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On Calabi–Yau supermanifolds

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Abstract

We prove that a Kähler supermetric on a supermanifold with one complex fermionic dimension admits a super Ricci-flat supermetric, if and only if the bosonic metric has vanishing scalar curvature. As a corollary, it follows that Yau's theorem does not hold for supermanifolds.

Calabi [**1**] proposed that if a Kähler manifold has vanishing first Chern class, that is, the Ricci-form obeys $R_{i\bar{j}}(g) = \partial_i \bar{v}_j - \bar{\partial}_j v_i$ for a globally defined 1-form v, or, equivalently, a complex n-dimensional Kähler manifold has a globally defined holomorphic top form $\Omega_{i_1,\ldots,i_n}$, then there exists a unique metric g' which is a smooth deformation of g and obeys $R_{i\bar{j}}(g') = 0$. Yau [**2**] proved this theorem for ordinary manifolds.

Recently, there has been a lot of interest in Calabi–Yau supermanifolds [**3**–**5**]; though these papers use only the topological properties of such spaces, it is interesting to ask whether they also admit Ricci-flat supermetrics. This paper studies the generalization of Calabi's conjecture to supermanifolds with one complex fermionic dimension. We find that such a Kähler supermanifold admits a Ricci-flat supermetric if and only if the bosonic metric has vanishing scalar curvature. For a given scalar-flat bosonic Kähler metric with Kähler potential K_{Bose}, the super-extension is unique, and has the

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super Kähler potential:

$$K(z^i, \bar{z}^j, \theta, \bar{\theta}) = K_{\text{Bose}}(z^i, \bar{z}^j) + \det\left(\frac{\partial^2}{\partial z^i \partial \bar{z}^j} K_{\text{Bose}}\right)\theta\bar{\theta}. \tag{1.1}$$

As complex projective spaces do not admit scalar-flat metrics, but do admit super Calabi–Yau extensions with one fermionic dimension, it follows that Yau's theorem does not hold for supermanifolds.

A supermanifold is a generalization of a usual manifold with *fermionic* as well as bosonic coordinates.[1] The bosonic coordinates are ordinary numbers, whereas the fermionic coordinates are grassmann numbers. Grassmann numbers are odd elements of a grassmann algebra and anti-commute: $\theta^1\theta^2 = -\theta^2\theta^1$ and $\theta^1\theta^1 = 0$.

On bosonic Kähler manifolds, the Ricci tensor

$$R_{i\bar{j}} = -(\ln\,\det(g))_{,i\bar{j}}. \tag{1.2}$$

For this to vanish, $\ln\det(g))$ (locally) must be the real part of a holomorphic function, and hence, $\det(g)) = |f(z)|^2$ for some holomorphic $f(z)$. This can always be absorbed by a holomorphic coordinate transformation, and hence a Kähler manifold is Ricci-flat if its Kähler potential K obeys the Monge–Ampère equation

$$\det(g) \equiv \det(K_{,i\bar{j}}) = 1. \tag{1.3}$$

On supermanifolds, because elements of g contain grassmann numbers, the determinant is not well-defined and a new definition of the determinant is needed. For any non-degenerate supermatrix

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.4}$$

where A and D are bosonic and B and C are fermionic,

$$\text{sdet}(g) \equiv \frac{\det(A)}{\det(D - CA^{-1}B)} = \frac{\det(A - BD^{-1}C)}{\det(D)}. \tag{1.5}$$

For arbitrary supermatrices X, Y, this definition is consistent with the basic relation $\text{sdet}(XY) = \text{sdet}(X)\,\text{sdet}(Y)$. In addition, the supertrace is defined as

$$\text{str}(g) = \text{tr}(A) - \text{tr}(D), \tag{1.6}$$

[1]More rigorous and technical definitions can be found in the literature (see e.g., [**6**, **7**], but this simple treatment suffices for our results.

which is consistent with $\mathrm{str}(XY) = \mathrm{str}(YX)$. These two definitions imply an identity that is useful in simplifying expressions that use grassmann numbers:

$$\ln \mathrm{sdet}(g) = \mathrm{str}\ln(g) \tag{1.7}$$

Simple examples [**5**] of Kähler supermanifolds are provided by superprojective spaces, $SP(m|n)$. These can be described in terms of $m+n+1$ homogeneous coordinates:

$$(z^1, z^2, \ldots, z^{m+1}|\theta^1, \ldots, \theta^n) \tag{1.8}$$

related by the equivalence relations $z^i \sim \lambda z^i$ and $\theta^i \sim \lambda\theta^i$. There are $m+1$ coordinate patches where $z^i \neq 0$ in the i-th coordinate patch. In the i-th patch, we can introduce inhomogeneous coordinates $\tilde{z}^j = z^j/z^i$. Other examples include weighted superprojective space, $WSP(k_1, \ldots, k_{m+1} |l_1, \ldots, l_n)$; the coordinates are identified under the equivalence relations $z^i \sim \lambda^{k_i} z^i$ and $\theta^i \sim \lambda^{l_i}\theta^i$. A direct calculation of the Ricci-form of the standard Fubini-Study metric reveals that $SP(m|m+1)$ are Calabi–Yau and have a vanishing Ricci-form, whereas $WSP(1, \ldots, 1|m)$ are Calabi–Yau but have a non-vanishing Ricci-form (see below).

We now show that for an arbitrary Kähler space with only one complex fermionic coordinate, $R_{i\bar{j}} = 0$ implies that the bosonic part of the Kähler potential yields a space with a Ricci scalar $s = 0$. Consider an arbitrary super Kähler potential K, on a supermanifold $\mathcal{M}(m|1)$ with one complex fermionic coordinate θ and m bosonic coordinates. The super Kähler potential can be written as $K = f^0 + f^1\theta\bar{\theta}$. We use the convention that holomorphic derivatives are taken from the left and anti-holomorphic derviatives are taken from the right. The supermetric $\mathbf{g}$ is the block matrix

$$\mathbf{g} = \begin{pmatrix} f^0_{,i\bar{j}} + f^1_{,i\bar{j}}\theta\bar{\theta} & f^1_{,i}\theta \\ & \\ f^1_{,\bar{j}}\bar{\theta} & f^1 \end{pmatrix}. \tag{1.9}$$

Its superdeterminant is

$$\begin{aligned} \mathrm{sdet}(\mathbf{g}) &= \frac{\det[f^0_{,i\bar{j}} + (f^1_{,i\bar{j}} - f^1_{,i}f^1_{,\bar{j}}/f^1)\theta\bar{\theta}]}{f^1} \\ &= \frac{\det(f^0_{,i\bar{j}})}{f^1}\ \det\!\left[\delta^k_i + g^{k\bar{j}}\left(f^1_{,i\bar{j}} - \frac{f^1_{,i}f^1_{,\bar{j}}}{f^1}\right)\theta\bar{\theta}\right], \end{aligned} \tag{1.10}$$

where $g^{i\bar{j}} \equiv (f^0_{,i\bar{j}})^{-1}$ is the inverse metric of the bosonic manifold. Using the identity (1.7), we can rewrite this as:

$$\mathrm{sdet}(\mathbf{g}) = \frac{\det(f^0_{,i\bar{j}})}{f^1}\left[1 + g^{i\bar{j}}\left(f^1_{,i\bar{j}} - \frac{f^1_{,i}f^1_{,\bar{j}}}{f^1}\right)\theta\bar{\theta}\right]. \tag{1.11}$$

On a super Ricci-flat manifold, the superdeterminant can be chosen to be 1. The θ-independent term of $\mathrm{sdet}(\mathbf{g}) = 1$ implies

$$f^1 = \det(f^0_{,i\bar{j}}). \tag{1.12}$$

The remaining term must vanish on a super Ricci-flat Kähler manifold. This implies

$$g^{i\bar{j}}\left(f^1_{,i\bar{j}} - \frac{f^1_{,i} f^1_{,\bar{j}}}{f^1}\right) = f^1 g^{i\bar{j}}[\ln(f^1)]_{,i\bar{j}} = 0. \tag{1.13}$$

Substituting (1.12) implies

$$g^{i\bar{j}} \ln \det(f^0_{,l\bar{k}})_{,i\bar{j}} \equiv g^{i\bar{j}} R_{i\bar{j}} = 0, \tag{1.14}$$

which is precisely the Ricci scalar of the bosonic space with Kähler potential f^0. This proves our main result: a Kähler supermanifold with one complex fermionic dimension admits a super Ricci-flat extension, if and only if the bosonic Kähler manifold that it is based on has vanishing scalar curvature s. Many such bosonic manifolds are known and have been studied; (see e.g., [**9**–**11**]; such spaces all admit supermanifolds with Calabi–Yau supermetrics. A simple example is the space $\mathbb{CP}^1 \times \Sigma$, where Σ is a Riemann surface with a metric with constant curvature chosen so that the total scalar curvature vanishes. The super Ricci-flat Kähler potential on such a space is

$$K = \ln(1 + z_1\bar{z}_1) - \ln(1 - z_2\bar{z}_2) + \frac{\theta\bar{\theta}}{(1 + z_1\bar{z}_1)^2(1 - z_2\bar{z}_2)^2}. \tag{1.15}$$

There are many other $s = 0$ metrics which can be studied this way.

A corollary of our result is that there are many Kähler supermanifolds with vanishing first Chern class that do not admit super Ricci-flat supermetrics, thus proving that Yau's theorem does not apply to supermanifolds. Clearly, since no projective space admits an $s = 0$ metric, no supermanifold with one complex fermionic coordinate that is based on projective space admits a super Ricci-flat supermetric. To find our counterexample, it suffices to prove that such supermanifolds may have vanishing first Chern class.

We now consider the explicit example $WSP(1, 1|2)$. The superprojective space $WSP(1, 1|2)$ has a bosonic base which is just $\mathbb{CP}^1$, and $\ln \det(\mathbf{g})$ of the Fubini-Study supermetric is the *globally defined* scalar, $\theta\bar{\theta}/(1 + z\bar{z})^2$. The gradient of this scalar is a globally defined vector that fulfills the conditions of the super Calabi–Yau conjecture. Equivalently, the top form $dz \wedge d\theta$ is a globally defined holomorphic top-form (the superdeterminant of the coordinate transformation $z \to -1/z,\ \theta \to \theta/z^2$ between the two patches that cover $\mathbb{CP}^1$ is 1). As the bosonic part of $WSP(1,1|2)$, $\mathbb{CP}^1$, has no metric with Ricci scalar $s = 0$, this space does not satisfy the super Calabi–Yau conjecture. This result can be generalized to $WSP(\underbrace{1,\ldots,1}_{m}|m)$. These spaces have a globally defined vector on them that fulfill the conditions of the super Calabi–Yau conjecture or equivlently they have globally defined holomorphic top-forms that exist in every coordinate patch. In [8], it is observed that $WSP(1,1|2)$ appears to violate the super Calabi–Yau conjecture, though no explicit proof is given, and it is conjectured that $WSP(1,\ldots,1|m)$ for $m > 2$ will satisfy the conjecture; here we have shown that no $WSP(1,\ldots,1|m)$ admits a super Ricci-flat supermetric.

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