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Conormal bundles to knots and the Gopakumar–Vafa conjecture

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Abstract

We offer a new construction of Lagrangian submanifolds for the Gopakumar–Vafa conjecture relating the Chern–Simons theory on the 3-sphere and the Gromov–Witten theory on the resolved conifold. Given a knot in the 3-sphere, its conormal bundle is perturbed to disconnect it from the zero-section and then pulled through the conifold transition. The construction produces totally real submanifolds of the resolved conifold that are Lagrangian in a perturbed symplectic structure and correspond to knots in a natural and explicit way. We prove that both the resolved conifold and the knot Lagrangians in it have bounded geometry, and that the moduli spaces of holomorphic curves ending on the Lagrangians are compact in the Gromov topology.

Introduction

In [26], Witten argues that the large N expansion of the $U(N)$ Chern–Simons theory on a 3-manifold M should be equivalent to an open string theory on T^*M. In the absence of knots (Wilson loops), the latter is supposed to describe pseudoholomorphic curves (strings) on T^*M with boundaries on the

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zero-section. When knots are present Ooguri and Vafa suggested that curves should additionally be allowed to end on conormal bundles to them [21]. Unfortunately, ordinary Gromov–Witten theory on T^*M is trivial, since there are no non-trivial pseudoholomorphic curves there. Neither closed surfaces nor surfaces ending on the zero-section or a conormal bundle may be pseudoholomorphic due to a vanishing theorem in [26] (see Remark 1.5). One way around this proposed by Witten himself is to use some degenerate "curves" (fat-graphs), but it is unclear how to formalize such a theory (see, however, [18]).

Another way around this difficulty was proposed by Gopakumar and Vafa in [10] for $M = S^3$. The idea is to change the topology of T^*S^3 so that Gromov–Witten theory on the resulting manifold is non-trivial and still equivalent to the Chern–Simons theory on S^3. The resulting manifold in this case is the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{CP}^1$ and it can be obtained from T^*S^3 by shrinking the S^3-cycle to a point and then inserting an S^2-cycle in its place. Thus, the $U(N)$ Chern–Simons theory on S^3 is predicted to be dual to the Gromov–Witten theory on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This is the Gopakumar–Vafa conjecture.

The midpoint in the transition is a singular variety $\mathcal{C}$ called the conifold and the change from T^*S^3 to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is called the conifold transition. In the physical literature, T^*S^3 is referred to as the deformed conifold and the resulting $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle $\widehat{\mathcal{C}}$ as the resolved conifold. Schematically,

$$\begin{tikzcd} T^*S^3 \arrow[rr, dashed] \arrow[dr, "F"'] & & \widehat{\mathcal{C}} \arrow[dl, "\pi_2"] \\ & \mathcal{C} & \end{tikzcd}$$

with F being the contraction map and π_2 projecting to $\mathbb{C}^4$ (see Section 1 for details). We note that π_2^{-1} is defined on $\mathcal{C} \setminus \{0\}$ so the dashed arrow is "almost" well defined and $\pi_2^{-1}(0) \simeq \mathbb{CP}^1$ is the exceptional S^2-cycle.

As the S^3-cycle represented by the zero-section in T^*S^3 shrinks the open curves that end on it become closed and then get lifted to $\widehat{\mathcal{C}}$. Although this picture has no mathematical meaning, note that unlike T^*S^3, the resolved conifold $\widehat{\mathcal{C}}$ does admit non-trivial closed holomorphic curves and one can talk about equivalence or duality between the Chern–Simons on S^3 and the Gromov–Witten on $\widehat{\mathcal{C}}$. In the case of closed curves, it was verified by a direct computation in [8].

When knots are present the geometric part of the Gopakumar–Vafa conjecture predicts that the conormal bundle to a knot undergoes the conifold transition and produces some Lagrangian submanifold L in $\widehat{\mathcal{C}}$ [21]. Then the Chern–Simons theory with knot observables (Wilson loops) on S^3 has to be dual to the open Gromov–Witten theory, where the curves end on L. Ooguri and Vafa were able to produce L explicitly in the case of the unknot using antiholomorphic involutions. This is a trick that does not generalize to any other knots. For this case, the conjecture has been verified in [13, 15] using some narrow definition of open Gromov–Witten invariants. Later Labastida, Mariño, and Vafa offered a way to construct Lagrangians for algebraic knots, in particular, torus knots [14]. This construction as explained by Taubes [25] begins with producing a two-dimensional Lagrangian surface in $\mathbb{C}^2$ that intersects spheres of large radii along a given knot. Then this surface is translated with twisting along the fibers of $\widehat{\mathcal{C}}$ over the equator of $\mathbb{CP}^1$ completing a half-twist after the full circle (analogous to the Möbius strip considered as a bundle over the circle). To match up the ends, the original surface in $\mathbb{C}^2$ must be centrally symmetric which imposes a restriction on admissible knots. Taubes came up with a generalization to non-algebraic knots and links, but the rather artificial symmetry restriction remained. In particular, it excludes something as simple as the trefoil knot. The main flaw of this construction though is that the Lagrangian submanifold constructed is entirely unrelated to the conormal bundle in T^*S^3 it is supposed to come from.

Our approach in contrast will be to obtain the corresponding Lagrangians directly by applying the conifold transition to the conormal bundle N_k^* of a knot k. As a result, the manifold L is produced for all knots in a uniform way and without restrictions. However, this approach presents its own difficulties. Since N_k^* intersects the zero-section of T^*S^3 which is being shrunk into the conifold singularity, it also acquires a singularity in the process. In general, this singularity is not resolved by subsequent lifting to $\widehat{\mathcal{C}}$. The intuitive idea held by physicists (e.g., C.Vafa) is that one needs to perturb N_k^* into $N_{k,\varepsilon}^*$ disconnected from the zero-section and only then perform the conifold transition to get $\mathrm{CT}(N_{k,\varepsilon}^*)$. The problem is that N_k^* is an exact Lagrangian and there is an obstruction to disconnecting it from the zero-section by a symplectic isotopy. It was discovered originally by Gromov for compact submanifolds [3, 11] and then generalized to non-compact ones by Oh using the Floer homology [20].

However, a smooth disconnecting isotopy can easily be found. The construction is very straightforward. Let $k : S^1 \hookrightarrow S^3$ be a naturally parametrized knot. Embed S^3 into $\mathbb{R}^4$ in the standard way then the embedding of $T^*S^3 \simeq TS^3$ into $\mathbb{R}^4 \times \mathbb{R}^4$ is also standard. The conormal bundle to k is realized as

$$N_k^* = \{(x, p) \in T^*S^3 \mid x = k(t),\ p \cdot \dot{k}(t) = 0\}.$$

To disconnect N_k^* from the zero-section we shift it in the fibers of T^*S^3 in the direction tangent to the knot, namely

$$N_{k,\varepsilon}^* := \{(x, p + \varepsilon\dot{k}(t)) \mid x = k(t),\ p \cdot \dot{k}(t) = 0\}.$$

Since $F(N_{k,\varepsilon}^*)$ misses the conifold singularity the conifold transition is given simply by

$$\mathrm{CT}(N_{k,\varepsilon}^*) := \pi_2^{-1} \circ F(N_{k,\varepsilon}^*).$$

The transition $\mathrm{CT}(N_{k,\varepsilon}^*)$ turns out to be a smooth submanifold in $\widehat{\mathcal{C}}$ and has the correct topology $S^1 \times \mathbb{R}^2$. Predictably, it fails to be Lagrangian in the standard Kähler structure of the resolved conifold.

We beleive, however, that this is a false problem. Although the standard Kähler structure on $\widehat{\mathcal{C}}$ is the simplest one, it is not in any way special from the physical point of view. The physically significant structure, if any, is the one induced by the Calabi-Yau metric. The Calabi-Yau metric on $\widehat{\mathcal{C}}$ is known almost explicitly [5], but no attempt has been made to check if even the Ooguri-Vafa submanifold for the unknot is Lagrangian in it. Moreover, the computations of open invariants in [13] only use the fact that it is Lagrangian in the standard metric on the resolved conifold. On the other hand, one does not need a necessarily Lagrangian submanifold to build a theory of open holomorphic curves. It suffices to have a totally real submanifold [19] with some uniformity conditions in non-compact cases [3, 23]. Conifold transitions of perturbed conormal bundles constructed in this paper do meet these conditions. Moreover, one can show that the moduli of holomorphic curves ending on these submanifolds are compact and thus suitable for defining open Gromov–Witten invariants (Theorem 5.12).

The paper is organized as follows. In Section 1, we briefly review the conifold transition and introduce a natural notion of the conifold transition for submanifolds of T^*S^3. In Section 2 we compute explicitly the conifold transitions of the conormal bundles to the unknot and torus knots. In the former case, we get the well-known Ooguri-Vafa Lagrangian [21], while in the latter a variety which is neither smooth nor Lagrangian. In Section 3, a perturbed conormal bundle is defined and we prove that its conifold transition is a tame Lagrangian in $\widehat{\mathcal{C}}$ (see Definition 3.4). In Section 4, we lay the geometric groundwork for the compactness result in Section 5. Namely, we use the technique of second fundamental forms and bi-Lipschitz maps to prove that the resolved conifold has bounded geometry, i.e., its sectional curvature is bounded from above and the injectivity radius is bounded from below. The key point is Lemma 4.8 which formalizes the idea that the conifold has cone-like geometry. Finally, in Section 5 the moduli spaces of open curves are introduced following [16] and the compactness of the moduli of curves ending on $\mathrm{CT}(N^*_{k,\varepsilon})$ is proved. In the end we present our conclusions.

1 The conifold transition

In this section, we first review some basic facts about the conifold transition [5, 6, 9, 24] and fix the notation used throughout the paper. Then we define the conifold transition for submanifolds in T^*S^3 and apply it to conormal bundles to knots in S^3.

It is convenient to think of S^3 as being embedded into $\mathbb{R}^4$ as the unit sphere and identify T^*S^3 with TS^3 via the standard metric. At each point x of the sphere the tangent space T_xS^3 is naturally identified with the tangent hyperplane in $\mathbb{R}^4$ at this point. Shifting it to the origin we get the subspace of $\mathbb{R}^4$ orthogonal to x and obtain a natural realization of the tangent bundle in $\mathbb{R}^4 \times \mathbb{R}^4$ as

$$T^*S^3 = TS^3 = \{(x,p) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |x| = 1,\, p \cdot x = 0\}.$$

Now introduce complex coordinates on $\mathbb{R}^4 \times \mathbb{R}^4 \simeq \mathbb{C}^4$ by $z_j := x_j + ip_j$. As realized above, T^*S^3 is not an algebraic submanifold of $\mathbb{C}^4$, but it is diffeomorphic to any member of the family with $a > 0$:

$$\mathcal{C}_a := \Big\{ z \in \mathbb{C}^4 \mid \textstyle\sum_j z_j^2 = a^2 \Big\} = \Big\{ (x,p) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |x|^2 - |p|^2 = a^2, p \cdot x = 0 \Big\}.$$

Any $\mathcal{C}_a$ is manifestly algebraic and the diffeomorphism with T^*S^3 is given by

$$\begin{aligned} T^*S^3 &\xrightarrow{F_a} \mathcal{C}_a \\ (x,p) &\longmapsto \big(x\sqrt{a^2 + |p|^2}, p\big). \end{aligned}$$

The standard Kähler form

$$\omega := \frac{i}{2} \sum_j dz_j \wedge \overline{dz_j} = \sum_j dx_j \wedge dp_j \tag{1.1}$$

on $\mathbb{C}^4$ restricts to a Kähler form on every $\mathcal{C}_a$. We can pull these forms back to T^*S^3 via F_a, i.e., $\omega_a := F_a^*\omega\,|_{\mathcal{C}_a}$. It is straightforward to check that the standard symplectic form on T^*S^3 is obtained as the limit $\omega_\infty := \lim_{a\to\infty} \omega_a/a$, but ω_∞ is no longer Kähler.

Let us look at the other limit $a \to 0$ (cf. [24], Section 3). The algebraic subvariety

$$\mathcal{C} := \mathcal{C}_0 = \left\{ z \in \mathbb{C}^4 \mid \sum_j z_j^2 = 0 \right\} = \left\{ (x,p) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |x| = |p|, p \cdot x = 0 \right\} \tag{1.2}$$

is called the **conifold** [5]. It is singular with a nodal (ordinary double) point at the origin [6, 9]. Equivalently, one can think of it as T^*S^3 with the singular form $\omega_0 := F_0^*\omega\,|_{\mathcal{C}}$ that degenerates along the zero-section. The manifolds $\mathcal{C}_a$ or equivalently (T^*S^3, ω_a) are called the **deformed conifolds**. The parameter a has a simple geometric interpretation as the radius of the 3-sphere $p = 0$ in $\mathcal{C}_a$, i.e., of the zero-section. As a goes to 0 this sphere collapses into the singular point in $\mathcal{C}$. The map

$$F(x,p) := F_0(x,p) = (|p|x, p) \tag{1.3}$$

contracts T^*S^3 onto the conifold (Figure 1).

An alternative to smooth deformation when dealing with singularities is resolution. The simplest type of resolution is blow-up [17] and blowing up the origin in $\mathbb{C}^4$ produces an algebraic submanifold of $\mathbb{CP}^3 \times \mathbb{C}^4$. Let $\pi_2 : \mathbb{CP}^3 \times \mathbb{C}^4 \to \mathbb{C}^4$ be the natural projection to the second factor. Then the proper transform $\widetilde{\mathcal{C}} := \overline{\pi_2^{-1}(\mathcal{C} \setminus \{0\})}$ is the blow-up or the large resolution of $\mathcal{C}$ (here and below the overline denotes the closure). The inverse image of the singular point $\pi_2^{-1}(0) \simeq \mathbb{CP}^2$ is called the exceptional divisor.

However, the conifold singularity admits a smaller resolution that only adds an exceptional curve $\mathbb{CP}^1$ instead of a whole divisor $\mathbb{CP}^2$. To describe this resolution it is convenient to use different coordinates on $\mathbb{C}^4$:

$$\begin{aligned} w_1 &= z_1 + \mathrm{i}z_2 \qquad & w_2 &= -z_3 + \mathrm{i}z_4 \\ w_3 &= z_3 + \mathrm{i}z_4 \qquad & w_4 &= z_1 - \mathrm{i}z_2. \end{aligned} \tag{1.4}$$

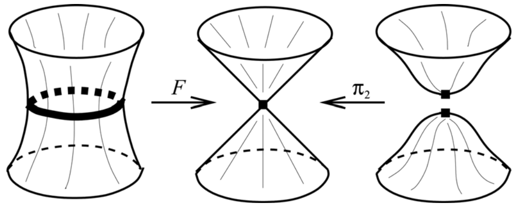

Figure 1: The "conifold" transition two dimensions down $S^1 \times \mathbb{R}^1 \rightsquigarrow S^0 \times \mathbb{R}^2$.

Up to the factor of $1/\sqrt{2}$, this is a unitary transformation of $\mathbb{C}^4$ and the difining equation (1.2) of the conifold becomes:

$$\mathcal{C} = \left\{ w \in \mathbb{C}^4 \mid w_1 w_4 - w_2 w_3 = \begin{vmatrix} w_1 & w_2 \\ w_3 & w_4 \end{vmatrix} = 0 \right\}.$$

In the w-coordinates the small resolution can be written explicitly as

$$\widehat{\mathcal{C}} := \left\{ ([u:v], w_1, w_2, w_3, w_4) \in \mathbb{CP}^1 \times \mathbb{C}^4 \,\middle|\, \begin{vmatrix} w_1 & w_2 \\ u & v \end{vmatrix} = \begin{vmatrix} w_3 & w_4 \\ u & v \end{vmatrix} = 0 \right\}, \tag{1.5}$$

and $\widehat{\mathcal{C}}$ is called the **resolved conifold**. It is obvious from the definition that $\widehat{\mathcal{C}} \to \mathbb{CP}^1$ is a holomorphic sub-bundle of the trivial $\mathbb{C}^4$-bundle over $\mathbb{CP}^1$, and therefore the total space is a smooth manifold. We denote its **zero-section** by $0(\widehat{\mathcal{C}})$. The resolution preserves the canonical class [6, 9] and $\widehat{\mathcal{C}}$ along with $\mathcal{C}$ and $\mathcal{C}_a$ is a Calabi-Yau threefold [12].

To understand the resolved conifold better, consider the tautological line bundle over $\mathbb{CP}^1$:

$$\mathcal{O}(-1) := \left\{ ([\lambda], w) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid w \in [\lambda] \right\}.$$

Here, we interpret $[\lambda]$ as a line in $\mathbb{C}^2$ and the fiber over it consists of all the points on this line (hence the name tautological). The letter $\mathcal{O}$ is traditional for holomorphic bundles in algebraic geometry and and the number -1 refers to the fact that the first Chern class of this bundle evaluates to -1 on the base $\mathbb{CP}^1$ [4, 17]. More explicitly,

$$\mathcal{O}(-1) = \left\{ ([u:v], w_1, w_2) \in \mathbb{CP}^1 \times \mathbb{C}^2 \,\middle|\, \begin{vmatrix} w_1 & w_2 \\ u & v \end{vmatrix} = 0 \right\}$$

and one can see by inspection that

$$\widehat{\mathcal{C}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Denoting by π_1, π_2 the natural projections to $\mathbb{CP}^1$ and $\mathbb{C}^4$, respectively, one notes that

$$\mathcal{C} = \pi_2(\widehat{\mathcal{C}}), \quad \widehat{\mathcal{C}} = \overline{\pi_2^{-1}(\mathcal{C} \setminus \{0\})}.$$

Moreover, the projection π_2 restricts to a biholomorphism from $\widehat{\mathcal{C}} \setminus \pi_2^{-1}(0)$ to $\mathcal{C} \setminus \{0\}$. The resolved conifold $\widehat{\mathcal{C}}$ admits a Kähler form $\pi_1^* \omega_{\mathrm{FS}} + \pi_2^* \omega$, where

ω_{FS} is the Fubini-Study form on $\mathbb{CP}^1$ and ω is the standard Kähler form (1.1) on $\mathbb{C}^4$.

The conifold transition from the deformed to the resolved conifold is streamlined in the following diagram

$$\begin{tikzcd} T^*S^3 \arrow[rr, dashed] \arrow[dr, "F"'] & & \widehat{\mathcal{C}} \arrow[r, hook] \arrow[dl, "\pi_2"] & \mathbb{CP}^1 \times \mathbb{C}^4 \\ & \mathcal{C} & & \end{tikzcd} \tag{1.6}$$

where $F(x,p) = (|p|x, p)$ is the contraction (1.3) that shrinks the zero-section to the conifold singularity at the origin. Since $\pi_1^{-1}(0) \simeq \mathbb{CP}^1$ the singularity gets replaced by a new S^2-cycle in $\widehat{\mathcal{C}}$. Topologically, we have the transition $S^3 \times \mathbb{R}^3 \rightsquigarrow S^2 \times \mathbb{R}^4$ (see Figure 1).

We will also be interested in the conifold transitions of certain submanifolds of T^*S^3. The most natural transition seems to be the contraction by F into $\mathcal{C}$ followed by the proper transform.

Definition 1.1. The **conifold transition of a submanifold** $N \subset T^*S^3$ is

$$\mathrm{CT}(N) := \overline{\pi_2^{-1}(F(N) \setminus \{0\})}. \tag{1.7}$$

In our case, $F(N)$ will be Lagrangian and not complex submanifolds so we cannot expect that the proper transform will produce smooth manifolds. In fact, as examples in the next section show, the conifold transitions are not smooth in general.

The submanifolds we are primarily interested in are conormal bundles to knots. We will show in Example 2.2 that our conifold transition of the conormal bundle to the unknot is the Ooguri-Vafa Lagrangian obtained in [13, 21] as the fixed locus of an antiholomorphic involution.

Definition 1.2. Let $S \hookrightarrow M$ be a submanifold. Then its **conormal bundle** in T^*M is

$$N_S^* := \{\ell \in T^*M \mid \pi(\ell) \in S,\ \ell\,|_{TS} = 0\}$$

Let $k : S^1 \to S^3$ be a knot. Then under the identification of T^*S^3 with the submanifold of $\mathbb{R}^4 \times \mathbb{R}^4$ one gets:

$$\begin{aligned} N_k^* &= \Big\{(k(t), p) \in T^*S^3 \mid t \in S^1,\ p \cdot \dot{k}(t) = 0\Big\} \\ &= \Big\{(k(t), p) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid t \in S^1,\ p \cdot k(t) = p \cdot \dot{k}(t) = 0\Big\}. \end{aligned} \tag{1.8}$$

Lemma 1.3. *N_k^* admits the following parametrization:*

$$\begin{aligned} S^1 \times \mathbb{R}^2 &\xrightarrow{K} T^*S^3 \\ (t, \alpha, \beta) &\longmapsto (k(t), \alpha p^1(t) + \beta p^2(t)), \end{aligned} \tag{1.9}$$

where $p^1(t)$, $p^2(t)$ are fundamental solutions in $\mathbb{R}^4$ to

$$\begin{aligned} k(t) \cdot p &= 0 \\ \dot{k}(t) \cdot p &= 0, \end{aligned}$$

and for every t, the vectors $k(t)$, $\dot{k}(t)$, $p^1(t)$, $p^2(t)$ form an orthonormal basis in $\mathbb{R}^4$. Moreover, p^1, p^2 are C^∞-smooth if k is.

Proof. Since $k(S^1) \subset S^3$, we have $|k(t)| = 1$ which implies $k(t) \cdot \dot{k}(t) = 0$. Choosing the natural parametrization for the knot we also get $|\dot{k}(t)| = 1$. This means that the system for p^1, p^2 is non-degenerate. Choose a stereographic projection $\sigma : S^3 \to \mathbb{R}^3 \cup \{\infty\}$, so that k passes neither through the north nor through the south pole. By transversality, we may also assume that $\sigma(k(t))$ and $\sigma_*\dot{k}(t)$ are linearly independent for every t. Set $\tilde{p}^1(t)$ to be the cross-product $\sigma(k(t)) \times \sigma_*\dot{k}(t)$ in $\mathbb{R}^3$ then $k(t)$, $\dot{k}(t)$, $\sigma_*^{-1}\tilde{p}^1(t)$ are linearly independent in $\mathbb{R}^4$. Now, one can get $p^1(t)$ by the Gram–Schmidt process. Finally, $p^2(t) := k(t) \times \dot{k}(t) \times p^1(t)$, the cross-product of three vectors in $\mathbb{R}^4$. The smoothness is obvious from the construction. $\square$

It turns out that N_k^* is an exact Lagrangian submanifold in T^*S^3. Let us recall the definition [3].

Definition 1.4. A symplectic manifold (X, ω) is exact, if ω has a primitive, i.e., there is a 1-form λ such that $\omega = d\lambda$. A Lagrangian submanifold $L \hookrightarrow X$ is exact, if $[\lambda|_L] = 0 \in H^1(L, \mathbb{R})$.

In particular, any $X = T^*M$ with the canonical symplectic structure is exact and λ is the canonical Liouville form. In Darboux coordinates, $\lambda = -\Sigma_j p_j dx_j$ and $\lambda\,|_{0(T^*M)} = 0$, where $0(T^*M)$ denotes the zero-section corresponding to $p = 0$. So the zero-section is always an exact Lagrangian submanifold.

Remark 1.5. Note that if $f : (\Sigma, \partial\Sigma) \to (T^*M, L)$ is a pseudoholomorphic open curve (see Section 5) ending on the zero-section or any other exact Lagrangian submanifold L then

$$\text{Area}(f) = \int_\Sigma f^*\omega = \int_\Sigma d(f^*\lambda) = \int_{\partial\Sigma} f^*\lambda = [\lambda\,|_L](f_*[\partial\Sigma]) = 0.$$

Thus f has to be a constant. Witten [26] calls this fact "the vanishing theorem".

In our case, the symplectic form on T^*S^3 was the restriction of $\omega = \Sigma_{j=1}^4 dx_j \wedge dp_j$ from $\mathbb{R}^4 \times \mathbb{R}^4$. If one sets $\lambda := -\Sigma_{j=1}^4 p_j dx_j$, then obviously $\omega = d\lambda$ and since d commutes with restrictions $\omega\,|_{T^*S^3} = d(\lambda\,|_{T^*S^3})$.

Lemma 1.6. *N_k^* is an exact Lagrangian submanifold in T^*S^3.*

Proof. In fact, we will show that $\lambda\,|_{N_k^*} = 0$. Identifying $T_{(x,p)}N_k^*$ with a subspace of $\mathbb{R}^4 \times \mathbb{R}^4$ in the usual way one gets:

$$\begin{aligned} K_*\partial_t &= (\dot{k}, \alpha\dot{p}^1 + \beta\dot{p}^2) \\ K_*\partial_\alpha &= (0, p^1) \\ K_*\partial_\beta &= (0, p^2) \end{aligned}$$

Thus

$$\lambda(K_*\partial_t) = (\alpha p^1 + \beta p^2) \cdot \dot{k} = \alpha p^1 \cdot \dot{k} + \beta p^2 \cdot \dot{k} = 0$$

and

$$\lambda(K_*\partial_\alpha) = (\alpha p^1 + \beta p^2) \cdot 0 = 0 = \lambda(K_*\partial_\beta).$$

Finally, N_k^* has the right dimension, $\dim N_k^* = 1/2 \dim T^*S^3 = 3$. $\square$

Intuitively, this corresponds to the fact that the only x-direction in N_k^* is the one orthogonal to its p-directions so $\lambda = -p \cdot dx$ vanishes on it "by definition".

Recall from (1.3) that the first half of the conifold transition is the contraction $F(x,p) = (|p|x, p)$. It turns out that although this map is not symplectic it does map exact Lagrangians into exact Lagrangians away from the singular locus.

Lemma 1.7. *$F(N_k^*) \setminus \{0\}$ is a Lagrangian submanifold in $\mathcal{C}$.*

Proof. On T^*S^3 one has $F^*\lambda = |p|\lambda, \quad F^*\omega = |p|\omega + d|p| \wedge \lambda$. Indeed,

$$F^*\lambda = F^*(-\sum_j p_j dx_j) = -\sum_j p_j d(|p|x_j)$$

$$= -|p| \sum_j p_j dx_j - \sum_j p_j x_j d|p|$$
$$= |p|\lambda - (p \cdot x) d|p| = |p|\lambda,$$

since $p \cdot x = 0$ on T^*S^3. The second relation follows from the first one:

$$F^*\omega = F^*d\lambda = dF^*\lambda = d(|p|\lambda)$$
$$= |p|d\lambda + d|p| \wedge \lambda.$$

$N_k^* \cap 0(T^*S^3)$ will be mapped into the conifold singularity at the origin, but away from that point $\lambda|_{F(N_k^*)} = F^*\lambda|_{N_k^*} = |p|\lambda|_{N_k^*} = 0$, i.e., $F(N_k^*) \setminus \{0\}$ is still a Lagrangian submanifold. $\square$

The last step is to lift $F(N_k^*)$ to the resolved conifold $\widehat{\mathcal{C}}$. The Kähler structure on $\widehat{\mathcal{C}}$ is induced by the product structure on $\mathbb{CP}^1 \times \mathbb{C}^4$, namely $\widehat{\omega} := (\pi_1^*\omega_{\mathrm{FS}} + \pi_2^*\omega)|_{\widehat{\mathcal{C}}}$ with $\omega = \frac{i}{2}\Sigma_{j=1}^4 dz_j \wedge \overline{dz_j} = i\Sigma_{j=1}^4 dw_j \wedge \overline{dw_j}$ in complex coordinates.

Theorem 1.8. *$\pi_2^{-1}(F(N_k^*) \setminus \{0\})$ is Lagrangian in $(\widehat{\mathcal{C}}, \widehat{\omega})$ if and only if it projects to a set of zero volume in $\mathbb{CP}^1$.*

Proof. Since $\pi_2^* : \widehat{\mathcal{C}} \setminus \pi_2^{-1}(0) \to \mathcal{C} \setminus \{0\}$ is a biholomorphism we have,

$$\widehat{\omega}\,|_{\pi_2^{-1}(F(N_k^*)\setminus\{0\})} = \pi_2^{-1*}\widehat{\omega}\,|_{F(N_k^*)\setminus\{0\}} = \pi_2^{-1*}(\pi_1^*\omega_{\mathrm{FS}} + \pi_2^*\omega)\,|_{F(N_k^*)\setminus\{0\}}$$
$$= (\pi_1 \circ \pi_2^{-1})^*\omega_{\mathrm{FS}}\,|_{F(N_k^*)\setminus\{0\}} + \omega\,|_{F(N_k^*)\setminus\{0\}}$$
$$= \omega_{\mathrm{FS}}\,|_{\pi_1(\pi_2^{-1}(F(N_k^*)\setminus\{0\}))'},$$

where we used the fact that $F(N_k^*) \setminus \{0\}$ is Lagrangian by Lemma 1.7. The left-hand side is 0 if and only if $\pi_2^{-1}(F(N_k^*) \setminus \{0\})$ is Lagrangian, while the right-hand side is 0 if and only if its projection to $\mathbb{CP}^1$ has 0 volume. $\square$

Recall that by our definition, $\mathrm{CT}(N_k^*) = \overline{\pi_2^{-1}(F(N_k^*) \setminus \{0\})}$ so this theorem does not guarantee that the conifold transition of a conormal bundle is a manifold even if it does project to a null set. However, if the closure is indeed a smooth manifold with the projection of zero volume, it will automatically be Lagrangian ($\widehat{\omega} = 0$ on the closure by continuity). Another remark is that our choice of $\widehat{\omega}$ on $\widehat{\mathcal{C}}$ is more or less arbitrary. From the physical point of view a more natural choice is ω_{CY}, the Kähler form induced by the Calabi-Yau metric on $\widehat{\mathcal{C}}$ [5]. But as we will see, $\mathrm{CT}(N_k^*)$ is not even a smooth manifold already for torus knots.

2 The unknot and torus knots

Here we use the parametrization of N_k^* from the previous section (Lemma 1.3) to compute $\mathrm{CT}(N_k^*)$ for k the unknot or a torus knot. For this computation, it is convenient to use a different complex structure on $\mathbb{R}^4 \times \mathbb{R}^4$ given by the new holomorphic coordinates

$$\xi = (\xi_1, \xi_2) = (x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4)$$
$$\eta = (\eta_1, \eta_2) = (p_1 + \mathrm{i}p_2, p_3 + \mathrm{i}p_4).$$

In these coordinates

$$T^*S^3 = \Big\{(\xi, \eta) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid |\xi| = 1,\ \mathrm{Re}(\xi\overline{\eta}) = 0\Big\}$$

and the change to w-coordinates is

$$w_1 = \xi_1 + \mathrm{i}\eta_1 \qquad w_2 = -(\overline{\xi}_2 + \mathrm{i}\overline{\eta}_2)$$
$$w_3 = \xi_2 + \mathrm{i}\eta_2 \qquad w_4 = \overline{\xi}_1 + \mathrm{i}\overline{\eta}_1$$

The parametrization of a conormal bundle has the same form as before

$$K(t, \alpha, \beta) = \big(k(t), \alpha p^1(t) + \beta p^2(t)\big)$$

but k, p^1, p^2 now are $\mathbb{C}^2$ vectors. Applying the contraction F one obtains

$$F \circ K(t, \alpha, \beta) = \Big(k(t)\sqrt{\alpha^2 + \beta^2}, \quad \alpha p^1(t) + \beta p^2(t)\Big),$$

which suggests the change to polar coordinates

$$r = \sqrt{\alpha^2 + \beta^2}, \quad \tan\theta = \frac{\beta}{\alpha},$$

where $F \circ K(t, r, \theta) = r(k(t), \wp(t, \theta))$, $\wp(t, \theta) := p^1(t)\cos\theta + p^2(t)\sin\theta$. Here $r \geq 0$ and $r = 0$ corresponds to the conifold singularity.

Example 2.1. For the unknot we have

$$k(t) = (e^{it}, 0) \in S^3 \hookrightarrow \mathbb{C}^2,$$
$$\dot{k}(t) = (ie^{it}, 0),$$
$$p^1(t) = (0, 1),$$
$$p^2(t) = (0, i).$$

Thus $\wp(t, \theta) = (0, 1)\cos\theta + (0, i)\sin\theta = (0, e^{i\theta})$ and $F \circ K(t, r, \theta) = r(e^{it}, e^{i\theta})$. In terms of this parametrization $F(N_k^*) \setminus \{0\} = F \circ K(S^1 \times \mathbb{R}_{>0} \times$

S^1). To find the proper transform, we change to w-coordinates:

$$w_1 = re^{it} \qquad w_2 = \overline{ire^{i\theta}} = -ire^{-i\theta}$$
$$w_3 = ire^{i\theta} \qquad w_4 = \overline{re^{it}} = re^{-it}$$

By definition of the resolved conifold

$$\pi_2^{-1}(w_1, w_2, w_3, w_4) = ([u : v], w_1, w_2, w_3, w_4),$$

where $\begin{vmatrix} w_1 & w_2 \\ u & v \end{vmatrix} = \begin{vmatrix} w_3 & w_4 \\ u & v \end{vmatrix} = 0$. Since on $F(N_k^*) \setminus \{0\}$ when $r > 0$ both w_1, w_2 are never 0, we can just set $[u : v] = [w_1 : w_2]$ and get:

$$\begin{aligned} \mathrm{CT}(N_k^*) &= \overline{\pi_2^{-1}(F(N_k^*) \setminus \{0\})} \\ &= \Big\{ ([e^{it} : -ie^{-i\theta}], re^{it}, -ire^{-i\theta}, ire^{i\theta}, re^{-it}) \mid t, \theta \in S^1,\ r \geq 0 \Big\} \\ &= \Big\{ ([ie^{i(t+\theta)} : 1], re^{it}, -ire^{i\theta}, ire^{i\theta}, re^{-it}) \mid t, \theta \in S^1, r \geq 0 \Big\} \end{aligned}$$

This becomes more transparent if one sets

$$\alpha := \mathrm{i}e^{\mathrm{i}(t+\theta)}$$
$$b := -\mathrm{i}r e^{-\mathrm{i}\theta}$$

so that

$$\mathrm{CT}(N_k^*) = \Big\{ ([\alpha : 1], \alpha b, b, \overline{b}, \overline{\alpha b}) \mid \alpha \in S^1, b \in \mathbb{C} \Big\}.$$

This is a smooth submanifold of $\widehat{\mathcal{C}}$ diffeomorphic to $S^1 \times \mathbb{C}$ and since $|\alpha| = 1$ it fibers over the equator of $\mathbb{CP}^1$ (Figure 2). By Theorem 1.8, this means that $\mathrm{CT}(N_k^*)$ is also Lagrangian. In fact, this is the same Lagrangian submanifold that was obtained in [21] as the fixed locus of an antiholomorphic involution and used in [13] to compute open Gromov–Witten invariants. Note that the topologies of N_k^* and $\mathrm{CT}(N_k^*)$ are the same, namely $S^1 \times \mathbb{R}^2$ even though T^*S^3 changes its topology from $S^3 \times \mathbb{R}^3$ to $S^2 \times \mathbb{R}^4$.

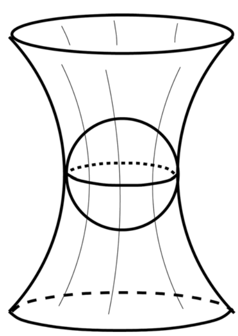

Figure 2: Conifold transition for the unknot (low-dimensional analog).

Example 2.2 (Torus knots).

There is a standard copy of a 2-torus sitting in $\mathbb{C}^2$: $\{(\xi_1, \xi_2) \mid |\xi_1| = |\xi_2| = 1\}$. If we change the normalization from 1 to $1/\sqrt{2}$ this torus will be placed inside $S^3 \hookrightarrow \mathbb{C}^2$. The embedding $k(t) = 1/\sqrt{2}\ (\mathrm{e}^{\mathrm{i}mt}, \mathrm{e}^{\mathrm{i}nt})$ obviously winds m times around one of the cycles in $\mathbb{T}^2$ and n times around the other one. Therefore, for relatively prime $(m,n) = 1$ it represents an (m,n) torus knot. We will assume, $m \neq n$, since $m = n = 1$ is the case of the unknot and otherwise m, n can not be relatively prime. We have

$$\begin{aligned}
k(t) &= \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}mt}, \mathrm{e}^{\mathrm{i}nt}),\\
\dot{k}(t) &= \frac{1}{\sqrt{2}}(\mathrm{i}m\mathrm{e}^{\mathrm{i}mt}, \mathrm{i}n\mathrm{e}^{\mathrm{i}nt}),\\
p^1(t) &= \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}mt}, -\mathrm{e}^{\mathrm{i}nt}),\\
p^2(t) &= \frac{1}{\sqrt{m^2+n^2}}(\mathrm{i}n\mathrm{e}^{\mathrm{i}mt}, -\mathrm{i}m\mathrm{e}^{\mathrm{i}nt}).
\end{aligned}$$

The parameter t here is obviously not the arclength, but the only effect this has is that $|\dot{k}|^2 = (m^2 + n^2/2) = \text{const}$ instead of 1 so the difference is insignificant.

$$\begin{aligned}
\wp(t,\theta) = \frac{1}{\sqrt{2}}\Bigg(&\mathrm{e}^{\mathrm{i}mt}\left(\cos\theta + \mathrm{i}\frac{n\sqrt{2}}{\sqrt{n^2+m^2}}\sin\theta\right),\\
&-\mathrm{e}^{\mathrm{i}nt}\left(\cos\theta + \mathrm{i}\frac{m\sqrt{2}}{\sqrt{n^2+m^2}}\sin\theta\right)\Bigg)
\end{aligned}$$

Writing $F \circ K(t,r,\theta) = r(k(t), \wp(t,\theta))$ and changing to w-coordinates one finds

$$\begin{aligned}
w_1 &= \frac{r\mathrm{e}^{\mathrm{i}mt}}{\sqrt{2}}\left(1 - \frac{n\sqrt{2}}{\sqrt{n^2+m^2}}\sin\theta + \mathrm{i}\cos\theta\right),\\
w_2 &= -\frac{r\mathrm{e}^{-\mathrm{i}nt}}{\sqrt{2}}\left(1 + \frac{m\sqrt{2}}{\sqrt{n^2+m^2}}\sin\theta - \mathrm{i}\cos\theta\right),\\
w_3 &= \frac{r\mathrm{e}^{\mathrm{i}nt}}{\sqrt{2}}\left(1 - \frac{m\sqrt{2}}{\sqrt{n^2+m^2}}\sin\theta - \mathrm{i}\cos\theta\right),\\
w_4 &= \frac{r\mathrm{e}^{-\mathrm{i}mt}}{\sqrt{2}}\left(1 + \frac{n\sqrt{2}}{\sqrt{n^2+m^2}}\sin\theta + \mathrm{i}\cos\theta\right).
\end{aligned}$$

Just as in the case of the unknot for $r > 0$ one can set $[u : v] = [w_1 : w_2] =$

$$= \left[\mathrm{e}^{imt} \left(1 - \frac{n\sqrt{2}}{\sqrt{n^2+m^2}} \sin\theta + \mathrm{i}\cos\theta \right) : \right.$$

$$\left. -\mathrm{e}^{-int} \left(1 + \frac{m\sqrt{2}}{\sqrt{n^2+m^2}} \sin\theta - \mathrm{i}\cos\theta \right) \right]$$

and since the last expression does not depend on r taking the closure is simply allowing $r = 0$ in the formulas for the w_j above. Since F and π_2^{-1} are diffeomorphisms away from the zero-section of T^*S^3 and the origin, respectively, $\mathrm{CT}(N_k^*)$ is a smooth manifold everywhere, but possibly at the zero-section of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \simeq \widehat{\mathcal{C}}$, where $r = 0$.

Now, we take a look at the $\mathbb{CP}^1$ projection of $\mathrm{CT}(N_k^*)$. It is parametrized by $[u : v](t, \theta)$ given above and u/v is a possible coordinate on the projection (unless $m = n$, which is the case we excluded). Hence

$$w(t,\theta) := \frac{u}{v} = -\mathrm{e}^{\mathrm{i}(m+n)t} \, \frac{1 - \left(n\sqrt{2}/\sqrt{n^2+m^2}\right)\sin\theta + \mathrm{i}\cos\theta}{1 + (m\sqrt{2}/\sqrt{n^2+m^2})\sin\theta - \mathrm{i}\cos\theta}$$

parametrizes the image of the projection in $\mathbb{C}$. The first factor is 1 in absolute value while the absolute value of the second one changes between two positive values less than and greater than 1.

This means that the $\mathbb{CP}^1$ trace of $\mathrm{CT}(N_k^*)$ is an annulus containing the equator. In the case of the unknot, the trace was just the equator itself, in particular it was one-dimensional (Figure 2). This is due to the fact that the unknot in S^3 can be flat, i.e., placed within a 2-plane, which is impossible for any non-trivial knot. As a result of replacing the 3-cycle by a 2-cycle, the conormal bundle to a non-planar knot "smashes" into an annulus on $\mathbb{CP}^1$ and $\mathrm{CT}(N_k^*)$ acquires a corner singularity along the edge of the annulus (see Figure 3). By Theorem 1.8, $\mathrm{CT}(N_k^*)$ is not Lagrangian even away from the

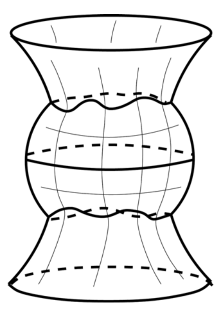

Figure 3: Corner singularity in the conifold transition for torus knots.

singularity. For the unknot, the flatness makes it possible for the conifold transited conormal bundle to just touch the equator.

3 Perturbed conormal bundles

The obvious reason for the conifold transition of a conormal bundle to be singular is that it intersects the zero-section of T^*S^3 which collapses into a singular point. The simplest way to avoid this is to perturb N_k^* so that it is disconnected from the zero-section. Of course, we would like to obtain an exact Lagrangian $\widetilde{N}_k^*$ after perturbation since this would make $\mathrm{CT}(\widetilde{N}_k^*)$ a Lagrangian submanifold of the resolved conifold. Unfortunately, there is an obstruction to such perturbation following from a theorem of Gromov-Oh: in a cotangent bundle every exact Lagrangian submanifold intersects the zero-section [3, 11, 20].

Thus we have to settle for an ordinary isotopy instead of a symplectic one. Even though $\mathrm{CT}(\widetilde{N}_k^*)$ will no longer be Lagrangian in $\widehat{\mathcal{C}}$, it will be good enough for the purposes of open Gromov–Witten theory. Specifically, it will be Lagrangian with respect to a different uniformly tame symplectic form on $\mathcal{C}$ (tame Lagrangian, see Definition 3.4).

As before we identify $T^*S^3 \simeq TS^3$. To separate N_k^* from the zero-section, we simply move it within each fiber in the direction tangent to the knot. Recall that $N_k^* = \{(k(t), p) \in T^*S^3 \mid t \in S^1, p \cdot \dot{k}(t) = 0\}$.

Definition 3.1. The perturbed conormal bundle is

$$N_{k,\varepsilon}^* := \{(k(t), p + \varepsilon\dot{k}(t)) \mid t \in S^1, \quad p \cdot \dot{k}(t) = 0\}. \tag{3.1}$$

Since $\dot{k}(t) \cdot k(t) = 0$ one has $N_{k,\varepsilon}^* \subset T^*S^3$ for all $\varepsilon \geq 0$. And $p \cdot \dot{k}(t) = 0$ implies $|p + \varepsilon\dot{k}(t)|^2 = |p|^2 + \varepsilon^2|\dot{k}(t)|^2 = |p|^2 + \varepsilon^2 \geq \varepsilon^2 > 0$ so $N_{k,\varepsilon}^*$ is, indeed, disjoint from the zero-section for any $\varepsilon > 0$.

Since $\mathrm{CT}(N_{k,\varepsilon}^*) = \pi_2^{-1} \circ F(N_{k,\varepsilon}^*)$, the proofs in this section split into two parts. First, we prove that $F(N_{k,\varepsilon}^*)$ is a tame Lagrangian in $\mathcal{C}$ and then that π_2^{-1} preserves this property. Lemma 3.10 is also used to prove bounded geometry of $\widehat{\mathcal{C}}$ in the next section. It is convenient to represent $N_{k,\varepsilon}^*$ as the image of N_k^* under an ambient isotopy in $\mathbb{R}^4 \times \mathbb{R}^4 \supset T^*S^3$. To this end, let ξ be a smooth vector field in $\mathbb{R}^4$ with compact support satisfying $\xi(k(t)) = \dot{k}(t)$.

Definition 3.2. Let

$$\begin{aligned} \mathbb{R}^4 \times \mathbb{R}^4 &\xrightarrow{\Phi_\varepsilon} \mathcal{C}_a \\ (x,p) &\longmapsto (x, p + \varepsilon\xi(x)). \end{aligned}$$

Φ_ε is an isotopy since $\Phi_\varepsilon^{-1}(x,p) = (x, p - \varepsilon\xi(x))$ and by construction of ξ, $\Phi_\varepsilon(N_k^*) = N_{k,\varepsilon}^*$.

Let us now look at the image of $N_{k,\varepsilon}^*$ under the contraction $F(x,p) = (|p|x, p)$, the first half of the conifold transition:

$$F(N_{k,\varepsilon}^*) = \left\{ \left(x\sqrt{|p|^2 + \varepsilon^2}, p + \varepsilon\dot{k}(t) \right) \mid t \in S^1, p \cdot \dot{k}(t) = 0 \right\}.$$

Recall the regularized contraction F_ε from Section 1: $F_\varepsilon(x,p) := (x\sqrt{|p|^2+\varepsilon^2}, p)$. The following diagram of maps commutes (see Figure 4):

$$\begin{tikzcd} N_k^* \arrow[r, "F_\varepsilon"] \arrow[d, "\Phi_\varepsilon"'] & \mathbb{R}^4 \times \mathbb{R}^4 \arrow[d, "\Phi_\varepsilon"] \\ \mathbb{R}^4 \times \mathbb{R}^4 \arrow[r, "F"] & \mathbb{R}^4 \times \mathbb{R}^4. \end{tikzcd}$$

Now we want to investigate in what sense $F(N_{k,\varepsilon}^*) = F(\Phi_\varepsilon(N_k^*)) = \Phi_\varepsilon(F_\varepsilon(N_k^*))$ is close to being Lagrangian in $\mathcal{C}$.

Lemma 3.3. *$F_\varepsilon(N_k^*)$ is isotropic in $\mathbb{R}^4 \times \mathbb{R}^4$, i.e., $\omega = \Sigma_i dx_i \wedge dp_i$ vanishes on it.*

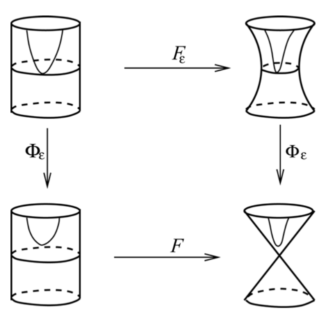

Figure 4: Disconnecting the conormal bundle from the zero-section.

Proof. From the parametrization of N_k^*, we have for $F_\varepsilon(N_k^*)$:

$$\begin{aligned} S^1 \times \mathbb{R}^2 &\overset{F_\varepsilon \circ K}{\longmapsto} \mathbb{R}^4 \times \mathbb{R}^4 \\ (t, \alpha, \beta) &\longmapsto \Big(k(t)\sqrt{\alpha^2+\beta^2+\varepsilon^2}, \alpha p^1(t) + \beta p^2(t)\Big). \end{aligned}$$

Thus the tangent bundle is spanned by

$$\begin{aligned} (F_\varepsilon \circ K)_* \partial_f &= \Big(\dot{k}(t)\sqrt{\alpha^2+\beta^2+\varepsilon^2}, \alpha \dot{p}^1(t) + \beta \dot{p}^2(t)\Big) \\ (F_\varepsilon \circ K)_* \partial_\alpha &= (k(t)\alpha(\alpha^2+\beta^2+\varepsilon^2)^{-1/2}, p^1(t)) \\ (F_\varepsilon \circ K)_* \partial_\beta &= (k(t)\beta(\alpha^2+\beta^2+\varepsilon^2)^{-1/2}, p^2(t)) \end{aligned}$$

Let J be the standard complex structure on $\mathbb{R}^4 \times \mathbb{R}^4 \simeq \mathbb{C}^4$, then

$$\begin{aligned} J(F_\varepsilon \circ K)_* \partial_t &= (-\alpha \dot{p}^1(t) - \beta \dot{p}^2(t), (\alpha^2+\beta^2+\varepsilon^2)^{1/2}\dot{k}(t)) \\ J(F_\varepsilon \circ K)_* \partial_\alpha &= (-p^1(t), \alpha(\alpha^2+\beta^2+\varepsilon^2)^{-1/2}k(t)) \\ J(F_\varepsilon \circ K)_* \partial_\beta &= (-p^2(t), \beta(\alpha^2+\beta^2+\varepsilon^2)^{-1/2}k(t)). \end{aligned}$$

Since $k(t)$, $\dot{k}(t)$, $p^1(t)$, $p^2(t)$ are pairwise orthogonal for each t one can see by inspection that $J(TF_\varepsilon(N_k^*)) \perp TF_\varepsilon(N_k^*)$, which is equivalent to isotropy. $\square$

However, $F(N_{k,\varepsilon}^*) = \Phi_\varepsilon(F_\varepsilon(N_k^*))$ will no longer be Lagrangian in $\mathcal{C}$. It does satisfy a weaker property that we now introduce [17, 23].

Definition 3.4. Let (M, J, g) be an almost Kähler manifold. A symplectic form $\widetilde{\omega}$ is called uniformly tame if there exists a constant $C \geq 1$ such that for any vector field X on M:

$$C^{-1}g(X,X) \leq \widetilde{\omega}(X, JX) \leq Cg(X,X) \tag{3.2}$$

A submanifold $L \hookrightarrow M$ is (uniformly) tame isotropic if there is a uniformly tame $\widetilde{\omega}$ defined in its neighborhood such that $\widetilde{\omega}\,|_{TL} = 0$. If in addition $\dim L = 1/2 \dim M$, then L is called (uniformly) tame Lagrangian.

Note that the Kähler form $\omega(X,Y) = g(JX, Y)$ is obviously uniformly tame with $C = 1$ so this is a generalization of the Lagrangian condition. Also, tame Lagrangian implies totally real, i.e., $J(TL) \cap TL = \{0\}$. Indeed, if $X \in J(TL) \cap TL$, then $JX \in TL$ since $J^2 = -I$ and $|X|^2 = g(X,X) \leq C\widetilde{\omega}(X, JX) = 0$ so $X = 0$. Most importantly for us, the property of being tame Lagrangian or isotropic is preserved under biholomorphisms. Namely, if $\Phi : (M, J, g) \to (\widetilde{M}, \widetilde{J}, \widetilde{g})$ is a biholomorphism and $L \hookrightarrow M$ is $\widetilde{\omega}$-Lagrangian then $\Phi(L)$ is $\Phi_*\widetilde{\omega} := \Phi^{-1*}\widetilde{\omega}$ -Lagrangian.

So far, we have not imposed any restrictions on the perturbation parameter ε. We will do so now to ensure uniform tameness. Let ξ be the perturbation vector field from Definition 3.2 and $D\xi$ be its Jacobian matrix. Set $\|D\xi\| := \sup_{x\in\mathbb{R}^4} \|D\xi(x)\|$

Lemma 3.5. *If $\varepsilon < (1/\|D\xi\|)$, then $\Phi_{\varepsilon *}\omega = \Phi_\varepsilon^{-1*}\omega$ is uniformly tame in $\mathbb{R}^4 \times \mathbb{R}^4 \simeq \mathbb{C}^4$.*

Proof.

$$\begin{aligned}\Phi_{\varepsilon *}\omega = \Phi_\varepsilon^{-1*}\omega &= \sum_i \Phi_\varepsilon^{-1*}(dx_i \wedge dp_i) = \sum_i dx_i \wedge d(p_i - \varepsilon\xi_i(x)) \\ &= \sum_i dx_i \wedge dp_i - \varepsilon \sum_{i,j} \frac{\partial \xi_i}{\partial x_j} dx_i \wedge dx_j\end{aligned}$$

Since $J\partial_{x_i} = \partial_{p_i}$ and $J\partial_{p_i} = -\partial_{x_i}$, we get

$$\begin{aligned}\Phi_{\varepsilon *}\omega(X, JX) &= \sum_i (dx_i(X)^2 + dp_i(X)^2) \\ &\quad - \varepsilon \sum_{i,j} \left(\frac{\partial \xi_i}{\partial x_j} - \frac{\partial \xi_i}{\partial x_j} \right) dp_i(X) dx_j(X) \\ &= |X|^2 - \varepsilon \big((D\xi - D\xi^T) dp(X), dx(X) \big)\end{aligned}$$

The absolute value of the second term is bounded by

$$2\varepsilon\|D\xi\||dp(X)||dx(X)| \le 2\varepsilon\|D\xi\| \frac{|dp(X)|^2 + |dx(X)|^2}{2} = \varepsilon\|D\xi\||X|^2$$

Therefore, $(1 - \varepsilon\|D\xi\|)|X|^2 \le \Phi_{\varepsilon *}\omega(X, JX) \le (1 + \varepsilon\|D\xi\|)|X|^2$ and if $\varepsilon < (1/\|D\xi\|)$, $\Phi_{\varepsilon *}\omega$ is uniformly tame with $C = (1 - \varepsilon\|D\xi\|)^{-1}$ $\square$

Corollary 3.6. *$F(N^*_{k,\varepsilon})$ is a tame Lagrangian in $\mathcal{C}$ for $\varepsilon < (1/\|D\xi\|)$.*

Proof. Let $\widetilde{\omega} = \Phi_{\varepsilon *}\omega$. Since Φ_ε is a diffeomorphism and $\Phi_\varepsilon(F_\varepsilon(N^*_k)) = F(N^*_{k,\varepsilon})$ vectors $\Phi_{\varepsilon *}X$ for $X \in TF_\varepsilon(N^*_k)$ span $TF(N^*_{k,\varepsilon})$. But

$$\widetilde{\omega}(\Phi_{\varepsilon *}X, \Phi_{\varepsilon *}Y) = \Phi_\varepsilon^{-1*}\omega(\Phi_{\varepsilon *}X, \Phi_{\varepsilon *}Y) = \omega(X, Y) = 0$$

since $F_\varepsilon(N^*_k)$ is isotropic by Lemma 3.3. $\square$

As $N^*_{k,\varepsilon}$ is disjoint from the zero-section of T^*S^3, its contraction $F(N^*_{k,\varepsilon})$ now avoids the singularity of the conifold at the origin. In fact, if $(x, p) \in F(N^*_{k,\varepsilon})$ then $|x|^2 + |p|^2 \ge 2\varepsilon^2 > 0$. Now the second half of the conifold transition (1.6) constitutes lifts $F(N^*_{k,\varepsilon})$ to $\widehat{\mathcal{C}}$. Recall from (1.4) that we changed the coordinates in $\mathbb{C}^4$ from $z = x + ip$ to $w = \sqrt{2}\,(Uz)$, where U is a unitary matrix. Therefore, along $F(N^*_{k,\varepsilon})$ we have $|w| \ge \sqrt{2} \cdot \sqrt{2}\,\varepsilon = 2\varepsilon$

and it is separated from 0. Since $\pi_2^{-1} : \mathcal{C} \setminus \{0\} \to \widehat{\mathcal{C}} \setminus 0(\widehat{\mathcal{C}})$ is a biholomorphism there is no need to take closure in the conifold transition (1.7) and we simply have

$$\mathrm{CT}(N^*_{k,\varepsilon}) = \pi_2^{-1}(F(N^*_{k,\varepsilon})).$$

To establish that $\mathrm{CT}(N^*_{k,\varepsilon})$ is a tame Lagrangian in $\widehat{\mathcal{C}}$ we need certain properties of π_2^{-1}.

Definition 3.7 [1]. A map $\Phi : (M, g) \to (\widetilde{M}, \widetilde{g})$ between Riemannian manifolds is (uniformly) bi-Lipschitz if there exists a constant $C \geq 1$ such that

$$C^{-1} g \leq \Phi^* \widetilde{g} \leq C g. \tag{3.3}$$

The map Φ does not have to be a diffeomorphism, e.g., any isometric immersion would satisfy this condition with $C = 1$. If Φ is a diffeomorphism then being uniformly bi-Lipschitz is equivalent to the norms of Φ_* and Φ_*^{-1} being bounded by a constant, i.e., $|\Phi_* X|_{\widetilde{g}} \leq C|X|_g$ and $|\Phi_*^{-1} Y|_g \leq C|Y|_{\widetilde{g}}$.

Lemma 3.8. *Let $\Phi : (M, J, g) \to (\widetilde{M}, \widetilde{J}, \widetilde{g})$ be a bi-Lipschitz biholomorphism between two almost Kähler manifolds. If $L \hookrightarrow M$ is a tame Lagrangian then so is $\widetilde{L} := \Phi(L)$.*

Proof. Let ω' be the corresponding uniformly tame symplectic form in the neighborhood of L in M, i.e., $\omega' |_{TL} = 0$, $C_1^{-1} g(X, X) \leq \omega'(X, JX) \leq C_1 g(X, X)$. Consider $\Phi_* \omega' = \Phi^{-1*} \omega' =: \widetilde{\omega}'$ in the neighborhood of $\widetilde{L}$. Since Φ is a biholomorphism:

$$\widetilde{\omega}'(\Phi_* X, J\Phi_* X) = \widetilde{\omega}'(\Phi_* X, \Phi_* JX) = \Phi^{-1*}\omega'(\Phi_* X, \Phi_* JX) = \omega'(X, JX).$$

Since Φ is bi-Lipschitz

$$C_2^{-1} g(X, X) \leq \widetilde{g}(\Phi_* X, \Phi_* X) \leq C_2 g(X, X).$$

Therefore, combining the inequalities

$$(C_1 C_2)^{-1} \widetilde{g}(\Phi_* X, \Phi_* X) \leq \widetilde{\omega}'(\Phi_* X, J\Phi_* X) \leq C_1 C_2 \, \widetilde{g}(\Phi_* X, \Phi_* X),$$

and $\widetilde{\omega}'$ is uniformly tame. Also if $X, Y \in TL$

$$\widetilde{\omega}'(\Phi_* X, \Phi_* Y) = \Phi^{-1*}\omega'(\Phi_* X, \Phi_* Y) = \omega'(X, Y) = 0$$

and $\widetilde{\omega}' |_{T\widetilde{L}} = 0$. $\square$

In view of Lemma 3.8 to prove that $\mathrm{CT}(N^*_{k,\varepsilon})$ is a tame Lagrangian, we have to show that π_2^{-1} is bi-Lipschitz away from the singularity in $\mathcal{C}$.

Recall that the metric on $\widehat{\mathcal{C}}$ is $\widehat{g} = \pi_1^* g_{\mathrm{FS}} + \pi_2^* g_{\mathrm{st}}$, where g_{FS} is the Fubinin-Study metric on $\mathbb{CP}^1$ and g_{st} is the standard metric on $\mathbb{C}^4$. Hence

$$\pi_2^{-1*}\widehat{g} = (\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} + g_{\mathrm{st}} \geq g_{\mathrm{st}}.$$

To establish the inverse inequality let us introduce convenient notation.

Definition 3.9. If α, β are 1-forms then $\alpha \odot \beta := \alpha \otimes \beta + \beta \otimes \alpha$ defines their symmetric product.

Then for instance $g_{\mathrm{FS}} = 1/2(dz \odot \overline{dz})/((1+|z|^2)^2)$, where $z = u/v$ and $[u : v]$ are homogeneous coordinates on $\mathbb{CP}^1$. One also has a Cauchy inequality:

$$|\alpha \odot \overline{\beta}| \leq \frac{1}{2}(\alpha \odot \overline{\alpha} + \beta \odot \overline{\beta}). \tag{3.4}$$

Lemma 3.10. *With the above notation*

$$(\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} \leq \frac{2}{|w|^2} g_{\mathrm{st}} \text{ for } w \in \mathcal{C} \setminus \{0\}. \tag{3.5}$$

Proof. If $([u : v], w_1, w_2, w_3, w_4) \in \widehat{\mathcal{C}}$ then $\begin{vmatrix} w_1 & w_2 \\ u & v \end{vmatrix} = \begin{vmatrix} w_3 & w_4 \\ u & v \end{vmatrix} = 0$. Therefore, $z = u/v = w_1/w_2$ if $w_2 \neq 0$ and w_3/w_4 if $w_4 \neq 0$. Assume for now that $w_2 \neq 0$ and $(\pi_1 \circ \pi_2^{-1})(w) = [w_1 : w_2]$. Then

$$\begin{aligned}
&(\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} \\
&= \frac{1}{2} \frac{d(w_1/w_2) \odot \overline{dw_1/w_2}}{(1+|w_1/w_2|^2)^2} \\
&= \frac{1}{2} \frac{(w_2 dw_1 - w_1 dw_2) \odot \overline{(w_2 dw_1 - w_1 dw_2)}}{(|w_1|^2 + |w_2|^2)^2} \\
&= \frac{1}{2} \frac{|w_2|^2 dw_1 \odot \overline{dw_1} + |w_1|^2 dw_2 \odot \overline{dw_2} - w_1 \overline{w_2} dw_2 \odot \overline{dw_1} - w_2 \overline{w_1} dw_1 \odot \overline{dw_2}}{(|w_1|^2 + |w_2|^2)^2}.
\end{aligned}$$

By the Cauchy inequality (3.4):

$$\begin{aligned}
|w_1 \overline{w_2} dw_2 \odot \overline{dw_1}| = |w_1 dw_2 \odot \overline{w_2 dw_1}| \leq \frac{1}{2}\big(&|w_1|^2 dw_2 \odot \overline{dw_2} \\
&+ |w_2|^2 dw_1 \odot \overline{dw_1}\big)
\end{aligned}$$

and the same estimate holds for the second cross-term. Hence

$$\begin{aligned}
(\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} &\leq \frac{|w_2|^2 dw_1 \odot \overline{dw_1} + |w_1|^2 dw_2 \odot \overline{dw_2}}{(|w_1|^2 + |w_2|^2)^2} \\
&\leq \frac{dw_1 \odot \overline{dw_1} + dw_2 \odot \overline{dw_2}}{(|w_1|^2 + |w_2|^2)},
\end{aligned}$$

or

$$(|w_1|^2 + |w_2|^2)(\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} \le dw_1 \odot \overline{dw_1} + dw_2 \odot \overline{dw_2}.$$

Analogously, if $w_4 \neq 0$

$$(|w_3|^2 + |w_4|^2)(\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} \le dw_3 \odot \overline{dw_3} + dw_4 \odot \overline{dw_4}.$$

Adding together the last two inequalities and taking into account that $g_{\mathrm{st}} = 1/2\,\Sigma_{i=1}^4\, dw_i \odot \overline{dw_i}$, one gets

$$|w|^2(\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} \le 2 g_{\mathrm{st}}.$$

Although we assumed $w_2 \neq 0$, $w_4 \neq 0$ in the process, the final inequality holds by continuity for any $w \in \mathcal{C}$ and for $w \neq 0$: $(\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} \le (2/|w|^2) g_{\mathrm{st}}$. □

Corollary 3.11. *π_2^{-1} is uniformly bi-Lipschitz on any $\mathcal{C}\backslash B_R(0)$ with $C = 1 + 2/R^2$.*

Proof. $g_{\mathrm{st}} \le g_{\mathrm{st}} + (\pi_1 \circ \pi_2^{-1})^* g_{\mathrm{FS}} = \pi_2^{-1*}\widehat{g} \le (1 + 2/R^2) g_{\mathrm{st}}$ by Lemma 3.10. □

Now we are ready to state the main result of this section.

Theorem 3.12. *Let $k : S^1 \to S^3 \hookrightarrow \mathbb{R}^4$ be a knot and $N_{k,\varepsilon}^*$ be its perturbed conormal bundle. If $\varepsilon < 1$ then $\mathrm{CT}(N_{k,\varepsilon}^*) \hookrightarrow \widehat{\mathcal{C}}$ is a tame Lagrangian in the resolved conifold. Moreover, the form $\widetilde{\omega}_\varepsilon := \pi_{2*}^{-1}\Phi_{\varepsilon *}\omega$ that makes it Lagrangian is exact on $\widehat{\mathcal{C}} \setminus 0(\widehat{\mathcal{C}})$.*

Proof. By Corollary 3.6 $F(N_{k,\varepsilon}^*)$ is a tame Lagrangian in $\mathcal{C}$ if $\varepsilon < (1/\|D\xi\|)$, where ξ extends $\dot{k}(t)$ from k to $\mathbb{R}^4$. Since $|\dot{k}(t)| = 1$ the extension can be carried out so that $\|D\xi\| = 1$. $F(N_{k,\varepsilon}^*) \subset \mathcal{C}\backslash B_{2\varepsilon}(0)$ and π_2^{-1} is a bi-Lipschitz biholomorphism on any $\mathcal{C}\backslash B_R(0)$ by Corollary 3.11. Since $\mathrm{CT}(N_{k,\varepsilon}^*) = \pi_2^{-1}(F(N_{k,\varepsilon}^*))$, the latter is a tame Lagrangian by Lemma 3.8. Since $\widetilde{\omega}_\varepsilon$ is a pushforward and therefore a pullback of an exact form on T^*S^3 it is exact. □

4 Geometry of the conifold

The ultimate goal of constructing Lagrangian or totally real submanifolds of the resolved conifold is to consider the moduli of holomorphic curves ending on them and to define open Gromov–Witten invariants. At the very least, one needs these moduli spaces to be compact. Since neither the resolved conifold itself nor the conormal bundles and their conifold transitions are compact certain uniform bounds are required to ensure compactness of the moduli. They are known as bounded geometry [3, 23] and generalize geometric properties of closed Riemannian manifolds and their submanifolds.

Definition 4.1. A Riemannian manifold (M, g) has bounded geometry (or is geometrically bounded) if its sectional curvature is bounded from above $\sec(X, Y) \leq K < \infty$ and its injectivity radius is bounded from below $i(M) \geq r_0 > 0$.

The main result of this section (Theorem 4.14) claims that the resolved conifold $\widehat{\mathcal{C}}$ is geometrically bounded. We use it to prove compactness of the moduli of holomorphic curves ending on $\mathrm{CT}(N^*_{k,\varepsilon})$ in the next section.

To obtain estimates on curvature it is convenient to use second fundamental forms. Recall the definition [22].

Definition 4.2. Let $L \hookrightarrow (M, g)$ be a smooth submanifold and $X, Y \in \Gamma(TL)$ be vector fields on it. Then

$$\mathrm{II}^{L/M}(X, Y) := \mathrm{pr}_{T^\perp L}(\nabla_X Y),$$

where ∇ is the Riemannian connection on M and $\mathrm{pr}_{T^\perp L}$ is the orthogonal projection to the normal bundle $T^\perp L$ of L in TM, is called the second fundamental form of L in M.

If $L \hookrightarrow Q \hookrightarrow (M, g)$, it follows from linear algebra that $\mathrm{II}^{L/M} = \mathrm{II}^{L/Q} + \mathrm{II}^{Q/M}$ and the terms on the right are orthogonal to each other. In particular,

$$|\,\mathrm{II}^{L/Q}\,| \leq |\,\mathrm{II}^{L/M}\,|. \tag{4.1}$$

When there is no confusion we drop the ambient manifold from notation and write simply II^L. The norm $\|\,\mathrm{II}^L\,\|$ is the smallest number C such that $|\,\mathrm{II}^L(X, Y)\,|_g \leq C|X|_g|Y|_g$. Our interest in the second fundamental forms is explained by the Gauss equation [22]. If $X, Y \in \Gamma(TL)$ and $\sec^L(X, Y)$, $\sec^M(X, Y)$ denote the sectional curvatures in L and M respectively:

$$\sec^L(X, Y) = \sec^M(X, Y) + g\big(\mathrm{II}^L(X, X), \mathrm{II}^L(Y, Y)\big) - |\,\mathrm{II}^L(X, Y)|_g^2. \tag{4.2}$$

Thus a bound on ambient curvature and second fundamental form yields one on the curvature of a submanifold. To consider the behavior of second fundamental forms under smooth maps we need the following notion.

Definition 4.3 [7]**.** Let $\Phi : (M, g) \to (\widetilde{M}, \widetilde{g})$ be a smooth map between two Riemannian manifolds. Let $\nabla, \widetilde{\nabla}$ be the respective Riemannian connections and $X, Y \in \Gamma(TM)$. The covariant Hessian $\nabla^2\Phi$ ("second fundamental form of a map" in [7]) is by definition

$$\nabla^2\Phi(X,Y) := \widetilde{\nabla}_{\Phi_* X}(\Phi_* Y) - \Phi_*(\nabla_X Y). \tag{4.3}$$

$\nabla^2\Phi \in \Gamma((T^*M)^{\otimes 2} \otimes \Phi^*(T\widetilde{M}))$ and it is straightforward to check that it is symmetric and tensorial. $\|\nabla^2\Phi\|$ is the smallest number C such that $|\nabla^2\Phi(X,Y)|_{\widetilde{g}} \leq C|X|_g|Y|_g$.

Lemma 4.4. *Let $X, Y \in \Gamma(TL)$ and*

$$\begin{tikzcd} (M,g) \arrow[r, "\Phi"] & (\widetilde{M},\widetilde{g}) \\ L \arrow[u, hook] \arrow[r, "\Phi"] & \tilde{L} \arrow[u, hook] \end{tikzcd}$$

Then

$$\mathrm{II}^{\tilde{L}}(\Phi_* X, \Phi_* Y) = \mathrm{pr}_{T^\perp L}(\Phi_* \, \mathrm{II}^L(X,Y) + \nabla^2\Phi(X,Y)) \tag{4.4}$$

Proof.

$$\begin{aligned} \mathrm{II}^{\tilde{L}}(\Phi_* X, \Phi_* Y) &= \mathrm{pr}_{T^\perp L}(\widetilde{\nabla}_{\Phi_* X}\Phi_* Y) \\ &= \mathrm{pr}_{T^\perp L}((\widetilde{\nabla}_{\Phi_* X}\Phi_* Y - \Phi_*(\nabla_X Y)) + \Phi_*(\nabla_X Y)) \\ &= \mathrm{pr}_{T^\perp L}(\nabla^2\Phi(X,Y) + \Phi_*(\mathrm{pr}_{T^\perp L}(\nabla_X Y) + \mathrm{pr}_{TL}(\nabla_X Y))) \\ &= \mathrm{pr}_{T^\perp L}(\nabla^2\Phi(X,Y) + \Phi_* \, \mathrm{II}^L(X,Y)) \\ &\quad + \mathrm{pr}_{T^\perp L}(\Phi_* \, \mathrm{pr}_{TL}(\nabla_X Y)) \\ &= \mathrm{pr}_{T^\perp L}(\Phi_* \, \mathrm{II}^L(X,Y) + \nabla^2\Phi(X,Y)) \end{aligned}$$

since $\Phi_*(TL) \subset T\tilde{L}$. □

Corollary 4.5. *If $\Phi : (M,g) \to (\widetilde{M},\widetilde{g})$ is a bi-Lipschitz map with a bounded covariant Hessian then images of submanifolds with bounded second fundamental forms have bounded second fundamental forms.*

Proof. Recall that by definition of a bi-Lipschitz map

$$C^{-1}g \leq \Phi^*\widetilde{g} \leq Cg$$

and, in particular, $|\Phi_* X|^2_{\widetilde{g}} = \widetilde{g}(\Phi_* X, \Phi_* X) = \Phi^*\widetilde{g}(X,X) \leq C|X|^2_g$ so $\|\Phi_*\| \leq C^{1/2}$. By Lemma 4.4

$$\begin{aligned} |\,\mathrm{II}^{\tilde{L}}(\Phi_* X, \Phi_* Y)|_g &\leq |\nabla^2\Phi(X,Y)|_{\widetilde{g}} + |\Phi_* \, \mathrm{II}^L(X,Y)\,|_{\widetilde{g}} \\ &\leq \|\nabla^2\Phi\||X|_g|Y|_g + \|\Phi_*\|\|\,\mathrm{II}^L\,\||X|_g|Y|_g \end{aligned}$$

$$\leq (\|\nabla^2\Phi\| + C^{1/2}\| \mathrm{II}^L \|)|X|_g|Y|_g$$
$$\leq C(\|\nabla^2\Phi\| + C^{1/2}\| \mathrm{II}^L \|)|\Phi_* X|_{\widetilde{g}}|\Phi_* Y|_{\widetilde{g}}.$$

Thus $\| \mathrm{II}^{\tilde{L}} \| \leq C(\|\nabla^2\Phi\| + C^{1/2}\| \mathrm{II}^L \|)$. □

Note that if Φ is a bi-Lipschitz embedding, then $L = M$, $\mathrm{II}^{L/M} = \mathrm{II}^{M/M} = 0$ and $\| \mathrm{II}^{\tilde{L}} \| \leq C\|\nabla^2\Phi\|$. In other words, second fundamental forms of embeddings are controlled by the bi-Lipschitz constants and covariant Hessians.

To obtain estimates on covariant Hessians, we need a coordinate representation. For convenience, we use the following notation:

$$\partial\Phi(X) = \frac{\partial\Phi}{\partial x_i}X^i, \quad \partial^2\Phi(X,Y) = \frac{\partial^2\Phi}{\partial x_i \partial x_j}X^iY^i,$$
$$\Gamma(X,Y) = \Gamma^k_{i_j}X^iY^j, \quad \widetilde{\Gamma}(\widetilde{X},\widetilde{Y}) = \widetilde{\Gamma}^k_{i_j}\widetilde{X}^i\widetilde{Y}^j,$$

where Γ^k_{ij}, $\widetilde{\Gamma}^k_{ij}$ are Christoffel symbols for ∇, $\widetilde{\nabla}$, respectively.

Lemma 4.6. *Let $\Phi : (M,g) \to (\widetilde{M},\widetilde{g})$. Then in local coordinates*

$$\nabla^2\Phi(X,Y) = \partial^2\Phi(X,Y) + \widetilde{\Gamma}(\partial\Phi(X),\partial\Phi(Y)) - \partial\Phi(\Gamma(X,Y)). \tag{4.5}$$

Proof. Since $\nabla^2\Phi$ is tensorial, we can ignore expressions containing derivatives of X, Y in the calculation:

$$\begin{aligned}
\nabla^2\Phi(X,Y) &= \widetilde{\nabla}_{X^i\Phi_*\partial_{x_i}}(Y^j\Phi_*\partial_{x_j}) - \Phi_*(\nabla_{X^i\partial_{x_i}}Y^j\partial_{x_j}) \\
&= X^iY^j\widetilde{\nabla}_{\Phi_*\partial_{x_i}}\Phi_*\partial_{x_j} - X^iY^j\Phi_*(\nabla_{\partial_{x_i}}\partial_{x_j}) \\
&= X^iY^j(\widetilde{\nabla}_{\Phi_*\partial_{x_i}}\left(\frac{\partial\Phi^\alpha}{\partial_{x_j}}\partial y_\alpha\right) - \Phi_*(\Gamma^k_{ij}\partial_{x_k})) \\
&= X^iY^j\left(\frac{\partial^2\Phi^\alpha}{\partial x_i\partial x_j}\partial_{y_\alpha} + \frac{\partial\Phi^\alpha}{\partial x_j}\widetilde{\nabla}_{\partial\Phi^\beta/\partial x_i}\partial_{y_\beta}\partial_{y_\alpha} - \frac{\partial\Phi^\alpha}{\partial x_k}\Gamma^k_{ij}\partial_{y_\alpha}\right) \\
&= X^iY^j(\left(\frac{\partial^2\Phi^\alpha}{\partial x_i\partial x_j}\partial_{y_\alpha} + \frac{\partial\Phi^\alpha}{\partial x_j}\frac{\partial\Phi^\beta}{\partial x_i}\widetilde{\Gamma}^\gamma_{\beta\alpha}\partial_{y_\gamma} - \frac{\partial\Phi^\alpha}{\partial x_k}\Gamma^k_{ij}\partial_{y_\alpha}\right) \\
&= \left(\frac{\partial^2\Phi^\alpha}{\partial x_i\partial x_j}X^iY^j + \widetilde{\Gamma}^\alpha_{\beta\gamma}\left(\frac{\partial\Phi^\beta}{\partial x_i}X^i\right)\left(\frac{\partial\Phi^\gamma}{\partial x_j}Y^j\right)\right. \\
&\quad \left. - \frac{\partial\Phi^\alpha}{\partial x_k}\left(\Gamma^k_{ij}X^iY^j\right)\right)\partial_{y_\alpha} \\
&= \partial^2\Phi(X,Y) + \widetilde{\Gamma}(\partial\Phi(X),\partial\Phi(Y)) - \partial\Phi(\Gamma(X,Y)).
\end{aligned}$$

□

Note that for a map between two flat spaces $\nabla^2\Phi$ turns into the usual Hessian $\partial^2\Phi$.

Now we turn to the geometry of the conifold

$$\mathcal{C} = \{w \in \mathbb{C}^4 \mid w_1w_4 = w_2w_3\}.$$

We start by finding convenient parametrizations for $\mathcal{C} \setminus \{0\}$.

Lemma 4.7. *Let $w \in \mathcal{C}\backslash\{0\}$. Then there exist $\xi, \eta \in \mathbb{C}$ and $z \in \overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ such that $w = (\xi, z\xi, \eta, z\eta)$ or $w = (z\xi, \xi, z\eta, \eta)$. Moreover,*

$$\begin{aligned} \mathbb{C} \times (\mathbb{C}^2\backslash\{0\}) &\xrightarrow{\Phi} \mathbb{C}^4 \\ (z, \xi, \eta) &\longmapsto (\xi, z\xi, \eta, z\eta) \end{aligned}$$

is an embedding.

Proof. Since $w \neq 0$ at least one of w_i is non-zero, let $w_1 \neq 0$. If $|w_2| > |w_1|$, then $w_2 \neq 0$ and we can set $\xi = w_2$, $z = (w_1/w_2)$ with $|z| < 1$. Otherwise, set $\xi = w_1$, $z = (w_2/w_1)$ with $|z| \leq 1$. Since the two cases are analogous let us consider just the latter one. In this case, $w_3 = 0$ implies $w_4 = 0$ since $w_1w_4 = w_2w_3$ and $w_1 \neq 0$. Therefore, $w = (\xi, z\xi, \eta, z\eta)$ with ξ, z as above and $\eta = 0$. If $w_3 \neq 0$, then $w_4/w_3 = w_2/w_1 = z$ and $w = (\xi, z\xi, \eta, z\eta)$ with $\eta = w_3$. The second possibility of $w = (z\xi, \xi, z\eta, \eta)$ arises when $|w_2| > |w_1|$.

The Jacobian of Φ is $\begin{pmatrix} 0 & \xi & 0 & \eta \\ 1 & z & 0 & 0 \\ 0 & 0 & 1 & z \end{pmatrix}$ and it obviously has full rank unless $\xi = \eta = 0$ so Φ is an immersion. Also, if $(\xi, z\xi, \eta, z\eta) = (\xi', z'\xi', \eta', z'\eta')$ then $\xi = \xi'$, $\eta = \eta'$ and one of them, say, $\xi = \xi' \neq 0$. But then $z' = z'\xi'/\xi' = z\xi/\xi = z$ and Φ is an embedding. $\square$

Lemma 4.7 implies that $\mathcal{C}\backslash\{0\}$ can be covered by two charts, each of them diffeomorphic (in fact, biholomorphic) to $2\mathbb{D} \times (\mathbb{C}^2\backslash\{0\})$. Since the corresponding parametrizations are the same up to permutation of coordinates, we may consider just one of them. First of all, we want to describe the induced metric on $\mathcal{C}$ in terms of z, ξ, η.

Lemma 4.8. *Let*

$$\begin{aligned} 2\mathbb{D} \times (\mathbb{C}^2\backslash\{0\}) &\xrightarrow{\Phi} \mathbb{C}^4 \\ (z, \xi, \eta) &\longmapsto (\xi, z\xi, \eta, z\eta) \end{aligned}$$

and

$$g := \frac{1}{2}\left((|\xi|^2 + |\eta|^2)dz \odot \overline{dz} + d\xi \odot \overline{d\xi} + dy \odot \overline{dy}\right)$$

be a metric on $2\mathbb{D} \times (\mathbb{C}^2\backslash\{0\})$. Then Φ is uniformly bi-Lipschitz.

Proof.

$$\begin{aligned}\Phi^* g_{\mathrm{st}} &= \frac{1}{2}\Big(d\xi \odot \overline{d\xi} + d(z\xi) \odot \overline{d(z\xi)} + d\eta \odot \overline{d\eta} + d(z\eta) \odot \overline{d(z\eta)}\Big)\\ &= \frac{1}{2}\left(d\xi \odot \overline{d\xi} + |\xi|^2 dz \odot \overline{dz} + |z|^2 d\xi \odot \overline{d\xi} + |\eta|^2 dz \odot \overline{dz} + |z|^2 d\eta \odot \overline{d\eta}\right.\\ &\quad \left. + \xi\overline{z} dz \odot \overline{d\xi} + \overline{\xi} z d\xi \odot \overline{dz} + \eta \overline{z} dz \odot \overline{d\eta} + \overline{\eta} z d\eta \odot \overline{dz}\right).\end{aligned}$$

Each of the cross-terms in the last line can be estimated using the Cauchy inequality (3.4), e.g.,

$$\begin{aligned}|\xi\overline{z} dz \odot \overline{d\xi}| = |\xi dz \odot \overline{z d\xi}| &\leq \frac{1}{2}(\xi dz \odot \overline{\xi dz} + z d\xi \odot \overline{z d\xi})\\ &= \frac{1}{2}(|\xi|^2 dz \odot \overline{dz} + |z|^2 d\xi \odot \overline{d\xi}).\end{aligned}$$

Therefore,

$$\begin{aligned}\Phi^* g_{\mathrm{st}} &\leq \frac{1}{2}((|\xi|^2 + |\eta|^2) dz \odot \overline{dz} + (1 + |z|^2) d\xi \odot \overline{d\xi} + (1 + |z|^2) d\eta \odot \overline{d\eta}\\ &\quad + |\xi|^2 dz \odot \overline{dz} + |z|^2 d\xi \odot \overline{d\xi} + |\eta|^2 dz \odot \overline{dz} + |z|^2 d\eta \odot \overline{d\eta})\\ &= \frac{1}{2}(2(|\xi|^2 + |\eta|^2) dz \odot \overline{dz} + (1 + 2|z|^2) d\xi \odot \overline{d\xi} + (1 + 2|z|^2) d\eta \odot \overline{d\eta})\\ &\leq 9g \quad \text{since } |z| < 2.\end{aligned}$$

To prove the inverse inequality let us go back to Lemma 3.10. There we proved that

$$\frac{1}{2}\frac{dz \odot \overline{dz}}{(1 + |z|^2)^2} \leq \frac{2}{|w|^2}\frac{1}{2}\sum_{i=1}^{4} dw_i \odot \overline{dw_i},$$

where $z = (w_2/w_1) = (w_4/w_3)$ when $w_1, w_3 \neq 0$. Since $w = (\xi, z\xi, \eta, z\eta)$ in terms of ξ, η, this gives

$$\begin{aligned}\frac{1}{2}\frac{dz \odot \overline{dz}}{(1 + |z|^2)^2} &\leq \frac{2}{|\xi|^2 + |z\xi|^2 + |\eta|^2 + |z\eta|^2}\Phi^* g_{\mathrm{st}},\\ \text{and} \quad \Phi^* g_{\mathrm{st}} &\geq \frac{1}{2}\ \frac{(|\xi|^2 + |\eta|^2)}{2}\ \frac{dz \odot \overline{dz}}{(1 + |z|^2)}.\end{aligned}$$

Also obviously

$$\Phi^* g_{\mathrm{st}} \geq \frac{1}{2}(d\xi \odot \overline{d\xi} + d\eta \odot \overline{d\eta})$$

Adding together the last two inequalities we obtain,

$$2\Phi^* g_{\mathrm{st}} \geq \frac{1}{2}\left(\frac{1}{2}(|\xi|^2+|\eta|^2)(1+|z|^2)^{-1}dz\odot\overline{dz}+d\xi\odot\overline{d\xi}+d\eta\odot\overline{d\eta}\right)$$

$$\Phi^* g_{\mathrm{st}} \geq \frac{1}{2\cdot 2\cdot 5}\cdot\frac{1}{2}(|\xi|^2+|\eta|^2)dz\odot\overline{dz}+d\xi\odot\overline{d\xi}+d\eta\odot\overline{d\eta})$$

$$= \frac{1}{20}g$$

Thus $20^{-1}g \leq \Phi^* g_{gs} \leq 20g$ and Φ is bi-Lipschitz. $\square$

The geometric meaning of this lemma is best described by considering a three-dimensional analog. Let $z^2 = x^2 + y^2$ be the standard cone in $\mathbb{R}^3$. If r, θ are the polar coordinates in the xy plane, then the induced metric on the cone is $2dr^2 + r^2 d\theta^2$. The metric g on the conifold has similar structure with θ replaced by z and r replaced by ξ, η. Sections with $r = \mathrm{const}$ describe circles of growing radii as $r \to \infty$ (Figure 5). In a similar way sections $|\xi|^2 + |\eta|^2 = \mathrm{const}$ describe disks of growing diameter along the fiber parameters ξ, η. Now recall that $2\mathbb{D} \times (\mathbb{C}\backslash\{0\})$ only parametrizes half of the conifold, the other half is also $2\mathbb{D} \times (\mathbb{C}\backslash\{0\})$ with disks overlapping over an annulus containing the unit circle. Put together the disks form $\mathbb{CP}^1$-like sections of increasing area and diameter as $\xi, \eta \to \infty$.

On the narrow end, just as circles collapse into the singular point of the usual cone at the origin, in the conifold $\mathbb{CP}^1$ sections collapse into the conifold singularity when $\xi = \eta = 0$. In the resolved conifold, this collapse is prevented by replacing the singular point with a copy of $\mathbb{CP}^1$ and adding the Fubini-Study term to the metric.

Corollary 4.9. *For any $\delta > 0$ the second fundamental form of $\mathcal{C}$ in $\mathbb{C}^4$ is uniformly bounded on $\mathcal{C} \setminus B_\delta(0)$.*

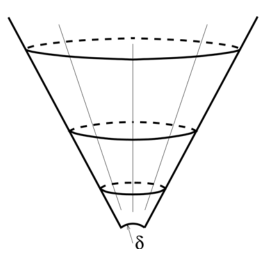

Figure 5: Horizontal circles in the cone correspond to $\mathbb{CP}^1$-like sections of the conifold.

Proof. If $w \in \mathcal{C}\backslash\{0\}$, then $w = \Phi(z, \xi, \eta)$ by Lemma 4.7. In fact, a whole neighborhood of (z, ξ, η) is mapped into a neighborhood of w in $\mathcal{C}$. Therefore, as was commented after Corollary 4.5 $\| \mathrm{II}^{\mathcal{C}/\mathbb{C}^4}(w)\| \leq C\|\nabla^2\Phi\|$, where $\nabla^2\Phi$ is computed in g_{st} on $\mathbb{C}^4$ and any metric g_1 on $2\mathbb{D} \times (\mathbb{C}^2\backslash\{0\})$ such that $C^{-1}g_1 \leq \Phi^* g_{\mathrm{st}}$. Let $g_1 = 1/2(dz \odot \overline{dz} + d\xi \odot \overline{d\xi} + d\eta \odot \overline{d\eta})$. If $|w|^2 = (1+|z|^2)(|\xi|^2 + |\eta|^2) \geq \delta^2$, then $(|\xi|^2 + |\eta|^2) \geq (\delta^2/1+|z|^2) \geq (\delta^2/5)$ and by Lemma 4.8

$$g_1 \leq \frac{5}{\delta^2} g \leq \frac{5}{\delta^2} \cdot 20\, \Phi^* g_{\mathrm{st}} = \frac{100}{\delta^2} \Phi^* g_{\mathrm{st}}.$$

Thus one can take $C = 100/\delta^2$. Both g_1, g_{st} are flat. Therefore by Lemma 4.6 $\nabla^2\Phi = \partial^2\Phi$, the usual Hessian. But the usual Hessian of Φ is in fact constant as one can see by inspection (all entries are polynomials in z, ξ, η of at most second degree). Therefore, $\nabla^2\Phi$ and $\mathrm{II}^{\mathcal{C}/\mathbb{C}^4}$ are uniformly bounded on $\mathcal{C}\backslash B_\delta(0)$. $\square$

Since $C = 100/\delta^2 \underset{\delta\to\infty}{\longrightarrow} 0$ the second fundamental form is not just bounded, it is asymptotically 0. This means that the conifold $\mathcal{C}$ is asymptotically locally flat just like the usual cone.

We now want to extend this conclusion to the resolved conifold $\widehat{\mathcal{C}}$. Recall that $\widehat{\mathcal{C}}\backslash 0(\widehat{\mathcal{C}})$ can be obtained from $\mathcal{C}\backslash\{0\}$ by applying the biholomorphism π_2^{-1}. To be able to use second fundamental forms in $\mathbb{C}^4$ and $\mathbb{CP}^1 \times \mathbb{C}^4$, we need to extend π_2^{-1} to $\mathbb{C}^4$. Unfortunately this is impossible, but we can partially extend π_2^{-1} into a neighborhood in $\mathbb{C}^4$ of every point $w \in \mathcal{C}\backslash\{0\}$. Namely,

$$\overline{\pi_2^{-1}}(w) := \begin{cases} ([w_1 : w_2], w) & \text{if} \quad |w_1|^2 + |w_2|^2 \neq 0, \\ ([w_3 : w_4], w) & \text{if} \quad |w_3|^2 + |w_4|^2 \neq 0. \end{cases}$$

If $|w|^2 > \delta^2$ then either $|w_1|^2 + |w_2|^2 > (\delta^2/2)$ or $|w_3|^2 + |w_4|^2 > (\delta^2/2)$ so one of the conditions is always satisfied and at least one of the extensions is defined around any point. Boundedness of $\mathrm{II}^{\widehat{\mathcal{C}}/(\mathbb{CP}^1\times\mathbb{C}^4)}$ follows from that of $\mathrm{II}^{\mathcal{C}/\mathbb{C}^4}$ by Corollary 4.5 and the next Lemma.

Lemma 4.10. *For any $\delta > 0$, the extension $\overline{\pi_2^{-1}}$ is uniformly bi-Lipschitz and has bounded covariant Hessian along $\mathcal{C}\backslash B_\delta(0)$.*

Proof. Due to symmetry it suffices to consider the case $|w_1|^2 + |w_2|^2 > (\delta^2/2)$ and $\overline{\pi_2^{-1}}(w) = ([w_1 : w_2], w) \in \mathbb{CP}^1 \times \mathbb{C}^4$. Introduce the standard coordinate charts on $\mathbb{CP}^1$ with coordinate maps $\varphi : [u : v] \mapsto (u/v) \in 2\mathbb{D}$ and $\mapsto (v/u) \in 2\mathbb{D}$. Again due to symmetry it suffices to consider just one. In coordinates we have

$$(\varphi, \mathrm{id}) \circ \overline{\pi_2^{-1}} : \quad \begin{array}{l} \mathbb{C}^4 \to 2\mathbb{D} \times \mathbb{C}^4 \\ w \mapsto \left(\frac{w_1}{w_2}, w\right) \end{array}.$$

Just as in Lemma 3.10 one gets

$$g_{\mathrm{st}} \leq (\varphi \circ \overline{\pi_2^{-1}})^* \widehat{g} \leq \left(1 + \frac{2}{|w_1|^2 + |w_2|^2}\right) g_{\mathrm{st}} \leq \left(1 + \frac{4}{\delta^2}\right) g_{\mathrm{st}}.$$

Now we take a look at the covariant Hessian. By Lemma 4.6

$$\nabla^2 \Phi(X, Y) = \partial^2 \Phi(X, Y) + \widetilde{\Gamma}(\partial\Phi(X), \partial\Phi(Y)) - \partial\Phi(\Gamma(X, Y)).$$

In our case $\Phi = (\varphi, \mathrm{id}) \circ \overline{\pi_2^{-1}}$ and $\Gamma = 0$ since $\mathbb{C}^4$ is flat. $\widetilde{\Gamma}$ only depends on $z \in 2\mathbb{D}$ since $\widehat{g} = \pi_1^* g_{\mathrm{FS}} + \pi_2^* g_{\mathrm{st}}$ and the second term is flat. Since $2\overline{\mathbb{D}}$ is compact and $\widetilde{\Gamma}$ extends smoothly to $\mathbb{C}$, it is uniformly bounded on $2\mathbb{D}$. Finally, since Φ is holomorphic we may consider just holomorphic derivatives.

Φ is linear in $\mathbb{C}^4$ variables, hence first derivatives are constant and second ones are 0. So the only part that matters is $\Phi_1(w) := \varphi \circ \overline{\pi_2^{-1}}(w) = (w_1/w_2)$. By a direct computation:

$$\partial\Phi_1 = \left(\frac{1}{w_2}, -\frac{w_1}{w_2^2}\right) = \left(\frac{1}{w_2}, -\frac{z}{w_2}\right),$$

where $z = \frac{w_1}{w_2}$ and

$$\partial^2 \Phi_1 = \begin{pmatrix} 0 & -\dfrac{1}{w_2^2} \\ -\dfrac{1}{w_2^2} & \dfrac{2w_1}{w_2^3} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{1}{w_2^2} \\ -\dfrac{1}{w_2^2} & \dfrac{2z}{w_2^2} \end{pmatrix}.$$

Now recall that by our choice of coordinates $|z| < 2$ so $|w_2| > (|w_1|/2)$, $5|w_2|^2 > |w_1|^2 + |w_2|^2 > (\delta^2/2)$ and hence $|w_2| > \delta/\sqrt{10}$. Therefore, both $\partial\Phi_1$, $\partial^2\Phi_1$ and therefore $\partial\Phi$, $\partial^2\Phi$ are uniformly bounded. Together with the previous remarks this implies the same for $\nabla^2\Phi$. $\square$

Corollary 4.11. *The resolved conifold $\widehat{\mathcal{C}}$ has bounded sectional curvature.*

Proof. Note that $\pi_2^{-1}(\mathcal{C} \bigcap \overline{B_\delta}(0))$ is a compact subset in $\widehat{\mathcal{C}}$. Its complement in $\widehat{\mathcal{C}}$ is the image under π_2^{-1} of $\mathcal{C} \backslash B_\delta(0)$. By Corollary 4.9 $\mathrm{II}^{\mathcal{C}/\mathbb{C}^4}$ is uniformly bounded on $\mathcal{C} \backslash B_\delta(0)$. Every point in $\mathcal{C} \backslash B_\delta(0)$ has a neighborhood in $\mathbb{C}^4$ such that π_2^{-1} extends to it and by Lemma 4.10 these extensions are uniformly bi-Lipschitz with bounded covariant Hessian. Therefore, by Corollary 4.5,

$\mathrm{II}^{\widehat{\mathcal{C}}/(\mathbb{CP}^1\times\mathbb{C}^4)}$ is uniformly bounded on $\pi_2^{-1}(\mathcal{C}\setminus B_\delta(0))$. Since its complement has compact closure $\mathrm{II}^{\widehat{\mathcal{C}}/(\mathbb{CP}^1\times\mathbb{C}^4)}$ is bounded on the whole $\widehat{\mathcal{C}}$. Since $\mathbb{CP}^1\times\mathbb{C}^4$ is a product of a closed manifold and a flat manifold with the product metric $\sec^{\mathbb{CP}^1\times\mathbb{C}^4}$ is bounded. Finally, by the Gauss equation (4.2):

$$\sec^{\widehat{\mathcal{C}}}(X,Y) = \sec^{\mathbb{CP}^1\times\mathbb{C}^4}(X,Y) + \widehat{g}(\mathrm{II}^{\widehat{\mathcal{C}}}(X,X),\mathrm{II}^{\widehat{\mathcal{C}}}(Y,Y)) - |\,\mathrm{II}^{\widehat{\mathcal{C}}}(X,Y)|^2_{\widehat{g}}$$

and all the terms on the right are bounded. Therefore so is $\sec^{\widehat{\mathcal{C}}}$. ☐

Now we want to establish that the injectivity radius $i(\widehat{\mathcal{C}})$ is strictly positive. It is convenient to use the following criterion.

Proposition 4.12 (Proposition 3.19 of [1])**.** *Let (M,g) be a Riemannian manifold with complete metric and bounded sectional curvature. Then three conditions are equivalent:*

(i) *$i(M)>0$ ($i(M)$ is the injectivity radius).*
(ii) *there exist numbers δ, $C>0$ such that every loop γ in M of length $\ell(\gamma)\le\delta$ bounds a disc D in M of $\mathrm{diam}(D)\le C\cdot\ell(\gamma)$.*
(iii) *every point in M has a neighborhood uniformly bi-Lipschitz to the flat unit ball.*

In particular, it follows directly from (iii) that.

Corollary 4.13. *Let $\Phi:(M,g)\to(\widetilde{M},\widetilde{g})$ be a bi-Lipschitz diffeomorphism between complete Riemannian manifolds with bounded sectional curvatures. Then $i(M)>0$ if and only if $i(\widetilde{M})>0$.*

$\widehat{\mathcal{C}}$ is obviously complete, since it is properly embedded in $\mathbb{CP}^1\times\mathbb{C}^4$ and it has bounded sectional curvature by Corollary 4.11.

Theorem 4.14. *The resolved conifold $\widehat{\mathcal{C}}$ has bounded geometry.*

Proof. Due to Corollary 4.11 it only remains to prove that $i(\widehat{\mathcal{C}})>0$. From Lemma 4.7 and the definition of $\widehat{\mathcal{C}}$ we have $\widehat{\mathcal{C}}$ covered by the parametrizations:

$$\begin{aligned} 2\mathbb{D}\times\mathbb{C}^2 &\xrightarrow{\Phi} \widehat{\mathcal{C}} \hookrightarrow \mathbb{CP}^1\times\mathbb{C}^4\\ (z,\xi,\eta) &\longmapsto ([1:z],\xi,z\xi,\eta,z\eta); \end{aligned}$$

$$\begin{aligned} 2\mathbb{D}\times\mathbb{C}^2 &\xrightarrow{\Phi} \widehat{\mathcal{C}} \hookrightarrow \mathbb{CP}^1\times\mathbb{C}^4\\ (z,\xi,\eta) &\longmapsto ([z:1],z\xi,\xi,z\eta,\eta); \end{aligned}$$

By Lemma 4.8 $\Phi^*\widehat{g}$ is bi-Lipschitz to

$$\Phi^*\pi_1^* g_{\mathrm{FS}} + g = \frac{1}{2}\frac{dz \odot \overline{dz}}{(1+|z|^2)} + \frac{1}{2}\left((|\xi|^2+|\eta|^2)dz\odot\overline{dz} + d\xi\odot\overline{d\xi} + d\eta\odot\overline{d\eta}\right)$$

$$\text{and also to} \quad \widetilde{g} := \frac{1}{2}\left((1+|\xi|^2+|\eta|^2)dz\odot\overline{dz} + d\xi\odot\overline{d\xi} + d\eta\odot\overline{d\eta}\right).$$

Let $(z_0,\xi_0,\eta_0) \in 2\mathbb{D}\times\mathbb{C}^2$. We may assume $|(\xi_0,\eta_0)| := \sqrt{|\xi_0|^2+|\eta_0|^2} \geq 3$, since the complement has compact closure. Consider the map

$$\begin{aligned}\mathbb{C}^3 \supset B_1(0) &\xrightarrow{f} \mathbb{C}\times\mathbb{C}^2\\ (\lambda,\alpha,\beta) &\longmapsto \left(z_0 + \frac{\lambda}{\sqrt{1+|\xi_0|^2+|\eta_0|^2}}, \xi_0+\alpha, \eta_0+\beta\right).\end{aligned}$$

We claim that f is uniformly bi-Lipschitz. Indeed,

$$f^*\widetilde{g} = \frac{1}{2}\left(\frac{1+|\xi|^2+|\eta|^2}{1+|\xi_0|^2+|\eta_0|^2} d\lambda\odot\overline{d\lambda} + d\alpha\odot\overline{d\alpha} + d\beta\odot\overline{d\beta}\right).$$

To prove that $f^*\widetilde{g}$ is equivalent to g_{st}, we need uniform estimates from above and below. Note that by definition of f, $|(\xi,\eta)-(\xi_0,\eta_0)| \leq 1$. Therefore,

$$\begin{aligned}\frac{1+|\xi|^2+|\eta|^2}{1+|\xi_0|^2+|\eta_0|^2} = \frac{1+|(\xi,\eta)|^2}{1+|(\xi_0,\eta_0)|^2} &\leq \frac{1+(|\xi_0,\eta_0)|+|(\xi,\eta)-(\xi_0,\eta_0)|)^2}{1+|(\xi_0,\eta_0)|^2}\\ &\leq \frac{(1+|(\xi_0,\eta_0)|^2)+2|(\xi_0,\eta_0)|+1}{1+|(\xi_0,\eta_0)|^2} \leq 4.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1+|\xi|^2+|\eta|^2}{1+|\xi_0,\eta_0|^2} &\geq \frac{(1+|(\xi_0,\eta_0)|^2)-2|(\xi_0,\eta_0)|)+1}{1+|(\xi_0,\eta_0)|^2}\\ &= 1-\frac{2|(\xi_0,\eta_0)|-1}{1+|(\xi_0,\eta_0)|^2} \geq \frac{1}{2} \quad \text{when } |(\xi_0,\eta_0)| \geq 3.\end{aligned}$$

Therefore regardless of the chosen point

$$C^{-1}g_{\mathrm{st}} \leq f^*\widetilde{g} \leq Cg_{\mathrm{st}}$$

with $C=4$. By Proposition 4.12, we now have $i(\widehat{\mathcal{C}})>0$ and $\widehat{\mathcal{C}}$ has bounded geometry. $\square$

This proof is not very illuminating as to why $i(\widehat{\mathcal{C}})>0$. It is useful to have in mind the analogy between the conifold and the usual cone in $\mathbb{R}^3$ described after Lemma 4.8. Let $\mathcal{C}_\delta = \mathcal{C}\backslash B_\delta(0)$ and $\widehat{\mathcal{C}}_\delta = \widehat{\mathcal{C}}\backslash\pi_2^{-1}(B_\delta(0))$. Asymptotically geometries of $\mathcal{C}_\delta$ and $\widehat{\mathcal{C}}_\delta$ are the same as one can see from expressions for g and $\widetilde{g}$ in the theorem because the Fubini-Study term becomes negligible as $|\xi|^2 + |\eta|^2 \to \infty$. More precisely, $\pi_2^{-1} : \mathcal{C}_\delta \to \widehat{\mathcal{C}_\delta}$ is bi-Lipschitz with the constant $C(\delta) \underset{\delta\to\infty}{\longrightarrow} 1$. Therefore "horizontal" sections of $\widehat{\mathcal{C}}$ are copies of $\mathbb{CP}^1$ with Kähler volume $\underset{\delta\to\infty}{\longrightarrow} \infty$ just as horizontal sections of the cone are circles of increasing diameters (see Figure 5). This means that cut points for points in $\widehat{\mathcal{C}_\delta}$ that are on the "other side" of the conifold are further and further away from them as $\delta \to \infty$. Thus not only is $i(\widehat{\mathcal{C}_\delta})$ bounded from below but in fact $i(\widehat{\mathcal{C}_\delta}) \underset{\delta\to\infty}{\longrightarrow} \infty$. Similarly as was noted after Corollary 4.9, $\sec(\mathcal{C}_\delta) \underset{\delta\to\infty}{\longrightarrow} 0$ and $\sec(\widehat{\mathcal{C}_\delta})$ behaves the same way by the Gauss equation since π_2^{-1} has bounded covariant Hessian. Summarizing, not only does $\widehat{\mathcal{C}}$ have bounded geometry, but it is in fact asymptotically globally flat.

5 Compactness of the moduli

In this section, we prove the main result of this paper on moduli compactness of open pseudoholomorphic curves ending on the conifold transited perturbed conormal bundles. After briefly recalling the notions of open stable maps and their Gromov convergence we state the Sikorav compactness theorem (Theorem 5.4) and proceed to verify its assumtions in our case.

Let (M, J, g) be an almost Kähler manifold, $L \hookrightarrow M$ be its totally real submanifold and Σ be a Riemann surface with boundary $\partial\Sigma$ and a complex structure j. A smooth open pseudoholomorphic curve in M ending on L (or with the boundary on L) is a map $f : (\Sigma, \partial\Sigma) \to (M, L)$ such that $f_*j = Jf_*$. If instead of smooth Riemann surfaces one considers complex one-dimensional varieties with at most nodal singularities (i.e., stable curves with boundary) then f is called an open stable map [16]. As is common in the literature, we often call stable maps pseudoholomorphic curves as well. Let $\partial\Sigma_j$ be the boundary components of Σ, $\partial\Sigma = \cup_i \partial\Sigma_i$ and let $\alpha \in H_2(M, L)$ and $\beta_i \in H_1(L)$ be integral homology classes.

Definition 5.1. The moduli space of open genus g curves with h boundary components is

$$\begin{aligned}\overline{\mathcal{M}}_{g,h}(M, L \mid \alpha, \beta_1, \dots, \beta_n) := \{f \text{ open stable map} \mid \text{genus}\,(\Sigma) = g, \#\{\partial\Sigma_i\} \\ = h, f_*[\Sigma] = \alpha, f_*[\partial\Sigma_i] = \beta_i\}.\end{aligned} \tag{5.1}$$

The appropriate topology on the moduli can be defined using the Gromov convergence. Since the domain of the limit curve may differ from that of the prelimit ones, one needs some kind of smooth resolution of nodes to pull back the maps to the same domain. For open curves, the definition of a resolution is worked out in [16], where the interested reader is directed.

Definition 5.2 (Gromov convergence)**.** One says that a sequence of stable maps (Σ_n, f_n) Gromov converges to a map (Σ, f) if there is a sequence of resolutions $\kappa_n : \Sigma_n \to \Sigma$ such that for any neighborhood V of the union of all nodes in Σ:

1) $f_n \circ \kappa_n^{-1} \underset{n\to\infty}{\longrightarrow} f$ in $C^\infty(\Sigma\backslash V)$;
2) $(\kappa_n^{-1})^* j_n \underset{n\to\infty}{\longrightarrow} j$ in $C^\infty(\Sigma\backslash V)$, where j_n, j are complex structures on Σ_n, Σ, respectively;
3) $\mathrm{Area}(f_n(\Sigma_n)) \underset{n\to\infty}{\longrightarrow} \mathrm{Area}(f(\Sigma))$.

These moduli spaces have been used to define open Gromov–Witten invariants in [13, 16]. However, for such definitions to work $\overline{\mathcal{M}}_{g,h}$ at least has to be compact. Proving compactness in the case $M = \widehat{\mathcal{C}}$ and $L = \mathrm{CT}(N^*_{k,\varepsilon})$ will be our goal in this section.

Compactness theorems for curves with boundary were considered by several authors [23, 27]. For our purposes, the most suitable result is due to Sikorav [23]. We restate it here in a form consistent with the terminology used throughout the paper.

Definition 5.3. $L \hookrightarrow (M, J, g)$ satisfies the 2-point estimate if there exist constants $C, \rho > 0$ such that for any two points $x, y \in L$ with $\mathrm{dist}^M(x, y) < \rho$ one has $\mathrm{dist}^L(x, y) \le C\,\mathrm{dist}^M(x, y)$.

Theorem 5.4 (Proposition 5.1.2 and Theorem 5.2.3 of [23])**.** *Assume that (M, J, g) has bounded geometry and $L \hookrightarrow M$ is a uniformly tame Lagrangian submanifold that satisfies the 2-point estimate. Let $f_n : (\Sigma_n, \partial\Sigma_n) \to (M, L)$ be a sequence of open curves with uniformly bounded areas such that $f(\Sigma_n) \cap K \neq \emptyset$ for some compact subset $K \subset M$. Then there exists a subsequence f_{n_k} that Gromov-converges to an open curve.*

We will now verify the assumptions of the Sikorav theorem in the order they are listed. Note that we already proved in Section 4 that $\widehat{\mathcal{C}}$ is geometrically bounded and in Section 3 that $L = \mathrm{CT}(N^*_{k,\varepsilon})$ is a tame Lagrangian in it.

The 2-point estimate is preserved under bi-Lipschitz maps, i.e., if $L \hookrightarrow M$ satisfies it and $f : M \to \widetilde{M}$ is bi-Lipschitz, then so does $f(L) \hookrightarrow \widetilde{M}$. Moreover, locally any submanifold satisfies it.

Lemma 5.5. *Let $L \hookrightarrow (M, g)$. Then every point of L has a neighborhood in L such that $\mathrm{dist}^L(q, q') \le C\,\mathrm{dist}^M(q, q')$ for q, q' in this neighborhood for some constant C.*

Proof. Let $q_0 \in L$. Choose a neighborhood U_{q_0} of q_0 in M with coordinate function φ such that $\varphi(q_0) = 0$, $\varphi(L \cap U_{q_0}) = \{x \in \mathbb{R}^m \mid x_{\ell+1} = \cdots = x_m = 0\}$, where $m := \dim M$, $\ell := \dim L$. For any ball $B_R \subset \mathbb{R}^m$, the map $\varphi : \varphi^{-1}(B_R) \to (\mathbb{R}^m, g_{\mathrm{st}})$ is bi-Lipschitz by a compactness argument. Fix R and let C be the corresponding bi-Lipschitz constant. Then

$$\begin{aligned}\mathrm{dist}^L(q,q') &\leq C^{1/2}\,\mathrm{dist}_{\mathbb{R}^l}(\varphi(q),\varphi(q')) = C^{1/2}\,\mathrm{dist}_{\mathbb{R}^m}(\varphi(q),\varphi(q'))\\ &\leq C^{1/2}\cdot C^{1/2}\,\mathrm{dist}^M(q,q') = C\,\mathrm{dist}^M(q,q')\end{aligned}$$

□

This implies of course that any compact submanifold satisfies the 2-point estimate. An example of a submanifold that does not satisfy it is given by the graph of $\sin(\pi e^x)$ in $\mathbb{R}^2$. This graph is the graph of sine that gets more and more compressed as $x \to \infty$ (see Figure 6). The distance in $\mathbb{R}^2$ between two consecutive zeros is $\ln(n+1) - \ln(n) \to 0$. However, the distance between them along the graph is ≥ 2, since one has to go along the arc of sine to get from one to the next.

Recall that $\mathrm{CT}(N^*_{k,\varepsilon}) = \pi_2^{-1} \circ \Phi_\varepsilon(F_\varepsilon(N^*_k))$, where $\Phi_\varepsilon(x,p) = (x, p + \varepsilon\xi(x))$ and $F_\varepsilon(x,p) = (x\sqrt{|p|^2+\varepsilon^2}, p)$. Since both Φ_ε and π_2^{-1} are uniformly bi-Lipschitz on the relevant sets it suffices to prove the 2-point estimate for $F_\varepsilon(N^*_k) = \{(k(t)\sqrt{|p|^2+\varepsilon^2}, p) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid p \cdot k(t) = p \cdot \dot{k}(t) = 0, t \in S^1\}$.

Lemma 5.6. *$F_\varepsilon(N^*_k) \hookrightarrow \mathbb{R}^4 \times \mathbb{R}^4$ and therefore $\mathrm{CT}(N^*_{k,\varepsilon}) \hookrightarrow \mathbb{CP}^1 \times \mathbb{C}^4$ satisfy the 2-point estimate.*

Proof. Despite the length this proof reduces to multiple applications of the triangle inequality. For convenience, it is split into three steps corresponding to pairs of points at a fixed distance from the zero-section, at a distance greater than 1 from it and finally in the general position.

Step 1. Let $S_r := \{(x,p) \in F_\varepsilon(N^*_k) \mid |p| = r\}$. In this step, we will prove that they satisfy the 2-point estimate in $\mathbb{R}^4 \times \mathbb{R}^4$ with constants C, ρ independent of $r \geq 1$.

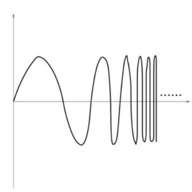

Figure 6: Submanifold in $\mathbb{R}^2$ that does not satisfy the 2-point estimate.

Since S_1 is compact in $\mathbb{R}^4 \times \mathbb{R}^4$, by Lemma 5.5 it satisfies the estimate with some $C_1, \rho_1 > 0$. We claim that the same ρ_1 and $C_1\sqrt{1+\varepsilon^2}$ work for all $r \geq 1$. Indeed, let $q = (x, p)$, $q' = (x', p') \in S_r$. Then $q_1 := ((x/\sqrt{r^2+\varepsilon^2}), (p/r))$, $q_1' := ((x'/\sqrt{r^2+\varepsilon^2}), (p'/r)) \in S_1$. Since

$$\operatorname{dist}^M(q_1, q_1') = \sqrt{\frac{1}{r^2+\varepsilon^2}|x-x'|^2 + \frac{1}{r^2}|p-p'|^2} \leq \operatorname{dist}^M(q_1, q_1') < \rho_1,$$

where $M = \mathbb{R}^4 \times \mathbb{R}^4$, we have $\operatorname{dist}^{S_1}(q_1, q) \leq C \cdot \operatorname{dist}^M(q_1, q_1')$.

Let $\gamma = (\gamma_x, \gamma_p)$ be any path in S_1 connecting q_1 and q_1'. Then $\gamma^r := (\gamma_x\sqrt{r^2+\varepsilon^2}, r\gamma_p)$ is a path in S_r connecting q and q'. Moreover, $r\ell(\gamma) \leq \ell(\gamma^r) \leq \sqrt{r^2+\varepsilon^2}\,\ell(\gamma)$. Minimizing over all such paths one gets

$$r \operatorname{dist}^{S_1}(q_1, q_1') \leq \operatorname{dist}^{S_r}(q, q') \leq \sqrt{r^2+\varepsilon^2} \operatorname{dist}^{S_1}(q_1, q_1')$$

Also obviously $r \operatorname{dist}^M(q_1, q_1') \leq \operatorname{dist}^M(q, q') \leq \sqrt{r^2+\varepsilon^2} \operatorname{dist}^M(q_1, q_1')$.

Thus

$$\begin{aligned}\operatorname{dist}^{S_r}(q, q') &\leq \sqrt{r^2+\varepsilon^2} \operatorname{dist}^{S_1}(q_1, q_1') \leq C\sqrt{r^2+\varepsilon^2} \operatorname{dist}^M(q_1, q_1') \\ &\leq \frac{C\sqrt{r^2+\varepsilon^2}}{r} \operatorname{dist}^M(q, q') \leq C_1\sqrt{1+\varepsilon^2} \operatorname{dist}^M(q, q')\end{aligned}$$

and $C_1\sqrt{1+\varepsilon^2}, \rho_1$ work for all $r \geq 1$.

Step 2. Let $q := (x\sqrt{|p|^2+\varepsilon^2}, p)$, $q' := (x'\sqrt{|p'|^2+\varepsilon^2}, p') \in F_\varepsilon(N_k^*)$ and $1 \leq |p| \leq |p'|$. Then the 2-point estimate holds with $\widetilde{C}_1 = \sqrt{2} + 3C_1\sqrt{1} + \varepsilon^2$, $\widetilde{\rho}_1 = \rho_1/3$. A proof follows.

Define $q'' := (x'\sqrt{|p|^2+\varepsilon^2}, |p|(p'/|p'|))$, then $q, q'' \in S_{|p|}$ (see Figure 7). Consider the following path:

$$\gamma(t) = (x'\sqrt{t^2|p'|^2+\varepsilon^2}, tp'),$$

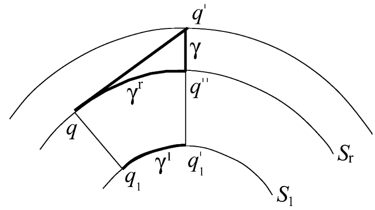

Figure 7: The 2-point estimate.

then $\gamma(1) = q'$, $\gamma(|p|/|p'|) = q''$ and $\gamma([|p|/|p'|, 1]) \subset F_\varepsilon(N_k^*)$. Essentially, q', q'' are on the same radial line and γ is the segment of the line connecting them.

$$\begin{aligned}
\dot\gamma(t) &= \left(\frac{tx'}{\sqrt{t^2|p'|^2+\varepsilon^2}}, p'\right) \\
\ell(\gamma) &= \int_{|p|/|p'|}^1 \sqrt{g_{\mathrm{st}}(\dot\gamma,\dot\gamma)}dt = \int_{|p|/|p'|}^1 \sqrt{\frac{t^2}{t^2|p'|^2+\varepsilon^2} + |p'|^2}\, dt \\
&\leq \int_{|p|/|p'|}^1 \sqrt{1+|p'|^2}\, dt \leq \sqrt{2}|p'||1 - \frac{|p|}{|p|'}| \\
&\leq \sqrt{2}|p-p'| \leq \sqrt{2}\,\mathrm{dist}^M(q,q').
\end{aligned}$$

Now we want to estimate $\mathrm{dist}^M(q,q'')$ in terms of $\mathrm{dist}^M(q,q')$.

$$\mathrm{dist}^M(q,q'') = \sqrt{(|p|^2+\varepsilon^2)|x-x'|^2 + |p - |p|\frac{p'}{|p'|}|^2}.$$

Let us consider separately each term under the square root:

$$\begin{aligned}
|x-x'|\sqrt{|p|^2+\varepsilon^2} &\leq |x\sqrt{|p|^2+\varepsilon^2} - x'\sqrt{|p'|^2+\varepsilon^2}| + |x\sqrt{|p'|^2+\varepsilon^2} \\
&\qquad - x'\sqrt{|p|^2+\varepsilon^2}| \\
&\leq \mathrm{dist}^M(q,q') + \frac{|(|p|+|p'|)(|p|-|p'|)}{\sqrt{|p'|^2+\varepsilon^2}+\sqrt{|p|^2+\varepsilon}} \\
&\leq \mathrm{dist}^M(q,q') + |p-p'| \leq 2\,\mathrm{dist}^M(q,q'); \\
|p - |p|\frac{p'}{|p'|}| &\leq \frac{|p|p'| - |p|p'|}{|p'|} + \frac{||p'|(p-p') + p'(|p'|-|p|)|}{|p'|} \\
&\leq \frac{2|p'||p-p'|}{|p'|} \leq 2\,\mathrm{dist}^M(q,q').
\end{aligned}$$

Therefore, $\mathrm{dist}^M(q,q'') \leq \sqrt{4+4}\ \mathrm{dist}^M(q,q') \leq 3\,\mathrm{dist}^M(q,q')$.

Finally, let $\mathrm{dist}^M(q,q') < \rho_1/3$, then $\mathrm{dist}^M(q,q'') < \rho_1$ and by the triangle inequality and Step 1 ($L = F_\varepsilon(N_k^*)$):

$$\begin{aligned}
\mathrm{dist}^L(q,q') &\leq \mathrm{dist}^L(q,q'') + \mathrm{dist}^L(q'',q') \\
&\leq C_1\sqrt{1+\varepsilon^2}\,\mathrm{dist}^M(q,q'') + \ell(\gamma) \\
&\leq 3C_1\sqrt{1+\varepsilon^2}\,\mathrm{dist}^M(q,q') + \sqrt{2}\,\mathrm{dist}^M(q,q') \\
&= (3C_1\sqrt{1+\varepsilon^2} + \sqrt{2})\,\mathrm{dist}^M(q,q').
\end{aligned}$$

Step 3. Now let $q, q' \in F_\varepsilon(N_k^*)$ be two arbitrary points with $\mathrm{dist}^M(q,q') < \rho_1/3$. If $|p|, |p'| \geq 1$ they are covered by Step 2. Otherwise, both are contained in $\{(x,p) \in F_\varepsilon(N_k^*) \mid |p| \leq 1 + \rho_1/3\}$. This set is compact and can be covered by a finite number of neighborhoods as in Lemma 5.5. So there are C_2, ρ_2 that realize the 2-point estimate there. Set $\rho := \min(\rho_2, \rho_1/3)$, $C := \max(C_2, 3C_1\sqrt{1+\varepsilon^2} + \sqrt{2})$. The entire $F_\varepsilon(N_k^*)$ now satisfies the 2-point estimate with these C, ρ. $\square$

Corollary 5.7 (2-point estimate)**.** $\mathrm{CT}(N_{k,\varepsilon}^*)$ *in* $\widehat{\mathcal{C}}$ *satisfies the 2-point estimate.*

Proof. By Lemma 5.6 this is true for $\mathrm{CT}(N_{k,\varepsilon}^*)$ in $\mathbb{CP}^1 \times \mathbb{C}^4$. Let C, ρ be the constants. If $\mathrm{dist}^{\widehat{\mathcal{C}}}(q,q') < \rho$, then $\mathrm{dist}^{\mathbb{CP}^1\times\mathbb{C}^4}(q,q') < \rho$ and $\mathrm{dist}^{\mathrm{CT}(N_{k,\varepsilon}^*)}$ $(q,q') \leq C\,\mathrm{dist}^{\mathbb{CP}^1\times\mathbb{C}^4}(q,q') \leq C\,\mathrm{dist}^{\widehat{\mathcal{C}}}(q,q')$ so the same constants work. $\square$

The next step in meeting the assumptions of Theorem 5.4 is to establish an area bound for curves in the moduli space. When the submanifold they are ending on, is Lagrangian with respect to a Kähler form on the ambient manifold any two curves in the same relative homology class have the same area. Indeed, let S be a chain realizing the relative homology. Then $\partial S = \Sigma_1 - \Sigma_2 + \partial S \cap L$ and

$$0 = \int_S d\omega = \int_{\partial S} \omega = \int_{\Sigma_1} \omega - \int_{\Sigma_2} \omega + \int_{\partial S \cap L} \omega = \int_{\Sigma_1} \omega - \int_{\Sigma_2} \omega$$

as $\omega|_L = 0$. But for pseudoholomorphic curves, $\mathrm{Area}(\Sigma) = \int_\Sigma \omega$ so $\mathrm{Area}(\Sigma_1) = \mathrm{Area}(\Sigma_2)$. In our case, we only have a symplectic form $\widetilde{\omega} := \widetilde{\omega}_\varepsilon$ defined on $\widehat{\mathcal{C}} \setminus 0(\widehat{\mathcal{C}})$ and uniformly tame on every $\widehat{\mathcal{C}}_\delta := \widehat{\mathcal{C}} \setminus \pi_2^{-1}(\mathcal{C} \cap B_\delta(0))$ that vanishes on $\mathrm{CT}(N_{k,\varepsilon}^*)$ (Theorem 3.12).

Lemma 5.8 (Area Bound)**.** *Let* $L = \mathrm{CT}(N_{k,\varepsilon}^*)$, $\beta \in H_2(\widehat{\mathcal{C}}, L)$. *There exists a constant* A_β *such that if* Σ *is an open curve ending on* L *with* $[\Sigma] = \beta$, *then* $\mathrm{Area}(\Sigma) \leq A_\beta$.

Proof. Define $\widetilde{g}(X,Y) := 1/2(\widetilde{\omega}(X,JY) + \widetilde{\omega}(Y,JX))$. Then it follows from the tameness condition for $\widetilde{\omega}$ that $\widetilde{g}$ is a metric on $\widehat{\mathcal{C}} \setminus 0(\widehat{\mathcal{C}})$ equivalent to $\widehat{g}$ on every $\widehat{\mathcal{C}}_\delta$. Let $\widetilde{\mathrm{Area}}$ denote the area with respect to this metric. Then $C(\delta)^{-1}\mathrm{Area} \leq \widetilde{\mathrm{Area}} \leq C(\delta)\mathrm{Area}$ by equivalence of metrics for surfaces in $\widehat{\mathcal{C}}_\delta$. Just as in the case of compatible forms, one proves that if Σ is pseudoholomorphic then $\widetilde{\mathrm{Area}}(\Sigma) = \int_\Sigma \widetilde{\omega}$ (see, e.g., [17]).

Recall from the discussion after Corollary 3.6 that $L = \mathrm{CT}(N_{k,\varepsilon}^*) \subset \widehat{\mathcal{C}}_{2\varepsilon}$. Let Σ_1, Σ_2 be two open curves ending on L with $[\Sigma_1] = [\Sigma_2] = \beta$ and let

S realize the relative homology, i.e., $[\partial S] = [\Sigma_1] - [\Sigma_2] \mod L$. Since S is three-dimensional, $0(\widehat{\mathcal{C}}) \simeq \mathbb{CP}^1$ is two-dimensional, and $\widehat{\mathcal{C}}$ is six-dimensional we can push S out of the 2ε-neighborhood of $0(\widehat{\mathcal{C}})$ by transversality. In other words, without loss of generality $S \subset \widehat{\mathcal{C}}_{2\varepsilon}$. Since $\widetilde{\omega}|_L = 0$ just as above:

$$0 = \int_S d\widetilde{\omega} = \int_{\partial S} \widetilde{\omega} = \int_{\Sigma_1} \widetilde{\omega} - \int_{\Sigma_2} \widetilde{\omega} = \widetilde{\mathrm{Area}}(\Sigma_1) - \widetilde{\mathrm{Area}}(\Sigma_2),$$

and $\widetilde{\mathrm{Area}}(\beta) := \widetilde{\mathrm{Area}}(\Sigma)$, $[\Sigma] = \beta$ only depends on the homology class. But by equivalence of metrics, $\mathrm{Area}(\Sigma) \leq C(2\varepsilon)\widetilde{\mathrm{Area}}(\beta) =: A_\beta$ for any $[\Sigma] = \beta$. □

For pseudoholomorphic curves, an area bound also implies a diameter bound due to the monotonicity lemma. In particular, we have the following property.

Lemma 5.9 (Area-diameter estimate, Proposition 4.4.1 of [23])**.** *Let (M, J, g) have bounded geometry, $L \hookrightarrow M$ be uniformly tame Lagrangian with taming constant C_1 and satisfy the 2-point estimate with constants C_2, ρ. Let $K \subset M$ be a compact subset and Σ a pseudoholomorphic curve, $\partial\Sigma \subset L$, $\Sigma \cap K \neq \emptyset$. Then $\Sigma \subset B_R(K)$ with $R = C\mathrm{Area}(\Sigma)$, $C := 8(C_1 + C_2 + 1)/(\pi \min(i(M), \rho))$. In particular,* $\mathrm{diam}(\Sigma) \leq C \cdot \mathrm{Area}(\Sigma)$.

Proof. Actually, in [23] this is proved for closed curves, but the same proof works for open ones if one uses the monotonicity lemma of Proposition 4.7.2 instead of that of Proposition 4.3.1. As for the second statement, it suffices to take $K = \{\sigma\}$, where $\sigma \in \Sigma$ is any point. □

Geometrically, this estimate means that pseudoholomorphic curves cannot be long and thin. But even when all assumptions of the last lemma are satisfied it does not follow that any sequence of curves with bounded area has a convergent subsequence.

Example 5.10. Let $M = \mathbb{C}^2$, $L = S^1 \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C} = \mathbb{C}^2$, $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. One can easily check that L is a Lagrangian submanifold with respect to the standard symplectic form on $\mathbb{C}^2$ that satisfies the 2-point estimate with $\rho = \infty$ and $C = \pi/2$. M of course has bounded geometry with $\sec^M = 0$, $i(M) = \infty$. Let $\Sigma = \overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $f_n : \Sigma \to M$, $f_n(z) = (z, n)$. $\mathrm{Area}(f_n) = \pi$ for all n, $\partial f_n(\Sigma) \subset L$, but f_n clearly does not have Gromov convergent subsequences (Figure 8).

The additional property which is missing here is some kind of convexity of L at ∞ that would force pseudoholomorphic curves to be anchored to a

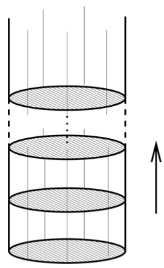

Figure 8: A sequence of holomorphic disks without Gromov-convergent subsequences.

compact subset. In terms of Theorem 5.4, there is no compact subset in $\mathbb{C}^2$ that all $f_n(\Sigma)$ meet.

In our case, anchoring to a compact subset follows from a simple observation below that generalizes the vanishing theorem of Witten (see Remark 1.5). Despite the simplicity it implies that all relevant pseudoholomorphic curves in the resolved conifold must intersect its zero-section.

Lemma 5.11 (Anchoring)**.** *Let M be a manifold with an almost complex structure J and an exact 2-form $\omega = d\lambda$ which is symplectic and tames J on $M\backslash Z$. If $L \subset M\backslash Z$ is an exact Lagrangian submanifold, then any non-constant pseudoholomorphic curve, closed or ending on L, intersects Z.*

Proof. Suppose not, then $f(\Sigma) \subset M\backslash Z$. Let g be the metric on $M\backslash Z$ determined by ω and J as in Lemma 5.8 and consider the corresponding area. Then as in Remark 1.5

$$\mathrm{Area}(f) = \int_\Sigma f^*\omega = \int_\Sigma d(f^*\lambda) = \int_{\partial\Sigma} f^*\lambda = [\lambda|_L](f_*[\partial\Sigma]) = 0 \tag{5.2}$$

and we arrive at a contradiction with non-constancy of f. □

Note that if $Z = \emptyset$, i.e., M is exact symplectic, the lemma implies that it has no non-constant pseudoholomorphic curves as observed by Witten for M a cotangent bundle [26]. In our case, $M = \widehat{\mathcal{C}}$, $Z = 0(\widehat{\mathcal{C}})$ is the zero-section, $L = \mathrm{CT}(N^*_{k,\varepsilon})$ and $\omega = \widetilde{\omega}_\varepsilon$ is supplied by Theorem 3.12. Note that here $Z \simeq \mathbb{CP}^1$ and $f(\Sigma)$ are two-dimensional and $\widehat{\mathcal{C}}$ is six-dimensional. Thus the non-empty intersection granted by Lemma 5.11 is a purely symplectic phenomenon that does not follow from a dimension count in differential topology. Now we are ready to prove the main result.

Theorem 5.12. *Moduli spaces $\overline{\mathcal{M}}_{g,h}(\widehat{\mathcal{C}}, \mathrm{CT}(N^*_{k,\varepsilon}) \mid \alpha, \beta_1, \ldots, \beta_n)$ are compact for $\alpha \neq 0 \in H_2(\widehat{\mathcal{C}}, \mathrm{CT}(N^*_{k,\varepsilon}))$ ($\alpha = 0$ corresponds to constant maps).*

Proof. Let f_n be a sequence of open curves in the above moduli space. By Lemma 5.6 $\mathrm{CT}(N^*_{k,\varepsilon})$ satisfies the 2-point estimate, by Lemma 5.8 $\mathrm{Area}(f_n) \leq A_\alpha < \infty$ and by Lemma 5.11 all f_n intersect $0(\widehat{\mathcal{C}}) \simeq \mathbb{CP}^1$ which is compact. Therefore, by Theorem 5.4 there exists a Gromov convergent subsequence. $\square$

As a bonus, Lemma 5.9 yields that for any pseudoholomorphic curve in the above moduli space $\mathrm{diam}(f) \leq C \cdot A_\alpha$ and we have them all contained in the ball $B_{C \cdot A_\alpha}(0(\widehat{\mathcal{C}}))$.

Conclusions

Whether the submanifolds constructed in this paper are suitable for the Gopakumar–Vafa conjecture remains to be seen. In addition to compactness, one also needs the open curves moduli to have virtual dimension zero [13]. For open curves ending on a Lagrangian submanifold, this is traditionally ensured by the special Lagrangian condition [12]: the imaginary part of the holomorphic volume form vanishes along the submanifold. Since the holomorphic volume form on a Calabi-Yau is nowhere zero this means, in particular, that its real part does not vanish along the submanifold. The special Lagrangian condition is far from being necessary. It suffices to have it satisfied only cohomologically: the Maslov class [2] of the submanifold should be trivial.

The last condition makes sense for totally real submanifolds as well as for Lagrangian ones. It holds, for example, if the real part of the holomorphic volume form does not vanish. Away from the zero-section, the holomorphic volume form for the resolved conifold can be obtained by deforming the one on the cotangent bundle. One can see that the real part of the latter is uniformly separated from zero along the conormal bundles to knots. For this reason, we beleive that non-vanishing along their conifold transitions can be proved by a perturbation argument akin to the one used for the symplectic form in Section 3.

Assuming that the Maslov class is zero one still needs to compare the Gromov–Witten invariants to the Chern–Simons knot invariants. A computational approach used so successfully for comparing the closed invariants [8, 9] does not seem to be feasible beyond the case of the unknot [13]. The problem is that the circle symmetry of the unknot is lacking in general and the standard localization techniques do not apply. It seems more likely that a proof will come from a deformation argument relating the Gromov–Witten theory on the resolved conifold to a degenerate string theory on the cotangent bundle postulated by Witten [26]. The recent work on contact knot homology [18] is an interesting step in this direction.

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References

[1] B. Amann, R. Lauter and V. Nistor, *On the geometry of Riemannian manifolds with a Lie structure at infinity*, Int. J. Math. Math. Sci. **2004** (1–14) (2004), 161–193, DG/0201202.

[2] A. Arsie, *Maslov class and minimality in Calabi–Yau manifolds*, J. Geom. Phys. **35**(2–3) (2000), 145–156.

[3] M. Audin, F. Lalonde and L.V. Polterovich, *Symplectic rigidity: Lagrangian submanifolds*, in 'Holomorphic Curves in Symplectic Geometry', Birkhäuser, Basel, 1994, 271–321.

[4] R. Bott and L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, **82**, Springer-Verlag, New York, Berlin, 1982.

[5] P. Candelas, and X. de la Ossa, *Comments on conifolds*, Nucl. Phys. B **342**(1) (1990), 246–268.

[6] C.H. Clemens, *Double solids*, Adv. Math. **47**(2) (1983), 107–230.

[7] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, AMS, Providence, 1983.

[8] C. Faber and R. Pandharipande, *Hodge integrals and Gromov–Witten theory*, Invent. Math. **139**(1)(2000), 173–199.

[9] A. Grassi and M. Rossi, *Large N dualities and transitions in geometry*, in Geometry and Physics of Branes (Como, 2001); Ser. High Energy Phys. Cosmol. Gravit., Bristol, 2003, 210–278, math.AG/0209044.

[10] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999), 1415-1443.

[11] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

[12] D. Joyce, *Riemannian holonomy groups and calibrated geometry*, in 'Lectures from the Summer School held in Nordfjordeid, June 2001' Springer-Verlag, Berlin, 2003, 1–68.

[13] S. Katz and M. Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions*, Adv. Theor. Math. Phys. **5**(1) (2001), 1–49.

[14] J.M. Labastida, M. Mariño and C. Vafa, *Knots, links and branes at large N*, JHEP, **5**(11) (2000), paper 7, 49 pp. hep-th/0010102.

[15] J. Li and Y.S. Song, *Open string instantons and relative stable morphisms*, Adv. Theor. Math. Phys. **5**(1) (2001), 67–91.

[16] M. Liu, *Moduli of J-Holomorphic Curves with Lagrangian Boundary Conditions and Open Gromov–Witten Invariants for an S^1-Equivariant Pair*, Harvard University, 2002, Preprint, math.SG/0210257.

[17] D. McDuff and D. Salomon, *Introduction to symplectic topology*, Clarendon Press, Oxford, 1998.

[18] L. Ng, Conormal bundles, contact homology, and knot invariants, Proceedings of the BIRS workshop *Interaction of Finite-Type and Gromov–Witten Invariants*, to appear, math.SG/0412330.

[19] Y.-G. Oh, *Fredholm theory of holomorphic disks under the perturbation of boundary conditions*, Math. Z. **222** (1996), 505–520.

[20] Y.-G. Oh, *Floer homology and its continuity for non-compact Lagrangian submanifolds*, Turkish J. Math. **25**(1) (2001), 103–124.

[21] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, Nucl. Phys. B **577** (2000), 419.

[22] M. Perdigao do Carmo, *Riemannian geometry*, Birkhäuser, Basel, Boston, Berlin, 1992.

[23] J.-C. Sikorav, *Some properties of holomorphic curves in almost complex manifolds*, in 'Holomorphic Curves in Symplectic Geometry', Birkhäuser, Basel, 1994, 165–189.

[24] I. Smith, and R. Thomas, *Symplectic surgeries from singularities*, Turkish J. Math. **27**(1) (2003), 231–250.

[25] C.-H. Taubes, *Lagrangians for the Gopakumar–Vafa conjecture*, Adv. Theor. Math. Phys. **5**(1) (2001), 139–163.

[26] E. Witten, *Chern–Simons gauge theory as a string theory*, in 'The Floer Memorial Volume', Birkhäuser, Basel, 1995, 637–678.

[27] R. Ye, *Gromov's compactness theorem for pseudoholomorphic curves*, Trans. AMS **342**(2) (1994), 671–694.