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Abelian homotopy Dijkgraaf–Witten theory

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Abstract

We construct a version of Dijkgraaf–Witten theory based on a compact abelian Lie group within the formalism of Turaev's homotopy quantum field theory. As an application we show that the 2+1-dimensional theory based on $U(1)$ classifies lens spaces up to homotopy type.

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1 Introduction

The central topic of this paper is Dijkgraaf–Witten (DW) invariants of closed, oriented $n+1$-manifolds based on a compact abelian gauge group, A. These may be defined as follows.

The "space of fields" on an $n+1$-manifold, W, is taken to be the moduli space $\mathcal{F}_W$ of isomorphism classes of A-bundles with flat connection. Since A is abelian there are identifications

$$\begin{aligned}\mathcal{F}_W &\cong \mathrm{Hom}(\pi_1(W), A)/\mathrm{conj} \cong \mathrm{Hom}(\pi_1(W), A)\\ &\cong \mathrm{Hom}(H_1(W;\mathbb{Z}), A) \cong H^1(W;A).\end{aligned}$$

The last isomorphism is an easy consequence of the universal coefficient theorem. If β is the first Betti number of W, we then see that

$$\mathcal{F}_W \cong A^\beta \times \mathrm{Tors},$$

where, Tors is a discrete abelian group of torsion and that we may therefore identify $\mathcal{F}_W$ with a compact abelian Lie group. Denote the normalized Haar measure on this group by μ_W. Note that $\mathcal{F}_W$ can also be identified with $[W, K_A]$, the set of based homotopy classes of maps from W to the Eilenberg–Mac Lane space $K_A = K(A, 1)$.

The "action" of the theory is defined by a cohomology class $[\theta]$ in the cohomology group $H^{n+1}(K_A; U(1))$ by

$$\begin{array}{ccc}\mathcal{F}_W & \longrightarrow & U(1)\\ \nu & \mapsto & \langle \nu^*([\theta]), [W]\rangle,\end{array}$$

where $[W]$ is the fundamental class of W and $\langle -, -\rangle$ is the evaluation pairing. Here we have ν^*: $H^{n+1}(K_A; U(1)) \to H^{n+1}(W; U(1))$, thinking of ν as a homotopy class of maps from W to K_A.

If the action is integrable with respect to the measure μ_W, the DW-invariant of W based on $[\theta]$ is defined to be

$$Z_A^{[\theta]}(W) = \int_{\nu\in\mathcal{F}_W} \langle \nu^*([\theta]), [W]\rangle d\mu_W. \tag{1.1}$$

Our interest in such invariants stems from the following. In general, a topological quantum field theory (TQFT) is either defined in a geometrically meaningful way via a non-rigorous path integral, or combinatorially, where the link to the underlying geometry is less clear. It has been a main goal of the subject for years to bring these two points of view closer together. DW TQFT, which begins with invariants defined by formula (1.1) where A

is replaced with a (not necessarily abelian) finite group G, is rigourously accessible from both perspectives because the path integral is a finite sum over $[W, BG] \cong \mathrm{Hom}(\pi_1 W, G)$. With this in mind one would like to extend DW theory to compact Lie groups, but in general the path integral becomes undefined. For the case of a compact abelian Lie group, however, the theory can still be approached from both points of view[1] .

Another reason why DW theories for continuous groups are interesting is that they can be viewed as state sum models in which the set of states over which one must sum to obtain the invariants is no longer finite or even discrete, but still finite-dimensional. Because of this, the path integral in these theories is at an intermediate level of difficulty between the finite sums of conventional state sum models and the infinite dimensional integrals that usually occur in non-topological models.

The original motivation for DW-invariants [4] is that they arise as the partition functions of Chern–Simons theories with finite gauge group. The physical states correspond to equivalence classes of principal G-bundles, and Dijkgraaf and Witten show that $H^3(K(G,1); U(1))$ classifies possible actions. The partition function is only one aspect of a full TQFT lying behind. One central feature of TQFTs is locality: a global invariant can be built up from local contributions. Locality and the problems associated with patching local information together make the rigorous construction of DW TQFT highly non-trivial. This programme was carried out by Freed and Quinn in [7] (and there is related work on $U(1)$–Chern–Simons theory by Manoliu in [10]). Turaev [17] has also recast the pre-path-integral structure arising in Freed and Quinn's work into a different axiomatic framework with his homotopy quantum field theories. This set-up is not specific to dimension 2+1, but works for any dimension.

It can immediately be seen from (1.1) that the DW-invariants only depend on the homotopy type of the manifold W. It is interesting to ask how good these invariants are as homotopy invariants. From a purely homotopical point of view locality is rather unnatural: the homotopy theory of local bits will overlook aspects of the global homotopy theory. If one is to proceed to understand the full theory, great care must be taken to work with respect to prescribed boundary conditions (on the local pieces) and then carefully analyse how to fit the pieces together. Nonetheless, the invariants themselves, being homotopy invariants, should be computable in a more natural

[1]For non-abelian gauge groups we would not get a group structure on $\mathcal{F}_W$, hence no Haar measure. The existence of a good measure on $\mathcal{F}_W$ (and on other spaces of field configurations to be defined later) is the main reason that we will look only at abelian groups. Most of the results that do not involve these measure theoretical problems are also valid for non-abelian gauge groups.

way from the point of view of classical techniques in algebraic topology. In this paper, we wish to follow something of a middle road, constructing a theory which is both simple and natural with regard to homotopy theory, at the expense of sacrificing full locality for a restricted version. The restriction is that we will only allow decompositions along *connected* submanifolds.

We adopt the formalism of Turaev's homotopy quantum field theories (HQFTs) and begin by recalling some background about these. The idea of integrating an HQFT to give a TQFT, is briefly discussed. Beginning with a compact abelian Lie group A and an HQFT of a certain type, we construct a version of DW theory based on A. We refer to this as *abelian homotopy DW theory* both to indicate the link with HQFT and to distinguish our theory from the Freed–Quinn formulation. Such a thing will consist of the following assignments.

- To each closed, oriented n-manifold M and $\alpha \in \mathcal{F}_M$ we assign a line $L_{M,\alpha}$.
- To each cobordism W with incoming (resp. outgoing) boundary M_0 (resp. M_1) and $(\alpha_0, \alpha_1) \in \mathcal{F}_{M_0} \times \mathcal{F}_{M_1}$ we assign a linear map

$$\mathbb{K}_W(\alpha_0, \alpha_1)\colon L_{M_0,\alpha_0} \to L_{M_1,\alpha_1}.$$

A notable feature is that the construction works in any dimension. We examine properties of such theories, in particular we prove a decomposition formula (Theorem 4.1) and examine invariants of products (Theorem 4.7). We devote the final section to calculations using both decomposition and product formulae but also showing how the more familiar combinatorial picture emerges for explicit calculation. We show for example in Theorem 5.3 that the DW-invariants with group $A = U(1)$ separate lens spaces up to homotopy type.

2 Background on HQFTs

2.1 What is an HQFT?

An HQFT may be seen as an axiomatic formulation of the "action" in a TQFT in which the spaces of fields on a closed $n+1$-manifold W, is the set of homotopy classes of maps from W to some auxiliary space X. Typically, X will be an Eilenberg-Mac Lane space for a discrete group and, hence, the spaces of fields is related to the moduli space of flat bundles with connection. This is, in fact, the motivating example and is a formulation of the "extended action" found in Freed and Quinn's work on Chern–Simons theory for finite gauge group [7]. HQFTs were defined by Turaev in [17] (and in a special case in [2] and further discussion of the connection between the two can be found in [14]).

To formulate the theory one considers smooth, oriented, closed n-manifolds and their diffeomorphisms and cobordisms between such. An $n+1$-dimensional cobordism (or $n+1$-cobordism for short) is a triple (M_0, W, M_1) where W is a smooth oriented $n+1$-manifold whose boundary is a disjoint union of n-manifolds, M_0 and M_1, such that M_1 has the induced orientation and M_0 the opposite orientation to the induced one. Now consider all manifolds and cobordisms to come equipped with characteristic maps, that is to say, maps to some auxiliary "background space" X. (Such manifolds and cobordisms are called X-manifolds and X-cobordisms, respectively.) Given a X-cobordism (M_0, W, M_1) note that by reversing the orientation of W we get a X-cobordism $(M_1, \overline{W}, M_0)$. It will sometimes be convenient to write $\overline{\sigma}$ for the characteristic map of this opposite cobordism, where $\sigma\colon W \to X$ is the characteristic map of (M_0, W, M_1).

The key ingredients of an HQFT are assignments as follows. To each n-manifold, M with characteristic map $\gamma\colon M \to X$, one assigns a finite dimensional vector space $V_{M,\gamma}$, and to each diffeomorphism one assigns an isomorphism of these vector spaces. To each cobordism (M_0, W, M_1) with characteristic map σ, one assigns a linear map $V_{M_0,\gamma_0} \to V_{M_1,\gamma_1}$, where γ_0 and γ_1 are the characteristic maps induced on the boundary. These assignments are subject to a list of axioms and the reader is asked to consult [17] for details. Key among the axioms is that the linear maps associated to X-cobordisms are invariant under homotopies of the characteristic map. It is also worth noting that Turaev's axiom 1.2.7 has a somewhat special status. (Here and elsewhere Turaev's axioms refer to the axioms in [17, Section. 1.2].) For a general background space it may be undesirable to impose this axiom as it reduces the theory to the one based on an Eilenberg–Mac Lane space. In this paper, the background space will be an Eilenberg–Mac Lane space and we will make use of this axiom.

2.2 Examples

The following class of examples is due to Turaev, cf. [17, Section. 1.3]. They are *rank one* in the sense that all vector spaces associated to n-manifolds are one-dimensional.

Example 2.1. (Turaev) Primitive cohomological HQFTs

Let X be any topological space and let $\theta \in C^{n+1}(X; U(1))$. For $\gamma\colon M \to X$ set

$$L_{M,\gamma} = \mathbb{C}\{a \in C_n M \mid [a] = [M]\}/a \sim \gamma^*\theta(e)b.$$

In the earlier expression, $e \in C_{n+1}M$ such that $\partial e = -a + b$. For $\sigma\colon W \to X$ define a homomorphism

$$E_{W,\sigma}\colon L_{M_0,\gamma_0} \to L_{M_1,\gamma_1}$$

on generators by

$$a_0 \mapsto \sigma^*\theta(f)a_1,$$

where $f \in C_{n+1}W$ satisfies $\partial f = -a_0 + a_1$ and is a representative of the fundamental class in $H_{n+1}(W, \partial W)$. Turaev shows that this construction is independent of any choices and that it indeed gives rise to an HQFT. Moreover, cohomologous cocycles give equivalent theories.

The following lemma is immediate from the definition just given.

Lemma 2.2. *A primitive cohomological HQFT satisfies*

$$E_{\overline{W},\overline{\sigma}} = E_{W,\sigma}^{-1}.$$

In the next section, we will restrict to the case where X is an Eilenberg–Mac Lane space $K(A,1)$ for a compact abelian Lie group A. Sometimes it will be convenient to consider group cocycles instead of singular cocycles which we can do using the fact that the cohomology of the space $K(A,1)$ is isomorphic to the group cohomology of the (discrete) group A. Recall that a *group n-cochain* with coefficients in $U(1)$ is a function $\omega\colon A^n \to U(1)$ and such functions form a group K^n under pointwise multiplication. The *coboundary operator* $\delta\colon K^n \to K^{n+1}$ is defined by

$$\delta\omega(x_1, \dots, x_{n+1}) = \omega(x_2, \dots, x_{n+1})\omega^{-1}(x_1, x_2x_3, x_4, \dots, x_{n+1}) \dots \\ \dots \omega^{(-1)^{n+1}}(x_1, \dots, x_nx_{n+1})\omega^{(-1)^{n+2}}(x_1, \dots, x_n).$$

We have $\delta^2 = 0$ and the group cohomology is defined as the homology of this cochain complex. A group n-cocycle ω is *normalized* if the function ω takes the value 1, whenever at least one entry is the identity.

Thus given $\theta \in H^{n+1}(K(A,1); U(1))$ we may choose a corresponding group cocycle ω. This group cocycle is not necessarily normalized, but within the cohomology class of θ one can always choose a normalized representative. Conversely, given a group cocycle one can choose a corresponding singular cocycle representing the same cohomology class.

Example 2.3. Taking $A = U(1)$ we define a 3-cocycle $\theta_k \in C^3(K(U(1),1); U(1))$ and the corresponding group cocycle $\omega_k \colon U(1)^3 \to U(1)$ for any integer k.

Noting that $H^1(K(U(1),1);U(1)) = [K(U(1),1), K(U(1),1)]$ pick a 1-cocycle representing the identity map. Lift this to a real cochain η:$C_1(K(U(1),1);\mathbb{Z}) \to \mathbb{R}$. This is not a cocycle but $\delta\eta$ takes integer values. Now consider the real valued 3-cochain $\eta \cup \delta\eta$ which again is not a cocycle, however $\delta(\eta \cup \delta\eta) = \delta\eta \cup \delta\eta$ which has integer values. We then define $\theta_k \in C^3(K(U(1),1);U(1)) = \mathrm{Hom}(C_3(K(U(1),1);\mathbb{Z}), U(1))$ by

$$\theta_k = e^{2\pi i k \eta \cup \delta\eta}.$$

Note that θ_k is independent of the lift η and is now a cocycle. To define the cocycle ω_k, let $g_1, g_2, g_3 \in A$ and write $g_1 = e^{2\pi i a}, g_2 = e^{2\pi i b}$ and $g_3 = e^{2\pi i c}$ with $0 \leq a, b, c < 1$. Then set

$$\omega_k(g_1, g_2, g_3) = e^{2\pi i k a(b+c-[b+c])}, \tag{2.1}$$

where the square bracket means addition modulo 1.

Example 2.4. When $A = U(1) \times U(1)$, we have the cocycles associated to the individual $U(1)$ factors, defined earlier, but we also get a second type of cocycles. These cocycles are also labelled by an integer $l \in \mathbb{Z}$ and we will call them ζ_l. The definition of ζ_l is very similar to that of θ_k. First, we define 1-cocycles corresponding to the identity maps of the first and second factors of $K(U(1) \times U(1), 1) \cong K(U(1),1) \times K(U(1),1)$. Then we lift these to real cochains η_1, η_2 and we note that the boundaries of these cochains take integer values. As a consequence, the same is true for the boundary of the real cochain $\eta_1 \cup \delta\eta_2$ and hence we may define the 3-cocycles ζ_l by

$$\zeta_l = e^{2\pi i l \eta_1 \cup \delta\eta_2}.$$

To write down the corresponding group cocycles ψ_l, we introduce a similar notation to the one used in formula (2.1). Let $g_1, g_2, g_3 \in A = U(1) \times U(1)$ and write $g_1 = (e^{2\pi i a_1}, e^{2\pi i a_2})$, $g_2 = (e^{2\pi i b_1}, e^{2\pi i b_2})$ and $g_3 = (e^{2\pi i c_1, 2\pi i c_2})$ with $0 \leq a_i, b_i, c_i < 1$. Then ψ_l is given by

$$\psi_l(g_1, g_2, g_3) = e^{2\pi i l a_1(b_2+c_2-[b_2+c_2])}, \tag{2.2}$$

where the square bracket means addition modulo 1 as in (2.1).

Clearly, we might also have reversed the roles of the first and second $U(1)$ factors in Example 2.4.

2.3 TQFTs and matrix elements

To obtain a TQFT from an HQFT one should perform some kind of integration. Although this may not be rigourously defined it is useful to keep it in mind and in the following brief digression we give an outline of the idea.

Recall that a rank one HQFT assigns a (complex) line $L_{M,\gamma}$ to each X-manifold (M,γ). One should think about the collection of these as a line bundle L_M over $\mathrm{Map}(M,X)$, the space of fields of M. The Hilbert space associated to M is the space of sections of this line bundle. The time evolution U_W along a cobordism W, denoted U_W is defined on a section ψ of L_{M_0} by

$$U_W(\psi)(\gamma_1) = \int_{\gamma_0 \in \mathrm{Map}(M_0,X)} K_W(\gamma_0,\gamma_1)(\psi(\gamma_0))\, d\gamma_0,$$

where $\gamma_1 \in \mathrm{Map}(M_1,X)$ and the $K_W(\gamma_0,\gamma_1)$ are the "matrix elements" of the theory. In this context, a matrix element is a linear map $K_W(\gamma_0,\gamma_1)\colon L_{M_0,\gamma_0} \to L_{M_1,\gamma_1}$ defined by

$$K_W(\gamma_0,\gamma_1)(x) = \int_{\sigma \in \mathrm{Map}(W,X;\gamma_0,\gamma_1)} E_{W,\sigma}(x)\, d\mu,$$

where $\mathrm{Map}(W,X;\gamma_0,\gamma_1)$ consists of maps $W \to X$ agreeing with the given γ_0 and γ_1 on the incoming and outgoing boundaries.

The reader should not require much convincing that in general much of this is ill defined. It is, however, worth noting that since the homomorphisms $E_{W,\sigma}$ are homotopy invariant, the "measure" $d\mu$ needs only defining on homotopy classes rather than the full mapping space, which may simplify the situation.

The fundamental property of locality can be expressed in terms of the matrix elements as follows. Suppose that W can be decomposed along M as $W = W' \cup_M W''$. Then locality is the requirement

$$K_{W' \cup W''}(\gamma_0,\gamma_1)(x) = \int_{\gamma \in \mathrm{Map}(M,X)} (K_{W''}(\gamma,\gamma_1) \circ K_{W'}(\gamma_0,\gamma))(x)\, d\gamma.$$

Numerical invariants of closed manifolds arise in the usual way: regard a closed, oriented $n+1$-manifold W as a cobordism from $\emptyset$ to $\emptyset$ in which case $K_W(\emptyset,\emptyset)(1) \in \mathbb{C}$ defines a numerical invariant of W.

3 The definition of abelian homotopy DW theory

We now turn to the central topic of the paper. *For the remainder of the paper, A will denote a compact abelian Lie group and K_A will denote the Eilenberg–Mac Lane space $K(A, 1)$.* The space K_A may be considered as the classifying space of A regarded as a discrete group. We note that A is isomorphic to the product of a torus and a finite abelian group (see e.g., [3, Corollary I.3.7]). We will freely use the fact that

$$H^1(W; A) \cong [W, K_A] \cong \mathrm{Hom}(H_1(W; \mathbb{Z}), A),$$

where the square bracket refers to based homotopy classes of maps.

Given an n-manifold M set

$$\mathcal{F}_M = H^1(M; A)$$

and similarly for an $n+1$-cobordism (M_0, W, M_1) set

$$\mathcal{F}_W = H^1(W; A).$$

These are the "fields" of the theory and can be identified with isomorphism classes of principal A-bundles with flat connection. There is a natural topology on $\mathcal{F}_W$ arising from the identification $H^1(-; A) \cong \mathrm{Hom}(H_1(-; \mathbb{Z}), A)$, which shows that $\mathcal{F}_W$ can be identified with the product of a number of copies of A and a discrete abelian group of torsion.

For any submanifold M of W the inclusion $i\colon M \to W$ induces a restriction map $i^*\colon \mathcal{F}_W \to \mathcal{F}_M$ which we will denote r^W_M.

Lemma 3.1. *The restriction map $r^W_M\colon \mathcal{F}_W \to \mathcal{F}_M$* is *continuous.*

Proof. It suffices to show that composition with the projection p onto each factor in $\mathcal{F}_M = A^l \times \mathrm{Tors}_M$ is continuous. Let B be such a factor and Z be the corresponding cyclic group factor in $H_1(M; \mathbb{Z})$ i.e., $B = \mathrm{Hom}(Z, A)$. Then $p \circ r\colon \mathcal{F}_W = A^k \times \mathrm{Tors}_W \to B$ maps $(a_1, \ldots, a_k, b_1, \ldots, b_q)$ to $a_1^{n_1} \cdots a_k^{n_k} b_1^{m_1} \cdots b_q^{m_q}$, where the map

$$Z \to H_1(M; \mathbb{Z}) \to H_1(W; \mathbb{Z}) = \mathbb{Z}^k \times \mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_q$$

takes 1 to $(n_1, \ldots, n_k, m_1, \ldots, m_q)$. Since multiplication in A is continuous this shows that $p \circ r$ is continuous. $\square$

For a cobordism (M_0, W, M_1) we will need to consider fields with prescribed boundary conditions. For a given pair $(\alpha_0, \alpha_1) \in \mathcal{F}_{M_0} \times \mathcal{F}_{M_1}$ of boundary fields we set

$$\mathcal{F}_W^{\alpha_0,\alpha_1} = \{\nu \in \mathcal{F}_W \mid r^W_{M_0}(\nu) = \alpha_0 \text{ and } r^W_{M_1}(\nu) = \alpha_1\}.$$

By Lemma 3.1, $\mathcal{F}_W^{\alpha_0,\alpha_1}$ is a closed, hence compact subset of $\mathcal{F}_W$ (perhaps empty).

3.1 An HQFT-like construction

Suppose we are given a primitive cohomological HQFT in dimension $n+1$ with background space $K_A = K(A,1)$. Let M be an n-manifold and let $\gamma, \gamma' \colon M \to K_A$.

Proposition 3.2. *If γ is homotopic to γ' then $L_{M,\gamma}$ is canonically isomorphic to $L_{M,\gamma'}$.*

Proof. Let $h \colon M \times I \to K_A$ be a homotopy. Regarding this as the characteristic map of a cobordism, the HQFT gives rise to an isomorphism

$$E_{M\times I,h} \colon L_{M,\gamma} \to L_{M,\gamma'}.$$

Given another homotopy $h' \colon M \times I \to K_A$ consider the map

$$h \cup \overline{h'} \colon M \times I \to K_A$$

defined by h on the first half of the cylinder and by $\overline{h'}$ on the second half. This map satisfies $h \cup \overline{h'}|_0 = \gamma$ and $h \cup \overline{h'}_1 = \gamma$ so by the axioms of HQFTs (and in particular Turaev's axiom 1.2.7 which holds since, our background space is an Eilenberg–Mac Lane space) we have

$$E_{M\times I,\overline{h'}} \circ E_{M\times I,h} = E_{M\times I,h\cup\overline{h'}} = Id.$$

Using Lemma 2.2 we conclude

$$E_{M\times I,h} = E^{-1}_{M\times I,\overline{h}} = E_{M\times I,h'}.$$

Hence the isomorphism given earlier is independent of the choice of homotopy finishing the proof. $\square$

This proposition means that given $\alpha \in \mathcal{F}_M$ we can define a one-dimensional vector space $L_{M,\alpha}$ by identifying the $L_{M,\gamma}$ given by the HQFT using the canonical isomorphisms earlier, i.e., denoting the isomorphisms earlier by $\sim$ set

$$L_{M,\alpha} = \bigoplus_{\{\gamma \mid [\gamma]=\alpha\}} L_{M,\gamma} / \sim .$$

Next, given $\alpha_0 \in \mathcal{F}_{M_0}, \alpha_1 \in \mathcal{F}_{M_1}$ and $\nu \in \mathcal{F}_W^{\alpha_0,\alpha_1}$ we wish to define

$$E_{W,\nu}\colon L_{M_0,\alpha_0} \to L_{M_1,\alpha_1}.$$

Suppose that $\sigma\colon W \to K_A, \gamma_0\colon M_0 \to K_A$ and $\gamma_1\colon M_1 \to K_A$ are maps representing ν, α_0 and α_1, respectively. Suppose, moreover that $\gamma_0 = \sigma|_{M_0}$ and $\gamma_1 = \sigma|_{M_1}$. Courtesy of the HQFT we have a map

$$E_{W,\sigma}\colon L_{M_0,\gamma_0} \to L_{M_1,\gamma_1},$$

which induces a map

$$L_{M_0,\alpha_0} \to L_{M_1,\alpha_1}.$$

Proposition 3.3. *The induced map given earlier depends only on the homotopy class of σ.*

Proof. Let σ' be another choice with $\sigma'|_{M_0} = \gamma_0'$ and $\sigma'|_{M_1} = \gamma_1'$. To prove the proposition, we must show that the following diagram commutes.

$$\begin{tikzcd}
L_{M_0,\gamma_0} \arrow[r, "E_{W,\sigma}"] \arrow[d, "c_{\gamma_0,\gamma_0'}"'] & L_{M_1,\gamma_1} \arrow[d, "c_{\gamma_1,\gamma_1'}"] \\
L_{M_0,\gamma_0'} \arrow[r, "E_{W,\sigma'}"'] & L_{M_1,\gamma_1'}
\end{tikzcd}$$

(the vertical maps are the canonical isomorphisms earlier). Let H be a homotopy from σ to σ' and let $h_0 = H|_{M_0\times I}$ and $h_1 = H|_{M_1\times I}$. Note that h_0 is a homotopy from γ_0 to γ_0' and that h_1 is a homotopy from γ_1 to γ_1'. Consider

$$W' = (M_0 \times I) \cup_{M_0} W \cup_{M_1} (M_1 \times I)$$

and let $g\colon W' \to K_A$ be defined by $g = h_0 \cup \sigma' \cup h_1^{-1}$. Using the HQFT and its properties we get

$$E_{W',g} = E_{M_1\times I,h_1^{-1}} \circ E_{W,\sigma'} \circ E_{M_0\times I,h_0} = c^{-1}_{\gamma_1,\gamma_1'} \circ E_{W,\sigma'} \circ c_{\gamma_0,\gamma_0'}.$$

Hence, in order to show that the previous diagram commutes, we need to show that $E_{W,\sigma} = E_{W',g}$.

Define $H_0\colon M_0 \times I \times I \to K_A$ by $H_0(x,s,t) = h_0(x, s(1-t))$ and define $H_1\colon M_1 \times I \times I \to K_A$ by $H_1(x,s,t) = h_1(x,(1-s)(1-t))$. Note that H_0 provides a homotopy between h_0 and κ_{γ_0}, where the latter is defined by

$\kappa_{\gamma_0}(x,s) = \gamma_0(x)$. Similarly, H_1 provides a homotopy between h_1^{-1} and κ_{γ_1}. Now define a map

$$\mathcal{H}\colon W' \times I = ((M_0 \times I) \cup_{M_0} W \cup_{M_1} (M_1 \times I)) \times I \to K_A$$

by $\mathcal{H} = H_0 \cup H^{-1} \cup H_1$. This map provides a homotopy between g and $f = \kappa_{\gamma_0} \cup \sigma \cup \kappa_{\gamma_1}$. Moreover, it is readily checked that on the boundary $\mathcal{H}$ is $\kappa_{\gamma_0} \sqcup \kappa_{\gamma_1}$ which is independent of t. Thus $\mathcal{H}$ provides a homotopy rel $\partial W'$ from g to f and hence, by the properties of an HQFT (see Turaev's axiom 1.2.8) we have

$$E_{W',g} = E_{W',f}.$$

There is a diffeomorphism $T\colon W' \to W$ making the following diagram commute.

$$\begin{tikzcd} W' \arrow[rr, "T"] \arrow[dr, "f"'] & & W \arrow[dl, "\sigma"] \\ & K_A & \end{tikzcd}$$

Hence, by Turaev's axiom 1.2.4 the following diagram commutes.

$$\begin{tikzcd} L_{M_0,\gamma_0} \arrow[r, "E_{W',f}"] \arrow[d, "id"'] & L_{M_1,\gamma_1} \arrow[d, "id"] \\ L_{M_0,\gamma_0} \arrow[r, "E_{W,\sigma}"'] & L_{M_1,\gamma_1} \end{tikzcd}$$

Thus, $E_{W,\sigma} = E_{W',f} = E_{W',g}$ which finishes the proof. □

Given $\alpha_0 \in \mathcal{F}_{M_0}$, $\alpha_1 \in \mathcal{F}_{M_1}$ and $\nu \in \mathcal{F}_W^{\alpha_0,\alpha_1}$, the earlier proposition shows that we have a well-defined map

$$E_{W,\nu}\colon L_{M_0,\alpha_0} \to L_{M_1,\alpha_1}$$

defined by $E_{W,\nu} = E_{W,\sigma}$, where σ is any representative of the class ν. As a corollary of Lemma 2.2 we have

$$E_{\overline{W},\nu} = E_{W,\nu}^{-1}. \tag{3.1}$$

If W is a closed manifold and $\sigma \in \mathcal{F}_W$ then $\sigma\colon W \to K_A$ may be considered as the classifying map of a principal A-bundle. The invariant $E_{W,\sigma}(1) \in \mathbb{C}^\times$ should correspond to Turaev's invariant $\tau_{\mathcal{C}}(W,\sigma)$ constructed (via surgery) in [18], where $\mathcal{C}$ is the modular A-category constructed from the cocycle θ.

3.2 Measures and abelian homotopy DW theory

In order to construct our analogue of matrix elements we need to integrate and in order to integrate we need to put measures on our spaces of fields.

We have identified $\mathcal{F}_W$ as the product of a number of copies of A and a discrete abelian group of torsion. Thus, we can equip $\mathcal{F}_W$ with the normalized Haar measure which we denote by μ_W. Since, A is the product of a torus and a finite abelian group we have that $\mathcal{F}_W$ is also the product of a torus T and a finite abelian group B and it follows by the defining properties of the Haar measure that the Haar measure on $\mathcal{F}_W$ is nothing but the product of the Lebesgue measure on T and the counting measure on B (normalized). For details on the Haar measure and the associated Haar integral we refer to [9, Chap. VIII], [11] and [12, Chap. 6]. Let us just remark here that the left invariant Haar measures on a Lie group G (which all differ by a scalar) are Borel measures, i.e., they are measures on the σ-algebra of all Borel sets of G. Moreover, if G is abelian or compact then the normalized left and right invariant Haar measures coincide and are just called the normalized Haar measure on G (see e.g., [9, Corollary 8.31] or [11, p. 81]).

Let us now define a measure on each of the spaces $\mathcal{F}_W^{\alpha_0,\alpha_1}$. It is tempting to define this measure using the restriction of the measure on $\mathcal{F}_W$, but we will not do this because it would yield the zero measure, whenever $\mathcal{F}_W^{\alpha_0,\alpha_1}$ has measure zero in $\mathcal{F}_W$. Instead, we will use the group structure of $\mathcal{F}_W$ to define a normalized measure on each of the $\mathcal{F}_W^{\alpha_0,\alpha_1}$. Firstly, note that $\mathcal{F}_W^{0,0}$ is a subgroup of $\mathcal{F}_W$ and being closed it is in fact a compact Lie subgroup of $\mathcal{F}_W$, hence we can endow $\mathcal{F}_W^{0,0}$ with its normalized Haar measure, which we will denote by $\mu_W^{0,0}$.

The set $\mathcal{F}_W^{\alpha_0,\alpha_1}$ is either empty or a coset of $\mathcal{F}_W^{0,0}$, hence the measure $\mu_W^{0,0}$ induces a normalized measure $\mu_W^{\alpha_0,\alpha_1}$ on $\mathcal{F}_W^{\alpha_0,\alpha_1}$, namely $\mu_W^{\alpha_0,\alpha_1}$ is nothing but the image measure of $\mu_W^{0,0}$ under the translation with an arbitrary element of $\mathcal{F}_W^{\alpha_0,\alpha_1}$. That is

$$\mu_W^{\alpha_0,\alpha_1}(S) = \mu_W^{0,0}(S - \nu)$$

for any $\nu \in \mathcal{F}_W^{\alpha_0,\alpha_1}$. (By translation invariance of the Haar measure this does not depend on the choice of ν.) We use here and in the following the standard definition of image measures. That is, given a measurable function $f\colon X \to Y$ between two measurable spaces and given a measure μ on X, we define the image measure of μ under f to be the measure ν on Y given by $\nu(S) = \mu(f^{-1}(S))$ for any measurable subset S of Y. It is then a standard result in integration theory that if $g\colon Y \to \mathbb{C}$ is measurable, then

$$\int_{y\in Y} g(y)\, d\nu = \int_{x\in X} g(f(x))\, d\mu \tag{3.2}$$

in the sense that if one of the integrals exists, then so does the other and the two integrals are equal.

If the measure of $\mathcal{F}_W^{0,0}$ in $\mathcal{F}_W$ is non-zero, then it follows from the defining properties of the Haar measure that the measure $\mu_W^{0,0}$ is nothing but the normalization of the measure obtained by restriction, hence the same is true for the measures $\mu_W^{\alpha_0,\alpha_1}$ in that case.

We now define homomorphisms

$$\mathbb{K}_W(\alpha_0,\alpha_1)\colon L_{M_0,\alpha_0} \to L_{M_1,\alpha_1}$$

by

$$\mathbb{K}_W(\alpha_0,\alpha_1)(a_0) = \int_{\nu\in\mathcal{F}_W^{\alpha_0,\alpha_1}} E_{W,\nu}(a_0)d\mu_W^{\alpha_0,\alpha_1}. \tag{3.3}$$

By convention, if $\mathcal{F}_W^{\alpha_0,\alpha_1} = \emptyset$, we take $\mathbb{K}_W(\alpha_0,\alpha_1)$ to be the zero map. Of course, for the integral in the definition to make sense, we must insist that the function $\mathcal{F}_W^{\alpha_0,\alpha_1} \to L_{M_1,\alpha_1}$ given by $\nu \mapsto E_{W,\nu}(a_0)$ is integrable. If all such functions are indeed integrable (and hence the $\mathbb{K}_W(\alpha_0,\alpha_1)$ are defined) we will refer to the defining primitive cohomological HQFT as *integrable.*

So, we start with a primitive cohomological HQFT based on a cocycle $\theta \in C^{n+1}(K_A; U(1))$ and define the associated *abelian homotopy DW theory* to consist of the assignments above. Namely,

- to each closed, oriented n-manifold M and $\alpha \in \mathcal{F}_M$, we assign the line $L_{M,\alpha}$,
- to each $n+1$-cobordism (M_0, W, M_1) and $(\alpha_0,\alpha_1) \in \mathcal{F}_{M_0} \times \mathcal{F}_{M_1}$, we assign the linear map

$$\mathbb{K}_W(\alpha_0,\alpha_1)\colon L_{M_0,\alpha_0} \to L_{M_1,\alpha_1}.$$

3.3 Invariants of closed manifolds

A closed oriented $n+1$-manifold may be regarded as a cobordism from $\emptyset$ to $\emptyset$ and thus $\mathbb{K}_W(\emptyset,\emptyset)$ is a map from $\mathbb{C}$ to $\mathbb{C}$. The DW-invariant of W is defined to be the image of 1 i.e.,

$$K_W(\emptyset,\emptyset)(1) = \int_{\nu\in\mathcal{F}_W} E_{W,\nu}(1)\, d\mu_W.$$

Note that for a given $\sigma\colon W \to K_A$ the map $E_{W,\sigma}\colon \mathbb{C} \to \mathbb{C}$ is given by $1 \mapsto \sigma^*\theta(f)$, where $f \in C_{n+1}W$ is a fundamental cycle for W. It follows that $E_{W,\sigma}(1)$ is a function of the cohomology class of θ only. Thus writing $\langle -, -\rangle$ for the evaluation map $H^{n+1}(W; U(1)) \otimes H_{n+1}(M;\mathbb{Z}) \to U(1)$ we get

$$K_W(\emptyset,\emptyset)(1) = \int_{\nu\in\mathcal{F}_W} \langle \nu^*([\theta]), [W]\rangle\, d\mu_W.$$

Writing $Z_A^{[\theta]}(W)$ for $K_W(\emptyset,\emptyset)(1)$ this is the expression (1.1) in the introduction. If we are given a group cocycle we will sometimes use the notation $Z_A^{[\omega]}$ instead.

Note that if W and W' are two closed, oriented $n+1$-manifolds then

$$Z_A^{[\theta]}(W \sqcup W') = Z_A^{[\theta]}(W) Z_A^{[\theta]}(W').$$

Example 3.4. *Spheres.* Let $\theta \in C^{n+1}(K_A; U(1))$ be a cocycle. For $n > 0$ we have $\mathcal{F}_{S^{n+1}} = H^1(S^{n+1}; A) = \{0\}$ and so

$$\begin{aligned} Z_A^{[\theta]}(S^{n+1}) &= \int_{\nu\in\mathcal{F}_{S^{n+1}}} \langle \nu^*([\theta]), [S^{n+1}]\rangle\, d\mu_{S^{n+1}} \\ &= \langle 0^*([\theta]), [S^{n+1}]\rangle = \langle 1, [S^{n+1}]\rangle = 1. \end{aligned}$$

For $n = 0$ suppose we have a corresponding group cocycle $\omega\colon A \to U(1)$. Noting that $\mathcal{F}_{S^1} = A$ we have

$$\begin{aligned} Z_A^{[\theta]}(S^1) &= \int_{\nu\in\mathcal{F}_{S^1}} \langle \nu^*([\theta]), [S^1]\rangle\, d\mu_{S^1} = \int_{\nu\in\mathcal{F}_{S^1}} \langle \theta, \nu_*[S^1]\rangle\, d\mu_{S^1} \\ &= \int_{a\in A} \omega(a)\, d\mu_{S^1}. \end{aligned}$$

Note here that the 1-cocycle ω is just a one-dimensional representation of A. Hence, the integral equals 0 unless ω is the trivial representation, in which case the integral equals 1.

4 Properties of abelian homotopy DW theory

4.1 Decompositions

In this section, we discuss the restricted version of locality satisfied by abelian homotopy DW theory. Suppose that we can decompose an $n+1$-cobordism (M_0, W, M_1) into two pieces W' and W'' along a connected n-manifold, M. Given such a decomposition and given a pair $(\alpha_0, \alpha_1) \in \mathcal{F}_{M_0} \times \mathcal{F}_{M_1}$, we define the space of *supporting fields* to be

$$\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1} = \{\alpha \in \mathcal{F}_M \mid \mathcal{F}_{W'}^{\alpha_0,\alpha} \times \mathcal{F}_{W''}^{\alpha,\alpha_1} \neq \emptyset\}.$$

Note that this depends on the decomposition. In this subsection, we construct a measure $\overline{\mu}_M^{\alpha_0,\alpha_1}$ on the space of supporting fields and we prove the following theorem.

Theorem 4.1. *Suppose we can decompose W as $W = W' \cup_M W''$, where M is a connected n-manifold and $W' \cap W'' = M$. Then for $\alpha_0 \in \mathcal{F}_{M_0}$ and $\alpha_1 \in \mathcal{F}_{M_1}$ we have*

$$\mathbb{K}_W(\alpha_0, \alpha_1)(x) = \int_{\alpha \in \overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}} \mathbb{K}_{W''}(\alpha, \alpha_1) \circ \mathbb{K}_{W'}(\alpha_0, \alpha)(x) d\bar{\mu}_M^{\alpha_0,\alpha_1}.$$

Before proving this theorem, we need to construct the measure $\overline{\mu}_M^{\alpha_0,\alpha_1}$ on $\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$. The connectedness of M which will be essential in the proof of this theorem will not be needed for the construction of the measure, so to begin with, we will not assume that M is connected.

By Lemma 3.1, we have a continuous (restriction) map $r_M^W \colon \mathcal{F}_W \to \mathcal{F}_M$, which restricts to a continuous surjection $r = r^{\alpha_0,\alpha_1} \colon \mathcal{F}_W^{\alpha_0,\alpha_1} \to \overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$. To see that r is surjective, apply the Mayer–Vietoris sequence for the triad $(W; W', W'')$, i.e., the exact sequence

$$\cdots \longrightarrow \tilde{H}^0(M) \longrightarrow H^1(W) \xrightarrow{a} H^1(W') \oplus H^1(W'') \xrightarrow{b} H^1(M) \longrightarrow \cdots, \tag{4.1}$$

where $a(\nu) = (r_{W'}^W(\nu), r_{W''}^W(\nu))$ and $b(\nu', \nu'') = r_M^{W'}(\nu') - r_M^{W''}(\nu'')$ (all cohomology groups having coefficients in A). In particular, $\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$ is a closed subset of $\mathcal{F}_M$. Given $\alpha \in \overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$ we will denote $r^{-1}(\alpha) \subset \mathcal{F}_W^{\alpha_0,\alpha_1}$ by $\mathcal{F}_W^{\alpha_0,\alpha,\alpha_1}$. We note that $\mathcal{F}_W^{0,0,0}$ is a compact Lie subgroup of $\mathcal{F}_W^{0,0}$.

As with the measures on the spaces $\mathcal{F}_W^{\alpha_0,\alpha_1}$ it turns out that one should not take the measure induced from the obvious inclusion (in this case into $\mathcal{F}_M$). To begin, let us instead note that $\overline{\mathcal{F}}_M^{0,0}$ is a Lie subgroup of $\mathcal{F}_M$ (as a closed subgroup of $\mathcal{F}_M$). We therefore let $\overline{\mu}_M^{0,0}$ be the normalized Haar measure on $\overline{\mathcal{F}}_M^{0,0}$. Next observe that $\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$ is either empty or a coset of $\overline{\mathcal{F}}_M^{0,0}$, hence we can (similar to the construction of $\mu_W^{\alpha_0,\alpha_1}$ in Section. 3.2) define $\overline{\mu}_M^{\alpha_0,\alpha_1}$ to be the image measure of $\overline{\mathcal{F}}_M^{0,0}$ under the translation by any element of $\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$.

Thinking about this quiet differently let $\pi = r^{0,0}\colon \mathcal{F}_W^{0,0} \to \overline{\mathcal{F}}_M^{0,0}$, which is a surjective Lie group homomorphism, which in turn induces a Lie group isomorphism $\bar{\pi}\colon \mathcal{F}_W^{0,0}/\mathcal{F}_W^{0,0,0} \to \overline{\mathcal{F}}_M^{0,0}$. Let $\bar{\mu}$ be the normalized Haar measure on the quotient $\mathcal{F}_W^{0,0}/\mathcal{F}_W^{0,0,0}$. Then $\bar{\mu}$ is also the image measure of $\mu_W^{0,0}$ under the canonical projection and $\overline{\mu}_M^{0,0}$ is the image measure of $\bar{\mu}$ under $\bar{\pi}$ and also the image measure of $\mu_W^{0,0}$ under π. We use here the obvious fact that if $f\colon G \to H$ is a surjective Lie group homomorphism and if μ_G and μ_H are the normalized left invariant Haar measures on respectively G and H, then μ_H equals the image measure of μ_G under f.

By (3.2) we then get

$$\int_{\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}} f d\bar{\mu}_M^{\alpha_0,\alpha_1} = \int_{\nu\in\mathcal{F}_W^{0,0}} f(\pi(\nu) + \rho_{\alpha_0,\alpha_1})\, d\mu_W^{0,0}, \tag{4.2}$$

for an integrable function f on $\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$, where ρ_{α_0,α_1} is an arbitrary element of $\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$. Moreover, if f is an integrable function on $\mathcal{F}_W^{0,0}$ we have

$$\int_{\mathcal{F}_W^{0,0}} f\, d\mu_W^{0,0} = \int_{p(g)\in\mathcal{F}_W^{0,0}/\mathcal{F}_W^{0,0,0}} \left(\int_{h\in\mathcal{F}_W^{0,0,0}} f(g+h)\, d\mu_W^{0,0,0} \right) d\bar{\mu}, \tag{4.3}$$

where p is the canonical projection and $\mu_W^{0,0,0}$ the normalized Haar measure on $\mathcal{F}_W^{0,0,0}$. We write things additively since we deal with abelian groups. The identity (4.3) simply follows by noting that both sides define a normalized integral which is left-invariant on the class of continuous functions (see also [3, Proposition I.5.16] and [9, Theorem 8.36] for a more general result).

Before proving the decomposition theorem we need one more result. To establish this result we must assume that M is connected. Let $a\colon \mathcal{F}_W \to \mathcal{F}_{W'} \times \mathcal{F}_{W''}$ be the continuous restriction map from the Mayer–Vietoris sequence (4.1).

Lemma 4.2. *Assume that M is connected. The map a restricts to a bijection*

$$\mathcal{F}_W^{\alpha_0,\alpha,\alpha_1} \cong \mathcal{F}_{W'}^{\alpha_0,\alpha} \times \mathcal{F}_{W''}^{\alpha,\alpha_1}$$

for any $\alpha_0 \in \mathcal{F}_{M_0}$, $\alpha_1 \in \mathcal{F}_{M_1}$ and $\alpha \in \overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}$. In particular, we have a Lie group isomorphism

$$\mathcal{F}_W^{0,0,0} \cong \mathcal{F}_{W'}^{0,0} \times \mathcal{F}_{W''}^{0,0}.$$

Proof. Let $\alpha_0 \in \mathcal{F}_{M_0}$ and $\alpha_1 \in \mathcal{F}_{M_1}$ be fixed. The map a clearly maps $\mathcal{F}_W^{\alpha_0,\alpha,\alpha_1}$ into $\mathcal{F}_{W'}^{\alpha_0,\alpha} \times \mathcal{F}_{W''}^{\alpha,\alpha_1}$. Consider the Mayer–Vietoris sequence (4.1). Since M is connected, a injects. Assume $(\nu',\nu'') \in \mathcal{F}_{W'}^{\alpha_0,\alpha} \times \mathcal{F}_{W''}^{\alpha,\alpha_1}$. Then $r_M^{W'}(\nu') = \alpha = r_M^{W''}(\nu'')$ so $(\nu',\nu'') \in \mathrm{Ker}(b)$. Hence there exists a $\nu \in \mathcal{F}_W$ such that $a(\nu) = (\nu',\nu'')$, and by the very definition of $\mathcal{F}_W^{\alpha_0,\alpha,\alpha_1}$ we see that $\nu \in \mathcal{F}_W^{\alpha_0,\alpha,\alpha_1}$. □

The bijections in the above lemma will all be denoted by a. We now prove Theorem 4.1.

Proof of Theorem 4.1. There is only something to prove in case $\mathcal{F}_W^{\alpha_0,\alpha_1}$, is non-empty, so this we assume in what follows. Let us start by introducing some notation. The statement of the theorem is for the linear maps $\mathbb{K}_W(\alpha_0,\alpha_1)$, but we can choose fixed basis vectors in the lines associated with M_0, M_1 and M and then replace these linear maps by their matrix elements $k_W, k_{W'}, k_{W''}$, which are just functions of the boundary configurations α_0, α and α_1. Similarly, we introduce the notation e_W, $e_{W'}$ and $e_{W''}$ for the matrix elements of the linear maps $E_{W,\sigma}$ that are integrated to give the maps $\mathbb{K}_W$. This means e_W is a function on $\mathcal{F}_W$ and analogously for $e_{W'}$ and $e_{W''}$. With this notation, we have (letting ρ_{α_0,α_1} be an arbitrary element of $\mathcal{F}_W^{\alpha_0,\alpha_1}$)

$$\begin{aligned} k_W(\alpha_0,\alpha_1) &= \int_{\nu\in\mathcal{F}_W^{\alpha_0,\alpha_1}} e_W(\nu)\, d\mu_W^{\alpha_0,\alpha_1} = \int_{\nu\in\mathcal{F}_W^{0,0}} e_W(\nu+\rho_{\alpha_0,\alpha_1})\, d\mu_W^{0,0} \\ &= \int_{p(\nu)\in\mathcal{F}_W^{0,0}/\mathcal{F}_W^{0,0,0}} \left(\int_{\sigma\in\mathcal{F}_W^{0,0,0}} e_W(\sigma+\nu+\rho_{\alpha_0,\alpha_1})\, d\mu_W^{0,0,0} \right) d\bar{\mu}, \end{aligned}$$

where the final equality follows by (4.3). Next we apply our Lie group isomorphism a from Lemma 4.2 to get

$$\begin{aligned} &\int_{\sigma\in\mathcal{F}_W^{0,0,0}} e_W(\sigma+\nu+\rho_{\alpha_0,\alpha_1})\, d\mu_W^{0,0,0} \\ &\qquad = \int_{(\sigma',\sigma'')\in\mathcal{F}_{W'}^{0,0}\times\mathcal{F}_{W''}^{0,0}} e_W(a^{-1}(\sigma',\sigma'')+\nu+\rho_{\alpha_0,\alpha_1})\, d\mu_{W'}^{0,0}\oplus\mu_{W''}^{0,0} \end{aligned}$$

noting that the product measure $\mu_{W'}^{0,0}\oplus\mu_{W''}^{0,0}$ is the normalized Haar measure on the product Lie group $\mathcal{F}_{W'}^{0,0}\times\mathcal{F}_{W''}^{0,0}$, hence the image measure of $\mu_W^{0,0,0}$ under a. Using the map $a\colon \mathcal{F}_W \to \mathcal{F}_{W'}\times\mathcal{F}_{W''}$ to write $(\nu',\nu'') = a(\nu) \in \mathcal{F}_{W'}^{0,\alpha_\nu}\times\mathcal{F}_{W''}^{\alpha_\nu,0}$ for $\nu\in\mathcal{F}_W^{0,0}$ and $(\rho'_{\alpha_0,\alpha_1},\rho''_{\alpha_0,\alpha_1}) = a(\rho_{\alpha_0,\alpha_1}) \in \mathcal{F}_{W'}^{\alpha_0,\beta}\times\mathcal{F}_{W''}^{\beta,\alpha_1}$, where $\beta = r_M^W(\rho_{\alpha_0,\alpha_1})$ and $\alpha_\nu = r_M^W(\nu)$, we get

$$e_W(a^{-1}(\sigma', \sigma'') + \nu + \rho_{\alpha_0,\alpha_1}) = e_{W'}(\sigma' + \nu' + \rho'_{\alpha_0,\alpha_1})e_{W''}(\sigma'' + \nu'' + \rho''_{\alpha_0,\alpha_1})$$

by the HQFT gluing property, hence

$$\begin{aligned} k_W(\alpha_0, \alpha_1) = \int_{p(\nu)\in\mathcal{F}_W^{0,0}/\mathcal{F}_W^{0,0,0}} &\left(\int_{\sigma'\in\mathcal{F}_{W'}^{0,0}} e_{W'}(\sigma' + \nu' + \rho'_{\alpha_0,\alpha_1})\, d\mu_{W'}^{0,0}\right) \\ &\times \left(\int_{\sigma''\in\mathcal{F}_{W''}^{0,0}} e_{W''}(\sigma'' + \nu'' + \rho''_{\alpha_0,\alpha_1})\, d\mu_{W''}^{0,0}\right) d\bar{\mu}, \end{aligned}$$

by Fubini's theorem. Here

$$\begin{aligned} \int_{\sigma'\in\mathcal{F}_{W'}^{0,0}} e_{W'}(\sigma' + \nu' + \rho'_{\alpha_0,\alpha_1})\, d\mu_{W'}^{0,0} &= \int_{x\in\mathcal{F}_{W'}^{\alpha_0,\alpha_\nu+\beta}} e_{W'}(x)\, d\mu_{W'}^{\alpha_0,\alpha_\nu+\beta} \\ &= k_{W'}(\alpha_0, \alpha_\nu + \beta) \end{aligned}$$

and

$$\begin{aligned} \int_{\sigma''\in\mathcal{F}_{W''}^{0,0}} e_{W''}(\sigma'' + \nu'' + \rho''_{\alpha_0,\alpha_1})\, d\mu_{W''}^{0,0} &= \int_{x\in\mathcal{F}_{W''}^{\alpha_\nu+\beta,\alpha_1}} e_{W''}(x)\, d\mu_{W''}^{\alpha_\nu+\beta,\alpha_1} \\ &= k_{W''}(\alpha_\nu + \beta, \alpha_1). \end{aligned}$$

Therefore, since $\alpha_\nu = \bar{\pi}(p(\nu))$,

$$k_W(\alpha_0, \alpha_1) = \int_{p(\nu)\in\mathcal{F}_W^{0,0}/\mathcal{F}_W^{0,0,0}} k_{W'}(\alpha_0, \beta + \bar{\pi}(p(\nu)))k_{W''}(\beta + \bar{\pi}(p(\nu)), \alpha_1)\, d\bar{\mu}.$$

By (3.2) and the remarks above (4.2) we then get

$$\begin{aligned} k_W(\alpha_0, \alpha_1) &= \int_{\alpha\in\overline{\mathcal{F}}_M^{0,0}} k_{W'}(\alpha_0, \beta + \alpha)k_{W''}(\beta + \alpha, \alpha_1)\, d\bar{\mu}_M^{0,0} \\ &= \int_{\alpha\in\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1}} k_{W'}(\alpha_0, \alpha)k_{W''}(\alpha, \alpha_1)\, d\bar{\mu}_M^{\alpha_0,\alpha_1} \end{aligned}$$

which is the desired result. □

We end this section with an important corollary to Theorem 4.1.

Corollary 4.3. *In the set up of* Theorem 4.1 *suppose that* $n > 1$ *and moreover that* $H_1(M; \mathbb{Z}) = \{0\}$. *Then either* $\mathbb{K}_W(\alpha_0, \alpha_1)$ *is trivial or*

$$\mathbb{K}_W(\alpha_0, \alpha_1) = \mathbb{K}_{W''}(0, \alpha_1) \circ \mathbb{K}_{W'}(\alpha_0, 0).$$

Proof. Follows immediately from the fact that $\overline{\mathcal{F}}_M^{\alpha_0,\alpha_1} \subset \mathcal{F}_M = \{0\}$. □

4.2 Connected sums

The decomposition theorem of the previous section allows us to calculate invariants of connected sums. First we need the following.

Lemma 4.4. *If D is an $n+1$-disk with ingoing boundary sphere then*

$$\mathbb{K}_{\overline{D}}(\emptyset, 0) \circ \mathbb{K}_D(0, \emptyset) = \mathrm{Id}\,.$$

Proof. Let a be a (representative of a) generator of $L_{S^n,0}$ and note that since D is contractible $\mathcal{F}_D^{0,\emptyset} = \{0\}$. Thus $\mathbb{K}_D(0, \emptyset)(a) = E_{D,\sigma}(a)$, where $[\sigma] = 0$ and similarly $\mathbb{K}_{\overline{D}}(\emptyset, 0)(1) = E_{\overline{D},\sigma}(1)$. Thus using Lemma 2.2 we have

$$\mathbb{K}_{\overline{D}}(\emptyset, 0)(\mathbb{K}_D(0, \emptyset)(a)) = \mathbb{K}_{\overline{D}}(\emptyset, 0)(E_{D,\sigma}(a)) = E_{\overline{D},\sigma}(E_{D,\sigma}(a)) = a.$$

□

Now for the result on connected sums.

Proposition 4.5. *If W' and W'' are closed, oriented connected $n+1$-manifolds then*

$$Z_A^{[\theta]}(W'\#W'') = Z_A^{[\theta]}(W')Z_A^{[\theta]}(W'').$$

Proof. Let $(\emptyset, V', S^n)$ be the $n+1$-cobordism obtained from W' by removing an $n+1$-disk D (creating a new outgoing boundary component), and similarly let $(S^n, V'', \emptyset)$ be the cobordism obtained from W'' again by removing an $n+1$-disk (this time creating a new incoming boundary component). We can then write $W'\#W'' = V' \cup_{S^n} V''$.

Also note that $W' = V' \cup_{S^n} D$. For this decomposition observe that $\overline{\mathcal{F}}_{S^n}^{\emptyset,\emptyset} = \{0\}$. This is immediate for $n > 1$ and for $n = 1$ we note that V' has the homotopy type of a wedge of circles and that the restriction map to $\mathcal{F}_{S^1}$ is given by a commutator map which is trivial, since A is abelian and thus $\mathcal{F}_{V'}^{\emptyset,\alpha}$ is only non-empty when $\alpha = 0$. Using this we see

$$Z_A^{[\theta]}(W') = \mathbb{K}_{V' \cup_{S^n} D}(\emptyset, \emptyset)(1) = \mathbb{K}_D(0, \emptyset) \circ \mathbb{K}_{V'}(\emptyset, 0)(1).$$

Similarly, $Z_A^{[\theta]}(W'') = \mathbb{K}_{V''}(0, \emptyset) \circ \mathbb{K}_{\overline{D}}(\emptyset, 0)(1)$. Thus by applying Theorem 4.1 we have

$$\begin{aligned}
Z_A^{[\theta]}(W' \# W'') &= \mathbb{K}_{V' \cup_{S^n} V''}(\emptyset, \emptyset)(1) \\
&= \int_{\alpha \in \overline{\mathcal{F}}_{S^n}^{\emptyset,\emptyset}} \mathbb{K}_{V''}(\alpha, \emptyset) \circ \mathbb{K}_{V'}(\emptyset, \alpha)(1)\, d\overline{\mu}_{S^n}^{\emptyset,\emptyset} \\
&= \mathbb{K}_{V''}(0, \emptyset) \circ \mathbb{K}_{V'}(\emptyset, 0)(1) \quad \text{since } \overline{\mathcal{F}}_{S^n}^{\emptyset,\emptyset} = \{0\} \\
&= \mathbb{K}_{V''}(0, \emptyset) \circ \mathrm{Id} \circ \mathbb{K}_{V'}(\emptyset, 0)(1) \\
&= \mathbb{K}_{V''}(0, \emptyset) \circ \mathbb{K}_{\overline{D}}(\emptyset, 0) \circ \mathbb{K}_D(0, \emptyset) \circ \mathbb{K}_{V'}(\emptyset, 0)(1) \\
&= Z_A^{[\theta]}(W') Z_A^{[\theta]}(W'').
\end{aligned}$$

The second to last equality is courtesy of Lemma 4.4. □

4.3 Invariants of products

In this section, we discuss the calculation of the invariants of the product of two closed manifolds. Let W and W' be closed, oriented and connected of dimensions $m+1$ and $n+1$, respectively.

Lemma 4.6. *There is an identification of measure spaces*

$$\mathcal{F}_{W \times W'} \cong \mathcal{F}_W \times \mathcal{F}_{W'}.$$

Proof. This follows from the fact that $H_1(W \times W'; \mathbb{Z}) \cong H_1(W; \mathbb{Z}) \oplus H_1(W'; \mathbb{Z})$ and from the fact that all of these field spaces are given the normalized Haar measure. □

Since K_A is a H-space there is a Pontrjagin slant product

$$\backslash \colon H^{m+n+2}(K_A; U(1)) \otimes H_{m+1}(K_A; \mathbb{Z}) \to H^{n+1}(K_A; U(1)).$$

If $[W] \in H_{m+1}(W; \mathbb{Z})$ is the fundamental class then given $\nu \in \mathcal{F}_W$, we have $\nu_*[W] \in H_{m+1}(K_A; \mathbb{Z})$.

Theorem 4.7. *Let $[\theta] \in H^{m+n+2}(K_A; U(1))$. Then*

$$Z_A^{[\theta]}(W \times W') = \int_{\nu \in \mathcal{F}_W} Z^{[\theta] \backslash \nu_*[W]}(W')\, d\mu_W.$$

Proof. First recall that the slant product satisfies

$$\langle a, b \bullet c\rangle = \langle a \setminus b, c\rangle,$$

where $\bullet$ denotes the Pontrjagin product. Thus for $v = (\nu, \nu') \in \mathcal{F}_{W\times W'} \cong \mathcal{F}_W \times \mathcal{F}_{W'}$ we have

$$\begin{aligned}\langle v^*[\theta], [W] \times [W']\rangle &= \langle [\theta], \nu_*[W] \bullet \nu'_*[W']\rangle \\ &= \langle [\theta] \setminus \nu_*[W], \nu'_*[W']\rangle = \langle \nu'^*([\theta] \setminus \nu_*[W]), [W']\rangle.\end{aligned}$$

Hence

$$\begin{aligned}Z_A^{[\theta]}(W \times W') &= \int_{v\in\mathcal{F}_{W\times W'}} \langle v^*[\theta], [W] \times [W']\rangle \, d\mu_{W\times W'} \\ &= \int_{\nu\in\mathcal{F}_W} \int_{\nu'\in\mathcal{F}_{W'}} \langle \nu'^*([\theta] \setminus \nu_*[W]), [W']\rangle \, d\mu_{W'} d\mu_W \\ &= \int_{\nu\in\mathcal{F}_W} Z^{[\theta]\setminus\nu_*[W]}(W') \, d\mu_W.\end{aligned}$$

□

Example 4.8. The product $M \times N$ where M is simply connected. *Let M and N be closed manifolds of dimension m and n, respectively, and let $\theta \in C^{m+n}(K_A; U(1))$ be a cocycle. For $0 \in H_m(K_A; \mathbb{Z})$, we have $[\theta] \setminus 0$ trivial so*

$$Z_A^{[\theta]}(M \times N) = \int_{\nu\in\mathcal{F}_M} Z_A^{[\theta]\setminus\nu_*[M]}(N) d\mu_M = Z_A^{[\theta]\setminus 0}(N) = 1.$$

Example 4.9. The product $S^1 \times M$. *Let M be an n-manifold and let $\omega\colon A^{n+1} \to U(1)$ be a group cocycle corresponding to $\theta \in C^{n+1}(K_A; U(1))$. Noting that $H_1(K_A; U(1)) \cong A$ the slant product takes the form*

$$\setminus\colon H^{n+1}(K_A; U(1)) \otimes A \to H^n(K_A; U(1))$$

and may be described in terms of group cohomology as follows. For $a \in A$ the slant product $\omega \setminus a\colon A^n \to U(1)$ is given by

$$(\omega \setminus a)(g_1, \ldots, g_n) = \prod_{i=0}^{n} \omega(g_1, \ldots, g_i, a, g_{i+1}, \ldots, g_n)^{(-1)^{\lambda_i}}. \tag{4.4}$$

In the previous expression λ_i is the sign of the permutation taking $(g_1, \ldots, g_n, a)$ to $(g_1, \ldots, g_i, a, g_{i+1}, \ldots, g_n)$. This arises by using the

Eilenberg–Zilber map given by shuffle product. Thus we have

$$Z_A^{[\omega]}(S^1 \times M) = \int_{a \in A} Z_A^{[\omega \backslash a]}(M)\, d\mu \tag{4.5}$$

and we can calculate an expression for the integrand using the expression (4.4) given earlier.

5 Calculations

Formulae such as that occurring in Theorems 4.1 and 4.7 are good tools for calculations. For example, we can fully compute all invariants in dimension $1+1$ with almost no further effort. Suppose, we have been given a normalized group 2-cocycle ω corresponding to the defining cocycle $\theta \in C^2(K_A; U(1))$. We have already computed the invariant for S^2. For T^2 we have

$$\begin{aligned} Z_A^{[\omega]}(T^2) = Z_A^{[\omega]}(S^1 \times S^1) &= \int_{a \in A} Z_A^{[\omega \backslash a]}(M)\, d\mu \quad \text{by (4.5)} \\ &= \int_{a \in A} \int_{b \in A} (\omega \setminus a)(b)\, d\mu \quad \text{by Example 3.4} \\ &= \int_{(a,b) \in A \times A} \omega(a,b)\overline{\omega}(b,a)\, d\mu \quad \text{by (4.4).} \end{aligned}$$

Finally, since a surface Σ_g of genus g is the connected sum of g tori, we use Proposition 4.5 to get

$$Z_A^{[\omega]}(\Sigma_g) = Z_A^{[\omega]}(T^2)^g.$$

One also needs to be able to make explicit calculations based on explicit choices of the various cycles and cocycles in the definitions. This takes us closer to the combinatorial view, but it is important to remember that from the point of view of this paper these are to be *deduced* not taken as definitions. This is in fact the way Dijkgraaf and Witten introduced their invariants: the path integral definition came first, followed by the combinatorial formulae used to make explicit calculations.

5.1 Δ-complexes

Everything in this section can be found elsewhere, but for convenience we reproduce the essentials. It is convenient for us to work with Δ-complexes, as defined by Hatcher [8], rather than simplicial complexes, since Δ-complexes will allow us to model manifolds with far fewer simplices.

Definition 5.1. Suppose we have a collection of simplices $\{\Delta_i\}$, together with an ordering (or numbering) of the vertices of each simplex. As a result, we also get orderings on the sets of vertices in the faces of the simplices Δ_i. We can now form a topological space by first taking the disjoint union of the Δ_i and then identifying certain chosen subsets F_j of the faces of the Δ_i using the canonical linear homeomorphisms that preserve the orderings of the vertices (all faces in a given set F_j are assumed to be of the same dimension). A space which is constructed in this way is called a Δ-*complex.*

Most of the "triangulations" of manifolds used in the existing literature on DW-invariants are in fact Δ-complexes rather than simplicial complexes. The same will apply in this paper, i.e., when we talk about a triangulation of a manifold M, we mean a Δ-complex homeomorphic to M. The main difference between a Δ-complex and a simplicial complex is that not every simplex of a Δ-complex has to be uniquely determined by the set consisting of its vertices. The numbering of the vertices in each simplex is needed to remove resulting ambiguities. Any simplicial complex can be turned into a Δ-complex by choosing an ordering of the vertices (this will induce an ordering of the vertices of each simplex). Conversely, any Δ-complex is homeomorphic to a simplicial complex, which can be constructed by subdivision of the simplices in the Δ-complex.

Homotopy classes of maps from a Δ-complex T to an Eilenberg–Mac Lane space can be understood in combinatorial terms as follows. A *colouring* of T by the group A is a map g from the set of oriented edges of T to A. If E is the oriented edge from the vertex labelled a to the vertex labelled b (with $a < b$), then we denote $g(E)$ also as g_{ab}. We will use the convention that $g_{ab} = g_{ba}^{-1}$ for all pairs of vertices a, b which are connected by an edge. Also, we impose a *flatness condition*, which requires that, for any triangle in T, the product of the colours on the boundary is unity. More precisely, denoting the vertices of the triangle by a, b and c, we require that $g_{ab}g_{bc} = g_{ac}$, or equivalently $g_{ab}g_{bc}g_{ca} = e$. We define a *gauge transformation* to be a map h from the set of vertices of T into G. We will often write h_a for $h(a)$. Gauge transformations form a group under pointwise multiplication (in fact this group is isomorphic to $G^{\mathcal{V}}$, where $\mathcal{V}$ is the number of vertices in T). The group of gauge transformations has an action on the set of colourings, given by

$$(h \cdot g)_{ab} = h_b g_{ab} (h_a)^{-1}. \tag{5.1}$$

The next proposition describes homotopy classes of maps from a Δ-complex to an Eilenberg–Mac Lane space $K_A = K(A, 1)$. Although it is well-known, we include a proof for completeness.

Proposition 5.2. *Let W be a manifold and T a triangulation of W, then the orbits of colourings of T under gauge transformations are in one to one correspondence with homotopy classes of maps from W into K_A. Moreover, homotopy classes of* based *maps from W to K_A are in one-to-one correspondence with orbits of colourings of T under gauge transformations which send a chosen vertex x_0 of T to the unit element of A.*

Proof. Start with a map $\sigma\colon W \to K_A$. After a suitable homotopy we can assume that σ maps all vertices of T to the same point of K_A. Hence all edges of T become loops in K_A, and since $\pi_1(K_A) \cong A$ we can color each edge of T with an element of A. All colourings of T induced in this way satisfy a flatness condition because the image of any triangle in K_A, and hence also the image of the loop which forms its boundary, is contractible. One should note that one may obtain different colourings of T from the same homotopy class of maps. It is easy to see why this happens. Suppose that we have two homotopic maps σ and σ' from W to K_A which both send all vertices of T to the base point for $\pi_1(K_A)$. Although σ and σ' are homotopic, the homotopy between them may move the vertices of T around non-contractible loops in K_A. If the vertex v gets moved around the loop labelled by $h \in A$, then the group elements of the edges of T which end at v get multiplied by h from the left, while the group elements on edges which begin at v get multiplied by h^{-1} from the right. This is exactly the effect of a gauge transformation at the vertex v. Thus, we do not get a well-defined map from homotopy classes of maps to colourings of T, but we do get a well-defined map from homotopy classes of maps to gauge orbits of colourings of T. This map is in fact invertible. To see injectivity, suppose that two maps σ and σ' induce the same gauge class of colourings of T. Then these maps are certainly homotopic on the 1-skeleton of T and, using the fact that K_A has trivial higher homotopy, we may extend the homotopy on the 1-skeleton to a homotopy on all of T, or W. For surjectivity, take any colouring of T satisfying the flatness condition. We may always construct a map from the 1-skeleton of T into K_A which induces this colouring and, because K_A has trivial higher homotopy, this map extends to a map from all of W to K_A. The statement about based maps follows in a similar way if we identify the base point of W with the chosen vertex x_0 of T. This vertex can now no longer be moved around K_A by homotopies and hence colourings, which differ by a non-trivial gauge transformation at x_0, do not correspond to the same homotopy class of based maps. $\square$

5.2 A formula for explicit calculation

We will assume that the HQFT defining the abelian homotopy DW theory is integrable (see Section. 3.2) and that we have a group cocycle ω corresponding to the defining singular cocycle θ. Given $\alpha_0 \in \mathcal{F}_{M_0}$ and $\alpha_1 \in \mathcal{F}_{M_1}$, we need to determine the effect of the maps $E_{W,\nu}\colon L_{M_0,\alpha_0} \to L_{M_1,\alpha_1}$ occurring in (3.3), where $\nu \in \mathcal{F}_W^{\alpha_0,\alpha_1}$. To do this, we must make some choices:

- choose a representative $\sigma\colon W \to K_A$ of the class ν,
- choose fundamental cycles $a_i \in C_n M_i$ for $i = 0, 1$ (giving generators of L_{M_i,γ_i}, where $\gamma_i = \sigma|_{M_i}$),
- choose $f \in C_{n+1}W$ representing the fundamental class in $H_{n+1}(W, \partial W)$.

Armed with these choices, we then compute $\sigma^*\theta(f)$ and hence are able to determine $E_{W,\sigma}(a_0) = \sigma^*\theta(f)a_1$.

Let us now suppose that T is a triangulation of W, which induces triangulations T_0 and T_1 of M_0 and M_1. Since T, T_0 and T_1 are Δ-complexes, they immediately give canonical representatives f, a_0 and a_1 for the fundamental classes of W, M_0 and M_1 and moreover these satisfy $\partial f = a_1 - a_0$. Explicitly, for $i = 0, 1$ we have

$$a_i = \sum_{t \in T_i} \epsilon_t [t], \tag{5.2}$$

where the sum runs over the n-simplices of T_i and $[t]$ denotes the inclusion map of the n-simplex t into T_i (i.e., the inclusion map into the set of disjoint simplices followed by the identification map). The signs ϵ_t express the orientation of the simplices when compared with that of the whole manifold. Note that the orientation of a simplex can be described in terms of the ordering of its vertices. Hence, the signs ϵ_t are fixed by the orientation of M and the chosen Δ-complex structure. Similarly, we have

$$f = \sum_{t \in T} \epsilon_t [t], \tag{5.3}$$

where here the sum is over the $n+1$-simplices of T.

Next, given $\nu \in \mathcal{F}_W$, we use Proposition 5.2 to choose a colouring of T (in general there may be many such colourings). Now define a map $\sigma\colon W \to K_A$ such that $[\sigma] = \nu$ as follows. Choose representatives for the elements of the fundamental group of K_A, or more precisely, for every $g \in A$ fix a map l_g from the standard 1-simplex onto a loop in K_A which corresponds to the element $g \in \pi_1(K_A) \cong A$. Using these, we can define σ on the 1-skeleton of the triangulation by mapping an edge labelled g into K_A by l_g. To fix σ on the 2-skeleton, one introduces standard maps from any coloured 2-simplex to K_A, such that these maps reduce to the standard maps for 1-simplices on the coloured boundary. One continues in this way for the higher skeleta until σ is defined (these map extensions are possible because K_A has trivial higher homotopy). It is clear by the proof of Proposition 5.2 that $[\sigma] = \nu$.

If t is an $n+1$-simplex in T then $\sigma^*\theta(t)$ is a function of the colouring chosen earlier and we can assume that θ and ω are related so that

$$\sigma^*\theta(t) = \omega(g^\sigma_{t,1}, \dots, g^\sigma_{t,n+1}),$$

where $g^\sigma_{t,1}, \dots, g^\sigma_{t,n+1}$ are the group elements which colour $n+1$ edges which does not lie in the same face (flatness then determines the others). We will take these $n+1$ edges to be the edges which connect the vertices of the simplex in ascending order[2] . Thus (using multiplicative notation for the group operation in $U(1)$) we have

$$\sigma^*\theta(f) = \sigma^*\theta\left(\sum_{t\in T}\epsilon_t[t]\right) = \prod_{t\in T}\sigma^*\theta(t)^{\epsilon_t} = \prod_{t\in T}\omega(g^\sigma_{t,1}, \dots, g^\sigma_{t,n+1})^{\epsilon_t}.$$

When W is closed, the number $\sigma^*\theta(f)$ does not depend on the chosen triangulation of W (which corresponds to a choice of f) or on the choice of g^σ in its gauge orbit. If W is not closed, then we will still have the same formula as above, but, since f has non-zero boundary in this case, the number $\sigma^*\theta(f)$ will now depend on the choice of σ, as well as on the choice of f, that is, of the triangulation. Nevertheless, one may check that any choice would still determine the same map $E_{W,\nu}$.

If we choose the same a_0, a_1 and f for each $\nu \in \mathcal{F}_W^{\alpha_0,\alpha_1}$ then $\mathbb{K}_W(\alpha_0,\alpha_1)$ is described by

$$\mathbb{K}_W(\alpha_0,\alpha_1)(a_0) = \left(\int_{\nu=[\sigma]\in\mathcal{F}_W^{\alpha_0,\alpha_1}}\prod_{t\in T}\omega(g^\sigma_{t,1}, \dots, g^\sigma_{t,n+1})^{\epsilon_t}\, d\mu_W^{\alpha_0,\alpha_1}\right)\, a_1. \tag{5.4}$$

For a closed $n+1$-manifold we get

$$Z_A^{[\theta]}(W) = \int_{\nu=[\sigma]\in\mathcal{F}_W}\prod_{t\in T}\omega(g^\sigma_{t,1}, \dots, g^\sigma_{t,n+1})^{\epsilon_t}\, d\mu_W. \tag{5.5}$$

Note that there is nothing in the above depending on any special property of the group A. As long as a good measure on the space of homotopy classes of based maps $\mathcal{F}_W = [W; K(A,1)]$ is available, the above formulae can be used

[2]Note that if we were only given σ, the procedure described here gives a way of determining a suitable ω.

to calculate the invariants. The reason for restricting to compact abelian Lie groups A is that we have good measures available as already stated in the introduction. Of course for finite A one also has a measure (the counting measure) available in case A is not abelian, and in the state sum approach one actually starts with the above formulae (5.4) and (5.5) for the invariants.

5.3 Dimension $2 + 1$

In this last section we take $A = U(1)$ and at level k we use the group cocycle ω_k defined in (2.1). We will write $Z^k(W)$ to mean $Z^{[\omega_k]}_{U(1)}(W)$ and by the "$U(1)$ homotopy DW-invariants" of a closed 3-manifold W, we mean the collection of numerical invariants $\{Z^k(W)\}_{k\geq 0}$. We will prove the following theorem.

Theorem 5.3. *The $U(1)$ homotopy DW invariants distinguish homotopy equivalence classes of lens spaces.*

Before proving this, let us recall certain facts about lens spaces. Lens spaces are a class of 3-manifolds parametrized by pairs of coprime integers (p, q), the lens space labelled by (p, q) being denoted $L(p, q)$. Since we are interested here in oriented and not only orientable lens spaces a bit of care is needed. Our orientation convention will be the standard one, i.e., $L(p, q)$ is the closed oriented 3-manifold obtained by surgery on S^3 along the unknot with surgery coefficient $-p/q$, where $L(p, q)$ is given the orientation induced by the standard right-handed orientation on S^3. We note that

- The lens spaces $L(p, q)$ and $L(p', q')$ are homeomorphic if and only if p is equal to p' and $q = \pm q' \bmod p$ or $qq' = \pm 1 \bmod p$.
- $L(p, q)$ and $L(p', q')$ are homotopy equivalent if and only if $p = p'$ and $qq' = \pm a^2 \bmod p$ for some integer a.

The first fact was proved by Reidemeister, cf. [13], and the second fact is due to Whitehead [19]. For a more recent source, see for instance [15, 16]. In all cases, the minus sign corresponds to a reversal of the orientation. We will be interested in homotopy classes of lens spaces using only orientation preserving homeomorphisms, since the DW-invariants depend on the orientation (e.g., they can have different values for, say, $L(p, q)$ and $L(p, p - q)$). Therefore, in the rest of the paper, when we say that two lens spaces $L(p, q)$ and $L(p, q')$ are homotopy equivalent, this means that $qq' = +a^2 \bmod p$ for some $a \in \mathbb{Z}$. We note that $L(0, \pm 1) = S^2 \times S^1$ with fundamental group $\mathbb{Z}$. All the abelian homotopy DW-invariants of this manifold are trivial by (4.5)

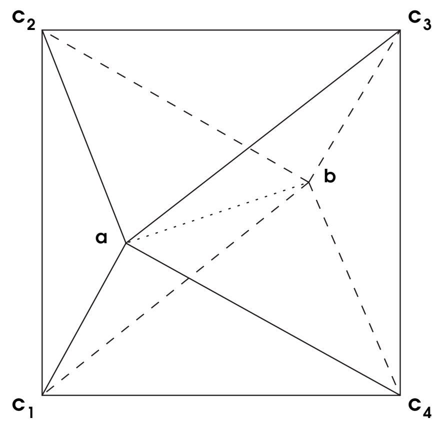

Figure 1: The polyhedron from which $L_{4,1}$ is formed by identification of each face on the front with the next face on the back.

and Example 3.4 (alternatively use Example 4.8). From now on we assume that $p \neq 0$. Note then that the fundamental group of $L(p,q)$ is $\mathbb{Z}/p$ and the other homotopy groups are isomorphic to those of the 3-sphere. Hence, the homotopy groups of a lens space do not determine its homotopy type.

The lens space $L(p,q)$ has a nice triangulation consisting of p tetrahedra with vertices a_i, b_i, c_i and d_i, $i = 1, \ldots, p$, illustrated for $p = 4$ in figure 1. The tetrahedra are first glued together along the abc–faces, i.e., we make the identification $(a_i, b_i, c_i) \equiv (a_{i+1}, b_{i+1}, c_{i+1})$ for all i with the convention that $a_{p+1} = a_1$, etc.

After these identifications there is one point corresponding to all the a_i, which we will call a and there is similarly one point corresponding to the b_i denoted b. To get the lens space $L_{p,q}$ from this polyhedron, one identifies each face on one side with the face which lies q steps clockwise removed on the other side, i.e., one makes the identification $(a, c_i, d_i) \equiv (b, c_{i+q}, d_{i+q})$, again with $c_{p+1} = c_1$, etc. The path ab has now become a loop and one may easily check that it is a generator of the fundamental group. One may number the vertices such that the signs ϵ_t, which occur in the formula for the fundamental cycle, are all positive.

For the U(1) homotopy DW theory the space of fields is

$$\mathcal{F}_{L(p,q)} = H^1(L(p,q); U(1)) = \mathrm{Hom}(\mathbb{Z}/p, U(1)) = \{\zeta \in U(1) \mid \zeta^p = 1\} =: \Lambda_p.$$

Colourings of the above triangulation were studied by Altschuler and Coste [1] (for finite groups which is sufficient here as $\Lambda_p \cong \mathbb{Z}/p$). Given $\nu \in \Lambda_p$ they provide a particularly nice colouring corresponding to ν by colouring the three independent edges (in ascending order) in the j'th tetrahedron t_j with the group elements $\nu, \nu^{j\bar{q}}$ and $\nu^{\bar{q}}$, respectively, where $\bar{q}$ is the inverse of q modulo p. Using (5.5) we then have

$$Z^k(L(p,q)) = \int_{\nu \in \mathcal{F}_{L(p,q)}} \prod_{j=1}^{p} \omega_k(\nu, \nu^{j\bar{q}}, \nu^{\bar{q}})\, d\mu_L = \frac{1}{p} \sum_{\nu \in \Lambda_p} \prod_{j=1}^{p} \omega_k(\nu, \nu^{j\bar{q}}, \nu^{\bar{q}}).$$

For $u \in U(1)$ let $\langle u \rangle$ be the unique number in the interval $[0,1)$ such that $u = e^{2\pi i \langle u \rangle}$. It is easy to see that $\sum_{j=1}^{p} \langle \nu^{j\bar{q}} \rangle = \sum_{j=1}^{p} \langle \nu^{\bar{q}(j+1)} \rangle$ and so we can write

$$\begin{aligned}
\prod_{j=1}^{p} \omega_k(\nu, \nu^{j\bar{q}}, \nu^{\bar{q}}) &= \prod_{j=1}^{p} e^{2\pi i k \langle \nu \rangle (\langle \nu^{j\bar{q}} \rangle + \langle \nu^{\bar{q}} \rangle - \langle \nu^{\bar{q}(j+1)} \rangle)} \\
&= e^{2\pi i k \langle \nu \rangle \sum_{j=1}^{p} (\langle \nu^{j\bar{q}} \rangle + \langle \nu^{\bar{q}} \rangle - \langle \nu^{\bar{q}(j+1)} \rangle)} \\
&= e^{2\pi i k \langle \nu \rangle p \langle \nu^{\bar{q}} \rangle} \\
&= e^{\frac{2\pi i k \bar{q} l^2}{p}},
\end{aligned}$$

where in the last equality we have written $\nu = e^{\frac{2\pi i l}{p}}$ for some $l = 1, \ldots, p$. Thus, we have

$$Z^k(L(p,q)) = \frac{1}{p} \sum_{l=1}^{p} e^{\frac{2\pi i k \bar{q} l^2}{p}}. \tag{5.6}$$

Let us recall formulas for the involved Gauss sums. For r, N relatively prime, let us write

$$G(r,N) := \sum_{l=1}^{N} e^{\frac{2\pi i r l^2}{N}}. \tag{5.7}$$

Dirichlet [5, 6] proved that

$$G(r=1,N) = \begin{cases} (1+i)\sqrt{N}, & N = 0 \bmod 4, \\ \sqrt{N}, & N = 1 \bmod 4, \\ 0, & N = 2 \bmod 4, \\ i\sqrt{N}, & N = 3 \bmod 4. \end{cases} \tag{5.8}$$

Futhermore, when N is an odd prime, there is a closed formula for $G(r,N)$ for all r,

$$G(r,N) = \begin{cases} (\frac{r}{N})\sqrt{N}, & N = 1 \bmod 4, \\ i(\frac{r}{N})\sqrt{N}, & N = 3 \bmod 4, \end{cases} \tag{5.9}$$

where (r/N) is the Legendre symbol for r modulo N, that is, (r/N) equals 1 if r is a square modulo N and -1 otherwise. Before proving Theorem 5.3, we require the following lemma.

Lemma 1. *Let $p = 2^k p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ be the prime decomposition of p (the p_i are odd primes, k is non-negative and the k_i are positive) and consider homotopy classes of lens spaces $L(p,q)$. We distinguish three cases.*

- $k = 0$ or $k = 1$. *There are 2^m homotopy classes which we can label by the string of signs $((\frac{q}{p_1}), \dots, (\frac{q}{p_m}))$.*
- $k = 2$. *There are 2^{m+1} homotopy classes which may be labelled by $q \bmod 4$ and the signs $\left(\frac{q}{p_i}\right)$. (Note that $q \bmod 4$ equals 1 or 3.)*
- $k > 2$. *There are 2^{m+2} homotopy classes labelled by $q \bmod 8$ and the signs $(\frac{q}{p_i})$. (Note that $q \bmod 4$ equals 1, 3, 5 or 7.)*

Proof. Recall that $\mathbb{Z}_p^*$, the multiplication group modulo p, decomposes as

$$\mathbb{Z}_p^* = \mathbb{Z}_{2^k}^* \times \mathbb{Z}_{p_1^{k_1}}^* \times \dots \times \mathbb{Z}_{p_m^{k_m}}^*. \tag{5.10}$$

Hence, if x is an element of $\mathbb{Z}_p^*$ we may write $x = (x_0, x_1, \dots, x_m)$ with $x_i \in \mathbb{Z}_{p_i^{k_i}}^*$ (with $p_0 = 2$, $k_0 = k$). In fact, we can take $x_i = x \bmod p_i^{k_i}$. From this decomposition it is clear that x will be a square modulo p if and only if x is a square modulo $p_i^{k_i}$ for $i = 0, 1, \dots, m$. Furthermore, it is not difficult to show that x is a square modulo 2^k if and only if $x = 1 \bmod 8$ and x is a square modulo $p_i^{k_i}$ if and only if x is a square modulo p_i, $i = 1, \dots, m$. To find the homotopy classes of lens spaces we must therefore find out which elements of $\mathbb{Z}_n^*$ give a square when they are multiplied together, n being any odd prime. Obviously the product of two squares is always a square. Also, using the fact that $\mathbb{Z}_n^*$ is cyclic, one sees that the product of two non-squares is a square in the $\mathbb{Z}_n^*$, while the product of a square and a non-square in $\mathbb{Z}_n^*$ is never a square. Finally we note that two elements multiply to a square in $\mathbb{Z}_{2^k}^*$ only if they are equal modulo the minimum of 8 and 2^k. □

Proof of Theorem 5.3. For any lens space $L(p,q)$ we have from (5.6) that $Z^0(L(p,q)) = 1$. The next value of k for which $Z^k(L(p,q)) = 1$ occurs when $k = p$ (essentially this is the triangle inequality for complex numbers), so this determines p.

Now fix p and write its prime decomposition as in Lemma 1. We need to show that the invariants Z^k determine the labels of the homotopy classes given in that lemma. Let p_i be one of the odd prime factors (if there are no odd prime factors, we only need to determine $q \bmod 4$ or $q \bmod 8$, see further on for that) and consider $k = p/p_i$. Filling in (5.6), we get

$$Z^{p/p_i}(L(p,q)) = \frac{1}{p} \sum_{l=1}^{p} \exp\left(\frac{2\pi i \bar{q} l^2}{p_i}\right) = \frac{1}{p_i} \sum_{l=1}^{p_i} \exp\left(\frac{2\pi i \bar{q} l^2}{p_i}\right) \tag{5.11}$$

and using (5.9), we see that

$$Z^{p/p_i}(L(p,q)) = \begin{cases} \frac{1}{\sqrt{p_i}}\left(\frac{q}{p_i}\right), & p_i = 1 \bmod 4, \\ \frac{i}{\sqrt{p_i}}\left(\frac{q}{p_i}\right), & p_i = 3 \bmod 4. \end{cases} \tag{5.12}$$

Thus these invariants determine the Legendre symbols $\left(\frac{q}{p_i}\right)$. This means they separate homotopy classes of lens spaces with p odd or $p = 2 \bmod 4$, the first case in Lemma 1. To settle the second case ($p = 4 \bmod 8$), we need to determine $q \bmod 4$. This is accomplished by taking $k = p/4$. We have

$$Z^{p/4}(L(p,q)) = \frac{1}{4}\sum_{l=1}^{4} e^{\frac{2\pi i \bar{q} l^2}{4}} = \frac{1}{2}(1 + i^{\bar{q}}) = \begin{cases} \frac{1}{2}(1+i), & q = 1 \bmod 4, \\ \frac{1}{2}(1-i), & q = 3 \bmod 4. \end{cases} \tag{5.13}$$

To deal with the final case ($p = 0 \bmod 8$), we have to determine $q \bmod 8$. This can be done using $k = p/8$:

$$\begin{aligned} Z^{p/8}(L(p,q)) &= \frac{1}{8}\sum_{l=1}^{8} e^{\frac{2\pi i \bar{q} a^2}{8}} \\ &= \frac{1}{4}(1 + (-1)^{\bar{q}} + 2e^{\frac{\pi i \bar{q} a^2}{4}}) = \begin{cases} \frac{1}{2}e^{i\pi/4}, & q = 1 \bmod 8, \\ \frac{1}{2}e^{3i\pi/4}, & q = 3 \bmod 8, \\ \frac{1}{2}e^{5i\pi/4}, & q = 5 \bmod 8, \\ \frac{1}{2}e^{7i\pi/4} & q = 7 \bmod 8. \end{cases} \end{aligned}$$

□

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