuming that θ_{ij} has the canonical form $\theta_{12} = -\theta_{21} = \theta_{34} = -\theta_{43} = \cdots =: \theta$. To simplify the notation, we will use the convention

$$n \equiv (n_1, n_2, n_3), \quad \underline{n} \equiv n_1 + n_2 + n_3 \tag{2.8}$$

throughout this paper, keeping in mind that n denotes a triple index. It turns out that we need

$$A = a_0 + a_1 J + a_2 J^2. \tag{2.9}$$

By suitable shifts[2] in the field ϕ

$$\tilde{\phi} = \phi + \frac{1}{i\lambda} J = X + \frac{1}{i\lambda} M \tag{2.10}$$

one can now either eliminate the linear term or the quadratic term in the action,

$$S = \mathrm{Tr}\left(-\frac{1}{2i\lambda} M^2 \tilde{\phi} + \frac{i\lambda}{3!} \tilde{\phi}^3 \right) = \mathrm{Tr}\left(\frac{1}{2} M X^2 + \frac{i\lambda}{3!} X^3 - \frac{1}{3(i\lambda)^2} M^3 \right), \tag{2.11}$$

where

$$M = \sqrt{J^2 + 2(i\lambda)^2 A} = xZ\sqrt{\tilde{J}^2 + d}, \tag{2.12}$$

$$\tilde{J} = \frac{J}{Z} + \frac{(i\lambda)^2 a_1}{x^2 Z}, \tag{2.13}$$

$$x = \sqrt{1 + 2(i\lambda)^2 a_2}, \tag{2.14}$$

$$d = -\left(\frac{(i\lambda)^2 a_1}{x^2 Z} \right)^2 + 2\frac{(i\lambda)^2 a_0}{x^2 Z^2}. \tag{2.15}$$

Equation (2.11) has precisely the form of the Kontsevich model [9]. The linear coupling of the field to the source M^2 resp. $\tilde{J}^2$ will be very useful for computing correlation functions. $\tilde{J}$ now has eigenvalues

$$\tilde{J}|n_1, n_2, n_3\rangle = 16\pi^3\theta^2 \left(n_1 + n_2 + n_3 + \frac{\mu_R^2\theta + 3}{2} \right) |n_1, n_2, n_3\rangle, \tag{2.16}$$

which will be finite after renormalization, and

$$\delta\mu^2\theta = \mu^2\theta - \mu_R^2\theta = -\frac{2}{16\pi^3\theta^2} \frac{(i\lambda)^2 a_1}{x^2 Z}. \tag{2.17}$$

2.1 Quantization, partition function and correlators

The quantization of model (2.6) resp. (2.11) is defined by an integral over all hermitian $N^3 \times N^3$ matrices ϕ, where N serves as a UV and IR cutoff. The partition function

$$Z(M) = \int D\tilde{\phi} \, \exp\left(-\mathrm{Tr}\left(-\frac{1}{2i\lambda} M^2 \tilde{\phi} + \frac{i\lambda}{3!} \tilde{\phi}^3 \right) \right) = \mathrm{e}^{F(M)} \tag{2.18}$$

defines the "free energy" $F(M)$, which is a function of the eigenvalues of M resp. $\tilde{J}$. Since N is finite, we can freely switch between the various

[2]For the quantization, the integral for the diagonal elements is then defined via analytical continuation, and the off-diagonal elements remain hermitian since J is diagonal.

parametrizations (2.6), (2.11) involving M, J, ϕ, or $\tilde{\phi}$. Correlators or "n-point functions" are defined through

$$\langle \phi_{i_1 j_1} \cdots \phi_{i_n j_n} \rangle = \frac{1}{Z(M)} \int D\phi \, \exp(-S) \phi_{i_1 j_1} \cdots \phi_{i_n j_n}, \tag{2.19}$$

keeping in mind that each index denotes a multi-index (2.8). Using the symmetry $Z(M) = Z(U^{-1}MU)$ for $U \in U(N^3)$, we can assume that M is diagonalized with ordered eigenvalues m_i. There is a residual $U(1) \times U(3) \times U(6) \times \cdots$ symmetry, reflecting the degeneracy of J. This implies certain obvious "index conservation laws", such as $\langle \phi_{kl} \rangle = \delta_{kl} \langle \phi_{ll} \rangle$, etc.

The nontrivial task is to show that all these n-point functions with finite indices have a well-defined and nontrivial limit $N \to \infty$, for a suitable scaling of the bare parameters. In addition, the index dependence of these n-point functions must be nontrivial. Recall that the wavefunction renormalization is already taken into account in (2.7), so that ϕ is the finite, physical field. The free parameters should in principle be determined by choosing renormalization conditions, such as $\langle \phi_{00} \phi_{00} \rangle = \frac{1}{2\pi} \frac{1}{\mu_R^2 \theta + 1}$, etc. These conditions can easily be solved using the explicit results given in Sections 4.1 and 4.2, relating the bare parameters with the "physical" quantities.

Noting that the field $\tilde{\phi}$ couples linearly to M^2 resp. $\tilde{J}^2$ in (2.18), one can compute the connected n-point functions by acting with the derivative operator $2i\lambda \frac{\partial}{\partial J^2} = 2i \frac{\lambda}{x^2 Z^2} \frac{\partial}{\partial \tilde{J}^2}$ on F. Anticipating some results below, we introduce the quantities

$$i\lambda_R = \frac{i\lambda}{x^2 Z^2}, \quad r = \frac{1}{x^2 Z}, \tag{2.20}$$

which turn out to be finite after renormalization. Then

$$\langle \tilde{\phi}_{i_1 j_1} \cdots \tilde{\phi}_{i_n j_n} \rangle_c = \left(2i\lambda_R \frac{\partial}{\partial \tilde{J}^2_{i_1 j_1}} \right) \cdots \cdot \left(2i\lambda_R \frac{\partial}{\partial \tilde{J}^2_{i_n j_n}} \right) F(\tilde{J}^2). \tag{2.21}$$

Since the connected n-point functions are independent of the shifts $\tilde{\phi} = \phi + \frac{J}{i\lambda}$ (2.10) for $n \geq 2$, the lhs coincides with the desired correlator $\langle \phi_{i_1 j_1} \cdots \phi_{i_n j_n} \rangle_c$ for $n \geq 2$. This strongly suggests that λ_R should be finite.

Using the Dyson–Schwinger equations for the path integral (2.18), one can derive a number of nontrivial identities for the n-point functions. Since their derivation was already given in [1], we simply quote them here with the appropriate substitutions. In particular, one finds for the propagator

$$\langle\tilde{\phi}_{kl}\tilde{\phi}_{lk}\rangle = \frac{2i\lambda}{m_k^2 - m_l^2}\langle\tilde{\phi}_{kk} - \tilde{\phi}_{ll}\rangle \tag{2.22}$$

for $k \neq l$ (no sum), where m_k denotes the eigenvalues of M. This gives

$$\langle\phi_{kl}\phi_{lk}\rangle = \frac{2r}{\tilde{J}_k + \tilde{J}_l} + \frac{2i\lambda_R}{\tilde{J}_k^2 - \tilde{J}_l^2}\langle\phi_{kk} - \phi_{ll}\rangle \tag{2.23}$$

using (2.12). The first term has the form of the free contribution, and the second is a quantum correction. The latter reduces to the 1-point functions, which will be obtained from the Kontsevich model. Similarly, one can show [1]

$$\left\langle\tilde{\phi}_{kl}\tilde{\phi}_{lk}\tilde{\phi}_{kk}\right\rangle = \frac{2i\lambda}{m_k^2 - m_l^2}\left\langle(\tilde{\phi}_{kk} - \tilde{\phi}_{ll})\tilde{\phi}_{kk} - \tilde{\phi}_{kl}\tilde{\phi}_{lk}\right\rangle \tag{2.24}$$

(no sum) for $k \neq l$. Using (2.23), this gives the connected part

$$\langle\phi_{kl}\phi_{lk}\phi_{kk}\rangle_c = \frac{2i\lambda}{m_k^2 - m_l^2}\Big(\langle(\phi_{kk} - \phi_{ll})\phi_{kk}\rangle_c - \langle\phi_{kl}\phi_{lk}\rangle\Big). \tag{2.25}$$

Clearly these relations can be generalized, greatly reducing the number of independent correlators. However, we will establish finiteness of the general correlation functions directly, by showing that the appropriate derivatives of the generating function $F(\tilde{J})$ are finite and well defined after renormalization.

3 The Kontsevich model: facts and background

The Kontsevich model is defined as a matrix integral

$$Z^{\text{Kont}}(\tilde{M}) = \mathrm{e}^{F^{\text{Kont}}} = \frac{\int dX\ \exp\{\mathrm{Tr}(-(\tilde{M}X^2/2) + i(X^3/6))\}}{\int dX\ \exp\{-\mathrm{Tr}(\tilde{M}X^2/2)\}} \tag{3.1}$$

over hermitian $N^3 \times N^3$ matrices X, where the parameter $\tilde{M}$ is some given hermitian $N^3 \times N^3$ matrix. This model has been introduced by Kontsevich [9] as a combinatorial way of computing certain topological quantities (intersection numbers) on moduli spaces of Riemann surfaces with punctures. It turns out to have an extremely rich structure related to integrable models (KdV flows) and Virasoro constraints. For our purpose, the most important results are those of [8, 9, 15], which provide explicit expressions for the genus expansion of the free energy. Note that λ can be introduced by rescaling the variables, writing $M = \lambda^{2/3}\tilde{M}$.

The matrix integral in (3.1) and its large N limit can be defined rigorously in terms of its asymptotic series. This involves the variables [9]

$$t_r := -(2r+1)!!\,\theta_{2r+1}, \quad \theta_r := \frac{1}{r}\mathrm{Tr}\,\tilde{M}^{-r}. \tag{3.2}$$

Then the large N limit of the partition function $Z^{\mathrm{Kont}}(\tilde{M})$ can be rigorously defined [8, 9], which turns out to be a function of these new variables only, $Z^{\mathrm{Kont}}(\tilde{M}) = Z^{\mathrm{Kont}}(\theta_i)$. In our case, θ_r is divergent for $r = 1, 2, 3$, which indicates that the model requires renormalization.

One can furthermore consider the genus expansion

$$\ln Z^{\mathrm{Kont}} = F^{\mathrm{Kont}} = \sum_{g\geq 0} F_g^{\mathrm{Kont}} \tag{3.3}$$

by drawing the Feynman diagrams on the appropriate Riemann surface, as usual for matrix models. In [8] it was shown that all F_g^{Kont} can be computed using the KdV equations and the Virasoro constraints, which allows to find closed expressions for any genus g. They are given in terms of the following variables

$$I_k(u_0, t_i) = \sum_{p\geq 0} t_{k+p}\frac{u_0^p}{p!} = -(2k-1)!!\sum_i \frac{1}{(\tilde{m}_i^2 - 2u_0)^{k+(1/2)}}, \tag{3.4}$$

where u_0 is given by the solution of the implicit equation

$$u_0 = I_0(u_0, t_i) = -\sum_i \frac{1}{\sqrt{\tilde{m}_i^2 - 2u_0}}. \tag{3.5}$$

These variables turn out to be more useful for our purpose. Using the KdV equations, [8] obtain the following explicit formulas:

$$\begin{aligned} F_0^{\mathrm{Kont}} &= \frac{1}{3}\sum_i \tilde{m}_i^3 - \frac{1}{3}\sum_i (\tilde{m}_i^2 - 2u_0)^{3/2} - u_0\sum_i (\tilde{m}_i^2 - 2u_0)^{1/2} \\ &\quad + \frac{u_0^3}{6} - \frac{1}{2}\sum_{i,k}\ln\left\{\frac{(\tilde{m}_i^2 - 2u_0)^{1/2} + (\tilde{m}_k^2 - 2u_0)^{1/2}}{\tilde{m}_i + \tilde{m}_k}\right\} \end{aligned} \tag{3.6}$$

$$F_1^{\mathrm{Kont}} = \frac{1}{24}\ln\frac{1}{1 - I_1}, \tag{3.7}$$

$$F_2^{\mathrm{Kont}} = \frac{1}{5760}\left[5\frac{I_4}{(1-I_1)^3} + 29\frac{I_3 I_2}{(1-I_1)^4} + 28\frac{I_2^3}{(1-I_1)^5}\right], \tag{3.8}$$

etc. This form of F_0 was first found in [15]. The sums over multi-indices (2.8) are to be interpreted as

$$\sum_i \equiv \sum_{i_1, i_2, i_3 = 0}^{N-1}$$

truncating the harmonic oscillators in (2.7). For our purpose, the most important result is that all F_g^{Kont} with $g \geq 2$ are given by polynomials $F_g^{\mathrm{Kont}} = \sum \chi_{\{l_k\}} x_2^{l_2} \cdots x_{3g-2}^{l_{3g-2}}$ in $x_k = I_k/(1-I_1)^{(2k+1)/3}$ with the constraint

$$\sum_{2 \leq k \leq 3g-2} (k-1) l_k = 3g-3, \tag{3.9}$$

where l_k is the power of x_k, and $\chi_{\{l_k\}}$ is some (rational) intersection number.

While many of these expressions are divergent as $N \to \infty$ for $\tilde{J}$ given by (2.16), the physically relevant observables will be convergent after renormalization.

4 The Kontsevich model applied to the ϕ^3 model

In order to apply the above results to the noncommutative ϕ^3 model, we need the following slightly modified version of the Kontsevich model, corresponding to the action (2.11):

$$Z(M) = \exp(F(M)) = Z^{\mathrm{Kont}}[\tilde{M}] Z^{\mathrm{free}}[\tilde{M}] \exp\left(\frac{1}{3(i\lambda)^2} \mathrm{Tr} M^3\right), \tag{4.1}$$

where

$$\begin{aligned} F_0 &:= F_0^{\mathrm{Kont}} + F_{\mathrm{free}} + \frac{1}{3(i\lambda)^2} \mathrm{Tr}\, M^3 \\ &= -\frac{1}{3} \sum_i \sqrt[3]{\tilde{m}_i^2 - 2u_0} - u_0 \sum_i \sqrt{\tilde{m}_i^2 - 2u_0} \\ &\quad + \frac{u_0^3}{6} - \frac{1}{2} \sum_{i,k} \ln\left(\sqrt{\tilde{m}_i^2 - 2u_0} + \sqrt{\tilde{m}_k^2 - 2u_0}\right). \end{aligned} \tag{4.2}$$

and $F_g = F_g^{\mathrm{Kont}}$ for $g \geq 1$. In the present case, the eigenvalues $\tilde{m}_i$ are given by (2.12)

$$\tilde{m}_i = \lambda^{-2/3} x Z \sqrt{\tilde{J}_i^2 + d}. \tag{4.3}$$

The model as it stands is ill-defined for $N \to \infty$, since u_0 and many of the above sums are divergent. However, we can recast (4.2) using more appropriate variables, which suggests how to renormalize the various parameters, i.e., how to scale them with N. Note that only the combinations $\sqrt{\tilde{m}_i^2 - 2u_0}$

enter in (3.4) and (4.2), which can be rewritten using (2.12) as

$$\sqrt{\tilde{m}_i^2 - 2u_0} = xZ\lambda^{-2/3}\sqrt{\tilde{J}_k^2 + 2b}. \tag{4.4}$$

Here b is defined through

$$\frac{x^3 Z^3}{(i\lambda)^2}\left(b - \frac{1}{2}d\right) = xZ\lambda^{-2/3}u_0 = -\sum_i \frac{1}{\sqrt{\tilde{J}_i^2 + 2b}} \tag{4.5}$$

using the constraint (3.5), which is replaced by (4.5) henceforth. Eliminating u_0 using (4.5) and expressing λ in terms of λ_R, F_0 takes the form

$$F_0 = \frac{1}{3(i\lambda_R)^2 xZ}\sum_i \sqrt[3]{\tilde{J}_i^2 + 2b} - \frac{b-(1/2)d}{(i\lambda_R)^2 xZ}\sum_i \sqrt{\tilde{J}_i^2 + 2b}$$
$$- \frac{(b-(1/2)d)^3}{6(i\lambda_R)^4 x^2 Z^2} - \frac{1}{2}\sum_{i,k} \ln \frac{1}{(\lambda_R^2 xZ)^{1/3}}\left(\sqrt{\tilde{J}_i^2 + 2b} + \sqrt{\tilde{J}_k^2 + 2b}\right). \tag{4.6}$$

The quantities $\tilde{J}$, b, and λ_R will be finite after renormalization, rendering the model well-defined. We consider $F = F(\tilde{J}^2)$ as a function of (the eigenvalues of) $\tilde{J}^2$ from now on, while b is implicitly determined by (4.5). Since the eigenvalues $\tilde{J}_k$ only enter through the combination $\sqrt{\tilde{J}_k^2 + 2b}$, we note that the eigenvalues can be analytically continued as long as this square-root is well-defined.

We can now compute various n-point functions by taking partial derivatives of $F = \sum_g F_g$ with respect to $\tilde{J}^2$, as indicated in Section 2.1. For the "diagonal" n-point functions $\langle\tilde{\phi}_{ii}\cdots\tilde{\phi}_{kk}\rangle_c$, this amounts to varying the eigenvalues $\tilde{J}_k^2$. In doing so, we must remember that b depends implicitly on $\tilde{J}_k^2$ through the constraint (4.5). However, some of these computations simplify recalling that the constraint (4.5) for b arises automatically through the e.o.m: using

$$\frac{\partial}{\partial b}F_0(\tilde{J}_i^2; b) = -\frac{1}{2}\left(\frac{b-(1/2)d}{(i\lambda_R)^2 xZ} + \sum_i \frac{1}{\sqrt{\tilde{J}_i^2 + 2b}}\right)^2 = 0 \tag{4.7}$$

we can write

$$\frac{d}{d\tilde{J}_i^2}F_0(\tilde{J}_i^2) = \frac{\partial}{\partial \tilde{J}_i^2}F_0(\tilde{J}_i^2; b) + \frac{\partial}{\partial b}F_0(\tilde{J}_i^2; b)\frac{\partial}{\partial \tilde{J}_i^2}b = \frac{\partial}{\partial \tilde{J}_i^2}F_0(\tilde{J}_i^2; b). \tag{4.8}$$

Thus for derivatives of order ≤ 2 w.r.t. $\tilde{J}_k^2$, we can simply ignore b and treat it as independent variable, since the omitted terms (4.7) vanish anyway once the constraint is imposed.

4.1 Renormalization

This section contains some computations required to show finiteness of the n-point functions for any genus g. This will establish Theorem 4.1 in Section 4.2.

The 1-point function We can now determine the required renormalization by considering the 1-point function. Using (4.6), (4.7), and (4.5), the genus 0 contribution is

$$\begin{aligned}\langle\phi_{kk}\rangle_{g=0} &= 2i\lambda_R \frac{\partial}{\partial \tilde{J}_k^2} F_0(\tilde{J}^2) - \frac{J_k}{i\lambda_R x^2 Z^2} \\ &= \frac{1}{i\lambda_R x Z} y_k + (i\lambda_R)\sum_j \frac{1}{y_k\sqrt{\tilde{J}_j^2+2b}+(\tilde{J}_j^2+2b)} \\ &\quad - \frac{r}{i\lambda_R}\tilde{J}_k + i\lambda_R Z^2 a_1 \\ &=: W(y_k) \end{aligned} \tag{4.9}$$

using (2.20), which must be finite and well defined as $N \to \infty$. Here we define

$$y_k = \sqrt{\tilde{J}_k^2 + 2b}. \tag{4.10}$$

We will find that $W(y)$ becomes a smooth function of y after renormalization, which amounts to the statement that the index dependence of the 1-point function is "smooth". This is a typical feature of matrix models, reflecting some "smooth" distribution of eigenvalues. This becomes here part of the statement of renormalizability. To proceed, we need to understand the function

$$f(y) := \sum_j \frac{1}{y\sqrt{\tilde{J}_j^2+2b}+(\tilde{J}_j^2+2b)} \tag{4.11}$$

which as it stands is ill-defined for $N \to \infty$.

4.1.1 Renormalization of $f(y)$

In order to make sense of $f(y)$, we consider the Taylor expansion of

$$f(y;\tilde{J}) = \sum_j \frac{1}{y\sqrt{\tilde{J}_j^2+2b}+(\tilde{J}_j^2+2b)} = f_0(\tilde{J}) + y f_1(\tilde{J}) + f_R(y;\tilde{J}) \tag{4.12}$$

in y, in analogy to the usual strategy in renormalization. Here

$$f_0(\tilde{J}) = \sum_j \frac{1}{\tilde{J}_j^2 + 2b} = f_0 + f_{0,R}(\delta\tilde{J}) \tag{4.13}$$

$$f_1(\tilde{J}) = -\sum_j \frac{1}{(\tilde{J}_j^2 + 2b)^{3/2}} = f_1 + f_{1,R}(\delta\tilde{J}), \tag{4.14}$$

$$f_R(y;\tilde{J}) = y^2 \sum_j \frac{1}{(\tilde{J}_j^2 + 2b)\left(y\sqrt{\tilde{J}_j^2 + 2b} + (\tilde{J}_j^2 + 2b)\right)}, \tag{4.15}$$

where f_1, f_2 are divergent constants obtained by fixing $\tilde{J}$ as in (2.16), while the $f_{i,R}(\delta\tilde{J})$ are regular (convergent) functions obtained by taking into account variations $\delta\tilde{J}$ of the eigenvalues. This is necessary e.g., to compute partial derivatives w.r.t. $\tilde{J}$. Thus

$$\begin{aligned} f_0 &= \frac{1}{(16\pi^3\theta^2)^2}\int_0^N dx_1 dx_2 dx_3 \frac{1}{(x_1 + x_2 + x_3 + (3 + \mu_R^2\theta)/2)^2} + \text{finite} \\ &= \frac{1}{(16\pi^3\theta^2)^2}\left((6\log(2) - 3\log(3))N - \frac{3 + \mu_R^2\theta}{2}\log(N)\right) + \text{finite}, \end{aligned} \tag{4.16}$$

and similarly

$$f_1 = -\frac{1}{2}\frac{1}{(16\pi^3\theta^2)^3}\log(N) + \text{finite}. \tag{4.17}$$

The remaining part $f_R(y;\tilde{J})$ is well defined and convergent, provided b and $\tilde{J}_k$ i.e., μ_R are finite, which will be assumed from now on.[3] To understand it better, we note that

$$f_R''(y) = f''(y) = 2\sum_j \frac{\tilde{J}_j^2 + 2b}{\left(y\sqrt{\tilde{J}_j^2 + 2b} + (\tilde{J}_j^2 + 2b)\right)^3}$$

is positive, and similarly $f_R'(y) \geq 0$. Hence $f_R(y)$ is a rather simple convex smooth function of $y > 0$, which satisfies $f_R(0) = f_R'(0) = 0$, and it remains only to determine its asymptotic behavior for large $y \approx \tilde{J}_N$, i.e., small x.

[3] We do not pursue the possibility of divergent μ_R, which does not appear to be interesting.

This can be obtained by writing

$$f_R(y) \approx \frac{y}{(16\pi^3\theta^2)^3} \int_{\tilde{J}_0/y}^{\tilde{J}_N/y} d^3x \frac{1}{(x^2+2b/y^2)\left(\sqrt{x^2+2b/y^2}+(x^2+2b/y^2)\right)}, \tag{4.18}$$

where $x = \frac{\tilde{J}}{y}$. This integral is convergent for large x, but logarithmically divergent at the origin i.e., for large $y \sim J_N$. We can hence replace the upper integration limit in (4.18) by $x = 1$, to obtain the asymptotic behavior for large y:

$$f_R(y) \approx \frac{y}{(16\pi^3\theta^2)^3} \int_{\tilde{J}_0/y}^{1} d^3x \frac{1}{(x^2+2b/y^2)^{3/2}} \approx \frac{1}{2}\frac{1}{(16\pi^3\theta^2)^3}\, y \log\left(\frac{y}{\tilde{J}_0}\right). \tag{4.19}$$

4.1.2 Renormalization of $\langle\phi_{kk}\rangle_{g=0}$

We have seen that only f_0 and f_1 are divergent in (4.12), while $f_R(y_k)$ is finite and well defined as $N \to \infty$. Then (4.9) becomes

$$\begin{aligned}
\langle\phi_{kk}\rangle_{g=0} &= \frac{y_k}{i\lambda_R x Z} - \frac{r\tilde{J}_k}{(i\lambda_R)} + i\lambda_R Z^2 a_1 + i\lambda_R(f_0(\tilde{J}) + y_k f_1(\tilde{J}) + f_R(y_k;\tilde{J})) \\
&= (i\lambda_R Z^2 a_1 + i\lambda_R f_0) + \left(\frac{1}{i\lambda_R x Z} + i\lambda_R f_1\right) y_k - \frac{r\tilde{J}_k}{i\lambda_R} \\
&\quad + i\lambda_R\left(f_R(y_k;\tilde{J}) + f_{0,R}(\delta\tilde{J}) + y_k f_{1,R}(\delta\tilde{J})\right) \\
&= c - \frac{y_k}{i\lambda_R\alpha} - \frac{r\tilde{J}_k}{i\lambda_R} + i\lambda_R\left(f_R(y_k) + f_{0,R}(\delta\tilde{J}) + y_k f_{1,R}(\delta\tilde{J})\right).
\end{aligned} \tag{4.20}$$

Here we define the quantities

$$\alpha = -\frac{xZ}{1+(i\lambda_R)^2 xZ f_1} = -\frac{1}{1/xZ + (i\lambda_R)^2 f_1}, \tag{4.21}$$

$$c = i\lambda_R(Z^2 a_1 + f_0). \tag{4.22}$$

Since $\tilde{J}_k$, y_k, and $f_R(y_k)$ are independent functions of k (as long as $b \neq 0$), it follows that $\langle\phi_{kk}\rangle_{g=0}$ is finite if and only if the four quantities

$$\left(i\lambda_R, c, \frac{1}{i\lambda_R\alpha}, \frac{r}{i\lambda_R}\right) \tag{4.23}$$

are finite. Remarkably, the condition that α be finite will also guarantee that the higher genus contributions are finite. The second form of (4.21) implies that this is possible only for real coupling $(i\lambda_R)$, since x and Z should be positive, while $f_1 \sim -\log N$.

4.1.3 Higher derivatives and higher genus contributions

The connected part of the n-point functions for diagonal entries $\langle\tilde{\phi}_{i_1 i_1}\cdots\tilde{\phi}_{i_n i_n}\rangle_c$ are obtained by taking higher derivatives $(i\lambda_R)^n \frac{\partial}{\partial \tilde{J}^2_{i_1}}\cdots\frac{\partial}{\partial \tilde{J}^2_{i_n}}$ of $F(\tilde{J}^2)$ resp. (4.20). Since the (infinite) shift $\tilde{\phi} = \phi + \frac{J}{i\lambda}$ drops out from the connected n-point function for $n \geq 2$, these coincide with $\langle\phi_{i_1 i_1}\cdots\phi_{i_n i_n}\rangle_c$ for $n \geq 2$, and we have to show that they are finite.

Consider first the genus 0 contributions. To compute higher derivatives of F_0 w.r.t. $\tilde{J}^2$, we must also take into account the implicit dependence of b on $\tilde{J}^2$. Indeed b is a smooth function of $\tilde{J}$ as shown in (4.32). Furthermore, recall that $\frac{\partial}{\partial b}F_0(\tilde{J}_i^2; b)$ vanishes through the constraint (4.7), however this is no longer true for the higher derivatives. In particular, taking derivatives of (4.7), we find

$$-\frac{\partial^2}{\partial b^2}F_0(\tilde{J}_i^2; b) = \frac{\partial}{\partial b}\left(\frac{b-(1/2)d}{(i\lambda_R)^2 xZ} + \sum_i \frac{1}{\sqrt{\tilde{J}_i^2+2b}}\right) = \frac{1-I_1}{(i\lambda_R)^2 xZ},$$

$$-\frac{\partial^2}{\partial \tilde{J}_k^2 \partial b}F_0(\tilde{J}_i^2; b) = \frac{\partial}{\partial \tilde{J}_k^2}\left(\frac{b-(1/2)d}{(i\lambda_R)^2 xZ} + \sum_i \frac{1}{\sqrt{\tilde{J}_i^2+2b}}\right) = \frac{1}{\sqrt[3]{\tilde{J}_i^2+2b}} \tag{4.24}$$

which are both finite and smooth using (4.28) below. Combining this with the explicit form (4.20) and using the results of Section 4.1.1, it follows that all higher derivatives of $F_0(\tilde{J}^2)$ w.r.t. $\tilde{J}_k^2$ are finite.

For the higher genus contributions, we also need the derivatives of the quantities

$$I_k(\tilde{J}_i^2, b) = -(2k-1)!!(\lambda_R^2 xZ)^{(2k+1)/3}\sum_i \frac{1}{(\tilde{J}_i^2+2b)^{k+(1/2)}}. \tag{4.25}$$

In particular,

$$\begin{aligned} I_1(\tilde{J}_k^2, b) &= -\lambda_R^2 xZ \sum_i \frac{1}{(\tilde{J}_i^2+2b)^{3/2}} = \lambda_R^2 xZ f_1(\tilde{J}) \\ &= \lambda_R^2 xZ(f_1 + f_{1,R}(\delta\tilde{J})), \end{aligned} \tag{4.26}$$

where $f_{1,R}(\delta\tilde{J})$ is a finite and smooth function (4.14) of $\tilde{J}_k^2$ and b (which vanishes for $\delta\tilde{J} = 0$). Hence for I_1 to be finite we need

$$\frac{1}{\lambda_R^2 xZ} \sim f_1 \sim \log(N), \tag{4.27}$$

which using (4.25) implies that all higher I_k vanish for $N \to \infty$ unless $(1 - I_1) \sim 0$. Hence the higher genus contributions can be nontrivial only if we carefully take a "double-scaling" limit[4] and require $\frac{I_k}{(1-I_1)^{(2k+1)/3}}$ to be finite, while $I_1 \to 1$. Using (4.26), we have

$$\frac{1}{\lambda_R^2 xZ}(1 - I_1) = \frac{1}{\alpha(i\lambda_R)^2} - f_{1,R}(\delta\tilde{J}) \tag{4.28}$$

where α is defined in (4.21). It follows that

$$\begin{aligned}\frac{I_k}{(1 - I_1)^{(2k+1)/3}} &= -\frac{(2k-1)!!}{\left(\frac{1}{(i\lambda_R)^2\alpha} - f_{1,R}(\delta\tilde{J})\right)^{(2k+1)/3}} \sum_i \frac{1}{(\tilde{J}_i^2 + 2b)^{k+(1/2)}} \\ &\sim -(2k-1)!!((i\lambda_R)^2\alpha)^{(2k+1)/3} \sum_i \frac{1}{(\tilde{J}_i^2 + 2b)^{k+(1/2)}} \end{aligned} \tag{4.29}$$

(the last form holds for $\delta\tilde{J} = 0$) is finite and nontrivial for $k \geq 2$, provided

$$\alpha = \text{finite}, \tag{4.30}$$

assuming that $i\lambda_R$ is finite. In particular, all derivatives of F_g for $g \geq 2$ w.r.t. $\tilde{J}_k^2$ are manifestly finite, and thus all genus $g \geq 2$ contributions to the diagonal n-point functions are finite. Using (4.28) and (3.7), this also holds for genus 1.

Finally, from the constraint (4.5), we derive

$$\begin{aligned}\frac{\partial}{\partial\tilde{J}_k^2} b &= -(i\lambda_R)^2 xZ \frac{\partial}{\partial\tilde{J}_k^2}\left(\sum_i \frac{1}{\sqrt{\tilde{J}_i^2 + 2b}}\right) \\ &= \frac{1}{2}(i\lambda_R)^2 xZ \left(\frac{1}{\sqrt[3]{\tilde{J}_k^2 + 2b}} + \sum_i \frac{2}{\sqrt[3]{\tilde{J}_k^2 + 2b}} \frac{\partial}{\partial\tilde{J}_k^2} b\right), \end{aligned} \tag{4.31}$$

which using (4.28) gives

$$\frac{\partial}{\partial\tilde{J}_k^2} b = -\frac{1}{2} \frac{1}{\sqrt[3]{\tilde{J}_k^2 + 2b}} \frac{1}{1/(i\lambda_R)^2\alpha - f_{1,R}(\delta\tilde{J})}. \tag{4.32}$$

This is again finite, using the above assumptions. Hence b depends smoothly on the variations in $\tilde{J}_k$ provided α is finite. If $\alpha = \infty$, then the constraint (4.5) cannot be solved any more for b as a function of the eigenvalues $\tilde{J}$ (and their variation), rendering the model nonrenormalizable. This implies that the genus 0 sector fully determines the required renormalization, as was found previously in two and four dimensions [1, 2].

[4] This is quite reminiscent of the double-scaling limit of matrix models in the context of 2D gravity.

4.2 Main result and renormalization group flow

We have now established all required formulas for the derivatives of $F(\tilde{J})$, and hence for the diagonal n-point functions. We also showed that all of these are finite and have a well-defined as $N \to \infty$, provided a few renormalization conditions hold. Let us collect these conditions and use them to determine the required scaling of the bare parameters.

Assume first that $\lambda_R = 0$. Imposing this exactly (i.e., independent of N) implies $\alpha = -xZ$ and $i\lambda = i\lambda_R \alpha^2 = 0$. This is the free case with x and Z finite,[5] as can be seen e.g., from (4.20) and (4.49).

Hence assume $\lambda_R \neq 0$. We assume also that $b \neq 0$ (we will see that this restriction is in fact not important), so that the four quantities in (4.23) are finite. By taking products, this implies that r and $\frac{1}{\alpha}$ are also finite, hence $\alpha \neq 0$. Finite α is in fact also required for the higher genus contributions to be finite. Using (4.21), we then obtain

$$\frac{1}{xZ} = -(i\lambda_R)^2 f_1 - \frac{1}{\alpha} = \frac{1}{2}\frac{(i\lambda_R)^2}{(16\pi^3\theta^2)^3}\log(N) - \frac{1}{\alpha}, \tag{4.33}$$

which implies[6]

$$\begin{aligned} i\lambda = i\lambda_R x^2 Z^2 &= \frac{(i\lambda_R)}{(-(1/\alpha) + (1/2)((i\lambda_R)^2 \log(N)/(16\pi^3\theta^2)^3))^2} \\ &\sim (i\lambda_R)^{-3}\log(N)^{-2} \end{aligned} \tag{4.34}$$

for large N. Note that in a perturbative approach i.e., if this is formally expanded in terms of λ_R, this is divergent and requires renormalization at each order. Nevertheless, the closed form (4.34) is rather simple, with leading behavior $(i\lambda_R)^{-3}\log(N)^{-2}$. Z is then determined through

$$Z = r\frac{i\lambda}{i\lambda_R} \sim \log(N)^{-2} \tag{4.35}$$

since $r = \frac{1}{x^2 Z}$ is finite, and similarly

$$x = \frac{1}{\sqrt{rZ}} \sim \log(N). \tag{4.36}$$

This gives a_2 using (2.14). a_1 is determined from (4.22),

$$Z^2 a_1 = \frac{c}{i\lambda_R} - f_0 \tag{4.37}$$

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[5]We shall not pursue here the possibility of other more subtle scaling limits.

[6]Note also that (4.33) implies that $(i\lambda_R)$ should be real, since xZ should be positive.

whose leading term is linearly divergent in N, and gives the mass renormalization

$$\delta\mu^2\theta = -\frac{2(i\lambda_R)^2}{16\pi^3\theta^2}\frac{1}{r}Z^2 a_1 \approx \frac{2(i\lambda_R)^2}{16\pi^3\theta^2}\frac{f_0}{r} \tag{4.38}$$

using (2.17) up to finite corrections. Its leading dependence on N is given by (4.16). Finally, d is determined by the constraint (4.5),

$$d = 2b + 2(i\lambda_R)^2 xZ\sum_i \frac{1}{\sqrt{\tilde{J}_i^2 + 2b}}, \tag{4.39}$$

which involves the further finite parameter b. Then a_0 follows from (2.15), which is quadratically divergent in N in agreement with the perturbative result (6.22).

Finally, (4.20) shows that $\alpha = 0$ marks some singularity or phase transition, dividing the moduli space into two disconnected components with $\alpha \gtrless 0$. The explicit form of the 2-point function (4.49) below shows that the phase with $\alpha < 0$ is the "physically relevant" one, while $\langle\phi_{ij}\phi_{ji}\rangle < 0$ for $\alpha > 0$ for small indices i, j. Since the matrices are supposed to be hermitian, this signals an instability or condensation for the low modes, i.e., some kind of tachyonic behavior. Observe also that e.g., $\langle\phi_{00}\rangle = 0$ can only be realized for $\alpha < 0$. Nevertheless, the renormalized n-point functions remain well defined for $\alpha > 0$. This phase transition is only seen in the genus 0 sector.

Finally consider the case $b = 0$, so that $y_k \equiv \tilde{J}_k$. Then finiteness of the 1-point function requires only that the three quantities

$$\left(i\lambda_R, c, \frac{1}{i\lambda_R\alpha} + \frac{r}{i\lambda_R}\right) \tag{4.40}$$

are finite. However, the genus 0 result for the 2-point function (4.49) shows that $\frac{1}{\alpha}$ must also be finite, and we are back to the previous analysis with finite $\alpha \neq 0$.

Putting these results together and recalling the structure of the higher genus contributions F_g stated below (3.8), we have established the following:

Theorem 4.1. *All derivatives of F_g w.r.t. $\tilde{J}_k^2$ for $g \geq 0$ (as well as all functions F_g for $g \geq 2$) are finite and have a well-defined limit $N \to \infty$, provided the six quantities $(i\lambda_R, \alpha, \mu_R^2, r, c, b)$ are finite and fixed, and $\alpha \neq 0$.*

This determines the scaling of the six bare parameters

$$(i\lambda(N), Z(N), \mu^2(N), a_0(N), a_1(N), a_2(N)) \tag{4.41}$$

as a function of N through (4.34), (4.35), (4.38), (4.39) resp. (2.15), (4.37), and (4.36). This defines a renormalization group flow in the space of free parameters. If desired, we could now impose some specific renormalization conditions such as $\langle\phi_{00}\rangle = 0$, etc. These would then determine the renormalized parameters $i\lambda_R$, etc., which in turn would determine the bare ones.

The most interesting "real" sector is given by $i\lambda_R \in \mathbb{R}, \alpha < 0, r > 0, \mu_R^2 > -3$, and $b \in \mathbb{R}$ such that $\tilde{J}_k^2 + 2b > 0$ for all k. Then $i\lambda \in \mathbb{R}$ for large enough N, and Z and x are positive. The case $\alpha = 0$ is a singularity of the genus 0 sector.

Since the connected n-point functions are given by the derivatives of $F = \sum_{g\geq 0} F_g$ w.r.t. $\tilde{J}$, this implies that all contributions in a genus expansion of the correlation functions for diagonal entries $\langle\phi_{kk}\cdots\phi_{ll}\rangle$ are finite and well defined. The general nondiagonal correlation functions are discussed in Section 4.2.2, and also turn out to be finite for arbitrary genus g under the same conditions. Putting these results together, we have established renormalizability of the model to all orders in a genus expansion, i.e.,

Theorem 4.2. *All connected genus g contributions to any given n-point function $\langle\phi_{i_1j_1}\cdots\phi_{i_nj_n}\rangle_c$ are finite and have a well-defined limit $N\to\infty$ for all g, under the above conditions.*

Moreover, they can in principle be computed explicitly using the above formulas. This immediately extends to nonconnected diagrams $\langle\phi_{i_1j_1}\cdots\phi_{i_nj_n}\rangle$. Furthermore, (4.61) shows that any contribution to F_g has order at least $((i\lambda_R)^2\alpha)^{2g-1}$. This implies (but is stronger than) perturbative renormalizability to all orders in λ_R. This might not seem surprising in view of the results in [3–5], however, note that the present model is more complicated than the ϕ^4 model where the beta-function was found to be zero at one loop for $\Omega = 1$ [16].

It is worth pointing out that only the genus 0 contribution requires renormalization, while all higher genus contributions are then automatically finite. This is very interesting because the genus 0 contribution can be obtained by various techniques in more general models, see e.g., [17]. A related approach was studied in [13, 14] without the oscillator potential.

It may appear surprising to find a well-defined ϕ^3 model for real coupling, where the action is not bounded from below. This is possible because we *first* quantize the model for imaginary coupling, where the Kontsevich model is well defined. The full genus expansion is then available, and the limit $N \to \infty$ is under control upon proper renormalization. This allows in a *second* step to define the model for real coupling i.e., real $i\lambda_R$, through analytic continuation. Similar behavior is well known in the context of pure matrix models, which typically allow extension to seemingly unstable potentials [18]; this is usually interpreted as suppression of tunneling from a local minimum.

4.2.1 Exact renormalization and fine tuning

The scaling (4.34) of the bare coupling is derived assuming that the 1-point function (4.20) and therefore its coefficients α and λ_R are kept fixed and independent of N. This is the usual way to proceed in perturbative renormalization, and it would lead to new infinite renormalization at each order of perturbation theory, as can be seen by expanding (4.34) formally in terms of λ_R.

On the other hand, we have the full renormalization in closed form available, which shows a simple *leading* scaling law for large N. It is then natural to ask what would happen if we scale the bare coupling in a simpler way, e.g., respecting only the leading scaling

$$i\lambda = c_\lambda \log(N)^{-2}, \quad Z = c_Z \log(N)^{-2}, \quad x = c_x \log(N) \tag{4.42}$$

for some constant c_λ, c_Z, c_x rather than the exact form (4.34), etc. It turns out that this is *not* sufficient, and the renormalization must respect the exact form of (4.34–4.39), rather than just their leading terms.

To see this, we determine the quantities $(\lambda_R, \alpha, r, \ldots)$ for the simpler scaling (4.42), and check whether they also converge as $N \to \infty$. This is obviously the case for $\lambda_R \sim \frac{c_\lambda}{c_x^2 c_Z^2}$ and $r \sim \frac{1}{c_x^2 c_Z}$, while α is given by (4.33)

$$\frac{1}{\alpha} = -\frac{1}{xZ} - (i\lambda_R)^2 f_1 = -\frac{1}{c_x c_Z} \log(N) + \frac{1}{2} \frac{c_\lambda^2}{c_x^4 c_Z^4} \frac{1}{(16\pi^3\theta^2)^3} \log(N). \tag{4.43}$$

For this to have a well-defined limit $N \to \infty$, the above scaling (4.42) is not sufficient, but the constants of proportionality must satisfy $\frac{1}{c_x c_Z} = \frac{1}{2} \frac{c_\lambda^2}{c_x^4 c_Z^4} \frac{1}{(16\pi^3\theta^2)^3}$, and moreover the scaling must be refined to give the sub-leading constant (4.43).

This shows that even this relatively simple, solvable asymptotically free model has a rather severe "fine-tuning problem", i.e., there is no obvious naturalness in the scaling of the bare parameters.

4.2.2 General n-point functions

Finally we show that all contributions in the genus expansion (and therefore perturbative expansion) of any n-point functions of the form

$$\langle \phi_{i_1 j_1} \cdots \cdot \phi_{i_n j_n} \rangle \tag{4.44}$$

have a well-defined and finite limit as $N \to \infty$, provided the above renormalization conditions hold. The argument is the same as in [2], which we repeat here for convenience.

Recall that the insertion of a factor $\tilde{\phi}_{ij}$ can be obtained by acting with $2i\lambda_R \frac{\partial}{\partial \tilde{J}^2_{ij}}$ on $Z(\tilde{J}^2)$, resp. $F_g(\tilde{J}^2)$. Now any given correlation function of type (4.44) involves only a finite set of indices $i, j, \ldots$ Thus taking derivatives w.r.t. $\tilde{J}^2_{ij}$ amounts to considering matrices $\tilde{J}$ of the form

$$\tilde{J} = \left(\begin{array}{c|c} \mathrm{diag}(\tilde{J}_1, \ldots, \tilde{J}_k) + \delta \tilde{J}_{k\times k} & 0 \\ \hline 0 & \mathrm{diag}(\tilde{J}_{k+1}, \ldots, \tilde{J}_N) \end{array} \right), \tag{4.45}$$

where k is chosen large enough such that all required variations are accommodated by the general hermitian $k \times k$ matrix

$$\tilde{J}_{k\times k} := (\mathrm{diag}(\tilde{J}_1, \ldots, \tilde{J}_k) + \delta \tilde{J}_{k\times k}) \tag{4.46}$$

in (4.45), while the higher eigenvalues $\tilde{J}_{k+1}, \ldots, \tilde{J}_N$ are fixed and given by (2.16). Therefore, we can restrict ourselves to this $k \times k$ matrix, which is independent of N. As was shown in Section 4.1, all F_g are in the limit $N \to \infty$ smooth (in fact analytic) symmetric functions of the first k eigenvalues squared, hence of the eigenvalues of $(\tilde{J}_{k\times k})^2$. Such a function can always be written as a smooth (analytic) function of some basis of symmetric polynomials in the $\tilde{J}^2_a$, in particular

$$F_g(\tilde{J}^2_1, \ldots, \tilde{J}^2_k) = f_g(\mathrm{Tr}(\tilde{J}^2_{k\times k}), \ldots, \mathrm{Tr}(\tilde{J}^{2k}_{k\times k})). \tag{4.47}$$

This can be seen by approximating the analytic function $F_g(z_1, .., z_k)$ at the point $z_i = \tilde{J}^2_i$ by a totally symmetric polynomial in the z_i, which correctly reproduces the partial derivatives up to some order n. According to a well-known theorem, that polynomial can be rewritten as polynomial in the elementary symmetric polynomials, or equivalently as a polynomial in the variables $s_n := \sum z_i^n$, $n = 1, 2, \ldots, k$. This amounts to the rhs of (4.47).

In the form (4.47), it is obvious that all partial derivatives $\frac{\partial}{\partial \tilde{J}^2_{ij}}$ of F_g exist to any given order and could be worked out in principle. This completes the proof that each genus g contribution to the general (connected) correlators $\langle \phi_{i_1 j_1} \cdots \phi_{i_n j_n} \rangle$ is finite and convergent as $N \to \infty$.

4.3 Asymptotic behavior of the correlation functions

4.3.1 $\langle\phi_{kk}\rangle$

Using (4.19), the behavior of $\langle\phi_{kk}\rangle$ for large k is dominated by $f_R(y_k)$, so that

$$\langle\phi_{kk}\rangle_{g=0} \sim \frac{1}{2}\frac{i\lambda_R}{(16\pi^3\theta^2)^3} y_k \log\left(\frac{y_k}{\tilde{J}_0}\right) \quad \text{for } k \to \infty. \tag{4.48}$$

4.3.2 $\langle\phi_{kl}\phi_{lk}\rangle$

We can use (2.23) to obtain the genus 0 contribution to the 2-point function $\langle\phi_{kl}\phi_{lk}\rangle$ for $k \neq l$. Using (4.20), we obtain the exact expression

$$\langle\phi_{kl}\phi_{lk}\rangle_{g=0} = -\frac{2\alpha^{-1}}{y_k + y_l} + 2(i\lambda_R)^2\frac{f_R(y_k) - f_R(y_l)}{y_k^2 - y_l^2}. \tag{4.49}$$

Note that the free case corresponds to $i\lambda_R = 0$ and $\alpha = -1$. Then indeed also the higher genus contributions vanish.

Consider the behavior for large $k \approx l$, which for large indices is dominated by the terms involving f_R:

$$\begin{aligned}\langle\phi_{kl}\phi_{lk}\rangle &\approx 2(i\lambda_R)^2\frac{f_R(y_k) - f_R(y_l)}{y_k^2 - y_l^2}\\ &\approx (i\lambda_R)^2\frac{1}{y_k}\frac{d}{dy_k}f_R(y_k)\\ &\approx (i\lambda_R)^2\frac{1}{2(16\pi^3\theta^2)^3}\frac{\log(y_k/J_0)}{y_k} \sim \frac{\log(k)}{k}\end{aligned} \tag{4.50}$$

for large $k \approx l$. It is worth pointing out that this 2-point function (and similar the 3-point function (4.56) below, etc.) is essentially determined by a function of a single variable y, which describes the dependence of the 1-point function $\langle\phi_{kk}\rangle$ on the index k. This is characteristic for the genus 0 sector, which is essentially determined by an eigenvalue distribution.

4.3.3 $\langle\phi_{ll}\phi_{kk}\rangle$

As a further example, consider the 2-point function $\langle\phi_{ll}\phi_{kk}\rangle$ for $k \neq l$, which vanishes in the free case. To compute it from the effective action, we need in principle

$$\langle\phi_{ll}\phi_{kk}\rangle_c = \langle\tilde{\phi}_{ll}\tilde{\phi}_{kk}\rangle - \langle\tilde{\phi}_{kk}\rangle\langle\tilde{\phi}_{ll}\rangle = (2i\lambda_R)^2\frac{\partial}{\partial\tilde{J}_l^2}\frac{\partial}{\partial\tilde{J}_k^2}(F_0 + F_1 + \cdots). \tag{4.51}$$

Even though this corresponds to a nonplanar diagram with external legs, it is obtained by taking derivatives of a closed genus 0 ring diagram. Therefore, we expect that only F_0 will contribute, and indeed the derivatives of F_1 contribute only to order λ_R^4. For $k \neq l$, the genus 0 contribution is

$$\begin{aligned}\langle \phi_{ll}\phi_{kk}\rangle_c &= (2i\lambda_R)^2 \frac{\partial}{\partial \tilde{J}_l^2}\frac{\partial}{\partial \tilde{J}_k^2} F_0 \\ &= (i\lambda_R)^2 \frac{1}{\sqrt{\tilde{J}_k^2+2b}}\frac{1}{\sqrt{\tilde{J}_l^2+2b}}\left(\frac{1}{\sqrt{\tilde{J}_k^2+2b}+\sqrt{\tilde{J}_l^2+2b}}\right)^2 \\ &\approx (i\lambda_R)^2\left(\frac{1}{\tilde{J}_k^2+2b}\right)^2 \end{aligned} \tag{4.52}$$

as $k \approx l \to \infty$, using again (4.7). For $k = l$, we need

$$\langle \phi_{kk}\phi_{kk}\rangle_c = (2i\lambda_R)^2 \frac{\partial^2}{\partial (\tilde{J}_k^2)^2} F_0 \approx \frac{(2i\lambda_R)^2}{2}\frac{\partial}{\partial \tilde{J}_k^2} f_R(y;\delta\tilde{J}), \tag{4.53}$$

since $f_R(y)$ is the leading contribution for large k. Writing

$$\begin{aligned}\frac{d}{d\tilde{J}_k^2} f_R(y;\delta\tilde{J}) &= \frac{dy}{d\tilde{J}_k^2}\frac{\partial}{\partial y} f_R(y;\delta\tilde{J}) \\ &\quad + \frac{\partial}{\partial \tilde{J}_k^2}\left(y_k^2 \frac{1}{(\tilde{J}_k^2+2b)\left(y_k\sqrt{\tilde{J}_k^2+2b}+(\tilde{J}_k^2+2b)\right)}\right),\end{aligned} \tag{4.54}$$

we note that the first term (involving the sum) is dominant, while the second is only one term in a large sum. This then gives the same asymptotic behavior as the usual propagator,

$$\langle \phi_{kk}\phi_{kk}\rangle_c \approx \frac{(2i\lambda_R)^2}{2}\frac{\partial}{\partial y_k^2} f_R(y_k;\delta\tilde{J}) \approx \langle \phi_{kl}\phi_{lk}\rangle \tag{4.55}$$

for large k, using (4.50). On the other hand, it is very different from $\langle \phi_{ll}\phi_{kk}\rangle_c$ given by (4.52). This should not be a surprise, since the latter corresponds to nonplanar diagrams as in figure 2.

Note that in (4.54), a derivative w.r.t. $\tilde{J}_k^2$ has essentially been replaced by a derivative w.r.t. $y \equiv y_k$. This is again characteristic for the genus 0 sector. In fact, F_0 can be obtained from the Dyson–Schwinger equation by making precisely this approximation.

4.4 The 3-point function

We can similarly use the exact expression (2.25) for the 3-point function to determine the asymptotic behavior of the 3-point function at genus 0:

$$\begin{aligned}
\langle\phi_{kl}\phi_{lk}\phi_{kk}\rangle_c &= 2i\lambda_R \frac{1}{\tilde{J}_k^2-\tilde{J}_l^2}(\langle\langle(\phi_{kk}-\phi_{ll})\phi_{kk}\rangle_c - \langle\phi_{kl}\phi_{lk}\rangle\rangle) \\
&\approx 2i\lambda_R \frac{1}{\tilde{J}_k^2-\tilde{J}_l^2}(\langle\langle\phi_{kk}\phi_{kk}\rangle_c - \langle\phi_{kl}\phi_{lk}\rangle\rangle) \\
&\approx (2i\lambda_R)^3 \frac{1}{y_k^2-y_l^2}\frac{1}{2}\left(\frac{\partial}{\partial y_k^2} f_R(y;\delta\tilde{J}) - \frac{f_R(y_k)-f_R(y_l)}{y_k^2-y_l^2}\right) \\
&\approx (i\lambda_R)^3 2\left(\frac{1}{2y_k}\frac{d}{dy_k}\right)^2 f_R(y_k;\delta\tilde{J}) \\
&\approx -(i\lambda_R)^3 \frac{1}{4}\frac{1}{(16\pi^3\theta^2)^3}\frac{1}{y_k^3}\log\left(\frac{y_k}{\tilde{J}_0}\right)
\end{aligned} \tag{4.56}$$

for large $k \approx l$, using (4.52), (4.50), and (4.19). The same behavior is found for

$$\langle\phi_{kk}\phi_{kk}\phi_{kk}\rangle_c = (2i\lambda_R)^3 \frac{\partial^3}{\partial(\tilde{J}_k^2)^3} F_0 \approx (2i\lambda_R)^3 \frac{1}{2}\frac{\partial^2}{\partial(y_k^2)^2} f_R(y_k), \tag{4.57}$$

up to a factor 2 which reflects the exchange symmetry between the external "legs" ϕ_{kl} and ϕ_{lk} for $k = l$. Combining (4.56) with (4.50) and (4.55), the 1PI vertex behaves like

$$\begin{aligned}
\langle\phi_{kl}\phi_{lk}\phi_{kk}\rangle_{1\text{PI}} &= \frac{1}{\langle\phi_{kl}\phi_{lk}\rangle_c}\frac{1}{\langle\phi_{kl}\phi_{lk}\rangle_c}\frac{1}{\langle\phi_{kk}\phi_{kk}\rangle_c}\langle\phi_{kl}\phi_{lk}\phi_{kk}\rangle_c \\
&\approx -2(i\lambda_R)^{-3}(16\pi^3\theta^2)^6\left(\frac{1}{\log(y_k/\tilde{J}_0)}\right)^2 \sim -\left(\frac{1}{\log\, k}\right)^2
\end{aligned} \tag{4.58}$$

for large $k \approx l$. This result is exact, because the higher genus contributions decay more rapidly as discussed in the next section. In particular, this establishes asymptotic freedom.

Several remarks are in order. First we note that this vertex function decays like $\frac{1}{(\log\, k)^2}$ rather than $\frac{1}{(\log\, k)^{1/2}}$, which would be found by a 1-loop computation as shown in Section 6. Moreover, (4.58) is in nice agreement with (4.34), demonstrating that the 1PI vertex approaches the bare coupling $-\frac{i\lambda}{2}$ for $k \to N$. Finally, note that the effective coupling constant for this asymptotic domain is $\frac{1}{\lambda_R^3}$ rather than λ_R. This is clearly a purely "quantum" effect, which is again due to higher order correction to the 1-loop RG result (6.14).

4.5 Higher genus contributions

We illustrate here the stronger decay behavior of the higher genus n-point functions, in the example of the 1-point function. Consider

$$\begin{aligned}\langle\phi_{kk}\rangle_{g=1} &= 2i\lambda_R\frac{\partial}{\partial\tilde{J}_l^2}F_1 = -\frac{2i\lambda_R}{24}\frac{\partial}{\partial\tilde{J}_k^2}\ln(1-I_1)\\ &= \frac{(i\lambda_R)^3\alpha}{12}\frac{\partial}{\partial\tilde{J}_k^2}f_{1,R}(\delta\tilde{J})\\ &= \frac{(i\lambda_R)^3\alpha}{8}\frac{1}{(\tilde{J}_k^2+2b)^{5/2}}\left(1+2\frac{\partial}{\partial\tilde{J}_k^2}b\right)\\ &= \frac{(i\lambda_R)^3\alpha}{8}\frac{1}{(\tilde{J}_k^2+2b)^{5/2}}\left(1-\frac{(i\lambda_R)^2\alpha}{\sqrt[3]{\tilde{J}_k^2+2b}}\right)\end{aligned} \tag{4.59}$$

using (4.28) and (4.32). This asymptotically behaves like

$$\langle\phi_{kk}\rangle_{g=1}\sim\frac{1}{k^5} \tag{4.60}$$

for $k\to\infty$. Looking at (3.8), it is obvious that the higher genus contributions are also decaying at least as rapidly as the genus 1 contribution (4.60), and similar results could be derived for the n-point functions.

Finally, it follows from (4.29) that each term in F_g contains the factor

$$F_g\sim((i\lambda_R)^2\alpha)^{2(g-1)+\sum l_p} \tag{4.61}$$

for $g\geq 2$, which is at least $\alpha^{2(g-1)+1}$. Hence $(i\lambda_R)^2\alpha$ is the parameter which controls the genus expansion.

5 Further aspects

Structural remarks

We briefly add some heuristic remarks which might shed new light on the formal results obtained above.

The model considered here is characterized by a function F of $\tilde{J}$ which is invariant under conjugation with $U(N^3)$. This implies in particular that $F=F_N(\tilde{J}_k)$ is a totally symmetric function of the eigenvalues $\tilde{J}_k$, i.e., it is a function on the quotient space $\mathbb{R}^{N^3}/\mathcal{S}_{N^3}$ where $\mathcal{S}_{N^3}$ denotes the permutation group. Since the observables are obtained by taking partial derivatives w.r.t. $\tilde{J}$, we are interested in

$$W_k(\tilde{J}) = \frac{\partial}{\partial \tilde{J}_k} F_N(\tilde{J}),$$
$$W_{k,l}(\tilde{J}) = \frac{\partial}{\partial \tilde{J}_k}\frac{\partial}{\partial \tilde{J}_l} F_N(\tilde{J}), \tag{5.1}$$

etc. Renormalizability requires that all these derivatives exist and have a well-defined limit $N \to \infty$, hence that $F_N(\tilde{J}_i)$ converges to an infinitely differentiable function $F(\tilde{J}_i)$. This must hold at the point $\tilde{J}$ with eigenvalues given by (2.16), or preferably in some "good" neighborhood $\mathcal{U} \ni \tilde{J}$ in $\mathbb{R}^{N^3}$, with an appropriate limit $N \to \infty$. Furthermore, these partial derivatives should have suitable decay properties such as indicated above.

Now consider the dependence of $W_k(\tilde{J})$ on the index k. It is a physical requirement that this dependence on the indices is mild.[7] This can be understood analytically as follows. Note that $W_k(\tilde{J})$ is related to $W_l(\tilde{J})$ by exchanging the arguments $\tilde{J}_k$ and $\tilde{J}_l$, i.e., by applying the permutation operator $\sigma_{k,l}$ on the space $\mathbb{R}^{N^3}$ of $\tilde{J}_k$,

$$W_k = W_l \circ \sigma_{k,l}. \tag{5.2}$$

This means that the index dependence may be traded for a "small" change (a permutation) of the eigenvalues. Since the eigenvalues are ordered and approach a simple distribution, it is plausible that $\sigma_{k,l}$ respects the neighborhood $\mathcal{U}$. Smoothness in $\mathcal{U}$ would then naturally imply that the dependence on the indices is mild. Indeed, we found explicitly that the index dependence at genus 0 becomes translated into the dependence of a smooth function $W(y)$ (4.9) on the variable y.

Extension to $\Omega \neq 1$

Once a suitable domain $\mathcal{U}$ is established where $F[\tilde{J}]$ is smooth with suitable decay properties of its partial derivatives, then the following strategy to extend our results to $\Omega \neq 1$ can be envisaged. The partition function for $0 \neq \Omega \neq 1$ can be obtained from the results for $\Omega = 1$ using

$$\begin{aligned} Z[\tilde{J}] &= \langle e^{\epsilon \operatorname{Tr}[\tilde{x}_i,\phi][\tilde{x}_i,\phi]}\rangle_{\Omega=1} = e^{\epsilon \operatorname{Tr}[\tilde{x}_i,\partial/\partial\tilde{J}^2][\tilde{x}_i,\partial/\partial\tilde{J}^2]} Z[\tilde{J}]_{\Omega=1} \\ &= e^{\epsilon \operatorname{Tr}[\tilde{x}_i,\partial/\partial\tilde{J}^2][\tilde{x}_i,\partial/\partial\tilde{J}^2]} e^{F[\tilde{J}]_{\Omega=1}} =: e^{F[\tilde{J}]} \end{aligned} \tag{5.3}$$

[7] Some mild singularities for coinciding indices might be allowed however.

for small ϵ, accompanied by a change of wavefunction normalization. Hence we should consider the operator

$$\Delta_J := \mathrm{Tr}\left[\tilde{x}_i, \frac{\partial}{\partial \tilde{J}^2}\right]\left[\tilde{x}_i, \frac{\partial}{\partial \tilde{J}^2}\right] \tag{5.4}$$

acting on functions of $\tilde{J}$. It does not commute with $U(N^3)$, hence $Z[\tilde{J}]$ will be a function of $\tilde{J}$ which no longer depends on the eigenvalues only. We have established in this paper that $Z[\tilde{J}]_{\Omega=1}$ is a well-defined, infinitely differentiable function (after subtracting a possible infinite constant from F). The question of renormalizability for $\Omega \neq 1$ is whether $F[\tilde{J}]$ is also a well-defined, infinitely differentiable function, possibly after further renormalization. To establish this using the methods of the present paper might therefore be feasible, by studying the operator Δ_J and establishing careful estimates on the partial derivatives of $F[\tilde{J}]_{\Omega=1}$ and their asymptotic behavior for large N. However, we will not attempt to do this in the present paper.

Remarks on the relation with string theory

The Kontsevich model has been related to string theory in [19] as follows. The matrix X_{ij} of the Kontsevich model is interpreted as coefficient of the open string field, more precisely the tachyon, connecting the (Liouville) D-brane with label i to the D-brane with label j. The eigenvalues $\tilde{M}_i$ of the external potential in the Kontsevich model are interpreted as boundary cosmological constants on the brane with label i, and the Kontsevich model (3.1) describes the (topological sector of) open string field theory in this situation.

Applying this interpretation to our model, we are led to a picture of open string tachyons in string field theory with N^3 D-branes, with specific cosmological constant $J_i \sim i_1 + i_2 + i_3$ on the brane $i = (i_1, i_2, i_3)$. The fact that our model is renormalizable implies that the correlators become essentially smooth functions of these three coordinates i_1, i_2, i_3, with a well-defined large N limit. This could be interpreted by saying that the endpoints i, j of the open strings effectively live in three dimensions with coordinates $i_1, i_2, i_3 \geq 0$, forming a three-dimensional wedge $\mathbb{R}^3_+$ with a potential determined by J_i. In other words, three extra dimensions appear from the string point of view by some kind of stringy "deconstruction" of dimensions. On the other hand, we have the interpretation as six-dimensional noncommutative field theory. This points to interesting directions for further studies.

6 Perturbative computations

We write the bare action (2.6) as

$$\begin{aligned} S &= \mathrm{Tr}\left(\frac{1}{4}(J\phi^2 + \phi^2 J) + \frac{i\lambda}{3!}\phi^3 - (i\lambda)A\phi\right) \\ &= \mathrm{Tr}\left(\frac{1}{2}\phi^i_j (G_R)^{j;l}_{i;k}\phi^k_l + \frac{i\lambda_R}{3!}\phi^3\right) + \delta S. \end{aligned} \tag{6.1}$$

The finite kinetic term $(G_R)^{j;l}_{i;k} = \frac{1}{2}\delta^i_l \delta^k_j (\tilde{J}_i + \tilde{J}_j)$ defines the renormalized propagator

$$\Delta^{i;k}_{j;l} = \langle \phi^i_j \phi^k_l \rangle = \delta^i_l \delta^k_j \frac{2}{\tilde{J}_i + \tilde{J}_j} = \delta^i_l \delta^k_j \frac{1/(4\pi^2\theta)}{\underline{i} + \underline{j} + (\mu_R^2\theta + 3)}, \tag{6.2}$$

corresponding to the finite (renormalized) matrix

$$\tilde{J}|n_1, n_2, n_3\rangle = 16\pi^3\theta^2 \left(\underline{n} + \frac{3 + \mu_R^2\theta}{2}\right)|n_1, n_2, n_3\rangle, \tag{6.3}$$

using the notation $\underline{n} = n_1 + n_2 + n_3$ (2.8). Here λ_R is the renormalized coupling constant, valid at a "scale" given by m, i.e., assuming that the indices of the interaction vertex approximately satisfy

$$\underline{i} \approx \underline{j} \approx \underline{k} = \underline{m}. \tag{6.4}$$

The counterterms are collected in

$$\delta S = \mathrm{Tr}\Big(-(i\lambda_R)Z_\lambda A\phi + \frac{1}{4}(\delta J\phi^2 + \phi^2\delta J) + \frac{i\delta\lambda}{3!}\phi^3\Big), \tag{6.5}$$

where

$$\begin{aligned} \delta J|n_1, n_2, n_3\rangle &= 16\pi^3\theta^2 \left(\left(Z\frac{\delta\mu^2\theta}{2}\right) + (Z-1)J_R\right)|n_1, n_2, n_3\rangle, \\ \lambda_R &= Z_\lambda^{-1}\lambda, \\ \delta\lambda &= \lambda_R(Z_\lambda - 1) \end{aligned} \tag{6.6}$$

is part of the counterterm, and $\delta\mu^2 = (\mu^2 - \mu_R^2)$. It is then easy to see that the usual power counting rules apply (with suitable extensions to the case of higher genus [20]), where N plays the role of the cutoff Λ^2. This can be seen in the following 1-loop examples.

3-point function In six dimensions, the interaction vertex requires renormalization, induced by the planar 1-loop 1PI graph (without external legs) in figure 1. This gives

$$\begin{aligned}\langle\phi_{ij}\phi_{jk}\phi_{ki}\rangle_{1\mathrm{PI}} &= -\frac{i\lambda_R}{2} + \left(-\frac{i\lambda_R}{2}\right)^3 \sum_{\underline{l}} \frac{2}{J_k^R + J_l^R}\frac{2}{J_i^R + J_l^R}\frac{2}{J_j^R + J_l^R} - \frac{i\delta\lambda}{2} \\ &\approx -\frac{i\lambda_R}{2} + (-i\lambda_R)^3 \sum_{\underline{l}} \frac{1}{(J_l^R + J_m^R)^3} - \frac{i\delta\lambda}{2} + \text{finite} \\ &\approx -\frac{i\lambda_R}{2} + \frac{(-i\lambda_R)^3}{(16\pi^3\theta^2)^3} \sum_{\underline{l}} \frac{1}{(\underline{l}+\underline{m})^3} - \frac{i\delta\lambda}{2} + \text{finite} \\ &\approx -\frac{i\lambda_R}{2} + \frac{(-i\lambda_R)^3}{(16\pi^3\theta^2)^3}\frac{1}{2}\log\frac{N+m}{8m} - \frac{i\delta\lambda}{2} + \text{finite}. \end{aligned} \tag{6.7}$$

All nonplanar contributions are finite. This gives the counterterm to the coupling constant

$$i\delta\lambda = -\frac{(i\lambda_R)^3}{(16\pi^3\theta^2)^3}\log\frac{N+m}{8m} + \text{finite}, \tag{6.8}$$

so that

$$(i\lambda) = (i\lambda_R)\left(1 + \frac{(i\delta\lambda)}{(i\lambda_R)}\right) = \frac{(i\lambda_R)}{1 + (i\lambda_R)^2/(16\pi^3\theta^2)^3\log(N/m)} + O(\lambda_R^5). \tag{6.9}$$

The minus sign indicates asymptotic freedom. However, the last formula (6.9) is only suggestive, and such a 1-loop result should be used with caution. Note that we could equally well write

$$(i\lambda) = \frac{(i\lambda_R)}{(1 + (1/2)((i\lambda_R)^2/(16\pi^3\theta^2)^3)\,\log(N/m))^2} + O(\lambda_R^5), \tag{6.10}$$

which in fact agrees much better with the exact scaling (4.34) of the bare coupling. A better way to understand the coupling constant renormalization

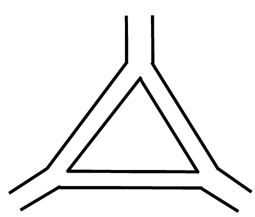

Figure 1: 1-loop contribution to 1PI vertex.

is to determine its dependence on the "scale" m, i.e., the running coupling constant

$$g(m) = -i\lambda_R(m) = 2\langle\phi_{ij}\phi_{jk}\phi_{ki}\rangle_{1PI} \tag{6.11}$$

for $\underline{i} \approx \underline{j} \approx \underline{k} = \underline{m}$ (6.4). An extra factor 2 is inserted for convenience. The corresponding 1-loop beta-function can be obtained from (6.7),

$$\beta := 2m\frac{\partial\langle\phi_{ij}\phi_{jk}\phi_{ki}\rangle_{\text{1PI}}}{\partial m} = \frac{g(m)^3}{(16\pi^3\theta^2)^3}m\frac{\partial \log(N/m)}{\partial m} = -\frac{g(m)^3}{(16\pi^3\theta^2)^3}, \tag{6.12}$$

indicating asymptotic freedom. This gives

$$\frac{dg(m)}{g(m)^3} = -\frac{1}{(16\pi^3\theta^2)^3}\frac{dm}{m}, \tag{6.13}$$

which can be integrated to give the running (1PI) coupling constant

$$g(m)^2 = \frac{g(m_0)^2}{1 + 2g(m_0)^2/(16\pi^3\theta^2)^3\log(m/m_0)}. \tag{6.14}$$

This decreases with increasing scale m, which means asymptotic freedom. However, note that the exact scale dependence $g(m) \sim -\frac{1}{(\log m)^2}$ for large m which was determined in (4.58) is not correctly reproduced by this 1-loop RG computation. This shows that the common practice of using the 1-loop RG results for the running coupling constant may not be sufficiently accurate for large scales.

It is worth pointing out that the coupling constant runs here even at 1-loop, in contrast to the case of the ϕ^4 model in four dimensions [16].

1-point function The 1-loop contribution to the 1-point function gives

$$\begin{aligned}\langle\phi_{ii}\rangle_{\text{1-loop}} &= \frac{i\lambda_R Z_\lambda}{\tilde{J}_i}A_i - \frac{i\lambda_R}{2}\frac{1}{\tilde{J}_i}\sum_k \frac{2}{\tilde{J}_i + \tilde{J}_k} \\ &= -\frac{i\lambda_R}{\tilde{J}_i}\left(-Z_\lambda A_i + \frac{1}{16\pi^3\theta^2}\sum_k \frac{1}{\underline{i}+\underline{k}+3+\mu_R^2\theta}\right).\end{aligned} \tag{6.15}$$

To proceed, we expand

$$h(\underline{i}) := \sum_k \frac{1}{\underline{i}+\underline{k}+3+\mu_R^2\theta} = h(0) + (\underline{i})h'(0) + \frac{1}{2}(\underline{i})^2 h''(0) + h_R(\underline{i}), \tag{6.16}$$

where $h_R(\underline{i})$ is a finite nontrivial function of $\underline{i}$, and

$$h(0) = \sum_k \frac{1}{\underline{k} + \mu_R^2\theta + 3} = \left(-6 \log 2 + \frac{9}{2}\log 3\right) N^2$$
$$+ (-6(\mu_R^2\theta + 3)\log 2 + 3(\mu_R^2\theta + 3)\log 3)N + \frac{1}{2}(\mu_R^2\theta + 3)^2 \log N,$$
$$h'(0) = -\sum_k \frac{1}{(\underline{k} + \mu_R^2\theta + 3)^2} = -(16\pi^3\theta^2)^2 f_0 - (16\pi^3\theta^2)^3(3 + \mu_R^2\theta) f_1$$
$$= -(6 \log(2) - 3\log(3))N + \frac{3 + \mu_R^2\theta}{2}\log(N), \tag{6.17}$$
$$h''(0) = 2\sum_k \frac{1}{(\underline{k} + \mu_R^2\theta + 3)^3} = -(16\pi^3\theta^2)^3 2 f_1 = \log(N) \tag{6.18}$$

up to finite corrections, using the results of Section 4.1.1. We note in particular that the i-dependent term $(\underline{i})h'(0) + \frac{1}{2}(\underline{i})^2 h''(0)$ in (6.15) forces us to introduce a corresponding counterterm to the action as in (2.9), $A = a_0 + a_1 J + a_2 J^2$. As discussed in Section 2, this is no longer equivalent to an infinite shift (2.10) of ϕ. Taking this into account, we have

$$\langle\phi_{ii}\rangle_{\text{1-loop}} = -\frac{i\lambda_R}{\tilde{J}_i}\Bigg(-(a_0 + a_1 J + a_2 J^2)Z_\lambda$$
$$+ \frac{1}{16\pi^3\theta^2}\left(h(0) + (\underline{i})h'(0) + \frac{1}{2}(\underline{i})^2 h''(0) + h_R(\underline{i})\right)\Bigg). \tag{6.19}$$

Finiteness and the condition $\langle\phi_{00}\rangle = 0$ implies to lowest order

$$a_2 = \frac{1}{(16\pi^3\theta^2)^3}\frac{1}{2}\log N = -f_1 + \text{finite}, \tag{6.20}$$
$$a_1 = -(16\pi^3\theta^2)(\mu_R^2\theta + 3)a_2 + \frac{1}{(16\pi^3\theta^2)^2}h'(0) = -f_0 + \text{finite}, \tag{6.21}$$
$$a_0 = \frac{1}{16\pi^3\theta^2}h(0) - (8\pi^3\theta^2)(\mu_R^2\theta + 3)a_1 - (8\pi^3\theta^2)^2(\mu_R^2\theta + 3)^2 a_2 \tag{6.22}$$

up to finite corrections. These renormalization conditions guarantee that the 1-point function $\langle\phi_{ii}\rangle$ has a well-defined and nontrivial limit $N \to \infty$. In particular, (6.20) and (2.14) imply

$$x = \sqrt{1 + \frac{(i\lambda_R)^2}{(16\pi^3\theta^2)^3}\log N} = 1 + \frac{1}{2}\frac{(i\lambda_R)^2}{(16\pi^3\theta^2)^3}\log N, \tag{6.23}$$

to lowest order, and (6.21) together with (2.17) gives

$$\delta\mu^2\theta = \frac{2(i\lambda_R)^2}{(16\pi^3\theta^2)} f_0 + \text{finite}. \tag{6.24}$$

This is consistent with the exact result (4.38).

Finally, it is interesting to consider the behavior of $\langle\phi_{ii}\rangle$ for large i. After imposing the above renormalization conditions, we observe that the dominating term in (6.19) is $\frac{i\lambda_R}{\tilde{J}_i}\frac{1}{2}(\underline{i})^2 h''(0)$. Here $h''(0) \sim \log(N)$ should more properly be replaced by $\log(N/\underline{i})$, which implies

$$\langle\phi_{ii}\rangle \sim \underline{i}\,\log(\underline{i}) \tag{6.25}$$

for large i. This is in agreement with (4.48).

2-point function Next we compute the leading contribution to the 2-point function $\langle\phi_{ll}\phi_{kk}\rangle$ for $l \neq k$, which vanishes at tree level. The leading contribution comes from the nonplanar graph in figure 2, which gives

$$\langle\phi_{ll}\phi_{kk}\rangle = \langle\phi_{kk}\rangle\langle\phi_{ll}\rangle + \frac{1}{4}\frac{(i\lambda_R)^2}{\tilde{J}_k\tilde{J}_l}\left(\frac{2}{\tilde{J}_k+\tilde{J}_l}\right)^2 \tag{6.26}$$

(for $l \neq k$) indicating the symmetry factors, where the disconnected contributions are given by (6.15). This is consistent with the result (4.52) obtained from the Kontsevich model approach. Note that the counterterm δJ does not enter here.

Similarly, the leading contribution to the 2-point function $\langle\phi_{kl}\phi_{lk}\rangle$ for $l \neq k$ has the contribution indicated in figure 3, involving also the

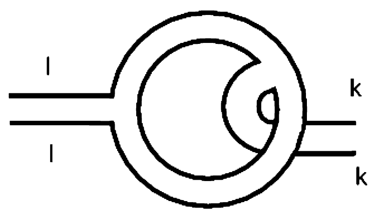

Figure 2: 1-loop contribution to $\langle\phi_{ll}\phi_{kk}\rangle$.

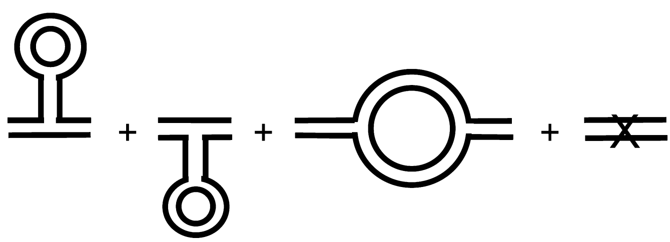

Figure 3: 1-loop contribution to $\langle\phi_{kl}\phi_{lk}\rangle$.

counterterm δJ. This gives

$$\langle\phi_{kl}\phi_{lk}\rangle_{\text{1-loop}} = \frac{2}{J_k^R + J_l^R} - 2(i\lambda_R)\frac{\langle\phi_{kk} + \phi_{ll}\rangle}{(J_k^R + J_l^R)^2} + \frac{4}{(J_k^R + J_l^R)^2}\left(\sum_j \frac{(i\lambda_R)^2}{J_k^R + J_j^R}\frac{1}{J_l^R + J_j^R} - \frac{\delta J_l + \delta J_k}{2}\right). \tag{6.27}$$

The first term is the free propagator, the next term the tadpole contributions, and the last them the 1-loop contribution in figure 3 with counterterm δJ.

We have to adjust the parameters such that the result is well defined and nontrivial. Using (6.17) and (6.18), we can write

$$\begin{aligned}\sum_j &\frac{1}{J_k^R + J_j^R}\frac{1}{J_l^R + J_j^R} \\ &= \frac{1}{(16\pi^3\theta^2)^2}\sum_j\left(\frac{1}{(\underline{j} + \mu_R^2\theta + 3)^2} - \frac{\underline{k} + \underline{l}}{(\underline{j} + \mu_R^2\theta + 3)^3}\right) + \text{finite} \\ &= \frac{1}{(16\pi^3\theta^2)^2}\left(-h'(0) - (\underline{k} + \underline{l})\frac{1}{2}h''(0) + \text{finite}\right).\end{aligned} \tag{6.28}$$

Therefore,

$$\delta J_l + \delta J_k = \frac{2(i\lambda_R)^2}{(16\pi^3\theta^2)^2}\left(-h'(0) - (\underline{k} + \underline{l})\frac{1}{2}h''(0)\right) + \text{finite}, \tag{6.29}$$

which using (6.17) and (6.18) imply

$$(Z - 1)(\underline{k} + \underline{l}) = 2(i\lambda_R)^2(\underline{k} + \underline{l})f_1 \tag{6.30}$$

and

$$Z(\delta\mu^2\theta) + (Z - 1)(3 + \underline{\mu}_R^2\theta) = \frac{2(i\lambda_R)^2}{(16\pi^3\theta^2)}f_0 + 2(i\lambda_R)^2(3 + \mu_R^2\theta)f_1 \tag{6.31}$$

up to finite corrections. Hence we obtain the lowest order mass and wave-function renormalization:

$$\begin{aligned} Z &= 1 + 2(i\lambda_R)^2 f_1 = \frac{1}{(1 + (1/2)((i\lambda_R)^2/(16\pi^3\theta^2)^3)\log N)^2} + O(\lambda_R^4) \\ \delta\mu^2\theta &= \frac{2(i\lambda_R)^2}{(16\pi^3\theta^2)}f_0 \end{aligned} \tag{6.32}$$

up to finite corrections, in agreement with (6.24).

Note that in eight or higher dimensions, divergent mixed terms $(\underline{k})(\underline{l})$ would occur in (6.28), which can no longer be absorbed. Then the model is no longer renormalizable, as in the commutative case.

7 Summary and discussion

We have shown that the self-dual NC ϕ^3 model in six dimensions can be renormalized and essentially solved in terms of a genus expansion, by using the Kontsevich model. This provides a model which contains essentially the full complexity of renormalization of a not super-renormalizable asymptotically free quantum field theory, while being solvable and hence fully under control. In principle, all n-point functions can be computed in a genus expansion, and we give explicit expressions for the 1-, 2-, and some 3-point functions.

In particular, we were able to determine exactly the RG flow of the bare parameters as a function of the cutoff N, as well as the running of the "physical" coupling constant i.e., the 1PI 3-point function. As in the case of two and four dimensions [1, 2], it turns out that the renormalization is fully determined by the genus 0 sector. In particular, we can compare the exact results with the standard perturbative methods. For example, it turns out that the 1-loop beta-function for the coupling constant gives roughly the correct behavior and correctly predicts asymptotic freedom, but wrongly gives a $(\log N)^{-1}$ dependence on the scale as opposed to the correct $(\log N)^{-2}$ dependence.

We also show that the model has a critical surface defined by $\alpha = 0$, which separates two different phases. One phase has the expected "physical" properties, while in the other some modes become unstable.

It is very remarkable that a nontrivial, asymptotically free six-dimensional NC ϕ^3 field theory allows such a detailed analytical description. There is no commutative analog where this has been achieved to our knowledge. Therefore, this model can serve as a testing ground for various ideas and methods for renormalization. It also shows that the noncommutative world in some cases is more accessible to analytical methods than the commutative case. While the techniques used in this paper are more-or-less restricted to the ϕ^3 interaction, it is worth pointing out that the renormalization is determined by the genus 0 contribution only, which is accessible in a wider class of models; see also [13, 14] in this context.

One of the open problems is the lack of control over the *sum* over all genera g. We have not shown that the sum over g converges in a suitable sense, which would amount to a full construction of the model. Furthermore, it would be extremely interesting to extend the analysis beyond the case of $\Omega = 1$. We propose a strategy in Section 5 how such an extension might be possible, and we also comment on a relation with open string field theory.

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