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Topological A-models on seamed Riemann surfaces

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Abstract

We define a class of topological A-models on a collection of Riemann surfaces, whose boundaries are sewn together along the seams. The target spaces for the Riemann surfaces are the Grassmannians $\mathrm{Gr}_{m_i,n}$ with the common value of n, and the boundary conditions at the seams demand that the spaces $\mathbb{C}^{m_i} \subset \mathbb{C}^n$ present the orthogonal decomposition of $\mathbb{C}^n$. The whole construction is a quantum field theory (QFT) interpretation of a part of Khovanov's categorification of the $sl(3)$ HOMFLY polynomial.

1 A QFT on a seamed Riemann surface

The idea of defining a two-dimensional theory on a "seamed" world-sheet is not exactly new. String theories, in which strings formed "networks", were considered by many authors (see, e.g., [1–3] and references therein). A similar idea was floated recently in [4] in relation to the study of the boundary conditions in conformal field theory (CFTs) (see also the review [5]). The present paper was inspired by Khovanov's construction [6], which includes both seamed surfaces (seamed along disjoint circles and called "foam")

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and a supply of the boundary conditions sufficient for defining an interesting A-model.

1.1 Seamed Riemann surfaces

Here is the definition of a seamed Riemann surface. First, we define it as a topological space. Let Σ be an oriented compact two-dimensional manifold (perhaps consisting of several connected components) with a boundary $\partial\Sigma$, which is a disjoint union of circles S^1. Let Γ be a graph, which we will call a *seam graph.* Γ may contain disjoint circles. A cycle on Γ is either a disjoint circle or a finite sequence of edges such that the beginning of the next edge coincides with the end of the previous edge, and the end of the final edge coincides with the beginning of the first one. A *seamed Riemann surface* (Σ, Γ) is constructed as a topological space by gluing the circles of $\partial\Sigma$ to some cycles of Γ. We assume that every edge of Γ is glued to at least one circle of $\partial\Sigma$, otherwise it can be removed from Γ without affecting the construction. A simple example of a seamed Riemann surface is depicted in figure 1.

Next, we endow (Σ, Γ) with a complex structure by choosing a neighborhood $U_P \subset (\Sigma, \Gamma)$ of every point $P \in (\Sigma, \Gamma)$ and specifying which complex-valued functions on those neighborhoods are called analytic. This can be done by selecting the maps

$$f_P \colon U_P \longrightarrow \mathbb{C} \tag{1.1}$$

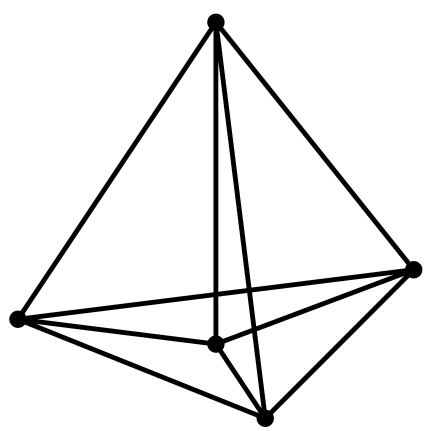

Figure 1: An example of a seamed Riemann surface. Every triangle in this picture represents a connected component Σ_i of a Riemann surface Σ.

and then defining the analytic functions on U_P as pull-backs of the analytic functions on $\mathbb{C}$. Of course, these definitions must be consistent on the intersections $U_P \cap U_{P'}$.

There are three different types of points of (Σ, Γ) depending on the structure of their neighborhoods: the internal points of Σ, the internal points of the edges of Γ, and the vertices of Γ. A complex structure in the neighborhoods of all internal points of Σ is defined simply by selecting a complex structure on $\Sigma \setminus \partial\Sigma$ compatible with its orientation. If P is an internal point of an edge e of Γ, then its neighborhood is depicted in figure 2. We draw the attached strips either above or below e depending on the orientation that they induce on it. Hence f_P maps the upper strips to the upper half-plane of $\mathbb{C}$, while mapping the lower strips to the lower half-plane of $\mathbb{C}$.

Note that for two strips S_1, S_2 attached to e, the map f_P defines locally an analytic map $f_{12}\colon S_1 \longrightarrow S_2$ by the condition that $f_P(f_{12}(P_1)) = f_P(P_1)$ for any point $P_1 \in S_1$. Obviously, $f_{12}(\partial S_1) = \partial S_2$. Although different maps f_P may lead to the same complex structure on U_P, the map f_{12} is determined by that complex structure uniquely, because it is analytic and its value on the boundary ∂S_1 is prescribed by the gluing.

The definition of the complex structure in the neighborhood of a vertex of Γ is slightly more complicated. We will use a general construction, which is also consistent with the definition of the complex structure for the first two types of points. For a point $P \in (\Sigma, \Gamma)$, we define its *local graph* γ_P as the intersection between (Σ, Γ) and a small sphere centered at P. If P is an internal point of Σ, then γ_P is a circle. If P is an internal point of an

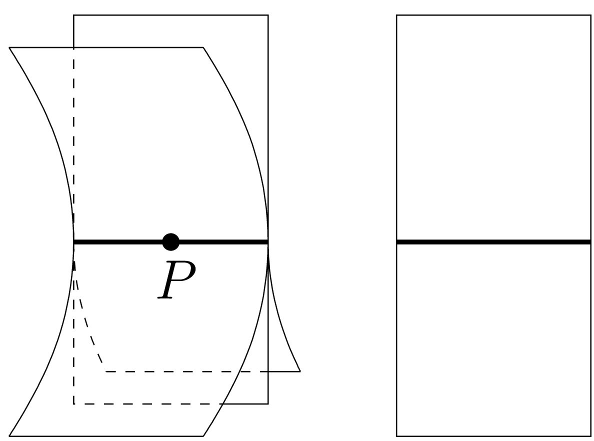

Figure 2: A neighborhood of an edge and the real line in the complex plane.

edge e of Γ, then γ_P has two vertices coming from the intersection of the small sphere with e, and the edges of γ_P correspond to the strips of Σ glued to e. If P is a vertex of Γ, then γ_P is a graph whose vertices correspond to the edges of Γ incident to P and whose edges correspond to the strips of Σ glued to the edges of Γ incident to P. The edges of γ are oriented according to the orientation of the corresponding strips of Σ.

The neighborhood U_P is isomorphic to the cone of γ_P, P being its vertex. The map f_P maps this cone to $\mathbb{C}$ as depicted in figure 3: the vertex of the cone maps to the origin of $\mathbb{C}$, the cones of the vertices of γ_P map to the rays emanating from the origin of $\mathbb{C}$, and the cones of the edges of γ_P map to the sectors of $\mathbb{C}$ bounded by the corresponding rays, so that the orientation of all the edges is counterclockwise (note that a sector may, in principle, wind many times around the origin).

1.2 Boundary conditions

Most types of two-dimensional QFTs, such as CFTs, $N = 2$ sigma models and topological sigma models, have two important properties. First, the theories of the same type can be "cross-multiplied", that is, if we put two theories T_1 and T_2 of the same type on the same world-sheet, then the resulting theory is again of the same type, and we denote it as $T_1 \times T_2$. Second, a theory can be complex-conjugated into a theory of the same type, that is, the original theory T can be equivalently described as a (possibly different) theory $\bar{T}$ of the same type defined relative to the conjugated complex structure on the same world-sheet. In case of the topological A-models, the crossing of theories results in the cross-product of the target spaces and

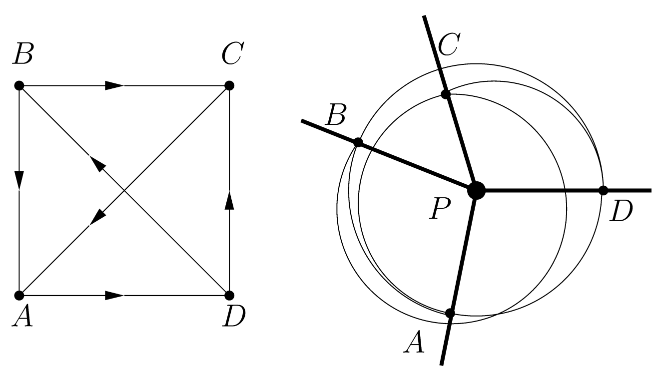

Figure 3: A local graph and its image in the complex plane.

the complex conjugation of a theory is equivalent to the conjugation of the complex structure of the target manifold.

Suppose that the Riemann surface Σ splits into a union of connected Riemann surfaces Σ_i. To every connected component Σ_i, we assign a two-dimensional QFT T_i of the same type. Now we have to formulate the boundary conditions, which serve as a "glue" holding the parts of the disjoint circles of $\partial\Sigma$ together at the edges of the seam graph Γ. Since formulating an admissible boundary condition is a local problem, we consider a neighborhood of an internal point of an edge of Γ, which is depicted in figure 2. The analytic map f_P identifies each strip of Σ attached to e with either the upper or the lower half-plane of $\mathbb{C}$, while preserving the gluing at e. Thus instead of looking for the boundary conditions of the QFTs on separate surface strips bounding the same edge, we may equivalently consider a sewing condition at the real axis of $\mathbb{C}$ for two QFTs T_{up} and T_{down}, which are the cross-products of the theories assigned to the strips that map to the corresponding half-planes. Now we can "fold" the lower half-plane (see, e.g., [4]) by conjugating its complex structure and then identifying it with the upper half-plane by the map $z \mapsto \bar{z}$, thus equating the sewing condition for T_{up} and T_{down} to the boundary condition for the single theory $T_{\text{up}} \times \bar{T}_{\text{down}}$ defined on the upper half-plane.

1.3 Observables

There are three ways in which a point-like operator-observable can be placed on a seamed Riemann surface: it can be placed either at an internal point of Σ, or at an internal point of an edge of Γ, or at a vertex of Γ. In the first two cases, the list of admissible operators is well known from the study of QFTs on Riemann surfaces with boundaries (in case of an operator on an edge, one has to consider the boundary operators of the theory $T_{\text{up}} \times \bar{T}_{\text{down}}$). The operators at the vertices may require a separate study (although a particular case of a vertex is well known: it is a point on the boundary of Σ, which separates two different D-branes). Since the list of admissible operators depends on the local properties of the theory, then in all three cases it should be determined by the local graph of the point.

1.4 Factorization

The topological two-dimensional QFTs have a simple and important factorization property. Namely, suppose that a Riemann surface Σ has two marked points P_1, P_2. Let Σ' be another Riemann surface constructed from Σ by cutting two small disks centered at P_1 and P_2 and gluing together the cutting boundaries. Then a correlator on Σ' splits into a sum of correlators on Σ with pairs of operators inserted at P_1 and P_2. More precisely, if $\{O_a \mid a \in \mathfrak{O}\}$ is a basis in the space of all admissible operators at a point of Σ ($\mathfrak{O}$ being the set, whose elements index these operators), then the sphere correlator defines a scalar product

$$g_{ab} = \langle O_a O_b \rangle_{S^2}, \tag{1.2}$$

and the factorization property reads

$$\langle \cdots \rangle_{\Sigma'} = \sum_{a,b \in \mathfrak{O}} g^{ab} \langle \cdots O_a(P_1)\, O_b(P_2) \rangle_{\Sigma}, \tag{1.3}$$

where $O(P)$ denotes an operator O placed at a point P.

A similar factorization property should hold for seamed Riemann surfaces. If we cut out a small neighborhood of a point of (Σ, Γ), then the boundary of the cut is its local graph. Suppose that for two points $P_1, P_2 \in (\Sigma, \Gamma)$ their local graphs are isomorphic and we denote them as γ. Then we can cut out small neighborhoods of P_1 and P_2 and glue the matching cut boundaries together, thus forming a new seamed Riemann surface (Σ', Γ'). Let $\{O_a \mid a \in \mathfrak{O}_\gamma\}$ be a basis in the space of admissible operators for the local graph γ. In order to define a scalar product on this space, we construct a special seamed Riemann surface $(\Sigma_\gamma, \Gamma_\gamma)$ by taking the cross-product $\gamma \times [0,1]$ and contracting its boundaries $\gamma \times \{0\}$ and $\gamma \times \{1\}$ to the points V_1 and V_2, which become the two vertices of Γ_γ. Thus the seam graph Γ_γ consists of two vertices V_1, V_2 connected by the edges, one edge per vertex of γ, and the Riemann surface Σ_γ consists of disjoint disks, one disk per edge of γ. The local graphs of V_1 and V_2 are isomorphic to γ, so we can insert the operators O_a there and define

$$g_{ab} = \langle O_a(V_1)\; O_b(V_2) \rangle_{(\Sigma_\gamma, \Gamma_\gamma)}. \tag{1.4}$$

Now the factorization property reads

$$\langle \cdots \rangle_{(\Sigma', \Gamma')} = \sum_{a,b \in \mathfrak{O}_\gamma} g^{ab} \langle \cdots O_a(P_1)\, O_b(P_2) \rangle_{(\Sigma, \Gamma)}. \tag{1.5}$$

2 A-models on seamed Riemann surfaces

2.1 Boundary conditions

Let us apply a general setup of a QFT on a seamed Riemann surface to topological A-models. A model of this type is specified by the choice of a compact Kähler manifold X as a target space, so we assign compact Kähler manifolds X_i to the connected components Σ_i of a Riemann surface Σ, which is a part of a seamed Riemann surface (Σ, Γ). A boundary condition for an A-model was established by Witten in [7]: the boundary of a world-sheet must be mapped to a Lagrangian submanifold $L \in X$. Thus, if n_e world-sheet strips joining the edge e of the seam graph Γ carry the A-models with target spaces $X_{i_1}, \ldots, X_{i_{n_e}}$, then the boundary condition at that edge is the selection of the Lagrangian submanifold $L_e \subset X_e = X_{i_1} \times \cdots \times X_{i_{n_e}}$ (actually, the Kähler manifolds assigned to the strips approaching the edge "from below" have to be complex-conjugated, which means that their Kähler forms change sign). Of course, L_e may factorize: $L_e = L_{i_1} \times \cdots \times L_{i_{n_e}}$, where $L_i \subset X_i$ are Lagrangian subspaces, but then the gluing of the world-sheet strips at e is purely formal, and this case is not interesting.

2.2 Observables

If P is an internal point of Σ_i, then it was established in [8] that for any cohomology class $\omega \in H^*(X_i)$ there is an admissible (BRST-closed) point-like operator-observable $O_\omega(P)$. Let us ignore the instanton corrections to the BRST operator. Then the analysis of [7] indicates that if P is an internal point of an edge $e \in \Gamma$, then the admissible operators $O_\omega(P)$ are determined by the cohomology classes of the corresponding Lagrangian submanifold: $\omega \in H^*(L_e)$. Similar considerations indicate that if P is a vertex of Γ, then the operators $O_\omega(P)$ are again determined by the cohomology classes $\omega \in H^*(\mathcal{M}_{\gamma_P})$, where γ_P is the local graph of P and $\mathcal{M}_\gamma$ is a special manifold determined by the local graph γ in the following way. Recall that the edges of γ correspond to the world-sheet strips of Σ and hence they are associated the Kähler manifolds X_i. Let $X_\gamma = X_{i_1} \times \cdots \times X_{i_{n_\gamma}}$ be the Kähler manifolds associated to n_γ edges of γ. The vertices of γ correspond to the edges of Γ, so let $L_{j_1}, \ldots, L_{j_{m_\gamma}}$ be the Lagrangian submanifolds corresponding to m_γ vertices of γ. Suppose that a vertex $v \in \gamma$ corresponds to an edge $e \in \Gamma$, then there is an obvious projection

$$p_e \colon X_\gamma \longrightarrow X_e \tag{2.1}$$

"forgetting" about the factors of X_γ, which do not participate in X_e. Let $\tilde{L}_e = p_e^{-1}(L_e)$ denote the pre-image of $L_e \subset X_e$. Then $\mathcal{M}_\gamma \subset X_\gamma$ is defined as the intersection

$$\mathcal{M}_\gamma = \tilde{L}_{j_1} \cap \cdots \cap \tilde{L}_{j_{m_\gamma}}. \tag{2.2}$$

Note that according to this definition, if P is an internal point of Σ_i, then $\mathcal{M}_{\gamma_P} = X_i$, and if P is an internal point of an edge $e \in \Gamma$, then $\mathcal{M}_{\gamma_P} = L_e$, so the identification between the space of admissible operators $O(P)$ and the cohomology space $H^*(\mathcal{M}_{\gamma_P})$ works for all three types of points of a seamed Riemann surface.

Following [7], we can also include the Chan–Paton factors associated with the edges of Γ. Suppose that we assign N_e Chan–Paton labels to an edge e. This means that we introduce a flat connection A_e in the associated $U(N_e)$ bundle over the Lagrangian submanifold L_e, whose fiber is $u(N_e)$ (the switching of the orientation of e changes the sign of this connection). If P is an internal point of e, then the space of admissible operators $O_\omega(P)$ is parametrized by the elements of the twisted cohomology $\omega \in H^*_{A_e}(L_e)$, which is defined on the sections of the bundle relative to the twisted differential $d + A_e$.

The spaces of observables at the vertices of Γ admit a similar description. Let us orient all edges of Γ. Let γ be the local graph of a vertex of Γ, and consider the sequence of maps

$$\mathcal{M}_\gamma \hookrightarrow \tilde{L}_e \xrightarrow{p_e} L_e, \tag{2.3}$$

where the first map is a natural embedding in view of equation (2.2) and the second map is the restriction of (2.1) to $\tilde{L}_e$. The composition of the maps (2.3) allows us to pull back the connection A_e on L_e to the connection $\tilde{A}_e$ on $\mathcal{M}_\gamma$. Now we introduce the connection $A_{\gamma_P} = \tilde{A}_{j_1} \oplus \cdots \oplus \tilde{A}_{j_{m_\gamma}}$ in the associated $U(N_{e_{j_1}}) \times \cdots \times U(N_{e_{j_{m_\gamma}}})$ bundle, whose fiber is the tensor product of the fundamental representations of these groups (in fact, the fundamental representation must be conjugated, if the oriented edge is directed into the vertex). The admissible operators at the vertex are parametrized by the corresponding twisted cohomology $H^*_{A_\gamma}(\mathcal{M}_\gamma)$.

2.3 Correlators

According to [8], a correlator of a topological A-model is determined by the contributions of the sets of holomorphic maps

$$\phi_{(i)} : \Sigma_i \longrightarrow X_i, \tag{2.4}$$

which satisfy the boundary conditions at the seam edges. We will neglect the Chan–Paton factors and provide the geometric interpretation for the contribution of the constant maps $\phi_{(i)}$. We denote this contribution as $\langle O_{\omega_1}(P_1) \cdots O_{\omega_n}(P_n) \rangle_{0,(\Sigma,\Gamma)}$. Recall that

$$\langle O_{\omega_1}(P_1) \cdots O_{\omega_n}(P_n) \rangle_{0,S^2} = \int_X \omega_1 \wedge \cdots \wedge \omega_n. \tag{2.5}$$

We assume for simplicity that the fermionic fields $\psi_z^{\bar{i}}, \psi_{\bar{z}}^i$ have no zero modes on the seamed Riemann surface (Σ, Γ). Hence the calculation of the contribution of the constant maps to the correlator is reduced to the integration over the constant modes of $\chi^i, \chi^{\bar{i}}$ and over the moduli space $\mathcal{M}_{0,(\Sigma,\Gamma)}$ of the constant maps (2.4). This means that the general correlator is again an intersection number:

$$\langle O_{\omega_1}(P_1)\cdots O_{\omega_n}(P_n)\rangle_{0,(\Sigma,\Gamma)} \;=\; \int_{\mathcal{M}_{0,(\Sigma,\Gamma)}} F_{P_1,*}\,\omega_1 \wedge \cdots \wedge F_{P_n,*}\,\omega_n, \quad (2.6)$$

where $F_{P_i,*}\,\omega_i$ is the pull-back of ω_i by a map

$$F_{P_i}\colon \mathcal{M}_{0,(\Sigma,\Gamma)} \longrightarrow \mathcal{M}_{\gamma(P_i)}, \quad (2.7)$$

which can be easily constructed for all three types of points $P \in (\Sigma,\Gamma)$ in the following way. Let $X_{(\Sigma,\Gamma)} = X_1 \times \cdots \times X_{n_\Sigma}$ be the product of all target spaces corresponding to the connected components $\Sigma_1,\ldots,\Sigma_{n_\Sigma}$ of Σ. Then for every component Σ_i, there is a natural projection

$$P_i\colon X_{(\Sigma,\Gamma)} \longrightarrow X_i. \quad (2.8)$$

Let e be an edge of Γ. We assume for simplicity that every component Σ_i bounds e at most once, so there is another natural projection

$$P_e\colon X_{(\Sigma,\Gamma)} \longrightarrow X_e, \quad (2.9)$$

which forgets about the factors of $X_{(\Sigma,\Gamma)}$, whose world-sheets Σ_i do not bound e. Then $\mathcal{M}_{0,(\Sigma,\Gamma)}$ is the intersection of the pre-images of the Lagrangian submanifolds $L_e \subset X_e$:

$$\mathcal{M}_{0,(\Sigma,\Gamma)} = \bigcap_{e\in E(\Gamma)} P_e^{-1}(L_e) \subset X_{(\Sigma,\Gamma)}. \quad (2.10)$$

Thus if P is an internal point of Σ_i, then we define F_P as the composition of maps

$$F_P:\ \mathcal{M}_{0,(\Sigma,\Gamma)} \hookrightarrow X_{(\Sigma,\Gamma)} \xrightarrow{P_i} X_i, \quad (2.11)$$

and if P is an internal point of an edge, then F_P is the composition of maps

$$F_P:\ \mathcal{M}_{0,(\Sigma,\Gamma)} \hookrightarrow P_e^{-1}(L_e) \xrightarrow{P_e} L_e. \quad (2.12)$$

Now let P be a vertex v of Γ. Assume for simplicity that every component Σ_i bounds v at most once. Then there is a natural projection $P_v\colon X_{(\Sigma,\Gamma)} \longrightarrow X_{\gamma_v}$ and $P_v(\mathcal{M}_{0,(\Sigma,\Gamma)}) \subset \mathcal{M}_{\gamma_v}$, so in this case we define F_P as the restriction $P_v|_{\mathcal{M}_{0,(\Sigma,\Gamma)}}$.

Note that the Lagrangian submanifolds $L_e \subset X_e$ must be selected together with their orientation. The Kähler manifolds X_i also have natural orientation coming from their complex structure. Hence the moduli space of constant maps $\mathcal{M}_{0,(\Sigma,\Gamma)}$ receives an orientation from formula (2.10), so the integral in the r.h.s. of equation (2.6) is well defined, provided that we choose the order in which we intersect the Lagrangian submanifolds.

3 Grassmannians

The general construction of two-dimensional theories on seamed Riemann surfaces looks rather abstract unless we provide interesting boundary conditions which mix multiple theories of the same class. Luckily, a wide class of Lagrangian submanifolds in the products of Kähler manifolds is implied by the construction in Khovanov's paper [6], which deals with the categorification of the $sl(3)$ HOMFLY polynomial.

3.1 Lagrangian submanifolds

A complex Grassmannian is the "moduli space" of m-dimensional subspaces of $\mathbb{C}^n$. A Grassmannian can be endowed with a Kähler structure. Namely, suppose that $\mathbb{C}^n$ has the standard Hermitian scalar product. Now if $V \subset \mathbb{C}^n$ ($\dim V = m$) represents a point $x \in \mathrm{Gr}_{m,n}$, then the tangent space $T_x\mathrm{Gr}_{m,n}$ is canonically isomorphic to the space of linear maps $\mathrm{Hom}(V, V^\perp)$ and the Kähler form ω_{K} evaluated on two tangent vectors $A, B \in \mathrm{Hom}(V, V^\perp)$ is

$$\omega_{\mathrm{K}}(A,B) = \frac{1}{2i}\,\mathrm{Tr}_V(B^*A - A^*B). \tag{3.1}$$

Consider a set of Grassmannians $\mathrm{Gr}_{m_1,n}, \ldots, \mathrm{Gr}_{m_k,n}$ such that

$$m_1 + \cdots + m_k = n. \tag{3.2}$$

Think of a point of their cross-product

$$X = \mathrm{Gr}_{m_1,n} \times \cdots \times \mathrm{Gr}_{m_k,n} \tag{3.3}$$

as a set of subspaces $V_1, \ldots, V_k \subset \mathbb{C}^n$ ($\dim V_i = m_i$) of the *same* complex space $\mathbb{C}^n$ endowed with a Hermitian scalar product. Then the condition that the subspaces V_i form an orthogonal decomposition of $\mathbb{C}^n$ specifies a Lagrangian submanifold $L \subset X$.

The Lagrangian nature of L can be verified by a straightforward calculation. If the subspaces V_i ($\dim V_i = m_i$) form an orthogonal decomposition of $\mathbb{C}^n$, then $V_i^\perp = \bigoplus_{j\neq i} V_j$, so that $\mathrm{Hom}(V_i, V_i^\perp) = \bigoplus_{j\neq i}\mathrm{Hom}(V_i, V_j)$ and the Kähler form on the tangent space of $\mathrm{Gr}_{m_i,n}$ is

$$\omega_{\mathrm{K}}^{(i)}(A,B) = \frac{1}{2i}\sum_{j\neq i}\mathrm{Tr}_V(B^*_{ij}A_{ij} - A^*_{ij}B_{ij}), \quad A_{ij}, B_{ij} \in \mathrm{Hom}(V_i, V_j). \tag{3.4}$$

The tangent space to the surface $L \subset X$ is specified by the conditions

$$A_{ij} = A^*_{ji} \quad \text{for all } 1 \leq i \neq j \leq k. \tag{3.5}$$

It is easy to see that these conditions halve the real dimension of the original manifold and make the total Kähler form $\omega_{\mathrm{K}} = \sum_{i=1}^{k} \omega_{\mathrm{K}}^{(i)}$ zero, so L is indeed a Lagrangian submanifold.

Note that condition (3.5) implies a simple model for the complex-conjugated Grassmannian $\overline{\mathrm{Gr}}_{m,n}$. Namely, a map $\mathrm{Gr}_{m,n} \to \mathrm{Gr}_{n-m,n}$, which maps every m-dimensional subspace $V \subset \mathbb{C}^n$ into its orthogonal complement, is an anti-holomorphic isomorphism, hence

$$\overline{\mathrm{Gr}}_{m,n} \cong \mathrm{Gr}_{n-m,n}. \tag{3.6}$$

A generalized version of the Lagrangian submanifold $L \subset X$ exists, if instead of equation (3.2) we have

$$m_1 + \cdots + m_k = Nn, \tag{3.7}$$

where N is a positive integer. In this case, a Lagrangian submanifold $L_{\mathbf{m}}$ is specified by a list of non-negative numbers

$$\mathbf{m} = \left(m_{ij},\ 1 \le i \le k,\ 1 \le j \le N \,|\, m_{ij} \ge 0,\ \sum_{j=1}^{N} m_{ij} = m_i \right). \tag{3.8}$$

A point of $\mathrm{Gr}_{m_1,n} \times \cdots \times \mathrm{Gr}_{m_k,n}$, specified by the set of k subspaces $V_i \subset \mathbb{C}^n$ $(1 \le i \le k)$, belongs to L if there exist the subspaces $V_{ij} \subset \mathbb{C}^n$ $(1 \le i \le k$, $1 \le j \le N)$ such that $\dim V_{ij} = m_{ij}$ and the spaces V_{ij} $(1 \le i \le k)$ form an orthogonal decomposition of $\mathbb{C}^n$ for every fixed j, while the spaces V_{ij} $(1 \le j \le N)$ form an orthogonal decomposition of V_i for every fixed i.

3.2 A-models

The Lagrangian submanifolds $L_{\mathbf{m}}$ allow us to construct topological A-models on seamed Riemann surfaces along the lines of Section 2. First, we pick a value of n. Then the Kähler manifolds X_i are the Grassmannians $\mathrm{Gr}_{m_i,n}$ for positive integers $m_i < n$, and the boundary conditions at the seam edges of a seamed Riemann surface are specified with the help of Lagrangian submanifolds $L_{\mathbf{m}}$.

A particular feature of this model is that all moduli spaces related to it have a $U(n)$ symmetry, which acts on the "master-space" $\mathbb{C}^n$. This $U(n)$ symmetry acts transitively on each Grassmannian $\mathrm{Gr}_{m_i,n}$, hence maps (2.1) and (2.8) are fiber-bundle projections. At the same time, the Grassmannian model indicates that some considerations of Section 1.3 are too naive: Khovanov and Kuperberg observed that the spaces $\mathcal{M}_\gamma$ may be singular. His simplest example is the moduli space associated with the "cube" graph

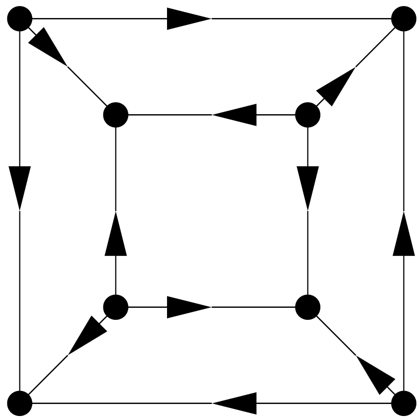

Figure 4: The cube graph produces a singular space $\mathcal{M}_\gamma$.

of figure 4 for the case of $n = 3$ when the projective spaces $\mathbb{CP}^2$ are associated to every edge. As a result of this singularity, the Poincare duality essential for the factorization property (1.5) is broken. This means that if $\mathcal{M}_\gamma$ is singular, then the identification of the space of observables with the cohomology space $H^*(\mathcal{M}_\gamma)$ has to be reconsidered.

4 Conclusion

One of the major selling points of the string theory is that its interactions are not arbitrary, but rather come from natural geometric principles, such as the "pants" world-sheet, describing the triple interaction between three closed strings. From this point of view, the idea of a seamed Riemann surface does not seem to be very attractive, since it reminds us of the intersecting world-lines of old QFTs. Those intersections of world-lines were responsible for the wild arbitrariness of the interaction coupling constants. However, the theories on seamed Riemann surfaces, in which the seams are due to non-factorizable D-brane type boundary conditions, seem geometric enough in order to be considered seriously. Moreover, the Lagrangian submanifolds of Section 3.1 provide enough boundary conditions in order to put the Grassmannian-based A-models on complicated seamed Riemann surfaces. Thus, one might conclude that seamed Riemann surfaces are as good as familiar Riemann surfaces for the purpose of building two-dimensional QFTs.

A mathematical implication of the Grassmannian-based A-models on seamed Riemann surfaces is that they lead to a Fukaya category on a family of manifolds rather than on a single Kähler manifold. One might expect that, due to the mirror symmetry, similar construction could exist for the categories of coherent sheaves. Also, it is worth noting that the Grassmannian-based A-models have an equivalent description as Landau–Ginzburg models and as G/G WZW models (see, e.g., [9]). It would be interesting to find the boundary conditions of those models, which correspond to the Lagrangian submanifolds $L_{\mathbf{m}}$.

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